Date: 18/10/2021

Q1. If $f(x) = |\cos x|$, find $f'\left(\frac{3\pi}{4}\right)$.

Q2. Given $f(x) = \frac{1}{x-1}$. Find the points of discontinuity of the composite function $y = f[f(x)]$.

Q3. If $f(x) = \begin{cases} \frac{x^3 + x^2 - 16x + 20}{(x-2)^2}, & x \neq 2 \\ k, & x = 2 \end{cases}$ is continuous at $x = 2$, find the value of k.

Q4. Discuss the continuity of the function $f(x) = \sin x \cdot \cos x$.

Q5. Examine the continuity of the function

$$f(x) = \begin{cases} \frac{|x-4|}{2(x-4)}, & \text{if } x \neq 4 \\ 0, & \text{if } x = 4 \end{cases} \text{ at } x = 4.$$

Q6. Examine the continuity of the function

$$f(x) = \begin{cases} 3x + 5, & \text{if } x \geq 2 \\ x^2, & \text{if } x < 2 \end{cases} \text{ at } x = 2.$$

Q7. Examine the continuity of the function $f(x) = x^3 + 2x^2 - 1$ at $x = 1$.

Q8. If $f(x) = |\cos x - \sin x|$, find $f'\left(\frac{\pi}{6}\right)$.

Q9. Examine the continuity of the function:

$$f(x) = \begin{cases} |x| \cos\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \text{ at } x = 0.$$

Q10. Examine the continuity of the function:

$$f(x) = \begin{cases} \frac{2x^2 - 3x - 2}{x-2}, & \text{if } x \neq 2 \\ 5, & \text{if } x = 2 \end{cases} \text{ at } x = 2.$$

Q11. Examine the continuity of the function

$$f(x) = \begin{cases} \frac{1 - \cos 2x}{x^2}, & \text{if } x \neq 0 \\ 5, & \text{if } x = 0 \end{cases} \text{ at } x = 0.$$

Q12. Find all points of discontinuity of the function $f(t) = \frac{1}{t^2 + t - 2}$, where $t = \frac{1}{x-1}$.

Q13. If the function $f(x) = \frac{1}{x+2}$, then find the points of discontinuity of the composite function $y = f\{f(x)\}$.

Q14. Prove that the function f defined by

$$f(x) = \begin{cases} \dfrac{x}{|x| + 2x^2}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$$

remains discontinuous at $x = 0$, regardless the choice of k.

Q15. For what value of k $f(x)$ is continuous at $x = 2$, where

$$f(x) = \begin{cases} \dfrac{2^{x+2} - 16}{4^x - 16}, & \text{if } x \neq 2 \\ k, & \text{if } x = 2 \end{cases}.$$

Q16. For what value of k $f(x)$ is continuous at $x = 5$, where

$$f(x) = \begin{cases} 3x - 8, & \text{if } x \leq 5 \\ 2k, & \text{if } x > 5 \end{cases}.$$

Q17. Examine the continuity of the function: $f(x) = |x| + |x - 1|$ at $x = 1$.

Q18. Examine the continuity of the function:

$$f(x) = \begin{cases} \dfrac{x^2}{2}, & \text{if } 0 \leq x \leq 1 \\ 2x^2 - 3x + \dfrac{3}{2}, & \text{if } 1 < x \leq 2 \end{cases} \text{ at } x = 1.$$

Q19. Examine the continuity of the function:

$$f(x) = \begin{cases} \dfrac{e^{1/x}}{1 + e^{1/x}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \text{ at } x = 0.$$

Q20. Examine the continuity of the function:

$$f(x) = \begin{cases} |x - a| \sin \dfrac{1}{x - a}, & \text{if } x \neq a \\ 0, & \text{if } x = a \end{cases} \text{ at } x = a.$$

Q21. Find $f'(x)$ if $f(x) = \sec^{-1}\left(\dfrac{1}{4x^3 - 3x}\right), 0 < x < \dfrac{1}{\sqrt{2}}$.

Q22. Find $f'(x)$ if $f(x) = \tan^{-1}\left(\dfrac{a \cos x - b \sin x}{b \cos x + a \sin x}\right), \dfrac{-\pi}{2} < x < \dfrac{\pi}{2}$ and $\dfrac{a}{b} \tan x > -1$.

Q23. Find $f'(x)$ if $f(x) = \cos^{-1}\left(\dfrac{\sin x + \cos x}{\sqrt{2}}\right), -\dfrac{\pi}{4} < x < \dfrac{\pi}{4}$.

Q24. Find $f'(x)$ if $f(x) = \sin^{-1} \dfrac{1}{\sqrt{x + 1}}$.

Q25. A function $f : R \to R$ satisfies the relation $f(x + y) = f(x) \cdot f(y)$ for all $x, y \in R$, $f(x) \neq 0$. Suppose that the function is differentiable at $x = 0$ and $f'(0) = 2$, then prove that $f'(x) = 2f(x)$.

Q26. Show that $f(x) = |x - 5|$ is continuous but not differentiable at $x = 5$.

Q27. Examine the differentiability of f, where f is defined by

$$f(x) = \begin{cases} 1 + x, & \text{if} \quad x \le 2 \\ 5 - x, & \text{if} \quad x > 2 \end{cases} \quad \text{at} \quad x = 2.$$

Q28. Examine the differentiability of f, where f is defined by

$$f(x) = \begin{cases} x[x], & \text{if} \quad 0 \le x < 2 \\ (x-1)x, & \text{if} \quad 2 \le x < 3 \end{cases} \quad \text{at} \quad x = 2.$$

Q29. Show that the function $f(x) = |\sin x + \cos x|$ is continuous at $x = \pi$.

Q30. If $y = \tan(x + y)$, find $\frac{dy}{dx}$.

Q31. Find $f'(x)$ if $f(x) = \sqrt{\tan \sqrt{x}}$.

Q32. If $x = e^{x/y}$, then prove that $\frac{dy}{dx} = \frac{x - y}{x \log x}$.

Q33. $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, then show that $\frac{dy}{dx} \cdot \frac{dx}{dy} = 1$.

Q34. Find $f'(x)$ if $f(x) = (x^2 + y^2)^2 = xy$.

Q35. Find $f'(x)$ if $f(x) = \sec(x + y) = xy$.

Q36. Find $f'(x)$ if $f(x) = \sin(xy) + \frac{x}{y} = x^2 - y$

Q37. Find $f'(x)$ if $f(x) = \sin^m x \cdot \cos^n x$.

Q38. Find $f'(x)$ if $f(x) = \sin x^2 + \sin^2 x + \sin^2(x^2)$.

Q39. Find $f'(x)$ if $f(x) = \cos(\tan\sqrt{x+1})$.

Q40. Find $f'(x)$ if $f(x) = \sin^n(ax^2 + bx + c)$.

Q41. Find $f'(x)$ if $f(x) = \sin\sqrt{x} + \cos^2\sqrt{x}$.

Q42. Find $\frac{dy}{dx}$, if $y = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right), -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$.

Q43. Find $f'(x)$ if $f(x) = \tan^{-1}(x^2 + y^2) = a$.

Q44. If $e^x + e^y = e^{x+y}$, prove that $\frac{dy}{dx} = -e^{y-x}$.

Q45. Find $f'(x)$ if $f(x) = 2^{\cos^2 x}$.

Q46. Find $f'(x)$ if $f(x) = \frac{8^x}{x^8}$.

Q47. Find $f'(x)$ if $f(x) = \log(x + \sqrt{x^2 + a})$.

Q48. Find $f'(x)$ if $f(x) = \text{Log}[\log(\log x^5)]$.

Q49. Find $f'(x)$ if $f(x) = (\sin x)^{\cos x}$.

Q50. Find $\frac{dy}{dx}$ if $x = t + \frac{1}{t}$, $y = t - \frac{1}{t}$.

Q51. Find $\frac{dy}{dx}$ if $x = 3\cos\theta - 2\cos^3\theta$, $y = 3\sin\theta - 2\sin^3\theta$.

Q52. Verify the Rolle's theorem for the function:

$$f(x) = x(x-1)^2 \text{ in } [0, 1].$$

Q53. Differentiate $\frac{x}{\sin x}$ w.r.t. $\sin x$.

Q54. Find $\frac{dy}{dx}$ if $x = \frac{1+\log t}{t^2}$, $y = \frac{3+2\log t}{t}$.

Q55. If $x = 3\sin t - \sin 3t$, $y = 3\cos t - \cos 3t$, then find $\frac{dy}{dx}$ at $t = \frac{\pi}{3}$.

Q56. Verify the Rolle's theorem for the function:

$$f(x) = \log(x^2 + 2) - \log 3 \text{ in } [-1, 1].$$

Q57. If $y = \tan^{-1} x$, then find $\frac{d^2y}{dx^2}$ in terms of y alone.

Q58. Verify the Rolle's theorem for the function:

$$f(x) = x(x+3)e^{-x/2} \text{ in } [-3, 0].$$

Q59. Find the points on the curve $y = (\cos x - 1)$ in $[0, 2\pi]$, where the tangent is parallel to X-axis.

Q60. Using Rolle's theorem, find the point on the curve $y = x(x-4)$, $x \in [0, 4]$, where the tangent is parallel to X-axis.

Q61. Verify Rolle's theorem for the function, $f(x) = \sin 2x$ in $\left[0, \frac{\pi}{2}\right]$.

Q62. Verify mean value theorem for the following function:

$$f(x) = \frac{1}{4x-1} \text{ in } [1, 4].$$

Q63. Verify mean value theorem for the following function:

$$f(x) = x^3 - 2x^2 - x + 3 \text{ in } [0, 1].$$

Q64. Verify mean value theorem for the following function:

$$f(x) = \sin x - \sin 2x \text{ in } [0, \pi].$$

Q65. Verify mean value theorem for the following function:

$$f(x) = \sqrt{25 - x^2} \text{ in } [1, 5].$$

Q66. Find a point on the curve $y = (x-3)^2$, where the tangent is parallel to the chord joining the points (3, 0) and (4, 1).

Q67. Using mean value theorem, prove that there is a point on the curve $y = 2x^2 - 5x + 3$ between the points $A(1, 0)$ and $B(2, 1)$, where tangent is parallel to the chord AB, Also, find that point.

Q68. Find the values of p and q, so that $f(x) = \begin{cases} x^2 + 3x + p, & \text{if } x \le 1 \\ qx + 2, & \text{if } x > 1 \end{cases}$ is differentiable at $x = 1$.

Q69. If $f(x) = \dfrac{\sqrt{2}\cos x - 1}{\cot x - 1}, x \neq \dfrac{\pi}{4}$ find the value of $f\left(\dfrac{\pi}{4}\right)$ so that $f(x)$ becomes continuous at $x = \dfrac{\pi}{4}$.

Q70. Find $f'(x)$ if $f(x) = \tan^{-1}\sqrt{\dfrac{1-\cos x}{1+\cos x}}, -\dfrac{\pi}{4} < x < \dfrac{\pi}{4}$.

Q71. Examine the differentiability of the function f defined by

$$f(x) = \begin{cases} 2x+3, & \text{if } -3 \le x < -2 \\ x+1, & \text{if } -2 \le x < 0 \\ x+2, & \text{if } 0 \le x \le 1 \end{cases}$$

Q72. If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, then prove that $\dfrac{dy}{dx} = \sqrt{\dfrac{1-y^2}{1-x^2}}$.

Q73. Find $f'(x)$ if $f(x) = (x+1)^2(x+2)^3(x+3)^4$.

Q74. If $\sin x = \dfrac{2t}{1+t^2}$, $\tan y = \dfrac{2t}{1-t^2}$, then prove that $\dfrac{dy}{dx} = 1$.

Q75. If $\sin x = \dfrac{2t}{1+t^2}$, $\tan y = \dfrac{2t}{1-t^2}$, then prove that $\dfrac{dy}{dx} = 1$.

Q76. If $x^m \cdot y^n = (x+y)^{m+n}$, prove that: (i) $\dfrac{dy}{dx} = \dfrac{y}{x}$ and (ii) $\dfrac{d^2y}{dx^2} = 0$.

Q77. Find the value of $\dfrac{dy}{dx}$, if $y = x^{\tan x} + \sqrt{\dfrac{x^2+1}{2}}$.

Date: 18/10/2021

S1. When $\frac{\pi}{2} < x < \pi$, cos $x < 0$ so that $|\cos x| = -\cos x$, *i.e.*, $f(x) = -\cos x$.

$\Rightarrow$ $$f'(x) = \sin x.$$

Hence, $$f'\left(\frac{3\pi}{4}\right) = \sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}.$$

S2. We know that $f(x) = \frac{1}{x-1}$ is discontinuous at $x = 1$.

Now, for $x \neq 1$, $$f(f(x)) = f\left(\frac{1}{x-1}\right) = \frac{1}{\frac{1}{x-1} - 1} = \frac{x-1}{2-x}$$

which is discontinuous at $x = 2$.

Hence, the points of discontinuity are $x = 1$ and $x = 2$.

S3. Given, $$f(2) = k.$$

Now, $$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = \lim_{x \to 2} \frac{x^3 + x^2 - 16x + 20}{(x-2)^2}$$

$$= \lim_{x \to 2} \frac{(x+5)(x-2)^2}{(x-2)^2} = \lim_{x \to 2} (x+5) = 7$$

As f is continuous at $x = 2$, we have

$$\lim_{x \to 2} f(x) = f(2)$$

$\Rightarrow$ $$k = 7.$$

S4. Since sin x and cos x are continuous functions and product of two continuous function is a continuous function, therefore $f(x) = \sin x \cdot \cos x$ is a continuous function.

S5. We have,

$$f(x) = \begin{cases} \frac{|x-4|}{2(x-4)}, & \text{if } x \neq 4 \\ 0, & \text{if } x = 4 \end{cases} \quad \text{at } x = 4.$$

At $x = 4$, $$\text{LHL} = \lim_{x \to 4^-} \frac{|x-4|}{2(x-4)}$$

$$= \lim_{h \to 0} \frac{|4 - h - 4|}{2[(4 - h) - 4]} = \lim_{h \to 0} \frac{|0 - h|}{(8 - 2h - 8)}$$

$$= \lim_{h \to 0} \frac{h}{-2h} = \frac{-1}{2} \quad \text{and} \quad f(4) = 0 \neq \text{LHL}$$

So, $f(x)$ is discontinuous at $x = 4$.

S6. We have,

$$f(x) = \begin{cases} 3x + 5, & \text{if} \quad x \geq 2 \\ x^2, & \text{if} \quad x < 2 \end{cases} \text{ at } x = 2.$$

At $x = 2$, LHL $= \lim_{x \to 2^-} (x)^2$

$$= \lim_{h \to 0} (2 - h)^2 = \lim_{h \to 0} (4 + h^2 - 4h) = 4$$

and RHL $= \lim_{x \to 2^+} (3x + 5)$

$$= \lim_{h \to 0} [3(1 + h) + 5] = 8$$

Since, LHL $\neq$ RHL at $x = 2$

So, $f(x)$ is discontinuous at $x = 2$.

S7. We have, $f(x) = x^3 + 2x^2 - 1$

$\therefore$ $\lim_{x \to 1^+} f(x) = \lim_{h \to 0} (1 + h)^3 + 2(1 + h)^2 - 1 = 2$

and $\lim_{x \to 1^-} f(x) = \lim_{h \to 0} (1 - h)^3 + 2(1 - h)^2 - 1 = 2$

$\therefore$ $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x)$

and $f(1) = 1 + 2 - 1 = 2.$

So, $f(x)$ is continuous at $x = 1$.

S8. When $0 < x < \frac{\pi}{4}$, $\cos x > \sin x$, so that $\cos x - \sin x > 0$, *i.e.*,

$$f(x) = \cos x - \sin x$$

$\Rightarrow$ $f'(x) = -\sin x - \cos x$

Hence, $f'\left(\frac{\pi}{6}\right) = -\sin\frac{\pi}{6} - \cos\frac{\pi}{6} = -\frac{1}{2}(1 + \sqrt{3}).$

S9. We have,

$$f(x) = \begin{cases} |x| \cos\frac{1}{x}, & \text{if} \quad x \neq 0 \\ 0, & \text{if} \quad x = 0 \end{cases} \quad \text{at} \quad x = 0.$$

At $x = 0$, LHL $= \lim_{x \to 0^-} |x| \cos\frac{1}{x}$

$$= \lim_{h \to 0} |0 - h| \cos\frac{1}{0 - h}$$

$$= \lim_{h \to 0} h \cos\left(\frac{-1}{h}\right)$$

$= 0 \times$ [an oscillating number between – 1 and 1] = 0

$$\text{RHL} = \lim_{x \to 0^+} |x| \cos \frac{1}{x}$$

$$= \lim_{h \to 0} |0 + h| \cos \frac{1}{(0 + h)}$$

$$= \lim_{h \to 0} h \cos \frac{1}{h}$$

$= 0 \times$ [an oscillating number between – 1 and 1] = 0

and $\quad f(0) = 0$

Since, $\quad$ LHL = RHL = $f(0)$

So, $f(x)$ is continuous at $x = 0$.

S10. We have,

$$f(x) = \begin{cases} \dfrac{2x^2 - 3x - 2}{x - 2}, & \text{if} \quad x \neq 2 \\ 5, & \text{if} \quad x = 2 \end{cases} \quad \text{at} \quad x = 2.$$

At $x = 2$, $\quad \text{LHL} = \lim_{x \to 2^-} \dfrac{2x^2 - 3x - 2}{x - 2}$

$$= \lim_{h \to 0} \frac{2(2 - h)^2 - 3(2 - h) - 2}{(2 - h) - 2}$$

$$= \lim_{h \to 0} \frac{8 + 2h^2 - 8h - 6 + 3h - 2}{-h}$$

$$= \lim_{h \to 0} \frac{2h^2 - 5h}{-h} = \lim_{h \to 0} \frac{h(2h - 5)}{-h} = 5$$

$$\text{RHL} = \lim_{x \to 2^+} \frac{2x^2 - 3x - 2}{x - 2}$$

$$= \lim_{h \to 0} \frac{2(2 + h)^2 - 3(2 + h) - 2}{(2 + h) - 2}$$

$$= \lim_{h \to 0} \frac{2h^2 + 5h}{h} = \lim_{h \to 0} \frac{h(2h + 5)}{h} = 5$$

and $\quad f(2) = 5$

$\therefore \quad$ LHL = RHL = $f(2)$

So, $f(x)$ is continuous at $x = 2$.

S11. We have,

$$f(x) = \begin{cases} \dfrac{1-\cos 2x}{x^2}, & \text{if} \quad x \neq 0 \\ 5, & \text{if} \quad x = 0 \end{cases} \quad \text{at} \quad x = 0.$$

At $x = 0$,

$$\text{LHL} = \lim_{x \to 0^-} \frac{1-\cos 2x}{x^2}$$

$$= \lim_{h \to 0} \frac{1-\cos 2(0-h)}{(0-h)^2}$$

$$= \lim_{h \to 0} \frac{1-\cos 2h}{h^2} \qquad [\because \cos(-\theta) = \cos\theta]$$

$$= \lim_{h \to 0} \frac{1-1+2\sin^2 h}{h^2} \qquad [\because \cos 2\theta = 1 - 2\sin^2\theta]$$

$$= \lim_{h \to 0} \frac{2(\sin h)^2}{(h)^2} = 2 \qquad \left[\because \lim_{h \to 0} \frac{\sin h}{h} = 1\right]$$

$$\text{RHL} = \lim_{x \to 0^+} \frac{1-\cos 2x}{x^2}$$

$$= \lim_{h \to 0} \frac{1-\cos 2(0+h)}{(0+h)^2}$$

$$= \lim_{h \to 0} \frac{2\sin^2 h}{h^2} = 2 \qquad \left[\because \lim_{h \to 0} \frac{\sin h}{h} = 1\right]$$

and $f(0) = 5$

Since, $\text{LHL} = \text{RHL} \neq f(0)$

So, $f(x)$ is not continuous at $x = 0$.

S12. We have,

$$f(t) = \frac{1}{t^2 + t - 2} \quad \text{and} \quad t = \frac{1}{x-1}$$

$$f(t) = \frac{1}{\left(\dfrac{1}{x^2+1-2x}\right) + \left(\dfrac{1}{x-1}\right) - \dfrac{2}{1}}$$

$$= \frac{1}{\left(\dfrac{1+x-1+[-2(x-1)^2]}{(x^2+1-2x)}\right)}$$

$$= \frac{x^2+1-2x}{x-2x^2-2+4x}$$

$$= \frac{x^2+1-2x}{-2x^2+5x-2}$$

$$= \frac{(x-1)^2}{-(2x^2-5x+2)}$$

$$= \frac{(x-1)^2}{(2x-1)(2-x)}$$

So, $f(t)$ is discontinuous at $2x - 1 = 0 \Rightarrow x = 1/2$

and $2 - x = 0 \Rightarrow x = 2.$

So, points of discontinuity of $f(t)$ is $\left\{\frac{1}{2}, 2\right\}$.

S13. We have,

$$f(x) = \frac{1}{x+2}$$

$$\therefore \quad y = f\{f(x)\}$$

$$= f\left(\frac{1}{x+2}\right) = \frac{1}{\frac{1}{x+2}+2}$$

$$= \frac{1}{1+2x+4}\cdot(x+2) = \frac{(x+2)}{(2x+5)}$$

So, the function y will not be continuous at those points, where it is not defined as it is a rational function.

Therefore, $y = \frac{(x+2)}{(2x+5)}$ is not defined, when $2x + 5 = 0$

$$\therefore \quad x = \frac{-5}{2}$$

Hence, y is discontinuous at $x = \frac{-5}{2}$.

S14. We have,

$$f(x) = \begin{cases} \frac{x}{|x|+2x^2}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$$

At $x = 0$, $\text{LHL} = \lim_{x\to 0^-} \frac{x}{|x|+2x^2} = \lim_{h\to 0} \frac{(0-h)}{|0-h|+2(0-h)^2}$

$$= \lim_{h\to 0} \frac{-h}{h+2h^2} = \lim_{h\to 0} \frac{-h}{h(1+2h)} = -1$$

$$\text{RHL} = \lim_{x\to 0^+} \frac{x}{|x|+2x^2} = \lim_{h\to 0} \frac{0+h}{|0+h|+2(0-h)^2}$$

$$= \lim_{h\to 0} \frac{h}{h+2h^2} = \lim_{h\to 0} \frac{h}{h(1+2h)} = 1$$

and $f(0) = k$

Since, LHL $\neq$ RHL for any value of k.

Hence, $f(x)$ is discontinuous at $x = 0$ regardless the choice of k.

S15. We have,

$$f(x) = \begin{cases} \dfrac{2^{x+2} - 16}{4^x - 16}, & \text{if } x \neq 2 \\ k, & \text{if } x = 2 \end{cases} \quad \text{at} \quad x = 2.$$

Since, $f(x)$ is continuous at $x = 2$.

$\therefore$ LHL = RHL = $f(2)$

At $x = 2$,

$$\lim_{x \to 2} \frac{2^x \cdot 2^2 - 2^4}{4^x - 4^2} = \lim_{x \to 2} \frac{4 \cdot (2^x - 4)}{(2^x)^2 - (4)^2}$$

$$= \lim_{x \to 2} \frac{4 \cdot (2^x - 4)}{(2^x - 4)(2^x + 4)} \qquad [\because\ a^2 - b^2 = (a + b)(a - b)]$$

$$= \lim_{x \to 2} \frac{4}{2^x + 4} = \frac{4}{8} = \frac{1}{2}$$

But $f(2) = k$

$\therefore$ $k = \dfrac{1}{2}$.

S16. We have,

$$f(x) = \begin{cases} 3x - 8, & \text{if } x \leq 5 \\ 2k, & \text{if } x > 5 \end{cases} \quad \text{at} \quad x = 5.$$

Since, $f(x)$ is continuous at $x = 5$.

$\therefore$ LHL = RHL = $f(5)$

Now, LHL $= \lim\limits_{x \to 5^-} (3x - 8) = \lim\limits_{h \to 0} [3(5 - h) - 8]$

$= \lim\limits_{h \to 0} [15 - 3h - 8] = 7$

RHL $= \lim\limits_{x \to 5^+} 2k = \lim\limits_{h \to 0} 2k$

and $f(5) = 3 \times 5 - 8 = 7$

$\therefore$ $2k = 7 \Rightarrow k = \dfrac{7}{2}$.

S17. We have $f(x) = |x| + |x - 1|$ at $x = 1$

At $x = 1$, LHL $= \lim\limits_{x \to 1^-} [|x| + |x - 1|]$

$= \lim\limits_{h \to 0} [|1 - h| + |1 - h - 1|] = 1 + 0 = 1$

and RHL $= \lim\limits_{x \to 1^+} [|x| + |x - 1|]$

$= \lim\limits_{h \to 0} [|1 + h| + |1 + h - 1|] = 1 + 0 = 1$

and $f(1) = |1| + |0| = 1$

$\therefore$ LHL = RHL = $f(1)$.

Hence, $f(x)$ is continuous at $x = 1$.

Note: Every modulus function is a continuous function at any real point.

S18. We have,

$$f(x) = \begin{cases} \dfrac{x^2}{2}, & \text{if } 0 \le x \le 1 \\ 2x^2 - 3x + \dfrac{3}{2}, & \text{if } 1 < x \le 2 \end{cases} \quad \text{at} \quad x = 1.$$

At $x = 1$,

$$\text{LHL} = \lim_{x \to 1^-} \frac{x^2}{2} = \lim_{h \to 0} \frac{(1-h)^2}{2}$$

$$= \lim_{h \to 0} \frac{1 + h^2 - 2h}{2} = \frac{1}{2}$$

$$\text{RHL} = \lim_{x \to 1^+} \left(2x^2 - 3x + \frac{3}{2}\right)$$

$$= \lim_{h \to 0} \left[2(1+h)^2 - 3(1+h) + \frac{3}{2}\right]$$

$$= \lim_{h \to 0} \left(2 + 2h^2 + 4h - 3 - 3h + \frac{3}{2}\right) = -1 + \frac{3}{2} = \frac{1}{2}$$

and $f(1) = \dfrac{1^2}{2} = \dfrac{1}{2}$ $[\because\ e^{-\infty} = 0]$

$\therefore$ LHL = RHL = $f(1)$ at $x = 1$.

Hence, $f(x)$ is continuous at $x = 1$.

S19. We have,

$$f(x) = \begin{cases} \dfrac{e^{1/x}}{1 + e^{1/x}}, & \text{if } x \ne 0 \\ 0, & \text{if } x = 0 \end{cases} \quad \text{at} \quad x = 0.$$

At $x = 0$,

$$\text{LHL} = \lim_{x \to 0^-} \frac{e^{1/x}}{1 + e^{1/x}} = \lim_{h \to 0} \frac{e^{1/0-h}}{1 + e^{1/0-h}}$$

$$= \lim_{h \to 0} \frac{e^{-1/h}}{1 + e^{-1/h}} = \lim_{h \to 0} \frac{1}{e^{1/h}(1 + e^{-1/h})}$$

$$= \lim_{h \to 0} \frac{1}{e^{1/h} + 1} = \frac{1}{e^{\infty} + 1} = \frac{1}{\infty + 1} \qquad [\because\ e^{\infty} = \infty]$$

$$= \frac{1}{\frac{1}{0}} = 0$$

$$\text{RHL} = \lim_{x \to 0^+} \frac{e^{1/x}}{1+e^{1/x}} = \lim_{h \to 0} \frac{e^{1/0+h}}{1+e^{1/0+h}}$$

$$= \lim_{h \to 0} \frac{e^{1/h}}{1+e^{1/h}} = \lim_{h \to 0} \frac{1}{e^{-1/h}+1} = \frac{1}{e^{-\infty}+1}$$

$$= \frac{1}{0+1} = 1 \qquad [\because \; e^{-\infty} = 0]$$

Hence, $\text{LHL} \neq \text{RHL}$ at $x = 0$.

So, $f(x)$ is discontinuous at $x = 0$.

S20. We have,

$$f(x) = \begin{cases} |x-a| \sin \dfrac{1}{x-a}, & \text{if} \quad x \neq a \\ 0, & \text{if} \quad x = a \end{cases} \quad \text{at} \quad x = a.$$

At $x = a$,

$$\text{LHL} = \lim_{x \to a^-} |x-a| \sin \frac{1}{x-a}$$

$$= \lim_{h \to 0} |a-h-a| \sin \left(\frac{1}{a-h-a} \right)$$

$$= \lim_{h \to 0} -h \sin \left(\frac{1}{h} \right) \qquad [\because \; \sin(-\theta) = -\sin\theta]$$

$= 0 \times$ [an oscillating number between -1 and 1] $= 0$

$$\text{RHL} = \lim_{x \to a^+} |x-a| \sin \frac{1}{x-a}$$

$$= \lim_{h \to 0} |a+h-a| \sin \left(\frac{1}{a+h-a} \right)$$

$$= \lim_{h \to 0} h \cos \frac{1}{h}$$

$= 0 \times$ [an oscillating number between -1 and 1] $= 0$

and $f(a) = 0$

Since, $\text{LHL} = \text{RHL} = f(a)$

So, $f(x)$ is continuous at $x = a$.

S21. Let

$$y = \sec^{-1} \left(\frac{1}{4x^3 - 3x} \right) \qquad \ldots \text{(i)}$$

On putting $x = \cos\theta$ in Eq. (i), we get

$$y = \sec^{-1} \frac{1}{4\cos^3\theta - 3\cos\theta}$$

$$= \sec^{-1} \frac{1}{\cos 3\theta}$$

$$= \sec^{-1} (\sec 3\theta) = 3\theta$$

$$= 3 \cos^{-1} x \qquad [\because \ \theta = \cos^{-1} x]$$

$$\therefore \qquad \frac{dy}{dx} = \frac{d}{dx} (3 \cos^{-1} x)$$

$$= 3 \cdot \frac{-1}{\sqrt{1-x^2}}.$$

S22. Let

$$y = \tan^{-1} \left(\frac{a \cos x - b \sin x}{b \cos x + a \sin x} \right)$$

$$= \tan^{-1} \left[\frac{\frac{a \cos x}{b \cos x} - \frac{b \sin x}{b \cos x}}{\frac{b \cos x}{b \cos x} + \frac{a \sin x}{b \cos x}} \right] = \tan^{-1} \left[\frac{\frac{a}{b} - \tan x}{1 + \frac{a}{b} \tan x} \right]$$

$$= \tan^{-1} \frac{a}{b} - \tan^{-1} \tan x \qquad \left[\because \ \tan^{-1} x - \tan^{-1} y = \tan^{-1} \left(\frac{x-y}{1+xy} \right) \right]$$

$$= \tan^{-1} \frac{a}{b} - x$$

$$\therefore \qquad \frac{dy}{dx} = \frac{d}{dx} \left(\tan^{-1} \frac{a}{b} \right) - \frac{d}{dx} (x)$$

$$= 0 - 1 \qquad \left[\because \ \frac{d}{dx} \left(\frac{a}{b} \right) = 0 \right]$$

$$= -1.$$

S23. Let

$$y = \cos^{-1} \left(\frac{\sin x + \cos x}{\sqrt{2}} \right)$$

$$\therefore \qquad \frac{dy}{dx} = \frac{d}{dx} \cos^{-1} \left(\frac{\sin x + \cos x}{\sqrt{2}} \right)$$

$$= \frac{-1}{\sqrt{1 - \left(\frac{\sin x + \cos x}{\sqrt{2}} \right)^2}} \cdot \frac{d}{dx} \left(\frac{\sin x + \cos x}{\sqrt{2}} \right) \qquad \left[\because \ \frac{d}{dx} (\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}} \right]$$

$$= \frac{-1}{\sqrt{1 - \frac{(\sin^2 x + \cos^2 x + 2 \sin x \cdot \cos x)}{2}}} \cdot \frac{1}{\sqrt{2}} (\cos x - \sin x)$$

$$= \frac{-1\cdot\sqrt{2}}{\sqrt{1-\sin 2x}}\cdot\frac{1}{\sqrt{2}}(\cos x - \sin x)$$

$[\because\; 1 - \sin 2x = (\cos x - \sin x)^2 = \cos^2 x + \sin^2 x - 2\sin x\cos x]$

$$= \frac{-1(\cos x - \sin x)}{(\cos x - \sin x)} = -1.$$

S24. Let $$y = \sin^{-1}\frac{1}{\sqrt{x+1}}$$

$\therefore$ $$\frac{dy}{dx} = \frac{d}{dx}\sin^{-1}\frac{1}{\sqrt{x+1}}$$

$$= \frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{x+1}}\right)^2}}\cdot\frac{d}{dx}\frac{1}{(x+1)^{1/2}} \qquad \left[\because\; \frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}\right]$$

$$= \frac{1}{\sqrt{\frac{x+1-1}{x+1}}}\cdot\frac{d}{dx}\cdot(x+1)^{-1/2}$$

$$= \sqrt{\frac{x+1}{x}}\cdot\frac{-1}{2}(x+1)^{-\frac{1}{2}-1}\cdot\frac{d}{dx}(x+1)$$

$$= \frac{(x+1)^{1/2}}{x^{1/2}}\cdot\left(-\frac{1}{2}\right)(x+1)^{-3/2} = \frac{-1}{2\sqrt{x}}\cdot\left(\frac{1}{x+1}\right).$$

S25. Let $f: R \to R$ satisfies the equation $f(x + y) = f(x)\cdot f(y),\ \forall x, y \in R,\ f(x) \neq 0$.

Let $f(x)$ is differentiable at $x = 0$ and $f'(0) = 2$.

$\Rightarrow$ $$f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0}$$

$\Rightarrow$ $$2 = \lim_{x\to 0}\frac{f(x)-f(0)}{x}$$

$\Rightarrow$ $$2 = \lim_{h\to 0}\frac{f(0+h)-f(0)}{0+h}$$

$\Rightarrow$ $$2 = \lim_{h\to 0}\frac{f(0)\cdot f(h)-f(0)}{h}$$

$\Rightarrow$ $$2 = \lim_{h\to 0}\frac{f(0)[f(h)-1]}{h} \qquad [\because\; f(0) = f(h)] \quad \ldots \text{(i)}$$

Also, $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x)\cdot f(h)-f(x)}{h} \qquad [\because\; f(x+y) = f(x)\cdot f(y)]$$

$$= \lim_{h \to 0} \frac{f(x)[f(h)-1]}{h} = 2f(x)$$ [Using Eq. (i)]

$\therefore$ $f'(x) = 2f(x)$.

S26. We have, $f(x) = |x - 5|$

$\therefore$ $$f(x) = \begin{cases} -(x-5), & \text{if} \quad x < 5 \\ x - 5, & \text{if} \quad x \geq 5 \end{cases}$$

For continuity at $x = 5$,

$$\text{LHL} = \lim_{x \to 5^-} (-x + 5)$$

$$= \lim_{h \to 0} [-(5-h)+5] = \lim_{h \to 0} h = 0$$

$$\text{RHL} = \lim_{x \to 5^+} (x - 5)$$

$$= \lim_{h \to 0} (5 + h - 5) = \lim_{h \to 0} h = 0$$

$\therefore$ $f(5) = 5 - 5 = 0$

$\Rightarrow$ LHL = RHL = $f(5)$

Hence, $f(x)$ is continuous at $x = 5$.

Now, $$Lf'(5) = \lim_{x \to 5^-} \frac{f(x)-f(5)}{x-5} = \lim_{x \to 5^-} \frac{-x+5-0}{x-5} = -1$$

$$Rf'(5) = \lim_{x \to 5^+} \frac{f(x)-f(5)}{x-5} = \lim_{x \to 5^+} \frac{x-5-0}{x-5} = 1$$

$\therefore$ $Lf'(5) \neq Rf'(5)$

So, $f(x) = |x - 5|$, is not differentiable at $x = 5$.

S27. We have,

$$f(x) = \begin{cases} 1+x, & \text{if} \quad x \leq 2 \\ 5-x, & \text{if} \quad x > 2 \end{cases} \quad \text{at} \quad x = 2.$$

For differentiability at $x = 2$,

$$Lf'(2) = \lim_{x \to 2^-} \frac{f(x)-f(2)}{x-2} = \lim_{x \to 2^-} \frac{(1+x)-(1+2)}{x-2}$$

$$= \lim_{h \to 0} \frac{(1+2-h)-3}{2-h-2} = \lim_{h \to 0} \frac{-h}{-h} = 1$$

$$Rf'(2) = \lim_{x \to 2^+} \frac{f(x)-f(2)}{x-2} = \lim_{x \to 2^+} \frac{(5-x)-3}{x-2}$$

$$= \lim_{h \to 0} \frac{5-(2+h)-3}{2+h-2}$$

$$= \lim_{h \to 0} \frac{5-2-h-3}{h} = \lim_{h \to 0} \frac{-h}{+h} = -1$$

$\because \quad Lf'(2) \neq Rf'(2)$

So, $f(x)$ is not differentiable at $x = 2$.

S28. We have,

$$f(x) = \begin{cases} x[x], & \text{if } 0 \le x < 2 \\ (x-1)x, & \text{if } 2 \le x < 3 \end{cases} \quad \text{at} \quad x = 2.$$

At $x = 2$,

$$Lf'(2) = \lim_{h \to 0} \frac{f(2-h) - f(2)}{-h}$$

$$= \lim_{h \to 0} \frac{(2-h)[2-h] - (2-1)2}{-h}$$

$$= \lim_{h \to 0} \frac{(2-h)(1) - 2}{-h}$$

$$= \lim_{h \to 0} \frac{2-h-2}{-h} = \lim_{h \to 0} \frac{-h}{-h} = 1$$

$$Rf'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{(2+h-1)(2+h) - (2-1)\cdot 2}{h}$$

$$= \lim_{h \to 0} \frac{(1+h)(2+h) - 2}{h}$$

$$= \lim_{h \to 0} \frac{2+h+2h+h^2-2}{h}$$

$$= \lim_{h \to 0} \frac{h^2+3h}{h} = \lim_{h \to 0} \frac{h(h+3)}{h} = 3$$

$\therefore \quad Lf'(2) \neq Rf'(2)$

So, $f(x)$, is not differentiable at $x = 2$.

S29. We have $\quad f(x) = |\sin x + \cos x| \quad \text{at} \quad x = \pi.$

Let $\quad g(x) = \sin x + \cos x$

and $\quad h(x) = |x|$

$\therefore \quad hog(x) = h[g(x)]$

$$= h(\sin x + \cos x)$$

$$= |\sin x + \cos x|$$

Since, $g(x) = \sin x + \cos x$ is a continuous function as it is forming with addition of two continuous functions $\sin x$ and $\cos x$.

Also, $h(x) = |x|$ is also a continuous function. Since, we know that composite functions of two continuous functions is also a continuous function.

Hence, $f(x) = |\sin x + \cos x|$ is a continuous function everywhere.

So, $f(x)$ is continuous at $x = \pi$.

S30. Given, $y = \tan(x + y)$.

Differentiating both sides w.r.t. x, we have

$$\frac{dy}{dx} = \sec^2(x+y)\frac{d}{dx}(x+y)$$

$$= \sec^2(x+y)\left(1+\frac{dy}{dx}\right)$$

or $$[1 - \sec^2(x + y)]\frac{dy}{dx} = \sec^2(x + y)$$

Therefore, $$\frac{dy}{dx} = \frac{\sec^2(x+y)}{1-\sec^2(x+y)} = -\operatorname{cosec}^2(x + y).$$

S31. Let, $$y = \sqrt{\tan\sqrt{x}}$$

Using chain rule, we have

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\tan\sqrt{x}}}\cdot\frac{d}{dx}(\tan\sqrt{x})$$

$$= \frac{1}{2\sqrt{\tan\sqrt{x}}}\cdot\sec^2\sqrt{x}\,\frac{d}{dx}(\sqrt{x})$$

$$= \frac{1}{2\sqrt{\tan\sqrt{x}}}(\sec^2\sqrt{x})\left(\frac{1}{2\sqrt{x}}\right)$$

$$= \frac{(\sec^2\sqrt{x})}{4\sqrt{x}\sqrt{\tan\sqrt{x}}}.$$

S32. We have, $x = e^{x/y}$

$$\therefore \quad \frac{d}{dx}x = \frac{d}{dx}e^{x/y}$$

$$\Rightarrow \quad 1 = e^{x/y}\cdot\frac{d}{dx}(x/y)$$

$$\Rightarrow \quad 1 = e^{x/y}\cdot\left[\frac{y\cdot 1 - x\cdot dy/dx}{y^2}\right]$$

$$\Rightarrow \quad y^2 = y\cdot e^{x/y} - x\cdot\frac{dy}{dx}\cdot e^{x/y}$$

$$\Rightarrow \quad x\cdot\frac{dy}{dx}\cdot e^{x/y} = ye^{x/y} - y^2$$

$\therefore \qquad \frac{dy}{dx} = \frac{y(e^{x/y} - y)}{x \cdot e^{x/y}}$

$$= \frac{(e^{x/y} - y)}{\frac{x}{y} \cdot e^{x/y}} \qquad \left[\because \; x = e^{x/y} \Rightarrow \log x = \frac{x}{y}\right]$$

$$= \frac{x - y}{x \cdot \log x}.$$

Hence proved.

S33. We have, $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ... (i)

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}(ax^2) + \frac{d}{dx}(2hxy) + \frac{d}{dx}(by^2) + \frac{d}{dx}(2gx) + \frac{d}{dx}(2fy) + \frac{d}{dx}(c) = 0$$

$$\Rightarrow \qquad 2ax + 2h\left(x \cdot \frac{dy}{dx} + y \cdot 1\right) + b \cdot 2y\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} + 0 = 0$$

$$\Rightarrow \qquad \frac{dy}{dx}[2hx + 2by + 2f] = -2ax - 2hy - 2g$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{-2(ax + hy + g)}{2(hx + by + f)}$$

$$= \frac{-(ax + hy + g)}{(hx + by + f)} \qquad \text{... (ii)}$$

Now, differentating Eq. (i) w.r.t. y, we get

$$\frac{d}{dy}(ax^2) + \frac{d}{dy}(2hxy) + \frac{d}{dy}(by^2) + \frac{d}{dy}(2gx) + \frac{d}{dy}(2fy) + \frac{d}{dy}(c) = 0$$

$$\Rightarrow \quad a \cdot 2x \cdot \frac{dx}{dy} + 2h \cdot \left(x \cdot \frac{d}{dy}y + y \cdot \frac{d}{dy}x\right) + b \cdot 2y + 2g \cdot \frac{dx}{dy} + 2f + 0 = 0$$

$$\Rightarrow \qquad \frac{dx}{dy}[2ax + 2hy + 2g] = -2hx - 2by - 2f$$

$$\Rightarrow \qquad \frac{dx}{dy} = \frac{-2(hx + by + f)}{2(ax + hy + g)} = \frac{-(hx + by + f)}{(ax + hy + g)} \qquad \text{... (iii)}$$

$$\therefore \qquad \frac{dy}{dx} \cdot \frac{dx}{dy} = \frac{-(ax + hy + g)}{(hx + by + f)} \cdot \frac{-(hx + by + f)}{(ax + hy + g)} \qquad \text{[Using Eqs. (ii) and (iii)]}$$

$$= 1.$$

Hence proved.

S34. We have, $(x^2 + y^2)^2 = xy$

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}(x^2 + y^2)^2 = \frac{d}{dx}(xy)$$

$$\Rightarrow \quad 2(x^2+y^2)\cdot\frac{d}{dx}(x^2+y^2) = x\cdot\frac{d}{dx}y + y\cdot\frac{d}{dx}x$$

$$\Rightarrow \quad 2(x^2+y^2)\cdot\left(2x+2y\frac{dy}{dx}\right) = x\frac{dy}{dx}+y$$

$$\Rightarrow \quad 2x^2\cdot 2x + 2x^2\cdot 2y\frac{dy}{dx} + 2y^2\cdot 2x + 2y^2\cdot 2y\frac{dy}{dx} = x\frac{dy}{dx}+y$$

$$\Rightarrow \quad \frac{dy}{dx}\,[4x^2y + 4y^3 - x] = y - 4x^3 - 4xy^2$$

$$\therefore \quad \frac{dy}{dx} = \frac{(y-4x^3-4xy^2)}{(4x^2y+4y^3-x)}.$$

S35. We have, $\sec(x+y) = xy$

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}\sec(x+y) = \frac{d}{dx}(xy)$$

$$\Rightarrow \quad \sec(x+y)\cdot\tan(x+y)\cdot\frac{d}{dx}(x+y) = x\cdot\frac{d}{dx}y + y\cdot\frac{d}{dx}x$$

$$\Rightarrow \quad \sec(x+y)\cdot\tan(x+y)\cdot\left(1+\frac{dy}{dx}\right) = x\frac{dy}{dx}+y$$

$$\Rightarrow \quad \sec(x+y)\tan(x+y) + \sec(x+y)\cdot\tan(x+y)\cdot\frac{dy}{dx} = x\frac{dy}{dx}+y$$

$$\Rightarrow \quad \frac{dy}{dx}\,[\sec(x+y)\cdot\tan(x+y) - x] = y - \sec(x+y)\cdot\tan(x+y)$$

$$\therefore \quad \frac{dy}{dx} = \frac{y-\sec(x+y)\cdot\tan(x+y)}{\sec(x+y)\cdot\tan(x+y)-x}.$$

S36. We have, $\sin(xy) + \frac{x}{y} = x^2 - y$

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}(\sin xy) + \frac{d}{dx}\left(\frac{x}{y}\right) = \frac{d}{dx}x^2 - \frac{d}{dy}y\cdot\frac{dy}{dx}$$

$$\Rightarrow \quad \cos xy\cdot\frac{d}{dx}(xy) + \frac{y\frac{d}{dx}x - x\cdot\frac{d}{dx}y}{y^2} = 2x - \frac{dy}{dx}$$

$$\Rightarrow \quad \cos xy\cdot\left[x\cdot\frac{d}{dx}y + y\cdot\frac{d}{dx}\cdot x\right] + \frac{y - x\frac{d}{dx}y}{y^2} = 2x - \frac{dy}{dx}$$

$\Rightarrow \quad x \cos xy \cdot \frac{dy}{dx} + y \cos xy + \frac{y}{y^2} - \frac{x}{y^2}\frac{dy}{dx} = 2x - \frac{dy}{dx}$

$\Rightarrow \quad \frac{dy}{dx}\left[x \cos xy - \frac{x}{y^2} + 1\right] = 2x - y \cos xy - \frac{y}{y^2}$

$\therefore \quad \frac{dy}{dx} = \left[\frac{2xy - y^2 \cos xy - 1}{y}\right]\left[\frac{y^2}{xy^2 \cos xy - x + y^2}\right]$

$= \frac{(2xy - y^2 \cos xy - 1)\, y}{(xy^2 \cos xy - x + y^2)}.$

S37. Let, $\quad y = \sin^m x \cdot \cos^n x$

$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}[(\sin x)^m \cdot (\cos x)^n]$

$= (\sin x)^m \cdot \frac{d}{dx}(\cos x)^n + (\cos x)^n \cdot \frac{d}{dx}(\sin x)^m$

$= (\sin x)^m \cdot n(\cos x)^{n-1} \cdot \frac{d}{dx}\cos x + (\cos x)^n \cdot m(\sin x)^{m-1} \cdot \frac{d}{dx}\sin x$

$= (\sin x)^m \cdot n(\cos x)^{n-1}(-\sin x) + (\cos x)^n \cdot m(\sin x)^{m-1}\cos x$

$= -n \sin^m x \cdot \cos^{n-1} x \cdot (\sin x) + m \cos^n x \cdot \sin^{m-1} x \cdot \cos x$

$= -n \cdot \sin^m x \cdot \sin x \cdot \cos^n x \cdot \frac{1}{\cos x} + m \cdot \sin^m x \cdot \frac{1}{\sin x} \cdot \cos^n x \cdot \cos x$

$= -n \cdot \sin^m x \cdot \cos^n x \cdot \tan x + m \sin^m x \cdot \cos^n x \cdot \cot x$

$= \sin^m x \cdot \cos^n x[-n \tan x + m \cot x].$

S38. Let, $\quad y = \sin x^2 + \sin^2 x + \sin^2(x^2)$

$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\sin(x^2) + \frac{d}{dx}(\sin x)^2 + \frac{d}{dx}(\sin x^2)^2$

$= \cos(x^2)\frac{d}{dx}(x^2) + 2\sin x \cdot \frac{d}{dx}\sin x + 2\sin x^2 \cdot \frac{d}{dx}\sin x^2$

$= (\cos x^2)2x + 2 \cdot \sin x \cdot \cos x + 2\sin x^2 \cos x^2 \cdot \frac{d}{dx}x^2$

$= 2x\cos(x)^2 + 2 \cdot \sin x \cdot \cos x + 2\sin x^2 \cos x^2 \cdot 2x$

$= 2x\cos(x)^2 + \sin 2x + \{\sin 2(x)^2\} \cdot 2x$

$= 2x\cos(x)^2 + 2x \cdot \sin 2(x)^2 + \sin 2x.$

S39. Let, $\quad y = \cos(\tan\sqrt{x+1})$

$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\cos(\tan\sqrt{x+1})$

$= -\sin(\tan\sqrt{x+1}) \cdot \frac{d}{dx}(\tan\sqrt{x+1})$

$$= -\sin(\tan\sqrt{x+1})\cdot\sec^2\sqrt{x+1}\cdot\frac{d}{dx}(x+1)^{1/2} \quad \left[\because \quad \frac{d}{dx}(\tan x) = \sec^2 x\right]$$

$$= -\sin(\tan\sqrt{x+1})\cdot(\sec\sqrt{x+1})^2\cdot\frac{1}{2}(x+1)^{-1/2}\cdot\frac{d}{dx}(x+1)$$

$$= \frac{-1}{2\sqrt{x+1}}\cdot\sin(\tan\sqrt{x+1})\cdot\sec^2(\sqrt{x+1}).$$

S40. Let, $y = \sin^n(ax^2 + bx + c)$

$\therefore$ $$\frac{dy}{dx} = \frac{d}{dx}[\sin(ax^2+bx+c)]^n$$

$$= n\cdot[\sin(ax^2+bx+c)]^{n-1}\cdot\frac{d}{dx}\sin(ax^2+bx+c)$$

$$= n\cdot\sin^{n-1}(ax^2+bx+c)\cdot\cos(ax^2+bx+c)\cdot\frac{d}{dx}(ax^2+bx+c)$$

$$= n\cdot\sin^{n-1}(ax^2+bx+c)\cdot\cos(ax^2+bx+c)\cdot(2ax+b)$$

$$= n\cdot(2ax+b)\cdot\sin^{n-1}(ax^2+bx+c)\cdot\cos(ax^2+bx+c)\cdot$$

S41. Let, $y = \sin\sqrt{x} + \cos^2\sqrt{x}$

$\therefore$ $$\frac{dy}{dx} = \frac{d}{dx}\sin(x^{1/2}) + \frac{d}{dx}[\cos(x^{1/2})]^2$$

$$= \cos x^{1/2}\cdot\frac{d}{dx}x^{1/2} + 2\cos(x^{1/2})\frac{d}{dx}[\cos(x^{1/2})]$$

$$= \cos(x^{1/2})\frac{1}{2}x^{-1/2} + 2\cdot\cos(x^{1/2})\cdot\left[-\sin(x^{1/2})\cdot\frac{d}{dx}x^{1/2}\right]$$

$$= \cos\sqrt{x}\cdot\frac{1}{2\sqrt{x}}[-2\cos(x^{1/2})]\cdot\sin x^{1/2}\cdot\frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}[\cos(\sqrt{x}) - \sin(2\sqrt{x})].$$

S42. Put $x = \tan\theta$, where $\frac{-\pi}{6} < \theta < \frac{\pi}{6}$.

Therefore, $$y = \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 3\theta)$$

$$= 3\theta$$

$$= 3\tan^{-1}x$$

Hence, $$\frac{dy}{dx} = \frac{3}{1+x^2}.$$

S43. We have, $\tan^{-1}(x^2 + y^2) = a$

On differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}\tan^{-1}(x^2+y^2) = \frac{d}{dx}(a)$$

$$\Rightarrow \quad \frac{1}{1+(x^2+y^2)^2}\cdot\frac{d}{dx}(x^2+y^2) = 0$$

$$\Rightarrow \quad 2x+2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 2y\cdot\frac{dy}{dx} = -2x$$

$$\therefore \quad \frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}.$$

S44. Given, $e^x + e^y = e^{x+y}$

Differentiating both sides w.r.t. x, we have

$$e^x + e^y\frac{dy}{dx} = e^{x+y}\left(1+\frac{dy}{dx}\right)$$

or $$(e^y - e^{x+y})\frac{dy}{dx} = e^{x+y} - e^x,$$

Which implies that $$\frac{dy}{dx} = \frac{e^{x+y}-e^x}{e^y-e^{x+y}} = \frac{e^x+e^y-e^x}{e^y-e^x-e^y} = -e^{y-x}.$$

S45. Let $y = 2^{\cos^2 x}$

$$\therefore \quad \log y = \log 2^{\cos^2 x} = \cos^2 x\cdot\log 2$$

On differentiating w.r.t. x, we get

$$\frac{d}{dy}\log y\cdot\frac{dy}{dx} = \frac{d}{dx}\log 2\cdot\cos^2 x$$

$$\Rightarrow \quad \frac{1}{y}\cdot\frac{dy}{dx} = \log 2\frac{d}{dx}(\cos x)^2$$

$$\Rightarrow \quad \frac{1}{y}\cdot\frac{dy}{dx} = \log 2\cdot[2\cos x]\cdot\frac{d}{dx}\cos x$$

$$= \log 2\cdot 2\cos x\cdot(-\sin x)$$

$$= \log 2\cdot[-(\sin 2x)]$$

$$\therefore \quad \frac{dy}{dx} = -y\cdot\log 2\,(\sin 2x)$$

$$= -2^{\cos^2 x}\cdot\log 2\,(\sin 2x).$$

S46. Let

$$y = \frac{8^x}{x^8} \quad \Rightarrow \quad \log y = \log \frac{8^x}{x^8}$$

$$\Rightarrow \quad \frac{d}{dy}\log y \cdot \frac{dy}{dx} = \frac{d}{dx}[\log 8^x - \log x^8]$$

$$\Rightarrow \quad \frac{1}{y}\cdot\frac{dy}{dx} = \frac{d}{dx}[x\cdot\log 8 - 8\cdot\log x]$$

On differentiating w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = \log 8\cdot 1 - 8\cdot\frac{1}{x}$$

$$\Rightarrow \quad \frac{1}{y}\cdot\frac{dy}{dx} = \log 8 - \frac{8}{x}$$

$$\therefore \quad \frac{dy}{dx} = y\left(\log 8 - \frac{8}{x}\right) = \frac{8^x}{x^8}\left(\log 8 - \frac{8}{x}\right).$$

S47. Let,

$$y = \log(x + \sqrt{x^2 + a})$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\log(x + \sqrt{x^2 + a})$$

$$= \frac{1}{(x + \sqrt{x^2 + a})}\cdot\frac{d}{dx}[x + \sqrt{x^2 + a}]$$

$$= \frac{1}{(x + \sqrt{x^2 + a})}\left[1 + \frac{1}{2}(x^2 + a)^{-1/2}\cdot 2x\right]$$

$$= \frac{1}{(x + \sqrt{x^2 + a})}\cdot\left(1 + \frac{x}{\sqrt{x^2 + a}}\right)$$

$$= \frac{(\sqrt{x^2 + a} + x)}{(x + \sqrt{x^2 + a})(\sqrt{x^2 + a})} = \frac{1}{(\sqrt{x^2 + a})}.$$

S48. Let,

$$y = \text{Log}\,[\log(\log x^5)]$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}[\log(\log\log x^5)]$$

$$= \frac{1}{\log\log x^5}\cdot\frac{d}{dx}(\log\cdot\log x^5)$$

$$= \frac{1}{\log\log x^5}\cdot\left(\frac{1}{\log x^5}\right)\cdot\frac{d}{dx}\log x^5$$

$$= \frac{1}{\log\log x^5}\cdot\frac{1}{\log x^5}\cdot\frac{d}{dx}(5\log x)$$

$$= \frac{5}{x\cdot\log(\log x^5)\cdot\log(x^5)}.$$

S49. Let
$$y = (\sin x)^{\cos x}$$
$\Rightarrow$
$$\log y = \log (\sin x)^{\cos x} = \cos x \log \sin x$$
$\therefore$
$$\frac{d}{dy}\log y \cdot \frac{dy}{dx} = \frac{d}{dx}(\cos x \cdot \log \sin x)$$
$\Rightarrow$
$$\frac{1}{y}\cdot\frac{dy}{dx} = \cos x \cdot \frac{d}{dx}\log \sin x + \log \sin x \cdot \frac{d}{dx}\cos x$$
$$= \cos x \cdot \frac{1}{\sin x}\cdot\frac{d}{dx}\sin x + \log \sin x \cdot (-\sin x)$$
$$= \cot x \cdot \cos x - \log (\sin x) \cdot \sin x \qquad \left[\because \cot x = \frac{\cos x}{\sin x}\right]$$
$\therefore$
$$\frac{dy}{dx} = y\left[\frac{\cos^2 x}{\sin x} - \sin x \cdot \log(\sin x)\right]$$
$$= \sin x^{\cos x}\left[\frac{\cos^2 x}{\sin x} - \sin x \cdot \log(\sin x)\right].$$

S50. $\because$
$$x = t + \frac{1}{t} \quad \text{and} \quad y = t - \frac{1}{t}$$
$\therefore$
$$\frac{dx}{dt} = \frac{d}{dt}\left(t + \frac{1}{t}\right) \quad \text{and} \quad \frac{dy}{dt} = \frac{d}{dt}\left(t - \frac{1}{t}\right)$$
$\Rightarrow$
$$\frac{dx}{dt} = 1 + (-1)\,t^{-2} \quad \text{and} \quad \frac{dy}{dt} = 1 - (-1)\,t^{-2}$$
$\Rightarrow$
$$\frac{dx}{dt} = 1 - \frac{1}{t^2} \quad \text{and} \quad \frac{dy}{dt} = 1 + \frac{1}{t^2}$$
$\Rightarrow$
$$\frac{dx}{dt} = \frac{t^2 - 1}{t^2} \quad \text{and} \quad \frac{dy}{dt} = \frac{t^2 + 1}{t^2}$$
$\therefore$
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{(t^2+1)/t^2}{(t^2-1)/t^2} = \frac{t^2+1}{t^2-1}.$$

S51. $\because$
$$x = 3\cos\theta - 2\cos^3\theta \quad \text{and} \quad y = 3\sin\theta - 2\sin^3\theta$$
$\therefore$
$$\frac{dx}{d\theta} = \frac{d}{d\theta}(3\cos\theta) - \frac{d}{d\theta}(2\cos^3\theta)$$
$$= 3\cdot(-\sin\theta) - 2\cdot 3\cos^2\theta \cdot \frac{d}{d\theta}\cdot\cos\theta$$
$$= -3\sin\theta + 6\cos^2\theta\sin\theta$$
and
$$\frac{dy}{d\theta} = 3\cos\theta - 2\cdot 3\sin^2\theta \cdot \frac{d}{d\theta}\cdot\sin\theta$$
$$= 3\cos\theta - 6\sin^2\theta\cdot\cos\theta$$

Now, $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3\cos\theta - 6\sin^2\theta\cos\theta}{-3\sin\theta + 6\cos^2\theta\sin\theta}$$

$$= \frac{3\cos\theta(1-2\sin^2\theta)}{3\sin\theta(-1+2\cos^2\theta)} = \cot\theta \cdot \frac{\cos 2\theta}{\cos 2\theta} = \cot\theta .$$

S52. We have, $f(x) = x(x-1)^2$ in [0, 1].

(i) Since, $f(x) = x(x-1)^2$

So, it is continuous in [0, 1].

(ii) Now, $$f'(x) = x \cdot \frac{d}{dx}(x-1)^2 + (x-1)^2 \frac{d}{dx} x$$

$$= x \cdot 2(x-1) \cdot 1 + (x-1)^2$$

$$= 2x^2 - 2x + x^2 + 1 - 2x$$

$$= 3x^2 - 4x + 1 \quad \text{which exists in } (0, 1)$$

So, $f(x)$ is differentiable in (0, 1).

(iii) Now, $f(0) = 0$ and $f(1) = 0 \Rightarrow f(0) = f(1)$

f satisfies the above conditions of Rolle's theorem.

Hence, by Rolle's theorem $\exists c \in (0, 1)$ such that

$$f'(x) = 0$$

$$\Rightarrow \quad 3c^2 - 4c + 1 = 0$$

$$\Rightarrow \quad 3c^2 - 3c - c + 1 = 0$$

$$\Rightarrow \quad 3c(c-1) - 1(c-1) = 0$$

$$\Rightarrow \quad (3c-1)(c-1) = 0$$

$$\Rightarrow \quad c = \frac{1}{3}, 1 \Rightarrow \frac{1}{3} \in (0, 1).$$

Thus, we see that there exists a real number c in the open interval (0, 1).

Hence, Rolle's theorem has been verified.

S53. Let $u = \frac{x}{\sin x}$ and $v = \sin x$

$$\therefore \quad \frac{du}{dx} = \frac{\sin x \cdot \frac{d}{dx} x - x \cdot \frac{d}{dx}\sin x}{(\sin x)^2}$$

$$= \frac{\sin x - x\cos x}{\sin^2 x} \qquad \ldots \text{(i)}$$

and $$\frac{dv}{dx} = \frac{d}{dx}\sin x = \cos x \qquad \ldots \text{(ii)}$$

$\therefore$ $$\frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{(\sin x - x\cos x)/\sin^2 x}{\cos x}$$

$$= \frac{\sin x - x\cos x}{\sin^2 x \cos x}$$

$$= \frac{\tan x - x}{\sin^2 x}.$$

S54. $\because$ $$x = \frac{1+\log t}{t^2} \quad \text{and} \quad y = \frac{3+2\log t}{t}$$

$\therefore$ $$\frac{dx}{dt} = \frac{t^2\cdot\frac{d}{dt}(1+\log t) - (1+\log t)\cdot\frac{d}{dt}t^2}{(t^2)^2}$$

$$= \frac{t^2\cdot\frac{1}{t} - (1+\log t)\cdot 2t}{t^4} = \frac{t-(1+\log t)\cdot 2t}{t^4}$$

$$= \frac{t}{t^4}[1-2(1+\log t)] = \frac{-1-2\log t}{t^3} \quad \text{... (i)}$$

and $$\frac{dy}{dt} = \frac{t\cdot\frac{d}{dt}(3+2\log t) - (3+2\log t)\cdot\frac{d}{dt}t}{t^2}$$

$$= \frac{t\cdot 2\cdot\frac{1}{t} - (3+2\log t)\cdot 1}{t^2}$$

$$= \frac{2-3-2\log t}{t^2} = \frac{-1-2\log t}{t^2} \quad \text{... (ii)}$$

$\therefore$ $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{(-1-2\log t)/t^2}{(-1-2\log t)/t^3} = t.$$

S55. $\because$ $$x = 3\sin t - \sin 3t \quad \text{and} \quad y = 3\cos t - \cos 3t$$

$\therefore$ $$\frac{dx}{dt} = 3\cdot\frac{d}{dt}\sin t - \frac{d}{dt}\sin 3t$$

$$= 3\cos t - \cos 3t\cdot\frac{d}{dt}3t = 3\cos t - 3\cos 3t$$

and $$\frac{dy}{dt} = 3\cdot\frac{d}{dt}\cos t - \frac{d}{dt}\cos 3t$$

$$= -3\sin t + \sin 3t\cdot\frac{d}{dt}3t$$

$$\frac{dy}{dt} = -3\sin t + 3t\sin 3t$$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3(\sin 3t - \sin t)}{3(\cos t - \cos 3t)}$

Now, $\left.\dfrac{dy}{dx}\right|_{t=\frac{\pi}{3}} = \dfrac{\sin\dfrac{3\pi}{3} - \sin\dfrac{\pi}{3}}{\left(\cos\dfrac{\pi}{3} - \cos\dfrac{3\pi}{3}\right)} = \dfrac{0 - \sqrt{3}/2}{\dfrac{1}{2} - (-1)}$

$$= \frac{-\sqrt{3}/2}{3/2} = \frac{-\sqrt{3}}{3} = \frac{-1}{\sqrt{3}}.$$

S56. We have, $f(x) = \log(x^2 + 2) - \log 3.$

(i) Logarithmic functions are continuous in their domain.

Hence, $f(x) = \log(x^2 + 2) - \log 3$ is continuous in $[-1, 1]$.

(ii) $f'(x) = \dfrac{1}{x^2+2}(2x - 0)$

$= \dfrac{2x}{x^2+2}$, which exists in $(-1, 1)$.

Hence, $f(x)$ is differentiable in $(-1, 1)$.

(iii) $f(-1) = \log[(-1)^2 + 2] - \log 3 = \log 3 - \log 3 = 0$ and

$f(1) = \log(1^2 + 2) - \log 3 = \log 3 - \log 3 = 0$

$\Rightarrow \quad f(-1) = f(1)$

Conditions of Rolle's theorem are satisfied.

Hence, there exists a real number c such that

$f'(c) = 0$

$\Rightarrow \quad \dfrac{2c}{c^2+c} = 0$

$\Rightarrow \quad c = 0 \in (-1, 1)$

Hence, Rolle's theorem has been verified.

S57. We have, $y = \tan^{-1} x$ [On differentiating w.r.t., x]

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{1+x^2}$ [Again differentiating w.r.t., x]

Now, $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}(1+x^2)^{-1}$

$= -1(1+x^2)^{-2} \cdot \dfrac{d}{dx}(1+x^2)$

$$= -\frac{1}{(1+x^2)^2}\cdot 2x$$

$$= \frac{-2\tan y}{(1+\tan^2 y)^2} \qquad [\because\ y = \tan^{-1} x \Rightarrow \tan y = x]$$

$$= \frac{-2\tan y}{(\sec^2 y)^2}$$

$$= -2\,\frac{\sin y}{\cos y}\cdot \cos^2 y\cdot \cos^2 y$$

$$= -\sin 2y \cdot \cos^2 y. \qquad [\because\ \sin 2x = 2\sin x\cos x]$$

S58. We have, $f(x) = x(x+3)e^{-x/2}$

(i) $f(x)$ is a continuous function. [Since, it is a combination of polynomial functions $x(x+3)$ and an exponential function $e^{-x/2}$ which are continuous functions].

So, $f(x) = x(x+3)e^{-x/2}$ is continuous in $[-3, 0]$.

(ii) $\therefore$

$$f'(x) = (x^2+3x)\cdot\frac{d}{dx}e^{-x/2} + e^{-x/2}\cdot\frac{d}{dx}(x^2+3x)$$

$$= (x^2+3x)\cdot e^{-x/2}\cdot\left(-\frac{1}{2}\right) + e^{-x/2}\cdot(2x+3)$$

$$= e^{-x/2}\left[2x+3-\frac{1}{2}\cdot(x^2+3x)\right]$$

$$= e^{-x/2}\left[\frac{4x+6-x^2-3x}{2}\right]$$

$$= e^{-x/2}\cdot\frac{1}{2}[-x^2+x+6]$$

$$= \frac{-1}{2}e^{-x/2}[x^2-x-6]$$

$$= \frac{-1}{2}e^{-x/2}[x^2-3x+2x-6]$$

$$= \frac{-1}{2}e^{-x/2}[(x+2)(x-3)] \text{ which exists in } (-3, 0).$$

Hence, $f(x)$ is differentiable in $(-3, 0)$.

(iii) $\therefore$ $f(-3) = -3(-3+3)e^{-3/2} = 0$

and $f(0) = 0(0+3)e^{-3/2} = 0$

$\Rightarrow$ $f(-3) = f(0)$

Since, conditions of Rolle's theorem are satisfied.

Hence, there exists a real number c such that $f'(c) = 0$

$\Rightarrow \qquad -\frac{1}{2} e^{-c/2}(c+2)(c-3) = 0$

$\Rightarrow \qquad c = -2, 3, \quad \text{where } -2 \in (-3, 0)$

Therefore, Rolle's theorem has been verified.

S59. The equation of the curve is $y = \cos x - 1$.

Now, we have to find a point of the curve in $[0, 2\pi]$.

where the tangent is parallel to X-axis *i.e.*, the tangent to the curve at $x = c$ has a slope 0, where $c \in]0, 2\pi[$.

Let us apply Rolle's theorem to get the point.

(i) $y = \cos x - 1$ is a continuous function in $[0, 2\pi]$.

[Since it is a combination of cosine function and a constant function]

(ii) $y' = -\sin x$, which exists in $(0, 2\pi)$.

Hence, y is differentiable in $(0, 2\pi)$.

(iii) $y(0) = \cos 0 - 1$ and $y(2\pi) = \cos 2\pi - 1 = 0$,

$\therefore \qquad y(0) = y(2\pi)$

Since, condition of Rolle's theorem are satisfied.

Hence, there exists a real number c such that

$$f'(c) = 0$$

$\Rightarrow \qquad -\sin c = 0$

$\Rightarrow \qquad c = \pi \quad \text{or} \quad 0, \text{ where } \pi \in (0, 2\pi)$

$\Rightarrow \qquad x = \pi$

$\therefore \qquad y = \cos \pi - 1 = -2$

Hence, the required point on the curve, where the tangent drawn is parallel to the X-axis is $(\pi, -2)$.

S60. We have, $\qquad y = x(x-4), \quad x \in [0, 4]$

(i) y is a continuous function since $x(x-4)$ is a polynomial function.

Hence, $y = x(x-4)$ is continuous in $[0, 4]$.

(ii) $y' = (x-4)\cdot 1 + x \cdot 1 = 2x - 4$ which exists in $(0, 4)$.

Hence, y is differentiable in $(0, 4]$.

(iii) $y(0) = 0(0-4) = 0$

and $\qquad y(4) = 4(4-4) = 0$

$\Rightarrow \qquad y(0) = y(4)$

Since, conditions of Rolle's theorem are satisfied.

Hence, there exists a point c such that

$$f'(c) = 0 \quad \text{in } (0, 4) \qquad [\because f'(x) = y']$$

$\Rightarrow \qquad 2c - 4 = 0$

$\Rightarrow \quad c = 2$

$\Rightarrow \quad x = 2; \quad y = 2(2-4) = -4.$

Thus, (2, –4) is the point on the curve at which the tangent drawn is parallel to X-axis.

S61. Consider $f(x) = \sin 2x$ in $\left[0, \frac{\pi}{2}\right]$.

Note that:

(i) The function f is continuous in $\left[0, \frac{\pi}{2}\right]$, as f is a sine function, which is always continuous.

(ii) $f'(x) = 2\cos 2x$, exists in $\left(0, \frac{\pi}{2}\right)$, hence f is differentiable in $\left(0, \frac{\pi}{2}\right)$.

(iii) $f(0) = \sin 0 = 0$ and $f\left(\frac{\pi}{2}\right) = \sin \pi = 0 \Rightarrow f(0) = f\left(\frac{\pi}{2}\right)$.

Conditions of Rolle's are satisfied. Hence there exists at least one $c \in \left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$. Thus,

$$2\cos 2c = 0 \quad \Rightarrow \quad 2c = \frac{\pi}{2} \quad \Rightarrow \quad c = \frac{\pi}{4}.$$

S62. We have, $f(x) = \frac{1}{4x-1}$ in [1, 4].

(i) $f(x)$ is continuous in [1, 4].

Also, at $x = \frac{1}{4}$, $f(x)$ is discontinuous

Hence, $f(x)$ is continuous in [1, 4].

(ii) $f'(x) = -\frac{4}{(4x-1)^2}$ which exists in (1, 4).

Since, conditions of mean value theorem are satisfied.

Hence, there exists a real number $c \in \,]1, 4[$ such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}$$

$$\Rightarrow \quad \frac{-4}{(4c-1)^2} = \frac{\frac{1}{16-1} - \frac{1}{4-1}}{4-1} = \frac{\frac{1}{15} - \frac{1}{3}}{3}$$

$$\Rightarrow \quad \frac{-4}{(4c-1)^2} = \frac{1-5}{45} = \frac{-4}{45}$$

$$\Rightarrow \quad (4c-1)^2 = 45$$

$$\Rightarrow \quad 4c - 1 = \pm 3\sqrt{5}$$

$$\Rightarrow \quad c = \frac{3\sqrt{5}+1}{4} \in (1, 4) \qquad \text{[Neglecting (–ve) value]}$$

Hence, mean value theorem has been verified.

S63. We have, $f(x) = x^3 - 2x^2 - x + 3$ in [0, 1].

(i) Since, $f(x)$ is a polynomial function.

Hence, $f(x)$ is continuous in [0, 1]

(ii) $f'(x) = 3x^2 - 4x - 1$, which exists in (0, 1).

Hence, $f(x)$ is differentiable in (0, 1).

Since, conditions of mean value theorem are satisfied.

Therefore, by mean value theorem $\exists c \in (0, 1)$, such that

$$f'(c) = \frac{f(1) - f(0)}{1 - 0}$$

$$\Rightarrow \quad 3c^2 - 4c - 1 = \frac{[1 - 2 - 1 + 3] - [0 + 3]}{1 - 0}$$

$$\Rightarrow \quad 3c^2 - 4c - 1 = \frac{-2}{1}$$

$$\Rightarrow \quad 3c^2 - 4c + 1 = 0$$

$$\Rightarrow \quad 3c^2 - 3c - c + 1 = 0$$

$$\Rightarrow \quad 3c(c - 1) - 1(c - 1) = 0$$

$$\Rightarrow \quad (3c - 1)(c - 1) = 0$$

$$\Rightarrow \quad c = 1/3, 2, \text{ where } \frac{1}{3} \in (0, 1).$$

Hence, the mean value theorem has been verified.

S64. We have, $f(x) = \sin x - \sin 2x$ in $[0, \pi]$.

(i) Since, we know that sine functions are continuous functions hence $f(x) = \sin x - \sin 2x$ is a continuous function in $[0, \pi]$.

(ii) $f'(x) = \cos x - \cos 2x \cdot 2 = \cos x - 2\cos 2x$, which exists in $(0, \pi)$.

So, $f'(x)$ is differentiable in $(0, \pi)$. Conditions of mean value theorem are satisfied.

Hence, $\exists c \in (0, \pi)$ such that,

$$f'(c) = \frac{f(\pi) - f(0)}{\pi - 0}$$

$$\Rightarrow \quad \cos c - 2\cos 2c = \frac{\sin\pi - \sin 2\pi - \sin 0 + \sin 2\cdot 0}{\pi - 0}$$

$$\Rightarrow \quad 2\cos 2c - \cos c = \frac{0}{\pi}$$

$$\Rightarrow \quad 2\cdot(2\cos^2 c - 1) - \cos c = 0$$

$$\Rightarrow \quad 4\cos^2 c - 2 - \cos c = 0$$

$$\Rightarrow \quad 4\cos^2 c - \cos c - 2 = 0$$

$$\Rightarrow \quad \cos c = \frac{1\pm\sqrt{1+32}}{8} = \frac{1\pm\sqrt{33}}{8}$$

$$\therefore \quad c = \cos^{-1}\left(\frac{1\pm\sqrt{33}}{8}\right)$$

Also $\cos^{-1}\left(\frac{1\pm\sqrt{33}}{8}\right) \in (0, \pi)$

Hence, the mean value theorem has been verified.

S65. We have, $f(x) = \sqrt{25 - x^2}$ in [1, 5].

(i) Since, $f(x) = (25 - x^2)^{1/2}$, where $25 - x^2 \geq 0$

$$\Rightarrow \quad x^2 \leq \pm 5 \quad \Rightarrow \quad -5 \leq x \leq 5$$

Hence, $f(x)$ is continuous in [1, 5].

(ii) $f(x) = \frac{1}{2}(25 - x^2)^{-1/2}\cdot -2x = \frac{-x}{\sqrt{25 - x^2}}$, which exists in (1, 5).

Hence, $f'(x)$ is differentiable in (1, 5).

Since, conditions of mean value theorem are satisfied.

By mean value theorem $\exists c \in (1, 5)$ such that

$$f'(c) = \frac{f(5) - f(1)}{5 - 1}$$

$$\Rightarrow \quad \frac{-c}{\sqrt{25 - c^2}} = \frac{\sqrt{24}}{4}$$

$$\Rightarrow \quad \frac{c^2}{25 - c^2} = \left(\frac{f(5) - f(1)}{5 - 1}\right)^2$$

$$\Rightarrow \quad 16c^2 = 600 - 24c^2$$

$$\Rightarrow \quad c^2 = \frac{600}{40} = 15$$

$\therefore \qquad c = \pm\sqrt{15}$

Also $\qquad c = \sqrt{15} \in (1, 5)$

Hence, the mean value theorem has been verified.

S66. We have, $y = (x-3)^2$, which is continuous in $x_1 = 3$ and $x_2 = 4$ *i.e.*, [3, 4].

Also, $y' = 2(x-3)\cdot 1 = 2(x-3)$ which exists in (3, 4).

Hence, by mean value theorem there exists a point on the curve at which tangent drawn is parallel to the chord joining the points (3, 0) and (4, 1).

Thus, $$f'(c) = \frac{f(4)-f(3)}{4-3}$$

$$\Rightarrow \qquad 2(c-3) = \frac{(4-3)^2-(3-3)^2}{4-3}$$

$$\Rightarrow \qquad 2c-6 = \frac{1-0}{1} \quad \Rightarrow \quad c = \frac{7}{2}$$

For $x = \frac{7}{2}$, $$y = \left(\frac{7}{2}-3\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

So, $\left(\frac{7}{2}, \frac{1}{4}\right)$ is the point on the curve at which tangent drawn is parallel to the chord joining the points (3, 0) and (4, 1).

S67. We have, $y = 2x^2 - 5x + 3$, which is continuous in [1, 2] as it is a polynomial function.

Also, $y' = 4x - 5$, which exists in (1, 2).

By mean value theorem, $\exists c \in (1, 2)$ at which drawn tangent is parallel to the chord AB, where A and B are (1, 0) and (2, 1), respectively.

$$f'(c) = \frac{f(2)-f(1)}{2-1}$$

$$\Rightarrow \qquad 4c-5 = \frac{(8-10+3)-(2-5+3)}{1}$$

$$\Rightarrow \qquad 4c-5 = 1$$

$$\therefore \qquad c = \frac{6}{4} = \frac{3}{2} \in (1, 2)$$

For $x = \frac{3}{2}$, $$y = 2\left(\frac{3}{2}\right)^2 - 5\left(\frac{3}{2}\right) + 3$$

$$= 2\times\frac{9}{4} - \frac{15}{2} + 3 = \frac{9-15+6}{2} = 0$$

Hence, $\left(\frac{3}{2}, 0\right)$ is the point on the curve $y = 2x^2 - 5x + 3$ between the points $A(1, 0)$ and $B(2, 1)$, where tangent is parallel to the chord AB.

S68. We have,

$$f(x) = \begin{cases} x^2 + 3x + p, & \text{if } x \le 1 \\ qx + 2, & \text{if } x > 1 \end{cases}$$ is differentiable at $x = 1$.

$$Lf'(1) = \lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x \to 1^-} \frac{(x^2 + 3x + p) - (1 + 3 + p)}{x - 1}$$

$$= \lim_{h \to 0} \frac{[(1-h)^2 + 3(1-h) + p] - [1 + 3 + p]}{(1-h) - 1}$$

$$= \lim_{h \to 0} \frac{[1 + h^2 - 2h + 3 - 3h + p] - [4 + p]}{-h}$$

$$= \lim_{h \to 0} \frac{[h^2 - 5h + p + 4 - 4 - p]}{-h} = \lim_{h \to 0} \frac{h[h-5]}{-h}$$

$$= \lim_{h \to 0} -[h - 5] = 5$$

$$Rf'(1) = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x \to 1^+} \frac{(qx + 2) - (1 + 3 + p)}{x - 1}$$

$$= \lim_{h \to 0} \frac{[q(1+h) + 2] - [4 + p]}{1 + h - 1}$$

$$= \lim_{h \to 0} \frac{[q + qh + 2 - 4 - p]}{h}$$

$$= \lim_{h \to 0} \frac{qh + (q - 2 - p)}{h}$$

$\Rightarrow$ $q - 2 - p = 0 \quad \Rightarrow \quad p - q = -2$ ... (i)

$\Rightarrow$ $\lim_{h \to 0} \frac{qh + 0}{h} = q$ [For existing the limit]

If $Lf'(1) = Rf'(1)$, then $5 = q$

$\Rightarrow$ $p - 5 = -2 \quad \Rightarrow \quad p = 3$

$\therefore$ $p = 3$ and $q = 5$.

S69. Given,

$$f(x) = \frac{\sqrt{2}\cos x - 1}{\cot x - 1}, \quad x \neq \frac{\pi}{4}$$

Therefore,

$$\lim_{x \to \frac{\pi}{4}} f(x) = \lim_{x \to \frac{\pi}{4}} \frac{\sqrt{2}\cos x - 1}{\cot x - 1}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{(\sqrt{2}\cos x - 1)\sin x}{\cos x - \sin x}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{(\sqrt{2}\cos x - 1)}{(\sqrt{2}\cos x + 1)} \cdot \frac{(\sqrt{2}\cos x + 1)}{(\cos x - \sin x)} \cdot \frac{(\cos x + \sin x)}{(\cos x + \sin x)} \cdot \sin x$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{2\cos^2 x - 1}{\cos^2 x - \sin^2 x} \cdot \frac{\cos x + \sin x}{\sqrt{2}\cos x + 1} \cdot (\sin x)$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{\cos 2x}{\cos 2x} \cdot \left(\frac{\cos x + \sin x}{\sqrt{2}\cos x + 1}\right) \cdot \sin x$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{(\cos x + \sin x)}{\sqrt{2}\cos x + 1} \sin x$$

$$= \frac{\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)}{\sqrt{2} \cdot \frac{1}{\sqrt{2}} + 1} = \frac{1}{2}$$

Thus,

$$\lim_{x \to \frac{\pi}{4}} f(x) = \frac{1}{2}$$

If we define $f\left(\frac{\pi}{4}\right) = \frac{1}{2}$, then $f(x)$ will become continuous at $x = \frac{\pi}{4}$. Hence for f to be continuous at $x = \frac{\pi}{4}$, $f\left(\frac{\pi}{4}\right) = \frac{1}{2}$.

S70. Let

$$y = \tan^{-1}\sqrt{\frac{1 - \cos x}{1 + \cos x}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\tan^{-1}\sqrt{\frac{1 - \cos x}{1 + \cos x}}$$

$$= \frac{1}{1 + \sqrt{\left(\frac{1 - \cos x}{1 + \cos x}\right)^2}} \cdot \frac{d}{dx}\left[\frac{1 - \cos x}{1 + \cos x}\right]^{1/2} \qquad \left[\because \ \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}\right]$$

$$= \frac{1}{1 + \frac{1 - \cos x}{1 + \cos x}} \cdot \frac{1}{2}\left[\frac{1 - \cos x}{1 + \cos x}\right]^{-1/2} \cdot \frac{d}{dx}\left(\frac{1 - \cos x}{1 + \cos x}\right)$$

$$= \frac{1}{\frac{1+\cos x+1-\cos x}{1+\cos x}} \cdot \frac{1}{2}\left[\frac{(1-\cos x)}{(1+\cos x)} \cdot \frac{(1-\cos x)}{(1-\cos x)}\right]^{-1/2} \cdot \frac{(1+\cos x)\cdot \sin x+(1-\cos x)\cdot \sin x}{(1+\cos x)^2}$$

$$= \frac{(1+\cos x)}{2} \cdot \frac{1}{2}\left[\frac{(1-\cos x)^2}{(1-\cos^2 x)}\right]^{-1/2}\left[\frac{\sin x\,(1+\cos x+1-\cos x)}{(1+\cos x)^2}\right]$$

$$= \frac{(1+\cos x)}{2} \cdot \frac{1}{2}\left[\frac{(1-\cos x)^2}{\sin x}\right]^{-1/2} \cdot \frac{2\sin x}{(1+\cos x)^2}$$

$$= \frac{(1+\cos x)}{2} \cdot \frac{1}{2}\,\frac{\sin x}{(1-\cos x)} \cdot \frac{2\sin x}{(1+\cos x)^2}$$

$$= \frac{2\sin^2 x}{4(1+\cos x)(1-\cos x)} = \frac{1}{2} \cdot \frac{\sin^2 x}{(1-\cos^2 x)} = \frac{1}{2} \cdot \frac{\sin^2 x}{\sin^2 x} = \frac{1}{2}$$

Alternate Method:

Let $y = \tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$

$$= \tan^{-1}\left(\sqrt{\frac{1-1+2\sin^2 \frac{x}{2}}{1+2\cos^2 \frac{x}{2}-1}}\right) \qquad \left[\because \ \cos x = 1-2\sin^2 \frac{x}{2} = 2\cos^2 \frac{x}{2}-1\right]$$

$$= \tan^{-1}\left(\tan \frac{x}{2}\right) = \frac{x}{2}$$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{2}.$$

S71. The only doubtful points for differentiability of $f(x)$ are $x = -2$ and $x = 0$.

Differentiability of $f(x)$ at $x = -2$.

Now, $$Lf'(-2) = \lim_{h \to 0^-} \frac{f(-2+h)-f(-2)}{h}$$

$$= \lim_{h \to 0^-} \frac{2(-2+h)+3-(-2+1)}{h} = \lim_{h \to 0^-} \frac{2h}{h} = \lim_{h \to 0^-} 2 = 2.$$

and $$Rf'(-2) = \lim_{h \to 0^+} \frac{f(-2+h)-f(-2)}{h}$$

$$= \lim_{h \to 0^+} \frac{-2+h+1-(-2+1)}{h}$$

$$= \lim_{h \to 0^+} \frac{h-1-(-1)}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1$$

Thus, $Rf'(-2) \neq Lf'(-2)$. Therefore f is not differentiable at $x = -2$. Similarly, for differentiability at $x = 0$, we have

$$L(f'(0) = \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0^-} \frac{0+h+1-(0+2)}{h}$$

$$= \lim_{h \to 0^-} \frac{h-1}{h} = \lim_{h \to 0^-} 1 - \frac{1}{h}$$

which does not exist. Hence f is not differentiable at $x = 0$.

S72. We have, $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$

On putting $x = \sin\alpha$ and $y = \sin\beta$, we get

$$\sqrt{1-\sin^2\alpha} + \sqrt{1-\sin^2\beta} = a(\sin\alpha - \sin\beta)$$

$$\Rightarrow \quad \cos\alpha + \cos\beta = a(\sin\alpha - \sin\beta)$$

$$\Rightarrow \quad 2\cos\frac{\alpha+\beta}{2} \cdot \cos\frac{\alpha-\beta}{2} = a\left(2\cos\frac{\alpha+\beta}{2} \cdot \sin\frac{\alpha-\beta}{2}\right)$$

$$\Rightarrow \quad \cos\frac{\alpha-\beta}{2} = a\sin\frac{\alpha-\beta}{2}$$

$$\Rightarrow \quad \cot\frac{\alpha-\beta}{2} = a$$

$$\Rightarrow \quad \frac{\alpha-\beta}{2} = \cot^{-1} a$$

$$\Rightarrow \quad \alpha - \beta = 2\cot^{-1} a$$

$$\Rightarrow \quad \sin^{-1} x - \sin^{-1} y = 2\cot^{-1} a \qquad [\because\ x = \sin\alpha \text{ and } y = \sin\beta]$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{\sqrt{1-x^2}} - \frac{1}{1-y^2}\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}} = \sqrt{\frac{1-y^2}{1-x^2}}$$

Hence proved.

S73. Let $y = (x+1)^2 (x+2)^3 (x+3)^4$

$$\therefore \quad \log y = \log\{(x+1)^2 \cdot (x+2)^3 (x+3)^4\}$$

$$= \log(x+1)^2 + \log(x+2)^3 + \log(x+3)^4$$

and $$\frac{d}{dy}\log y \cdot \frac{dy}{dx} = \frac{d}{dx}[2\log(x+1)] + \frac{d}{dx}[3\log(x+2)] + \frac{d}{dx}[4\log(x+3)]$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{2}{(x+1)}\cdot\frac{d}{dx}(x+1)+3\cdot\frac{1}{(x+2)}\cdot\frac{d}{dx}(x+2)+4\cdot\frac{1}{(x+3)}\cdot\frac{d}{dx}(x+3)$$

$$\left[\because \quad \frac{d}{dx}(\log x)=\frac{1}{x}\right]$$

$$= \left[\frac{2}{(x+1)}+\frac{3}{(x+2)}+\frac{4}{(x+3)}\right]$$

$\therefore$
$$\frac{dy}{dx} = y\left[\frac{2}{(x+1)}+\frac{3}{(x+2)}+\frac{4}{(x+3)}\right]$$

$$= (x+1)^2\cdot(x+2)^3\cdot(x+3)^4\left[\frac{2}{(x+1)}+\frac{3}{(x+2)}+\frac{4}{(x+3)}\right]$$

$$= (x+1)^2\cdot(x+2)^3\cdot(x+3)^4\left[\frac{2(x+2)(x+3)+3(x+1)(x+3)+4(x+1)(x+2)}{(x+1)(x+2)(x+3)}\right]$$

$$= \frac{(x+1)^2(x+2)^3(x+3)^4}{(x+1)(x+2)(x+3)}\,[2(x^2+5x+6)+3(x^2+4x+3)+4(x^2+3x+2)]$$

$$= (x+1)(x+2)^2(x+3)^3[2x^2+10x+12+3x^2+12x+9+4x^2+12x+8]$$

$$= (x+1)(x+2)^2(x+3)^3[9x^2+34x+29].$$

S74. $\because$
$$\sin x = \frac{2t}{1+t^2} \quad \text{... (i)}$$

and
$$\tan y = \frac{2t}{1-t^2} \quad \text{... (ii)}$$

$\therefore$
$$\frac{d}{dx}\sin x\cdot\frac{dx}{dt} = \frac{d}{dt}\left(\frac{2t}{1+t^2}\right)$$

$\Rightarrow$
$$\cos x\,\frac{dx}{dt} = \frac{(1+t^2)\cdot\frac{d}{dt}(2t)-(2t)\cdot\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$= \frac{2(1+t^2)-2t\cdot 2t}{(1+t^2)^2} = \frac{2+2t^2-4t^2}{(1+t^2)^2}$$

$\Rightarrow$
$$\frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2}\cdot\frac{1}{\cos x}$$

$\Rightarrow$
$$\frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2}\cdot\frac{1}{\sqrt{1-\sin^2 x}} = \frac{2(1-t^2)}{(1+t^2)^2}\cdot\frac{1}{\sqrt{1-\left(\frac{2t}{1+t^2}\right)^2}}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2} \cdot \frac{(1+t^2)}{(1-t^2)} = \frac{2}{1+t^2} \quad \text{... (iii)}$$

Also,

$$\frac{d}{dy}\tan y \cdot \frac{dy}{dt} = \frac{d}{dt}\left(\frac{2t}{1-t^2}\right)$$

$$\sec^2 y \frac{dy}{dt} = \frac{(1-t^2)\frac{d}{dt}\cdot(2t) - 2t\cdot\frac{d}{dt}(1-t^2)}{(1-t^2)^2}$$

$$\frac{dy}{dt} = \frac{2-2t^2+4t^2}{(1-t^2)^2} \cdot \frac{1}{\sec^2 y}$$

$$= \frac{2(1+t^2)}{(1-t^2)^2} \cdot \frac{1}{(1+\tan^2 y)} = \frac{2(1+t^2)}{(1-t^2)^2} \cdot \frac{1}{1+\frac{4t^2}{(1-t^2)^2}}$$

$$= \frac{2(1+t^2)}{(1-t^2)^2} \cdot \frac{(1-t^2)^2}{(1+t^2)^2} = \frac{2}{1+t^2} \quad \text{... (iv)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2/(1+t^2)}{2/(1+t^2)} = 1. \quad \text{[from Eqs. (iii) and (iv)]}$$

S75. $\because$

$$\sin x = \frac{2t}{1+t^2} \quad \text{... (i)}$$

and

$$\tan y = \frac{2t}{1-t^2} \quad \text{... (ii)}$$

$$\therefore \quad \frac{d}{dx}\sin x \cdot \frac{dx}{dt} = \frac{d}{dt}\left(\frac{2t}{1+t^2}\right)$$

$$\Rightarrow \quad \cos x \frac{dx}{dt} = \frac{(1+t^2)\cdot\frac{d}{dt}(2t) - (2t)\cdot\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$= \frac{2(1+t^2) - 2t\cdot 2t}{(1+t^2)^2} = \frac{2+2t^2-4t^2}{(1+t^2)^2}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2} \cdot \frac{1}{\cos x}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2} \cdot \frac{1}{\sqrt{1-\sin^2 x}} = \frac{2(1-t^2)}{(1+t^2)^2} \cdot \frac{1}{\sqrt{1-\left(\frac{2t}{1+t^2}\right)^2}}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2} \cdot \frac{(1+t^2)}{(1-t^2)} = \frac{2}{1+t^2} \quad \text{... (iii)}$$

Also, $$\frac{d}{dy}\tan y \cdot \frac{dy}{dt} = \frac{d}{dt}\left(\frac{2t}{1-t^2}\right)$$

$$\sec^2 y \frac{dy}{dt} = \frac{(1-t^2)\frac{d}{dt}\cdot(2t) - 2t\cdot\frac{d}{dt}(1-t^2)}{(1-t^2)^2}$$

$$\frac{dy}{dt} = \frac{2-2t^2+4t^2}{(1-t^2)^2}\cdot\frac{1}{\sec^2 y}$$

$$= \frac{2(1+t^2)}{(1-t^2)^2}\cdot\frac{1}{(1+\tan^2 y)} = \frac{2(1+t^2)}{(1-t^2)^2}\cdot\frac{1}{1+\frac{4t^2}{(1-t^2)^2}}$$

$$= \frac{2(1+t^2)}{(1-t^2)^2}\cdot\frac{(1-t^2)^2}{(1+t^2)^2} = \frac{2}{1+t^2} \quad \text{... (iv)}$$

$\therefore$ $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2/(1+t^2)}{2/(1+t^2)} = 1. \quad \text{[from Eqs. (iii) and (iv)]}$$

S76. We have, $$x^m \cdot y^n = (x+y)^{m+n} \quad \text{... (i)}$$

(i) Differentiating Eq. (i), w.r.t. x, we get

$$\frac{d}{dx}(x^m \cdot y^n) = \frac{d}{dx}(x+y)^{m+n}$$

$$\Rightarrow \quad x^m \cdot \frac{d}{dy}y^n \cdot \frac{d}{dx} + y^n \cdot \frac{d}{dx}x^m = (m+n)(x+y)^{m+n-1}\frac{d}{dx}(x+y)$$

$$\Rightarrow \quad x^m \cdot ny^{n-1}\frac{dy}{dx} + y^n \cdot mx^{m-1} = (m+n)(x+y)^{m+n-1}\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{dy}{dx}[x^m \cdot ny^{n-1} - (m+n)\cdot(x+y)^{m+n-1}] = (m+n)(x+y)^{m+n-1} - y^n mx^{m-1}$$

$$\Rightarrow \quad \frac{dy}{dx}[nx^m y^{n-1} - (m+n)(x+y)^{m+n-1}] = (m+n)\cdot(x+y)^{m+n-1} - \frac{y^{n-1}\cdot y\cdot mx^m}{x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{(m+n)(x+y)^{m+n}}{(x+y)} - \frac{y^{n-1}\cdot y\cdot mx^m}{x}}{\frac{nx^m y^n}{y} - (m+n)(x+y)^{m+n}\frac{1}{(x+y)}}$$

$$= \frac{\frac{x(m+n)(x+y)^{m+n} - (x+y)\cdot y^{n-1}\cdot y\cdot mx^m}{(x+y)\cdot x}}{\frac{(x+y)nx^m y^n - y(m+n)(x+y)^{m+n}}{(x+y)\cdot y}}$$

$$= \frac{\frac{x(m+n)\cdot x^m\cdot y^n - m(x+y)y^n x^m}{(x+y)\cdot x}}{\frac{(x+y)nx^m\cdot y^n - y(m+n)\cdot x^m\cdot y^n}{(x+y)\cdot y}} \quad [\because (x+y)^{m+n} = x^m\cdot y^n]$$

$$= \frac{x^m y^n [mx + nx - mx - my] \cdot (x+y) y}{x^m y^n [nx + ny - my - ny] \cdot (x+y) \cdot x}$$

$$= \frac{y}{x} \qquad \text{... (ii)}$$

(ii) Further, differentiating Eq. (ii), *i.e.*, $\frac{dy}{dx} = \frac{y}{x}$ on both the sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{x \cdot \frac{dy}{dx} - y \cdot 1}{x^2}$$

$$= \frac{x \cdot \frac{y}{x} - y}{x^2} \qquad \left[\because \frac{dy}{dx} = \frac{y}{x}\right]$$

$$= 0.$$

Hence proved.

S77. We have,

$$y = x^{\tan x} + \sqrt{\frac{x^2+1}{2}} \qquad \text{... (i)}$$

Taking $u = x^{\tan x}$ and $v = \sqrt{\frac{x^2+1}{2}}$

$$\log u = \tan x \log x \qquad \text{... (ii)}$$

and $$v^2 = \frac{x^2+1}{2} \qquad \text{... (iii)}$$

On, differentiating Eq. (ii) w.r.t. x, we get

$$\frac{1}{u} \cdot \frac{du}{dx} = \tan x \cdot \frac{1}{x} + \log x \cdot \sec^2 x$$

$$\Rightarrow \quad \frac{du}{dx} = u\left[\frac{\tan x}{x} + \log x \cdot \sec^2 x\right]$$

$$= x^{\tan x}\left[\frac{\tan x}{x} + \log x \cdot \sec^2 x\right] \qquad \text{... (iv)}$$

Also, differentiating Eq. (iii) w.r.t. x, we get

$$2v \cdot \frac{dv}{dx} = \frac{1}{2}(2x) \Rightarrow \frac{dv}{dx} = \frac{1}{4v} \cdot (2x)$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{1}{4 \cdot \sqrt{\frac{x^2+1}{2}}} \cdot 2x = \frac{x \cdot \sqrt{2}}{2\sqrt{x^2+1}}$$

$\Rightarrow$ $$\frac{dv}{dx} = \frac{x}{\sqrt{2(x^2+1)}} \quad \text{... (iv)}$$

Now, $$y = u + v$$

$\therefore$ $$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$= x^{\tan x}\left[\frac{\tan x}{x} + \log x \cdot \sec^2 x\right] + \frac{x}{\sqrt{2(x^2+1)}}.$$