

DETERMINANTS

4.1 Overview

To every square matrix $A = [a_{ij}]$ of order n , we can associate a number (real or complex) called determinant of the matrix A , written as $\det A$, where a_{ij} is the (i, j) th element of A .

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then determinant of A , denoted by $|A|$ (or $\det A$), is given by

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Remarks

- (i) Only square matrices have determinants.
- (ii) For a matrix A , $|A|$ is read as determinant of A and not, as modulus of A .

4.1.1 Determinant of a matrix of order one

Let $A = [a]$ be the matrix of order 1, then determinant of A is defined to be equal to a .

4.1.2 Determinant of a matrix of order two

Let $A = [a_{ij}] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix of order 2. Then the determinant of A is defined as: $\det(A) = |A| = ad - bc$.

4.1.3 Determinant of a matrix of order three

The determinant of a matrix of order three can be determined by expressing it in terms of second order determinants which is known as expansion of a determinant along a row (or a column). There are six ways of expanding a determinant of order 3 corresponding to each of three rows (R_1, R_2 and R_3) and three columns (C_1, C_2 and C_3) and each way gives the same value.

Consider the determinant of a square matrix $A = [a_{ij}]_{3 \times 3}$, i.e.,

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expanding $|A|$ along C_1 , we get

$$\begin{aligned} |A| &= a_{11} (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{21} (-1)^{2+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} (-1)^{3+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ &= a_{11}(a_{22} a_{33} - a_{23} a_{32}) - a_{21}(a_{12} a_{33} - a_{13} a_{32}) + a_{31}(a_{12} a_{23} - a_{13} a_{22}) \end{aligned}$$

Remark In general, if $A = kB$, where A and B are square matrices of order n , then $|A| = k^n |B|$, $n = 1, 2, 3$.

4.1.4 Properties of Determinants

For any square matrix A , $|A|$ satisfies the following properties.

- (i) $|A'| = |A|$, where $A' =$ transpose of matrix A .
- (ii) If we interchange any two rows (or columns), then sign of the determinant changes.
- (iii) If any two rows or any two columns in a determinant are identical (or proportional), then the value of the determinant is zero.
- (iv) Multiplying a determinant by k means multiplying the elements of only one row (or one column) by k .
- (v) If we multiply each element of a row (or a column) of a determinant by constant k , then value of the determinant is multiplied by k .
- (vi) If elements of a row (or a column) in a determinant can be expressed as the sum of two or more elements, then the given determinant can be expressed as the sum of two or more determinants.

- (vii) If to each element of a row (or a column) of a determinant the equimultiples of corresponding elements of other rows (columns) are added, then value of determinant remains same.

Notes:

- (i) If all the elements of a row (or column) are zeros, then the value of the determinant is zero.
- (ii) If value of determinant ' Δ ' becomes zero by substituting $x = \alpha$, then $x - \alpha$ is a factor of ' Δ '.
- (iii) If all the elements of a determinant above or below the main diagonal consists of zeros, then the value of the determinant is equal to the product of diagonal elements.

4.1.5 Area of a triangle

Area of a triangle with vertices (x_1, y_1) , (x_2, y_2) and (x_3, y_3) is given by

$$\Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

4.1.6 Minors and co-factors

- (i) Minor of an element a_{ij} of the determinant of matrix A is the determinant obtained by deleting i^{th} row and j^{th} column, and it is denoted by M_{ij} .
- (ii) Co-factor of an element a_{ij} is given by $A_{ij} = (-1)^{i+j} M_{ij}$.
- (iii) Value of determinant of a matrix A is obtained by the sum of products of elements of a row (or a column) with corresponding co-factors. For example

$$|A| = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}.$$

- (iv) If elements of a row (or column) are multiplied with co-factors of elements of any other row (or column), then their sum is zero. For example,

$$a_{11} A_{21} + a_{12} A_{22} + a_{13} A_{23} = 0.$$

4.1.7 Adjoint and inverse of a matrix

- (i) The adjoint of a square matrix $A = [a_{ij}]_{n \times n}$ is defined as the transpose of the matrix

$[a_{ij}]_{n \times n}$, where A_{ij} is the co-factor of the element a_{ij} . It is denoted by $adj A$.

$$\text{If } A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \text{ then } adj A = \begin{vmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{vmatrix}, \text{ where } A_{ij} \text{ is co-factor of } a_{ij}.$$

- (ii) $A (adj A) = (adj A) A = |A| I$, where A is square matrix of order n .
- (iii) A square matrix A is said to be singular or non-singular according as $|A| = 0$ or $|A| \neq 0$, respectively.
- (iv) If A is a square matrix of order n , then $|adj A| = |A|^{n-1}$.
- (v) If A and B are non-singular matrices of the same order, then AB and BA are also nonsingular matrices of the same order.
- (vi) The determinant of the product of matrices is equal to product of their respective determinants, that is, $|AB| = |A| |B|$.
- (vii) If $AB = BA = I$, where A and B are square matrices, then B is called inverse of A and is written as $B = A^{-1}$. Also $B^{-1} = (A^{-1})^{-1} = A$.
- (viii) A square matrix A is invertible if and only if A is non-singular matrix.
- (ix) If A is an invertible matrix, then $A^{-1} = \frac{1}{|A|} (adj A)$

4.1.8 System of linear equations

- (i) Consider the equations:

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3, \end{aligned}$$

In matrix form, these equations can be written as $A X = B$, where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

- (ii) Unique solution of equation $AX = B$ is given by $X = A^{-1}B$, where $|A| \neq 0$.

- (iii) A system of equations is consistent or inconsistent according as its solution exists or not.
- (iv) For a square matrix A in matrix equation $AX = B$
- If $|A| \neq 0$, then there exists unique solution.
 - If $|A| = 0$ and $(adj A) B \neq 0$, then there exists no solution.
 - If $|A| = 0$ and $(adj A) B = 0$, then system may or may not be consistent.

4.2 Solved Examples

Short Answer (S.A.)

Example 1 If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & 5 \\ 8 & 3 \end{vmatrix}$, then find x .

Solution We have $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & 5 \\ 8 & 3 \end{vmatrix}$. This gives

$$2x^2 - 40 = 18 - 40 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3.$$

Example 2 If $\Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$, $\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ yz & zx & xy \\ x & y & z \end{vmatrix}$, then prove that $\Delta + \Delta_1 = 0$.

Solution We have $\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ yz & zx & xy \\ x & y & z \end{vmatrix}$

Interchanging rows and columns, we get

$$\Delta_1 = \begin{vmatrix} 1 & yz & x \\ 1 & zx & y \\ 1 & xy & z \end{vmatrix} = \frac{1}{xyz} \begin{vmatrix} x & xyz & x^2 \\ y & xyz & y^2 \\ z & xyz & z^2 \end{vmatrix}$$

$$= \frac{xyz}{xyz} \begin{vmatrix} x & 1 & x^2 \\ y & 1 & y^2 \\ z & 1 & z^2 \end{vmatrix} \quad \text{Interchanging } C_1 \text{ and } C_2$$

$$= (-1) \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = -\Delta$$

$$\Rightarrow \Delta_1 + \Delta = 0$$

Example 3 Without expanding, show that

$$\Delta = \begin{vmatrix} \operatorname{cosec}^2\theta & \cot^2\theta & 1 \\ \cot^2\theta & \operatorname{cosec}^2\theta & -1 \\ 42 & 40 & 2 \end{vmatrix} = 0.$$

Solution Applying $C_1 \rightarrow C_1 - C_2 - C_3$, we have

$$\Delta = \begin{vmatrix} \operatorname{cosec}^2\theta - \cot^2\theta - 1 & \cot^2\theta & 1 \\ \cot^2\theta - \operatorname{cosec}^2\theta + 1 & \operatorname{cosec}^2\theta & -1 \\ 0 & 40 & 2 \end{vmatrix} = \begin{vmatrix} 0 & \cot^2\theta & 1 \\ 0 & \operatorname{cosec}^2\theta & -1 \\ 0 & 40 & 2 \end{vmatrix} = 0$$

Example 4 Show that $\Delta = \begin{vmatrix} x & p & q \\ p & x & q \\ q & q & x \end{vmatrix} = (x-p)(x^2 + px - 2q^2)$

Solution Applying $C_1 \rightarrow C_1 - C_2$, we have

$$\Delta = \begin{vmatrix} x-p & p & q \\ p-x & x & q \\ 0 & q & x \end{vmatrix} = (x-p) \begin{vmatrix} 1 & p & q \\ -1 & x & q \\ 0 & q & x \end{vmatrix}$$

$$=(x-p) \begin{vmatrix} 0 & p+x & 2q \\ -1 & x & q \\ 0 & q & x \end{vmatrix} \quad \text{Applying } R_1 \rightarrow R_1 + R_2$$

Expanding along C_1 , we have

$$\Delta = (x-p)(px + x^2 - 2q^2) = (x-p)(x^2 + px - 2q^2)$$

Example 5 If $\Delta = \begin{vmatrix} 0 & b-a & c-a \\ a-b & 0 & c-b \\ a-c & b-c & 0 \end{vmatrix}$, then show that Δ is equal to zero.

Solution Interchanging rows and columns, we get $\Delta = \begin{vmatrix} 0 & a-b & a-c \\ b-a & 0 & b-c \\ c-a & c-b & 0 \end{vmatrix}$

Taking '-1' common from R_1, R_2 and R_3 , we get

$$\Delta = (-1)^3 \begin{vmatrix} 0 & b-a & c-a \\ a-b & 0 & c-b \\ a-c & b-c & 0 \end{vmatrix} = -\Delta$$

$$\Rightarrow 2\Delta = 0 \quad \text{or} \quad \Delta = 0$$

Example 6 Prove that $(A^{-1})' = (A')^{-1}$, where A is an invertible matrix.

Solution Since A is an invertible matrix, so it is non-singular.

We know that $|A| = |A'|$. But $|A| \neq 0$. So $|A'| \neq 0$ i.e. A' is invertible matrix.

Now we know that $AA^{-1} = A^{-1}A = I$.

Taking transpose on both sides, we get $(A^{-1})' A' = A' (A^{-1})' = (I)' = I$

Hence $(A^{-1})'$ is inverse of A' , i.e., $(A')^{-1} = (A^{-1})'$

Long Answer (L.A.)

Example 7 If $x = -4$ is a root of $\Delta = \begin{vmatrix} x & 2 & 3 \\ 1 & x & 1 \\ 3 & 2 & x \end{vmatrix} = 0$, then find the other two roots.

Solution Applying $R_1 \rightarrow (R_1 + R_2 + R_3)$, we get

$$\begin{vmatrix} x+4 & x+4 & x+4 \\ 1 & x & 1 \\ 3 & 2 & x \end{vmatrix}.$$

Taking $(x + 4)$ common from R_1 , we get

$$\Delta = (x+4) \begin{vmatrix} 1 & 1 & 1 \\ 1 & x & 1 \\ 3 & 2 & x \end{vmatrix}$$

Applying $C_2 \rightarrow C_2 - C_1$, $C_3 \rightarrow C_3 - C_1$, we get

$$\Delta = (x+4) \begin{vmatrix} 1 & 0 & 0 \\ 1 & x-1 & 0 \\ 3 & -1 & x-3 \end{vmatrix}.$$

Expanding along R_1 ,

$$\Delta = (x+4) [(x-1)(x-3) - 0]. \text{ Thus, } \Delta = 0 \text{ implies}$$

$$x = -4, 1, 3$$

Example 8 In a triangle ABC, if

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A + \sin^2 A & \sin B + \sin^2 B & \sin C + \sin^2 C \end{vmatrix} = 0,$$

then prove that ΔABC is an isosceles triangle.

Solution Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A + \sin^2 A & \sin B + \sin^2 B & \sin C + \sin^2 C \end{vmatrix}$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 + \sin A & 1 + \sin B & 1 + \sin C \\ -\cos^2 A & -\cos^2 B & -\cos^2 C \end{vmatrix} \quad R_3 \rightarrow R_3 - R_2$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1 + \sin A & \sin B - \sin A & \sin C - \sin B \\ -\cos^2 A & \cos^2 A - \cos^2 B & \cos^2 B - \cos^2 C \end{vmatrix} \cdot (C_3 \rightarrow C_3 - C_2 \text{ and } C_2 \rightarrow C_2 - C_1)$$

Expanding along R_1 , we get

$$\Delta = (\sin B - \sin A)(\sin^2 C - \sin^2 B) - (\sin C - \sin B)(\sin^2 B - \sin^2 A)$$

$$= (\sin B - \sin A)(\sin C - \sin B)(\sin C - \sin A) = 0$$

$$\Rightarrow \text{either } \sin B - \sin A = 0 \text{ or } \sin C - \sin B \text{ or } \sin C - \sin A = 0$$

$$\Rightarrow A = B \text{ or } B = C \text{ or } C = A$$

i.e. triangle ABC is isosceles.

Example 9 Show that if the determinant $\Delta = \begin{vmatrix} 3 & -2 & \sin 3\theta \\ -7 & 8 & \cos 2\theta \\ -11 & 14 & 2 \end{vmatrix} = 0$, then $\sin \theta = 0$ or $\frac{1}{2}$.

Solution Applying $R_2 \rightarrow R_2 + 4R_1$ and $R_3 \rightarrow R_3 + 7R_1$, we get

$$\begin{vmatrix} 3 & -2 & \sin 3\theta \\ 5 & 0 & \cos 2\theta + 4\sin 3\theta \\ 10 & 0 & 2 + 7\sin 3\theta \end{vmatrix} = 0$$

$$\text{or } 2 [5(2 + 7\sin 3\theta) - 10(\cos 2\theta + 4\sin 3\theta)] = 0$$

$$\text{or } 2 + 7\sin 3\theta - 2\cos 2\theta - 8\sin 3\theta = 0$$

$$\text{or } 2 - 2\cos 2\theta - \sin 3\theta = 0$$

$$\sin \theta (4\sin^2 \theta + 4\sin \theta - 3) = 0$$

or $\sin\theta = 0$ or $(2\sin\theta - 1) = 0$ or $(2\sin\theta + 3) = 0$

or $\sin\theta = 0$ or $\sin\theta = \frac{1}{2}$ (Why?).

Objective Type Questions

Choose the correct answer from the given four options in each of the Example 10 and 11.

Example 10 Let $\Delta = \begin{vmatrix} Ax & x^2 & 1 \\ By & y^2 & 1 \\ Cz & z^2 & 1 \end{vmatrix}$ and $\Delta_1 = \begin{vmatrix} A & B & C \\ x & y & z \\ zy & zx & xy \end{vmatrix}$, then

(A) $\Delta_1 = -\Delta$

(B) $\Delta \neq \Delta_1$

(C) $\Delta - \Delta_1 = 0$

(D) None of these

Solution (C) is the correct answer since $\Delta_1 = \begin{vmatrix} A & B & C \\ x & y & z \\ zy & zx & xy \end{vmatrix} = \begin{vmatrix} A & x & yz \\ B & y & zx \\ C & z & xy \end{vmatrix}$

$$= \frac{1}{xyz} \begin{vmatrix} Ax & x^2 & xyz \\ By & y^2 & xyz \\ Cz & z^2 & xyz \end{vmatrix} = \frac{xyz}{xyz} \begin{vmatrix} Ax & x^2 & 1 \\ By & y^2 & 1 \\ Cz & z^2 & 1 \end{vmatrix} = \Delta$$

Example 11 If $x, y \in \mathbf{R}$, then the determinant $\Delta = \begin{vmatrix} \cos x & -\sin x & 1 \\ \sin x & \cos x & 1 \\ \cos(x+y) & -\sin(x+y) & 0 \end{vmatrix}$ lies

in the interval

(A) $[-\sqrt{2}, \sqrt{2}]$

(B) $[-1, 1]$

(C) $[-\sqrt{2}, 1]$

(D) $[-1, -\sqrt{2},]$

Solution The correct choice is A. Indeed applying $R_3 \rightarrow R_3 - \cos y R_1 + \sin y R_2$, we get

$$\Delta = \begin{vmatrix} \cos x & -\sin x & 1 \\ \sin x & \cos x & 1 \\ 0 & 0 & \sin y - \cos y \end{vmatrix}.$$

Expanding along R_3 , we have

$$\begin{aligned} \Delta &= (\sin y - \cos y) (\cos^2 x + \sin^2 x) \\ &= (\sin y - \cos y) = \sqrt{2} \left[\frac{1}{\sqrt{2}} \sin y - \frac{1}{\sqrt{2}} \cos y \right] \\ &= \sqrt{2} \left[\cos \frac{\pi}{4} \sin y - \sin \frac{\pi}{4} \cos y \right] = \sqrt{2} \sin \left(y - \frac{\pi}{4} \right) \end{aligned}$$

Hence $-\sqrt{2} \leq \Delta \leq \sqrt{2}$.

Fill in the blanks in each of the Examples 12 to 14.

Example 12 If A, B, C are the angles of a triangle, then

$$\Delta = \begin{vmatrix} \sin^2 A & \cot A & 1 \\ \sin^2 B & \cot B & 1 \\ \sin^2 C & \cot C & 1 \end{vmatrix} = \dots\dots\dots$$

Solution Answer is 0. Apply $R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 - R_1$.

Example 13 The determinant $\Delta = \begin{vmatrix} \sqrt{23} + \sqrt{3} & \sqrt{5} & \sqrt{5} \\ \sqrt{15} + \sqrt{46} & 5 & \sqrt{10} \\ 3 + \sqrt{115} & \sqrt{15} & 5 \end{vmatrix}$ is equal to

Solution Answer is 0. Taking $\sqrt{5}$ common from C_2 and C_3 and applying $C_1 \rightarrow C_1 - \sqrt{3} C_2$, we get the desired result.

Example 14 The value of the determinant

$$\Delta = \begin{vmatrix} \sin^2 23^\circ & \sin^2 67^\circ & \cos 180^\circ \\ -\sin^2 67^\circ & -\sin^2 23^\circ & \cos^2 180^\circ \\ \cos 180^\circ & \sin^2 23^\circ & \sin^2 67^\circ \end{vmatrix} = \dots\dots\dots$$

Solution $\Delta = 0$. Apply $C_1 \rightarrow C_1 + C_2 + C_3$.

State whether the statements in the Examples 15 to 18 is **True** or **False**.

Example 15 The determinant

$$\Delta = \begin{vmatrix} \cos(x+y) & -\sin(x+y) & \cos 2y \\ \sin x & \cos x & \sin y \\ -\cos x & \sin x & \cos y \end{vmatrix}$$

is independent of x only.

Solution True. Apply $R_1 \rightarrow R_1 + \sin y R_2 + \cos y R_3$, and expand

Example 16 The value of

$$\begin{vmatrix} 1 & 1 & 1 \\ {}^n C_1 & {}^{n+2} C_1 & {}^{n+4} C_1 \\ {}^n C_2 & {}^{n+2} C_2 & {}^{n+4} C_2 \end{vmatrix} \text{ is } 8.$$

Solution True

Example 17 If $A = \begin{bmatrix} x & 5 & 2 \\ 2 & y & 3 \\ 1 & 1 & z \end{bmatrix}$, $xyz = 80$, $3x + 2y + 10z = 20$, then

$$A \text{ adj. } A = \begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix}.$$

Solution : False.

Example 18 If $A = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 2 & x \\ 2 & 3 & 1 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} \frac{1}{2} & -4 & \frac{5}{2} \\ -\frac{1}{2} & 3 & -\frac{3}{2} \\ \frac{1}{2} & y & \frac{1}{2} \end{bmatrix}$

then $x = 1, y = -1$.

Solution True

4.3 EXERCISE

Short Answer (S.A.)

Using the properties of determinants in Exercises 1 to 6, evaluate:

1. $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

2. $\begin{vmatrix} a + x & y & z \\ x & a + y & z \\ x & y & a + z \end{vmatrix}$

3. $\begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & zy^2 & 0 \end{vmatrix}$

4. $\begin{vmatrix} 3x & -x + y & -x + z \\ x - y & 3y & z - y \\ x - z & y - z & 3z \end{vmatrix}$

5. $\begin{vmatrix} x + 4 & x & x \\ x & x + 4 & x \\ x & x & x + 4 \end{vmatrix}$

6. $\begin{vmatrix} a - b - c & 2a & 2a \\ 2b & b - c - a & 2b \\ 2c & 2c & c - a - b \end{vmatrix}$

Using the properties of determinants in Exercises 7 to 9, prove that:

7. $\begin{vmatrix} y^2z^2 & yz & y + z \\ z^2x^2 & zx & z + x \\ x^2y^2 & xy & x + y \end{vmatrix} = 0$

8. $\begin{vmatrix} y + z & z & y \\ z & z + x & x \\ y & x & x + y \end{vmatrix} = 4xyz$

9.
$$\begin{vmatrix} a^2 + 2a & 2a + 1 & 1 \\ 2a + 1 & a + 2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a - 1)^3$$

10. If $A + B + C = 0$, then prove that
$$\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix} = 0$$

11. If the co-ordinates of the vertices of an equilateral triangle with sides of length

' a ' are $(x_1, y_1), (x_2, y_2), (x_3, y_3)$, then
$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = \frac{3a^4}{4}.$$

12. Find the value of θ satisfying
$$\begin{bmatrix} 1 & 1 & \sin 3\theta \\ -4 & 3 & \cos 2\theta \\ 7 & -7 & -2 \end{bmatrix} = 0.$$

13. If
$$\begin{vmatrix} 4 - x & 4 + x & 4 + x \\ 4 + x & 4 - x & 4 + x \\ 4 + x & 4 + x & 4 - x \end{vmatrix} = 0,$$
 then find values of x .

14. If $a_1, a_2, a_3, \dots, a_r$ are in G.P., then prove that the determinant

$$\begin{vmatrix} a_{r+1} & a_{r+5} & a_{r+9} \\ a_{r+7} & a_{r+11} & a_{r+15} \\ a_{r+11} & a_{r+17} & a_{r+21} \end{vmatrix}$$
 is independent of r .

15. Show that the points $(a + 5, a - 4), (a - 2, a + 3)$ and (a, a) do not lie on a straight line for any value of a .

16. Show that the ΔABC is an isosceles triangle if the determinant

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 + \cos A & 1 + \cos B & 1 + \cos C \\ \cos^2 A + \cos A & \cos^2 B + \cos B & \cos^2 C + \cos C \end{vmatrix} = 0.$$

17. Find A^{-1} if $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and show that $A^{-1} = \frac{A^2 - 3I}{2}$.

Long Answer (L.A.)

18. If $A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{bmatrix}$, find A^{-1} .

Using A^{-1} , solve the system of linear equations $x - 2y = 10$, $2x - y - z = 8$, $-2y + z = 7$.

19. Using matrix method, solve the system of equations $3x + 2y - 2z = 3$, $x + 2y + 3z = 6$, $2x - y + z = 2$.

20. Given $A = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$, find BA and use this to solve the

system of equations $y + 2z = 7$, $x - y = 3$, $2x + 3y + 4z = 17$.

21. If $a + b + c \neq 0$ and $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$, then prove that $a = b = c$.

22. Prove that $\begin{vmatrix} bc - a^2 & ca - b^2 & ab - c^2 \\ ca - b^2 & ab - c^2 & bc - a^2 \\ ab - c^2 & bc - a^2 & ca - b^2 \end{vmatrix}$ is divisible by $a + b + c$ and find the

quotient.

23. If $x + y + z = 0$, prove that
$$\begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix} = xyz \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

Objective Type Questions (M.C.Q.)

Choose the correct answer from given four options in each of the Exercises from 24 to 37.

24. If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$, then value of x is

- (A) 3 (B) ± 3
(C) ± 6 (D) 6

25. The value of determinant
$$\begin{vmatrix} a-b & b+c & a \\ b-a & c+a & b \\ c-a & a+b & c \end{vmatrix}$$

- (A) $a^3 + b^3 + c^3$ (B) $3bc$
(C) $a^3 + b^3 + c^3 - 3abc$ (D) none of these

26. The area of a triangle with vertices $(-3, 0)$, $(3, 0)$ and $(0, k)$ is 9 sq. units. The value of k will be

- (A) 9 (B) 3
(C) -9 (D) 6

27. The determinant
$$\begin{vmatrix} b^2 - ab & b - c & bc - ac \\ ab - a^2 & a - b & b^2 - ab \\ bc - ac & c - a & ab - a^2 \end{vmatrix}$$
 equals

- (A) $abc(b-c)(c-a)(a-b)$ (B) $(b-c)(c-a)(a-b)$
(C) $(a+b+c)(b-c)(c-a)(a-b)$ (D) None of these

28. The number of distinct real roots of $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$ in the interval

$$-\frac{\pi}{4} \leq x \leq \frac{\pi}{4} \text{ is}$$

- (A) 0 (B) 2
(C) 1 (D) 3
29. If A, B and C are angles of a triangle, then the determinant

$$\begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix} \text{ is equal to}$$

- (A) 0 (B) -1
(C) 1 (D) None of these

30. Let $f(t) = \begin{vmatrix} \cos t & t & 1 \\ 2 \sin t & t & 2t \\ \sin t & t & t \end{vmatrix}$, then $\lim_{t \rightarrow 0} \frac{f(t)}{t^2}$ is equal to

- (A) 0 (B) -1
(C) 2 (D) 3

31. The maximum value of $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 + \sin \theta & 1 \\ 1 + \cos \theta & 1 & 1 \end{vmatrix}$ is (θ is real number)

- (A) $\frac{1}{2}$ (B) $\frac{\sqrt{3}}{2}$
(C) $\sqrt{2}$ (D) $\frac{2\sqrt{3}}{4}$

32. If $f(x) = \begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix}$, then
- (A) $f(a) = 0$ (B) $f(b) = 0$
 (C) $f(0) = 0$ (D) $f(1) = 0$
33. If $A = \begin{bmatrix} 2 & \lambda & -3 \\ 0 & 2 & 5 \\ 1 & 1 & 3 \end{bmatrix}$, then A^{-1} exists if
- (A) $\lambda = 2$ (B) $\lambda \neq 2$
 (C) $\lambda \neq -2$ (D) None of these
34. If A and B are invertible matrices, then which of the following is not correct?
- (A) $\text{adj } A = |A| \cdot A^{-1}$ (B) $\det(A)^{-1} = [\det(A)]^{-1}$
 (C) $(AB)^{-1} = B^{-1} A^{-1}$ (D) $(A + B)^{-1} = B^{-1} + A^{-1}$
35. If x, y, z are all different from zero and $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = 0$, then value of $x^{-1} + y^{-1} + z^{-1}$ is
- (A) $x y z$ (B) $x^{-1} y^{-1} z^{-1}$
 (C) $-x - y - z$ (D) -1
36. The value of the determinant $\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$ is
- (A) $9x^2(x+y)$ (B) $9y^2(x+y)$
 (C) $3y^2(x+y)$ (D) $7x^2(x+y)$

37. There are two values of a which makes determinant, $\Delta = \begin{vmatrix} 1 & -2 & 5 \\ 2 & a & -1 \\ 0 & 4 & 2a \end{vmatrix} = 86$, then

sum of these number is

- (A) 4 (B) 5
(C) -4 (D) 9

Fill in the blanks

38. If A is a matrix of order 3×3 , then $|3A| = \underline{\hspace{2cm}}$.

39. If A is invertible matrix of order 3×3 , then $|A^{-1}| = \underline{\hspace{2cm}}$.

40. If $x, y, z \in \mathbb{R}$, then the value of determinant $\begin{vmatrix} (2^x + 2^{-x})^2 & (2^x - 2^{-x})^2 & 1 \\ (3^x + 3^{-x})^2 & (3^x - 3^{-x})^2 & 1 \\ (4^x + 4^{-x})^2 & (4^x - 4^{-x})^2 & 1 \end{vmatrix}$ is

equal to $\underline{\hspace{2cm}}$.

41. If $\cos 2\theta = 0$, then $\begin{vmatrix} 0 & \cos \theta & \sin \theta \\ \cos \theta & \sin \theta & 0 \\ \sin \theta & 0 & \cos \theta \end{vmatrix}^2 = \underline{\hspace{2cm}}$.

42. If A is a matrix of order 3×3 , then $(A^2)^{-1} = \underline{\hspace{2cm}}$.

43. If A is a matrix of order 3×3 , then number of minors in determinant of A are $\underline{\hspace{2cm}}$.

44. The sum of the products of elements of any row with the co-factors of corresponding elements is equal to $\underline{\hspace{2cm}}$.

45. If $x = -9$ is a root of $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$, then other two roots are $\underline{\hspace{2cm}}$.

46. $\begin{vmatrix} 0 & xyz & x-z \\ y-x & 0 & y-z \\ z-x & z-y & 0 \end{vmatrix} = \underline{\hspace{2cm}}$.

47. If $f(x) = \begin{vmatrix} (1+x)^{17} & (1+x)^{19} & (1+x)^{23} \\ (1+x)^{23} & (1+x)^{29} & (1+x)^{34} \\ (1+x)^{41} & (1+x)^{43} & (1+x)^{47} \end{vmatrix} = A + Bx + Cx^2 + \dots$, then

A = _____.

State True or False for the statements of the following Exercises:

48. $(A^3)^{-1} = (A^{-1})^3$, where A is a square matrix and $|A| \neq 0$.

49. $(aA)^{-1} = \frac{1}{a}A^{-1}$, where a is any real number and A is a square matrix.

50. $|A^{-1}| \neq |A|^{-1}$, where A is non-singular matrix.

51. If A and B are matrices of order 3 and $|A| = 5$, $|B| = 3$, then $|3AB| = 27 \times 5 \times 3 = 405$.

52. If the value of a third order determinant is 12, then the value of the determinant formed by replacing each element by its co-factor will be 144.

53. $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = 0$, where a, b, c are in A.P.

54. $adj. A = |A|^2$, where A is a square matrix of order two.

55. The determinant $\begin{vmatrix} \sin A & \cos A & \sin A + \cos B \\ \sin B & \cos A & \sin B + \cos B \\ \sin C & \cos A & \sin C + \cos B \end{vmatrix}$ is equal to zero.

56. If the determinant $\begin{vmatrix} x+a & p+u & l+f \\ y+b & q+v & m+g \\ z+c & r+w & n+h \end{vmatrix}$ splits into exactly K determinants of

order 3, each element of which contains only one term, then the value of K is 8.

57. Let $\Delta = \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = 16$, then $\Delta_1 = \begin{vmatrix} p+x & a+x & a+p \\ q+y & b+y & b+q \\ r+z & c+z & c+r \end{vmatrix} = 32$.

58. The maximum value of $\begin{vmatrix} 1 & 1 & 1 \\ 1 & (1+\sin\theta) & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$ is $\frac{1}{2}$.



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