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Q1. Evaluate: $\begin{bmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{bmatrix}$.

Q2. Evaluate: $\begin{bmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & zy^2 & 0 \end{bmatrix}$.

Q3. If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & 5 \\ 8 & 3 \end{vmatrix}$, then find x.

Q4. Without expanding, show that

$$\Delta = \begin{vmatrix} \operatorname{cosec}^2\theta & \cot^2\theta & 1 \\ \cot^2\theta & \operatorname{cosec}^2\theta & -1 \\ 42 & 40 & 2 \end{vmatrix} = 0.$$

Q5. If $\Delta = \begin{vmatrix} 0 & b-a & c-a \\ a-b & 0 & c-b \\ a-c & b-c & 0 \end{vmatrix}$, then show that Δ is equal to zero.

Q6. Prove that $(A^{-1})' = (A')^{-1}$, where A is an invertible matrix.

Q7. Calculate $\begin{vmatrix} x+4 & x & x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix}$.

Q8. Prove that, $\begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix} = 4xyz$.

Q9. Prove that, $\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$.

Q10. Show that the points $(a+5, a-4)$, $(a-2, a+3)$ and (a, a) do not lie on a straight line for any value of a.

Q11. Show that $\Delta = \begin{vmatrix} x & p & q \\ p & x & q \\ q & q & x \end{vmatrix} = (x-p)(x^2+px-2q^2)$

Q12. If $a+b+c \neq 0$ and $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$, then prove that $a = b = c$.

Q13. If $\Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$, $\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ yz & zx & xy \\ x & y & z \end{vmatrix}$, then prove that $\Delta + \Delta_1 = 0$.

Q14. If $A+B+C=0$, then prove that $\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix} = 0$.

Q15. If the coordinates of the vertices of an equilateral triangle with sides of length 'a' are (x_1, y_1), (x_2, y_2) and (x_3, y_3), then prove that

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = \frac{3a^4}{4}.$$

Q16. IF $x = -4$ is a root of $\Delta = \begin{vmatrix} x & 3 & 3 \\ 1 & x & 1 \\ 3 & 2 & x \end{vmatrix} = 0$, then find the other two roots.

Q17. If $x + y + z = 0$, then prove that $\begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix} = xyz \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$.

Q18. Prove that $\begin{vmatrix} bc - a^2 & ca - b^2 & ab - c^2 \\ ca - b^2 & ab - c^2 & bc - a^2 \\ ab - c^2 & bc - a^2 & ca - b^2 \end{vmatrix}$ is divisible by $(a + b + c)$ and find the quotient.

Q19. If $a_1, a_2, a_3, ..., a_r$ are in G.P., then prove that the determinant $\begin{vmatrix} a_{r+1} & a_{r+5} & a_{r+9} \\ a_{r+7} & a_{r+11} & a_{r+15} \\ a_{r+11} & a_{r+17} & a_{r+21} \end{vmatrix}$ is independent of r.

Q20. Find the value of θ satisfying $\begin{bmatrix} 1 & 1 & \sin 3\theta \\ -4 & 3 & \cos 2\theta \\ 7 & -7 & -2 \end{bmatrix} = 0$

Q21. If $\begin{bmatrix} 4 - x & 4 + x & 4 + x \\ 4 + x & 4 - x & 4 + x \\ 4 + x & 4 + x & 4 - x \end{bmatrix} = 0$, then find the value of x.

Q22. Show that if the determinant $\Delta = \begin{vmatrix} 3 & -2 & \sin 3\theta \\ -7 & 8 & \cos 2\theta \\ -11 & 14 & 2 \end{vmatrix} = 0$, then $\sin\theta = 0$ or $\frac{1}{2}$.

Q23. In a triangle ABC, if

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 + \sin A & 1 + \sin B & 1 + \sin C \\ \sin A + \sin^2 A & \sin B + \sin^2 B & \sin C + \sin^2 C \end{vmatrix} = 0,$$

then prove that ΔABC is an isoceles triangle.

Q24. Using matrix method, solve the system of equations $3x + 2y - 2z = 3$, $x + 2y + 3z = 6$ and $2x - y + z = 2$.

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S1. We have, $\begin{bmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{bmatrix} = \begin{bmatrix} a & -a & 0 \\ 0 & a & -a \\ x & y & a+z \end{bmatrix}$ $[\because R_1 \to R_1 - R_2 \text{ and } R_2 \to R_2 - R_3]$

$$= \begin{bmatrix} a & 0 & 0 \\ 0 & a & -a \\ x & x+y & a+z \end{bmatrix} \qquad [\because C_2 \to C_2 - C_1]$$

$$= a\,(a^2 + az + ax + ay)$$
$$= a^2(a + z + x + y).$$

S2. We have, $\begin{bmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & zy^2 & 0 \end{bmatrix} = x^2y^2z^2 \begin{bmatrix} 0 & x & x \\ y & 0 & y \\ z & z & 0 \end{bmatrix}$

[taking x^2, y^2 and z^2 common from C_1, C_2 and C_3, respectively]

$$= x^2y^2z^2 \begin{bmatrix} 0 & 0 & x \\ y & -y & y \\ z & z & 0 \end{bmatrix} \qquad [\because C_2 \to C_2 - C_3]$$

$$= x^2y^2z^2\,[x\,(yz + yz)]$$
$$= x^2y^2z^2 \cdot 2xyz = 2x^3y^3z^3.$$

S3.

$$\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & 5 \\ 8 & 3 \end{vmatrix}$$

$\Rightarrow \quad 2x \times x - 8 \times 5 = 6 \times 3 - 8 \times 5$

$\Rightarrow \quad x^2 = 9 \quad \Rightarrow \quad x = \pm 3.$

S4. Applying $C_1 \to C_1 - C_2 - C_3$, we have

$$\Delta = \begin{vmatrix} \text{cosec}^2\theta - \cot^2\theta - 1 & \cot^2\theta & 1 \\ \cot^2\theta - \text{cosec}^2\theta + 1 & \text{cosec}^2\theta & -1 \\ 0 & 40 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & \cot^2\theta & 1 \\ 0 & \text{cosec}^2\theta & -1 \\ 0 & 40 & 2 \end{vmatrix} = 0.$$

S5. Interchanging rows and column, we get

$$\Delta = \begin{vmatrix} 0 & b-a & c-a \\ a-b & 0 & c-b \\ a-c & b-c & 0 \end{vmatrix} = -\Delta$$

$\Rightarrow$ $2\Delta = 0 \quad \text{or} \quad \Delta = 0.$

S6. SInce A is an invertible matrix, so, it is non-singular.

We know that $|A| = |A'|$. But $|A| \neq 0$. So $|A'| \neq 0$. *i.e.*, A' is invertible matrix.

Now we know that, $AA^{-1} = A^{-1}A = I$.

Taking transposde on both sides, we get

$$(A^{-1})'A' = A'(A^{-1})' = (I)' = I.$$

Hence, $(A^{-1})'$ is inverse of A', *i.e.*, $(A')^{-1} = (A^{-1})'$.

S7. We have, $\begin{vmatrix} x+4 & x & x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix} = \begin{vmatrix} 2x+4 & 2x+4 & 2x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix}$ $[\because\ R_1 \to R_1 + R_2]$

$$= \begin{vmatrix} 2x & 2x & 2x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix} + \begin{vmatrix} 4 & 4 & 0 \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix}$$

[Here, given determinant is expressed in sum of two determinants]

$$= 2x\begin{vmatrix} 1 & 1 & 1 \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix} + 4\begin{vmatrix} 1 & 1 & 0 \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix}$$

[Taking $2x$ common from first row of first determinant and 4 from first row of second determinant]

Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$ in first and applying $C_1 \to C_1 - C_2$ in second, we get

$$= 2x\begin{vmatrix} 0 & 0 & 1 \\ 0 & 4 & x \\ -4 & -4 & x+4 \end{vmatrix} + 4\begin{vmatrix} 0 & 1 & 0 \\ -4 & x+4 & x \\ 0 & x & x+4 \end{vmatrix}$$

Expanding both the along first column, we get

$$2x[-4(-4)] + 4[4(x+4-0)] = 2x \times 16 + 16(x+4)$$

$= 32x + 16x + 64$

$= 16\,(3x + 4).$

S8. We have to prove,

$$\begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix} = 4xyz$$

$\therefore$

$$\text{L.H.S.} = \begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix}$$

$$= \begin{vmatrix} y+z+z+y & z & y \\ z+z+x+x & z+x & x \\ y+x+x+y & x & x+y \end{vmatrix} \qquad [\because\ C_1 \to C_1 + C_2 + C_3]$$

$$= 2\begin{vmatrix} (y+z) & z & y \\ (z+x) & z+x & x \\ (x+y) & x & x+y \end{vmatrix} \qquad [\text{Taking 2 common from } C_1]$$

$$= 2\begin{vmatrix} y & z & y \\ 0 & z+x & x \\ y & x & x+y \end{vmatrix} \qquad [\because\ C_1 \to C_1 - C_2]$$

$$= 2\begin{vmatrix} 0 & z-x & -x \\ 0 & z+x & x \\ y & x & x+y \end{vmatrix} \qquad [\because\ R_1 \to R_1 - R_3]$$

$= 2\,[y\,(xz - x^2 + xz + x^2)]$

$= 4xyz = \text{R.H.S.}$ **Hence proved.**

S9. We have to prove,

$$\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$$

$\therefore$ $$\text{L.H.S.} = \begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a^2+2a-2a-1 & 2a+1-a-2 & 0 \\ 2a+1-3 & a+2-3 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

[$\because$ $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$]

$$= \begin{vmatrix} (a-1)(a+1) & (a-1) & 0 \\ 2(a-1) & (a-1) & 0 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^2 \begin{vmatrix} (a+1) & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

[Taking $(a - 1)$ common from R_1 and R_2 each]

$$= (a-1)^2 [1(a+1) - 2] = (a-1)^3$$

= R.H.S. **Hence proved.**

S10. Given, the points are $(a + 5, a - 4)$, $(a - 2, a + 3)$ and (a, a).

$\therefore$ $$\Delta = \frac{1}{2}\begin{vmatrix} a+5 & a-4 & 1 \\ a-2 & a+3 & 1 \\ a & a & 1 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 5 & -4 & 0 \\ -2 & 3 & 0 \\ a & a & 1 \end{vmatrix}$$ [$\because$ $R_1 \to R_1 - R_3$ and $R_2 \to R_2 + R_3$]

$$= \frac{1}{2}[1(15-8)]$$

$\Rightarrow$ $$= \frac{7}{2} \neq 0.$$

Hence, given points form a triangle *i.e.*, points do not lie in a straight line.

S11. Applying $C_1 \to C_1 - C_2$, we have

$$\Delta = \begin{vmatrix} x-p & p & q \\ p-x & x & q \\ 0 & q & x \end{vmatrix} = (x-p)\begin{vmatrix} 1 & p & q \\ -1 & x & q \\ 0 & q & x \end{vmatrix}$$

$$= (x-p)\begin{vmatrix} 0 & p+x & 2q \\ -1 & x & q \\ 0 & q & x \end{vmatrix}$$ [Applying $R_1 \to R_1 - R_2$]

Expanding Along C_1, we have

$$\Delta = (x-p)(px + x^2 - 2q^2)$$
$$= (x-p)(x^2 + px - 2q^2).$$

S12. Let

$$A = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix} \qquad [\because\ R_1 \to R_1 + R_2 + R_3]$$

$$= (a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ b-a & c-a & a \\ c-b & a-b & b \end{vmatrix} \qquad [\because\ C_1 \to C_1 - C_3 \text{ and } C_2 \to C_2 + C_3]$$

Expanding along R_1,

$$= (a+b+c)[1(b-a)(a-b) - (c-a)(c-b)]$$
$$= (a+b+c)(ba - b^2 - a^2 + ab - c^2 + cb + ac - ab)$$
$$= \frac{-1}{2}(a+b+c) \times (-2)(-a^2 - b^2 - c^2 - ab - bc - ca)$$
$$= \frac{-1}{2}(a+b+c)[a^2 + b^2 + c^2 - 2ab - 2bc - 2ca + a^2 + b^2 + c^2]$$
$$= \frac{-1}{2}(a+b+c)[a^2 + b^2 - 2ab + b^2 + c^2 - 2bc + c^2 + a^2 - 2ac]$$
$$= \frac{-1}{2}(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2]$$

Also, $A = 0$

$$= \frac{-1}{2}(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2] = 0$$
$$(a-b)^2 + (b-c)^2 + (c-a)^2 = 0 \qquad [\because\ a+b+c \neq 0, \text{ given}]$$
$$a - b = b - c = c - a = 0$$
$$a = b = c$$

Hence proved.

S13. We have,

$$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ yz & zx & xy \\ x & y & z \end{vmatrix}$$

Interchanging rows and column, we get

$$\Delta_1 = \begin{vmatrix} 1 & yz & x \\ 1 & zx & y \\ 1 & xy & z \end{vmatrix} = \frac{1}{xyz}\begin{vmatrix} x & xyz & x^2 \\ y & xyz & y^2 \\ z & xyz & z^2 \end{vmatrix}$$

$$= \frac{xyz}{xyz}\begin{vmatrix} x & 1 & x^2 \\ y & 1 & y^2 \\ z & 1 & z^2 \end{vmatrix} \qquad \text{[Interchanging } C_1 \text{ and } C_2\text{]}$$

$$= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = -\Delta$$

$\Rightarrow$ $\Delta_1 + \Delta = 0.$

S14. We have to prove,

$$\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix} = 0$$

$\therefore$

$$\text{L.H.S.} = \begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix}$$

$$= 1(1 - \cos^2 A) - \cos C(\cos C - \cos A \cdot \cos B) + \cos B(\cos C \cdot \cos A - \cos B)$$

$$= \sin^2 A - \cos^2 C + \cos A \cdot \cos B \cdot \cos C + \cos A \cdot \cos B \cdot \cos C - \cos^2 B$$

$$= \sin^2 A - \cos^2 B + 2\cos A \cdot \cos B \cdot \cos C - \cos^2 C$$

$$= -\cos(A + B) \cdot \cos(A - B) + 2\cos A \cdot \cos B \cdot \cos C - \cos^2 C$$

$$[\because \ \cos^2 B - \sin^2 A = \cos(A + B) \cdot \cos(A - B)]$$

$$= -\cos(-C) \cdot \cos(A - B) + \cos C(2\cos A \cdot \cos B - \cos C)$$

$$[\because \ \cos(-\theta) = \cos\theta]$$

$$= -\cos C(\cos A \cdot \cos B + \sin A \cdot \sin B - 2\cos A \cdot \cos B + \cos C)$$

$$= \cos C(\cos A \cdot \cos B - \sin A \cdot \sin B - \cos C)$$

$$= \cos C[\cos(A + B) - \cos C]$$

$$= \cos C(\cos C - \cos C) = 0 = \text{RHS.}$$

Hence proved.

S15. Since, we know that area of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3), is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\Rightarrow \quad \Delta^2 = \frac{1}{4}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 \quad \text{... (i)}$$

We know that, area of an equilateral triangle with side *a*,

$$\Delta = \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)a^2 = \frac{\sqrt{3}}{4}a^2$$

$$\Rightarrow \quad \Delta^2 = \frac{3}{16}a^4 \quad \text{... (ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{3}{16}a^4 = \frac{1}{4}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2$$

$$\Rightarrow \quad \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = \frac{3}{4}a^4$$ **Hence proved.**

S16. Applying $R_1 \to (R_1 + R_2 + R_3)$, we get

$$\begin{vmatrix} x+4 & x+4 & x+4 \\ 1 & x & 1 \\ 3 & 2 & x \end{vmatrix}.$$

Taking $(x + 4)$ common from R_1, we get

$$\Delta = (x+4)\begin{vmatrix} 1 & 1 & 1 \\ 1 & x & 1 \\ 3 & 2 & x \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we get

$$\Delta = (x+4)\begin{vmatrix} 1 & 0 & 0 \\ 1 & x-1 & 0 \\ 3 & -1 & x-3 \end{vmatrix}.$$

Expanding along R_1,

$$\Delta = (x + 4)\,[(x - 1)\,(x - 3)\,(x - 3) - 0].$$

Thus, $\Delta = 0$ implies

$$x = -4, 1, 3.$$

Hence, other two roots are 1 & 3.

S17. Since, $x + y + z = 0$ also we have to prove

$$\begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix} = xyz \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

$\therefore$ $$\text{L.H.S.} = \begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix}$$

$$= xa\,(za \cdot ya - xb \cdot xc) - yb\,(yc \cdot ya - xb \cdot zb) + zc\,(yc \cdot xc - za \cdot zb)$$
$$= xa\,(a^2yz - x^2bc) - yb\,(y^2ac - b^2xz) + zc\,(c^2xy - z^2ab)$$

$$= xyza^3 - x^3abc - y^3abc + b^3xyz + c^3xyz - z^3abc$$
$$= xyz\,(a^3 + b^3 + c^3) - abc\,(x^3 + y^3 + z^3)$$
$$= xyz\,(a^3 + b^3 + c^3) - abc\,(3xyz) \quad [\because\ x + y + z = 0 \Rightarrow x^3 + y^3 + z^3 - 3xyz]$$
$$= xyz\,(a^3 + b^3 + c^3 - 3abc) \qquad \ldots \text{(i)}$$

Now, $$\text{R.H.S.} = xyz \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = xyz \begin{vmatrix} a+b+c & b & c \\ a+b+c & a & b \\ a+b+c & c & a \end{vmatrix} \quad [\because\ C_1 \to C_1 - C_2 + C_3]$$

$$= xyz\,(a + b + c) \begin{vmatrix} 1 & b & c \\ 1 & a & b \\ 1 & c & a \end{vmatrix} \quad [\text{Taking } (a + b + c) \text{ common from } C_1]$$

$$= xyz\,(a + b + c) \begin{vmatrix} 0 & b-c & c-a \\ 0 & a-c & b-a \\ 1 & c & a \end{vmatrix} \quad [\because\ R_1 \to R_1 - R_3 \text{ and } R_2 \to R_2 - R_3]$$

Expanding along C_1,

$$= xyz\,(a + b + c)\,[1\,(b - c)\,(b - a) - (a - c)\,(c - a)]$$
$$= xyz\,(a + b + c)\,(b^2 - ab - bc + ac + a^2 + c^2 - 2ac)$$
$$= xyz\,(a + b + c)\,(a^2 + b^2 + c^2 - ab - bc - ca)$$
$$= xyz\,(a^3 + b^3 + c^3 - 3abc) \qquad \ldots \text{(ii)}$$

From Eqs. (i) and (ii), we get

L.H.S. = R.H.S.

$$\Rightarrow \begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix} = xyz \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

Hence proved.

S18. Let,

$$\Delta = \begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$$

$$= \begin{vmatrix} bc-a^2-ca+b^2 & ca-b^2-ab+c^2 & ab-c^2 \\ ca-b^2-ab+c^2 & ab-c^2-bc+a^2 & bc-a^2 \\ ab-c^2-bc+a^2 & bc-a^2-ca+b^2 & ca-b^2 \end{vmatrix}$$

[$\because$ $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$]

$$= \begin{vmatrix} (b-a)(a+b+c) & (c-b)(a+b+c) & ab-c^2 \\ (c-b)(a+b+c) & (a-c)(a+b+c) & bc-a^2 \\ (a-c)(a+b+c) & (b-a)(a+b+c) & ca-b^2 \end{vmatrix}$$

$$= (a+b+c)^2 \begin{vmatrix} b-a & c-b & ab-c^2 \\ c-b & a-c & bc-a^2 \\ a-c & b-a & ca-b^2 \end{vmatrix}$$

[Taking $(a+b+c)$ common from C_1 and C_2 each]

$$= (a+b+c)^2 \begin{vmatrix} 0 & 0 & ab+bc+ca-(a^2+b^2+c^2) \\ c-b & a-c & bc-a^2 \\ a-c & b-a & ca-b^2 \end{vmatrix}$$

[$\because$ $R_1 \to R_1 + R_2 + R_3$]

Now, expanding along R_1,

$$= (a+b+c)^2 [ab+bc+ca-(a^2+b^2+c^2)]\,(c-b)\,(b-a)\,(a-c)^2]$$

$$= (a+b+c)^2 (ab+bc+ca-a^2-b^2-c^2)(cb-ac-b^2+ab-a^2-c^2+2ac)$$

$$= (a+b+c)^2 (a^2+b^2+c^2-ab-bc-ca)(a^2+b^2+c^2-ac-ab-bc)$$

$$= \frac{1}{2}(a+b+c)\,[(a+b+c)(a^2+b^2+c^2-ab-bc-ca)]\,[(a-b)^2+(b-c)^2+(c-a)^2]$$

$$= \frac{1}{2}(a+b+c)(a^3+b^3+c^3-3abc)\,[(a-b)^2+(b-c)^2+(c-a)^2]$$

Hence, given determinant is divisible by $(a+b+c)$ and quotient is

$$(a^3+b^3+c^3-3abc)\,[(a-b)^2+(b-c)^2+(c-a)^2].$$

S19. We know that, $a_{r+1} = AR^{(r+1)-1} = AR^r$

where $r = r^{th}$ term of a G.P., A = First term of a G.P. and R = Common ratio of G.P.

We have, $\begin{vmatrix} a_{r+1} & a_{r+5} & a_{r+9} \\ a_{r+7} & a_{r+11} & a_{r+15} \\ a_{r+11} & a_{r+17} & a_{r+21} \end{vmatrix} = \begin{vmatrix} AR^r & AR^{r+4} & AR^{r+8} \\ AR^{r+6} & AR^{r+10} & AR^{r+14} \\ AR^{r+10} & AR^{r+16} & AR^{r+20} \end{vmatrix}$

$$= AR^r \cdot AR^{r+6} \cdot AR^{r+10} \begin{vmatrix} 1 & AR^4 & AR^8 \\ 1 & AR^4 & AR^8 \\ 1 & AR^6 & AR^{10} \end{vmatrix}$$

$= 0$ [Since, R_1 and R_2 are identicals]

[Taking AR^r, AR^{r+6} and AR^{r+10} common from R_1, R_2 and R_3, respectively]

Hence, given Δ is Independent of θ.

S20. We have,

$$\begin{vmatrix} 1 & 1 & \sin 3\theta \\ -4 & 3 & \cos 2\theta \\ 7 & -7 & -2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 0 & 1 & \sin 3\theta \\ -7 & 3 & \cos 2\theta \\ 14 & -7 & -2 \end{vmatrix} = 0 \qquad [\because C_1 \to C_1 - C_2]$$

$$\Rightarrow 7\begin{vmatrix} 0 & 1 & \sin 3\theta \\ -1 & 3 & \cos 2\theta \\ 2 & -7 & -2 \end{vmatrix} = 0 \qquad \text{[Taking 7 common from } C_1]$$

$\Rightarrow$ $7\,[0 - 1\,(2 - 2\cos 2\theta) + \sin 3\theta\,(7 - 6) = 0$ [Expanding along R_1]

$\Rightarrow$ $7\,[-2\,(1 - \cos 2\theta) + \sin 3\theta) = 0$

$\Rightarrow$ $-14 + 14\cos 2\theta + 7\sin 3\theta = 0$

$\Rightarrow$ $14\cos 2\theta + 7\sin 3\theta = 14$

$\Rightarrow$ $14\,(1 - 2\sin^2\theta) + 7\,(3\sin\theta - 4\sin^3\theta) = 14$

$\Rightarrow$ $-28\sin^2\theta + 14 + 21\sin\theta - 28\sin^3\theta = 14$

$\Rightarrow$ $-28\sin^3\theta + 28\sin^3\theta + 21\sin\theta = 0$

$\Rightarrow$ $28\sin^3\theta + 28\sin^2\theta + 21\sin\theta = 0$

$\Rightarrow$ $4\sin^3\theta + 4\sin^2\theta - 3\sin\theta = 0$

$\Rightarrow$ $\sin\theta\,(4\sin^2\theta + 4\sin\theta - 3) = 0$

$\Rightarrow$ Either $\sin\theta = 0$,

$\Rightarrow$ $\theta = n\pi$ or $4\sin^2\theta + 4\sin\theta - 3 = 0$

$\therefore$ $\sin\theta = \dfrac{-4 \pm \sqrt{16+48}}{8} = \dfrac{-4 \pm \sqrt{64}}{8}$

$= \dfrac{-4 \pm 8}{8} = \dfrac{4}{8}, \dfrac{-12}{8}$

$\sin\theta = \dfrac{1}{2}, \dfrac{-3}{2}$

If $\sin\theta = \dfrac{1}{2} = \sin\dfrac{\pi}{6}$, then

$$\theta = n\pi + (-1)^n \frac{\pi}{6}$$

Since, $\sin\theta = \frac{-3}{2}$ [Not possible because $-1 \le \sin\theta \le 1$]

S21. Given,

$$\begin{vmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0,$$

$$\Rightarrow \begin{vmatrix} 12-x & 12+x & 12+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0 \qquad [\because R_1 \to R_1 + R_2 + R_3]$$

$$\Rightarrow (12+x)\begin{vmatrix} 1 & 1 & 1 \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0 \qquad [\text{Taking } (12+x) \text{ common from } R_1]$$

$$\Rightarrow (12+x)\begin{vmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0 \qquad [\because C_1 \to C_1 - C_3 \text{ and } C_2 \to C_2 + C_3]$$

$$\Rightarrow (12+x)[1 \cdot (-16x) = 0$$

$$\Rightarrow (12+x)(-16x) = 0$$

$$\therefore x = -12, 0.$$

S22. Applying $R_2 \to R_2 + 4R_1$ and $R_3 \to R_3 + 7R_1$, we get

$$\begin{vmatrix} 3 & -2 & \sin 3\theta \\ 5 & 0 & \cos 2\theta + 4\sin 3\theta \\ 10 & 0 & 2 + 7\sin 3\theta \end{vmatrix} = 0$$

or $2[5(2 + 7\sin 3\theta) - 10(\cos 2\theta + 4\sin 3\theta)] = 0$

or $2 + 7\sin 3\theta - 2\cos 2\theta - 8\sin 3\theta = 0$

or $2 - 2\cos 2\theta - \sin 3\theta = 0$

$\sin\theta(4\sin^2\theta + 4\sin\theta - 3) = 0$

or $\sin\theta = 0$ or $(2\sin\theta - 1) = 0$ or $(2\sin\theta + 3) = 0$

or $\sin\theta = 0$ or $\sin\theta = \frac{1}{2}$ [Since, $\sin\theta \ne -3/2$]

S23. Let,

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A + \sin^2 A & \sin B + \sin^2 B & \sin C + \sin^2 C \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ -\cos^2 A & -\cos^2 B & -\cos^2 C \end{vmatrix} \qquad [R_3 \to R_3 - R_2]$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ -\cos^2 A & \cos^2 A - \cos^2 B & \cos^2 B - \cos^2 C \end{vmatrix} \quad [C_1 \to C_3 - C_2 \text{ and } C_2 \to C_2 - C_1]$$

Expanding along R_1, we get

$$\Delta = (\sin B - \sin A)\,(\sin^2 C - \sin^2 B) - (\sin C - \sin B)\,(\sin^2 B - \sin^2 A)$$

$$= (\sin B - \sin A)\,(\sin C - \sin B)\,(\sin C - \sin A) = 0$$

$\Rightarrow$ either $\sin B - \sin A = 0$ or $\sin C - \sin B$ or $\sin C - \sin A = 0$

$\Rightarrow$ $A = B$ or $B = C$ or $C = A$.

i.e., triangle ABC is isoceles.

S24. Given system of equations is $3x + 2y - 2z = 3$, $x + 2y + 3z = 6$ and $2x - y + z = 2$.

In the form of $AX = B$,

$$\begin{bmatrix} 3 & 2 & -2 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$$

For A^{-1}, $\quad |A| = |\,3(5) - 2(1-6) + (-2)(-5)\,|$

$$= |\,15 + 10 + 10\,| = |\,35\,| \neq 0$$

$\therefore \quad A_{11} = 5, A_{12} = 5, A_{13} = -5, A_{21} = 0, A_{22} = 7, A_{23} = 7, A_{31} = 10, A_{32} = -11$ and $A_{33} = 4$

$$\therefore \qquad \text{adj } A = \begin{bmatrix} 5 & 5 & -5 \\ 0 & 7 & 7 \\ 10 & -11 & 4 \end{bmatrix}^T = \begin{bmatrix} 5 & 0 & 10 \\ 5 & 7 & -11 \\ -5 & 7 & 4 \end{bmatrix}$$

Now, $$A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{35}\begin{bmatrix} 5 & 0 & 10 \\ 5 & 7 & -11 \\ -5 & 7 & 4 \end{bmatrix}$$

For $X = A^{-1}B$, $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{35}\begin{bmatrix} 5 & 0 & 10 \\ 5 & 7 & -11 \\ -5 & 7 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$$

$$= \frac{1}{35}\begin{bmatrix} 15+20 \\ 15+42-22 \\ -15+42+8 \end{bmatrix} = \frac{1}{35}\begin{bmatrix} 35 \\ 35 \\ 35 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore \qquad x = 1, \quad y = 1 \quad$ and $\quad z = 1$.