Date: 18/10/2021

Q1. For the curve $y = 5x - 2x^3$, if x increases at the rate of 2 units/sec, then how fast is the slope of cure changing when $x = 3$?

Q2. If the area of a circle increases at a uniform rate, then prove that perimeter varies inversely as the radius.

Q3. A spherical ball of salt is dissolving in water in such a manner that the rate of decrease of the volume at any instant is proportional to the surface area, Prove that the radius is decreasing at constant rate.

Q4. The volume of a cube increases at a constant rate. Prove that the increase in its surface area varies inversely as the length of the side.

Q5. A kite is moving horizontally at a height of 151.5 m. If the speed of kite is 10 m/s, how fast is the string being let out, when the kite is 250 m away from the boy who is flying the kite, if the height of boy is 1.5 m?

Q6. A swimming pool is to be drained for cleaning. If L represents the number of litres of water in the pool, t seconds after the pool has been plugged off to drain and $L = 200\,(10 - t)^2$. How fast is the water running out at the end of 5 s and what is the average rate at which the water flows out during the first 5 s?

Q7. If x and y are the sides of two squares such that $y = x - x^2$, then find the rate of change of the area of second square with respect to the area of first square.

Q8. Find the angle of intersection of the curves $y^2 = x$ and $x^2 = y$.

Q9. Prove that the curves $y^2 = 4x$ and $x^2 + y^2 - 6x + 1 = 0$ touch each other at the point (1, 2).

Q10. Find the angle of intersection of the curves $y = 4 - x^2$ and $y = x^2$.

Q11. Find the coordinates of the point on the curve $\sqrt{x} + \sqrt{y} = 4$ at which tangent is equally inclined to the axes.

Q12. Prove that the curves $xy = 4$ and $x^2 + y^2 = 8$ touch each other.

Q13. Determine for which values of x, the function $y = x^4 - \frac{4x^3}{3}$ is increasing and for which values, it is decreasing.

Q14. Show that for $a \geq 1$, $f(x) = \sqrt{3}\sin x - \cos x - 2ax + b$ is decreasing in R.

Q15. Show that $f(x) = 2x + \cot^{-1} x + \log(\sqrt{1+x^2} - x)$ is increasing in R.

Q16. Find the condition for the curves $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$; $xy = c^2$ to intersect orthogonally.

Q17. Find the approximate value of $(1.999)^5$.

Q18. Find the approximate volume of metal in a holow spheircal shell whose internal and external radii are 3 cm and 3.0005 cm, respectively.

Q19. Find the points of local maxima, local minima and the points of inflection of the function $f(x) = x^5 - 5x^4 + 5x^3 - 1$. Also, find the corresponding local maximum and local minimum values.

Q20. Prove that $f(x) = \sin x + \sqrt{3}\cos x$ has maximum value at $x = \frac{\pi}{6}$.

Q21. At what point, the slope of the curve $y = -x^3 + 3x^2 + 9x - 27$ is maximum? Also, find the maximum slope.

Q22. Using differentials, find the approaximate value of $\sqrt{0.082}$.

Q23. Show that the function $f(x) = 4x^3 - 18x^2 + 27x - 7$ has neither maxima nor minima.

Q24. Show that the local maximum value of $x + \frac{1}{x}$ is less than local minimaum value.

Q25. Find the maximum and minimum values of

$$f(x) = \sec x + 2\log\cos x, \quad 0 < x < 2\pi$$

Q26. Find the difference between the greatest and least values of the function $f(x) = \sin 2x - x$, on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Q27. An isosceles triangle of vertical angle 2θ is inscribed in a circle of radius a. Show that the area of triangle is maximum when $\theta = \frac{\pi}{6}$.

Q28. Two men A and B start with velocities v at the same time from the junction of two roads inclined at 45° to each other. If they travel by different roads, then find the rate at which they are being separated.

Q29. At what points on the curve $x^2 + y^2 - 2x - 4y + 1 = 0$, the tangents are parallel to the Y-axis?

Q30. Find the condition that curves $2x = y^2$ and $2xy = k$ intersect orthogonaly.

Q31. Water is dripping out at a steady rate of 1 cu cm/sec through a tiny hole at the vertex of the conical vessel, whose axis is vertical. When the slant height of water in the vessel is 4 cm, find the rate of decrease of slant height, where the vertical angle of the conical vessel is $\frac{\pi}{6}$.

Q32. A man, 2 m tall, walks at the rate of $1\frac{2}{3}$ m/s towards a street light which is $5\frac{1}{3}$ m above the ground. At what rate is the tip of his shadow moving and at what rate is the length of the shadow changing when he is $3\frac{1}{3}$ m from the base of the light?

Q33. If the straight line $x\cos\alpha + y\sin\alpha = p$ touches the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then prove that $a^2\cos^2\alpha + b^2\sin^2\alpha = p^2$.

Q34. A metal box with a square base and vertical sides is to contain 1024 cm^3. If the material for the top and bottom costs Rs. 5 per cm^2 and the material for the sides costs Rs. 2.50 per cm^2. Then, find the least cost of the box.

Q35. If the sum of the surface areas of cube and a sphere is constant, what is the ratio of an edge of the cube to the diameter of the sphere, when the sum of their volumes is minimum?

Q36. Find the dimensions of the rectangle of perimeter 36 cm which will sweep out a volume as large as possible, when revolved about one of its sides. Also, find the maximum volume.

Q37. A telephone company in a town has 500 subscribers on its list and collects fixed charges of Rs. 300 per subscriber per year. The company proposes to increase the annual subscription and it is believed that for every increase of Rs. 1 per one subscriber will discontinue the service. Find what increase will bring maximum profit?

Q38. Show that the equation of normal at any point on the curve $x = 3\cos\theta - \cos^3\theta$, $y = 3\sin\theta - \sin^3\theta$ is $4(y\cos^3\theta - x\sin^3\theta) = 3\sin 4\theta$.

Q39. The sum of surface areas of a rectangular parallelopiped with sides x, $2x$ and $\frac{x}{3}$ and a sphere is given to be constant. Prove that the sum of their volumes is minimum, if x is equal to three times the radius of the sphere. Also, find the minimum value of the sum of their volumes.

Date: 18/10/2021

S1. Slope of curve = $\frac{dy}{dx} = 5 - 6x^2$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{dy}{dx}\right) = -12x \cdot \frac{dx}{dt}$$

$$= -12 \cdot (3) \cdot (2)$$

$$= -72 \text{ units/sec.}$$

Thus, slope of curve is decreasing at the rate of 72 units/sec when x is increasing at the rate of 2 units/sec.

S2. Let the radius of circle = r and area of the circle, $A = \pi r^2$

$$\therefore \quad \frac{d}{dt}A = \frac{d}{dt}\pi r^2$$

$$\Rightarrow \quad \frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt} \quad \text{... (i)}$$

Since, the area of a circle increases at a uniform rate, the

$$\frac{dA}{dt} = k \quad \text{... (ii)}$$

where, k is a constant

From Eqs. (i) and (ii), we get

$$2\pi r \cdot \frac{dr}{dt} = k$$

$$\Rightarrow \quad \frac{dr}{dt} = \frac{k}{2\pi r} = \frac{k}{2\pi} \cdot \left(\frac{1}{r}\right) \quad \text{... (iii)}$$

Let the perimeter, $P = 2\pi r$

$$\therefore \quad \frac{dP}{dt} = \frac{d}{dt} \cdot 2\pi r \Rightarrow \frac{dP}{dt} = 2\pi \cdot \frac{dr}{dt}$$

$$= 2\pi \cdot \frac{k}{2\pi} \cdot \frac{1}{r} = \frac{k}{r} \quad \text{[Using Eq. (iii)]}$$

$$\Rightarrow \quad \frac{dP}{dt} \propto \frac{1}{r}$$

Hence Proved.

S3. We have, rate of decrease of the volume of spherical ball of salt at any instant is $\propto$ surface.

Let the radius of the spherical ball of the salt be r.

$\therefore$ Volume of the ball (V) = $\frac{4}{3}\pi r^3$

and Surface area (S) = $4\pi r^2$

$\because$ $\frac{dV}{dt} \propto S$ $\Rightarrow$ $\frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) \propto \pi r^2$

$\Rightarrow$ $\frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt} \propto 4\pi r^2$ $\Rightarrow$ $\frac{dr}{dt} \propto \frac{4\pi r^2}{4\pi r^2}$

$\Rightarrow$ $\frac{dr}{dt} = k \cdot 1$ [where, k is the proportionality constant]

$\Rightarrow$ $\frac{dr}{dt} = k$

Hence, the radius of ball is decreasing at a constant rate.

S4. Let the side of a cube be x unit.

$\therefore$ Volume of cube (V) = x^3

On differentiating both side w.r.t. t, we get

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt} = k \quad \text{[Constant]}$$

$$\Rightarrow \frac{dx}{dt} = \frac{k}{3x^2} \quad \text{... (i)}$$

Also, surface area of cube, S = $6x^2$

On differentiating w.r.t. t, we get

$$\frac{dS}{dt} = 12x \cdot \frac{dx}{dt}$$

$$\Rightarrow \frac{dS}{dt} = 12x \cdot \frac{k}{3x^2} \quad \text{[Using Eq. (i)]}$$

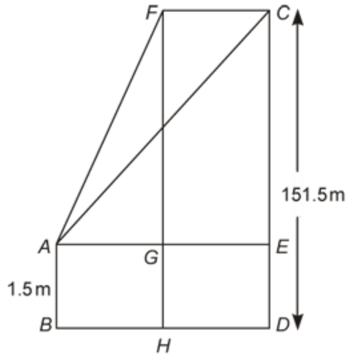
$$\Rightarrow \frac{dS}{dt} = \frac{12k}{3x} = 4\left(\frac{k}{x}\right)$$

$$\Rightarrow \frac{dS}{dt} \propto \frac{1}{x}$$

Hence, the surface area of the cube varies inversely as the length of the side.

S5. We have, height (h) = 151.5 m speed of kite (v) = 10 m/s.

Let CD be the height of kite and AB be the height of boy.



Let $DB = xm = EA$ and $AC = 250\,m$

$\therefore \quad \frac{dx}{dt} = 10\,m/s$

From the figure, we see that

$$EC = 151.5 - 1.5 = 150\,m$$

and $AE = x$

Also, $AC = 250$ m

In right angled $\triangle CEA$,

$$AE^2 + EC^2 = AC^2$$

$$\Rightarrow \quad x^2 + (150)^2 = y^2$$

$$\Rightarrow \quad x^2 + (150)^2 = (250)^2 \quad \text{... (i)}$$

$$\Rightarrow \quad x^2 = (250)^2 - (150)^2$$

$$= (250 + 150)(250 - 150)$$

$$= 400 \times 100$$

$$\therefore \quad x = 20 \times 10 = 200$$

From Eq. (i), on differentiating w.r.t. t, we get

$$2x \cdot \frac{dx}{dt} = 2y\frac{dy}{dt}$$

$$\Rightarrow \quad 2y\frac{dy}{dt} = 2x\frac{dx}{dt}$$

$$\therefore \quad \frac{dy}{dt} = \frac{x}{y} \cdot \frac{dx}{dt}$$

$$= \frac{200}{250} \cdot 10 = 8\,m/s \qquad \left[\because \frac{dx}{dt} = 10\,m/s\right]$$

So, the required rate at which the string is being let out is 8 m/s.

S6. Let L represents the number of litres of water in the pool, t seconds after the pool has been plugged off to drain, then

$$L = 200\,(10 - t)^2$$

$\therefore$ Rate at which the water is running out

$$= -\frac{dL}{dt}$$

$$\frac{dL}{dt} = -200.2\,(10 - t)\cdot(-1)$$
$$= 400\,(10 - t)$$

Rate at which the water is running out at the end of 5 s

$$= 400\,(10 - 5)$$
$$= 2000 \text{ L/s} = \text{Final rate}$$

Since, Initial rate = $-\left(\frac{dL}{dt}\right)_{t=0}$ = 4000 L/s

$\therefore$ Average rate during 5 s = $\frac{\text{Initial rate} + \text{Final rate}}{2}$

$$= \frac{4000 + 2000}{2}$$
$$= 3000 \text{ L/s.}$$

S7. Since, x and y are the sides of two squares such that $y = x - x^2$.

$\therefore$ Area of the first square $(A_1) = x^2$

and area of the second square $(A_2) = y^2 = (x - x^2)^2$

$$\therefore \quad \frac{dA_2}{dt} = \frac{d}{dt}(x - x^2)^2 = 2(x - x^2)\left(\frac{dx}{dt} - 2x\cdot\frac{dx}{dt}\right)$$

$$= \frac{dx}{dt}(1 - 2x)2(x - x^2)$$

and $$\frac{dA_1}{dt} = \frac{d}{dt}x^2 = 2x\cdot\frac{dx}{dt}$$

$$\therefore \quad \frac{dA_2}{dA_1} = \frac{dA_2/dt}{dA_1/dt} = \frac{\frac{dx}{dt}\cdot(1 - 2x)(2x - 2x^2)}{2x\cdot\frac{dx}{dt}}$$

$$= \frac{(1 - 2x)2x(1 - x)}{2x}$$

$$= (1 - 2x)(1 - x)$$
$$= 1 - x - 2x + 2x^2$$
$$= 2x^2 - 3x + 1.$$

S8. Solving the given equations, we have $y^2 = x$ and $x^2 = y \Rightarrow x^4 = x$ or $x^4 - x = 0 \Rightarrow x(x^3 - 1) = 0 \Rightarrow$ $x = 0$, $x = 1$.

Therefore, $y = 0$, $y = 1$

i.e., points of intersection are (0, 0) and (1, 1).

Further $y^2 = x \Rightarrow 2y\dfrac{dy}{dx} = 1 \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2y}$

and $x^2 = y \Rightarrow \dfrac{dy}{dx} = 2x$.

At (0, 0), the slope of the tangent to the curve $y^2 = x$ is parallel to y-axis and the tangent to the curve $x^2 = y$ is parallel to x-axis.

$\Rightarrow$ angle of intersection = $\dfrac{\pi}{2}$.

At (1, 1), slope of the tangent to the curve $y^2 = x$ is equal to $\dfrac{1}{2}$ and that of $x^2 = y$ is 2.

$$\tan\theta = \left|\frac{2-\frac{1}{2}}{1+1}\right| = \frac{3}{4} \Rightarrow \theta = \tan^{-1}\left(\frac{3}{4}\right).$$

S9. We have, $y^2 = 4x$ and $x^2 + y^2 - 6x + 1 = 0$

Since, both the curves touch each other at (1, 2) i.e., curves are passing through (1, 2).

$\therefore$ $2y\cdot\dfrac{dy}{dx} = 4$

and $2x + 2y\dfrac{dy}{dx} = 6$

$\Rightarrow$ $\dfrac{dy}{dx} = \dfrac{4}{2y}$

and $\dfrac{dy}{dx} = \dfrac{6-2x}{2y}$

$\Rightarrow$ $\left(\dfrac{dy}{dx}\right)_{(1,\,2)} = \dfrac{4}{4} = 1$

and $\left(\dfrac{dy}{dx}\right)_{(1,\,2)} = \dfrac{6-2\cdot 1}{2\cdot 2} = \dfrac{4}{4} = 1$

$\Rightarrow$ $m_1 = 1$ and $m_2 = 1$

Thus, we see that slope of both the curves are equal to each other *i.e.*, $m_1 = m_2 = 1$ at the point (1, 2).

Hence, both the curves touch each other.

S10. We have, $y = 4 - x^2$ ... (i)

and $y = x^2$ ... (ii)

$\Rightarrow$ $\dfrac{dy}{dx} = -2x$

and $\frac{dy}{dx} = 2x$

$\Rightarrow$ $m_1 = -2x$

and $m_2 = 2x$

From Eq.s (i) and (ii), $x^2 = 4 - x^2$

$\Rightarrow$ $2x^2 = 4$

$\Rightarrow$ $x^2 = 2$

$\Rightarrow$ $x = \pm\sqrt{2}$

$\therefore$ $y = x^2 = (\pm\sqrt{2})^2 = 2$

So, the points of intersection are ($\sqrt{2}$, 2) and ($-\sqrt{2}$, 2).

For point ($+\sqrt{2}$, 2). $m_1 = -2x = -2\cdot\sqrt{2} = -2\sqrt{2}$

and for point ($\sqrt{2}$, 2) $m_2 = 2x = 2\sqrt{2}$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{-2\sqrt{2} - 2\sqrt{2}}{1 - 2\sqrt{2}\cdot 2\sqrt{2}}\right| = \left|\frac{-4\sqrt{2}}{-7}\right|$$

$\therefore$ $\theta = \tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$.

S11. We have, $\sqrt{x} + \sqrt{y} = 4$ ... (i)

$\Rightarrow$ $x^{1/2} + y^{1/2} = 4$

$\Rightarrow$ $\frac{1}{2}\cdot\frac{1}{x^{1/2}} + \frac{1}{2}\cdot\frac{1}{y^{1/2}}\cdot\frac{dy}{dx} = 0$

$\therefore$ $\frac{dy}{dx} = -\frac{1}{2}\cdot x^{-1/2}\cdot 2\cdot y^{1/2}$

$= -\sqrt{\frac{y}{x}}$

Since, tangent is equally inclined to the axes.

$\therefore$ $\frac{dy}{dx} = \pm 1$

$\Rightarrow$ $-\sqrt{\frac{y}{x}} = \pm 1$

$\Rightarrow$ $\frac{y}{x} = 1 \quad = \quad y = x$

From Eq. (i), $\sqrt{x} + \sqrt{y} = 4$

$\Rightarrow$ $2\sqrt{y} = 4$

$\Rightarrow \qquad 4y = 16$

$\therefore \qquad y = 4 \quad \text{and} \quad x = 4$

When $y = 4$, then $x = 4$.

So, the required coordinates are (4, 4).

S12. Given, equation of curves are $xy = 4$ ... (i)

and $x^2 + y^2 = 8$ ... (ii)

$$\Rightarrow \qquad x \cdot \frac{dy}{dx} + y = 0$$

$$\text{and} \qquad 2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{-y}{x}$$

$$\text{and} \qquad \frac{dy}{dx} = \frac{-2x}{2y}$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{-y}{x} = m_1 \qquad \text{[Say]}$$

$$\text{and} \qquad \frac{dy}{dx} = \frac{-x}{y} = m_2 \qquad \text{[Say]}$$

Since, both the curves should have same slope.

$$\therefore \qquad \frac{-y}{x} = \frac{-x}{y} \quad \Rightarrow \quad -y^2 = -x^2$$

$$\Rightarrow \qquad x^2 = y^2 \qquad \text{... (iii)}$$

Using the value of x^2 in Eq. (ii), we get

$$y^2 + y^2 = 8$$

$$\Rightarrow \qquad y^2 = 4 \quad \Rightarrow \quad y = \pm 2$$

For $y = 2$, $\qquad x = \frac{4}{2} = 2$

and for $y = -2$, $\qquad x = \frac{4}{-2} = -2$

Thus, the required points of intersection are (2, 2) and $(-2, -2)$.

For (2, 2) $\qquad m_1 = \frac{-y}{x} = \frac{-2}{2} = -1$

and $\qquad m_2 = \frac{-x}{y} = \frac{-2}{2} = -1$

$\because \qquad m_1 = m_2$

For $(-2, -2)$ $\qquad m_1 = \frac{-y}{x} = \frac{-(-2)}{-2} = -1$

and $$m_2 = \frac{-x}{y} = \frac{-(-2)}{-2} = -1$$

Thus, for both the intersection points, we see that slope of both the curves are same. Hence, the curves touch each other.

S13. Giving, $$y = x^4 - \frac{4x^3}{3} \quad \Rightarrow \quad \frac{dy}{dx} = 4x^3 - 4x^2 = 4x^2(x-1)$$

Now, $$\frac{dy}{dx} = 0 \quad \Rightarrow \quad x = 0, \quad x = 1.$$

Since $f'(x) < 0 \ \forall\, x \in (-\infty, 0) \cup (0, 1)$ and f is continuous in $(-\infty, 0]$ and $[0, 1]$. Therefore f is decreasing in $(-\infty, 1]$ and f is increasing in $[1, \infty)$.

Note: Here f is strictly decreasing $(-\infty, 0) \cup (0, 1)$ and is strictly increaasing $(1, \infty)$.

S14. We have, $$f(x) = \sqrt{3}\sin x - \cos x - 2ax + b \qquad a \ge 1$$

$\therefore$
$$f'(x) = \sqrt{3}\cos x - (-\sin x) - 2a$$
$$= \sqrt{3}\cos x + \sin x - 2a$$
$$= 2\left[\frac{\sqrt{3}}{2}\cdot\cos x + \frac{1}{2}\cdot\sin x\right] - 2a$$
$$= 2\left[\cos\frac{\pi}{6}\cdot\cos x + \sin\frac{\pi}{6}\cdot\sin x\right] - 2a$$
$$= 2\cos\left(\frac{\pi}{6} - x\right) - 2a$$

[$\because \ \cos(A - B) = \cos A \cdot \cos B + \sin A \cdot \sin B$]

$$= 2\left[\cos\left(\frac{\pi}{6} - x\right) - a\right]$$

We know that, $\cos x \in [-1, 1]$

and $a \ge 1$

So, $$2\left[\cos\left(\frac{\pi}{6} - x\right) - a\right] \le 0$$

$\therefore$ $f'(x) \le 0$

Hence, $f(x)$ is a decreasing function in R.

S15. We have, $f(x) = 2x + \cot^{-1} x + \log(\sqrt{1+x^2} - x)$

$$\therefore \quad f'(x) = 2 + \left(\frac{-1}{1+x^2}\right) + \frac{1}{(\sqrt{1+x^2}-x)}\left(\frac{1}{2\sqrt{1+x^2}}\cdot 2x - 1\right)$$

$$= 2 - \frac{1}{1+x^2} + \frac{1}{(\sqrt{1+x^2}-x)}\cdot\frac{(x-\sqrt{1+x^2})}{\sqrt{1+x^2}}$$

$$= 2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{1+x^2}}$$

$$= \frac{2+2x^2-1-\sqrt{1+x^2}}{1+x^2} = \frac{1+2x^2-\sqrt{1+x^2}}{1+x^2}$$

For increasing function, $f'(x) \geq 0$

$$\Rightarrow \quad \frac{1+2x^2-\sqrt{1+x^2}}{1+x^2} \geq 0$$

$$\Rightarrow \quad 1 + 2x^2 \geq \sqrt{1+x^2}$$

$$\Rightarrow \quad (1+2x^2)^2 \geq 1 + x^2$$

$$\Rightarrow \quad 1 + 4x^4 + 4x^2 \geq 1 + x^2$$

$$\Rightarrow \quad 4x^4 + 3x^2 \geq 0$$

$$\Rightarrow \quad x^2(4x^2+3) \geq 0$$

which is true for any real value of x.

Hence, $f(x)$ is increasing in R.

S16. Let the curves intersect at (x_1, y_1). Therefore.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{b^2x}{a^2y}$$

$\Rightarrow$ slope of tangent at the point of intersection $(m_1) = \dfrac{b^2x_1}{a^2y_1}$.

Again, $xy = c^2 \Rightarrow x\dfrac{dy}{dx} + y = 0 \Rightarrow \dfrac{dy}{dx} = \dfrac{-y}{x} \Rightarrow m_2 = \dfrac{-y_1}{x_1}$

For orthogonality, $m_1 \times m_2 = -1 \Rightarrow \dfrac{b^2}{a^2} = 1$ or $a^2 - b^2 = 0$.

S17. Let $x = 2$

and $\Delta x = -0.001$ $[\because\ 2 - 0.001 = 1.999]$

Let $y = x^5$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 5x^4$$

Now, $\Delta y = \frac{dy}{dx} \cdot \Delta x = 5x^4 \times \Delta x$

$= 5 \times 2^4 \; [-0.001]$

$= -80 \times 0.001 = -0.080$

$\therefore$ $(1.999)^5 = y + \Delta y$

$= 2^5 + (-0.080)$

$= 32 - 0.080 = 31.920.$

S18. Let internal rdius = r and external radius = R

$\therefore$ Volume of hollow spherical shell, $V = \frac{4}{3}\pi(R^3 - r^3)$

$\Rightarrow$ $V = \frac{4}{3}\pi[(3.0005)^3 - (3)^3]$ ... (i)

Now, we shall use differentiation to get approximate value of $(3.0005)^3$.

Let $(3.0005)^3 = y + \Delta y$

and $x = 3, \quad \Delta x = 0.0005$

Also, let $y = x^3$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 3x^2$$

$\Delta y = \frac{dy}{dx} \times \Delta x = 3x^2 \times 0.0005$

$= 3 \times 3^2 \times 0.0005$

$= 27 \times 0.0005 = 0.0135$

Also, $(3.0005)^3 = y + \Delta y$

$= 3^3 + 0.0135 = 27.0135$

$\therefore$ $V = \frac{4}{3}\pi[27.0135 - 27.000]$ [Using Eq. (i)]

$= \frac{4}{3}\pi[0.0135] = 4\pi \times (0.0045)$

$= 0.0180\,\pi\,\text{cm}^3.$

S19. Given that, $f(x) = x^5 - 5x^4 + 5x^3 - 1$

On differentiating w.r.t. x, we get

$f'(x) = 5x^4 - 20x^3 + 15x^2$

For maxima or minima, $f'(x) = 0$

$\Rightarrow \quad 5x^4 - 20x^3 + 15x^2 = 0$

$\Rightarrow \quad 5x^2(x^2 - 4x + 3) = 0$

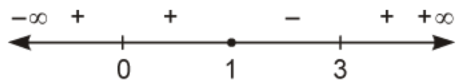
$\Rightarrow \quad 5x^2(x^2 - 3x - x + 3) = 0$

$\Rightarrow \quad 5x^2[x(x - 3) - 1(x - 3)] = 0$

$\Rightarrow \quad 5x^2[(x - 1)(x - 3)] = 0$

$\therefore \quad x = 0, 1, 3$

Sign scheme for $\dfrac{dy}{dx} = 5x^2(x - 1)(x - 3)$



So, y has maximum value at $x = 1$ and minimum value at $x = 3$.

At $x = 0$, y has neither maximum nor minimum value.

$\therefore \quad$ Maximum value of $y = 1 - 5 + 5 - 1 = 0$

and $\quad$ Minimum value $= (3)^5 - 5(3)^4 + 5(3)^3 - 1$

$= 243 - 81 \times 5 - 27 \times 5 - 1 = -298.$

S20. We have, $\quad f(x) = \sin x + \sqrt{3}\cos x$

$\therefore \quad f'(x) = \cos x + \sqrt{3}(-\sin x)$

$= \cos x - \sqrt{3}\sin x$

For $\quad f'(x) = 0, \quad \cos x = \sqrt{3}\sin x$

$\Rightarrow \quad \tan x = \dfrac{1}{\sqrt{3}} = \tan\dfrac{\pi}{6}$

$\Rightarrow \quad x = \dfrac{\pi}{6}$

Again, differentiating $f'(x)$, we get

$f''(x) = -\sin x - \sqrt{3}\cos x$

At $\quad x = \dfrac{\pi}{6}, \quad f''(x) = -\sin\dfrac{\pi}{6} - \sqrt{3}\cos\dfrac{\pi}{6}$

$= -\dfrac{1}{2} - \sqrt{3}\cdot\dfrac{\sqrt{3}}{2}$

$= -\dfrac{1}{2} - \dfrac{3}{2} = -2 < 0.$

Hence, at $x = \dfrac{\pi}{6}$, $f(x)$ has maximum value $\dfrac{\pi}{6}$ is the point of local maxima.

S21. We have, $y = -x^3 + 3x^2 + 9x - 27$

$\therefore$ $\dfrac{dy}{dx} = -3x^3 + 6x + 9$ = Slope of tangent to the curve

Now, $\dfrac{d^2y}{dx^2} = -6x + 6$

For $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = 0$,

$$-6x + 6 = 0$$

$\Rightarrow$ $x = \dfrac{-6}{-6} = 1$

$\therefore$ $\dfrac{d}{dx}\left(\dfrac{d^2y}{dx^2}\right) = -6 < 0$

So, the slope of tangent to the curve is maximum, when $x = 1$.

For $x = 1$, $\left(\dfrac{dy}{dx}\right)_{(x=1)} = -3 \cdot 1^2 + 6 \cdot 1 + 9 = 12.$

which is maximum slope.

Also, for $x = 1,\ y = -1^3 + 3 \cdot 1^2 + 9 \cdot 1 - 27$

$$= -1 + 3 + 9 - 27$$
$$= -16$$

So, the required point is $(1, -16)$.

S22. Let, $f(x) = \sqrt{x}$

Using $f(x + \Delta x); f(x) + \Delta x \cdot f'(x)$, taking $x = .09$ and $\Delta x = -0.008$.

We get, $f(0.09 - 0.008) = f(0.09) + (-0.008)\, f'(0.09)$

$\Rightarrow$ $\sqrt{0.082} = \sqrt{0.09} - 0.008 \cdot \left(\dfrac{1}{2\sqrt{0.09}}\right) = 0.3 - \dfrac{0.008}{0.6}$

$$= 0.3 - 0.0133 = 0.2867.$$

S23. Let, $f(x) = 4x^3 - 18x^2 + 27x - 7$

$$f'(x) = 12x^2 - 36x + 27$$
$$= 3(4x^2 - 12x + 9)$$
$$= 3(2x - 3)^2$$

$$f'(x) = 0 \quad \Rightarrow \quad x = \frac{3}{2} \quad \text{(critical point)}$$

Since, $f'(x) > 0$ for all $x < \dfrac{3}{2}$ and for all $x > \dfrac{3}{2}$.

Hence, $x = \frac{3}{2}$ is a point of inflextion *i.e.*, neither a point of maxima nor a point of minima $x = \frac{3}{2}$ is the only critical point, and f has neither maxima nor minima.

S24. Let

$$y = x + \frac{1}{x} \Rightarrow \frac{dy}{dx} = 1 - \frac{1}{x^2},$$

$$\frac{dy}{dx} = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1.$$

$$\frac{d^2y}{dx^2} = \frac{2}{x^3},$$

Therefore, $\frac{d^2y}{dx^2}$ (at $x = 1$) > 0 and $\frac{d^2y}{dx^2}$ (at $x = -1$) < 0.

Hence, local maximum value of y is at $x = -1$ and the local maximum value = -2.

Local minimum value of y is at $x = 1$ and local minimum value = 2.

Therefore, local maximum value (-2) is less than local minimum value 2.

S25. Let,

$$f(x) = \sec x + 2 \log \cos x$$

Therefore

$$f'(x) = \sec x \tan x - 2 \tan x = \tan x (\sec x - 2)$$

$$f'(x) = 0 \Rightarrow \tan x = 0 \quad \text{or} \quad \sec x = 2 \quad \text{or} \quad \cos x = \frac{1}{2}$$

Therefore, possible values of x are

$$x = 0, \quad \text{or} \quad x = \pi \quad \text{and} \quad x = \frac{\pi}{3} \quad \text{or} \quad x = \frac{5\pi}{3}$$

Again,

$$f''(x) = \sec^2 x (\sec x - 2) + \tan x (\sec x \tan x)$$

$$= \sec^3 x + \sec x \tan^2 x - 2 \sec^2 x$$

$$= \sec x (\sec^2 x + \tan^2 x - 2 \sec x).$$

Since,

$f''(0) = 1 (1 + 0 - 2) = -1 < 0.$ Therefore, $x = 0$ is a point of maxima.

$f''(\pi) = -1 (1 + 0 + 2) = -3 < 0.$ Therefore, $x = \pi$ is a point of maxima.

$f''\left(\frac{\pi}{3}\right) = 2 (4 + 3 - 4) = 6 > 0.$ Therefore, $x = \frac{\pi}{3}$ is a point of minima.

$f''\left(\frac{5\pi}{3}\right) = 2 (4 + 3 - 4) = 6 > 0.$ Therefore, $x = \frac{5\pi}{3}$ is a point of minima.

Maximum Value of y at $x = 0$ is $1 + 0 = 1$

Maximum Value of y at $x = \pi$ is $-1 + 0 = -1$

Minimum Value of y at $x = \frac{\pi}{3}$ is $2 + 2 \log \frac{1}{2} = 2 (1 - \log 2)$

Minimum Value of y at $x = \frac{5\pi}{3}$ is $\quad 2 + 2\log\frac{1}{2} = 2(1 - \log 2)$

S26.

$$f(x) = \sin 2x - x$$

$$f'(x) = 2\cos 2x - 1$$

Therefore, $\quad f'(x) = 0 \quad \Rightarrow \quad \cos 2x = \frac{1}{2} \quad \Rightarrow \quad 2x$ is $\frac{-\pi}{3}$ or $\frac{\pi}{3} \quad \Rightarrow \quad x = -\frac{\pi}{6}$ or $\frac{\pi}{6}$

$$f\left(-\frac{\pi}{2}\right) = \sin(-\pi) + \frac{\pi}{2} = \frac{\pi}{2}$$

$$f\left(-\frac{\pi}{6}\right) = \sin\left(-\frac{2\pi}{6}\right) + \frac{\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{\pi}{6}$$

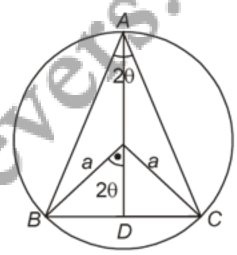
$$f\left(\frac{\pi}{6}\right) = \sin\left(\frac{2\pi}{6}\right) - \frac{\pi}{6} = \frac{\sqrt{3}}{2} - \frac{\pi}{6}$$

$$f\left(\frac{\pi}{2}\right) = \sin(\pi) - \frac{\pi}{2} = -\frac{\pi}{2}$$

Clearly, $\frac{\pi}{2}$ is the greatest value and $-\frac{\pi}{2}$ is the least.

Therefore, difference = $\frac{\pi}{2} + \frac{\pi}{2} = \pi$.

S27. Let ABC be an isosceles triangle inscribed in the circle with radius a such that $AB = AC$.



$$AD = AO + OD = a + a\cos 2\theta$$

and $\quad BC = 2BD = 2a\sin 2\theta$ (see figure)

Therefore, area of the triangle ABC *i.e.*, $\Delta = \frac{1}{2} BC \cdot AD$

$$= \frac{1}{2} 2a\sin 2\theta \cdot (a + a\cos 2\theta)$$

$$= a^2 \sin 2\theta\,(1 + \cos 2\theta)$$

$\Rightarrow \quad \Delta = a^2 \sin 2\theta + \frac{1}{2} a^2 \sin 4\theta$

Therefore, $\quad \frac{d\Delta}{d\theta} = 2a^2 \cos 2\theta + 2a^2 \cos 4\theta$

$$= 2a^2 (\cos 2\theta + \cos 4\theta)$$

$$\frac{d\Delta}{d\theta} = 0 \quad \Rightarrow \quad \cos 2\theta = -\cos 4\theta = \cos(\pi - 4\theta)$$

Therefore, $\quad 2\theta = \pi - 4\theta \quad \Rightarrow \quad \theta = \frac{\pi}{6}$

$$\frac{d^2\Delta}{d\theta^2} = 2a^2 (-2 \sin 2\theta - 4 \sin 4\theta) < 0 \quad \left(\text{at } \theta = \frac{\pi}{6}\right).$$

Therefore, area of triangle is maximum when $\theta = \pi/6$.

S28. Let two men start from the point C with velocity v each at the same time.

Also, $\angle BCA = 45°$

Since, A and B are moving with same velocity v, so they will cover same distance in same time.

Therefore, ΔABC is an isosceles triangle with $AC = BC$.

Now, draw $CD \perp AB$.

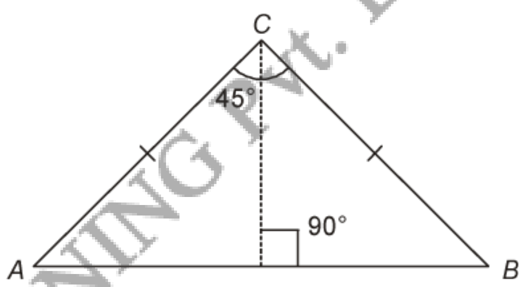
Let at any instant t, the distance between them is AB.

Let $AC = BC = x$ and $AB = y$

In ΔACD and ΔDCB,

$$\angle CAD = \angle CBD \qquad [\because\ AC = BC]$$

$$\angle CDA = \angle CDB = 90°$$



$\therefore$ $\angle ACD = \angle DCB$

or $\angle ACD = \frac{1}{2} \times \angle ACB$

$\Rightarrow$ $\angle ACD = \frac{1}{2} \times 45°$

$\Rightarrow$ $\angle ACD = \frac{\pi}{8}$

$\therefore$ $\sin\frac{\pi}{8} = \frac{AD}{AC}$

$\Rightarrow$ $\sin\frac{\pi}{8} = \frac{y/2}{x} \qquad [\because\ AD = y/2]$

$\Rightarrow$ $\frac{y}{2} = x \sin\frac{\pi}{8}$

$\Rightarrow$ $y = 2x \cdot \sin\frac{\pi}{8}$

Now, differentiating both sides w.r.t. t, we get

$$\frac{dy}{dt} = 2 \cdot \sin\frac{\pi}{8} \cdot \frac{dx}{dt}$$

$$= 2 \cdot \sin\frac{\pi}{8} \cdot v \qquad \left[\because\ v = \frac{dx}{dt}\right]$$

$$= 2v \cdot \frac{\sqrt{2-\sqrt{2}}}{2} \qquad \left[\because\ \sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}\right]$$

$$= \sqrt{2-\sqrt{2}}\, v \text{ unit/s}$$

which is the rate at which A and B are being separated.

S29. Given, equation of curve is

$$x^2 + y^2 - 2x - 4y + 1 = 0 \quad \text{... (i)}$$

$$\Rightarrow \quad 2x + 2y\frac{dy}{dx} - 2 - 4\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx}(2y - 4) = 2 - 2x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2(1-x)}{2(y-2)}$$

Since, the tangents are parallel to the Y-axis, *i.e.*, $\tan\theta = \tan 90° = \frac{dy}{dx}$.

$$\therefore \quad \frac{1-x}{y-2} = \frac{1}{0}$$

$$\Rightarrow \quad y - 2 = 0$$

$$\Rightarrow \quad y = 2$$

Putting $y = 2$ in Eq. (i), we get

$$x^2 + 2^2 - 2x - 4 \times 2 + 1 = 0$$

$$\Rightarrow \quad x^2 - 2x - 3 = 0$$

$$\Rightarrow \quad x^2 - 3x + x - 3 = 0$$

$$\Rightarrow \quad x(x-3) + 1(x-3) = 0$$

$$\Rightarrow \quad (x+1)(x-3) = 0$$

$$\therefore \quad x = -1, \quad x = 3$$

So, the required points are (– 1, 2) and (3, 2).

S30. Given, equation of curves are $2x = y^2$ ... (i)

and $2xy = k$ ... (ii)

$$\Rightarrow \quad y = \frac{k}{2x} \quad \text{[From Eq. (ii)]}$$

From Eq. (i), $2x = \left(\frac{k}{2x}\right)^2$

$$\Rightarrow \quad 8x^3 = k^2$$

$$\Rightarrow \quad x^3 = \frac{1}{8}k^2$$

$\Rightarrow$ $x = \frac{1}{2}k^{2/3}$

$\therefore$ $y = \frac{k}{2x} = \frac{k}{2 \cdot \frac{1}{2}k^{2/3}} = k^{1/3}$

Thus, we get point of intersection of curves which is $\left(\frac{1}{2}k^{2/3}, k^{1/3}\right)$.

From Eq. (i), and (ii), $2 = 2y\frac{dy}{dx}$

and $2\left[x \cdot \frac{dy}{dx} + y \cdot 1\right] = 0$

$\Rightarrow$ $\frac{dy}{dx} = \frac{1}{y}$

and $\left(\frac{dy}{dx}\right) = \frac{-2y}{2x} = -\frac{y}{x}$

$\Rightarrow$ $\left(\frac{dy}{dx}\right)_{\left(\frac{1}{2}k^{2/3}, k^{1/3}\right)} = \frac{1}{k^{1/3}}$ [Say m_1]

and $\left(\frac{dy}{dx}\right)_{\left(\frac{1}{2}k^{2/3}, k^{1/3}\right)} = \frac{-k^{1/3}}{\frac{1}{2}k^{2/3}} = -2k^{-1/3}$ [Say m_2]

Since, the curves intersect orthogonally.

i.e., $m_1 \cdot m_2 = -1$

$\Rightarrow$ $\frac{1}{k^{1/3}} \cdot (-2k^{-1/3}) = -1$

$\Rightarrow$ $-2k^{-2/3} = -1$

$\Rightarrow$ $\frac{2}{k^{2/3}} = 1$

$\Rightarrow$ $k^{2/3} = 2$

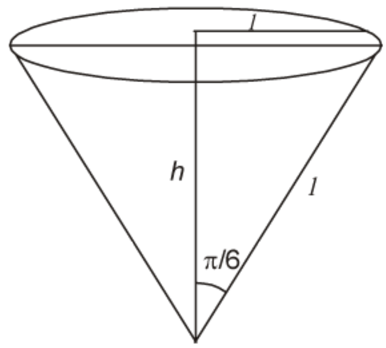
$\therefore$ $k^2 = 8$

which is the reuired condition.

S31. Given that $\frac{dv}{dt} = 1\text{ cm}^3/\text{s}$, where v is the volume of water in the conical vessel.

From the figure, $l = 4$ cm, $h = l\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}l$ and $r = l\sin\frac{\pi}{6} = \frac{l}{2}$.

Therefore, $v = \frac{1}{3}\pi r^2 h = \frac{\pi}{3}\frac{l^2}{4}\frac{\sqrt{3}}{2} l = \frac{\sqrt{3}\pi}{24} l^3$



$$\frac{dv}{dt} = \frac{\sqrt{3}\pi}{8} l^2 \frac{dl}{dt}$$

Therefore, $1 = \frac{\sqrt{3}\pi}{8} 16 \cdot \frac{dl}{dt}$

= $\frac{dl}{dt} = \frac{1}{2\sqrt{3}\pi}$ cm/s.

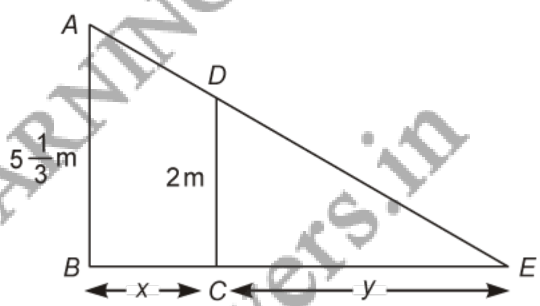
Therefore, the rate of decrease of slant height = $\frac{1}{2\sqrt{3}\pi}$ cm/s.

S32. Let AB be the street light post and CD be the height of man *i.e.*, $CD = 2$ m.

Let, $BC = x$ m, $CE = y$ m and $\frac{dx}{dt} = \frac{-5}{3}$ m/s

From ΔABE and ΔDCE, we see that

$\Delta ABE \sim \Delta DCE$ [by *AAA* similarity]



$\therefore$ $\frac{AB}{DC} = \frac{BE}{CE} \Rightarrow \frac{\frac{16}{3}}{2} = \frac{x+y}{y}$

$\Rightarrow$ $\frac{16}{6} = \frac{x+y}{y}$

$\Rightarrow$ $16y = 6x + 6y \Rightarrow 10y = 6x$

$\Rightarrow$ $y = \frac{3}{5}x$

On differentiating both sides w.r.t. t, we get

$$\frac{dy}{dt} = \frac{3}{5}\cdot\frac{dx}{dt} = \frac{3}{5}\cdot\left(-1\frac{2}{3}\right)$$

[Since, man is moving towards the light post]

$$= \frac{3}{5}\cdot\left(\frac{-5}{3}\right) = -1 \text{ m/s}$$

Let $z = x + y$

Now, differentiating both sides w.r.t. t, we get

$$\frac{dz}{dt} = \frac{dx}{dt} + \frac{dy}{dt} = -\left(\frac{5}{3} + 1\right)$$

$$= -\frac{8}{3} = -2\frac{2}{3}\text{ m/s}$$

Hence, the tip of shadow is moving at the rate of $2\frac{2}{3}$ m/s towards the light source and length of the shadow is decreasing at the rate of 1 m/s.

S33. Given, line is $x\cos\alpha + y\sin\alpha = p$ ... (i)

and curve is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\Rightarrow$ $b^2x^2 + a^2y^2 = a^2b^2$ ... (ii)

Now, differentiating Eq. (ii) w.r.t. x, we get

$$b^2 \cdot 2x + a^2 \cdot 2y \cdot \frac{dy}{dx} = 0$$

$\Rightarrow$ $\frac{dy}{dx} = \frac{-2b^2x}{2a^2y} = \frac{-xb^2}{ya^2}$ ... (iii)

From Eq. (i), $y\sin\alpha = p - x\cos\alpha$

$\Rightarrow$ $y = -x\cot\alpha + \frac{p}{\sin\alpha}$

Thus, slope of the line is $(-\cot\alpha)$.

So, the given equation of line will be tangent to the Eq. (ii), if $\left(-\frac{x}{y}\cdot\frac{b^2}{a^2}\right) = (-\cot\alpha)$

$\Rightarrow$ $\frac{x}{a^2\cos\alpha} = \frac{y}{b^2\sin\alpha} = k$ [Say]

$\Rightarrow$ $x = ka^2\cos\alpha$

and $y = b^2k\sin\alpha$

So, the line $x\cos\alpha + y\sin\alpha = p$ will touch the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at point $(ka^2\cos\alpha, kb^2\sin\alpha)$.

From Eq. (i), $ka^2\cos^2\alpha + kb^2\sin^2\alpha = p$

$\Rightarrow$ $a^2\cos^2\alpha + b^2\sin^2\alpha = \frac{p}{k}$

$\Rightarrow$ $(a^2\cos^2\alpha + b^2\sin^2\alpha)^2 = \frac{p^2}{k^2}$ ... (iv)

From Eq. (ii), $b^2k^2a^4\cos^2\alpha + a^2k^2b^4\sin^2\alpha = a^2b^2$

$\Rightarrow$ $k^2(a^2\cos^2\alpha + b^2\sin^2\alpha) = 1$

$\Rightarrow$ $(a^2\cos^2\alpha + b^2\sin^2\alpha)^2 = \frac{1}{k^2}$ ... (v)

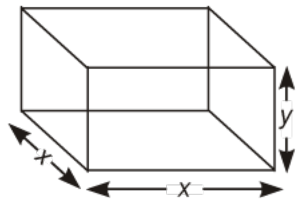
On dividing Eq. (iv) by Eq. (v), we get

$a^2\cos^2\alpha + b^2\sin^2\alpha = p^2$ **Hence proved.**

S34. Since, Volume of the box = 1024 cm^3.

Let length of the side of square base be x cm and height of the box be y cm.

$\therefore$ Volume of the box $(V) = x^2 \cdot y = 1024$



Since, $x^2 y = 1024 \Rightarrow y = \dfrac{1024}{x^2}$

Let C denotes the cost of the box.

$$\therefore \quad C = 2x^2 \times 5 + 4xy \times 2.50$$

$$= 10x^2 + 10xy = 10x(x + y)$$

$$= 10x\left(x + \frac{1024}{x^2}\right)$$

$$= \frac{10x}{x^2}(x^3 + 1024)$$

$$\Rightarrow \quad C = 10x^2 + \frac{10240}{x} \quad \text{... (i)}$$

On differentiating both sides w.r.t. x, we get

$$\frac{dC}{dx} = 20x + 10240\,(-x)^{-2} \quad \text{... (ii)}$$

$$= 20x - \frac{10240}{x^2}$$

Now, $\dfrac{dC}{dx} = 0$

$$\Rightarrow \quad 20x = \frac{10240}{x^2}$$

$$\Rightarrow \quad 20x^3 = 10240$$

$$\Rightarrow \quad x^3 = 512 = 8^3 \Rightarrow x = 8$$

Again, differentiating Eq. (ii) w.r.t. x, we get

$$\frac{d^2C}{dx^2} = 20 - 10240(-2)\cdot\frac{1}{x^3}$$

$$= 20 + \frac{20480}{x^3} > 0$$

$$\therefore \quad \left(\frac{d^2C}{dx^2}\right)_{x=8} = 20 + \frac{20480}{512} = 60 > 0$$

For $x = 8$, cost is minimum and the corresponding least cost of the box,

$$C(8) = 10 \cdot 8^2 + \frac{10240}{8}$$

$$= 640 + 1280 = 1920$$

$\therefore$ Least cost = Rs. 1920.

S35. Let length of one edge of cube be x units and radius of sphere be r units

$\therefore$ Surface Area of cube = $6x^2$

and, Surface area of sphere = $4\pi r^2$

Also, $6x^2 + 4\pi r^2 = k$ [Constant, given]

$$\Rightarrow \quad 6x^2 = k - 4\pi r^2$$

$$\Rightarrow \quad x^2 = \frac{k - 4\pi r^2}{6}$$

$$\Rightarrow \quad x = \left[\frac{k - 4\pi r^2}{6}\right]^{1/2} \quad \text{... (i)}$$

Now, Volume of cube = x^3

and Volume of sphere = $\frac{4}{3}\pi r^3$

Let sum of volume of the cube and volume of the sphere be given by

$$S = x^3 + \frac{4}{3}\pi r^3 = \left[\frac{k - 4\pi r^2}{6}\right]^{3/2} + \frac{4}{3}\pi r^3$$

On differentiating both sides w.r.t. r, we get

$$\frac{dS}{dr} = \frac{3}{2}\left[\frac{k - 4\pi r^2}{6}\right]^{1/2} \cdot \left(\frac{-8\pi r}{6}\right) + \frac{12}{3}\pi r^2$$

$$= -2\pi r\left[\frac{k - 4\pi r^2}{6}\right]^{1/2} + 4\pi r^2 \quad \text{... (ii)}$$

$$= -2\pi r\left[\left\{\frac{k - 4\pi r^2}{6}\right\}^{1/2} - 2r\right]$$

Now, $\frac{dS}{dr} = 0$

$$\Rightarrow \quad r = 0 \quad \text{or} \quad 2r = \left(\frac{k - 4\pi r^2}{6}\right)^{1/2}$$

$$\Rightarrow \quad 4r^2 = \frac{k - 4\pi r^2}{6} \Rightarrow 24r^2 = k - 4\pi r^2$$

$$\Rightarrow \quad 24r^2 + 4\pi r^2 = k \Rightarrow r^2[24 + 4\pi] = \text{k}$$

$$\therefore \quad r = 0 \quad \text{or} \quad r = \sqrt{\frac{k}{24 + 4\pi}} = \frac{1}{2}\sqrt{\frac{k}{6 + \pi}}$$

We know that, $r \neq 0$

$$\therefore \qquad r = \frac{1}{2}\sqrt{\frac{k}{6+\pi}}$$

Again, differentiating w.r.t r in Eq. (ii), we get

$$\frac{d^2S}{dr^2} = \frac{d}{dr}\left[-2\pi r\left\{\left(\frac{k-4\pi r^2}{6}\right)^{1/2} + 4\pi r^2\right\}\right]$$

$$= -2\pi\left[r\cdot\frac{1}{2}\left(\frac{k-4\pi r^2}{6}\right)^{-1/2}\cdot\left(\frac{-8\pi r}{6}\right) + \left(\frac{k-4\pi r^2}{6}\right)^{1/2}\cdot 1\right] + 4\pi\cdot 2r$$

$$= -2\pi\left[r\cdot\frac{1}{2\sqrt{\frac{k-4\pi r^2}{6}}}\cdot\left(\frac{-8\pi r}{6}\right) + \sqrt{\frac{k-4\pi r^2}{6}}\right] + 8\pi r$$

$$= -2\pi\left[\frac{-8\pi r^2 + 12\left(k - \frac{4\pi r^2}{6}\right)}{12\sqrt{\frac{k-4\pi r^2}{6}}}\right] + 8\pi r$$

$$= -2\pi\left[\frac{-48\pi r^2 + 72k - 48\pi r^2}{72\sqrt{\frac{k-4\pi r^2}{6}}}\right] + 8\pi r = -2\pi\left[\frac{-96\pi r^2 + 72k}{72\sqrt{\frac{k-4\pi r^2}{6}}}\right] + 8\pi r > 0$$

For $r = \frac{1}{2}\sqrt{\frac{k}{6+\pi}}$, then the sum of their volume is minimum.

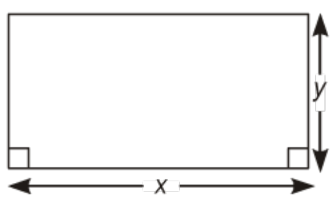
For $r = \frac{1}{2}\sqrt{\frac{k}{6+\pi}}$, $\qquad x = \left[\frac{k - 4\pi\cdot\frac{1}{4}\frac{k}{6+\pi}}{6}\right]^{1/2}$

$$= \left[\frac{(6+\pi)k - \pi k}{6(6+\pi)}\right]^{1/2} = \left[\frac{k}{6+\pi}\right]^{1/2} = 2r$$

Since, the sum of their volume is minimum when $x = 2r$.

Hence, the ratio of an edge of cube to the diameter of the sphere is 1 : 1.

S36. Let breadth and length of the rectangle be x and y, respectively.



$\therefore$ Perimeter of the rectangle = 36 cm

$\Rightarrow \qquad 2x + 2y = 36$

$\Rightarrow \qquad x + y = 18$

$\Rightarrow \quad y = 18 - x \quad \text{... (i)}$

Let the rectangle is being revolved about its length y.

Then, volume (V) of resultant cylinder = $\pi x^2 \cdot y$

$$\Rightarrow \quad V = \pi x^2 \cdot (18 - x) \quad [\because \; V = \pi r^2 h] \quad \text{[Using Eq. (i)]}$$

$$= 18\pi x^2 - \pi x^3 = \pi [18x^2 - x^3]$$

On differentiating both sides w.r.t. x, we get

$$\frac{dV}{dx} = \pi (36x - 3x^2)$$

Now, $\frac{dV}{dx} = 0$

$\Rightarrow \quad 36x = 3x^2$

$\Rightarrow \quad 3x^2 - 36x = 0$

$\Rightarrow \quad 3(x^2 - 12x) = 0$

$\Rightarrow \quad 3x(x - 12) = 0$

$\Rightarrow \quad x = 0, \quad x = 12$

$\therefore \quad x = 12 \quad [\because \; x \neq 0]$

Again, differentiating w.r.t. x, we get

$$\frac{d^2V}{dx^2} = \pi (36 - 6x)$$

$$\Rightarrow \quad \left(\frac{d^2V}{dx^2}\right)_{x = 12} = \pi (36 - 6 \times 12) = -36\pi < 0$$

At $x = 12$, volume of the resultant cylinder is maximum.

So, the dimensions of rectangle are 12 cm and 6 cm, respectively [Using Eq. (i)]

$\therefore$ Maximum volume of resultant cylinder,

$$(V)_{x = 12} = \pi [18 \cdot (12)^2 - (12)^3]$$

$$= \pi [12^2(18 - 12)]$$

$$= \pi \times 144 \times 6$$

$$= 864\pi \text{ cm}^3.$$

S37. Consider that company increases the annual subscription by Rs. x.

So, x subscribes will discontinue the service.

$\therefore$ Total revenue of company after the increment is given by

$$R(x) = (500 - x)(300 + x)$$

$$= 15 \times 10^4 + 500x - 300x - x^2$$

$$= -x^2 + 200x + 150000$$

On differentiating both sides w.r.t. x, we get

$$R'(x) = -2x + 200$$

Now, $R'(x) = 0$

$\Rightarrow$ $2x = 200 \Rightarrow x = 100$

$\therefore$ $R''(x) = -2 < 0$

So, $R(x)$ is maximum when $x = 100$.

Hence, the company should increase the subscription fee by Rs. 100, so that it has maximum profit.

S38. W have, $x = 3\cos\theta - \cos^3\theta$

Therefore,
$$\frac{dx}{d\theta} = -3\sin\theta + 3\cos^2\theta\sin\theta$$
$$= -3\sin\theta(1-\cos^2\theta) = -3\sin^3\theta.$$

$$\frac{dy}{d\theta} = 3\cos\theta - 3\sin^2\theta\cos\theta$$
$$= 3\cos\theta(1-\sin^2\theta) = 3\cos^3\theta$$

$$\frac{dy}{dx} = -\frac{\cos^3\theta}{\sin^3\theta}.$$

Therefore, Slope of normal = $\dfrac{\sin^3\theta}{\cos^3\theta}$

Hence, the equation of normal is

$$y - (3\sin\theta - \sin^3\theta) = \frac{\sin^3\theta}{\cos^3\theta}[x - (3\cos\theta - \cos^3\theta)]$$

$\Rightarrow$ $y\cos^3\theta - 3\sin\theta\cos^3\theta + \sin^3\theta\cos^3\theta = x\sin^3\theta - 3\sin^3\theta\cos\theta + \sin^3\theta\cos^3\theta$

$\Rightarrow$ $y\cos^3\theta - x\sin^3\theta = 3\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta)$

$$= \frac{3}{2}\sin 2\theta \cdot \cos 2\theta$$

$$= \frac{3}{4}\sin 4\theta$$

or $4(y\cos^3\theta - x\sin^3\theta) = 3\sin 4\theta.$

S39. We have given that, the sum of the surface areas of a rectangular parallelopiped with sides x, $2x$ and $\dfrac{x}{3}$ and a sphere is constant.

Let S be the sum of both the surface area.

$\therefore$
$$S = 2\left(x \cdot 2x + 2x \cdot \frac{x}{3} + \frac{x}{3} \cdot x\right) + 4\pi r^2 = k$$

$$k = 2\left(2x^2 + \frac{2x^2}{3} + \frac{x^2}{3}\right) + 4\pi r^2$$

$$= 2[3x^2] + 4\pi r^2 = 6x^2 + 4\pi r^2$$

$$\Rightarrow \quad 4\pi r^2 = k - 6x^2$$

$$\Rightarrow \quad r^2 = \frac{k - 6x^2}{4\pi}$$

$$\Rightarrow \quad r = \sqrt{\frac{k - 6x^2}{4\pi}} \quad \text{... (i)}$$

Let V denotes the volume of both the parallelopiped and the sphere.

Then, $$V = 2x \cdot x \cdot \frac{x}{3} + \frac{4}{3}\pi r^3 = \frac{2}{3}x^3 + \frac{4}{3}\pi r^3$$

$$= \frac{2}{3}x^3 + \frac{4}{3}\pi\left(\frac{k - 6x^2}{4\pi}\right)^{3/2}$$

$$= \frac{2}{3}x^3 + \frac{4}{3}\pi \cdot \frac{1}{8\pi^{3/2}}(k - 6x^2)^{3/2}$$

$$= \frac{2}{3}x^3 + \frac{1}{6\sqrt{\pi}}(k - 6x^2)^{3/2} \quad \text{... (ii)}$$

On differentiating both sides w.r.t. x, we get

$$\frac{dV}{dx} = \frac{2}{3} \cdot 3x^2 + \frac{1}{6\sqrt{\pi}} \cdot \frac{3}{2}(k - 6x^2)^{1/2} \cdot (-12x)$$

$$= 2x^2 - \frac{12x}{4\sqrt{\pi}}\sqrt{k - 6x^2}$$

$$= 2x^2 - \frac{3x}{\sqrt{\pi}}(k - 6x^2)^{1/2} \quad \text{... (iii)}$$

$$\because \quad \frac{dV}{dx} = 0$$

$$\Rightarrow \quad 2x^2 = \frac{3x}{\sqrt{\pi}}(k - 6x^2)^{1/2}$$

$$\Rightarrow \quad 4x^4 = \frac{9x^2}{\pi}(k - 6x^2)$$

$$\Rightarrow \quad 4\pi x^4 = 9kx^2 - 54x^4$$

$$\Rightarrow \quad 4\pi x^4 + 54x^4 = 9kx^2$$

$$\Rightarrow \quad x^4[4\pi + 54] = 9 \cdot k \cdot x^2$$

$$\Rightarrow \qquad x^2 = \frac{9k}{4\pi + 54}$$

$$\Rightarrow \qquad x = 3 \cdot \sqrt{\frac{k}{4\pi + 54}} \qquad \text{... (iv)}$$

Again, differenting Eq. (iii) w.r.t. x, we get

$$\frac{d^2V}{dx^2} = 4x - \frac{3}{\sqrt{\pi}}\left[x \cdot \frac{1}{2}(k - 6x^2)^{-1/2} \cdot (-12x) + (k - 6x^2)^{1/2} \cdot 1\right]$$

$$= 4x - \frac{3}{\sqrt{\pi}}\left[-6x^2 \cdot (k - 6x^2)^{-1/2} + (k - 6x^2)^{1/2}\right]$$

$$= 4x - \frac{3}{\sqrt{\pi}}\left[\frac{-6x^2 + k - 6x^2}{\sqrt{k - 6x^2}}\right]$$

$$= 4x - \frac{3}{\sqrt{\pi}}\left[\frac{k - 12x^2}{\sqrt{k - 6x^2}}\right]$$

Now,

$$\left(\frac{d^2V}{dx^2}\right)_{x = 3 \cdot \sqrt{\frac{k}{4\pi + 54}}} = 4 \cdot 3\sqrt{\frac{k}{4\pi + 54}} - \frac{3}{\sqrt{\pi}}\left[\frac{k - 12 \cdot 9 \cdot \frac{k}{4\pi + 54}}{\sqrt{k - \frac{6 \cdot 9 \cdot k}{4\pi + 54}}}\right]$$

$$= 12\sqrt{\frac{k}{4\pi + 54}} - \frac{3}{\sqrt{\pi}}\left[\frac{k - \frac{108k}{4\pi + 54}}{\sqrt{k - \frac{54k}{4\pi + 54}}}\right]$$

$$= 12\sqrt{\frac{k}{4\pi + 54}} - \frac{3}{\sqrt{\pi}}\left[\frac{4k\pi + 54k - 108k/4\pi + 54}{\sqrt{4k\pi + 54k - 54k/4\pi + 54}}\right]$$

$$= 12\sqrt{\frac{k}{4\pi + 54}} - \frac{3}{\sqrt{\pi}}\left[\frac{4k\pi - 54\,k}{\sqrt{4k\pi}\,\sqrt{4\pi + 54}}\right]$$

$$= 12\sqrt{\frac{k}{4\pi + 54}} - \frac{6}{\sqrt{\pi}}\left[\frac{k(2\pi - 27)}{\sqrt{k}\,\sqrt{16\pi^2 + 216\pi}}\right]$$

$$\left[\text{since, } (2\pi - 27) < 0 \Rightarrow \frac{d^2V}{dx^2} > 0; k > 0\right]$$

For $x = 3\sqrt{\dfrac{k}{4\pi + 54}}$, then the sum of volumes is minimum.

For $x = 3\sqrt{\dfrac{k}{4\pi + 54}}$, then $r = \sqrt{\dfrac{k - 6x^2}{4\pi}}$ [Using Eq. (i)]

$$= \frac{1}{2\sqrt{\pi}}\sqrt{k - 6 \cdot \frac{9k}{4\pi + 54}}$$

$$= \frac{1}{2\sqrt{\pi}} \cdot \sqrt{\frac{4k\pi + 54k - 54k}{4\pi + 54}}$$

$$= \frac{1}{2\sqrt{\pi}}\sqrt{\frac{4k\pi}{4\pi + 54}} = \frac{\sqrt{k}}{\sqrt{4\pi + 54}} = \frac{1}{3}x$$

$\Rightarrow$ $x = 3r$ **Hence proved.**

$\therefore$ Minimum sum of volume,

$$V_{\left(x = 3 \cdot \sqrt{\frac{k}{4\pi + 54}}\right)} = \frac{2}{3}x^3 + \frac{4}{3}\pi r^3 = \frac{2}{3}x^3 + \frac{4}{3}\pi \cdot \left(\frac{1}{3}x\right)^3$$

$$= \frac{2}{3}x^3 + \frac{4}{3}\pi \cdot \frac{x^3}{27} = \frac{2}{3}x^3\left(1 + \frac{2\pi}{27}\right).$$