Date: 19/10/2021

Q1. **Verify the following:** $\int \frac{2x-1}{2x+3}\,dx = x - \log|(2x+3)^2| + C.$

Q2. **Evaluate:** $\int \frac{dx}{1+\cos x}.$

Q3. **Evaluate:** $\int \frac{\sin x + \cos x}{\sqrt{1+\sin 2x}}\,dx.$

Q4. **Evaluate:** $\int \sqrt{1+\sin x}\,dx$

Q5. **Evaluate:** $\int \left(\frac{2a}{\sqrt{x}} - \frac{b}{x^2} + 3c\sqrt[3]{x^2}\right)dx.$

Q6. **Verify the following:** $\int \frac{2x+3}{x^2+3x}\,dx = \log|x^2+3x| + C$

Q7. **Verify the following:** $\int \frac{(x^2+2)}{x+1}\,dx$

Q8. **Verify the following:** $\int \frac{(1+\cos x)}{x+\sin x}\,dx.$

Q9. **Verify the following:** $\int \tan^2 x \sec^4 x\,dx$

Q10. **Verify the following:** $\int_0^1 \frac{x}{\sqrt{1+x^2}}\,dx.$

Q11. **Evaluate:** $\int \frac{3ax}{b^2+c^2x^2}\,dx.$

Q12. **Evaluate:** $\int \left(\frac{e^{6\log x} - e^{5\log x}}{e^{4\log x} - e^{3\log x}}\right)dx$

Q13. **Evaluate:** $\int \frac{\sqrt{1+x^2}}{x^4}\,dx.$

Q14. **Verify the following using the concept of integration as an antiderivative.**

$$\int \frac{x^3 dx}{x+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \log|x+1| + C$$

Q15. **Evaluate:** $\int \sqrt{\frac{1+x}{1-x}}\,dx, \quad x \neq 1.$

Q16. **Evaluate:** $\int \tan^8 x \sec^4 x\,dx$

Q17. **Evaluate:** $\int \frac{x}{\sqrt{x}+1}\,dx.$

Q18. Evaluate: $\int \frac{dx}{\sqrt{16-9x^2}}$.

Q19. Evaluate: $\int \frac{dt}{\sqrt{3t-2t^2}}$.

Q20. Evaluate: $\int \frac{3x-1}{\sqrt{x^2+9}}\,dx$.

Q21. Evaluate: $\int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}, \quad \beta > \alpha$.

Q22. Find $\int \frac{dx}{2\sin^2 x + 5\cos^2 x}$.

Q23. Verify the following: $\int \frac{(\cos 5x + \cos 4x)}{1-2\cos 3x}\,dx$.

Q24. Verify the following: $\int e^{\tan^{-1} x}\left(\frac{1+x+x^2}{1+x^2}\right)dx$.

Q25. Evaluate: $\int \frac{x^2\,dx}{x^4+x^2-2}$.

Q26. Verify the following: $\int \sqrt{5-2x+x^2}\,dx$

Q27. Verify the following: $\int \sqrt{2ax-x^2}\,dx$

Q28. Find $\int \sqrt{10-4x+4x^2}\,dx$.

Q29. Verify the following: $\int \frac{x}{x^4-1}\,dx$.

Q30. Verify the following: $\int \frac{x^2}{1-x^4}\,dx$.

Q31. Verify the following: $\int \frac{dx}{x\sqrt{x^4-1}}$.

Q32. Evaluate: $\int \frac{x^3+x}{x^4-9}\,dx$.

Q33. Evaluate: $\int_1^2 \frac{dx}{\sqrt{(x-1)(2-x)}}$.

Q34. Evaluate: $\int_0^{\pi} \frac{x}{1+\sin x}$.

Q35. Evaluate: $\int_{\pi/3}^{\pi/2} \frac{\sqrt{1+\cos x}}{(1-\cos x)^{5/2}}\,dx$.

Q36. Evaluate: $\int_2^8 \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}\,dx$.

Q37. Evaluate $\int_{-1}^{2} f(x)\,dx$, where $f(x) = |x+1| + |x| + |x-1|$.

Q38. Evaluate: $$\int_0^{\frac{\pi}{4}} \sqrt{1+\sin 2x}\, dx .$$

Q39. Evaluate: $$\int \frac{x^{1/2}}{1+x^{3/4}}\, dx .$$

Q40. Verify the following: $$\int \frac{x^2}{x^4-x^2-12}\, dx .$$

Q41. Verify the following: $\int e^{-3x} \cos^3 x\, dx$.

Q42. Evaluate $\int_{-1}^{2}(7x-5)\, dx$ **as a limit of sums.**

Q43. Verify the following: $\int_0^2 e^x\, dx$.

Q44. Verify the following: $\int_0^2 (x^2+3)\, dx$.

Q45. Verify the following: $\int \sqrt{\tan x}\, dx$.

Q46. Verify the following: $$\int \frac{x^2}{(x^2+a^2)(x^2+b^2)}\, dx .$$

Q47. Evaluate: $\int_0^{\pi/2} \frac{\tan x}{1+m^2 \tan^2 x}\, dx$.

Q48. Evaluate: $\int_0^{1/2} \frac{dx}{(1+x^2)\sqrt{1-x^2}}$.

Q49. Evaluate: $\int_0^1 x \log(1+2x)\, dx$.

Q50. Evaluate: $\int_0^{\pi} x \sin x \cos^2 x\, dx$.

Q51. Evaluate: $\int_0^{\pi} x \log \sin x\, dx$.

Q52. Evaluate: $$\int_0^{\frac{\pi}{2}} \frac{\tan^7 x}{\cot^7 x+\tan^7 x}\, dx .$$

Q53. Evaluate: $\int_0^{\pi/2} \frac{dx}{(a^2\cos^2 x+b^2\sin^2 x)^2}$.

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S1. Let,

$$I = \int \frac{2x-1}{2x+3}\,dx = \int \frac{2x+3-3-1}{2x+3}\,dx$$

$$= \int 1dx - 4\int \frac{1}{2x+3}\,dx = x - \int \frac{4}{2\left(x+\frac{3}{2}\right)}\,dx$$

$$= x - 2\log x\left|\left(x+\frac{3}{2}\right)\right| C = x - 2\log\left|\left(\frac{2x+3}{2}\right)\right| + C'$$

$$= x - 2\log|(2x+3)| + 2\log 2\, C' \quad \left[\because \log\frac{m}{n}\log m - \log n\right]$$

$$= x - \log|(2x+3)^2| + C.$$

S2. Let,

$$I = \int \frac{dx}{1+\cos x} = \int \frac{dx}{1+2\cos^2\frac{x}{2}-1}$$

$$= \frac{1}{2}\int \frac{1}{\cos^2\frac{x}{2}}\,dx = \frac{1}{2}\int \sec^2\frac{x}{2}\,dx$$

$$= \frac{1}{2}\cdot\left(\tan\frac{x}{2}\right)\cdot 2 + C = \tan\frac{x}{2} + C \qquad \left[\because \int \sec^2 x\,dx = \tan x\right]$$

S3. Let,

$$I = \int \frac{\sin x + \cos x}{\sqrt{1+\sin 2x}}\,dx$$

$$= \int \frac{(\sin x + \cos x)}{\sqrt{\sin^2 x + \cos^2 x + 2\sin x\cos x}}\,dx$$

$$= \int \frac{\sin x + \cos x}{\sqrt{(\sin x + \cos x)^2}}\,dx = \int 1dx = x + C.$$

S4. Let,

$$I = \int \sqrt{1+\sin x}\,dx$$

$$= \int \sqrt{\sin^2\frac{x}{2} + \cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}\,dx \quad \left[\because \sin^2\frac{x}{2} + \cos^2\frac{x}{2} = 1\right]$$

$$= \int \sqrt{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2}\,dx = \int\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)dx.$$

$$= -\cos\frac{x}{2}\cdot 2 + \sin\frac{x}{2}\cdot 2 + C = -2\cos\frac{x}{2} + 2\sin\frac{x}{2} + C.$$

S5. $$\int\left(\frac{2a}{\sqrt{x}} - \frac{b}{x^2} + 3c\sqrt[3]{x^2}\right) = \int 2a(x)^{\frac{-1}{2}}dx - \int bx^{-2}dx + \int 3cx^{\frac{2}{3}}dx$$

$$= 4a\sqrt{x} + \frac{b}{x} + \frac{9cx^{\frac{2}{3}}}{5} + C.$$

S6. Let, $$I = \int\frac{2x+3}{x^2+3x}dx$$

Put, $x^2 + 3x = t$

$\Rightarrow$ $(2x + 3)\,dx = dt$

$\therefore$ $$I = \int\frac{1}{t}dt = \log|t| + C$$

$$= \log|(x^2 + 3x)| + C.$$

S7. Let, $$I = \int\frac{x^2+2}{x+1}dx$$

$$= \int\left(x - 1 + \frac{3}{x+1}\right)dx$$

$$= \int(x-1)dx + 3\int\frac{1}{x+1}dx$$

$$= \frac{x^2}{2} - x + 3\log|(x+1)| + C$$

S8. Consider that, $$I = \int\frac{(1+\cos x)}{(x+\sin x)}dx$$

Let $x + \sin x = t \;\Rightarrow\; (1 + \cos x)\,dx = dt$

$\therefore$ $$I = \int\frac{1}{t}dt = \log|t| + C$$

$$= \log|(x + \sin x)| + C$$

S9. Let, $$I = \int\tan^2 x\sec^4 x\,dx$$

Put, $\tan x = t \;\Rightarrow\; \sec^2 x\,dx = dt$

$\therefore$ $$I = \int t^2(1+t^2)\,dt$$

$$= \int (t^2 + t^4)\, dt$$

$$= \frac{t^3}{3} + \frac{t^5}{5} + C$$

$$= \frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} + C.$$

S10. Let, $$I = \int_0^1 \frac{x}{\sqrt{1+x^2}}\, dx$$

Put $1 + x^2 = t^2$

$\Rightarrow$ $2x\, dx = 2t\, dt$

$\Rightarrow$ $x\, dx = t\, dt$

$\therefore$ $$I = \int_1^{\sqrt{2}} \frac{t\, dt}{t}$$

$$= [t]_1^{\sqrt{2}} = \sqrt{2} - 1.$$

S11. Let, $v = b^2 + c^2x^2$, then $dv = 2c^2x dx$

Therefore, $$\int \frac{3ax}{b^2 + c^2x^2}\, dx = \frac{3a}{2c^2} \int \frac{dv}{v}$$

$$= \frac{3a}{2c^2} \log |b^2 + c^2x^2| + C.$$

S12. Let, $$I = \int \left(\frac{e^{6 \log x} - e^{5 \log x}}{e^{4 \log x} - e^{3 \log x}} \right) dx$$

$$= \int \left(\frac{e^{\log x^6} - e^{\log x^5}}{e^{\log x^4} - e^{\log x^3}} \right) dx$$ [$\because$ $a \log b = \log b^a$]

$$= \int \left(\frac{x^6 - x^5}{x^4 - x^3} \right) dx$$ [$\because$ $e^{\log x} = x$]

$$= \int \left(\frac{x^3 - x^2}{x - 1} \right) dx$$

$$= \int \frac{x^2(x-1)}{x-1}\, dx$$

$$= \int x^2 dx = \frac{x^3}{3} + C$$

S13. Let,

$$I = \int \frac{\sqrt{1+x^2}}{x^4}\,dx = \int \frac{\sqrt{1+x^2}}{x}\cdot\frac{1}{x^3}\,dx$$

$$= \int \sqrt{\frac{1+x^2}{x^2}}\cdot\frac{1}{x^3}\,dx = \int \sqrt{\frac{1}{x^2}+1}\cdot\frac{1}{x^3}\,dx$$

Put, $1+\frac{1}{x^2} = t^2 \quad \Rightarrow \quad \frac{-2}{x^3}\,dx = 2t\,dt$

$\Rightarrow$ $-\frac{1}{x^3} = t\,dt$

$\therefore$ $I = -\int t^2\,dt = -\frac{t^3}{3}+C = -\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{3/2}+C.$

S14. $\frac{d}{dx}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\log|x+1|+C\right) = 1-\frac{2x}{2}+\frac{3x^2}{3}-\frac{1}{x+1}$

$$= 1-x+x^2-\frac{1}{x+1} = \frac{x^3}{x+1}$$

Thus, $\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\log|x+1|+C\right) = \int \frac{x^3}{x+1}\,dx.$

S15. Let $I = \int \sqrt{\frac{1+x}{1-x}}\,dx$

$$= \int \frac{1}{\sqrt{1-x^2}}\,dx + \int \frac{x\,dx}{\sqrt{1-x^2}} = \sin^{-1}x + I_1$$

where $I_1 = \int \frac{x\,dx}{\sqrt{1-x^2}}$

Put $1-x^2 = t^2 \Rightarrow -2x\,dx = 2t\,dt.$

Therefore, $I_1 = -\int dt = -t+C = -\sqrt{1-x^2}+C$

Hence, $I = \sin^{-1}x - \sqrt{1-x^2}+C.$

S16. Let $I = \int \tan^8 x \sec^4 x\,dx$

$$= \int \tan^8 x\,(\sec^2 x)\sec^2 x\,dx$$

$$= \int \tan^8 x\,(\tan^2 x+1)\sec^2 x\,dx$$

$$= \int \tan^{10} x \sec^2 x\,dx + \int \tan^8 x \sec^2 x\,dx$$

$$= \frac{\tan^{11}x}{11}+\frac{\tan^9 x}{9}+C$$

S17. Let,
$$I = \int \frac{x}{\sqrt{x}+1}\,dx$$

Put,
$$\sqrt{x} = t \quad \Rightarrow \quad \frac{1}{2\sqrt{x}}\,dx = dt$$

$\Rightarrow$
$$dx = 2\sqrt{x}\,dt$$

$\therefore$
$$I = 2\int\left(\frac{x\sqrt{x}}{t+1}\right)dt = 2\int\frac{t^2\cdot t}{t+1}\,dt = 2\int\frac{t^3}{t+1}\,dt$$

$$= 2\int\frac{(t^3+1-1)}{t+1}\,dt - 2\int\frac{(t+1)(t^2-t+1)}{t+1}\,dt - 2\int\frac{1}{t+1}\,dt$$

$$= 2\int(t^2-t+1)\,dt - 2\int\frac{1}{t+1}\,dt$$

$$= 2\left[\frac{t^3}{3}-\frac{t^2}{2}+t-\log|(t+1)|\right]+C$$

$$= 2\left[\frac{x\sqrt{x}}{3}-\frac{x}{2}+\sqrt{x}-\log|(\sqrt{x}+1)|\right]+C.$$

S18. Let,
$$I = \int\frac{dx}{\sqrt{16-9x^2}}$$

$$= \int\frac{dx}{\sqrt{(4)^2-(3x)^2}}$$

$$= \frac{1}{3}\sin^{-1}\left(\frac{3x}{4}\right)+C$$

S19. Let,
$$I = \int\frac{dt}{\sqrt{3t-2t^2}} = \frac{1}{\sqrt{2}}\int\frac{dt}{\sqrt{-\left(t^2-\frac{3}{2}t\right)}}$$

$$= \frac{1}{\sqrt{2}}\int\frac{dt}{\sqrt{-\left[\left(t^2-2\cdot\frac{1}{2}\cdot\frac{3}{2}t\right)+\left(\frac{3}{4}\right)^2-\left(\frac{3}{4}\right)^2\right]}}$$

$$= \frac{1}{\sqrt{2}}\int\frac{dt}{\sqrt{-\left[\left(t-\frac{3}{4}\right)^2-\left(\frac{3}{4}\right)^2\right]}}$$

$$= \frac{1}{\sqrt{2}}\int\frac{dt}{\sqrt{\left(\frac{3}{4}\right)^2-\left(t-\frac{3}{4}\right)^2}}$$

$$= \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{t-\frac{3}{4}}{\frac{3}{4}}\right) + C = \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{4t-3}{3}\right) + C.$$

S20. Let,
$$I = \int \frac{3x-1}{\sqrt{x^2+9}}\, dx$$
$$= \int \frac{3x}{\sqrt{x^2+9}}\, dx - \int \frac{1}{\sqrt{x^2+9}}\, dx$$
$$I = I_1 - I_2$$

Now,
$$I_1 = \int \frac{3x}{\sqrt{x^2+9}}$$

Put,
$$x^2 + 9 = t^2 \Rightarrow 2x\,dx = 2t\,dt \Rightarrow x\,dx = t\,dt$$

$\therefore$
$$I_1 = 3\int \frac{t}{t}\, dt$$
$$= 3\int dt = 3t + C_1 = 3\sqrt{x^2+9} + C_1$$

and
$$I_2 = \int \frac{1}{\sqrt{x^2+9}}\, dx = \int \frac{1}{\sqrt{x^2+(3)^2}}\, dx$$
$$= \log|x + \sqrt{x^2+9}| + C_2$$
$$I = 3\sqrt{x^2+9} + C_1 - \log|x+\sqrt{x^2+9}| - C_2$$
$$= 3\sqrt{x^2+9} - \log|x+\sqrt{x^2+9}| + C \qquad [\because C = C_1 - C_2]$$

S21. Put, $x - \alpha = t^2$. Then $\beta - x = \beta - (t^2 + \alpha) = \beta - t^2 - \alpha = -t^2 - \alpha + \beta$ and $dx = 2tdt$. Now,
$$I = \int \frac{2t\,dt}{\sqrt{t^2(\beta-\alpha-t^2)}}$$
$$= \int \frac{2dt}{\sqrt{(\beta-\alpha-t^2)}}$$
$$= 2\int \frac{dt}{\sqrt{k^2-t^2}}$$

where, $k^2 = \beta - \alpha$
$$= 2\sin^{-1}\frac{t}{k} + C = 2\sin^{-1}\sqrt{\frac{x-\alpha}{\beta-\alpha}} + C.$$

S22. Dividing numerator and denominator by $\cos^2 x$, we have
$$I = \int \frac{\sec^2 x\, dx}{2\tan^2 x + 5}$$

Put, $\tan x = t$ so that $\sec^2 x\, dx = dt$. Then

$$I = \int \frac{dt}{2t^2+5} = \frac{1}{2}\int \frac{dt}{t^2 + \left(\sqrt{\frac{5}{2}}\right)^2}$$

$$= \frac{1}{2}\frac{\sqrt{2}}{\sqrt{5}}\tan^{-1}\left(\frac{\sqrt{2}\,t}{\sqrt{5}}\right) + C$$

$$= \frac{1}{\sqrt{10}}\tan^{-1}\left(\frac{\sqrt{2}\tan x}{\sqrt{5}}\right) + C.$$

S23. Let,

$$I = \int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x}\,dx$$

$$= \int \frac{2\cos\frac{9x}{2}\cdot\cos\frac{x}{2}}{1 - 2\left(2\cos^2\frac{3x}{2} - 1\right)}\,dx$$

$$\left[\because\ \cos C + \cos D = 2\cos\frac{C+D}{2}\cdot\cos\frac{C-D}{2} \text{ and } \cos 2x = 2\cos^2 x - 1\right]$$

$\therefore$

$$I = \int \frac{2\cos\frac{9x}{2}\cdot\cos\frac{x}{2}}{3 - 4\cos^2\frac{3x}{2}}\,dx = -\int \frac{2\cos\frac{9x}{2}\cdot\cos\frac{x}{2}}{4\cos^2\frac{3x}{2} - 3}\,dx$$

$$= -\int \frac{2\cos\frac{9x}{2}\cdot\cos\frac{x}{2}\cdot\cos\frac{3x}{2}}{4\cos^3\frac{3x}{2} - 3\cos\frac{3x}{2}}\,dx \qquad \left[\text{Multiply and divide by } \cos\frac{3x}{2}\right]$$

$$= -\int \frac{2\cos\frac{9x}{2}\cdot\cos\frac{x}{2}\cdot\cos\frac{3x}{2}}{\cos 3\cdot\frac{3x}{2}}\,dx$$

$$= -\int 2\cos\frac{3x}{2}\cdot\cos\frac{x}{2}\,dx$$

$$= -\int\left\{\cos\left(\frac{3x}{2}+\frac{x}{2}\right) + \cos\left(\frac{3x}{2}-\frac{x}{2}\right)\right\}dx$$

$$= -\int(\cos 2x + \cos x)\,dx$$

$$= -\left[\frac{\sin 2x}{2} + \sin x\right] + C$$

$$= -\frac{1}{2}\sin 2x - \sin x + C.$$

S24. Let
$$I = \int e^{\tan^{-1} x}\left(\frac{1+x+x^2}{1+x^2}\right)dx$$
$$= \int e^{\tan^{-1} x}\left(\frac{1+x^2}{1+x^2}+\frac{x}{1+x^2}\right)dx$$
$$= \int e^{\tan^{-1} x}\,dx + \int \frac{xe^{\tan^{-1} x}}{1+x^2}\,dx$$
$$I = I_1 + I_2 \qquad \ldots \text{(i)}$$

Now, $$I_2 = \int \frac{xe^{\tan^{-1} x}}{1+x^2}\,dx$$

Put $\tan^{-1} x = t \;\Rightarrow\; x = \tan t$

$\Rightarrow$ $$\frac{1}{1+x^2}\,dx = dt$$

$\therefore$ $$I_2 = \int \underset{\text{I}}{\tan t}\cdot \underset{\text{II}}{e^t}\,dt$$
$$= \tan t\cdot e^t - \int \sec^2 t\cdot e^t dt + C$$
$$= \tan t\cdot e^t - \int (1+\tan^2 t)\,e^t dt + C \qquad [\because\ \sec^2\theta = 1+\tan^2\theta]$$
$$I_2 = \tan t\cdot e^t - \int (1+x^2)\cdot\frac{e^{\tan^{-1}x}}{1+x^2}\,dx + C$$
$$I_2 = \tan t\cdot e^t - \int e^{\tan^{-1}x}\,dx + C$$

$\therefore$ $$I = \int e^{\tan^{-1}x}\,dx + \tan t\cdot e^t - \int e^{\tan^{-1}x}\,dx + C$$
$$= \tan t\cdot e^t + C$$
$$= xe^{\tan^{-1} x} + C.$$

S25. Let $x^2 = t$. Then
$$\frac{x^2}{x^4+x^2-2} = \frac{t}{t^2+t-2} = \frac{t}{(t+2)(t-1)} = \frac{A}{t+2}+\frac{B}{t-1}$$

So, $$t = A(t-1) + B(t+2)$$

Comparing coefficients, we get
$$A = \frac{2}{3},\quad B = \frac{1}{3}$$

So, $$\frac{x^2}{x^4+x^2-2} = \frac{2}{3}\frac{1}{x^2+2}+\frac{1}{3}\frac{1}{x^2-1}$$

Therefore, $$\int\frac{x^2}{x^4+x^2-2}\,dx = \frac{2}{3}\int\frac{1}{x^2+2}\,dx+\frac{1}{3}\int\frac{dx}{x^2-1}$$

$$= \frac{2}{3}\frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}+\frac{1}{6}\log\left|\frac{x-1}{x+1}\right|+C.$$

S26. Let, $$I = \int\sqrt{5-2x+x^2}\,dx = \int\sqrt{x^2\ -2x\ +1\ +4}\,dx$$

$$= \int\sqrt{(x-1)^2+(2)^2}\,dx = \int\sqrt{(2)^2\ +(x-1)^2}\,dx$$

$$= \frac{x-1}{2}\sqrt{2^2+(x-1)^2}\,dx+2\log|x-1+\sqrt{2^2\ +(x-1)^2}\,|+C$$

$$= \frac{x-1}{2}\sqrt{5-2x+x^2}\ +2\log|x-1+\sqrt{5-2x+x^2}\,|+C.$$

S27. Let, $$I = \int\sqrt{2ax-x^2}\,dx = \int\sqrt{-(x^2-2ax)}\,dx$$

$$= \int\sqrt{-(x^2-2ax+a^2-a^2)}\,dx = \int\sqrt{-\{(x-a)^2-a^2\}}\,dx$$

$$= \int\sqrt{a^2\ -(x-a)^2}\,dx$$

$$= \frac{x-a}{2}\sqrt{a^2-(x-a)^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right)+C$$

$$= \frac{x-a}{2}\sqrt{2ax-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right)+C.$$

S28. We have $$I = \int\sqrt{10-4x+4x^2}\,dx$$

$$= \int\sqrt{(2x-1)^2+(3)^2}\,dx$$

Put $t = 2x - 1$, then $dt = 2dx$.

Therefore, $$I = \frac{1}{2}\int\sqrt{t^2+(3)^2}\,dt$$

$$= \frac{1}{2}t\frac{\sqrt{t^2+9}}{2}+\frac{9}{4}\log\left|t+\sqrt{t^2+9}\right|+C$$

$$= \frac{1}{4}(2x-1)\sqrt{(2x-1)^2+9}+\frac{9}{4}\log\left|(2x-1)+\sqrt{(2x-1)^2+9}\right|+C.$$

S29. Let, $$I = \int \frac{x}{x^4 - 1}\,dx$$

Put $$x^2 = t \;\Rightarrow\; 2x\,dx = dt \;\Rightarrow\; x\,dx = \frac{1}{2}\,dt$$

$\therefore$ $$I = \frac{1}{2}\int \frac{dt}{t^2 - 1} = \frac{1}{2}\cdot\frac{1}{2}\log\left|\frac{t-1}{t+1}\right| + C \qquad \left[\because \int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C\right]$$

$$= \frac{1}{4}\left[\log|x^2 - 1| - \log|x^2 + 1|\right] + C$$

S30. Let, $$I = \int \frac{x^2}{1 - x^4}\,dx$$

$$= \int \frac{\left(\frac{1}{2} + \frac{x^2}{2} - \frac{1}{2} + \frac{x^2}{2}\right)}{(1 - x^2)(1 + x^2)}\,dx \qquad [\because a^2 - b^2 = (a + b)(a - b)]$$

$$= \int \frac{\frac{1}{2}(1 + x^2) - \frac{1}{2}(1 - x^2)}{(1 - x^2)(1 + x^2)}\,dx$$

$$= \int \frac{\frac{1}{2}(1 + x^2)}{(1 - x^2)(1 + x^2)}\,dx - \frac{1}{2}\int \frac{(1 - x^2)}{(1 - x^2)(1 + x^2)}\,dx$$

$$= \frac{1}{2}\int \frac{1}{1 - x^2}\,dx - \frac{1}{2}\int \frac{1}{1 + x^2}\,dx$$

$$= \frac{1}{2}\cdot\frac{1}{2}\log\left|\frac{1 + x}{1 - x}\right| + C_1 - \frac{1}{2}\tan^{-1}x + C_2$$

$$= \frac{1}{4}\log\left|\frac{1 + x}{1 - x}\right| - \frac{1}{2}\tan^{-1}x + C. \qquad [\because C = C_1 + C_2]$$

S31. Let, $$I = \int \frac{dx}{x\sqrt{x^4 - 1}}$$

Put, $$x^2 = \sec\theta \;\Rightarrow\; \theta = \sec^{-1}x^2$$

$\Rightarrow$ $$2x\,dx = \sec\theta\cdot\tan\theta\,d\theta$$

$\therefore$ $$I = \frac{1}{2}\int \frac{\sec\theta\cdot\tan\theta}{\sec\theta\tan\theta}\,d\theta = \frac{1}{2}\int d\theta = \frac{1}{2}\theta + C$$

$$= \frac{1}{2}\sec^{-1}(x^2) + C.$$

S32. We have, $I = \int \frac{x^3 + x}{x^4 - 9}\,dx = \int \frac{x^3}{x^4 - 9}\,dx + \frac{x\,dx}{x^4 - 9} = I_1 + I_2$

Now, $I_1 = \int \frac{x^3}{x^4 - 9}\,dx$

Put $t = x^4 - 9$ so that $4x^3dx = dt$

Therefore $I_1 = \frac{1}{4}\int \frac{dt}{t} = \frac{1}{4}\log|t| + C_1 = \frac{1}{4}\log|x^4 - 9| + C_1$

Again, $I_2 = \int \frac{x\,dx}{x^4 - 9}$.

Put, $x^2 = u$ so that $2x\,dx = du$

Then, $I_2 = \frac{1}{2}\int \frac{du}{u^2 - (3)^2} = \frac{1}{2 \times 6}\log\left|\frac{u - 3}{u + 3}\right| + C_2$

$$= \frac{1}{12}\log\left|\frac{x^2 - 3}{x^2 + 3}\right| + C_2$$

Thus, $I = I_1 + I_2$

$$= \frac{1}{4}\log\left|x^4 - 9\right| + \frac{1}{12}\log\left|\frac{x^2 - 3}{x^2 + 3}\right| + C.$$

S33. Let, $I = \int_1^2 \frac{dx}{\sqrt{(x - 1)(2 - x)}}$

$$= \int_1^2 \frac{dx}{\sqrt{-(x^2 - 3x + 2)}}$$

$$= \int_1^2 \frac{dx}{\sqrt{-\left[x^2 - 2 \cdot \frac{3}{2}x + \left(\frac{3}{2}\right)^2 + 2 - \frac{9}{4}\right]}}$$

$$= \int_1^2 \frac{dx}{\sqrt{-\left\{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right\}}}$$

$$= \int_1^2 \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2}} = \left[\sin^{-1}\left(\frac{x - \frac{3}{2}}{\frac{1}{2}}\right)\right]_1^2$$

$$= \left[\sin^{-1}(2x - 3)\right]_1^2 = \sin^{-1}1 - \sin^{-1}(-1)$$

$$= \frac{\pi}{2} + \frac{\pi}{2} \qquad \left[\because \sin\frac{\pi}{2} = 1 \text{ and } \sin(-\theta) = -\sin\theta\right]$$

$$= \pi.$$

S34. Let,

$$I = \int_0^{\pi} \frac{x}{1+\sin x}\,dx \qquad \dots \text{(i)}$$

$$= \int_0^{\pi} \frac{\pi - x}{1+\sin(\pi - x)}\,dx$$

$$= \int_0^{\pi} \frac{\pi - x}{1+\sin x}\,dx \qquad \dots \text{(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2I = \pi \int_0^{\pi} \frac{1}{1+\sin x}\,dx$$

$$= \pi \int_0^{\pi} \frac{(1-\sin x)\,dx}{(1+\sin x)(1-\sin x)}$$

$$= \pi \int_0^{\pi} \frac{(1-\sin x)\,dx}{\cos^2 x}$$

$$= \pi \int_0^{\pi} (\sec^2 x - \tan x \cdot \sec x)\,dx$$

$$= \pi \int_0^{\pi} \sec^2 x\,dx - \pi \int_0^{\pi} \sec x \cdot \tan x\,dx$$

$$= \pi\left[\tan x\right]_0^{\pi} - \pi\left[\sec x\right]_0^{\pi}$$

$$= \pi\left[\tan x - \sec x\right]_0^{\pi}$$

$$= \pi\,[\tan\pi - \sec\pi - \tan 0 - \sec 0]$$

$\Rightarrow$ $$2I = \pi\,[0 + 1 - 0 + 1]$$

$$2I = 2\pi$$

$\therefore$ $$I = \pi.$$

S35.

Let

$$I = \int_{\pi/3}^{\pi/2} \frac{\sqrt{1+\cos x}}{(1-\cos x)^{5/2}}\,dx = \int_{\pi/3}^{\pi/2} \frac{\left(2\cos^2\frac{x}{2}\right)^{1/2}}{\left(2\sin^2\frac{x}{2}\right)^{5/2}}\,dx$$

$$= \frac{\sqrt{2}}{4\sqrt{2}} \int_{\pi/3}^{\pi/2} \frac{\cos\left(\frac{x}{2}\right)}{\sin^5\left(\frac{x}{2}\right)}\,dx = \frac{1}{4}\int_{\pi/3}^{\pi/2} \frac{\cos\left(\frac{x}{2}\right)}{\sin^5\left(\frac{x}{2}\right)}\,dx$$

Put $\sin\frac{x}{2} = t$

$\Rightarrow$ $\cos\frac{x}{2}\cdot\frac{1}{2}dx = dt$

$\Rightarrow$ $\cos\frac{x}{2}dx = 2dt$

As $x \to \frac{\pi}{3}$, then $t \to \frac{1}{2}$

and $x \to \frac{\pi}{2}$, then $t \to \frac{1}{\sqrt{2}}$

$$\therefore \quad I = \frac{2}{4}\int_{1/2}^{1/\sqrt{2}}\frac{dt}{t^5} = \frac{1}{2}\left[\frac{t^{-5+1}}{-5+1}\right]_{1/2}^{1/\sqrt{2}}$$

$$= -\frac{1}{8}\left[\frac{1}{\left(\frac{1}{\sqrt{2}}\right)^4} - \frac{1}{\left(\frac{1}{2}\right)^4}\right]$$

$$= -\frac{1}{8}(4-16) = \frac{12}{8} = \frac{3}{2}.$$

Note: If we integrate the trigonometric function in different ways [using different identities] then, we can get different answers.

S36. We have $$I = \int_2^8 \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}dx \qquad \text{... (i)}$$

$$= \int_2^8 \frac{\sqrt{10-(10-x)}}{\sqrt{10-x}+\sqrt{10-(10-x)}}dx \qquad \left[\because \int_a^b f(x)dx = \int_a^b f(a+b-x)\,dx\right]$$

$$\Rightarrow \quad I = \int_2^8 \frac{\sqrt{x}}{\sqrt{10-x}+\sqrt{x}}dx \qquad \text{... (ii)}$$

Adding Eq. (i) and (ii), we get

$$2I = \int_2^8 1\,dx = 8-2 = 6$$

Hence, $I = 3$.

S37. We can redefine f as

$$f(x) = \begin{cases} 2-x, & \text{if } -1 < x \le 0 \\ x+2, & \text{if } 0 < x \le 1 \\ 3x, & \text{if } 1 < x \le 2 \end{cases}$$

Therefore, $$\int_{-1}^{2} f(x)\,dx = \int_{-1}^{0}(2-x)\,dx + \int_{0}^{1}(x+2)\,dx + \int_{1}^{2}3x\,dx$$

$$= \left(2x - \frac{x^2}{2}\right)_{-1}^{0} + \left(\frac{x^2}{2}+2x\right)_{0}^{1} + \left(\frac{3x^2}{2}\right)_{1}^{2}$$

$$= 0 - \left(-2-\frac{1}{2}\right) + \left(\frac{1}{2}+2\right) + 3\left(\frac{4}{2}-\frac{1}{2}\right)$$

$$= \frac{5}{2}+\frac{5}{2}+\frac{9}{2} = \frac{19}{2}$$

S38. We have $$I = \int_{0}^{\frac{\pi}{4}} \sqrt{1+\sin 2x}\,dx$$

$$= \int_{0}^{\frac{\pi}{4}} \sqrt{(\sin x + \cos x)^2}\,dx$$

$$= \int_{0}^{\frac{\pi}{4}} (\sin x + \cos x)\,dx$$

$$= \left| -\cos x + \sin x \right|_{0}^{\frac{\pi}{4}}$$

Hence, $I = 1$.

S39. Let, $$I = \int \frac{x^{1/2}}{1+x^{3/4}}\,dx$$

Put $x = t^4 \;\Rightarrow\; dx = 4t^3dt$

$$I = 4\int \frac{t^2(t^3)}{1+t^3}\,dt = 4\int\left(t^2 - \frac{t^2}{1+t^3}\right)dt$$

$$I = 4\int t^2dt - 4\int \frac{t^2}{1+t^3}\,dt$$

$$I = I_1 - I_2$$

$$I_1 = 4\int t^2dt = 4\cdot\frac{t^3}{3} + C_1 = \frac{4}{3}x^{3/4} + C_1$$

Now, $$I_2 = 4\int \frac{t^2}{1+t^3}\,dt$$

Again, put $1 + t^3 = z \;\Rightarrow\; 3t^2dt = dz$

$\Rightarrow$ $t^2dt = \frac{1}{3}dz = \frac{4}{3}\int\frac{1}{z}dz$

$= \frac{4}{3}\log|z| + C_2 = \frac{4}{3}\log|(1+t^3)| + C_2$

$= \frac{4}{3}\log|(1+x^{3/4})| + C_2$

$\therefore$ $I = \frac{4}{3}x^{3/4} + C_1 - \frac{4}{3}\log|(1+x^{3/4})| - C_2$

$= \frac{4}{3}\left\{x^{3/4} - \log|(1+x^{3/4})|\right\} + C.$ $[\because\ C = C_1 - C_2]$

S40. Let $I = \int\frac{x^2}{x^4 - x^2 - 12}dx$

$= \int\frac{x^2}{x^4 - 4x^2 + 3x^2 - 12}dx$

$= \int\frac{x^2dx}{x^2(x^2-4) + 3(x^2-4)}$

$= \int\frac{x^2dx}{(x^2-4)(x^2+3)}$

Now, $\frac{x^2}{(x^2-4)(x^2+3)}$ [Let $x^2 = t$]

$\Rightarrow$ $\frac{t}{(t-4)(t+3)} = \frac{A}{t-4} + \frac{B}{t+3}$

$\Rightarrow$ $t = A(t+3) + B(t-4)$

On comparing the coefficient of t on both sides, we get

$A + B = 1$ ... (i)

$\Rightarrow$ $3A - 4B = 0$ ... (ii)

$\Rightarrow$ $3(1-B) - 4B = 0$

$\Rightarrow$ $3 - 3B - 4B = 0$

$\Rightarrow$ $7B = 3$

$\Rightarrow$ $B = \frac{3}{7}$

If $B = \frac{3}{7}$, then $A + \frac{3}{7} = 1$

$\Rightarrow$ $A = 1 - \frac{3}{7} = \frac{4}{7}$

$$\frac{x^2}{(x^2-4)(x^2+3)} = \frac{4}{7(x^2-4)} + \frac{3}{7(x^2+3)}$$

$\therefore$
$$I = \frac{4}{7}\int\frac{1}{x^2-(2)^2}\,dx + \frac{3}{7}\int\frac{1}{x^2+(\sqrt{3})^2}\,dx$$

$$= \frac{4}{7}\cdot\frac{1}{2\cdot 2}\log\left|\frac{x-2}{x+2}\right| + \frac{3}{7}\cdot\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + C$$

$$= \frac{1}{7}\log\left|\frac{x-2}{x+2}\right| + \frac{\sqrt{3}}{7}\tan^{-1}\frac{x}{\sqrt{3}} + C.$$

S41. Let
$$I = \int \underset{\text{II}}{e^{-3x}}\ \underset{\text{I}}{\cos^3 x}\,dx$$

$$= \cos^3 x\int e^{-3x}\,dx - \int\left(\frac{d}{dx}\cos^3 x\int e^{-3x}\,dx\right)dx$$

$$= \cos^3 x\cdot\frac{e^{-3x}}{-3} - \int(-3\cos^2 x)\sin x\cdot\frac{e^{-3x}}{-3}\,dx$$

$$= -\frac{1}{3}\cos^3 xe^{-3x} - \int\cos^2 x\sin xe^{-3x}\,dx$$

$$= -\frac{1}{3}\cos^3 xe^{-3x} - \int(1-\sin^2 x)\sin xe^{-3x}\,dx$$

$$= -\frac{1}{3}\cos^3 xe^{-3x} - \int\sin xe^{-3x}\,dx + \int\underset{\text{I}}{\sin^3 x}\,\underset{\text{II}}{e^{-3x}}\,dx$$

$$= -\frac{1}{3}\cos^3 xe^{-3x} - \int\sin xe^{-3x}\,dx + \sin^3 x\cdot\frac{e^{-3x}}{-3} - \int 3\sin^2 x\cos x\cdot\frac{e^{-3x}}{-3}\,dx$$

$$= -\frac{1}{3}\cos^3 xe^{-3x} - \int\sin xe^{-3x}\,dx - \frac{1}{3}\sin^3 xe^{-3x} + \int(1-\cos^2 x)\cos xe^{-3x}\,dx$$

$$I = -\frac{1}{3}\cos^3 xe^{-3x} - \int\underset{\text{I}}{\sin x}\,\underset{\text{II}}{e^{-3x}} - \frac{1}{3}\sin^3 xe^{-3x} + \int\cos xe^{-3x}\,dx - \int\cos^3 xe^{-3x}\,dx$$

$$2I = \frac{e^{-3x}}{3}[\cos^3 x + \sin^3 x] - \left[\sin x\cdot\frac{e^{-3x}}{-3} - \int\cos x\cdot\frac{e^{-3x}}{-3}\,dx\right] + \int\cos xe^{-3x}\,dx$$

$$2I = \frac{e^{-3x}}{-3}[\cos^3 x + \sin^3 x] + \frac{1}{3}\sin x\cdot e^{-3x} - \frac{1}{3}\int\cos x\cdot e^{-3x}dx + \int\cos x\,e^{-3x}dx$$

$$2I = \frac{e^{-3x}}{-3}[\cos^3 x + \sin^3 x] + \frac{1}{3}\sin xe^{-3x} + \frac{2}{3}\int\cos xe^{-3x}\,dx$$

Now, let $I_1 = \int \underset{\text{I}}{\cos x}\, \underset{\text{II}}{e^{-3x}} dx$

$$I_1 = \cos x \cdot \frac{e^{-3x}}{-3} - \int (-\sin x) \cdot \frac{e^{-3x}}{-3}\, dx$$

$$I_1 = \frac{-1}{3} \cos x \cdot e^{-3x} - \frac{1}{3} \int \sin x \cdot e^{-3x}\, dx$$

$$= -\frac{1}{3} \cos x \cdot e^{-3x} - \frac{1}{3}\left[\sin x \cdot \frac{e^{-3x}}{-3} - \int \cos x \cdot \frac{e^{-3x}}{-3}\, dx\right]$$

$$= -\frac{1}{3} \cos x \cdot e^{-3x} + \frac{1}{9} \sin x \cdot e^{-3x} - \frac{1}{9} \int \cos x \cdot e^{-3x}\, dx$$

$$I_1 + \frac{1}{9} I_1 = -\frac{1}{3} e^{-3x} \cdot \cos x + \frac{1}{9} \sin x \cdot e^{-3x}$$

$$\left(\frac{10}{9}\right) I_1 = -\frac{1}{3} e^{-3x} \cdot \cos x + \frac{1}{9} \sin x \cdot e^{-3x}$$

$$I_1 = \frac{-3}{10} e^{-3x} \cdot \cos x + \frac{1}{10} e^{-3x} \sin x$$

$$2I = -\frac{1}{3} e^{-3x} [\sin^3 x + \cos^3 x] + \frac{1}{3} \sin x \cdot e^{-3x} - \frac{3}{10} e^{-3x} \cdot \cos x + \frac{1}{10} e^{-3x} \cdot \sin x + C$$

$$I = -\frac{1}{6} e^{-3x} [\sin^3 x + \cos^3 x] + \frac{13}{30} e^{-3x} \cdot \sin x - \frac{3}{10} e^{-3x} \cdot \cos x + C$$

[$\because$ $\sin 3x = 3 \sin x - 4 \sin^3 x$ and $\cos 3x = 4 \cos^3 x - 3 \cos x$]

$$= \frac{e^{-3x}}{24} [\sin 3x - \cos 3x] + \frac{3e^{-3x}}{40} [\sin x - 3 \cos x] + C.$$

S42. Here $a = -1$, $b = 2$ and $h = \dfrac{2+1}{n}$, *i.e.*, $nh = 3$ and $f(x) = 7x - 5$.

Now, we have

$$\int_{-1}^{2} (7x - 5)\, dx = \lim_{h \to 0} h[f(-1) + f(-1+h) + f(-1+2h) + \ldots + f(-1+(n-1)h)]$$

Note that

$$f(-1) = -7 - 5 = -12$$

$$f(-1 + h) = -7 + 7h - 5 = -12 + 7h$$

$$f(-1 + (n-1)h) = 7(n-1)h - 12.$$

Therefore,

$$\int_{-1}^{2} (7x - 5)\, dx = \lim_{h \to 0} h[(-12) + (7h - 12) + (14h - 12) + \ldots + (7(n-1)h - 12)]$$

$$= \lim_{h \to 0} h[7h[1 + 2 + \ldots + (n-1)] - 12n]$$

$$= \lim_{h \to 0} h\left[7h\frac{(n-1)n}{2} - 12n\right] = \lim_{h \to 0}\left[\frac{7}{2}(nh)(nh-h) - 12nh\right]$$

$$= \frac{7}{2}(3)(3-0) - 12 \times 3 = \frac{7 \times 9}{2} - 36 = \frac{-9}{2}.$$

S43. Let, $I = \int_0^2 e^x\,dx$

Here, $a = 0$ and $b = 2$

$\therefore$ $h = \dfrac{b-a}{n}$

$\Rightarrow$ $nh = 2$ and $f(x) = e^x$

Now $\int_0^2 e^x\,dx = \lim_{h \to 0} h[f(0) + f(0+h) + f(0+2h) + \ldots + f\{0 + (n-1)h\}]$

$\therefore$ $I = \lim_{h \to 0} h[1 + e^h + e^{2h} + \ldots + e^{(n-1)h}]$

$$= \lim_{h \to 0} h\left[\frac{1 \cdot (e^h)^n - 1}{e^h - 1}\right] = \lim_{h \to 0} h\left(\frac{e^{nh} - 1}{e^h - 1}\right)$$

$$= \lim_{h \to 0} h\left(\frac{e^2 - 1}{e^h - 1}\right)$$

$$= e^2 \lim_{h \to 0}\frac{h}{e^h - 1} - \lim_{h \to 0}\frac{h}{e^h - 1} \qquad \left[\because \lim_{h \to 0}\frac{h}{e^h - 1} = 1\right]$$

$$= e^2 - 1.$$

S44. Let, $I = \int_0^2 (x^2 + 3)\,dx$

Here, $a = 0$, $b = 2$ and $h = \dfrac{b-a}{n} = \dfrac{2-0}{n}$

Put, $h = \dfrac{2}{n} \Rightarrow nh = 2$ here $f(x) = (x^2 + 3)$

Now, $\int_0^2 (x^2 + 3)\,dx = \lim_{h \to 0} h[f(0) + f(0+h) + f(0+2h) + \ldots + f\{0 + (n-1)h\}]$ ... (i)

$\therefore$ $f(0) = 3$

$\Rightarrow$ $f(0 + h) = h^2 + 3$, $f(0 + 2h) = 4h^2 + 3 = 2^2h^2 + 3$

$f[0 + (n-1)h] = (n^2 - 2n + 1)h + 3 = (n-1)^2h + 3$

From Eq. (i),

$$\int_0^2 (x^2 + 3)\,dx = \lim_{h \to 0} h[3 + h^2 + 3 + 2^2h^2 + 3 + 3^2h^2 + 3 + \ldots + (n-1)^2h^2 + 3]$$

$$= \lim_{h \to 0} h[3n + h^2\{1^2 + 2^2 + \ldots + (n-1)^2\}]$$

$$= \lim_{h \to 0} h\left[3n + h^2\left(\frac{(n-1)(2n-2+1)(n-1+1)}{6}\right)\right] \qquad \left[\because \sum n^2 = \frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{h \to 0} h\left[3n + h^2\left(\frac{(n^2-n)(2n-1)}{6}\right)\right]$$

$$= \lim_{h \to 0} h\left[3n + \frac{h^2}{6}(2n^3 - n^2 - 2n^2 + n)\right]$$

$$= \lim_{h \to 0}\left[3nh + \frac{2n^3h^3 - 3n^2h^2 \cdot h + nh \cdot h^2}{6}\right]$$

$$= \lim_{h \to 0}\left[3 \cdot 2 + \frac{2 \cdot 8 - 3 \cdot 2^2 \cdot h + 2 \cdot h^2}{6}\right]$$

$$= \lim_{h \to 0}\left[6 + \frac{16 - 12h + 2h^2}{6}\right]$$

$$= 6 + \frac{16}{6} = 6 + \frac{8}{3} = \frac{26}{3}$$

S45. Let $\qquad I = \int \sqrt{\tan x}\, dx$

Put $\qquad \tan x = t^2 \;\Rightarrow\; \sec^2 x\, dx = 2t\, dt$

$\therefore \qquad I = \int t \cdot \frac{2t}{\sec^2 x}\, dt = 2\int \frac{t^2}{1+t^4}\, dt$

$$= \int \frac{(t^2+1)+(t^2-1)}{(1+t^4)}\, dt$$

$$= \int \frac{t^2+1}{1+t^4}\, dt + \int \frac{t^2-1}{1+t^4}\, dt$$

$$= \int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}\, dt + \int \frac{1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}}\, dt$$

$$= \int \frac{1-\left(-\frac{1}{t^2}\right)}{\left(t-\frac{1}{t}\right)^2+2}\, dt + \int \frac{1+\left(-\frac{1}{t^2}\right)}{\left(t+\frac{1}{t}\right)^2-2}\, dt$$

Put $\qquad u = t - \frac{1}{t} \;\Rightarrow\; du = \left(1+\frac{1}{t^2}\right) dt$

and $\qquad v = t + \frac{1}{t} \;\Rightarrow\; dv = \left(1-\frac{1}{t^2}\right) dt$

$\therefore$ $$I = \int \frac{du}{u^2 + (\sqrt{2})^2} + \int \frac{dv}{v^2 - (\sqrt{2})^2}$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{u}{\sqrt{2}} + \frac{1}{2\sqrt{2}} \log \left| \frac{v - \sqrt{2}}{v + \sqrt{2}} \right| + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\tan x - 1}{\sqrt{2 \tan x}} \right) + \frac{1}{2\sqrt{2}} \log \left| \frac{\tan x - \sqrt{2 \tan x} + 1}{\tan x + \sqrt{2 \tan x} + 1} \right| + C.$$

S46. Let $$I = \int \frac{x^2}{(x^2 + a^2)(x^2 + b^2)}\, dx$$

Now, $$\frac{x^2}{(x^2 + a^2)(x^2 + b^2)}$$ [Let $x^2 = t$]

$\Rightarrow$ $$\frac{t}{(t + a^2)(t + b^2)} = \frac{A}{(t + a^2)} + \frac{B}{(t + b^2)}$$

$\Rightarrow$ $$t = A(t + b^2) + B(t + a^2)$$

On comparing the coefficient of t, we get

$$A + B = 1 \quad \text{... (i)}$$

$$b^2A - a^2B = 0 \quad \text{... (ii)}$$

$\Rightarrow$ $$b^2(1 - B) + a^2B = 0$$

$\Rightarrow$ $$b^2 - b^2B + a^2B = 0$$

$\Rightarrow$ $$b^2 + (a^2 - b^2)B = 0$$

$\Rightarrow$ $$B = \frac{-b^2}{a^2 - b^2} = \frac{b^2}{b^2 - a^2}$$

From Eq. (i)

$$A + \frac{b^2}{b^2 - a^2} = 1$$

$\Rightarrow$ $$A = \frac{b^2 - a^2 - b^2}{b^2 - a^2} = \frac{-a^2}{b^2 - a^2}$$

$\therefore$ $$I = \int \frac{-a^2}{(b^2 - a^2)(x^2 + a^2)}\, dx + \int \frac{b^2}{b^2 - a^2} \cdot \frac{1}{x^2 + b^2}\, dx$$

$$= \frac{-a^2}{(b^2 - a^2)} \int \frac{1}{x^2 + a^2}\, dx + \frac{b^2}{b^2 - a^2} \int \frac{1}{x^2 + b^2}\, dx$$

$$= \frac{-a^2}{b^2 - a^2} \cdot \frac{1}{a} \tan^{-1} \frac{x}{a} + \frac{b^2}{b^2 - a^2} \cdot \frac{1}{b} \tan^{-1} \frac{x}{b}$$

$$= \frac{1}{b^2 - a^2} \left[-a \tan^{-1} \frac{x}{a} + b \tan^{-1} \frac{x}{b} \right]$$

$$= \frac{1}{a^2 - b^2} \left[a \tan^{-1} \frac{x}{a} - b \tan^{-1} \frac{x}{b} \right].$$

S47. Let,
$$I = \int_0^{\pi/2} \frac{\tan x\,dx}{1+m^2\tan^2 x}\,dx$$
$$= \int_0^{\pi/2} \frac{\frac{\sin x}{\cos x}}{1+m^2\cdot\frac{\sin^2 x}{\cos^2 x}}\,dx$$
$$= \int_0^{\pi/2} \frac{\frac{\sin x}{\cos x}}{\frac{\cos^2 x+m^2\sin^2 x}{\cos^2 x}}\,dx$$
$$= \int_0^{\pi/2} \frac{\sin x\cos x\,dx}{1-\sin^2 x+m^2\sin^2 x}\,dx$$
$$= \int_0^{\pi/2} \frac{\sin x\cos x}{1-\sin^2 x(1-m^2)}\,dx$$

Put, $\sin^2 x = t$

$\Rightarrow$ $2\sin x\cos x\,dx = dt$

$\therefore$
$$I = \frac{1}{2}\int_0^1 \frac{dt}{1-t(1-m^2)}$$
$$= \frac{1}{2}\left[-\log|1-t(1-m^2)|\cdot\frac{1}{1-m^2}\right]_0^1$$
$$= \frac{1}{2}\left[-\log|1-1+m^2|\cdot\frac{1}{1-m^2}+\log|1|\cdot\frac{1}{1-m^2}\right]$$
$$= \frac{1}{2}\left[-\log|m^2|\cdot\frac{1}{1-m^2}\right] = \frac{2}{2}\cdot\frac{\log m}{(m^2-1)}$$
$$= \log\frac{m}{m^2-1}.$$

S48. Let
$$I = \int_0^{1/2} \frac{dx}{(1+x^2)\sqrt{1-x^2}}$$

Put $x = \sin\theta$

$\Rightarrow$ $dx = \cos\theta\,d\theta$

As $x\to 0$, then $\theta\to 0$

and $x\to\frac{1}{2}$, then $\theta\to\frac{\pi}{6}$

$\therefore \qquad I = \int_0^{\pi/6} \frac{\cos\theta}{(1+\sin^2\theta)\cos\theta}\, d\theta$

$= \int_0^{\pi/6} \frac{1}{1+\sin^2\theta}\, d\theta$

$= \int_0^{\pi/6} \frac{\sec^2\theta}{\sec^2\theta + \tan^2\theta}\, d\theta$

$= \int_0^{\pi/6} \frac{\sec^2\theta}{1+\tan^2\theta + \tan^2\theta}\, d\theta$

$= \int_0^{\pi/6} \frac{\sec^2\theta}{1+2\tan^2\theta}\, d\theta$

Again, put $\tan\theta = t$

$\Rightarrow \qquad \sec^2\theta\, d\theta = dt$

As $\theta \to 0$, then $t \to 0$

and $\theta \to \frac{\pi}{6}$, then $t \to \frac{1}{\sqrt{3}}$

$\therefore \qquad I = \int_0^{1/\sqrt{3}} \frac{dt}{1+2t^2} = \frac{1}{2}\int_0^{1/\sqrt{3}} \frac{dt}{\left(\frac{1}{\sqrt{2}}\right)^2 + t^2}$

$= \frac{1}{2}\cdot\frac{1}{1/\sqrt{2}}\left[\tan^{-1}\frac{t}{\frac{1}{\sqrt{2}}}\right]_0^{1/\sqrt{3}} = \frac{1}{\sqrt{2}}\left[\tan^{-1}(\sqrt{2}t)\right]_0^{1/\sqrt{3}}$

$= \frac{1}{\sqrt{2}}\left[\tan^{-1}\sqrt{\frac{2}{3}} - 0\right] = \frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{\frac{2}{3}}\right)$

S49. Let, $I = \int_0^1 x\log(1+2x)\, dx$

$= \left[\log(1+2x)\,\frac{x^2}{2}\right]_0^1 - \int \frac{1}{1+2x}\cdot 2\cdot\frac{x^2}{2}\, dx$

$= \frac{1}{2}[x^2\log(1+2x)]_0^1 - \int \frac{x^2}{1+2x}\, dx$

$= \frac{1}{2}[1\log 3 - 0] - \left[\int_0^1 \left(\frac{x}{2} - \frac{\frac{x}{2}}{1+2x}\right) dx\right]$

$$= \frac{1}{2}\log 3 - \frac{1}{2}\int_0^1 x\,dx - \frac{1}{2}\int_0^1 \frac{x}{1+2x}\,dx$$

$$= \frac{1}{2}\log 3 - \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{2}\int_0^1 \frac{\frac{1}{2}(2x+1-1)}{(2x+1)}\,dx$$

$$= \frac{1}{2}\log 3 - \frac{1}{2}\left[\frac{1}{2}-0\right] + \frac{1}{4}\int_0^1 dx - \frac{1}{4}\int_0^1 \frac{1}{1+2x}\,dx$$

$$= \frac{1}{2}\log 3 - \frac{1}{4} + \frac{1}{4}[x]_0^1 - \frac{1}{8}[\log|1+2x|]_0^1$$

$$= \frac{1}{2}\log 3 - \frac{1}{4} + \frac{1}{4} - \frac{1}{8}[\log 3 - \log 1]$$

$$= \frac{1}{2}\log 3 - \frac{1}{8}\log 3$$

$$= \frac{3}{8}\log 3$$

S50. Let $\quad I = \int_0^{\pi} x \sin x \cos^2 x\,dx \quad \ldots \text{(i)}$

and $\quad I = \int_0^{\pi} (\pi - x)\sin(\pi - x)\cos^2(\pi - x)\,dx$

$\Rightarrow \quad I = \int_0^{\pi} (\pi - x)\sin x \cos^2 x\,dx \quad \ldots \text{(ii)}$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi} \pi \sin x \cos^2 x\,dx$$

Put $\quad \cos x = t$

$\Rightarrow \quad -\sin x\,dx = dt$

As $\quad x \to 0,\ \text{then}\ t \to 1$

and $\quad x \to \pi,\ \text{then}\ t \to -1$

$\therefore \quad I = -\pi \int_1^{-1} t^2 dt \ \Rightarrow\ I = -\pi\left[\frac{t^3}{3}\right]_1^{-1}$

$\Rightarrow \quad 2I = -\frac{\pi}{3}[-1-1] \ \Rightarrow\ 2I = \frac{2\pi}{3}$

$\therefore \quad I = \frac{\pi}{3}.$

S51. Let $$I = \int_0^{\pi} x \log \sin x \, dx \quad \ldots \text{(i)}$$

$$I = \int_0^{\pi} (\pi - x) \log \sin (\pi - x) \, dx$$

$$= \int_0^{\pi} (\pi - x) \log \sin x \, dx \quad \ldots \text{(ii)}$$

$$2I = \pi \int_0^{\pi} \log \sin x \, dx \quad \ldots \text{(iii)}$$

$$2I = 2\pi \int_0^{\pi/2} \log \sin x \, dx \quad \left[\because \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx\right]$$

$$I = \pi \int_0^{\pi/2} \log \sin x \, dx \quad \ldots \text{(iv)}$$

Now, $$I = \pi \int_0^{\pi/2} \log \sin (\pi/2 - x) \, dx \quad \ldots \text{(v)}$$

On adding Eqs. (iv) and (v), we get

$$2I = \pi \int_0^{\pi/2} (\log \sin x + \log \cos x) \, dx$$

and $$2I = \pi \int_0^{\pi/2} \log \sin x \cos x \, dx$$

$$= \pi \int_0^{\pi/2} \log \frac{2 \sin x \cos x}{2} \, dx$$

$$2I = \pi \int_0^{\pi/2} (\log \sin 2x - \log 2) \, dx$$

$$2I = \pi \int_0^{\pi/2} \log \sin 2x \, dx - \pi \int_0^{\pi/2} \log 2 \, dx$$

Put $$2x = t \Rightarrow dx = \frac{dt}{2}$$

As $x \to 0$, then $t \to 0$

and $x \to \frac{\pi}{2}$, then $t \to \pi$

$\therefore$ $$2I = \frac{\pi}{2} \int_0^{\pi} \log \sin t \, dt - \frac{\pi^2}{2} \log 2$$

$\Rightarrow$ $$2I = \frac{\pi}{2} \int_0^{\pi} \log \sin x \, dx - \frac{\pi^2}{2} \log 2$$

$\Rightarrow$ $$2I = I - \frac{\pi^2}{2}\log 2 \quad \text{[From Eq. (iii)]}$$

$\therefore$ $$I = -\frac{\pi^2}{2}\log 2 = \frac{\pi^2}{2}\log\left(\frac{1}{2}\right)$$

S52. We have $$I = \int_0^{\frac{\pi}{2}} \frac{\tan^7 x}{\cot^7 x + \tan^7 x}\,dx \quad \text{... (i)}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\tan^7\left(\frac{\pi}{2} - x\right)}{\cot^7\left(\frac{\pi}{2} - x\right) + \tan^7\left(\frac{\pi}{2} - x\right)}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\tan^7 (x)\,dx}{\cot^7 x + \tan^7 x} \quad \text{... (ii)}$$

Adding Eq. (i) and (ii), we get

$$2I = \int_0^{\frac{\pi}{2}} \left(\frac{\tan^7 x + \cot^7 x}{\tan^7 x + \cot^7 x}\right) dx$$

$$= \int_0^{\frac{\pi}{2}} dx \quad \text{which gives } I = \frac{\pi}{4}.$$

S53. Let $$I = \int_0^{\pi/2} \frac{dx}{(a^2\cos^2 x + b^2\sin^2 x)^2}$$

Divide numerator and denominator by $\cos^4 x$, we get

$$I = \int_0^{\pi/2} \frac{\sec^4 x\,dx}{(a^2 + b^2\tan^2 x)^2}$$

$$= \int_0^{\pi/2} \frac{(1 + \tan^2 x)\sec^2 x\,dx}{(a^2 + b^2\tan^2 x)^2}$$

Put $\tan x = t$

$\Rightarrow$ $\sec^2 x\,dx = dt$

As $x \to 0$, then $t \to 0$

and $x \to \frac{\pi}{2}$, then $t \to \infty$

$$I = \int_0^{\infty} \frac{(1 + t^2)}{(a^2 + b^2 t^2)^2}$$

Now, $$\frac{1 + t^2}{(a^2 + b^2 t^2)^2} \quad \text{[Let } t^2 = u\text{]}$$

$$\frac{1+u}{(a^2+b^2u)^2} = \frac{A}{(a^2+b^2u)} + \frac{B}{(a^2+b^2u)^2}$$

$\Rightarrow$ $$1 + u = A\,(a^2 + b^2u) + B$$

On comparing the coefficient of x and constant term on both sides, we get

$$a^2A + B = 1 \quad \text{... (i)}$$

and $$b^2A = 1 \quad \text{... (ii)}$$

$\therefore$ $$A = \frac{1}{b^2}$$

Now, $$\frac{a^2}{b^2} + B = 1$$

$\Rightarrow$ $$B = 1 - \frac{a^2}{b^2} = \frac{b^2-a^2}{b^2}$$

$\therefore$ $$I = \int_0^\infty \frac{(1+t^2)}{(a^2+b^2t^2)^2}$$

$$= \frac{1}{b^2}\int_0^\infty \frac{dt}{a^2+b^2t^2} + \frac{b^2-a^2}{b^2}\int_0^\infty \frac{dt}{(a^2+b^2t^2)^2}$$

$$= \frac{1}{b^2}\int_0^\infty \frac{dt}{b^2\left(\frac{a^2}{b^2}+t^2\right)} + \frac{b^2-a^2}{b^2}\int_0^\infty \frac{dt}{(a^2+b^2t^2)^2}$$

$$= \frac{1}{ab^3}\left[\tan^{-1}\left(\frac{tb}{a}\right)\right]_0^\infty + \frac{b^2-a^2}{b^2}\left(\frac{\pi}{4}\cdot\frac{1}{a^3b}\right)$$

$$= \frac{1}{ab^3}[\tan^{-1}\infty - \tan^{-1}0] + \frac{\pi}{4}\cdot\frac{b^2-a^2}{(a^3b^3)}$$

$$= \frac{\pi}{2ab^3} + \frac{\pi}{4}\cdot\frac{b^2-a^2}{(a^3b^3)}$$

$$= \pi\left(\frac{2a^2+b^2-a^2}{4a^3b^3}\right) = \frac{\pi}{4}\left(\frac{a^2+b^2}{a^3b^3}\right).$$