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Q1. The x-coordinate of a point on the line joining the points $A(2, 2, 1)$ and $R(5, 1, -2)$ is 4. Find its z-coordinate.

Q2. Find the direction cosines of the line passing through the points $P(2, 3, 5)$ and $Q(-1, 2, 4)$.

Q3. If the direction ratios of a line are 1, 1, 2, find the direction cosines of the line.

Q4. If a line makes an angle of 30°, 60°, 90° with the positive direction of x, y, z-axes, respectively, then find its direction cosines.

Q5. Find the vector equation of the line which is parallel to the vector $3\hat{i} - 2\hat{j} + 6\hat{k}$ and which passes through the point $(1, -2, 3)$.

Q6. Find the position vector of a point A in space such that $\vec{OA}$ is inclined at 60° to OX and 45° to OY and $|\vec{OA}| = 10$ units

Q7. Find the angle between the lines

$$\vec{r} = 3\hat{i} - 2\hat{j} + 6\hat{k} + \lambda(2\hat{i} + \hat{j} + 2\hat{k}) \text{ and } \vec{r} = (2\hat{j} - 5\hat{k}) + \mu(6\hat{i} + 3\hat{j} + 2\hat{k}).$$

Q8. Find the angle between the lines whose direction cosines are given by the equation $l + m + n = 0$ and $l^2 + m^2 - n^2 = 0$.

Q9. Find the distance of a point $(2, 4, -1)$ from the line

$$\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9}$$

Q10. Find the distance of the point $(-2, 4, -5)$ from the line $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$.

Q11. Find the shortest distance between the lines gives by

$$\vec{r} = (8 + 3\lambda)\hat{i} - (9 + 16\lambda)\hat{j} + (10 + 7\lambda)\hat{k}$$

and

$$\vec{r} = 15\hat{i} + 29\hat{j} + 5\hat{k} + \mu(3\hat{i} + 8\hat{j} - 5\hat{k})$$

Q12. Find the equation of a plane which bisects perpendicularly the line joining the points $A(2, 3, 4)$ and $B(4, 5, 8)$ at right angles.

Q13. Find the equation of a plane which is at a distance $3\sqrt{3}$ units from origin and the normal to which is equally inclined to coordinate axis.

Q14. Find the length and the foot of perpendicular from the point $\left(1, \frac{3}{2}, 2\right)$ to the plane $2x - 2y + 4z + 5 = 0$.

Q15. Two systems of rectangular axis have the same origin. If a plane cuts them at distances a, b, c and a', b', c', respectively from the origin, then prove that

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}.$$

Q16. If O is the origin and A is (a, b, c), then find the direction cosines of the line OA and the equation of plane through A at right angle to OA.

Q17. If the line drawn from the point $(-2, -1, -3)$ meets a plane at right angle at the point $(1, -3, 3)$, then find the equation of the plane.

Q18. Find the equation of the plane through the points $(2, 1, 0)$, $(3, -2, -2)$ and $(3, 1, 7)$.

Q19. Find the equation of the line passing through the point $(3, 0, 1)$ and parallel to the planes $x + 2y = 0$ and $3y - z = 0$.

Q20. Find the distance of the point whose position vector is $(2\hat{i} + \hat{j} - \hat{k})$ from the plane $\vec{r} \cdot (\hat{i} - 2\hat{j} + 4\hat{k}) = 9$.

Q21. Show that the points $(\hat{i} - \hat{j} + 3\hat{k})$ and $3(\hat{i} + \hat{j} + \hat{k})$ are equidistant from the plane $\vec{r} \cdot (5\hat{i} + 2\hat{j} - 7\hat{k}) + 9 = 0$ and lies on opposite side of it.

Q22. Find the coordinates of the point where the line through $(3, -4, -5)$ and $(2, -3, 1)$ crosses the plane passing through three points $(2, 2, 1)$, $(3, 0, 1)$ and $(4, -1, 0)$.

Q23. Find the distance of the point $(-1, -5, -10)$ from the point of intersection of the line $\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$.

Q24. Show that the straight lines whose direction cosines are given by $2l + 2m - n = 0$ and $mn + nl + lm = 0$ are at right angles.

Q25. Find the foot of perpendicular from the point $(2, 3, -8)$ to the line $\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$. Also, find the perpendicular distance from the given point to the line.

Q26. If a variable line in two adjacent positions has direction cosines l, m, n and $l + \delta l, m + \delta m, n + \delta n$ then show that the small angle $\delta\theta$ between the two positions is given by $\delta\theta^2 = \delta l^2 + \delta m^2 + \delta n^2$.

Q27. Find the equations of the two lines passing through the origin which intersect the line $\frac{x-3}{2} = \frac{y-3}{1} = \frac{z}{1}$ at angles of $\frac{\pi}{3}$ each.

Q28. Prove that the lines $x = py + q$, $z = ry + s$ and $x = p'y + q'$, $z = r'y + s'$ are perpendicular, if $pp' + rr' + 1 = 0$.

Q29. $\vec{AB} = 3\hat{i} - \hat{j} + \hat{k}$ and $\vec{CD} = -3\hat{i} + 2\hat{j} + 4\hat{k}$ are two vectors. The position vectors of the points A and C are $6\hat{i} + 7\hat{j} + 4\hat{k}$ and $-9\hat{i} + 2\hat{k}$, respectively. Find the position vector of a point P on the line AB and a point Q on the line CD such that $\vec{PQ}$ is perpendicular to $\vec{AB}$ and $\vec{CD}$ both.

Q30. Prove that the line through $A\,(0, -1, -1)$ and $B\,(4, 5, 1)$ intersects the line passing through $C\,(3, 9, 4)$ and $D\,(-4, 4, 4)$.

Q31. A plane meets the coordinates axis in A, B, C such that the centroid of the ΔABC is the point (α, β, γ). Show that the equation of the plane is $\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 3$.

Q32. Find the equation of the plane through the intersection of the planes $\vec{r} \cdot (\hat{i} + 3\hat{j}) - 6 = 0$ and $\vec{r} \cdot (3\hat{i} - \hat{j} - 4\hat{k}) = 0$, whose perpendicular distance from origin is unity.

Q33. If the plane $ax + by = 0$ is rotated about its line of intersection with the plane $z = 0$ through an angle α, then prove that the equation of the plane in its new position is $ax + by \pm (\sqrt{a^2 + b^2} \tan \alpha) z = 0$.

Q34. Find the equation of the plane which is perpendicular to the plane $5x + 3y + 6z + 8 = 0$ and which contains the points of intersection of the planes $x + 2y + 3z - 4 = 0$ and $2x + y - z + 5 = 0$.

Q35. Find the image of the point (1, 6, 3) in the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$.

Q36. Find the coordinates of the foot of perpendicular drawn from the point $A(1, 8, 4)$ to the line joining the points $B(0, -1, 3)$ and $C(2, -3, -1)$.

Q37. Find the angle between the lines whose direction cosines are given by the equations: $3l + m + 5n = 0$ and $6mn - 2nl + 5lm = 0$.

Q38. If $l_1, m_1, n_1, l_2, m_2, n_2$ and l_3, m_3, n_3 are the direction cosines of three mutually perpendicular lines, then prove that the line whose direction cosines are proportional to $l_1 + l_2 + l_3$, $m_1 + m_2 + m_3$ and $n_1 + n_2 + n_3$ makes equal angles with them.

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S1. Let the point P divide QR in the ratio $\lambda : 1$, then the coordinate of P are

$$\left(\frac{5\lambda+2}{\lambda+1}, \frac{\lambda+2}{\lambda+1}, \frac{-2\lambda+1}{\lambda+1}\right)$$

But x-coordintate of P is 4. Therefore,

$$\frac{5\lambda+2}{\lambda+1} = 4 \quad \Rightarrow \quad \lambda = 2$$

Hence, the z-coordinate of P is $\frac{-2\lambda+1}{\lambda+1} = -1$.

S2. The direction cosines of a line passing through the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are

$$\frac{x_2 - x_1}{PQ}, \quad \frac{y_2 - y_1}{PQ}, \quad \frac{z_2 - z_1}{PQ}$$

Here,
$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$
$$= \sqrt{(-1-2)^2 + (2-3)^2 + (4-5)^2}$$
$$= \sqrt{9+1+1} = \sqrt{11}$$

Hence, DC's are

$$\pm\left(\frac{-3}{\sqrt{11}}, \frac{-1}{\sqrt{11}}, \frac{-1}{\sqrt{11}}\right) \text{ or } \pm\left(\frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}}\right)$$

S3. The direction cosines are given by

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \frac{b}{\sqrt{a^2+b^2+c^2}}, \quad n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

Here a, b, c are 1, 1, 2, respectively.

Therefore
$$l = \frac{1}{\sqrt{1^2+1^2+2^2}}, \quad m = \frac{1}{\sqrt{1^2+1^2+2^2}}, \quad n = \frac{2}{\sqrt{1^2+1^2+2^2}}$$

i.e., $l = \frac{1}{\sqrt{6}}$, $m = \frac{1}{\sqrt{6}}$, $n = \frac{2}{\sqrt{6}}$ *i.e.,* $\pm\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$ are DC's of the line.

S4. The direction cosines of a line which makes an angle of α, β, γ with the axes, are $\cos\alpha$, $\cos\beta$, $\cos\gamma$.

Therefore, DC's of the line are cos 30°, cos 60°, cos 90° *i.e.*, $\pm\left(\frac{\sqrt{3}}{2}, \frac{1}{2}, 0\right)$.

S5. Let $\vec{a} = 3\hat{i} - 2\hat{j} + 6\hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$.

So, vector equation of the line, which is parallel to the vector $\vec{a} = 3\hat{i} - 2\hat{j} + 6\hat{k}$ and passes through the vector $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$ is $\vec{r} = \vec{b} + \lambda\vec{a}$.

$$\therefore \quad \vec{r} = (\hat{i} - 2\hat{j} + 3\hat{k}) + \lambda(3\hat{i} - 2\hat{j} + 6\hat{k})$$

$$\Rightarrow \quad (x\hat{i} + y\hat{j} + z\hat{k}) - (\hat{i} - 2\hat{j} + 3\hat{k}) = \lambda(3\hat{i} - 2\hat{j} + 6\hat{k})$$

$$\Rightarrow \quad (x - 1)\hat{i} + (y + 2)\hat{j} + (z - 3)\hat{k} = \lambda(3\hat{i} - 2\hat{j} + 6\hat{k})$$

S6. SInce, $\vec{OA}$ is inclined at 60° to OX and at 45° to OY. Let $\vec{OA}$ makes angle α with *OZ*.

$$\therefore \quad \cos^2 60° + \cos^2 45° + \cos^2\alpha = 1$$

$$\Rightarrow \quad \left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cos^2\alpha = 1 \qquad [\because\ l^2 + m^2 + n^2 = 1]$$

$$\Rightarrow \quad \frac{1}{4} + \frac{1}{2} + \cos^2\alpha = 1$$

$$\Rightarrow \quad \cos^2\alpha = 1 - \left(\frac{1}{2} + \frac{1}{4}\right)$$

$$\Rightarrow \quad \cos^2\alpha = 1 - \left(\frac{6}{8}\right)$$

$$\Rightarrow \quad \cos^2\alpha = \frac{1}{4}$$

$$\Rightarrow \quad \cos\alpha = \frac{1}{2}$$

$$\therefore \quad \alpha = 60°$$

$$\therefore \quad \vec{OA} = |\vec{OA}|\left(\frac{1}{2}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + \frac{1}{2}\hat{k}\right)$$

$$= 10\left(\frac{1}{2}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + \frac{1}{2}\hat{k}\right) \qquad [\because\ |\vec{OA}| = 10$$

$$= 5\hat{i} + 5\sqrt{2}\hat{j} + 5\hat{k}$$

S7. We have, $\vec{r} = 3\hat{i} - 2\hat{j} + 6\hat{k} + \lambda(2\hat{i} + \hat{j} + 2\hat{k})$

and $\vec{r} = (2\hat{j} - 5\hat{k}) + \mu(6\hat{i} + 3\hat{j} + 2\hat{k})$

where, $\vec{a}_1 = 3\hat{i} - 2\hat{j} + 6\hat{k}, \quad \vec{b}_1 = 2\hat{i} + \hat{j} + 2\hat{k}$

and $\vec{a}_2 = 2\hat{j} - 5\hat{k}, \quad \vec{b}_2 = 6\hat{i} + 3\hat{j} + 2\hat{k}$

If θ is angle between the lines, then

$$\cos\theta = \frac{|\vec{b}_1 \cdot \vec{b}_2|}{|\vec{b}_1|\cdot|\vec{b}_2|}$$

$$= \frac{|(2\hat{i} + \hat{j} + 2\hat{k})\cdot(6\hat{i} + 3\hat{j} + 2\hat{k})|}{|2\hat{i} + \hat{j} + 2\hat{k}||6\hat{i} + 3\hat{j} + 2\hat{k}|}$$

$$= \frac{|12 + 3 + 4|}{\sqrt{9}\sqrt{49}} = \frac{19}{21}$$

$\therefore \qquad \theta = \cos^{-1}\frac{19}{21}.$

S8. Eliminating n from both the equations, we have

$$l^2 + m^2 - (l - m)^2 = 0$$

$\Rightarrow \qquad l^2 + m^2 - l^2 - m^2 + 2ml = 0 \quad\Rightarrow\quad 2lm = 0$

$\Rightarrow \qquad lm = 0 \quad\Rightarrow\quad (-m - n) = 0 \qquad [\because\ l = -m - n]$

$\Rightarrow \qquad (m + n)\,m = 0$

$\Rightarrow \qquad m = -n \quad\Rightarrow\quad m = 0$

$\Rightarrow \qquad l = 0, \quad l = -n$

Thus, Dr's two lines are proportional to 0, $-n$, n and $-n$, 0, n *i.e.*, 0, -1, 1 and -1, 0, 1.

So, the vector parallel to these given lines are $\vec{a} = -\hat{j} + \hat{k}$ and $\vec{b} = -\hat{i} + \hat{k}$

Now, $\qquad \cos\theta = \frac{\vec{a}\vec{b}}{|\vec{a}||\vec{b}|} = \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} \quad\Rightarrow\quad \cos\theta = \frac{1}{2}$

$\therefore \qquad \theta = \frac{\pi}{3} \qquad \left[\because\ \cos\frac{\pi}{3} = \frac{1}{2}\right]$

S9. We have, equation of the line as $\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9} = \lambda$

$\therefore \qquad x = \lambda - 5, \quad y = 4\lambda - 3, \quad z = 6 - 9\lambda$

Let the coordinates of L be $(\lambda - 5, 4\lambda - 3, 6 - 9\lambda)$, then Dr's of PL are $(\lambda - 7, 4\lambda - 7, 7 - 9\lambda)$.

Also, the direction ratios of given line are proportional to 1, 4, -9.

Since, PL is perpendicular to the given line.

$\therefore \qquad (\lambda - 7)\cdot 1 + (4\lambda - 7)\cdot 4 + (7 - 9\lambda)\cdot(-9) = 0$

$\Rightarrow \qquad \lambda - 7 + 16\lambda - 28 + 81\lambda - 63 = 0$

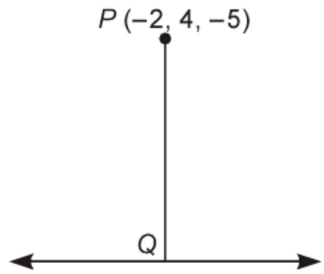
$\Rightarrow \qquad 98\lambda = 98 \quad\Rightarrow\quad \lambda = 1$

So, the coordinates of L are $(-4, 1, -3)$.

$\therefore$ Required distance, $PL = \sqrt{(-4-2)^2+(1-4)^2+(-3+1)^2}$

$$= \sqrt{36+9+4} = 7 \text{ units.}$$

S10. Here $P(-2, 4, -5)$ is the given point.



Any point Q on the line is given by $(3\lambda - 3)$, $(5\lambda + 4)$, $(6\lambda - 8)$,

$$\overrightarrow{PQ} = (3\lambda - 1)\hat{i} + 5\lambda\hat{j} + (6\lambda - 3)\hat{k}$$

Since, $\overrightarrow{PQ} \perp (3\hat{i} + 5\hat{j} + 6\hat{k})$, we have

$$3(3\lambda - 1) + 5(5\lambda) + 6(6\lambda - 3) = 0$$

$$9\lambda + 25\lambda + 36\lambda = 21, \quad \textit{i.e.,} \quad \lambda = \frac{3}{10}$$

Thus, $$\overrightarrow{PQ} = -\frac{1}{10}\hat{i} + \frac{15}{10}\hat{j} - \frac{12}{10}\hat{k}$$

Hence $$|\overrightarrow{PQ}| = \frac{1}{10}\sqrt{1+225+144} = \sqrt{\frac{37}{10}}$$

S11. We have, $$\vec{r} = (8+3\lambda)\hat{i} - (9+16\lambda)\hat{j} + (10+7\lambda)\hat{k}$$

$$= 8\hat{i} - 9\hat{j} + 10\hat{k} + 3\lambda\hat{i} - 16\lambda\hat{j} + 7\lambda\hat{k}$$

$$= 8\hat{i} - 9\hat{j} + 10\hat{k} + \lambda(3\hat{i} - 16\hat{j} + 7\hat{k})$$

$\Rightarrow$ $$\vec{a}_1 = 8\hat{i} - 9\hat{j} + 10\hat{k} \quad \text{and} \quad \vec{b}_1 = 3\hat{i} - 16\hat{j} + 7\hat{k} \qquad \ldots \text{(i)}$$

Also, $$\vec{r} = 15\hat{i} + 29\hat{j} + 5\hat{k} + \mu(3\hat{i} + 8\hat{j} - 5\hat{k})$$

$\Rightarrow$ $$\vec{a}_2 = 15\hat{i} + 29\hat{j} + 5\hat{k} \quad \text{and} \quad \vec{b}_2 = 3\hat{i} + 8\hat{j} - 5\hat{k} \qquad \ldots \text{(ii)}$$

Now, shortest distance between two lines is given by $\left|\dfrac{(\vec{b}_1 \times \vec{b}_2)\cdot(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right|$

$\therefore$ $$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix}$$

$$= \hat{i}(80-56) + \hat{j}(-15-21) - \hat{k}(24-48)$$

$$= 24\hat{i} + 36\hat{j} + 72\hat{k}$$

Now, $$|\vec{b}_1 \times \vec{b}_2| = \sqrt{(24)^2 + (36)^2 + (72)^2}$$

$$= 12\sqrt{2^2 + 3^2 + 6^2} = 84$$

and $$(\vec{a}_2 - \vec{a}_1) = (15-8)\hat{i} + (29+9)\hat{j} - (5-10)\hat{k}$$

$$= 7\hat{i} + 38\hat{j} - 5\hat{k}$$

$\therefore$ $$\text{Shortest distance} = \left|\frac{(24\hat{i} + 36\hat{j} + 72\hat{k})\cdot(7\hat{i} + 38\hat{j} - 5\hat{k})}{84}\right|$$

$$= \left|\frac{168 + 1368 - 360}{84}\right| = \left|\frac{1176}{84}\right| = 14 \text{ units}$$

S12. Since, the equation of a plane is bisecting perpendicular the line joining the points $A(2, 3, 4)$ and $B(4, 5, 8)$ at right angles.

So, mid-point of AB is $\left(\frac{2+4}{2}, \frac{3+5}{2}, \frac{4+8}{2}\right)$ *i.e.*, (3, 4, 6).

Also $\vec{N} = (4-2)\hat{i} + (5-3)\hat{j} + (8-4)\hat{k}$

$= 2\hat{i} + 2\hat{j} + 4\hat{k}$

So, the required equation of the plane is $(\vec{r} - \vec{a}) \cdot \vec{N} = 0$

$\Rightarrow \quad [(x-3)\hat{i} + (y-4)\hat{j} + (z-6)\hat{k}] \cdot (2\hat{i} + 2\hat{j} + 4\hat{k}) = 0 \qquad [\because \ \vec{a} = 3\hat{i} + 4\hat{j} + 6\hat{k}]$

$\Rightarrow \quad 2x - 6 + 2y - 8 + 4z - 24 = 0$

$\Rightarrow \quad 2x + 2y + 4z = 38$

$\therefore \quad x + y + 2z = 19.$

S13. Since, normal to the plane is equally inclined to the coordinate axis.

Therefore, $\cos\alpha = \cos\beta = \cos\gamma = \frac{1}{\sqrt{3}}$

So, the normal is $\vec{N} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$ and plane is at a distance of $3\sqrt{3}$ units from origin.

The equation plane is $\vec{r} \cdot \hat{N} = 3\sqrt{3}$. $\qquad \left[\because \ \hat{N} = \frac{\vec{N}}{|N|}\right]$

[Since, vector equation of the plane at a distance p from the origin is $\vec{r} \cdot \hat{N} = \rho$]

$\Rightarrow \quad (x\hat{i} + y\hat{j} + z\hat{k}) \frac{\left(\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}\right)}{1} = 3\sqrt{3}$

$\Rightarrow \quad \frac{x}{\sqrt{3}}\hat{i} + \frac{y}{\sqrt{3}}\hat{j} + \frac{z}{\sqrt{3}}\hat{k} = 3\sqrt{3}$

$\therefore \quad x + y + z = 3\sqrt{3} \cdot \sqrt{3} = 9$

So, the required equation of plane is $x + y + z = 9$.

S14. Equation of the given plane is $2x - 2y + 4z + 5 = 0$ ... (i)

$\Rightarrow \quad \vec{n} = 2\hat{i} - 2\hat{j} + 4\hat{k}$

So, the equation of line through $\left(1, \frac{3}{2}, 2\right)$ and parallel to $\vec{n}$ is given by

$$\frac{x-1}{2} = \frac{y-3/2}{-2} = \frac{z-2}{4} = \lambda$$

$\Rightarrow \quad x = 2\lambda + 1, \quad y = -2\lambda + \frac{3}{2} \quad$ and $\quad z = 4\lambda + 2.$

If this point lies on the given plane, then

$$2(2\lambda+1) - 2\left(-2\lambda + \frac{3}{2}\right) + 4(4\lambda+2) + 5 = 0 \qquad \text{[Using Eq. (i)]}$$

$\Rightarrow \quad 4\lambda + 2 + 4\lambda - 3 + 16\lambda + 8 + 5 = 0$

$\Rightarrow \quad 24\lambda = -12 \quad \Rightarrow \quad \lambda = \frac{-1}{2}$

$\therefore$ Required foot of perpendicular

$$= \left[2 \times \left(\frac{-1}{2}\right) + 1, -2 \times \left(\frac{-1}{2}\right) + \frac{3}{2}, 4 \times \left(\frac{-1}{2}\right) + 2\right] \text{ i.e., } \left(0, \frac{5}{2}, 0\right)$$

$\therefore$ Required length of perpendicular

$$= \sqrt{(1-0)^2 + \left(\frac{3}{2} - \frac{5}{2}\right)^2 + (2-0)^2}$$

$$= \sqrt{1+1+4} = \sqrt{6} \text{ units.}$$

S15. Consider OX, OY, OZ and ox, oy, oz are two system of rectangular axes.

Let their corresponding equation of plane be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad \text{... (i)}$$

$$\frac{x}{a'} + \frac{y}{b'} + \frac{z}{c'} = 1 \qquad \text{... (ii)}$$

Also, the length of perpendicular from origin to Eq.s (i) and (ii) must be same.

$$\therefore \quad \frac{\frac{0}{a} + \frac{0}{b} + \frac{0}{c} - 1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}} = \frac{\frac{0}{a'} + \frac{0}{b'} + \frac{0}{c'} - 1}{\sqrt{\frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}}}$$

$\Rightarrow \quad \sqrt{\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}} = \sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}$

$\Rightarrow \quad \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} = \frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}.$

S16. Since, DC's of line *OA* are $\frac{a}{\sqrt{a^2+b^2+c^2}}$, $\frac{b}{\sqrt{a^2+b^2+c^2}}$ and $\frac{c}{\sqrt{a^2+b^2+c^2}}$.

Also $\quad \vec{n} = \vec{OA} = \vec{a} = a\hat{i} + b\hat{j} + c\hat{k}$

The equation of plane passes through (*a*, *b*, *c*) and perpendicular to *OA* is given by

$$[\vec{r} - \vec{a}]\cdot\vec{n} = 0$$

$\Rightarrow \quad \vec{r}\cdot\vec{n} = \vec{a}\cdot\vec{n}$

$\Rightarrow \quad [(x\hat{i} + y\hat{j} + z\hat{k})\cdot(a\hat{i} + b\hat{j} + c\hat{k})] = [(a\hat{i} + b\hat{j} + c\hat{k})\cdot(a\hat{i} + b\hat{j} + c\hat{k})]$

$\Rightarrow \quad ax + by + cz = a^2 + b^2 + c^2.$

S17. Since, the line drawn from the point (– 2, – 1, – 3) meets a plane at right angle at the point (1, –3, 3). So, the plane passes through the point (1, –3, 3) and normal to plane is $(-3\hat{i} + 2\hat{j} - 6\hat{k})$.

$\Rightarrow \quad \vec{a} = \hat{i} - 3\hat{j} + 3\hat{k}$

and $\quad \vec{N} = -3\hat{i} + 2\hat{j} - 6\hat{k}$

So, the equation of required plane is $(\vec{r} - \vec{a})\cdot\vec{N} = 0$

$\Rightarrow \quad [(x\hat{i} + y\hat{j} - z\hat{k}) - (\hat{i} - 3\hat{j} + 3\hat{k})]\cdot(-3\hat{i} + 2\hat{j} - 6\hat{k}) = 0$

$\Rightarrow \quad [(x-1)\hat{i} + (y+3)\hat{j} + (z-3)\hat{k}]\cdot(-3\hat{i} + 2\hat{j} - 6\hat{k}) = 0$

$\Rightarrow \quad -3x + 3 + 2y + 6 - 6z + 18 = 0$

$\Rightarrow \quad -3x + 2y - 6z = -27$

$\therefore \quad 3x - 2y + 6z - 27 = 0.$

S18. We know that, the equation of a plane passing through three non-collinear points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x - 2 & y - 1 & z - 0 \\ 3 - 2 & -2 - 1 & -2 - 0 \\ 3 - 2 & 1 - 1 & 7 - 0 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x - 2 & y - 1 & z \\ 1 & -3 & -2 \\ 1 & 0 & 7 \end{vmatrix} = 0$$

$\Rightarrow \quad (x-2)(-21+0)-(y-1)(7+2)+z(3)=0$

$\Rightarrow \quad -21x+42-9y+9+3z=0$

$\Rightarrow \quad -21x-9y+3z=-51$

$\therefore \quad 7x+3y-z=17.$

So, the required equation of plane is $7x + 3y - z = 17$.

S19. Equation of the two planes are $x + 2y = 0$ and $3y - z = 0$.

Let $\vec{n}_1$ and $\vec{n}_2$ are the normals to the two planes, respectively

$\therefore \quad \vec{n}_1 = \hat{i} + 2\hat{j} \quad \text{and} \quad \vec{n}_2 = 3\hat{j} - \hat{k}$

Since, required line is parallel to the given two planes.

Therefore, $\quad \vec{b} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 0 \\ 0 & 3 & -1 \end{vmatrix}$

$= \hat{i}(-2) - \hat{j}(-1) + \hat{k}(3)$

$= -2\hat{i} + \hat{j} + 3\hat{k}$

So, the equation of the lines through the point (3, 0, 1) and parallel to the given two planes are

$$(x-3)\hat{i} + (y-0)\hat{j} + (z-1)\hat{k} + \lambda(-2\hat{i} + \hat{j} + 3\hat{k}) = 0$$

$\Rightarrow \quad (x-3)\hat{i} + y\hat{j} + (z-1)\hat{k} + \lambda(-2\hat{i} + \hat{j} + 3\hat{k}) = 0$

S20. Here, $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}, \quad \vec{n} = \hat{i} - 2\hat{j} + 4\hat{k}$ and $d = 9$

So, the required distance is $\dfrac{|(2\hat{i} + \hat{j} - \hat{k}) \cdot (\hat{i} - 2\hat{j} + 4\hat{k}) - 9|}{\sqrt{1+4+16}}$

$= \dfrac{|2-2-4-9|}{\sqrt{21}} = \dfrac{13}{\sqrt{21}}.$

S21. To show that these given points $(\hat{i} - \hat{j} + 3\hat{k})$ and $3(\hat{i} + \hat{j} + \hat{k})$ are equidistant from the plane $\vec{r} \cdot (5\hat{i} + 2\hat{j} - 7\hat{k}) + 9 = 0$, we first find out the mid-point of the points which is $2\hat{i} + \hat{j} + 3\hat{k}$.

On substituting $\vec{r}$ by the mid-point in plane, we get

$\text{L.H.S.} = (2\hat{i} + \hat{j} + 3\hat{k}) \cdot (5\hat{i} + 2\hat{j} - 7\hat{k}) + 9$

$= 10 + 2 - 21 + 9 = 0$

$= \text{R.H.S.}$

Hence, the two points lie on opposite sides of the plane are equidistant from the plane.

S22. Equation of plane through three points (2, 2, 1), (3, 0, 1) and (4, – 1, 0) is

$$\left[(\vec{r} - (2\hat{i} + 2\hat{j} + \hat{k}))\right] \cdot \left[(\hat{i} - 2\hat{j}) \times (\hat{i} - \hat{j} - \hat{k})\right] = 0$$

i.e., $\vec{r} \cdot (2\hat{i} + \hat{j} + \hat{k}) = 7$ or $2x + y + z - 7 = 0$ ... (i)

Equation of line through (3, –4, –5) and (2, –3, 1) is

$$\frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6} \quad \text{... (ii)}$$

Any point on line (ii) is $(-\lambda + 3, \lambda - 4, 6\lambda - 5)$. This point lies on plane (i). Therefore $2(-\lambda + 3) + (\lambda - 4) + (6\lambda - 5) - 7 = 0$, *i.e*, $\lambda = -2$.

Hence, the required point is (1, –2, 7).

S23. We have, $\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$.

Solving these two equations, we get $[(2\hat{i} - \hat{j} + 2\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})] \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$ which gives $\lambda = 0$.

Therefore, the point of intersection of line and the plane is (2, – 1, 2) and the other given point is (– 1, – 5, – 10). Hence the distance between these two points is

$$\sqrt{[2-(-1)]^2 + [-1+5]^2 + [2-(-10)]^2}, \quad i.e., \ 13.$$

S24. We have, $2l + 2m - n = 0$ ... (i)

and $mn + nl + lm = 0$ ... (ii)

Eliminating m from the both equations, we get

$$m = \frac{n - 2l}{2} \quad \text{[From Eq. (i)]}$$

$$\Rightarrow \quad \left(\frac{n-2l}{2}\right)n + nl + l\left(\frac{n-2l}{2}\right) = 0$$

$$\Rightarrow \quad \frac{n^2 - 2nl + 2nl + nl - 2l^2}{2} = 0$$

$$\Rightarrow \quad n^2 + nl - 2l^2 = 0$$

$$\Rightarrow \quad n^2 + 2nl - nl - 2l^2 = 0$$

$$\Rightarrow \quad (n + 2l)(n - l) = 0$$

$$\Rightarrow \quad n = -2l \quad \text{and} \quad n = l$$

$$\therefore \quad m = \frac{-2l - 2l}{2}, \quad m = \frac{l - 2l}{2}$$

$$\Rightarrow \quad m = -2l, \quad m = \frac{-l}{2}$$

Thus, the direction ratios of two lines are proportional to $l, -2l, -2$ and $l, \frac{-l}{2}, l$.

$\Rightarrow \quad 1, -2, -2$ and $1, \frac{-1}{2}, 1$

$\Rightarrow \quad 1, -2, -2$ and $2, -1, 2$

Also, the vectors parallel to these lines are $\vec{a} = \hat{i} - 2\hat{j} - 2\hat{k}$ and $\vec{b} = 2\hat{i} - \hat{j} + 2\hat{k}$, respectively.

$$\therefore \quad \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{(\hat{i} - 2\hat{j} - 2\hat{k})\cdot(2\hat{i} - \hat{j} + 2\hat{k})}{3\cdot 3}$$

$$= \frac{2+2-4}{9} = 0$$

$$\Rightarrow \quad \theta = \frac{\pi}{2} \qquad \left[\because \cos\frac{\pi}{2} = 0\right]$$

S25. We have, equation of line as

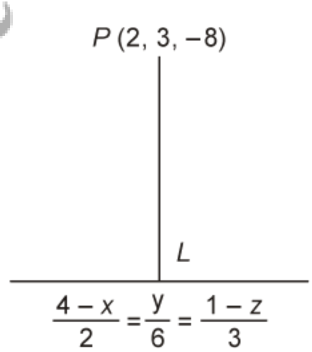
$$\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$$

$$\Rightarrow \quad \frac{x-4}{-2} = \frac{y}{6} = \frac{z-1}{-3} = \lambda$$

$$\Rightarrow \quad x = -2\lambda + 4y, \quad y = 6\lambda \quad \text{and} \quad z = -3\lambda + 1$$

Let the coordinates of L be $(4 - 2\lambda, 6\lambda, 1 - 3\lambda)$ and direction ratios of PL are proportional to $(4 - 2\lambda - 2, 6\lambda - 3, 1 - 3\lambda + 8)$ *i.e.*, $(2 - 2\lambda, 6\lambda - 3, 9 - 3\lambda)$.

Also, direction ratios are proportional to $-2, 6, -3$. Since, PL is perpendicular to give line.



$\therefore \quad -2(2 - 2\lambda) + 6(6\lambda - 3) - 3(9 - 3\lambda) = 0$

$\Rightarrow \quad -4 + 4\lambda + 36\lambda - 18 - 27 + 9\lambda = 0$

$\Rightarrow \quad 49\lambda = 49 \quad \Rightarrow \quad \lambda = 1$

So, the coordinates of L are $(4 - 2\lambda, 6\lambda, 1 - 3\lambda)$ *i.e.*, $(2, 6, -2)$.

Also, $\quad$ Length of $PL = \sqrt{(2-2)^2 + (6-3)^2 + (-2+8)^2}$

$= \sqrt{0 + 9 + 36} = 3\sqrt{5}$ units.

S26. We have l, m, n and $l + \delta l, m + \delta m, n + \delta n$ as direction cosines of a variable line in two different positions.

$$\therefore \quad l^2 + m^2 + n^2 = 1 \qquad \dots (i)$$

$$\text{and} \quad (l + \delta l)^2 + (m + \delta m)^2 + (n + \delta n)^2 = 1 \qquad \dots (ii)$$

$\Rightarrow \quad l^2 + m^2 + n^2 + \delta l^2 + \delta m^2 + \delta n^2 + 2\,(l\delta l + m\delta m + n\delta n) = 1$

$\Rightarrow \quad \delta l^2 + \delta m^2 + \delta n^2 = -2\,(l\delta l + m\delta m + n\delta n)$

$[\because\ l^2 + m^2 + n^2 \Rightarrow 1]$

$\Rightarrow \quad l\delta l + m\delta m + n\delta n = \frac{-1}{2}\,(\delta l^2 + \delta m^2 + \delta n^2)$ ... (iii)

Now, $\vec{a}$ and $\vec{b}$ are unit vectors along a line with direction cosines l, m, n and $(l + \delta l)$, $(m + \delta m)$, $(n + \delta n)$, respectively

$\therefore \quad \vec{a} = l\hat{i} + m\hat{j} + n\hat{k}$

and $\quad \vec{b} = (l + \delta l)\,\hat{i} + (m + \delta m)\,\hat{j} + (n + \delta n)\,\hat{k}$

$\Rightarrow \quad \cos\delta\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \vec{a}\cdot\vec{b}$ $\quad [\because\ |\vec{a}| = |\vec{b}| = 1]$

$\Rightarrow \quad \cos\delta\theta = l\,(l + \delta l) + m\,(m + \delta m) + n\,(n + \delta n)$

$= (l^2 + m^2 + n^2) + (l\delta l + m\delta m + n\delta n)$

$= 1 - \frac{1}{2}\,(\delta l^2 + \delta m^2 + \delta n^2)$ $\quad$ [Using Eq. (iii)]

$\Rightarrow \quad 2\,(1 - \cos\delta\theta) = (\delta l^2 + \delta m^2 + \delta n^2)$

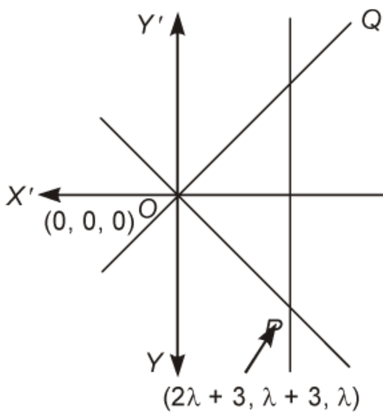
$\Rightarrow \quad 2\cdot 2\sin^2\frac{\delta\theta}{2} = \delta l^2 + \delta m^2 + \delta n^2$ $\quad \left[\because\ 1 - \cos\theta = 2\sin^2\frac{\theta}{2}\right]$

$\Rightarrow \quad 4\left(\frac{\delta\theta}{2}\right)^2 = \delta l^2 + \delta m^2 + \delta n^2$ $\quad \left[\text{Since, } \frac{\delta\theta}{2} \text{ is small, then } \sin\frac{\delta\theta}{2} = \frac{\delta\theta}{2}\right]$

$\therefore \quad \delta\theta^2 = \delta l^2 + \delta m^2 + \delta n^2.$

S27. Given equation of the line is $\frac{x-3}{2} = \frac{y-3}{1} = \frac{z}{1} = \lambda$ ... (i)

So, DR's of the line are 2, 1, 1 and DC's of the given line are $\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{16}}$



Also, the required lines make angle $\frac{\pi}{3}$ with the given line.

From Eq. (i), $\quad x = (2\lambda + 3), \quad y = (\lambda + 3) \quad$ and $\quad z = \lambda$

$\because \quad \cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$

$\Rightarrow \quad \cos\frac{\pi}{3} = \frac{(4\lambda + 6) + (\lambda + 3) + (\lambda)}{\sqrt{6}\sqrt{(2\lambda + 3)^2 + (\lambda + 3)^2 + \lambda^2}}$

$$\Rightarrow \quad \frac{1}{2} = \frac{6\lambda + 9}{\sqrt{6}\sqrt{(4\lambda^2 + 9 + 12\lambda + \lambda^2 + 9 + 6\lambda + \lambda^2)}}$$

$$\Rightarrow \quad \frac{\sqrt{6}}{2} = \frac{6\lambda + 9}{\sqrt{6\lambda^2 + 18\lambda + 18}}$$

$\Rightarrow \quad 6\sqrt{(\lambda^2 + 3\lambda + 3)} = 2\,(6\lambda + 9)$

$\Rightarrow \quad 36\,(\lambda^2 + 3\lambda + 3) = 36\,(4\lambda^2 + 9 + 12\lambda)$

$\Rightarrow \quad \lambda^2 + 3\lambda + 3 = 4\lambda^2 + 9 + 12\lambda$

$\Rightarrow \quad 3\lambda^2 + 9\lambda + 6 = 0$

$\Rightarrow \quad \lambda^2 + 3\lambda + 2 = 0$

$\Rightarrow \quad \lambda\,(\lambda + 2) + 1\,(\lambda + 2) = 0$

$\Rightarrow \quad (\lambda + 1)\,(\lambda + 2) = 0$

$\therefore \quad \lambda = -1, -2$

So, the DC's are 1, 2, – 1 and – 1, 1, – 2.

Also both the required lines passes through origin.

So, the equations of required lines are $\frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$ and $\frac{x}{-1} = \frac{y}{1} = \frac{z}{-2}$.

S28. We have, $\quad x = py + q \quad \Rightarrow \quad y = \frac{x - q}{p}$ ... (i)

and $\quad z = ry + s \quad \Rightarrow \quad y = \frac{z - s}{r}$ ... (ii)

$\Rightarrow \quad \frac{x - q}{p} = \frac{y}{1} = \frac{z - s}{r}$ [Using Eqs. (i) and (ii)] ... (iii)

Similarly, $\quad \frac{x - q'}{p'} = \frac{y}{1} = \frac{z - s'}{r'}$ ... (iv)

From Eqs. (iii) and (iv), we get

$a_1 = p, \quad b_1 = 1, \quad c_1 = r$

and $\quad a_2 = p', \quad b_2 = 1, \quad c_2 = r'$

If these given lines are perpendicular to each other, then

$a_1a_2 + b_1b_2 + c_1c_2 = 0$

$\Rightarrow \quad pp' + 1 + rr' = 0$

which is the required condition.

S29. We have, $\vec{AB} = 3\hat{i} - \hat{j} + \hat{k}$ and $\vec{CD} = -3\hat{i} + 2\hat{j} + 4\hat{k}$

Also, the position vectors of A and C are $6\hat{i} + 7\hat{j} + 4\hat{k}$ and $-9\hat{i} + 2\hat{k}$, respectively. Since $\vec{PQ}$ is perpendicular to both $\vec{AB}$ and $\vec{CD}$.

So, P and Q will be foot of perpendicular to both the lines through A and C.

Now, equation of the line through A and parallel to the vector $\vec{AB}$ is,

$$\vec{r} = (6\hat{i} + 7\hat{j} + 4\hat{k}) + \lambda(3\hat{i} - \hat{j} + \hat{k})$$

and the line through C and parallel to the vector $\vec{CD}$ is given by

$$\vec{r} = -9\hat{j} + 2\hat{k} + \mu(-3\hat{i} + 2\hat{j} + 4\hat{k}) \quad \text{... (i)}$$

Let $\vec{r} = (6\hat{i} + 7\hat{j} + 4\hat{k}) + \lambda(3\hat{i} - \hat{j} + \hat{k})$

and $\vec{r} = -9\hat{j} + 2\hat{k} + \mu(-3\hat{i} + 2\hat{j} + 4\hat{k}) \quad \text{... (ii)}$

Let $P(6 + 3\lambda, 7 - \lambda, 4 + \lambda)$ is any point on the first line and Q be any point on second line is given by $(-3\mu, -9 + 2\mu, 2 + 4\mu)$.

$$\Rightarrow \vec{PQ} = (-3\mu - 6 - 3\lambda)\hat{i} + (-9 + 2\mu - 7 + \lambda)\hat{j} + (2 + 4\mu - 4 - \lambda)\hat{k}$$

$$= (-3\mu - 6 - 3\lambda)\hat{i} + (2\mu + \lambda - 16)\hat{j} + (4\mu - \lambda - 2)\hat{k}$$

If $\vec{PQ}$ is perpendicular to the first line, then

$\Rightarrow$ $3(-3\mu - 6 - 3\lambda) - (2\mu + \lambda - 16) + (4\mu - \lambda - 2) = 0$

$\Rightarrow$ $-9\mu - 18 - 9\lambda - 2\mu - \lambda + 16 + 4\mu - \lambda - 2 = 0$

$\Rightarrow$ $-7\mu - 11\lambda - 4 = 0 \quad \text{... (iii)}$

If $\vec{PQ}$ is perpendicular to the second line, then

$\Rightarrow$ $-3(-3\mu - 6 - 3\lambda) + 2(2\mu + \lambda - 16) + 4(4\mu - \lambda - 2) = 0$

$\Rightarrow$ $9\mu + 18 + 9\lambda + 4\mu + 2\lambda - 32 + 16\mu - 4\lambda - 8 = 0$

$\Rightarrow$ $29\mu + 7\lambda - 22 = 0 \quad \text{... (iv)}$

On solving Eqs. (iii) and (iv), we get

$-49\mu - 77\lambda - 28 = 0$

$\Rightarrow$ $319\mu + 77\lambda - 242 = 0$

$\Rightarrow$ $270\mu - 270 = 0$

$\Rightarrow$ $\mu = 1$

Using μ in Eq. (iii), we get

$-7(1) - 11\lambda - 4 = 0$

$\Rightarrow$ $-7 - 11\lambda - 4 = 0$

$\Rightarrow$ $-11 - 11\lambda = 0$

$\Rightarrow$ $\lambda = -1$

$\therefore \quad \vec{PQ} = [-3(1) - 6 - 3(-1)]\hat{i} + [2(1) + (-1) - 16]\hat{j} + [4(1) - (-1) - 2]\hat{k}$

$= -6\hat{i} - 15\hat{j} + 3\hat{k}$.

S30. We know that, the cartesian equation of a line that passes through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Hence, the cartesian equation of line passes through $A(0, -1, -1)$ and $B(4, 5, 1)$ is

$$\frac{x - 0}{4 - 0} = \frac{y + 1}{5 + 1} = \frac{z + 1}{1 + 1}$$

$$\Rightarrow \quad \frac{x}{4} = \frac{y + 1}{6} = \frac{z + 1}{2} \quad \text{... (i)}$$

and cartesian equation of the line passes through $C(3, 9, 4)$ and $D(-4, 4, 4)$ is

$$\frac{x - 3}{-4 - 3} = \frac{y - 9}{4 - 9} = \frac{z - 4}{4 - 4}$$

$$\Rightarrow \quad \frac{x - 3}{-7} = \frac{y - 9}{-5} = \frac{z - 4}{0} \quad \text{... (ii)}$$

If the lines intersect, then shortest distance between both of them should be zero.

$\therefore$ Shortest distance between the lines

$$= \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}$$

$$= \frac{\begin{vmatrix} 3 - 0 & 9 + 1 & 4 + 1 \\ 4 & 6 & 2 \\ -7 & -5 & 0 \end{vmatrix}}{\sqrt{(6 \cdot 0 + 10)^2 + (-14 - 0)^2 + (-20 + 42)^2}}$$

$$= \frac{\begin{vmatrix} 3 & 10 & 5 \\ 4 & 6 & 2 \\ -7 & -5 & 0 \end{vmatrix}}{\sqrt{100 + 196 + 484}}$$

$$= \frac{30(0 + 10) - 10(14) + 5(-20 + 42)}{\sqrt{780}}$$

$$= \frac{30 - 140 + 110}{\sqrt{780}} = 0$$

So, the given lines intersect.

S31. Let the equation of the plane be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1$$

Then the coordinate of A, B, C are $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ respectively. Centroid of the ΔABC is

$$\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3},\frac{z_1+z_2+z_3}{3}\right) \quad i.e., \left(\frac{a}{3},\frac{b}{3},\frac{c}{3}\right)$$

But coordinates of the centroid of the ΔABC are (α, β, γ) (given).

Therefore, $\alpha = \frac{a}{3}, \quad \beta = \frac{b}{3}, \quad \gamma = \frac{c}{3}, \qquad i.e., \quad a = 3\alpha, \quad b = 3\beta, \quad c = 3\gamma$

Thus, the equation of plane is

$$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma} = 3.$$

S32. We have, $\vec{n}_1 = (\hat{i}+3\hat{j})$, $d_1 = 6$ and $\vec{n}_2 = (3\hat{i}-\hat{j}-4\hat{k})$, $d_2 = 0$

Using the relation, $\vec{r}\cdot(\vec{n}_1+\lambda\vec{n}_2) = d_1 + d_2\lambda$

$\Rightarrow \quad \vec{r}\cdot[(\hat{i}+3\hat{j})+\lambda(3\hat{i}-\hat{j}-4\hat{k})] = 6 + 0\cdot\lambda$

$\Rightarrow \quad \vec{r}\cdot[(1+3\lambda)\hat{i}+(3-\lambda)\hat{j}+\hat{k}(-4\lambda)] = 6 \qquad \text{... (i)}$

On dividing both sides by $\sqrt{(1+3\lambda)^2+(3-\lambda)^2+(-4\lambda)^2}$, we get

$$\frac{\vec{r}\cdot[(1+3\lambda)\hat{i}+(3-\lambda)\hat{j}+\hat{k}(-4\lambda)]}{\sqrt{(1+3\lambda)^2+(3-\lambda)^2+(-4\lambda)^2}} = \frac{6}{\sqrt{(1+3\lambda)^2+(3-\lambda)^2+(-4\lambda)^2}}$$

Since, the perpendicular distance from origin is unity.

$\therefore \quad \frac{6}{\sqrt{(1+3\lambda)^2+(3-\lambda)^2+(-4\lambda)^2}} = 1$

$\Rightarrow \quad (1 + 3\lambda)^2 + (3-\lambda)^2 + (-4\lambda)^2 = 36$

$\Rightarrow \quad 1 + 9\lambda^2 + 6\lambda + 9 + \lambda^2 - 6\lambda + 16\lambda^2 = 36$

$\Rightarrow \quad 26\lambda^2 + 10 = 36$

$\Rightarrow \quad \lambda^2 = 1$

$\therefore \quad \lambda = \pm 1$

Using Eq. (i), the required equation of plane is

$$\vec{r}\cdot[(1\pm 3)\hat{i}+(3\mp 1)\hat{j}+(\mp 4)\hat{k}] = 6$$

$\Rightarrow \quad \vec{r}\cdot[(1+3)\hat{i}+(3-1)\hat{j}+(-4)\hat{k}] = 6$

and $\vec{r}\cdot[(1-3)\hat{i}+(3+1)\hat{j}+4\hat{k}] = 6$

$\Rightarrow$ $\vec{r}\cdot(4\hat{i}+2\hat{j}-4\hat{k}) = 6$

and $\vec{r}\cdot(-2\hat{i}+4\hat{j}+4\hat{k}) = 6$

$\Rightarrow$ $4x + 2y - 4z - 6 = 0$

and $-2x + 4y + 4z - 6 = 0$

S33. Equation of the plane is $ax + by = 0.$ ... (i)

$\therefore$ Equation of the plane after new position is

$$\frac{ax\cos\alpha}{\sqrt{a^2+b^2}}+\frac{by\cos\alpha}{\sqrt{a^2+b^2}}\pm z\sin\alpha = 0$$

$\Rightarrow$ $$\frac{ax}{\sqrt{a^2+b^2}}+\frac{by}{\sqrt{a^2+b^2}}\pm z\tan\alpha = 0$$ [On dividing by $\cos\alpha$]

$\Rightarrow$ $$ax+by\pm z\tan\alpha\sqrt{a^2+b^2} = 0$$ [On multiplying with $\sqrt{a^2+b^2}$]

Alternative Method:

Given, planes are $ax + by = 0$ ... (i)

and $z = 0$ ... (ii)

Therefore, the equation of any plane passing through the line of intersection of planes (i) and (ii) may be taken as

$$ax + by + k = 0. \quad \text{... (iii)}$$

Then, direction cosines of a normal to the plane (iii) are $\frac{a}{\sqrt{a^2+b^2+k^2}}$, $\frac{b}{\sqrt{a^2+b^2+k^2}}$, $\frac{c}{\sqrt{a^2+b^2+k^2}}$ and direction cosines of the normal to the plane (i) are $\frac{a}{\sqrt{a^2+b^2}}$, $\frac{b}{\sqrt{a^2+b^2}}$, 0.

Since, the angle between the planes (i) and (ii) is α,

$\therefore$ $$\cos\alpha = \frac{a\cdot a+b\cdot b+k\cdot 0}{\sqrt{a^2+b^2+k^2}\sqrt{a^2+b^2}}$$

$$= \sqrt{\frac{a^2+b^2}{a^2+b^2+k^2}}$$

$\Rightarrow$ $$k^2\cos^2\alpha = a^2(1-\cos^2\alpha) + b^2(1-\cos^2\alpha)$$

$\Rightarrow$ $$k^2 = \frac{(a^2+b^2)\sin^2\alpha}{\cos^2\alpha}$$

$$k = \pm\sqrt{a^2+b^2}\tan\alpha$$

On putting this value in plane (iii), we get the equation of the plane as

$$ax + by + z\sqrt{a^2 + b^2}\tan\alpha = 0$$

S34. The equation of a plane through the line of intersection of the planes $x + 2y + 3z - 4 = 0$ and $2x + y - z + 5 = 0$ is

$$(x + 2y + 3z - 4) + \lambda(2x + y - z + 5) = 0$$

$$\Rightarrow \quad x(1 + 2\lambda) + y(2 + \lambda) + z(-\lambda + 3) - 4 + 5\lambda = 0 \quad \text{... (i)}$$

Also, this is perpendicular to the plane $5x + 3y + 6z + 8 = 0$.

$$\therefore \quad 5(1 + 2\lambda) + 3(2 + \lambda) + 6(3 - \lambda) = 0 \quad [\because\ a_1a_2 + b_1b_2 + c_1c_2 = 0]$$

$$\Rightarrow \quad 5 + 10\lambda + 6 + 3\lambda + 18 - 6\lambda = 0$$

$$\therefore \quad \lambda = -29/7$$

From Eq. (i),

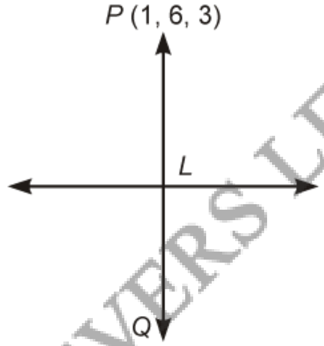
$$x\left[1 + 2\left(\frac{-29}{7}\right)\right] + y\left(2 - \frac{29}{7}\right) + z\left(\frac{29}{7} + 3\right) - 4 + 5\left(\frac{-29}{7}\right) = 0$$

$$\Rightarrow \quad x(7 - 58) + y(14 - 29) + z(29 + 21) - 28 - 145 = 0$$

$$\Rightarrow \quad -51x - 15y + 50z - 173 = 0$$

So, the required equation of plane is $51x + 15y - 50z + 173 = 0$.

S35. Let $P(1, 6, 3)$ be the given point and let L be the foot of perpendicular from P to the given line.



The coordinates of a general point on the given line are

$$\frac{x - 0}{1} = \frac{y - 1}{2} = \frac{z - 2}{3} = \lambda, \qquad i.e., \quad x = \lambda, \quad y = 2\lambda + 1, \quad z = 3\lambda + 2.$$

If the coordinates of L are $(\lambda, 2\lambda + 1, 3\lambda + 2)$, then the direction ratios of PL are $\lambda - 1, 2\lambda - 5, 3\lambda - 1$.

But the direction ratios of given line which is perpendicular of PL are 1, 2, 3. Therefore, $(\lambda - 1)1 + (2\lambda - 5)2 + (3\lambda - 1)3 = 0$, which gives $\lambda = 1$. Hence coordinates of L are (1, 3, 5).

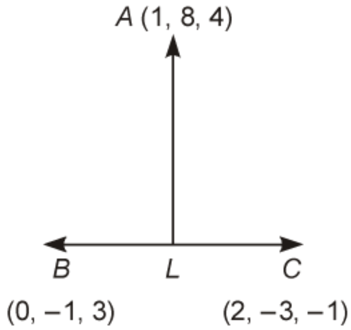
Let $Q(x_1, y_1, z_1)$ be the image of $P(1, 6, 3)$ in the given line. Then L is the mid-point of PQ. Therefore,

$$\frac{x_1 + 1}{2} = 1, \quad \frac{y_1 + 6}{2} = 3, \quad \frac{z_1 + 3}{2} = 5$$

$$\Rightarrow \quad x_1 = 1, \quad y_1 = 0, \quad z_1 = 7$$

Hence, the image of (1, 6, 3) in the given line is (1, 0, 7).

S36. Let L be the foot of perpendicular drawn from the points $A(1, 8, 4)$ to the line passing through B and C as shown in the figure. The equation of line BC by using formula $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$, the equation of the line BC is



$$\vec{r} = (-\hat{j} + 3\hat{k}) + \lambda(2\hat{i} - 2\hat{j} - 4\hat{k})$$

$$\Rightarrow \quad x\hat{i} + y\hat{i} + z\hat{k} = 2\lambda\hat{i} - (2\lambda + 1)\hat{i} + \lambda(3 - 4\lambda)\hat{k}$$

Comparing both sides, we get

$$x = 2\lambda, \quad y = -(2\lambda + 1), \quad z = 3 - 4\lambda \qquad \dots \text{(i)}$$

Thus, the coordinate of L are $(2\lambda), -(2\lambda + 1), (3 - 4\lambda)$,

So that the direction ratios of the line AL are

$$(1 - 2\lambda), 8 + (2\lambda + 1), 4 - (3 - 4\lambda), \textit{ i.e.},$$

$$1 - 2\lambda, \quad 2\lambda + 9, \quad 1 + 4\lambda$$

Since, AL is perpendicular to BC, we have

$$(1 - 2\lambda)(2 - 0) + (2\lambda + 9)(-3 + 1) + (4\lambda + 1)(-1 - 3) = 0$$

$$\Rightarrow \quad \lambda = \frac{-5}{6}$$

The required point is obtained by substituting the value of λ, in (i), which is

$$\left(\frac{-5}{3}, \frac{2}{3}, \frac{19}{3}\right).$$

S37. Eliminating m from the given two equations, we get

$$\Rightarrow \quad 2n^2 + 2ln + l^2 = 0$$

$$\Rightarrow \quad (n + l)(2n + l) = 0$$

$$\Rightarrow \quad \text{either } n = -l \quad \text{or} \quad l = -2n$$

Now, $l = -n$, then $m = -2n$.

and if $l = -2n$, then $m = n$.

Thus, the direction ratios of two lines are proportional to $-n, -2n, n$ and $-2n, n, n$.

i.e., $1, 2, -1$ and $-2, 1, 1$.

So, vectors parallel to these lines are

$$\vec{a} = \hat{i} + 2\hat{j} - \hat{k} \quad \text{and} \quad \vec{b} = -2\hat{i} + \hat{j} + \hat{k}, \text{ respectively.}$$

If θ is the angle between the lines, then

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

$$= \frac{(\hat{i} + 2\hat{j} - \hat{k}) \cdot (-2\hat{i} + \hat{j} + \hat{k})}{\sqrt{1^2 + 2^2 + (-1)^2}\sqrt{(-2)^2 + 1^2 + 1^2}} = -\frac{1}{6}$$

Hence $\theta = \cos^{-1}\left(-\frac{1}{6}\right)$.

S38. Let

$$\vec{a} = l_1\hat{i} + m_1\hat{j} + n_1\hat{k}$$

$$\vec{b} = l_2\hat{i} + m_2\hat{j} + n_2\hat{k}$$

$$\vec{c} = l_3\hat{i} + m_3\hat{j} + n_3\hat{k}$$

$$\vec{d} = (l_1 + l_2 + l_3)\hat{i} + (m_1 + m_2 + m_3)\hat{j} + (n_1 + n_2 + n_3)\hat{k}$$

Also, let α, β and γ are the angles between $\vec{a}$ and $\vec{d}$, $\vec{b}$ and $\vec{d}$, $\vec{c}$ and $\vec{d}$.

$\therefore$ $\cos\alpha = l_1(l_1 + l_2 + l_3) + m_1(m_1 + m_2 + m_3) + n_1(n_1 + n_2 + n_3)$

$= l_1^2 + l_1l_2 + l_1l_3 + m_1^2 + m_1m_2 + m_1m_3 + n_1^2 + n_1n_2 + n_1n_3$

$= (l_1^2 + m_1^2 + n_1^2) + (l_1l_2 + l_1l_3 + m_1m_2 + m_1m_3 + n_1n_2 + n_1n_3)$

$= 1 + 0 = 1$

[$\because$ $l_1^2 + m_1^2 + n_1^2 = 1$ and $l_1 \perp l_2$, $l_1 \perp l_3$, $m_1 \perp m_2$, $m_1 \perp m_3$, $n_1 \perp n_2$, $n_1 \perp n_3$]

Similarly, $\cos\beta = l_2(l_1 + l_2 + l_3) + m_2(m_1 + m_2 + m_3) + n_2(n_1 + n_2 + n_3)$

$= 1 + 0$ and $\cos\gamma = 1 + 0$

$\Rightarrow$ $\cos\alpha = \cos\beta = \cos\gamma$

$\Rightarrow$ $\alpha = \beta = \gamma$

So, the line whose direction cosines are proportional to $l_1 + l_2 + l_3$, $m_1 + m_2 + m_3$ and $n_1 + n_2 + n_3$ makes equal angles with the three mutually perpendicular lines whose direction cosines are l_1, m_1, n_1, l_2, m_2, n_2 and l_3, m_3, n_3 respectively.