Limits

Single Correct Answer Type

1. If a_n and b_n are positive integers and $a_n+\sqrt{2}b_n=\left(2+\sqrt{2}\right)^n$, then $\lim\limits_{n\to\infty}\left(\dfrac{a_n}{b_n}\right)=$

A) $\sqrt{2}$ B) 2 C) $e^{\sqrt{2}}$ D) e^2

Key. A

Sol. We have $a_n+\sqrt{2}b_n=\left(2+\sqrt{2}\right)^n$

$$\Rightarrow a_n-\sqrt{2}b_n=\left(2-\sqrt{2}\right)^n$$

Therefore $a_n=\dfrac{1}{2}\left[\left(2+\sqrt{2}\right)^n+\left(2-\sqrt{2}\right)^n\right]$

And $b_n=\dfrac{\left[\left(2+\sqrt{2}\right)^n-\left(2-\sqrt{2}\right)^n\right]}{2\sqrt{2}}$

Therefore $\dfrac{a_n}{b_n}=\sqrt{2}\dfrac{\left[\left(2+\sqrt{2}\right)^n+\left(2-\sqrt{2}\right)^n\right]}{\left[\left(2+\sqrt{2}\right)^n-\left(2-\sqrt{2}\right)^n\right]}$

$$=\sqrt{2}\frac{\left[1+\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)^n\right]}{\left[1-\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)^n\right]}$$

Hence $\lim\limits_{n\to\infty}\left(\dfrac{a_n}{b_n}\right)=\sqrt{2}\left(\dfrac{1+0}{1-0}\right)\left(\because \dfrac{2-\sqrt{2}}{2+\sqrt{2}}<1\right)=\sqrt{2}$

2. If $f(0)=0$ and that 'f' is differentiable at x = 0, and 'k' is a positive integer. Then $\underset{x\to 0}{\text{lt}}\dfrac{1}{x}\left[f(x)+f\left(\dfrac{x}{2}\right)+f\left(\dfrac{x}{3}\right)+----+f\left(\dfrac{x}{k}\right)\right]$

(A) $K.f^1(0)$ (B) $\left(\sum\limits_{r=1}^{K}\dfrac{1}{r}.\right)f^1(0)$ (C) $\sum\limits_{r=1}^{K}\dfrac{1}{r}$ (D) does not exist

Key. B

Sol. $l=\underset{x\to 0}{\text{lt}}\left\{\dfrac{f(x)-f(0)}{x-0}+\dfrac{f\left(\dfrac{x}{2}\right)-f(0)}{x-0}+---\right.$

$$- - - + \frac{f\left(\frac{x}{k}\right) - f(0)}{x - 0}\Bigg\}$$

$$= \left(1 + \frac{1}{2} + \frac{1}{3} + - - - + \frac{1}{k}\right) f^1(0).$$

3. $\underset{x \to 0}{Lt}\left(\sum_{r=1}^{n} r^{\cos ec^2 x}\right)^{\sin^2 x} =$

A. 0 B. ∞ C. n D. $\frac{1}{n}$

Key. C

Sol. $L = \underset{x \to 0}{Lt} (1^{\cos ec^2 n} + 2^{\cos ec^2 n} + \_\_\_\_ + n^{\cos ec^2 n})^{\sin^2 n}$

$\underset{x \to 0}{Lt}((\frac{1}{n})^{\cos ec^- b} + (\frac{2}{n})^{\cos ec^- n} + \_\_\_\_(\frac{n-1}{n})^{\cos ec^- n} + 1)^{\sin^- n}.n$

$= (0 + 0 + 0 + \_\_\_\_\_ + 1)^0.n = n$

4. For each positive integer *n*, let $s_n = \frac{3}{1.2.4} + \frac{4}{2.3.5} + \frac{5}{3.4.6} + ... + \frac{n+2}{n(n+1)(n+3)}$. Then $\lim_{n \to \infty} s_n$ equals

A) $\frac{29}{6}$ B) $\frac{29}{36}$ C) 0 D) $\frac{29}{18}$

Key. B

Sol. Let $u_k = \frac{k+2}{k(k+1)(k+3)}$

$$= \frac{(k+2)^2}{k(k+1)(k+2)(k+3)}$$

$$= \frac{k^2 + 4k + 4}{k(k+1)(k+2)(k+3)}$$

$$= \frac{k(k+1) + 3k + 4}{k(k+1)(k+2)(k+3)}$$

$$= \frac{1}{(k+2)(k+3)} + \frac{3}{(k+1)(k+2)(k+3)} + \frac{4}{k(k+1)(k+2)(k+3)}$$

$$=\left(\frac{1}{k+2}-\frac{1}{k+3}\right)-\frac{3}{2}\left[\frac{1}{(k+2)(k+3)}-\frac{1}{(k+1)(k+2)}\right]$$

$$-\frac{4}{3}\left[\frac{1}{(k+1)(k+2)(k+3)}-\frac{1}{k(k+1)(k+2)}\right]$$

Now, put $k=1,2,3,....,n$ and add. Thus

$$s_u=u_1+u_2+.....+u_n$$

$$=\left(\frac{1}{3}-\frac{1}{n+3}\right)-\frac{3}{2}\left[\frac{1}{(n+2)(n+3)}-\frac{1}{2.3}\right]$$

$$-\frac{4}{3}\left[\frac{1}{(n+1)(n+2)(n+3)}-\frac{1}{1.2.3}\right]$$

Therefore $\lim\limits_{n\to\infty} s_n=\frac{1}{3}+\frac{3}{12}+\frac{4}{18}=\frac{29}{36}$

5. $\lim\limits_{x\to 0}\frac{a^{\tan x}-a^{\sin x}}{\tan x-\sin x}$ is equal to $(a>0)$

A) $\log_e a$ B) 1 C) 0 D) ∞

Key. A

Sol. We have $\lim\limits_{x\to 0}\frac{a^{\tan x}-a^{\sin x}}{\tan x-\sin x}=\lim\limits_{x\to 0} a^{\sin x}\left(\frac{a^{\tan x-\sin x}-1}{\tan x-\sin x}\right)$

$=\lim\limits_{x\to 0}\left(a^{\sin x}\right)\times\lim\limits_{t\to 0}\left(\frac{a^t-1}{t}\right)$ (where $t=\tan x-\sin x$)

$=a^0\times\log_e a=\log_e a$

6. $\lim\limits_{x\to\frac{\pi}{2}}\frac{(1-\sin x)\left(8x^3-\pi^3\right)\cos x}{(\pi-2x)^4}$

A) $-\frac{\pi^2}{16}$ B) $\frac{3\pi^2}{16}$ C) $\frac{\pi^2}{16}$ D) $-\frac{3\pi^2}{16}$

Key. D

Sol. Let $f(x)=\frac{(1-\sin x)\left(8x^3-\pi^3\right)\cos x}{(\pi-2x)^4}$

$$=\frac{(1-\sin x)\cos x(2x-\pi)\left(4x^2+2\pi x+\pi^2\right)}{(2x-\pi)^4}$$

$$=\frac{(1-\sin x)\cos x\left(4x^2+2\pi x+\pi^2\right)}{(2x-\pi)^3}$$

Therefore $\lim\limits_{x\to\frac{\pi}{2}} f(x)=\lim\limits_{x\to\frac{\pi}{2}}\frac{(1-\sin x)\cos x}{(2x-\pi)^3}.\left(3\pi^2\right)$

$$\lim_{x\to\frac{\pi}{2}} f(x) = \lim_{x\to\frac{\pi}{2}} \frac{(1-\sin x)\cos x}{(2x-\pi)^3}.(3\pi^2) \qquad ----(1.62)$$

Put $2x-\pi = y$ so that $y\to 0$ as $x\to\pi/2$. Therefore now

$$\frac{(1-\sin x)\cos x}{(2x-\pi)^3} = \frac{\left[1-\sin\left(\frac{\pi+y}{2}\right)\right]\cos\left(\frac{\pi+y}{2}\right)}{y^3}$$

$$= \frac{\left(1-\cos\frac{y}{2}\right)\left(-\sin\frac{y}{2}\right)}{y^3}$$

$$= -\left(\frac{2\sin^2\frac{y}{4}}{y^2}\right)\left(\frac{\sin\frac{y}{2}}{y}\right)$$

$$= -2\left(\frac{\sin\frac{y}{4}}{y/4}\right)^2.\frac{1}{16}.\left(\frac{\sin\frac{y}{2}}{y/2}\right).\frac{1}{2}$$

$$= \frac{-1}{16}\left(\frac{\sin\frac{y}{4}}{y/4}\right)^2\left(\frac{\sin\frac{y}{2}}{y/2}\right) \qquad ----(1.63)$$

Therefore from Eqs. (1.62) and (1.63)

$$\lim_{x\to\frac{\pi}{2}} f(x) = \frac{-3\pi^2}{16}\times 1\times 1.$$

7. Let $f: R^+ \to R^+$ be a function satisfying the relation $f(x.f(y)) = f(xy)+x$ for all $x, y\in R^+$. Then $\lim_{x\to 0}\left(\frac{(f(x))^{1/3}-1}{(f(x))^{1/2}-1}\right) =$

(A) 1 (B) $\frac{1}{2}$ (C) $\frac{2}{3}$ (D) $\frac{3}{2}$

Key. C

Sol. Given relation is $f(x.f(y)) = f(xy)+x$ (1.56)

Interchanging x and y in Eq. (1.56), we have

$$f(y.f(x)) = f(yx)+y \qquad (1.57)$$

Again replacing *x* with $f(x)$ in Eq. (1.56) we get

$$f(f(x).f(y)) = f(y.f(x)) + f(x) \qquad (1.58)$$

Therefore, Eqs. $(1.56)-(1.58)$ imply

$$f(f(x).f(y)) = f(xy)+y+f(x) \qquad (1.59)$$

Again interchanging *x* and *y* in Eq. (1.59), we have

$$f\left(f\left(y\right).f\left(x\right)\right)=f\left(yx\right)+x+f\left(y\right) \qquad (1.60)$$

Equations (1.59) and (1.60) imply

$$f\left(xy\right)+y+f\left(x\right)=f\left(yx\right)+x+f\left(y\right) \qquad (1.61)$$

Suppose $f\left(x\right)-x=f\left(y\right)-y=\lambda$

Substituting $f\left(x\right)=\lambda+x$ in Eq. (1.56), we have

$$x.f\left(y\right)+\lambda=\left(xy+\lambda\right)+x$$

$$\Rightarrow x.f\left(y\right)=xy+x$$

Therefore $x\left(y+\lambda\right)=xy+x \quad \left[\because f\left(y\right)=\lambda+y\right]$

$$\Rightarrow \lambda x=x$$

$$\Rightarrow \lambda=1 \quad \left(\because x>0\right)$$

So $f\left(x\right)=x+\lambda=x+1$

Hence $\lim_{x\to0}\frac{\left(f\left(x\right)\right)^{1/3}-1}{\left(f\left(x\right)\right)^{1/2}-1}=\lim_{x\to0}\frac{\left(1+x\right)^{1/3}-1}{\left(1+x\right)^{1/2}-1}$

$$=\lim_{x\to0}\left(\frac{\left(1+x\right)^{1/3}-1}{1+x-1}\right).\left(\frac{1+x-1}{\left(1+x\right)^{1/2}-1}\right)$$

$$=\frac{1/3}{1/2}=\frac{2}{3}$$

8. Let $x_1=1$ and $x_{n+1}=\frac{4+3x_n}{3+2x_n}$ for $n\geq1$. If $\lim_{n\to\infty}x_n$ exists finitely, then the limit is equal to

(A) $\sqrt{2}$ (B) 1 (C) 2 (D) $\sqrt{2}+1$

Key. A

Sol. We have $x_1=1, x_2=\frac{4+3}{3+2}=\frac{7}{5}$

$$x_3=\frac{4+3x_2}{3+2x_2}=\frac{4+3\left(\frac{7}{5}\right)}{3+2\left(\frac{7}{5}\right)}=\frac{41}{29}>x_2$$

We can easily verify that $x_n<x_{n+1}$ and hence $\left\{x_n\right\}$ is strictly increasing sequence of positive terms. Let $\lim_{n\to\infty}x_n=l$. Therefore

$$l=\lim_{n\to\infty}x_{n+1}$$

$$=\lim_{n\to\infty}\left(\frac{4+3x_n}{3+2x_n}\right)$$

$$=\frac{4+3\lim_{n\to\infty}x_n}{3+2\lim_{n\to\infty}x_n}$$

$$= \frac{4+3l}{3+2l}$$

Hence $3l + 2l^2 = 4 + 3l$

or $l^2 = 2 \Rightarrow l = \sqrt{2} \quad (\because x_n > 0 \ \forall \ n)$.

9. Let $f(x) = x^3\left\{\sqrt{x^2+\sqrt{x^4+1}} - x\sqrt{2}\right\}$. Then $\lim_{x\to\infty} f(x)$ is equal to

(A) $\frac{1}{2\sqrt{2}}$ (B) $\frac{1}{4\sqrt{2}}$ (C) $\frac{3}{4\sqrt{2}}$ (D) does not exist

Key. B

Sol. We have $f(x) = \frac{x^3\left\{x^2+\sqrt{x^4+1}-2x^2\right\}}{\sqrt{x^2+\sqrt{x^4+1}}+x\sqrt{2}}$

$$= \frac{x^3\left\{\sqrt{x^4+1}-x^2\right\}}{\sqrt{x^2+\sqrt{x^4+1}}+x\sqrt{2}}$$

$$= \frac{x^3\left(x^4+1-x^4\right)}{\left[\sqrt{x^2+\sqrt{x^4+1}}+x\sqrt{2}\right]\left[\sqrt{x^4+1}+x^2\right]}$$

$$= \frac{x^3}{\left[\sqrt{x^2+\sqrt{x^4+1}+}+x\sqrt{2}\right]\left[\sqrt{x^4+1}+x^2\right]}$$

$$= \frac{1}{\left[\sqrt{1+\sqrt{1+\frac{1}{x^4}}}+\sqrt{2}\right]\left[\sqrt{1+\frac{1}{x^4}}+1\right]}$$

$$= \frac{1}{\left(\sqrt{1+\sqrt{1}}+\sqrt{2}\right)\left(\sqrt{1}+1\right)}$$

$$= \frac{1}{2\sqrt{2}(2)} = \frac{1}{4\sqrt{2}}.$$

10. $\lim_{x\to\frac{-1}{3}^-} \frac{1}{x}\left[\frac{-1}{x}\right]$ [.] → denotes greatest integer function

1) -9 2) -12 3) -6 4) 0

Key. 3

Sol. $x < -\frac{1}{3}$

$$\frac{1}{x} > -3 \Rightarrow -\frac{1}{x} < 3 \Rightarrow \left[-\frac{1}{3}\right] = 2$$

$$\lim_{x\to -\frac{1}{3}} \frac{1}{x}\left[-\frac{1}{x}\right] = (-3)(2) = -6$$

11. $\lim_{x\to\infty}\left(x-\log_e(\cosh x)\right) =$

1) 1 2) 0 3) $\log_e 2$ 4) ∞

Key. 3

Sol. $\lim_{x\to\infty} x-\log_e\left(\frac{e^x+e^{-x}}{2}\right)$

$\lim_{x\to\infty} x-\log_e e^x\left(\frac{1+e^{-2x}}{2}\right)$

$\lim_{x\to\infty} x-x-\log_e\left(\frac{1+e^{-2x}}{2}\right)$

$\lim_{x\to\infty} -\log_e\left(\frac{1}{2}\right) = \log_e 2$

12. If α is a root of the equation $\sin x+1 = x$ then $\lim_{x\to\alpha}\left[\frac{\min(\sin x, \{x\})}{x-1}\right]$ is

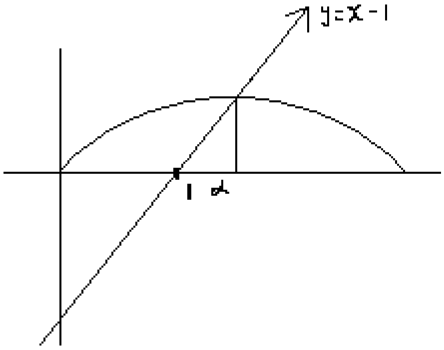
Where [.] $\rightarrow$ denotes greatest integer function

{x}$\rightarrow$ fractional part of x.

1) 1 2) 0 3) does not exist 4) -1

Key. 3

Sol. LHL :



$\lim_{x\to\alpha-}\left[\frac{\min(\sin x,\ x-[x]}{(x-1)}\right]$

When $1 < x < \alpha$

$\{x\} = x-1 < \sin x$

$\min\{\sin x,\ x-1\} = x-1$

Required limit = $\lim_{x\to\infty-}\left[\frac{x-1}{x-1}\right] = 1$

RHL :

$\lim_{x\to\alpha+}\left[\frac{\sin x}{x-1}\right] = 0$

$x\to\alpha+$

$\sin x < x-1$

$\frac{\sin x}{x-1} < 1$

Hence $LHL \neq RHL$ $\left[\frac{\sin x}{x-1}\right]=0$

Limit does not exist

13. If a_1 is the greatest value of $f(x)$ where $f(x)=\frac{1}{2+[\sin x]}$ and $a_{n+1}=\frac{(-1)^{n+2}}{n+1}+a_n$

Then $\lim_{n\to\infty} a_n =$ _______

1) 0 2) e 3) 1 4) $\log_e 2$

Key. 4

Sol. $a_1=1, a_2=1-\frac{1}{2}, a_3=1-\frac{1}{2}+\frac{1}{3}.........a_n=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+......+(-1)^{n-1}.\frac{1}{n}$

$\lim_{n\to\infty} a_n = \log_e 2$

14. $\lim_{x\to\frac{\pi}{2}}\left[\frac{[\sin x]-[\cos x]+1}{3}\right]=$

[.] $\rightarrow$ denotes greatest integer function

1) 0 2) 1 3) -1 4) does not exist

Key. 1

Sol. LHL = RHL = 0

15. $\lim_{x\to 0}\left(\frac{1+2x}{1+3x}\right)^{\frac{1}{x^2}}.e^{\frac{1}{x}}=$ _______

1) $e^{\frac{5}{2}}$ 2) e^2 3) 4) 1

Key. 1

Sol. $\lim_{x\to 0} e^{\frac{1}{x^2}(\log(1+2x)-\log(1+3x)+\frac{1}{x}}$

$$e^{\lim_{x\to 0}\frac{(\log(1+2x)-\log(1+3x)+x}{x^2}} = e^{\frac{5}{2}}$$

16. $\lim_{n\to\infty}\sum_{r=1}^{n}\cot^{-1}\left(r^2+\frac{3}{4}\right)=$

1) $\tan^{-1}(2)$ 2) $\frac{\pi}{4}$ 3) $\frac{\pi}{2}$ 4) $\tan^{-1}(3)$

Key. 1

Sol. $\cot^{-1}\left(r^2+\frac{3}{4}\right)=\tan^{-1}\left(\frac{1}{r^2+\frac{3}{4}}\right)$

$=\tan^{-1}\left(\frac{1}{1+\left(r^2-\frac{1}{4}\right)}\right)$

$=\tan^{-1}\left(\frac{1}{1+\left(r+\frac{1}{2}\right)\left(r-\frac{1}{2}\right)}\right)$

$=\tan^{-1}\left(\frac{\left(r+\frac{1}{2}\right)-\left(r-\frac{1}{2}\right)}{1+\left(r^2+\frac{1}{4}\right)}\right)$

$=\tan^{-1}\left(r+\frac{1}{2}\right)-\tan^{-1}\left(r-\frac{1}{2}\right)$

17. $\lim\limits_{x\to\infty}\sqrt[3]{x}\left(\sqrt[3]{(x+1)^2}-\sqrt[3]{(x-1)^2}\right)=$

1) $\frac{1}{3}$ 2) $\frac{2}{3}$ 3) 1 4) $\frac{4}{3}$

Key. 4

Sol. $\lim\limits_{x\to\infty}x^{1/3}\left\{(x+1)^{1/3}+(x-1)^{1/3}\right\}\left\{(x+1)^{1/3}-(x-1)^{1/3}\right\}$

Rationalise $\lim\limits_{x\to\infty}\frac{x^{1/3}\left\{(x+1)^{1/3}+(x-1)^{1/3}\right\}2}{\left\{(x+1)^{2/3}+(x^2-1)^{1/3}+(x-1)^{2/3}\right\}}$

$$\lim_{x\to\infty}\frac{2.x^{2/3}\left\{\left(1+\frac{1}{x}\right)^{1/3}+\left(1-\frac{1}{x}\right)^{1/3}\right\}2}{x^{2/3}\left\{\left(1+\frac{1}{x}\right)^{2/3}+\left(1-\frac{1}{x}\right)^{1/3}+\left(1-\frac{1}{x}\right)^{2/3}\right\}}=\frac{2\text{x}2}{3}=\frac{4}{3}$$

18. If $a>0,\ b>0$ then $\lim\limits_{n\to\infty}\left(\frac{a-1+b^{\frac{1}{n}}}{a}\right)^n=$

1) $b^{\frac{1}{a}}$ 2) $a^{\frac{1}{b}}$ 3) a^b 4) b^a

Key. 1

Sol. Let $\frac{1}{n}=x,\quad \Rightarrow x\to 0\ \ as\ \ n\to\infty$ then required limit $\underset{x\to 0}{Lt}\left(\frac{a-1+b^x}{a}\right)^{\frac{1}{x}}\ =e^{\underset{x\to 0}{Lt}\frac{b^x-1}{x^a}}$

$$= e^{\frac{1}{a}\log e^{b}} = \left(b^{\frac{1}{a}}\right)$$

19. If $S_n = \frac{1}{1.2.3.4} + \frac{1}{2.3.4.5} + ... + \frac{1}{n(n+1)(n+2)(n+3)}$ then $\lim_{n\to\infty} S_n =$

1) $\frac{5}{18}$ 2) $\frac{1}{9}$ 3) $\frac{7}{18}$ 4) $\frac{1}{18}$

Key. 4

Sol. $S_n = c - \frac{1}{(n+1)(n+2)(n+3).3}$

$n = 1 \Rightarrow s_1 = c - \frac{1}{2.3.4.3} \Rightarrow c = \frac{1}{1.2.3.4} + \frac{1}{2.3.4.3}$

$c = \frac{1}{2.3.4}\left(1 + \frac{1}{3}\right)$

$= \frac{1}{18}$ Now as $n \to \infty,\ S_n \to c = \frac{1}{18}$

20. $\lim_{x\to\infty}\left(\frac{x^2+5x+3}{x^2+x+2}\right)^x =$

1) e^2 2) e^4 3) e^3 4) e

Key. 2

Sol. $\lim_{x\to\infty}\left(\frac{x^2+5x+3}{x^2+x+2}\right)^x = e^{\lim_{x\to 0}\left(\frac{4x+1}{x^2+x+2}\right)x} = e^4$

21. If a_n and b_n are positive integers and $a_n + \sqrt{2}b_n = \left(2+\sqrt{2}\right)^n$, then $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) =$

A) $\sqrt{2}$ B) 2 C) $e^{\sqrt{2}}$ D) e^2

Key. A

Sol. We have $a_n + \sqrt{2}b_n = \left(2+\sqrt{2}\right)^n$

$\Rightarrow a_n - \sqrt{2}b_n = \left(2-\sqrt{2}\right)^n$

Therefore $a_n = \frac{1}{2}\left[\left(2+\sqrt{2}\right)^n + \left(2-\sqrt{2}\right)^n\right]$

And $b_n = \frac{\left[\left(2+\sqrt{2}\right)^n - \left(2-\sqrt{2}\right)^n\right]}{2\sqrt{2}}$

Therefore $\frac{a_n}{b_n} = \sqrt{2}\frac{\left[\left(2+\sqrt{2}\right)^n + \left(2-\sqrt{2}\right)^n\right]}{\left[\left(2+\sqrt{2}\right)^n - \left(2-\sqrt{2}\right)^n\right]}$

$$= \sqrt{2}\frac{\left[1+\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)^n\right]}{\left[1-\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)^n\right]}$$

Hence $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \sqrt{2}\left(\frac{1+0}{1-0}\right)\left(\because \frac{2-\sqrt{2}}{2+\sqrt{2}} < 1\right) = \sqrt{2}$

22. $\lim_{n\to\infty}\frac{(n!)^{\frac{1}{n}}}{n}$ equals

a) e b) e^{-1}

c) e^{-2} d) e^2

Key. B

Sol. $let \quad P = \frac{(n!)^{\frac{1}{n}}}{n}$

$$= \left(\frac{(n!)}{n^n}\right)^{\frac{1}{n}}$$

$$\log P = \frac{1}{n}\sum_{r=1}^{n}\log\left(\frac{r}{n}\right)$$

23. The value of $\lim_{x\to 0}\frac{(1+x)^{1/x} - e}{x}$ is

a) $\frac{e}{2}$ b) $-\frac{e}{2}$

c) $\frac{3e}{2}$ d) $-\frac{2e}{3}$

Key. B

Sol. $(1+x)^{\frac{1}{x}} = e^{\frac{1}{x}\log(1+x)}$

$$= e^{(1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}......)}$$

24. $\underset{n\to\infty}{Lt}\left[\left(1+\frac{1}{n}\right)^{n}-\left(1+\frac{1}{n}\right)\right]^{-n}=$

1) 1 2) $\frac{1}{e-1}$ 3) $1-e^{-1}$ 4) 0

Key. 4

Sol. $\underset{n\to\infty}{Lt}\left[\left(1+\frac{1}{n}\right)^{n}-\left(1+\frac{1}{n}\right)\right]=e-1>1$

25. Let $f(x)=\frac{\tan x}{x}$, then $\log_e\left(\lim_{x\to 0}\left(\left[f(x)\right]+x^2\right)^{\frac{1}{\{f(x)\}}}\right)$ is equal, (where $[\cdot]$ denotes greatest integer function and $\{\cdot\}$ fractional part)

(A) 1 (B) 2 (C) 3 (D) 4

Key. C

Sol. $\lim_{x\to 0}\left[f(x)\right]=\lim_{x\to 0}\left[\frac{\tan x}{x}\right]=1$

$\lim_{x\to 0}\left(\left[f(x)\right]+x^2\right)^{\frac{1}{\{f(x)\}}}=\lim_{x\to 0}\left(1+x^2\right)^{\frac{1}{\{f(x)\}}}\ \left(1^{\infty}\ \text{form}\right)$

Again, $f(x)=\frac{\tan x}{x}=\frac{x+\frac{x^3}{3}+\frac{2}{15}x^5+\ldots}{x}$

$=1+\frac{x^2}{3}+\frac{2}{15}x^4+\ldots$

$\{f(x)\}=\frac{x^2}{3}+\frac{2}{15}x^4+\ldots$

(i) becomes,

$\log_e\left(e^{\lim_{x\to 0}x^2\times\frac{1}{\{f(x)\}}}\right)=e^{\lim_{x\to 0}\frac{x^2}{\frac{x^2}{3}+\frac{2}{15}x^4+\ldots}}=3$

$\therefore$ (C) is the correct answer.

26. Let $x>0$ then $\underset{x\to 0}{Lt}\left(\sqrt{\tan x}\right)^{\sqrt{x}}+\left(\sec x\right)^{\frac{1}{x}}=$

(A) $1/e$ (B)1 (C) $\frac{1}{e^2}$ (D) 2

Key. D

Sol. $\underset{x\to 0^+}{Lt}\left(\sqrt{\tan x}\right)^{\sqrt{x}}+\underset{x\to 0^+}{Lt}\left(\cos x\right)^{-1/x}$

$e^{\underset{x\to 0^+}{Lt}\frac{\log_e\left(\sqrt{\tan x}\right)}{\frac{1}{\sqrt{x}}}}\left(\frac{-\infty}{\infty}\right)=e^0=1,\ \underset{x\to 0^+}{Lt}\left(\cos x\right)^{-1/x}=1\ as\ \ 0<\cos x<1$

27. $\lim\limits_{x\to 0}\frac{\sin[\cos x]}{1+[\cos x]}$, ([x] denotes the greatest integer less than or equal to)

(A) sin 1 (B) 0

(C) Does not exist (D) $\frac{\sin 1}{2}$

Key. B

Sol. $\text{LHL} = \lim\limits_{x\to 0^-} f(0-h) = \lim\limits_{h\to 0}\frac{\sin[\cos h]}{1+[\cos h]}$

$=\frac{\sin(0)}{1+0}=0 \qquad \begin{pmatrix} \because\ h>0 \\ \therefore\ \cos h<1 \end{pmatrix}$

$\text{RHL} = \lim\limits_{x\to 0^+} f(x) = \lim\limits_{h\to 0} f(0+h)$

$=\lim\limits_{h\to 0}\frac{\sin[\cos h]}{1+[\cos h]}$

$=\frac{\sin(0)}{1+0}=0 \qquad \begin{pmatrix} \because\ h>0 \\ \therefore\ \cos h<1 \end{pmatrix}$

$\therefore \lim\limits_{x\to 0}\frac{\sin[\cos x]}{1+[\cos x]}=0$

28. If $\lim\limits_{x\to a}\left(2-\frac{a}{x}\right)^{a\tan\left(\frac{\pi x}{2a}\right)}=e$, then '$a$' is equal to

A) $-\pi$ B) $\frac{-\pi}{2}$ C) $\frac{\pi}{2}$ D) $\frac{-2}{\pi}$

Key. B

Sol. $\mathop{\text{Lim}}\limits_{x\to a}\left(2-\frac{a}{x}\right)^{a\tan\left(\frac{\pi x}{2a}\right)}=e$

$\Rightarrow e^{\mathop{\text{Lim}}\limits_{x\to a} a\tan\left(\frac{\pi x}{2a}\right)\left(1-\frac{a}{x}\right)}$

$\Rightarrow e^{\mathop{\text{Lim}}\limits_{x\to a}\frac{a\left(1-\frac{a}{x}\right)}{\cot\left(\frac{\pi x}{2a}\right)}}=e$

$\therefore \mathop{\text{Lim}}\limits_{x\to a}\frac{a\left(\frac{-x}{a}\right)\left(1-\frac{x}{a}\right)}{\text{Tan}\frac{\pi}{2}\left(1-\frac{x}{a}\right)}=1$

$$\lim_{x \to a} \frac{\frac{-2x}{\pi}\left(1-\frac{x}{a}\right)\frac{\pi}{2}}{\operatorname{Tan}\frac{\pi}{2}\left(1-\frac{x}{a}\right)} = 1$$

$$\frac{-2a}{\pi} = 1 \Rightarrow a = \frac{-\pi}{2}$$

29. If $f\left(x\right) = \left(\frac{|x|}{|x|+2}\right)^{-x}$ then

A) $\lim_{x \to -\infty} f\left(x\right) = e^{2}$

B) $\lim_{x \to -\infty} f\left(x\right) = 0$

C) $\lim_{x \to 1} f\left(x\right) = \frac{1}{3}$

D) $\lim_{x \to \infty} f\left(x\right) = e^{2}$

Key. D

Sol. $\lim_{x \to \infty} \left(\frac{|x|}{|x|+2}\right)^{-x}$

$= \lim_{x \to -\infty} \left(\frac{2-x-2}{2-x}\right)^{x}$

$= \lim_{x \to -\infty} \left(1-\frac{2}{2-x}\right)^{x}$

$x \to -\infty \Rightarrow |x| = -x$

$x = -\frac{1}{y},\ y \to 0$

$\underset{y \to 0}{Lt} \left(1-\frac{2}{2+\frac{1}{y}}\right)^{\frac{1}{y}}$

$= \underset{y \to 0}{Lt} \left(1-\frac{y}{2y+1}\right)^{\frac{1}{y}}$; 1^{∞} form

$= e^{\underset{y \to 0}{Lt} \frac{1}{y}\left(1-\frac{y}{2y+1}-1\right)}$

$= e^{\underset{y \to 0}{Lt}\left(\frac{1}{2y+1}\right)} = e^{1}$

30. The value of $\lim_{x\to 0}\frac{\cos(\sin^2 x)-\cos(x^2)}{x^6}$ is

(A) 0 (B) 1/2

(C) 1/3 (D) 3/4

Key. C

Sol. $\lim_{x\to 0}\frac{\cos(\sin^2 x)-\cos(x^2)}{x^6}$

$$= \lim_{x\to 0}\frac{2\sin\left(\frac{\sin^2 x + x^2}{2}\right).\sin\left(\frac{x^2-\sin^2 x}{2}\right)}{x^6}$$

$$= \lim_{x\to 0}\frac{2\sin\left(\frac{\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+.....\right)^2+x^2}{2}\right).\sin\left(\frac{x^2-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+......\right)^2}{2}\right)}{x^6}$$

$$= \lim_{x\to 0}\frac{2\sin\left(\frac{2x^2-\frac{2x^4}{6}....}{2}\right)\sin\left(\frac{x^4}{6}......\right)}{x^2\times 6.\frac{x^4}{6}}$$

$$= \lim_{x\to 0}\frac{2\sin\left(x^2-\frac{x^4}{6}.....\right)}{x^2}.\frac{1}{6}=\frac{1}{3}$$

31. $\lim_{x\to 0}\frac{\sin^{-1}x-\tan^{-1}x}{x^3}$ is equal to

(A) $\frac{1}{6}$ (B) $\frac{1}{2}$

(C) 2 (D) $-\frac{1}{2}$

Key. B

Sol. $p=\lim_{x\to 0}\frac{\sin^{-1}x-\tan^{-1}x}{x^3} = \lim_{x\to 0}\left(\frac{1}{\sqrt{1-x^2}}-\frac{1}{1+x^2}\right).\frac{1}{3x^2}$

$$=\frac{1}{3}\lim_{x\to 0}\frac{1+x^2-\sqrt{1-x^2}}{x^2}.\frac{1}{\sqrt{1-x^2}(1+x^2)}$$

$$=\frac{1}{3}\lim_{x\to 0}\frac{(1+x^2)^2-(1-x^2)}{x^2}.\frac{1}{1+x^2+\sqrt{1-x^2}}.\frac{1}{\sqrt{1-x^2}(1+x^2)}$$

$$=\frac{1}{3}.3.\frac{1}{2}.\frac{1}{1}=\frac{1}{2}$$

32. Let $f\left(x\right)=\lim\limits_{n\to\infty}\dfrac{\left(2\sin x\right)^{2n}}{3^{n}-\left(2\cos x\right)^{2n}}; n\in I$,then which of the following is not true?

(A) at $x=n\pi\pm\dfrac{\pi}{6}$, f(x) is discontinuous (B) $f\left(\dfrac{\pi}{3}\right)=1$

(C) f(0) = 0 (D) $f\left(\dfrac{\pi}{2}\right)=1$

Key. D

Sol.

33. If $\lim\limits_{x\to e^{3}}\dfrac{(\ln x-3)^{n}}{\ln((\cos^{m}(\ln x-3)))}$ = –1 (n, m∈N) then n/m is equal to

(A) 3 (B) 4

(C) 9 (D) 1

Key. D

Sol. Let ln x – 3 = t

$\Rightarrow \lim\limits_{t\to 0}\dfrac{t^{n}}{\ln(\cos^{m} t)}\left(\dfrac{0}{0}\text{form}\right)$ = –1

$\Rightarrow \lim\limits_{t\to 0}\dfrac{nt^{n-1}}{-m\tan t}$ = –1

$\Rightarrow$ n – 1 = 1 & $-\dfrac{n}{m}=-1\Rightarrow$ n = m = 2.

34. $\underset{x\to 0}{Lt}\dfrac{\tan\left(\left[-\pi^{2}\right]x^{2}\right)-x^{2}\tan\left(\left[-\pi^{2}\right]\right)}{\sin^{2}x}$ where $[.]$ denote g.i.f

a) $\tan 10+10$ b) $\tan 10-10$ c) $10-\tan 10$ d) none of these

Key. B

Sol. $\pi=3.14, then\left[-\pi^{2}\right]=-10$

$\underset{x\to 0}{Lt}\dfrac{\tan\left(\left[-\pi^{2}\right]x^{2}\right)-\tan\left(\left[-\pi^{2}\right]\right)x^{2}}{\sin^{2}x}$ dilute by x^{2} we get

$\underset{x\to 0}{Lt}\dfrac{\dfrac{-\tan 10x^{2}}{x^{2}}+\tan 10}{\dfrac{\sin^{2}x}{x^{2}}}=\tan 10-10$

35. $\underset{x\to 0}{\text{Lim}}\, x^{2}\left(1+2+3+......\left[\dfrac{1}{|x|}\right]\right)$ is equal to, where [.] is greatest integer function

(A) 1 (B) 3/2 (C) 1/2 (D) 2

Key. C

Sol. $x^2\left(1+2+3+.....\left[\frac{1}{|x|}\right]\right)$

$$\frac{x^2\left(1+\left[\frac{1}{|x|}\right]\right)}{2}\left[\frac{1}{|x|}\right]$$

Now using the property that

$$\frac{1}{|x|}-1<\left[\frac{1}{|x|}\right]\le\frac{1}{|x|}$$

we get

$$\frac{1}{2}|x|<\frac{x^2\left(1+\left[\frac{1}{|x|}\right]\right)}{2}\left[\frac{1}{|x|}\right]\le\frac{1}{2}(1+|x|)$$

Now applying sandwich theorem the required limit is $\frac{1}{2}$

36. If ‘f’ be a bounded, differentiable and increasing function then

$\lim_{x\to 0}\left[f(\sin x.\tan x)-f(x^2)\right]$, where [.] is greatest integer function is equal to

(A) 1 (B) 0 (C) –1 (D) does not exists

Key. B

Sol. since sin x . tan x $> x^2\ \forall\ x\in(0,\pi/2)$

so, f(sin x . tanx) > f(x^2)

hence required limit is 0.

37. If $\underset{x\to 0}{\mathrm{Lt}}\frac{\left((a-n)nx-\tan x\right)\sin nx}{x^2}=0$ where n is a non zero real number then a is equal to

a) 0 b) $\frac{n+1}{n}$ c) n d) $n+\frac{1}{n}$

Key: D

Hint $\underset{x\to 0}{\mathrm{Lt}}\left((a-n)n-\frac{\tan x}{x}\right)\frac{\sin nx}{x}=0$

$\Rightarrow\left((a-n)n-1\right)n=0$

$\Rightarrow a=n+\frac{1}{n}$

38. Let $x>0$ then $\underset{x\to 0}{Lt}\left(\sqrt{\tan x}\right)^{\sqrt{x}}+\left(\sec x\right)^{\frac{1}{x}}=$

(A) $1/e$ (B)1 (C) $\frac{1}{e^2}$ (D) 2

Key: D

Hint: $\underset{x\to 0^+}{Lt}\left(\sqrt{\tan x}\right)^{\sqrt{x}}+\underset{x\to 0^+}{Lt}\left(\cos x\right)^{-1/x}$

$$e^{\underset{x\to 0^+}{Lt}\frac{\log_e\left(\sqrt{\tan x}\right)}{\frac{1}{\sqrt{x}}}\left(\frac{-\infty}{\infty}\right)}=e^0=1,\ \underset{x\to 0^+}{Lt}\left(\cos x\right)^{-1/x}=1\ as\ \ 0<\cos x<1$$

39. Let $f\left(x\right)=\begin{cases}\underset{n\to\infty}{Lt}\dfrac{x^n-\sin\left(x^n\right)}{x^n+\sin\left(x^n\right)} & ,\ if\ \ x>0,\ x\neq 1\\ 1 & ,\ if\ \ \ \ x=1\end{cases}$. Then, at x = 1,

A) f is continuous

B) f has removable discontinuity (i.e., $\underset{x\to 1}{Lt}f\left(x\right)$ exists, but this limit is different from f(1))

C) f has finite (jump) discontinuity (i.e., f (1+) and f (1-) both exist finitely, but they are different)

D) f has infinite or oscillatory discontinuity (for eg like $\sin\frac{1}{x}$ at $x=0$ and $\tan x$ at $x=\frac{\pi}{2}$)

Key: C

Hint: $0<x<1\Rightarrow x^n\to 0\ as\ n\to\infty\Rightarrow f\left(x\right)=0$ and

$x>1\Rightarrow x^n\to+\infty\ \ as\ n\to\infty\Rightarrow f\left(x\right)=1$

$\therefore\ f$ has a jump (finite) discontinuity at $x=1$

40. $\underset{n\to\infty}{Lt}\left[\left(1+\frac{1}{n}\right)^n-\left(1+\frac{1}{n}\right)\right]^{-n}=$

A) 1 B) $\frac{1}{e-1}$ C) $1-e^{-1}$ D) 0

Ans: D

Hint: $\underset{n\to\infty}{Lt}\left[\left(1+\frac{1}{n}\right)^n-\left(1+\frac{1}{n}\right)\right]=e-1>1$

41. Let $f(x)=\frac{\tan x}{x}$, then $\log_e\left(\lim_{x\to 0}\left(\left[f(x)\right]+x^2\right)^{\frac{1}{\{f(x)\}}}\right)$ is equal, (where [·] denotes greatest integer function and {·} fractional part)

(A) 1 (B) 2 (C) 3 (D) 4

Key: C

Hint: $\lim_{x\to 0}\left[f(x)\right]=\lim_{x\to 0}\left[\frac{\tan x}{x}\right]=1$

$$\lim_{x\to 0}\left(\left[f(x)\right]+x^2\right)^{\frac{1}{\{f(x)\}}}=\lim_{x\to 0}\left(1+x^2\right)^{\frac{1}{\{f(x)\}}}\left(1^{\infty}\text{ form}\right)$$

Again, f(x) = $\frac{\tan x}{x}=\frac{x+\frac{x^3}{3}+\frac{2}{15}x^5+\ldots}{x}$

$$=1+\frac{x^2}{3}+\frac{2}{15}x^4+\ldots$$

$$\{f(x)\}=\frac{x^2}{3}+\frac{2}{15}x^4+\ldots$$

(i) becomes,

$$\log_e\left(e^{\lim_{x\to 0}x^2\times\frac{1}{\{f(x)\}}}\right)=e^{\lim_{x\to 0}\frac{x^2}{\frac{x^2}{3}+\frac{2}{15}x^4+\ldots}}=3$$

$\therefore$ (C) is the correct answer.

42. If $\lim_{x\to\infty}x\left(\tan^{-1}\left(\frac{x+\lambda}{x+\mu}\right)-\frac{\pi}{4}\right)=1$ then ordered pair(s) (λ,μ) can be

(A) (2000,2011) (B) (0,1)

(C) (5,3) (D) (1,0)

Key: C

Hint: $\lim_{x\to ki}\frac{\tan^{-1}\left(\frac{x+\lambda}{x+\mu}\right)-\frac{\pi}{4}}{\frac{1}{x}}=1$

Apply L' hospital rule and simplifying we get

$$\lim_{x\to\infty}\frac{(\lambda-\mu)x^2}{2x^2+2x(\lambda+\mu)+\left(\mu^2+\lambda^2\right)}=1$$

$$\Rightarrow\frac{\lambda-\mu}{2}=1$$

$$\Rightarrow\lambda-\mu=2$$

$\therefore(\lambda,\mu)$ can be (5,3)

43. Consider the function $f(x)=\begin{cases}\frac{p(x)}{x-2}; & x\neq 2\\ 7\ ; & x=2\end{cases}$ where P(x) is a polynomial such that $p'''(x)$ is identically equal to 0 and $p(3)=9.$ If f(x) is continuous at x = 2, then p(x) is

(A) $2x^2+x+6$ (B) $2x^2-x-6$ (C) x^2+3 (D) x^2-x+7

Key: B

Hint: Since $P'''(x)=0$

Let $p(x)=ax^2+bx+c$

$p(2)=0$

$4a+2b+c=0$ ....(1)

$9a+3b+c=9$ ....(2)

$p'(2)=7$

$\Rightarrow 4a+b=7$

Solve 1,2 and 3 to get a,b,c

44. $\lim_{n\to\infty}\frac{(n!)^{\frac{1}{n}}}{n}$ equals

a) e b) e^{-1} c) e^{-2} d) e^2

KEY : B

$$let \quad P=\frac{(n!)^{\frac{1}{n}}}{n}$$

Sol. $$=\left(\frac{(n!)}{n^n}\right)^{\frac{1}{n}}$$

$$\log P=\frac{1}{n}\sum_{r=1}^{n}\log\left(\frac{r}{n}\right)$$

45. $\underset{x\to 0}{\text{Lim}}\, x^2\left(1+2+3+......\left[\frac{1}{|x|}\right]\right)$ is equal to, where [.] is greatest integer function

(A) 1 (B) 3/2
(C) 1/2 (D) 2

Key. C

Sol. $$x^2\left(1+2+3+.....\left[\frac{1}{|x|}\right]\right)$$

$$\frac{x^2\left(1+\left[\frac{1}{|x|}\right]\right)}{2}\left[\frac{1}{|x|}\right]$$

Now using the property that

$$\frac{1}{|x|}-1<\left[\frac{1}{|x|}\right]\le\frac{1}{|x|}$$

we get

$$\frac{1}{2}|x| < \frac{x^2\left(1+\left[\frac{1}{|x|}\right]\right)}{2}\left[\frac{1}{|x|}\right] \le \frac{1}{2}(1+|x|)$$

Now applying sandwich theorem the required limit is $\frac{1}{2}$

46. If 'f' be a bounded, differentiable and increasing function then $\lim_{x\to 0}\left[f(\sin x.\tan x)-f(x^2)\right]$, where [.] is greatest integer function is equal to

(A) 1 (B) 0
(C) –1 (D) does not exists

Key. B

Sol. since sin x . tan x $> x^2\ \forall\ x \in (0, \pi/2)$
so, f(sin x . tanx) > f(x^2)
hence required limit is 0.

47. $\lim_{x\to 0}\frac{\sin[\cos x]}{1+[\cos x]}$, ([x] denotes the greatest integer less than or equal to)

(A) sin 1 (B) 0

(C) Does not exist (D) $\frac{\sin 1}{2}$

Key. B

Sol. $\text{LHL} = \lim_{x\to 0^-} f(0-h) = \lim_{h\to 0}\frac{\sin[\cos h]}{1+[\cos h]}$

$= \frac{\sin(0)}{1+0} = 0 \qquad \begin{pmatrix}\because\ h>0\\ \therefore\ \cos h<1\end{pmatrix}$

$\text{RHL} = \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h)$

$= \lim_{h\to 0}\frac{\sin[\cos h]}{1+[\cos h]}$

$= \frac{\sin(0)}{1+0} = 0 \qquad \begin{pmatrix}\because\ h>0\\ \therefore\ \cos h<1\end{pmatrix}$

$\therefore \lim_{x\to 0}\frac{\sin[\cos x]}{1+[\cos x]} = 0$

48. $\underset{x\to 0}{\text{Lt}}\left(\frac{1}{x^2}-\operatorname{cosec}^2 x\right) =$

a) $\frac{1}{3}$ b) $\frac{2}{3}$ c) $\frac{-1}{3}$ d) $\frac{-2}{3}$

Key. C

Sol. Apply, L-H rule

49. If a_n and b_n are positive integers and $a_n+\sqrt{2}b_n = \left(2+\sqrt{2}\right)^n$, then $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) =$

A) 2 B) $\sqrt{2}$ C) $e^{\sqrt{2}}$ D) e^{2}

Key. B

Sol. We have $a_n+\sqrt{2}b_n=\left(2+\sqrt{2}\right)^n$

$$\Rightarrow a_n-\sqrt{2}b_n=\left(2-\sqrt{2}\right)^n$$

Therefore $a_n=\frac{1}{2}\left[\left(2+\sqrt{2}\right)^n+\left(2-\sqrt{2}\right)^n\right]$

And $b_n=\frac{\left[\left(2+\sqrt{2}\right)^n-\left(2-\sqrt{2}\right)^n\right]}{2\sqrt{2}}$

Therefore $\frac{a_n}{b_n}=\sqrt{2}\frac{\left[\left(2+\sqrt{2}\right)^n+\left(2-\sqrt{2}\right)^n\right]}{\left[\left(2+\sqrt{2}\right)^n-\left(2-\sqrt{2}\right)^n\right]}$

$$=\sqrt{2}\frac{\left[1+\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)^n\right]}{\left[1-\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)^n\right]}$$

Hence $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)=\sqrt{2}\left(\frac{1+0}{1-0}\right)\left(\because \frac{2-\sqrt{2}}{2+\sqrt{2}}<1\right)=\sqrt{2}$

50. The value of $\lim_{x\to 0} f(x)$ where $f(x)=\frac{\cos(\sin x)-\cos x}{x^4}$, is

(A) 2 (B) $1/6$ (C) $2/3$ (D) $-1/3$

Key. B

Sol. $\lim_{x\to 0} f(x)=\lim_{x\to 0}\frac{\cos(\sin x)-\cos x}{x^4}$

$$=\lim_{x\to 0}\frac{-2\sin\frac{\sin x+x}{2}\sin\frac{\sin x-x}{2}}{x^4}$$

$$=-\frac{1}{2}\lim_{x\to 0}\frac{\sin\left(\frac{\sin x+x}{2}\right)}{\left(\frac{\sin x+x}{2}\right)}\frac{\sin\left(\frac{\sin x-x}{2}\right)}{\left(\frac{\sin x-x}{2}\right)}\times\frac{\sin x+x}{x}\times\frac{\sin x-x}{x^3}$$

$$=-\frac{1}{2}\lim_{u\to 0}\frac{\sin u}{u}\lim_{v\to 0}\frac{\sin v}{v}\lim_{x\to 0}\left(\frac{\sin x}{x}+1\right)$$

$$\times\frac{-\frac{x^3}{3!}+\frac{x^5}{5!}+....}{x^3}\left(u=\frac{\sin x+x}{2},v=\frac{\sin x-x}{2}\right)$$

$$= -\frac{1}{2}\times 1\times 1\times 2\times \frac{-1}{3!} = \frac{1}{6}.$$

51. $\lim\limits_{n\to\infty} \frac{\{x\}+\{2x\}+\{3x\}+....+\{nx\}}{n^2} =$

[Where $\{x\} = x - [x]$ denotes the fractional part of x]

A) 1 B) 0 C) $\frac{1}{2}$ D) None of these

Key. B

Sol. $0 \le \{nx\} < 1,\ for\ n = 1,2,3,......,n$

$$\Rightarrow 0 \le \sum_{n=1}^{n}\{nx\} < n \qquad \Rightarrow \frac{0}{n^2} \le \frac{\sum_{n=1}^{n}\{nx\}}{n^2} < \frac{1}{n}$$

$$\Rightarrow \underset{x\to\infty}{Lt}\frac{0}{n^2} \le \underset{n\to\infty}{Lt}\frac{\sum_{n=1}^{n}\{nx\}}{n^2} \le \underset{n\to\infty}{Lt}\frac{1}{n} \qquad \Rightarrow 0 \le \underset{n\to\infty}{Lt}\frac{\sum_{n=1}^{n}\{nx\}}{n^2} \le 0$$

$$\Rightarrow \underset{n\to\infty}{Lt}\frac{\{x\}+\{2x\}+...+\{nx\}}{n^2} = 0$$

52. For x > 0; $\lim\limits_{x\to 0}\left\{(\sin x)^{1/x} + \left(\frac{1}{x}\right)^{\sin x}\right\}$ is __________

(1) 0 (2) -1 (3) 1 (4) 2

Key. 3

Sol. $\underset{x\to 0}{Lt}(\sin x)^{1/2} = 0 \quad \left(0 < \sin x < 1;\ \frac{1}{x} \to \infty\right)$

And $\log y = \sin x.\log x$

53. $\lim\limits_{x\to 0}\frac{\cos(\sin x) - \cos x}{x^4} =$ ____

(1) $\frac{1}{5}$ (2) $\frac{1}{6}$ (3) $\frac{1}{4}$ (4) $\frac{1}{2}$

Key. 2

Sol. $$\underset{x\to 0}{Lt}\frac{2\sin\left(\frac{x+\sin x}{2}\right)}{\frac{\sin x + x}{2}}\left(\frac{\sin x + x}{2}\right).\underset{x\to 0}{Lt}\frac{2\sin\left(\frac{x-\sin x}{2}\right)}{\frac{x-\sin x}{2}}.\underset{x\to 0}{Lt}\frac{1}{2}(x-\sin x)$$

$$\underset{x\to 0}{Lt}\left(\frac{\sin x + x}{2x^4}\right).\frac{1}{2}\left[x-\left(x-\frac{x^3}{13}+\frac{x^5}{15}+....\infty\right)\right]$$

54. $\lim\limits_{x\to 0}\left\{\frac{7}{10}+\frac{29}{10^2}+\frac{133}{10^3}+\ldots+\frac{5^n+2^n}{10^n}\right\}=$ _____

(1) $\frac{3}{4}$ (2) 2 (3) $\frac{5}{4}$ (4) $\frac{1}{2}$

Key. 3

Sol. $\frac{5+2}{10}+\frac{5^2+2^2}{10^2}+.....+\frac{5^w+2^w}{10^n}$

(use G.P; s_∞)

55. $\lim\limits_{x\to 0}\frac{729^x-243^x-81^x+9^x+3^x-1}{x^3}=K(\log 3)^3 \quad \Rightarrow$ K = _______

(1) 4 (2) 5 (3) 6 (4) 7

Key. 3

Sol. $\underset{x\to 0}{Lt}\frac{(3^x-1)}{x}\frac{(9^x-1)}{x}\left(\frac{27^x-1}{x}\right)$

56. $\lim\limits_{x\to\infty}\left(1+\frac{a}{x}+\frac{b}{x^2}\right)^{2x}=e^2$ then ____

(1) $a\in R;\ b\in R$ (2) $a=1;\ b\in R$ (3) $a\in R;\ b=2$ (4) $a=1;\ b=2$

Key. 2

Sol. $Lt\ f(x)^{g(x)}$ is of form $1^\infty \Rightarrow e^{\underset{x\to 0}{Lt}\, g(x)\{f(x)-1\}}$

57. $\lim\limits_{\theta\to 0}\left\{\left[\frac{n\sin\theta}{\theta}\right]+\left[\frac{n\tan\theta}{\theta}\right]\right\}=$ _____ where [x] is greatest integer $\le x$ and $n\in I$

(1) 2n (2) 2n + 1 (3) 2n – 1 (4) 0

Key. 3

Sol. $\frac{\sin\theta}{\theta}\to 1 as\,\theta\to 0$ but < 1

$\therefore\left[\frac{n\sin\theta}{\theta}\right]=n-1$

$\left[n\frac{Tan\theta}{\theta}\right]=n$ $\quad \frac{Tan\theta}{\theta}\to 1\ as\ \theta\to 0$ but > 1

58. If $f(x)=\underset{n\to\infty}{Lt}\left\{\frac{x}{x+1}+\frac{x}{(x+1)(2x+1)}+\frac{x}{(2x+1)(3x+1)}+\ldots..\right\}$ to n terms; then range of f(x) is _____

(1) $[0, 1]$ (2) $[-1, 1]$ (3) $\{0, 1\}$ (4) $\{-1, 0, 1\}$

Key. 3

Sol. $1-\frac{1}{1+nx}$

$Lt\ nx=\infty \quad for\ \ x>0$

$Lt\ nx=-\infty \quad for\ \ x<0$

$Lt\ nx=0 \quad for\ \ x=0$

$\underset{n\to\infty}{Lt}S_w=1;\ 0$

59. $\lim\limits_{x\to 0}\left\{Tan\left(\frac{\pi}{4}+x\right)\right\}^{1/x} =$ ____

(1) 1 (2) -1 (3) e^2 (4) e

Key. 3

Sol. 1^∞ form $\Rightarrow e^{\underset{x\to 0}{Lt}\, g(x)(f(x)-1)}$

60. $\lim\limits_{x\to\infty} x\left\{Tan^{-1}\frac{x+1}{x+2}-Tan^{-1}\frac{x}{x+2}\right\} =$ ____

(1) 1 (2) -1 (3) $\frac{1}{2}$ (4) $-\frac{1}{2}$

Key. 3

Sol. $Tan^{-1}x - Tan^{-1}y = Tan^{-1}\frac{x-y}{1-xy}$

$$\underset{x\to\infty}{Lt}\; x\left(\frac{Tan^{-1}\dfrac{x+2}{2x^2+5x+4}}{\dfrac{x+2}{2x^2+5x+4}}\right)\left(\frac{x+2}{2x^2+5x+4}\right)$$

61. If 0 < b < a then $\lim\limits_{n\to\infty}\frac{a^n+b^n}{a^n-b^n} =$

(1) 0 (2) 1 (3) –1 (4) none of these

Key. 2

Sol. $0<\frac{b}{a}<1; \left(\frac{b}{a}\right)^n \to 0$ as $n\to\infty$

62. If $a_1 = 1 and\; a_n = n(1+a_{n-1})\,\forall n\ge 2,$ then $\lim\limits_{n\to\infty}\left(1+\frac{1}{a_1}\right)\left(1+\frac{1}{a_2}\right)......\left(1+\frac{1}{a^n}\right) =$

(1) e (2) $\log e^2$ (3) $e^{\frac{1}{2}}$ (4) $\log_2 e$

Key. 1

Sol. $\underset{n\to\infty}{Lt}\left(\frac{a_1+1}{a_1}\right)\left(\frac{a_2+1}{a_2}\right).......\left(\frac{a_n+1}{a_n}\right)$

$$\underset{n\to\infty}{Lt}\left(\frac{a_2}{2}\right)\left(\frac{a_3}{3}\right)\left(\frac{a_4}{4}\right)...\left(\frac{a_{n+1}}{n+1}\right)\frac{1}{a_1a_2....a_n}$$

$$= \underset{n\to\infty}{Lt}\;\frac{a_{n+1}}{(n+1)!} = \underset{n\to\infty}{Lt}\frac{1+a_n}{n!} = \underset{n\to\infty}{Lt}\left(\frac{1}{n!}+\frac{a_n}{n!}\right)$$

$= \underset{n\to\infty}{\text{Lt}} \left(\frac{1}{n!} + \frac{1}{(n-1)!} + \frac{a_{n-1}}{(n-1)!}\right) = e$

63. The integer n for which $\lim_{x\to 0}\left(\frac{(\cos x - 1)(\cos x - e^{x})}{x^{n}}\right)$ is a finite non zero number is

(1) 1 (2) 2 (3) 3 (4) 4

$\lim_{x\to 0}\left(\left[\frac{100x}{\sin x}\right] + \left[\frac{99\sin x}{x}\right]\right)$

Key. 3

Sol. Conceptual

64. The value of where [.] represents greatest integral function, is

(1) 199 (2) 198 (3) 0 (4) none of these

Key. 2

Sol. We know that $\lim_{x\to 0}\frac{\sin x}{x} \to I^{-}$ and $\lim_{x\to 0}\frac{x}{\sin x} \to I^{+}$

So, $\lim_{x\to 0}\left[100\frac{x}{\sin x}\right] + \lim_{x\to 0}\left[99\frac{\sin x}{x}\right] = 100 + 98 = 198$

65. If $\sum_{r=1}^{k}\cos^{-1}\beta_r = \frac{k\pi}{2}$ for any $k \ge 1$ where $\beta_r \ge 0\,\forall r$ and $A = \sum_{r=1}^{k}(\beta_r)^{r}$. Then

$\lim_{x\to A}\frac{(1+x^2)^{1/3} - (1-2x)^{1/4}}{x + x^2} =$

A) $\frac{1}{2}$ B) 0 C) 3/2 D) $\frac{\pi}{2}$

Key. A

Sol. Given $\cos^{-1}\beta_1 + \cos^{-1}\beta_2 + ... + \cos^{-1}\beta_k = k\frac{\pi}{2}$ We know that $\cos^{-1} x \le \frac{\pi}{2}\,\forall r \ge 0$

$\therefore \cos^{-1}\beta_r \le \frac{\pi}{2}\,\forall r = 1, 2, 3...., k \qquad \Rightarrow \sum_{r=1}^{k}\cos^{-1}\beta_r \le \frac{k\pi}{2}$

So the given equality holds only if

$\cos^{-1}\beta_1 = \cos^{-1}\beta_2 = ... = \cos^{-1}\beta_k = \frac{\pi}{2}$

$\Rightarrow \beta_1 = \beta_2 = ... = \beta_k = 0$

Thus $A = \sum_{r=1}^{k}(\beta_r)^{r} = 0$

Rrequired limit = $\lim_{x\to 0} \frac{(1+x^2)^{1/3}-(1-2x)^{1/4}}{x+x^2}$

= $\lim_{x\to 0} \frac{\frac{1}{3}(1+x^2)^{-2/3}(2x)-\frac{1}{4}(1-2x)^{3/4}(-2)}{1+2x}$ (L' Hospital Rule)

= $\frac{1}{2}$

66. If [x] and {x} represent integral and fractional parts of x respectively and a is any real number, then $\lim_{x\to[a]} \frac{e^{\{x\}}-\{x\}-1}{\{x\}^2} =$

A) a B) $\{a\}$ C) $\frac{1}{2}$ D) Does not exist

Key. D

Sol. Let $P = \lim_{x\to[a]} = \frac{e^{\{x\}}-\{x\}-1}{\{x\}^2}$

Put $x=[a]+h$, $h>0$

Then $P = \lim_{h\to 0} \frac{e^{\{[a]+h\}}-\{[a]+h\}-1}{\{[a]+h\}^2}$

$P = \lim_{h\to 0} \frac{e^h-h-1}{h^2}$

$\lim_{h\to 0} \frac{e^h-1}{2h} = \frac{1}{2}$ [Using L Hospital Rule]

Next put $x=[a]-h, h>0$

then $P = \lim_{h\to 0} \frac{e^{\{[a]-h\}}-\{[a]-h\}-1}{\{[a]-h\}^2}$

$= \lim_{h\to 0} \frac{e^{1-h}-(1-h)-1}{(1-h)^2} = \lim_{h\to 0} \frac{e^{1-h}+h-2}{(1-h)^2} = e-2$

$\therefore$ Limit does not exist

67. Let $f: R^+ \to R^+$ be a function satisfying the relation $f(x.f(y)) = f(xy)+x$ for all $x, y \in R^+$. Then $\lim_{x\to 0}\left(\frac{(f(x))^{1/3}-1}{(f(x))^{1/2}-1}\right) =$

(A) 1 (B) $\frac{1}{2}$ (C) $\frac{2}{3}$ (D) $\frac{3}{2}$

Key. C

Sol. Given relation is $f\left(x.f\left(y\right)\right)=f\left(xy\right)+x$ (1.56)

Interchanging x and y in Eq. (1.56), we have

$$f\left(y.f\left(x\right)\right)=f\left(yx\right)+y \quad (1.57)$$

Again replacing *x* with $f\left(x\right)$ in Eq. (1.56) we get

$$f\left(f\left(x\right).f\left(y\right)\right)=f\left(y.f\left(x\right)\right)+f\left(x\right) \quad (1.58)$$

Therefore, Eqs. $(1.56)-(1.58)$ imply

$$f\left(f\left(x\right).f\left(y\right)\right)=f\left(xy\right)+y+f\left(x\right) \quad (1.59)$$

Again interchanging *x* and *y* in Eq. (1.59), we have

$$f\left(f\left(y\right).f\left(x\right)\right)=f\left(yx\right)+x+f\left(y\right) \quad (1.60)$$

Equations (1.59) and (1.60) imply

$$f\left(xy\right)+y+f\left(x\right)=f\left(yx\right)+x+f\left(y\right) \quad (1.61)$$

Suppose $f\left(x\right)-x=f\left(y\right)-y=\lambda$

Substituting $f\left(x\right)=\lambda+x$ in Eq. (1.56), we have

$$x.f\left(y\right)+\lambda=\left(xy+\lambda\right)+x$$

$$\Rightarrow x.f\left(y\right)=xy+x$$

Therefore $x\left(y+\lambda\right)=xy+x \quad \left[\because f\left(y\right)=\lambda+y\right]$

$$\Rightarrow \lambda x=x$$

$$\Rightarrow \lambda=1 \quad \left(\because x>0\right)$$

So $f\left(x\right)=x+\lambda=x+1$

Hence $$\lim_{x\to 0}\frac{\left(f\left(x\right)\right)^{1/3}-1}{\left(f\left(x\right)\right)^{1/2}-1}=\lim_{x\to 0}\frac{\left(1+x\right)^{1/3}-1}{\left(1+x\right)^{1/2}-1}$$

$$=\lim_{x\to 0}\left(\frac{\left(1+x\right)^{1/3}-1}{1+x-1}\right).\left(\frac{1+x-1}{\left(1+x\right)^{1/2}-1}\right)$$

$$=\frac{1/3}{1/2}=\frac{2}{3}$$

68. The value of $\lim_{x\to 0}f\left(x\right)$ where $f\left(x\right)=\frac{\cos\left(\sin x\right)-\cos x}{x^4}$, is

(A) 2 (B) $1/6$ (C) $2/3$ (D) $-1/3$

Key. B

Sol. $$\lim_{x\to 0}f\left(x\right)=\lim_{x\to 0}\frac{\cos\left(\sin x\right)-\cos x}{x^4}$$

$$=\lim_{x\to 0}\frac{-2\sin\frac{\sin x+x}{2}\sin\frac{\sin x-x}{2}}{x^4}$$

$$= -\frac{1}{2}\lim_{x\to 0}\frac{\sin\left(\frac{\sin x + x}{2}\right)}{\left(\frac{\sin x + x}{2}\right)}\frac{\sin\left(\frac{\sin x - x}{2}\right)}{\left(\frac{\sin x - x}{2}\right)}\times\frac{\sin x + x}{x}\times\frac{\sin x - x}{x^3}$$

$$= -\frac{1}{2}\lim_{u\to 0}\frac{\sin u}{u}\lim_{v\to 0}\frac{\sin v}{v}\lim_{x\to 0}\left(\frac{\sin x}{x}+1\right)$$

$$\times\frac{-\frac{x^3}{3!}+\frac{x^5}{5!}+....}{x^3}\left(u=\frac{\sin x + x}{2}, v=\frac{\sin x - x}{2}\right)$$

$$= -\frac{1}{2}\times 1\times 1\times 2\times\frac{-1}{3!}=\frac{1}{6}.$$

69. Let $x_1 = 1$ and $x_{n+1} = \frac{4+3x_n}{3+2x_n}$ for $n \geq 1$. If $\lim_{n\to\infty} x_n$ exists finitely, then the limit is equal to

(A) $\sqrt{2}$ (B) 1 (C) 2 (D) $\sqrt{2}+1$

Key. A

Sol. We have $x_1 = 1, x_2 = \frac{4+3}{3+2} = \frac{7}{5}$

$$x_3 = \frac{4+3x_2}{3+2x_2} = \frac{4+3\left(\frac{7}{5}\right)}{3+2\left(\frac{7}{5}\right)} = \frac{41}{29} > x_2$$

We can easily verify that $x_n < x_{n+1}$ and hence $\{x_n\}$ is strictly increasing sequence of positive terms. Let $\lim_{n\to\infty} x_n = l$. Therefore

$$l = \lim_{n\to\infty} x_{n+1}$$

$$= \lim_{n\to\infty}\left(\frac{4+3x_n}{3+2x_n}\right)$$

$$= \frac{4+3\lim_{n\to\infty} x_n}{3+2\lim_{n\to\infty} x_n}$$

$$= \frac{4+3l}{3+2l}$$

Hence $3l + 2l^2 = 4 + 3l$

or $l^2 = 2 \Rightarrow l = \sqrt{2}$ $(\because x_n > 0\ \forall\ n)$.

70. Let $f(x) = x^3\left\{\sqrt{x^2+\sqrt{x^4+1}} - x\sqrt{2}\right\}$. Then $\lim_{x\to\infty} f(x)$ is equal to

(A) $\frac{1}{2\sqrt{2}}$ (B) $\frac{1}{4\sqrt{2}}$ (C) $\frac{3}{4\sqrt{2}}$ (D) does not exist

Key. B

Sol. We have $f\left(x\right)=\frac{x^3\left\{x^2+\sqrt{x^4+1}-2x^2\right\}}{\sqrt{x^2+\sqrt{x^4+1}}+x\sqrt{2}}$

$$=\frac{x^3\left\{\sqrt{x^4+1}-x^2\right\}}{\sqrt{x^2+\sqrt{x^4+1}}+x\sqrt{2}}$$

$$=\frac{x^3\left(x^4+1-x^4\right)}{\left[\sqrt{x^2+\sqrt{x^4+1}}+x\sqrt{2}\right]\left[\sqrt{x^4+1}+x^2\right]}$$

$$=\frac{x^3}{\left[\sqrt{x^2+\sqrt{x^4+1+}}+x\sqrt{2}\right]\left[\sqrt{x^4+1}+x^2\right]}$$

$$=\frac{1}{\left[\sqrt{1+\sqrt{1+\frac{1}{x^4}}}+\sqrt{2}\right]\left[\sqrt{1+\frac{1}{x^4}}+1\right]}$$

$$=\frac{1}{\left(\sqrt{1+\sqrt{1}}+\sqrt{2}\right)\left(\sqrt{1}+1\right)}$$

$$=\frac{1}{2\sqrt{2}\left(2\right)}=\frac{1}{4\sqrt{2}}.$$

71. If a_n and b_n are positive integers and $a_n+\sqrt{2}b_n=\left(2+\sqrt{2}\right)^n$, then $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)=$

A) 2 B) $\sqrt{2}$ C) $e^{\sqrt{2}}$ D) e^2

Key. B

Sol. We have $a_n+\sqrt{2}b_n=\left(2+\sqrt{2}\right)^n$

$$\Rightarrow a_n-\sqrt{2}b_n=\left(2-\sqrt{2}\right)^n$$

Therefore $a_n=\frac{1}{2}\left[\left(2+\sqrt{2}\right)^n+\left(2-\sqrt{2}\right)^n\right]$

And $b_n=\frac{\left[\left(2+\sqrt{2}\right)^n-\left(2-\sqrt{2}\right)^n\right]}{2\sqrt{2}}$

Therefore $\frac{a_n}{b_n}=\sqrt{2}\frac{\left[\left(2+\sqrt{2}\right)^n+\left(2-\sqrt{2}\right)^n\right]}{\left[\left(2+\sqrt{2}\right)^n-\left(2-\sqrt{2}\right)^n\right]}$

$$=\sqrt{2}\,\frac{\left[1+\left(\dfrac{2-\sqrt{2}}{2+\sqrt{2}}\right)^{n}\right]}{\left[1-\left(\dfrac{2-\sqrt{2}}{2+\sqrt{2}}\right)^{n}\right]}$$

Hence $\lim\limits_{n\to\infty}\left(\dfrac{a_n}{b_n}\right)=\sqrt{2}\left(\dfrac{1+0}{1-0}\right)\left(\because \dfrac{2-\sqrt{2}}{2+\sqrt{2}}<1\right)=\sqrt{2}$

72. If $\lim\limits_{x\to 0}\dfrac{\left((a-n)nx-\tan x\right)\sin nx}{x^2}=0$, where $n\in R\sim\{0\}$, then *a* is equal to

A) 0 B) $\dfrac{n}{n+1}$ C) *n* D) $n+\dfrac{1}{n}$

Key. D

Sol. The given limit can be written as

$$\lim_{x\to 0}\left(\frac{\sin nx}{nx}\right)(n)\left((a-n)n-\frac{\tan x}{x}\right)=0$$

$$\Rightarrow (1)(n)\left((a-n)n-1\right)=0$$

$$\Rightarrow (a-n)n-1=0\Rightarrow a=n+1/n$$

73. For each positive integer *n*, let $s_n=\dfrac{3}{1.2.4}+\dfrac{4}{2.3.5}+\dfrac{5}{3.4.6}+.....+\dfrac{n+2}{n(n+1)(n+3)}$. Then $\lim\limits_{n\to\infty}s_n$ equals

A) $\dfrac{29}{6}$ B) $\dfrac{29}{36}$ C) 0 D) $\dfrac{29}{18}$

Key. B

Sol. Let $u_k=\dfrac{k+2}{k(k+1)(k+3)}$

$$=\frac{(k+2)^2}{k(k+1)(k+2)(k+3)}$$

$$=\frac{k^2+4k+4}{k(k+1)(k+2)(k+3)}$$

$$=\frac{k(k+1)+3k+4}{k(k+1)(k+2)(k+3)}$$

$$=\frac{1}{(k+2)(k+3)}+\frac{3}{(k+1)(k+2)(k+3)}+\frac{4}{k(k+1)(k+2)(k+3)}$$

$$=\left(\frac{1}{k+2}-\frac{1}{k+3}\right)-\frac{3}{2}\left[\frac{1}{(k+2)(k+3)}-\frac{1}{(k+1)(k+2)}\right]$$

$$-\frac{4}{3}\left[\frac{1}{(k+1)(k+2)(k+3)}-\frac{1}{k(k+1)(k+2)}\right]$$

Now, put $k=1,2,3,....,n$ and add. Thus

$$s_u = u_1 + u_2 + ... + u_n$$

$$= \left(\frac{1}{3} - \frac{1}{n+3}\right) - \frac{3}{2}\left[\frac{1}{(n+2)(n+3)} - \frac{1}{2.3}\right]$$

$$-\frac{4}{3}\left[\frac{1}{(n+1)(n+2)(n+3)} - \frac{1}{1.2.3}\right]$$

Therefore $\lim_{n\to\infty} s_n = \frac{1}{3} + \frac{3}{12} + \frac{4}{18} = \frac{29}{36}$

74. $\lim_{x\to 0} \frac{a^{\tan x} - a^{\sin x}}{\tan x - \sin x}$ is equal to $(a>0)$

A) $\log_e a$ B) 1 C) 0 D) ∞

Key. A

Sol. We have $\lim_{x\to 0} \frac{a^{\tan x} - a^{\sin x}}{\tan x - \sin x} = \lim_{x\to 0} a^{\sin x}\left(\frac{a^{\tan x - \sin x} - 1}{\tan x - \sin x}\right)$

$= \lim_{x\to 0}\left(a^{\sin x}\right) \times \lim_{t\to 0}\left(\frac{a^t - 1}{t}\right)$ (where $t = \tan x - \sin x$)

$= a^0 \times \log_e a = \log_e a$

75. $\lim_{x\to\frac{\pi}{2}} \frac{(1-\sin x)\left(8x^3 - \pi^3\right)\cos x}{(\pi - 2x)^4}$

A) $-\frac{\pi^2}{16}$ B) $\frac{3\pi^2}{16}$ C) $\frac{\pi^2}{16}$ D) $-\frac{3\pi^2}{16}$

Key. D

Sol. Let $f(x) = \frac{(1-\sin x)\left(8x^3 - \pi^3\right)\cos x}{(\pi - 2x)^4}$

$$= \frac{(1-\sin x)\cos x(2x-\pi)\left(4x^2 + 2\pi x + \pi^2\right)}{(2x-\pi)^4}$$

$$= \frac{(1-\sin x)\cos x\left(4x^2 + 2\pi x + \pi^2\right)}{(2x-\pi)^3}$$

Therefore $\lim_{x\to\frac{\pi}{2}} f(x) = \lim_{x\to\frac{\pi}{2}} \frac{(1-\sin x)\cos x}{(2x-\pi)^3}.\left(3\pi^2\right)$

$$\lim_{x\to\frac{\pi}{2}} f(x) = \lim_{x\to\frac{\pi}{2}} \frac{(1-\sin x)\cos x}{(2x-\pi)^3}.\left(3\pi^2\right) \qquad ----(1.62)$$

Put $2x - \pi = y$ so that $y \to 0$ as $x \to \pi/2$. Therefore now

$$\frac{(1-\sin x)\cos x}{(2x-\pi)^3} = \frac{\left[1 - \sin\left(\frac{\pi+y}{2}\right)\right]\cos\left(\frac{\pi+y}{2}\right)}{y^3}$$

$$= \frac{\left(1-\cos\frac{y}{2}\right)\left(-\sin\frac{y}{2}\right)}{y^3}$$

$$= -\left(\frac{2\sin^2\frac{y}{4}}{y^2}\right)\left(\frac{\sin\frac{y}{2}}{y}\right)$$

$$= -2\left(\frac{\sin\frac{y}{4}}{y/4}\right)^2 \cdot \frac{1}{16} \cdot \left(\frac{\sin\frac{y}{2}}{y/2}\right) \cdot \frac{1}{2}$$

$$= \frac{-1}{16}\left(\frac{\sin\frac{y}{4}}{y/4}\right)^2 \left(\frac{\sin\frac{y}{2}}{y/2}\right) \qquad ----(1.63)$$

Therefore from Eqs. (1.62) and (1.63)

$$\lim_{x\to\frac{\pi}{2}} f(x) = \frac{-3\pi^2}{16} \times 1 \times 1.$$

76. If a_1 is the greatest value of $f(x)$ where $f(x) = \frac{1}{2+[\sin x]}$ and $a_{n+1} = \frac{(-1)^{n+2}}{n+1} + a_n$

Then $\lim_{n\to\infty} a_n =$ _______

1) 0 2) e 3) 1 4) $\log_e 2$

Key. 4

Sol. $a_1 = 1, a_2 = 1-\frac{1}{2}, a_3 = 1-\frac{1}{2}+\frac{1}{3}.........a_n = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+......+(-1)^{n-1}.\frac{1}{n}$

$\lim_{n\to\infty} a_n = \log_e 2$

77. $\lim_{x\to\frac{\pi}{2}}\left[\frac{[\sin x]-[\cos x]+1}{3}\right] =$

[.] $\rightarrow$ denotes greatest integer function

1) 0 2) 1 3) -1 4) does not exist

Key. 1

Sol. LHL = RHL = 0

78. $\lim_{x\to 0}\left(\frac{1+2x}{1+3x}\right)^{\frac{1}{x^2}} .e^{\frac{1}{x}} =$ _______

1) $e^{\frac{5}{2}}$ 2) e^2 3) 4) 1

Key. 1

Sol. $\lim\limits_{x\to 0} e^{\frac{1}{x^2}(\log(1+2x)-\log(1+3x)+\frac{1}{x}}$

$$e^{\lim\limits_{x\to 0}\frac{(\log(1+2x)-\log(1+3x)+x}{x^2}} = e^{\frac{5}{2}}$$

79. $\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\cot^{-1}\left(r^2+\frac{3}{4}\right)=$

1) $\tan^{-1}(2)$ 2) $\frac{\pi}{4}$ 3) $\frac{\pi}{2}$ 4) $\tan^{-1}(3)$

Key. 1

Sol. $\cot^{-1}\left(r^2+\frac{3}{4}\right)=\tan^{-1}\left(\frac{1}{r^2+\frac{3}{4}}\right)$

$=\tan^{-1}\left(\frac{1}{1+\left(r^2-\frac{1}{4}\right)}\right)$

$=\tan^{-1}\left(\frac{1}{1+\left(r+\frac{1}{2}\right)\left(r-\frac{1}{2}\right)}\right)$

$=\tan^{-1}\left(\frac{\left(r+\frac{1}{2}\right)-\left(r-\frac{1}{2}\right)}{1+\left(r^2+\frac{1}{4}\right)}\right)$

$=\tan^{-1}\left(r+\frac{1}{2}\right)-\tan^{-1}\left(r-\frac{1}{2}\right)$

80. $\lim\limits_{x\to\infty}\sqrt[3]{x}\left(\sqrt[3]{(x+1)^2}-\sqrt[3]{(x-1)^2}\right)=$

1) $\frac{1}{3}$ 2) $\frac{2}{3}$ 3) 1 4) $\frac{4}{3}$

Key. 4

Sol. $\lim\limits_{x\to\infty} x^{1/3}\left\{(x+1)^{1/3}+(x-1)^{1/3}\right\}\left\{(x+1)^{1/3}-(x-1)^{1/3}\right\}$

Rationalise $\lim\limits_{x\to\infty}\frac{x^{1/3}\left\{(x+1)^{1/3}+(x-1)^{1/3}\right\}2}{\left\{(x+1)^{2/3}+(x^2-1)^{1/3}+(x-1)^{2/3}\right\}}$

$$\lim_{x\to\infty}\frac{2.x^{2/3}\left\{\left(1+\frac{1}{x}\right)^{1/3}+\left(1-\frac{1}{x}\right)^{1/3}\right\}2}{x^{2/3}\left\{\left(1+\frac{1}{x}\right)^{2/3}+\left(1-\frac{1}{x}\right)^{1/3}+\left(1-\frac{1}{x}\right)^{2/3}\right\}}=\frac{2\text{x}2}{3}=\frac{4}{3}$$

81. If $a>0,\ b>0$ then $\lim\limits_{n\to\infty}\left(\frac{a-1+b^{\frac{1}{n}}}{a}\right)^{n}=$

1) $b^{\frac{1}{a}}$ 2) $a^{\frac{1}{b}}$ 3) a^{b} 4) b^{a}

Key. 1

Sol. Let $\frac{1}{n}=x,\quad \Rightarrow x\to 0\ \ as\ \ n\to\infty$ then required limit $\underset{x\to 0}{Lt}\left(\frac{a-1+b^{x}}{a}\right)^{\frac{1}{x}}=e^{\underset{x\to 0}{Lt}\frac{b^{x}-1}{x^{a}}}$

$$=e^{\frac{1}{a}\log e^{b}}=\left(b^{\frac{1}{a}}\right)$$

82. If $S_n=\frac{1}{1.2.3.4}+\frac{1}{2.3.4.5}+.........+\frac{1}{n(n+1)(n+2)(n+3)}$ then $\lim\limits_{n\to\infty}S_n=$

1) $\frac{5}{18}$ 2) $\frac{1}{9}$ 3) $\frac{7}{18}$ 4) $\frac{1}{18}$

Key. 4

Sol. $S_n=c-\frac{1}{(n+1)(n+2)(n+3).3}$

$n=1\Rightarrow s_1=c-\frac{1}{2.3.4.3}\Rightarrow c=\frac{1}{1.2.3.4}+\frac{1}{2.3.4.3}$

$$c=\frac{1}{2.3.4}\left(1+\frac{1}{3}\right)$$

$$=\frac{1}{18}$$

Now as $n\to\infty,\ S_n\to c=\frac{1}{18}$

83. $\lim\limits_{x\to\infty}\left(\frac{x^2+5x+3}{x^2+x+2}\right)^{x}=$

1) e^{2} 2) e^{4} 3) e^{3} 4) e

Key. 2

Sol. $$\lim_{x\to\infty}\left(\frac{x^2+5x+3}{x^2+x+2}\right)^{x}=e^{\lim\limits_{x\to 0}\left(\frac{4x+1}{x^2+x+2}\right)x}=e^{4}$$

84. $\lim\limits_{x\to\frac{-1}{3}^-}\frac{1}{x}\left[\frac{-1}{x}\right]$ [.] $\to$ denotes greatest integer function

1) -9 2) -12 3) -6 4) 0

Key. 3

Sol. $x < -\frac{1}{3}$

$\frac{1}{x} > -3 \Rightarrow -\frac{1}{x} < 3 \Rightarrow \left[-\frac{1}{3}\right] = 2$

$\lim_{x \to -\frac{1}{3}} \frac{1}{x}\left[-\frac{1}{x}\right] = (-3)(2) = -6$

85. $\lim_{x \to \infty}\left(x - \log_e(\cosh x)\right) =$

1) 1 2) 0 3) $\log_e 2$ 4) ∞

Key. 3

Sol. $\lim_{x \to \infty} x - \log_e\left(\frac{e^x + e^{-x}}{2}\right)$

$\lim_{x \to \infty} x - \log_e e^x\left(\frac{1+e^{-2x}}{2}\right)$

$\lim_{x \to \infty} x - x - \log_e\left(\frac{1+e^{-2x}}{2}\right)$

$\lim_{x \to \infty} -\log_e\left(\frac{1}{2}\right) = \log_e 2$

86. If f(x) = 0 be a quadratic equation such that $f(-\pi) = f(\pi) = 0$ and $f\left(\frac{\pi}{2}\right) = \frac{-3\pi^2}{4,}$ then

$\lim_{x \to -\pi} \frac{f(x)}{\sin(\sin x)}$ is equal to

a) 0 b) π c) $+2\pi$ d) None

Key. C

Sol. From given data $f(x) = x^2 - \pi^2$

$\lim_{x \to -\pi} \frac{x^2 - \pi^2}{-\sin(\sin x)} = 2\pi$.

$\lim_{h \to o} \frac{-2h\pi + h^2}{-\sin(\sinh)} = 2\pi$.

87. If the normal to the curve y = f(x) at x = 0 be given by the equation 3x – y + 1 = 0 then the value of $\lim_{x \to 0} x^2\left\{f(x^2) - 5f(4x^2) + 4f(7x^2)\right\}^{-1}$ is

(A) $\frac{1}{3}$ (B) $\frac{2}{3}$

(C) $-\frac{2}{3}$ (D) $-\frac{1}{3}$

Key. D

SOL. SLOPE OF TANGENT AT X = 0 IS $-\frac{1}{3}$

$\Rightarrow$ $f'(x) = -\frac{1}{3}$

$$\lim_{x\to 0}\frac{x^2}{f(x^2)-5f(4x^2)+4f(7x^2)} \div \text{(USE L.H. RULE)}$$

$$\Rightarrow \quad \lim_{x\to 0}\frac{1}{f'(x^2)-20f'(4x^2)+28f'(7x^2)}=-\frac{1}{3}$$

88. $f(x)$ is a polynomial function and $\left(f(\alpha)\right)^2+\left(f'(\alpha)\right)^2=0$ then the value of $\underset{x\to\alpha}{\text{lt}}\frac{f(x)}{f'(x)}\left[\frac{f'(x)}{f(x)}\right]$ (where $[.]$ denotes greatest integer function) is __

a) 0 b) 1 c) -1 d) 2

Key. B

Sol. Clearly, α is repeated root of $f(x)=0$

$$\underset{x\to\alpha}{\text{lt}}\frac{f(x)}{f'(x)}\left(\frac{f'(x)}{f(x)}-\left\{\frac{f'(x)}{f(x)}\right\}\right)\Rightarrow \underset{x\to\alpha}{\text{lt}}\left(1-\frac{f(x)}{f'(x)}\left\{\frac{f'(x)}{f(x)}\right\}\right)$$

$\left(\underset{x\to\alpha}{\text{lt}}\frac{f(x)}{f'(x)}=0\ \&\ \left\{\frac{f'(x)}{f(x)}\right\}\right.$ is bounded function)

89. $\lim_{x\to a^-}\left(\frac{|x|^3}{a}-\left[\frac{x}{a}\right]^3\right)(a>0)$, [.] GIF, is

A) a^2-2 B) a^2-1 C) a^2 D) a^2+1

Key. C

Sol. For $a-1<x<a\Rightarrow\left[\frac{x}{a}\right]=0$

$$\lim_{x\to a^-}\left(\frac{|x|^3}{a}-0\right)=\frac{a^3}{a}=a^2$$

90. $\underset{x\to\frac{\pi}{2}}{Lt}\frac{\sin x-(\sin x)^{\sin x}}{1-\sin x+\ln(\sin x)}=$

(A) 1 (B) 0 (C) 2 (D) – 1

Key. C

Sol. $\underset{x\to 1}{Lt}\frac{t-t^t}{1-t+\log t}$

91. $\underset{x\to 1}{Lt}\left(\tan^{-1}x.\frac{4}{\pi}\right)^{\frac{1}{x^2-1}}=$

(A) e^{π} (B) $e^{\frac{1}{\pi}}$ (C) $\frac{1}{e^{\pi}}$ (D) $e^{-\frac{1}{\pi}}$

Key. B

Sol. $e^{\underset{x\to\pi}{Lt}\left(\frac{4}{\pi}\tan^{-1}x-1\right)\frac{1}{x^2-1}}$

92. Value of $f\left(\frac{\pi}{2}\right)$ so that the function is continuous at $x=\frac{\pi}{2}$ is, if

$$f(x)=\frac{(1-\sin x)\ln\sin x}{(\pi-2x)^2\ln\left(1+\pi^2-4\pi x+4x^2\right)}$$

a) $\frac{1}{8}$ b) $\frac{1}{16}$ c) $-\frac{1}{32}$ d) $-\frac{1}{64}$

Key. D

Sol. Put $x=\frac{\pi}{2}+h$

$$\Rightarrow \underset{h\to 0}{Lt}\frac{(1-\cosh)\ln(\cosh)}{4h^2\ln\left(1+4h^2\right)}$$

Simplify to get $-\frac{1}{64}$

93. S_1 : If $\underset{x\to a}{Lim}f(x)+g(x)$ and $\underset{x\to a}{Lim}f(x)-g(x)$ exist : then it is not necessary that $\underset{x\to a}{Lim}f(x)$ and $\underset{x\to a}{Lim}g(x)$ both exist separately

S_2 : If $\underset{x\to a}{Lim}f(x)g(x)$ exists then it is necessary that $\underset{x\to a}{Lim}f(x)$ and $\underset{x\to a}{Lim}g(x)$ both exist separately

S_3 : $\underset{x\to a}{Lim}\left(f(x)\right)^{g(x)}=e^{\underset{x\to a}{Lim}g(x)(f(x)-1)}$

S_4 : $\underset{x\to 0^+}{Lt}\frac{e^{x\ln x}-e^{[\cos x]}}{x\ln x}=1$, where [ ] represents greatest integer function state in order, whether S_1,S_2,S_3,S_4 are true or false.

a) FTTT b) FFFF c) TTTT d) FFFT

Key. D

Sol. S_3 is applied only for form $(\to 1)^{\infty}$

94. $\underset{n\to\infty}{Lt}\frac{2^3-1^3}{2^3+1^3}\cdot\frac{3^3-1^3}{3^3+1^3}\,..........\,\frac{n^3-1^3}{n^3+1^3}$ is equal to

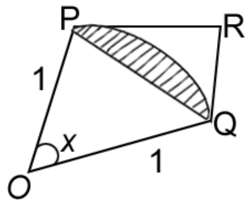
a) $\frac{1}{3}$ b) $\frac{1}{2}$ c) $\frac{2}{3}$ d) None of these

Key. C

Sol. Conceptual

95.

A circular arc of radius '1' subtends an angle of '*x*' radians, $0 < x < \frac{\pi}{2}$ as shown in the figure. The point 'R' is the point of intersection of the two tangent lines at P & Q. Let T(x) be the area of triangle PQR and S(*x*) be area of the shaded region. Then $\underset{x\to 0}{Lim}\frac{T(x)}{S(x)} =$



a) 2 b) $\frac{1}{2}$ c) $\frac{3}{4}$ d) $\frac{3}{2}$

Key. D

Sol. $T(x) = \frac{1}{2}.PR.RQ\sin(\pi - x)$

$= \frac{1}{2}\left(\tan^2\frac{x}{2}\right).\sin x = \tan\frac{x}{2} - \frac{\sin x}{2}$

$s(x) =$ **area of sector OPQ – area of** ΔOPQ

$= \frac{1}{2}(1)^2.x - \frac{1}{2}(1)^2\sin x$

$\underset{x\to 0}{Lim}\frac{\tan\frac{x}{2} - \sin\frac{x}{2}}{\frac{x - \sin x}{2}} = \frac{3}{2}$

96. $\underset{x\to 0}{Lt}\left(\frac{\sin hx}{x}\right)^{\frac{1}{x^2}}$

(a) $e^{\frac{1}{2}}$ (b) 1 (c) $e^{\frac{1}{6}}$ (d) $e^{\frac{1}{3}}$

Key. C

Sol. $Let\, l = \underset{x\to 0}{Lt}\left(\frac{\sin hx}{x}\right)^{\frac{1}{x^2}}$

$\log l = \underset{x\to 0}{Lt}\frac{1}{x^2}\log\left(\frac{\sin hx}{x}\right)$ by L' Hospital Rule $\Rightarrow l = e^{\frac{1}{6}}$