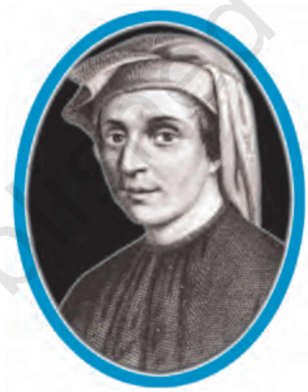
Chapter 9

SEQUENCES AND SERIES

❖*Natural numbers are the product of human spirit. – DEDEKIND* ❖

9.1 Introduction



Fibonacci
(1175-1250)

In mathematics, the word, "*sequence*" is used in much the same way as it is in ordinary English. When we say that a collection of objects is listed in a sequence, we usually mean that the collection is ordered in such a way that it has an identified first member, second member, third member and so on. For example, population of human beings or bacteria at different times form a sequence. The amount of money deposited in a bank, over a number of years form a sequence. Depreciated values of certain commodity occur in a sequence. Sequences have important applications in several spheres of human activities.

Sequences, following specific patterns are called *progressions.* In previous class, we have studied about *arithmetic progression* (A.P). In this Chapter, besides discussing more about A.P.; *arithmetic mean, geometric mean, relationship between A.M. and GM., special series in forms of sum to n terms of consecutive natural numbers, sum to n terms of squares of natural numbers and sum to n terms of cubes of natural numbers* will also be studied.

9.2 Sequences

Let us consider the following examples:

Assume that there is a generation gap of 30 years, we are asked to find the number of ancestors, i.e., parents, grandparents, great grandparents, etc. that a person might have over 300 years.

Here, the total number of generations $= \dfrac{300}{30} = 10$

The number of person's ancestors for the first, second, third, …, tenth generations are 2, 4, 8, 16, 32, …, 1024. These numbers form what we call a *sequence*.

Consider the successive quotients that we obtain in the division of 10 by 3 at different steps of division. In this process we get 3,3.3,3.33,3.333, ... and so on. These quotients also form a sequence. The various numbers occurring in a sequence are called its *terms*. We denote the terms of a sequence by a_1, a_2, a_3, …, a_n, …, etc., the subscripts denote the position of the term. The n^{th} term is the number at the n^{th} position of the sequence and is denoted by a_n. The n^{th} term is also called the *general term* of the sequence.

Thus, the terms of the sequence of person's ancestors mentioned above are:

$$a_1 = 2,\ a_2 = 4,\ a_3 = 8,\ \ldots,\ a_{10} = 1024.$$

Similarly, in the example of successive quotients

$$a_1 = 3,\ a_2 = 3.3,\ a_3 = 3.33,\ \ldots,\ a_6 = 3.33333, \text{ etc.}$$

A sequence containing finite number of terms is called a *finite sequence*. For example, sequence of ancestors is a finite sequence since it contains 10 terms (a fixed number).

A sequence is called *infinite*, if it is not a finite sequence. For example, the sequence of successive quotients mentioned above is an *infinite sequence*, infinite in the sense that it never ends.

Often, it is possible to express the rule, which yields the various terms of a sequence in terms of algebraic formula. Consider for instance, the sequence of even natural numbers 2, 4, 6, …

Here $a_1 = 2 = 2 \times 1 \qquad a_2 = 4 = 2 \times 2$

$a_3 = 6 = 2 \times 3 \qquad a_4 = 8 = 2 \times 4$

....

....

$a_{23} = 46 = 2 \times 23$, $a_{24} = 48 = 2 \times 24$, and so on.

In fact, we see that the n^{th} term of this sequence can be written as $a_n = 2n$, where n is a natural number. Similarly, in the sequence of odd natural numbers 1,3,5, …, the n^{th} term is given by the formula, $a_n = 2n - 1$, where n is a natural number.

In some cases, an arrangement of numbers such as 1, 1, 2, 3, 5, 8,.. has no visible pattern, but the sequence is generated by the recurrence relation given by

$$a_1 = a_2 = 1$$
$$a_3 = a_1 + a_2$$
$$a_n = a_{n-2} + a_{n-1},\ n > 2$$

This sequence is called *Fibonacci sequence*.

In the sequence of primes 2,3,5,7,…, we find that there is no formula for the n^{th} prime. Such sequence can only be described by verbal description.

In every sequence, we should not expect that its terms will necessarily be given by a specific formula. However, we expect a theoretical scheme or a rule for generating the terms $a_1, a_2, a_3,\ldots,a_n,\ldots$ in succession.

In view of the above, *a sequence can be regarded as a function whose domain is the set of natural numbers or some subset of it of the type* $\{1, 2, 3...k\}$. *Sometimes, we use the functional notation a(n) for* a_n.

9.3 Series

Let $a_1, a_2, a_3,\ldots,a_n$, be a given sequence. Then, the expression

$$a_1 + a_2 + a_3 +,\ldots + a_n + \ldots$$

is called the *series associated with the given sequence* .The series is finite or infinite according as the given sequence is finite or infinite. Series are often represented in compact form, called *sigma notation*, using the Greek letter $\sum$ (sigma) as means of indicating the summation involved. Thus, the series $a_1 + a_2 + a_3 + \ldots + a_n$ is abbreviated as $\sum_{k=1}^{n} a_k$.

Remark When the series is used, it refers to the indicated sum not to the sum itself. For example, 1 + 3 + 5 + 7 is a finite series with four terms. When we use the phrase "*sum of a series*," we will mean the number that results from adding the terms, the sum of the series is 16.

We now consider some examples.

Example 1 Write the first three terms in each of the following sequences defined by the following:

(i) $a_n = 2n + 5$, (ii) $a_n = \dfrac{n-3}{4}$.

Solution (i) Here $a_n = 2n + 5$

Substituting $n = 1, 2, 3$, we get

$$a_1 = 2(1) + 5 = 7,\ a_2 = 9,\ a_3 = 11$$

Therefore, the required terms are 7, 9 and 11.

(ii) Here $a_n = \dfrac{n-3}{4}$. Thus, $a_1 = \dfrac{1-3}{4} = -\dfrac{1}{2},\ a_2 = -\dfrac{1}{4},\ a_3 = 0$

Hence, the first three terms are $-\frac{1}{2}, -\frac{1}{4}$ and 0.

Example 2 What is the 20th term of the sequence defined by

$$a_n = (n - 1)(2 - n)(3 + n) ?$$

Solution Putting $n = 20$, we obtain

$$a_{20} = (20 - 1)(2 - 20)(3 + 20)$$
$$= 19 \times (-18) \times (23) = -7866.$$

Example 3 Let the sequence a_n be defined as follows:

$$a_1 = 1, \ a_n = a_{n-1} + 2 \text{ for } n \geq 2.$$

Find first five terms and write corresponding series.

Solution We have

$$a_1 = 1, a_2 = a_1 + 2 = 1 + 2 = 3, a_3 = a_2 + 2 = 3 + 2 = 5,$$
$$a_4 = a_3 + 2 = 5 + 2 = 7, a_5 = a_4 + 2 = 7 + 2 = 9.$$

Hence, the first five terms of the sequence are 1,3,5,7 and 9. The corresponding series is 1 + 3 + 5 + 7 + 9 +...

EXERCISE 9.1

Write the first five terms of each of the sequences in Exercises 1 to 6 whose n^{th} terms are:

1. $a_n = n(n + 2)$ **2.** $a_n = \frac{n}{n+1}$ **3.** $a_n = 2^n$

4. $a_n = \frac{2n-3}{6}$ **5.** $a_n = (-1)^{n-1} 5^{n+1}$ **6.** $a_n = n\frac{n^2+5}{4}$.

Find the indicated terms in each of the sequences in Exercises 7 to 10 whose n^{th} terms are:

7. $a_n = 4n - 3$; a_{17}, a_{24} **8.** $a_n = \frac{n^2}{2^n}$; a_7

9. $a_n = (-1)^{n-1} n^3$; a_9 **10.** $a_n = \frac{n(n-2)}{n+3}$; a_{20}.

Write the first five terms of each of the sequences in Exercises 11 to 13 and obtain the corresponding series:

11. $a_1 = 3, a_n = 3a_{n-1} + 2$ for all $n > 1$ **12.** $a_1 = -1, a_n = \frac{a_{n-1}}{n}, n \geq 2$

13. $a_1 = a_2 = 2, a_n = a_{n-1} - 1, n > 2$

14. The Fibonacci sequence is defined by

$$1 = a_1 = a_2 \text{ and } a_n = a_{n-1} + a_{n-2}, n > 2.$$

Find $\frac{a_{n+1}}{a_n}$, for $n = 1, 2, 3, 4, 5$

9.4 Arithmetic Progression (A.P.)

Let us recall some formulae and properties studied earlier.

A sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ is called *arithmetic sequence or arithmetic progression* if $a_{n+1} = a_n + d, n \in \mathbf{N}$, where a_1 is called the *first term* and the constant term d is called the *common difference* of the A.P.

Let us consider an A.P. (in its standard form) with first term a and common difference d, i.e., $a, a + d, a + 2d, \ldots$

Then the n^{th} term (*general term*) of the A.P. is $a_n = a + (n - 1) d$.

We can verify the following simple properties of an A.P. :

(i) If a constant is added to each term of an A.P., the resulting sequence is also an A.P.

(ii) If a constant is subtracted from each term of an A.P., the resulting sequence is also an A.P.

(iii) If each term of an A.P. is multiplied by a constant, then the resulting sequence is also an A.P.

(iv) If each term of an A.P. is divided by a non-zero constant then the resulting sequence is also an A.P.

Here, we shall use the following notations for an arithmetic progression:

a = the first term, l = the last term, d = common difference,

n = the number of terms.

S_n = the sum to n terms of A.P.

Let $a, a + d, a + 2d, \ldots, a + (n - 1) d$ be an A.P. Then

$$l = a + (n - 1) d$$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

We can also write, $S_n = \frac{n}{2}[a + l]$

Let us consider some examples.

Example 4 In an A.P. if m^{th} term is n and the n^{th} term is m, where $m \neq n$, find the pth term.

Solution We have $a_m = a + (m-1)\ d = n,$ ... (1)

and $a_n = a + (n-1)\ d = m$ ... (2)

Solving (1) and (2), we get

$(m - n)\ d = n - m$, or $d = -1$, ... (3)

and $a = n + m - 1$ ... (4)

Therefore $a_p = a + (p-1)d$

$= n + m - 1 + (p-1)(-1) = n + m - p$

Hence, the p^{th} term is $n + m - p$.

Example 5 If the sum of n terms of an A.P. is $n\text{P} + \frac{1}{2}n(n-1)\text{Q}$, where P and Q are constants, find the common difference.

Solution Let $a_1, a_2, \ldots a_n$ be the given A.P. Then

$$S_n = a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n = n\text{P} + \frac{1}{2}n\ (n-1)\ \text{Q}$$

Therefore $S_1 = a_1 = \text{P}$, $S_2 = a_1 + a_2 = 2\text{P} + \text{Q}$

So that $a_2 = S_2 - S_1 = \text{P} + \text{Q}$

Hence, the common difference is given by $d = a_2 - a_1 = (\text{P} + \text{Q}) - \text{P} = \text{Q}$.

Example 6 The sum of n terms of two arithmetic progressions are in the ratio $(3n + 8) : (7n + 15)$. Find the ratio of their 12^{th} terms.

Solution Let a_1, a_2 and d_1, d_2 be the first terms and common difference of the first and second arithmetic progression, respectively. According to the given condition, we have

$$\frac{\text{Sum to } n \text{ terms of first A.P.}}{\text{Sum to } n \text{ terms of second A.P.}} = \frac{3n+8}{7n+15}$$

or $$\frac{\frac{n}{2}[2a_1+(n-1)d_1]}{\frac{n}{2}[2a_2+(n-1)d_2]}=\frac{3n+8}{7n+15}$$

or $$\frac{2a_1+(n-1)d_1}{2a_2+(n-1)d_2}=\frac{3n+8}{7n+15} \qquad \text{... (1)}$$

Now $$\frac{12^{\text{th}}\text{ term of first A.P.}}{12^{\text{th}}\text{ term of second A.P}}=\frac{a_1+11d_1}{a_2+11d_2}$$

$$\frac{2a_1+22d_1}{2a_2+22d_2}=\frac{3\times 23+8}{7\times 23+15} \qquad \text{[By putting } n = 23 \text{ in (1)]}$$

Therefore $$\frac{a_1+11d_1}{a_2+11d_2}=\frac{12^{\text{th}}\text{ term of first A.P.}}{12^{\text{th}}\text{ term of second A.P.}}=\frac{7}{16}$$

Hence, the required ratio is 7 : 16.

Example 7 The income of a person is Rs. 3,00,000, in the first year and he receives an increase of Rs.10,000 to his income per year for the next 19 years. Find the total amount, he received in 20 years.

Solution Here, we have an A.P. with a = 3,00,000, d = 10,000, and n = 20.
Using the sum formula, we get,

$$S_{20}=\frac{20}{2}[600000+19\times 10000] = 10\,(790000) = 79{,}00{,}000.$$

Hence, the person received Rs. 79,00,000 as the total amount at the end of 20 years.

9.4.1 ***Arithmetic mean*** Given two numbers a and b. We can insert a number A between them so that a, A, b is an A.P. Such a number A is called the *arithmetic mean* (A.M.) of the numbers a and b. Note that, in this case, we have

$$A - a = b - A, \quad \text{i.e., } A = \frac{a+b}{2}$$

We may also interpret the A.M. between two numbers a and b as their average $\frac{a+b}{2}$. For example, the A.M. of two numbers 4 and 16 is 10. We have, thus constructed an A.P. 4, 10, 16 by inserting a number 10 between 4 and 16. The natural question now arises : Can we insert two or more numbers between given two numbers so that the resulting sequence comes out to be an A.P. ? Observe that two numbers 8 and 12 can be inserted between 4 and 16 so that the resulting sequence 4, 8, 12, 16 becomes an A.P.

More generally, given any two numbers a and b, we can insert as many numbers as we like between them such that the resulting sequence is an A.P.

Let $A_1, A_2, A_3, \ldots, A_n$ be n numbers between a and b such that $a, A_1, A_2, A_3, \ldots, A_n, b$ is an A.P.

Here, b is the $(n + 2)^{\text{th}}$ term, i.e., $b = a + [(n + 2) - 1]d = a + (n + 1) d$.

This gives $$d = \frac{b-a}{n+1}.$$

Thus, n numbers between a and b are as follows:

$$A_1 = a + d = a + \frac{b-a}{n+1}$$

$$A_2 = a + 2d = a + \frac{2(b-a)}{n+1}$$

$$A_3 = a + 3d = a + \frac{3(b-a)}{n+1}$$

....

....

$$A_n = a + nd = a + \frac{n(b-a)}{n+1}.$$

Example 8 Insert 6 numbers between 3 and 24 such that the resulting sequence is an A.P.

Solution Let A_1, A_2, A_3, A_4, A_5 and A_6 be six numbers between 3 and 24 such that

$3, A_1, A_2, A_3, A_4, A_5, A_6, 24$ are in A.P. Here, $a = 3$, $b = 24$, $n = 8$.

Therefore, $24 = 3 + (8 - 1) d$, so that $d = 3$.

Thus $A_1 = a + d = 3 + 3 = 6$; $A_2 = a + 2d = 3 + 2 \times 3 = 9$;

$A_3 = a + 3d = 3 + 3 \times 3 = 12$; $A_4 = a + 4d = 3 + 4 \times 3 = 15$;

$A_5 = a + 5d = 3 + 5 \times 3 = 18$; $A_6 = a + 6d = 3 + 6 \times 3 = 21$.

Hence, six numbers between 3 and 24 are 6, 9, 12, 15, 18 and 21.

EXERCISE 9.2

1. Find the sum of odd integers from 1 to 2001.
2. Find the sum of all natural numbers lying between 100 and 1000, which are multiples of 5.
3. In an A.P., the first term is 2 and the sum of the first five terms is one-fourth of the next five terms. Show that 20th term is –112.
4. How many terms of the A.P. $-6, -\frac{11}{2}, -5, \ldots$ are needed to give the sum –25?
5. In an A.P., if p^{th} term is $\frac{1}{q}$ and q^{th} term is $\frac{1}{p}$, prove that the sum of first pq terms is $\frac{1}{2}(pq+1)$, where $p \neq q$.
6. If the sum of a certain number of terms of the A.P. 25, 22, 19, … is 116. Find the last term.
7. Find the sum to n terms of the A.P., whose k^{th} term is $5k + 1$.
8. If the sum of n terms of an A.P. is $(pn + qn^2)$, where p and q are constants, find the common difference.
9. The sums of n terms of two arithmetic progressions are in the ratio $5n + 4 : 9n + 6$. Find the ratio of their 18th terms.
10. If the sum of first p terms of an A.P. is equal to the sum of the first q terms, then find the sum of the first $(p + q)$ terms.
11. Sum of the first p, q and r terms of an A.P. are a, b and c, respectively.

    Prove that $\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)=0$
12. The ratio of the sums of m and n terms of an A.P. is $m^2 : n^2$. Show that the ratio of m^{th} and n^{th} term is $(2m - 1) : (2n - 1)$.
13. If the sum of n terms of an A.P. is $3n^2 + 5n$ and its m^{th} term is 164, find the value of m.
14. Insert five numbers between 8 and 26 such that the resulting sequence is an A.P.
15. If $\frac{a^n+b^n}{a^{n-1}+b^{n-1}}$ is the A.M. between a and b, then find the value of n.
16. Between 1 and 31, m numbers have been inserted in such a way that the resulting sequence is an A. P. and the ratio of 7th and $(m-1)^{\text{th}}$ numbers is 5 : 9. Find the value of m.

17. A man starts repaying a loan as first instalment of Rs. 100. If he increases the instalment by Rs 5 every month, what amount he will pay in the 30th instalment?

18. The difference between any two consecutive interior angles of a polygon is 5°. If the smallest angle is 120° , find the number of the sides of the polygon.

9.5 Geometric Progression (G. P.)

Let us consider the following sequences:

$$\text{(i) } 2,4,8,16,\ldots, \text{ (ii) } \frac{1}{9}, \frac{-1}{27}, \frac{1}{81}, \frac{-1}{243} \ldots \text{ (iii) } .01,.0001,.000001,\ldots$$

In each of these sequences, how their terms progress? We note that each term, except the first progresses in a definite order.

In (i), we have and so on.

In (ii), we observe, $a_1 = \frac{1}{9}, \frac{a_2}{a_1} = \frac{-1}{3}, \frac{a_3}{a_2} = \frac{-1}{3}, \frac{a_4}{a_3} = \frac{-1}{3}$ and so on.

Similarly, state how do the terms in (iii) progress? It is observed that in each case, every term except the first term bears a constant ratio to the term immediately preceding

$$a_1 = 2, \frac{a_2}{a_1} = 2, \frac{a_3}{a_2} = 2, \frac{a_4}{a_3} = 2$$

it. In (i), this constant ratio is 2; in (ii), it is $-\frac{1}{3}$ and in (iii), the constant ratio is 0.01. Such sequences are called *geometric sequence* or *geometric progression* abbreviated as G.P.

A sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ is called *geometric progression*, if each term is non-zero and $\frac{a_{k+1}}{a_k} = r$ (constant), for $k \geq 1$.

By letting $a_1 = a$, we obtain a geometric progression, $a, ar, ar^2, ar^3, \ldots$, where a is called the *first term* and r is called the *common ratio* of the G.P. Common ratio in geometric progression (i), (ii) and (iii) above are 2, $-\frac{1}{3}$ and 0.01, respectively.

As in case of arithmetic progression, the problem of finding the n^{th} term or sum of n terms of a geometric progression containing a large number of terms would be difficult without the use of the formulae which we shall develop in the next Section. We shall use the following notations with these formulae:

a = the first term, r = the common ratio, l = the last term,

n = the numbers of terms,

S_n = the sum of first n terms.

9.5.1 General term of a G.P. Let us consider a G.P. with first non-zero term 'a' and common ratio 'r'. Write a few terms of it. The second term is obtained by multiplying a by r, thus $a_2 = ar$. Similarly, third term is obtained by multiplying a_2 by r. Thus, $a_3 = a_2r = ar^2$, and so on.

We write below these and few more terms.

1st term = $a_1 = a = ar^{1-1}$, 2nd term = $a_2 = ar = ar^{2-1}$, 3rd term = $a_3 = ar^2 = ar^{3-1}$
4th term = $a_4 = ar^3 = ar^{4-1}$, 5th term = $a_5 = ar^4 = ar^{5-1}$
Do you see a pattern? What will be 16th term?

$$a_{16} = ar^{16-1} = ar^{15}$$

Therefore, the pattern suggests that the n^{th} term of a G.P. is given by

$a_n = ar^{n-1}$.

Thus, a, G.P. can be written as $a, ar, ar^2, ar^3, \ldots ar^{n-1}$; $a, ar, ar^2, \ldots, ar^{n-1} \ldots$;according as G.P. is *finite* or *infinite*, respectively.

The series $a + ar + ar^2 + \ldots + ar^{n-1}$ or $a + ar + ar^2 + \ldots + ar^{n-1} + \ldots$ are called *finite* or *infinite geometric series*, respectively.

9.5.2. Sum to n terms of a G.P. Let the first term of a G.P. be a and the common ratio be r. Let us denote by S_n the sum to first n terms of G.P. Then

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1} \quad \ldots (1)$$

Case 1 If $r = 1$, we have $S_n = a + a + a + \ldots + a$ (n terms) $= na$

Case 2 If $r \neq 1$, multiplying (1) by r, we have

$$rS_n = ar + ar^2 + ar^3 + \ldots + ar^n \quad \ldots (2)$$

Subtracting (2) from (1), we get $(1 - r) S_n = a - ar^n = a(1 - r^n)$

This gives $S_n = \dfrac{a(1-r^n)}{1-r}$ or $S_n = \dfrac{a(r^n-1)}{r-1}$

Example 9 Find the 10th and nth terms of the G.P. 5, 25,125,… .

Solution Here $a = 5$ and $r = 5$. Thus, $a_{10} = 5(5)^{10-1} = 5(5)^9 = 5^{10}$
and $a_n = ar^{n-1} = 5(5)^{n-1} = 5^n$.

Example10 Which term of the G.P., 2,8,32, ... up to n terms is 131072?

Solution Let 131072 be the n^{th} term of the given G.P. Here $a = 2$ and $r = 4$.
Therefore $131072 = a_n = 2(4)^{n-1}$ or $65536 = 4^{n-1}$
This gives $4^8 = 4^{n-1}$.
So that $n - 1 = 8$, i.e., $n = 9$. Hence, 131072 is the 9th term of the G.P.

Example 11 In a G.P., the 3[rd] term is 24 and the 6[th] term is 192. Find the 10[th] term.

Solution Here, $a_3 = ar^2 = 24$ ... (1)

and $a_6 = ar^5 = 192$ ... (2)

Dividing (2) by (1), we get $r = 2$. Substituting $r = 2$ in (1), we get $a = 6$.

Hence $a_{10} = 6\,(2)^9 = 3072$.

Example 12 Find the sum of first n terms and the sum of first 5 terms of the geometric series $1+\frac{2}{3}+\frac{4}{9}+...$

Solution Here $a = 1$ and $r = \frac{2}{3}$. Therefore

$$S_n = \frac{a\,(1-r^n)}{1-r} = \frac{\left[1-\left(\frac{2}{3}\right)^n\right]}{1-\frac{2}{3}} = 3\left[1-\left(\frac{2}{3}\right)^n\right]$$

In particular, $S_5 = 3\left[1-\left(\frac{2}{3}\right)^5\right] = 3\times\frac{211}{243} = \frac{211}{81}$.

Example 13 How many terms of the G.P. $3, \frac{3}{2}, \frac{3}{4}, ...$ are needed to give the sum $\frac{3069}{512}$?

Solution Let n be the number of terms needed. Given that $a = 3$, $r = \frac{1}{2}$ and $S_n = \frac{3069}{512}$

Since $$S_n = \frac{a\,(1-r^n)}{1-r}$$

Therefore $$\frac{3069}{512} = \frac{3(1-\frac{1}{2^n})}{1-\frac{1}{2}} = 6\left(1-\frac{1}{2^n}\right)$$

or $$\frac{3069}{3072} = 1 - \frac{1}{2^n}$$

or $$\frac{1}{2^n} = 1 - \frac{3069}{3072} = \frac{3}{3072} = \frac{1}{1024}$$

or $2^n = 1024 = 2^{10}$, which gives $n = 10$.

Example 14 The sum of first three terms of a G.P. is $\frac{13}{12}$ and their product is – 1. Find the common ratio and the terms.

Solution Let $\frac{a}{r}$, a, ar be the first three terms of the G.P. Then

$$\frac{a}{r} + ar + a = \frac{13}{12} \qquad \text{... (1)}$$

and $$\left(\frac{a}{r}\right)(a)\,(ar) = -1 \qquad \text{... (2)}$$

From (2), we get $a^3 = -1$, i.e., $a = -1$ (considering only real roots)

Substituting $a = -1$ in (1), we have

$$-\frac{1}{r} - 1 - r = \frac{13}{12} \text{ or } 12r^2 + 25r + 12 = 0.$$

This is a quadratic in r, solving, we get $r = -\frac{3}{4}$ or $-\frac{4}{3}$.

Thus, the three terms of GP. are : $\frac{4}{3}, -1, \frac{3}{4}$ for $r = \frac{-3}{4}$ and $\frac{3}{4}, -1, \frac{4}{3}$ for $r = \frac{-4}{3}$,

Example15 Find the sum of the sequence 7, 77, 777, 7777, ... to n terms.

Solution This is not a GP., however, we can relate it to a GP. by writing the terms as

$$S_n = 7 + 77 + 777 + 7777 + \ldots \text{ to } n \text{ terms}$$

$$= \frac{7}{9}[9 + 99 + 999 + 9999 + \ldots \text{to } n \text{ term}]$$

$$= \frac{7}{9}[(10-1) + (10^2-1) + (10^3-1) + (10^4-1) + \ldots n \text{ terms}]$$

$$= \frac{7}{9}[(10+10^2+10^3+\ldots n \text{ terms}) - (1+1+1+\ldots n \text{ terms})]$$

$$= \frac{7}{9}\left[\frac{10(10^n-1)}{10-1} - n\right] = \frac{7}{9}\left[\frac{10(10^n-1)}{9} - n\right].$$

Example 16 A person has 2 parents, 4 grandparents, 8 great grandparents, and so on. Find the number of his ancestors during the ten generations preceding his own.

Solution Here $a = 2$, $r = 2$ and $n = 10$

Using the sum formula $\quad S_n = \dfrac{a(r^n-1)}{r-1}$

We have $\quad S_{10} = 2(2^{10} - 1) = 2046$

Hence, the number of ancestors preceding the person is 2046.

9.5.3 ***Geometric Mean (G.M.)*** The geometric mean of two positive numbers a and b is the number $\sqrt{ab}$. Therefore, the geometric mean of 2 and 8 is 4. We observe that the three numbers 2,4,8 are consecutive terms of a G.P. This leads to a generalisation of the concept of geometric means of two numbers.

Given any two positive numbers a and b, we can insert as many numbers as we like between them to make the resulting sequence in a G.P.

Let $G_1, G_2, \ldots, G_n$ be n numbers between positive numbers a and b such that $a, G_1, G_2, G_3, \ldots, G_n, b$ is a G.P. Thus, b being the $(n+2)^{\text{th}}$ term, we have

$$b = ar^{n+1}, \quad \text{or} \quad r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}.$$

Hence $\quad G_1 = ar = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}, \quad G_2 = ar^2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}, \quad G_3 = ar^3 = a\left(\frac{b}{a}\right)^{\frac{3}{n+1}},$

$$G_n = ar^n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

Example17 Insert three numbers between 1 and 256 so that the resulting sequence is a G.P.

Solution Let G_1, G_2, G_3 be three numbers between 1 and 256 such that 1, G_1, G_2, G_3, 256 is a G.P.

Therefore $256 = r^4$ giving $r = \pm 4$ (Taking real roots only)

For $r = 4$, we have $G_1 = ar = 4$, $G_2 = ar^2 = 16$, $G_3 = ar^3 = 64$

Similarly, for $r = -4$, numbers are $-4, 16$ and -64.

Hence, we can insert 4, 16, 64 between 1 and 256 so that the resulting sequences are in GP.

9.6 Relationship Between A.M. and G.M.

Let A and G be A.M. and G.M. of two given positive real numbers a and b, respectively. Then

$$A = \frac{a+b}{2} \text{ and } G = \sqrt{ab}$$

Thus, we have

$$A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2}$$

$$= \frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{2} \geq 0 \qquad \text{... (1)}$$

From (1), we obtain the relationship $A \geq G$.

Example 18 If A.M. and G.M. of two positive numbers a and b are 10 and 8, respectively, find the numbers.

Solution Given that $\quad A.M. = \frac{a+b}{2} = 10 \qquad \text{... (1)}$

and $\quad G.M. = \sqrt{ab} = 8 \qquad \text{... (2)}$

From (1) and (2), we get

$$a + b = 20 \qquad \text{... (3)}$$

$$ab = 64 \qquad \text{... (4)}$$

Putting the value of a and b from (3), (4) in the identity $(a - b)^2 = (a + b)^2 - 4ab$, we get

$$(a - b)^2 = 400 - 256 = 144$$

or $\quad a - b = \pm 12$

... (5)

Solving (3) and (5), we obtain

$$a = 4, b = 16 \text{ or } a = 16, b = 4$$

Thus, the numbers a and b are 4, 16 or 16, 4 respectively.

EXERCISE 9.3

1. Find the 20th and n^{th} terms of the G.P. $\frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \ldots$
2. Find the 12th term of a G.P. whose 8th term is 192 and the common ratio is 2.
3. The 5th, 8th and 11th terms of a G.P. are p, q and s, respectively. Show that $q^2 = ps$.
4. The 4th term of a G.P. is square of its second term, and the first term is – 3. Determine its 7th term.
5. Which term of the following sequences:

   (a) $2, 2\sqrt{2}, 4, \ldots$ is 128 ? (b) $\sqrt{3}, 3, 3\sqrt{3}, \ldots$ is 729 ?

   (c) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$ is $\frac{1}{19683}$?

6. For what values of x, the numbers $-\frac{2}{7}, x, -\frac{7}{2}$ are in G.P.?

Find the sum to indicated number of terms in each of the geometric progressions in Exercises 7 to 10:

7. 0.15, 0.015, 0.0015, ... 20 terms.
8. $\sqrt{7}, \sqrt{21}, 3\sqrt{7}, \ldots$ n terms.
9. $1, -a, a^2, -a^3, \ldots$ n terms (if $a \neq -1$).
10. $x^3, x^5, x^7, \ldots$ n terms (if $x \neq \pm 1$).
11. Evaluate $\sum_{k=1}^{11} (2+3^k)$.
12. The sum of first three terms of a G.P. is $\frac{39}{10}$ and their product is 1. Find the common ratio and the terms.
13. How many terms of G.P. $3, 3^2, 3^3, \ldots$ are needed to give the sum 120?
14. The sum of first three terms of a G.P. is 16 and the sum of the next three terms is 128. Determine the first term, the common ratio and the sum to n terms of the G.P.
15. Given a G.P. with $a = 729$ and 7th term 64, determine S_7.
16. Find a G.P. for which sum of the first two terms is – 4 and the fifth term is 4 times the third term.

17. If the 4^{th}, 10^{th} and 16^{th} terms of a GP. are x, y and z, respectively. Prove that x, y, z are in G.P.

18. Find the sum to n terms of the sequence, 8, 88, 888, 8888… .

19. Find the sum of the products of the corresponding terms of the sequences 2, 4, 8, 16, 32 and 128, 32, 8, 2, $\frac{1}{2}$.

20. Show that the products of the corresponding terms of the sequences a, ar, ar^2, $\ldots ar^{n-1}$ and A, AR, AR^2, … AR^{n-1} form a G.P, and find the common ratio.

21. Find four numbers forming a geometric progression in which the third term is greater than the first term by 9, and the second term is greater than the 4^{th} by 18.

22. If the p^{th}, q^{th} and r^{th} terms of a GP. are a, b and c, respectively. Prove that

$$a^{q-r}\, b^{r-p} c^{P-q} = 1.$$

23. If the first and the n^{th} term of a G.P. are a and b, respectively, and if P is the product of n terms, prove that $P^2 = (ab)^n$.

24. Show that the ratio of the sum of first n terms of a GP. to the sum of terms from $(n + 1)^{th}$ to $(2n)^{th}$ term is $\frac{1}{r^n}$.

25. If a, b, c and d are in G.P. show that

$(a^2 + b^2 + c^2)\,(b^2 + c^2 + d^2) = (ab + bc + cd)^2$.

26. Insert two numbers between 3 and 81 so that the resulting sequence is G.P.

27. Find the value of n so that $\frac{a^{n+1} + b^{n+1}}{a^n + b^n}$ may be the geometric mean between a and b.

28. The sum of two numbers is 6 times their geometric mean, show that numbers are in the ratio $\left(3+2\sqrt{2}\right):\left(3-2\sqrt{2}\right)$.

29. If A and G be A.M. and G.M., respectively between two positive numbers, prove that the numbers are $A \pm \sqrt{(A+G)(A-G)}$.

30. The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria present in the culture originally, how many bacteria will be present at the end of 2^{nd} hour, 4^{th} hour and n^{th} hour ?

31. What will Rs 500 amounts to in 10 years after its deposit in a bank which pays annual interest rate of 10% compounded annually?

32. If A.M. and G.M. of roots of a quadratic equation are 8 and 5, respectively, then obtain the quadratic equation.

9.7 Sum to *n* Terms of Special Series

We shall now find the sum of first n terms of some special series, namely;

(i) $1 + 2 + 3 + \ldots + n$ (sum of first n natural numbers)

(ii) $1^2 + 2^2 + 3^2 + \ldots + n^2$ (sum of squares of the first n natural numbers)

(iii) $1^3 + 2^3 + 3^3 + \ldots + n^3$ (sum of cubes of the first n natural numbers).

Let us take them one by one.

(i) $S_n = 1 + 2 + 3 + \ldots + n$, then $S_n = \dfrac{n(n+1)}{2}$ (See Section 9.4)

(ii) Here $S_n = 1^2 + 2^2 + 3^2 + \ldots + n^2$

We consider the identity $k^3 - (k-1)^3 = 3k^2 - 3k + 1$

Putting $k = 1, 2\ldots, n$ successively, we obtain

$$1^3 - 0^3 = 3(1)^2 - 3(1) + 1$$
$$2^3 - 1^3 = 3(2)^2 - 3(2) + 1$$
$$3^3 - 2^3 = 3(3)^2 - 3(3) + 1$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$n^3 - (n-1)^3 = 3(n)^2 - 3(n) + 1$$

Adding both sides, we get

$$n^3 - 0^3 = 3(1^2 + 2^2 + 3^2 + \ldots + n^2) - 3(1 + 2 + 3 + \ldots + n) + n$$

$$n^3 = 3\sum_{k=1}^{n} k^2 - 3\sum_{k=1}^{n} k + n$$

By (i), we know that $\sum_{k=1}^{n} k = 1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$

Hence $S_n = \sum_{k=1}^{n} k^2 = \dfrac{1}{3}\left[n^3 + \dfrac{3n(n+1)}{2} - n\right] = \dfrac{1}{6}(2n^3 + 3n^2 + n)$

$$= \frac{n(n+1)(2n+1)}{6}$$

(iii) Here $S_n = 1^3 + 2^3 + \ldots + n^3$

We consider the identity, $(k+1)^4 - k^4 = 4k^3 + 6k^2 + 4k + 1$

Putting $k = 1, 2, 3\ldots n$, we get

$$2^4 - 1^4 = 4(1)^3 + 6(1)^2 + 4(1) + 1$$
$$3^4 - 2^4 = 4(2)^3 + 6(2)^2 + 4(2) + 1$$
$$4^4 - 3^4 = 4(3)^3 + 6(3)^2 + 4(3) + 1$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$(n-1)^4 - (n-2)^4 = 4(n-2)^3 + 6(n-2)^2 + 4(n-2) + 1$$
$$n^4 - (n-1)^4 = 4(n-1)^3 + 6(n-1)^2 + 4(n-1) + 1$$
$$(n+1)^4 - n^4 = 4n^3 + 6n^2 + 4n + 1$$

Adding both sides, we get

$$(n+1)^4 - 1^4 = 4(1^3 + 2^3 + 3^3 + \ldots + n^3) + 6(1^2 + 2^2 + 3^2 + \ldots + n^2) + 4(1 + 2 + 3 + \ldots + n) + n$$

$$= 4\sum_{k=1}^{n} k^3 + 6\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k + n \quad \ldots (1)$$

From parts (i) and (ii), we know that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Putting these values in equation (1), we obtain

$$4\sum_{k=1}^{n} k^3 = n^4 + 4n^3 + 6n^2 + 4n - \frac{6n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} - n$$

or $\quad 4S_n = n^4 + 4n^3 + 6n^2 + 4n - n(2n^2 + 3n + 1) - 2n(n+1) - n$

$$= n^4 + 2n^3 + n^2$$
$$= n^2(n+1)^2.$$

Hence, $\quad S_n = \dfrac{n^2(n+1)^2}{4} = \dfrac{[n(n+1)]^2}{4}$

Example 19 Find the sum to n terms of the series: 5 + 11 + 19 + 29 + 41…

Solution Let us write

$$S_n = 5 + 11 + 19 + 29 + \ldots + a_{n-1} + a_n$$

or $$S_n = \quad 5 + 11 + 19 + \ldots + a_{n-2} + a_{n-1} + a_n$$

On subtraction, we get

$$0 = 5 + [6 + 8 + 10 + 12 + \ldots(n-1) \text{ terms}] - a_n$$

or $$a_n = 5 + \frac{(n-1)[12+(n-2)\times 2]}{2}$$

$$= 5 + (n-1)(n+4) = n^2 + 3n + 1$$

Hence $$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n}(k^2 + 3k + 1) = \sum_{k=1}^{n} k^2 + 3\sum_{1}^{n} k + n$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2} + n = \frac{n(n+2)(n+4)}{3}.$$

Example 20 Find the sum to n terms of the series whose n^{th} term is $n(n+3)$.

Solution Given that $a_n = n(n+3) = n^2 + 3n$

Thus, the sum to n terms is given by

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2} = \frac{n(n+1)(n+5)}{3}.$$

EXERCISE 9.4

Find the sum to n terms of each of the series in Exercises 1 to 7.

1. $1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + \ldots$

2. $1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + \ldots$

3. $3 \times 1^2 + 5 \times 2^2 + 7 \times 3^2 + \ldots$

4. $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \ldots$

5. $5^2 + 6^2 + 7^2 + \ldots + 20^2$

6. $3 \times 8 + 6 \times 11 + 9 \times 14 + \ldots$

7. $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \ldots$

Find the sum to n terms of the series in Exercises 8 to 10 whose n^{th} terms is given by

8. $n(n+1)(n+4)$. **9.** $n^2 + 2^n$

10. $(2n-1)^2$

Miscellaneous Examples

Example21 If p^{th}, q^{th}, r^{th} and s^{th} terms of an A.P. are in G.P, then show that $(p-q)$, $(q-r)$, $(r-s)$ are also in G.P.

Solution Here

$$a_p = a + (p-1)\,d \qquad \text{... (1)}$$

$$a_q = a + (q-1)\,d \qquad \text{... (2)}$$

$$a_r = a + (r-1)\,d \qquad \text{... (3)}$$

$$a_s = a + (s-1)\,d \qquad \text{... (4)}$$

Given that a_p, a_q, a_r and a_s are in G.P.,

So $$\frac{a_q}{a_p} = \frac{a_r}{a_q} = \frac{a_q - a_r}{a_p - a_q} = \frac{q-r}{p-q} \text{ (why ?)} \qquad \text{... (5)}$$

Similarly $$\frac{a_r}{a_q} = \frac{a_s}{a_r} = \frac{a_r - a_s}{a_q - a_r} = \frac{r-s}{q-r} \text{ (why ?)} \qquad \text{... (6)}$$

Hence, by (5) and (6)

$$\frac{q-r}{p-q} = \frac{r-s}{q-r}, \text{ i.e., } p-q,\ q-r \text{ and } r-s \text{ are in G.P.}$$

Example 22 If a, b, c are in G.P. and $a^{\frac{1}{x}} = b^{\frac{1}{y}} = c^{\frac{1}{z}}$, prove that x, y, z are in A.P.

Solution Let $a^{\frac{1}{x}} = b^{\frac{1}{y}} = c^{\frac{1}{z}} = k$ Then

$$a = k^x,\ b = k^y \text{ and } c = k^z. \qquad \text{... (1)}$$

Since a, b, c are in G.P., therefore,

$$b^2 = ac \qquad \text{... (2)}$$

Using (1) in (2), we get

$$k^{2y} = k^{x+z}, \text{ which gives } 2y = x + z.$$

Hence, x, y and z are in A.P.

Example 23 If a, b, c, d and p are different real numbers such that

$(a^2 + b^2 + c^2)p^2 - 2(ab + bc + cd)\, p + (b^2 + c^2 + d^2) \leq 0$, then show that a, b, c and d are in G.P.

Solution Given that

$$(a^2 + b^2 + c^2)\, p^2 - 2\,(ab + bc + cd)\, p + (b^2 + c^2 + d^2) \leq 0 \qquad \text{... (1)}$$

But L.H.S.

$$= (a^2p^2 - 2abp + b^2) + (b^2p^2 - 2bcp + c^2) + (c^2p^2 - 2cdp + d^2),$$

which gives $(ap - b)^2 + (bp - c)^2 + (cp - d)^2 \geq 0$ ... (2)

Since the sum of squares of real numbers is non negative, therefore, from (1) and (2), we have, $(ap - b)^2 + (bp - c)^2 + (cp - d)^2 = 0$

or $ap - b = 0,\ bp - c = 0,\ cp - d = 0$

This implies that $\dfrac{b}{a} = \dfrac{c}{b} = \dfrac{d}{c} = p$

Hence a, b, c and d are in G.P.

Example 24 If p,q,r are in G.P. and the equations, $px^2 + 2qx + r = 0$ and $dx^2 + 2ex + f = 0$ have a common root, then show that $\dfrac{d}{p}, \dfrac{e}{q}, \dfrac{f}{r}$ are in A.P.

Solution The equation $px^2 + 2qx + r = 0$ has roots given by

$$x = \frac{-2q \pm \sqrt{4q^2 - 4rp}}{2p}$$

Since p, q, r are in G.P. $q^2 = pr$. Thus $x = \dfrac{-q}{p}$ but $\dfrac{-q}{p}$ is also root of $dx^2 + 2ex + f = 0$ (Why ?). Therefore

$$d\left(\frac{-q}{p}\right)^2 + 2e\left(\frac{-q}{p}\right) + f = 0,$$

or $dq^2 - 2eqp + fp^2 = 0$ ... (1)

Dividing (1) by pq^2 and using $q^2 = pr$, we get

$$\frac{d}{p}-\frac{2e}{q}+\frac{fp}{pr}=0, \text{ or } \qquad \frac{2e}{q}=\frac{d}{p}+\frac{f}{r}$$

Hence $\frac{d}{p},\frac{e}{q},\frac{f}{r}$ are in A.P.

Miscellaneous Exercise On Chapter 9

1. Show that the sum of $(m + n)^{\text{th}}$ and $(m - n)^{\text{th}}$ terms of an A.P. is equal to twice the m^{th} term.

2. If the sum of three numbers in A.P., is 24 and their product is 440, find the numbers.

3. Let the sum of n, $2n$, $3n$ terms of an A.P. be S_1, S_2 and S_3, respectively, show that $S_3 = 3(S_2 - S_1)$

4. Find the sum of all numbers between 200 and 400 which are divisible by 7.

5. Find the sum of integers from 1 to 100 that are divisible by 2 or 5.

6. Find the sum of all two digit numbers which when divided by 4, yields 1 as remainder.

7. If f is a function satisfying $f(x + y) = f(x) f(y)$ for all $x, y \in \mathbf{N}$ such that

$f(1) = 3$ and $\sum_{x=1}^{n} f(x) = 120$, find the value of n.

8. The sum of some terms of G.P. is 315 whose first term and the common ratio are 5 and 2, respectively. Find the last term and the number of terms.

9. The first term of a G.P. is 1. The sum of the third term and fifth term is 90. Find the common ratio of G.P.

10. The sum of three numbers in G.P. is 56. If we subtract 1, 7, 21 from these numbers in that order, we obtain an arithmetic progression. Find the numbers.

11. A G.P. consists of an even number of terms. If the sum of all the terms is 5 times the sum of terms occupying odd places, then find its common ratio.

12. The sum of the first four terms of an A.P. is 56. The sum of the last four terms is 112. If its first term is 11, then find the number of terms.

13. If $\frac{a+bx}{a-bx}=\frac{b+cx}{b-cx}=\frac{c+dx}{c-dx}\,(x\neq 0)$, then show that a, b, c and d are in G.P.

14. Let S be the sum, P the product and R the sum of reciprocals of n terms in a G.P. Prove that $P^2R^n = S^n$.

15. The p^{th}, q^{th} and r^{th} terms of an A.P. are a, b, c, respectively. Show that

$$(q-r)a + (r-p)b + (p-q)c = 0$$

16. If $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in A.P., prove that a, b, c are in A.P.

17. If a, b, c, d are in G.P, prove that $(a^n + b^n), (b^n + c^n), (c^n + d^n)$ are in G.P.

18. If a and b are the roots of $x^2 - 3x + p = 0$ and c, d are roots of $x^2 - 12x + q = 0$, where a, b, c, d form a G.P. Prove that $(q + p) : (q - p) = 17:15$.

19. The ratio of the A.M. and G.M. of two positive numbers a and b, is $m : n$. Show that $a:b = \left(m+\sqrt{m^2-n^2}\right):\left(m-\sqrt{m^2-n^2}\right)$.

20. If a, b, c are in A.P.; b, c, d are in G.P. and $\frac{1}{c}, \frac{1}{d}, \frac{1}{e}$ are in A.P. prove that a, c, e are in G.P.

21. Find the sum of the following series up to n terms:
(i) 5 + 55 +555 + … (ii) .6 +. 66 +. 666+…

22. Find the 20th term of the series $2 \times 4 + 4 \times 6 + 6 \times 8 + ... + n$ terms.

23. Find the sum of the first n terms of the series: 3+ 7 +13 +21 +31 +…

24. If S_1, S_2, S_3 are the sum of first n natural numbers, their squares and their cubes, respectively, show that $9S_2^2 = S_3(1 + 8S_1)$.

25. Find the sum of the following series up to n terms:

$$\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+...$$

26. Show that $$\frac{1\times 2^2+2\times 3^2+...+n\times(n+1)^2}{1^2\times 2+2^2\times 3+...+n^2\times(n+1)}=\frac{3n+5}{3n+1}.$$

27. A farmer buys a used tractor for Rs 12000. He pays Rs 6000 cash and agrees to pay the balance in annual instalments of Rs 500 plus 12% interest on the unpaid amount. How much will the tractor cost him?

28. Shamshad Ali buys a scooter for Rs 22000. He pays Rs 4000 cash and agrees to pay the balance in annual instalment of Rs 1000 plus 10% interest on the unpaid amount. How much will the scooter cost him?

29. A person writes a letter to four of his friends. He asks each one of them to copy the letter and mail to four different persons with instruction that they move the chain similarly. Assuming that the chain is not broken and that it costs 50 paise to mail one letter. Find the amount spent on the postage when 8th set of letter is mailed.

30. A man deposited Rs 10000 in a bank at the rate of 5% simple interest annually. Find the amount in 15th year since he deposited the amount and also calculate the total amount after 20 years.

31. A manufacturer reckons that the value of a machine, which costs him Rs. 15625, will depreciate each year by 20%. Find the estimated value at the end of 5 years.

32. 150 workers were engaged to finish a job in a certain number of days. 4 workers dropped out on second day, 4 more workers dropped out on third day and so on. It took 8 more days to finish the work. Find the number of days in which the work was completed.

Summary

- By a *sequence*, we mean an arrangement of number in definite order according to some rule. Also, we define a sequence as a function whose domain is the set of natural numbers or some subsets of the type {1, 2, 3, ...k}. A sequence containing a finite number of terms is called a *finite sequence*. A sequence is called *infinite* if it is not a finite sequence.
- Let $a_1, a_2, a_3, ...$ be the sequence, then the sum expressed as $a_1 + a_2 + a_3 + ...$ is called *series*. A series is called *finite series* if it has got finite number of terms.
- An arithmetic progression (A.P.) is a sequence in which terms increase or decrease regularly by the same constant. This constant is called *common difference of the* A.P. Usually, we denote the first term of A.P. by a, the common difference by d and the last term by l. The *general term* or the n^{th} *term* of the A.P. is given by $a_n = a + (n - 1)\, d$.

  The sum S_n of the first n terms of an A.P. is given by

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right] = \frac{n}{2}(a+l).$$

- The *arithmetic mean A* of any two numbers a and b is given by $\frac{a+b}{2}$ i.e., the sequence a, A, b is in A.P.

- A sequence is said to be a *geometric progression* or *G.P.*, if the ratio of any term to its preceding term is same throughout. This constant factor is called the *common ratio*. Usually, we denote the first term of a G.P. by a and its common ratio by r. The general or the n^{th} term of G.P. is given by $a_n = ar^{n-1}$. The sum S_n of the first n terms of G.P. is given by

$$S_n = \frac{a\left(r^n - 1\right)}{r-1} \text{ or } \frac{a\left(1-r^n\right)}{1-r}, \text{ if } r \neq 1$$

- The geometric mean (G.M.) of any two positive numbers a and b is given by $\sqrt{ab}$ i.e., the sequence a, G, b is G.P.

Historical Note

Evidence is found that Babylonians, some 4000 years ago, knew of arithmetic and geometric sequences. According to Boethius (510), arithmetic and geometric sequences were known to early Greek writers. Among the Indian mathematician, Aryabhatta (476) was the first to give the formula for the sum of squares and cubes of natural numbers in his famous work Aryabhatiyam, written around 499. He also gave the formula for finding the sum to n terms of an arithmetic sequence starting with p^{th} term. Noted Indian mathematicians Brahmgupta (598), Mahavira (850) and Bhaskara (1114-1185) also considered the sum of squares and cubes. Another specific type of sequence having important applications in mathematics, called *Fibonacci sequence*, was discovered by Italian mathematician Leonardo Fibonacci (1170-1250). Seventeenth century witnessed the classification of series into specific forms. In 1671 James Gregory used the term infinite series in connection with infinite sequence. It was only through the rigorous development of algebraic and set theoretic tools that the concepts related to sequence and series could be formulated suitably.