Tangent & Normals

Single Correct Answer Type

1. The points of contact of the tangents drawn from the origin to the curve $y = x^2 + 3x + 4$ are

1. (2, 14), (-2, 12) 2. (2, 12), (-2, 2) 3. (2, 14), (-2, 2) 4. (2, 12), (-2, 14)

Key. 3

Sol. Let $P(x_1, y_1)$ be a point on the curve $y = x^2 + 3x + 4$

$\Rightarrow y_1 = x_1^2 + 3x_1 + 4$ ….(1)

$(\frac{dy}{dx})_{at(x_1, y_1)} = 2x_1 + 3$

Equation of tangent is : $y - y_1 = m(x - x_1)$

It is passes through (0, 0)

Then $y_1 = 2x_1^2 + 3x_1$ …..(2)

From (1) & (2) $x_1 = \pm 2$

$\therefore$ the points are $(2,14) \& (-2, 2)$

2. If $3x + 2y = 1$ acts as a tangent to $y = f(x)$ at $x = 1/2$ and if

$$p = \lim_{x \to 0} \frac{x(x-1)}{f\left(\frac{e^{2x}}{2}\right) - f\left(\frac{e^{-2x}}{2}\right)}, \text{ then, } \sum_{r=1}^{\infty} p^r = \_\_\_\_\_$$

a) 1/2 b) 1/3 c) 1/6 d) 1/7

Key. A

Sol. slope of $3x + 2y = 1$ is $\frac{-3}{2}$

$$\Rightarrow f^1\left(\frac{1}{2}\right) = \frac{-3}{2}$$

$$p = \lim_{x \to 0} \frac{x(x-1)}{f\left(\frac{e^{2x}}{2}\right) - f\left(\frac{e^{-2x}}{2}\right)}\left(\frac{0}{0}\right) = \frac{-1}{f^1\left(\frac{1}{2}\right) + f^1\left(\frac{1}{2}\right)} = \frac{1}{3}$$

$$\therefore \sum_{r=1}^{\infty} p^r = \frac{1}{3} + \frac{1}{3^2} + \ldots..\infty = \frac{1/3}{1 - 1/3} = \frac{1/3}{2/3} = \frac{1}{2}$$

3. If the tangent drawn at $P\left(t=\frac{\pi}{4}\right)$ to the curve $x=\sec^2 t,\ y=\cot t$ meets the curve again at R, then, PR=________

a) $\frac{3\sqrt{5}}{2}$ b) $\frac{2\sqrt{5}}{3}$ c) $\frac{5\sqrt{5}}{4}$ d) $\frac{4\sqrt{5}}{5}$

Key. A

Sol. At $t=\frac{\pi}{4}, x=2, y=1 \Rightarrow P$ is $(2,1)$

$$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}}=\frac{-\operatorname{cosec}^2 t}{2\sec t.\sec t.\tan t}=-1/2$$

$\therefore$ tangent at $P(2,1)$ is, $y=\frac{4-x}{2}$

Elimating 't' curve equation is, $x=2,5 \Rightarrow R(5,-1/2) \Rightarrow PR=\frac{3}{2}\sqrt{5}$

4. If the points of contact of tangents to $y=\sin x$, drawn from origin always lie on $\frac{a}{y^2}-\frac{b}{x^2}=c$, then,

a) a,b,c are in AP, but not in GP and HP
b) a,b,c are in GP, but not in HP and AP
c) a,b,c are in HP, but not in AP and GP
d) a,b,c are in AP,GP and HP

Key. D

Sol. Let $P(h,k)$ be any point on $y=\sin x$

$\Rightarrow k=\sin h$. tangent P is $y-k=\cos h(x-h)$

$(0,0) \Rightarrow -k=\cos h.(0-n) \Rightarrow \cos h=\frac{k}{h}$

$\Rightarrow \frac{1}{y^2}-\frac{1}{x^2}=1 \Rightarrow a=1, b=1, c=1$

5. $A(1,0), B(e,1)$ are two points on the curve $y=\log_e x$. If P is a point on the curve at which the tangent to the curve is parallel to the chord AB, then, abscissa of P, is

a) $\frac{e-1}{2}$ b) $\frac{e+1}{2}$ c) $e-1$ d) $e+1$

Key. C

Sol. By LMVT, applied to $f(x)=\log_e x$ on $[1,e], \exists$ an $x_0 \in (1,e) \ni f^1(x_0)=\frac{f(e)-f(1)}{e-1}$

$\Rightarrow x_0=e-1$

6. The abscissa of the points. Where the tangent to the curve $y=x^3-3x^2-9x+5$ is parallel to x-axis is

1) 0 and 0 2) x=1 and -1 3) x=1 and -3 4) x=-1 and 3

Key. 4

Sol. Tangent is parallel to x-axis $\Rightarrow \frac{dy}{dx} = 0 \Rightarrow x = -1, 3$

7. Co-ordinates of a point on the curve $y = x\log x$ at which the normal is parallel to the line $2x - 2y = 3$, are

1) (0,0) 2) (e,e) 3) $\left(e^{-2}, 2e^{-2}\right)$ 4) $\left(e^{-2}, -2e^{-2}\right)$

Key. 4

Sol. Slope of the normal $= \frac{-1}{1+\log x} \Rightarrow \frac{-1}{1+\log x} = 1 \Rightarrow x = e^{-2}$

8. If the point on $y = x\tan\alpha - \frac{ax^2}{2u^2\cos^2\alpha}\left(0 < \alpha < \frac{\pi}{2}\right)$ where the tangent is parallel to y=x has an ordinate $\frac{u^2}{4a}$ then the value of α is

1) $\frac{\pi}{2}$ 2) $\frac{\pi}{6}$ 3) $\frac{\pi}{3}$ 4) $\frac{\pi}{4}$

Key. 3

Sol. Given m=1 $\Rightarrow \tan\alpha - \frac{ax}{u^2\cos^2\alpha} = 1 \Rightarrow x = \frac{(\tan\alpha - 1)}{a}u^2\cos^2\alpha$ substitute x and y values in given equation $\frac{u^2}{4a} = \frac{u^2}{a}\left[\sin^2\alpha - \frac{1}{2}\right] \Rightarrow \alpha = \frac{\pi}{3}$

9. If at each point of the curve $y = x^3 - ax^2 + x + 1$ the tangent is inclined at an acute angle with the positive direction of the x-axis, then a lies in the interval

1) $[-3,3]$ 2) $[-2,2]$ 3) $\left[-\sqrt{3},\sqrt{3}\right]$ 4) R

Key. 3

Sol. $\frac{dy}{dx} = 3x^2 - 2ax + 1, \frac{dy}{dx} > 0$ $3x^2 - 2ax + 1 > 0$

10. The number of tangents to the curve $x^{3/2} + y^{3/2} = a^{3/2}$, where the tangents are equally inclined to the axes, is

1) 2 2) 1 3) 0 4) 4

Key. 2

Sol. $\Rightarrow \frac{dy}{dx} = -\frac{x^{1/2}}{y^{1/2}}$

$\therefore \left(\frac{dy}{dx}\right)_{\alpha,\beta} = 1 \Rightarrow \alpha^{1/2} + \beta^{1/2} = 0$

$\alpha^{3/2} + \beta^{3/2} = a^{3/2} \quad \{\because (\alpha, \beta) \text{ is on the curve}\}$

$\left(\frac{dy}{dx}\right)_{\alpha,\beta} = -1 \quad \Rightarrow \alpha^{1/2} = \beta^{1/2}$

$\therefore \alpha = \beta = \frac{a}{2^{2/3}}$

there is only one point

11. The tangent at any point on the curve $x = a\cos^3\theta, y = a\sin^3\theta$ meets the axes in P and Q. The locus of the mid point of PQ is

1) $x^2 + y^2 = a^2$ 2) $2\left(x^2 + y^2\right) = a^2$ 3) $4\left(x^2 + y^2\right) = a^2$ 4) $x^2 + y^2 = 4a^2$

Key. 3

Sol. Equation of tangent at θ is $\Rightarrow P = (a\cos\theta, 0), Q = (0, a\sin\theta)$. Locus of midpoint of PQ is $4\left(x^2 + y^2\right) = a^2$

12. If the curves $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{l^2} - \frac{y^2}{m^2} = 1$ cut each other orthogonally then .

1) $a^2 + b^2 = l^2 + m^2$ 2) $a^2 - b^2 = l^2 - m^2$ 3) $a^2 - b^2 = l^2 + m^2$ 4) $a^2 + b^2 = l^2 - m^2$

Key. 3

Sol. If the curves $a_1x^2 + b_1y^2 = 1$, $a_2x^2 + b_2y^2 = 1$ cut each other orthogonally then apply

$\frac{1}{a} - \frac{1}{b} = \frac{1}{a_1} - \frac{1}{b_1}$

13. If the relation between the sub-normal and sub-tangent at any point on the curve $y^2 = (x+a)^3$ is $p\,(S.N) = q\,(S.T)^2$ $then\,\frac{p}{q} =$

1) $\frac{8}{27}$ 2) $\frac{27}{8}$ 3) $\frac{4}{9}$ 4) $\frac{9}{4}$

Key. 1

Sol. Length of sub normal $= |y_1 m|$

Length of sub tangent $= \left|\frac{y_1}{m}\right|$

14. The sum of the lengths of subtangent and tangent to the curve

$x = c\left[2\cos\theta - \log\left(\operatorname{cosec}\theta + \cot\theta\right)\right], y = c\sin 2\theta \ at\ \theta = \frac{\pi}{3}$ is

1) $\frac{c}{2}$ 2) $2c$ 3) $\frac{3c}{2}$ 4) $\frac{5c}{2}$

Key. 3

Sol. Length of tangent $= \left|\frac{y_1\sqrt{1+m^2}}{m}\right|$

Length of sub-tangent $= \left|\frac{y_1}{m}\right|$

15. The curves $C_1 : y = x^2 - 3$; $C_2 : y = kx^2, k < 1$ intersect each other at two different points. The tangent drawn to C_2, at one of the points of intersection $A = (a, y_1)(a > 0)$ meets C_1 again at $B(1, y_2)$. $(y_1 \neq y_2)$. Then value of $a =$ ___?

a) 4 b) 3 c) 2 d) 1

Sol : ans: b

solving

$C_1 \& C_2 \Rightarrow A\left(\sqrt{\frac{3}{1-k}}, \frac{3k}{1-k}\right) = \left(a, ka^2\right) \equiv \left(a, a^2 - 3\right).$

$\text{tangent } l \text{ to } C_2 \text{ at A is } y + a^2 - 3 = 2kx - - - - - (1)$

$\Rightarrow B = (1, -2)\ (A \neq 1).$

$\text{from expression (1) } -2 + a^2 - 3 = 2a\left(1 - \frac{3}{a^2}\right).$

$\Rightarrow a = 3, a = -2, a = 1$

$\therefore a = 3$

16. Let $f\left(\frac{x+y}{2}\right) = \frac{1}{2}(f(x) + f(y))$ for real x and y. If $f'(0)$ exists and equals to -1 and f(0)=1 then the value of f(2) is

a) 1 b) -1 c) $\frac{1}{2}$ d) 2

KEY : B

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{f(2x)+f(2h)}{2}-f(x)}{h}$$

$$f'(x)=-1 \qquad ; f(2x)=2f(x)-1$$

$$\Rightarrow f(x)=1-x$$

17. If the length of subnormal is equal to length of sub-tangent at point $(3,4)$ on the curve $y=f(x)$ and the tangent at $(3,4)$ to $y=f(x)$ meets the coordinate axes at A and B, then maximum area of the ΔOAB where O is origin, is

(A) $\frac{45}{2}$ sq.units (B) $\frac{49}{2}$ sq.units

(C) $\frac{51}{2}$ sq.units (D) $\frac{81}{2}$ sq.units

KEY : B

Sol : Length of subnormal = length of subtangent

$$\Rightarrow \left|y_1\left(\frac{dy}{dx}\right)_{(x_1y_1)}\right|=\left|\frac{y_1}{\left(\frac{dy}{dx}\right)_{(x_1y_1)}}\right|$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\pm 1$$

If $\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=1$

Then the equation of tangent is $y-x=1$ and area of $\Delta OAB=\frac{1}{2}\times 1\times 1=\frac{1}{2}$

If $\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=-1$

Then the equation of tangent is $x+y=7$ and area of $\Delta OAB=\frac{1}{2}\times 7\times 7=\frac{49}{2}$

18. The equation of normal to the curve $x + y = x^y$, where it cuts the x-axis is

(A) y = x - 1 (B) x + y = 1

(C) 12x + y + 2 = 0 (D) 3x + y = 3

Key: A

Sol: At x-axis, y = 0 $\Rightarrow$ x = 1

$x + y = x^y \Rightarrow \ln(x + y) = y \ln x$

$$\frac{1}{x+y}\left(1+\frac{dy}{dx}\right)=\frac{y}{x}+\frac{dy}{dx}\ln x$$

$\left(\frac{dy}{dx}(1,0\right)$ = -1

So equation of normal y - 0 = x - 1.

19. Maximum no. of parallel tangents of curves $y = x^3 - x^2 - 2x + 5$ and $y = x^2 - x + 3$ is

(A) 2 (B) 3

(C) 4 (D) none of these

Key: D

Sol: Let m be slope is common tangent

Then $m = 2x - 1$ and $m = 3x^2 - 2x - 2$,

So, infinite common tangents

20. The equation of the straight lines which are both tangent and normal to the curve $27x^2 = 4y^3$ are

a) $x = \pm\sqrt{2}(y-2)$ b) $x = \pm\sqrt{3}(y-2)$

c) $x = \pm\sqrt{2}(y-3)$ d) $x = \pm\sqrt{3}(y-3)$

Key. A

Sol. $x = 2t^3, y = 3t^2 \Rightarrow$ tangent at t is $x - yt = -t^3$ Normal at t_1 is, $xt_1 + y = 2t_1^4 + 3t_1^2$

$\Rightarrow \frac{1}{t_1} = -t = \frac{-t^3}{2t_1^4 + 3t_1^2} \Rightarrow t^6 - 3t^2 - 2 = 0 \Rightarrow t^2 = 2 \Rightarrow t = \pm\sqrt{2}$

$\therefore$ lines are $x = \pm\sqrt{2}(y-2)$

21. If $f(x)+f(y) = f(x)f(y)+f(xy), f(1) = 0, f^1(1) = -2$ then, equation to the tangent, drawn to the curve $y = f(x)$ at $x = \sqrt{2}$ is,

a) $2\sqrt{2}x - y - 3 = 0$ b) $2\sqrt{2}x + y - 3 = 0$

c) $2\sqrt{2}x + y + \sqrt{3} = 0$ d) $2\sqrt{2}x + 2y - 3 = 0$

Key. B

Sol. Clearly $f(x) = 1 - x^2$ at $x = \sqrt{2}, y = -1 \Rightarrow$ tangent at $(\sqrt{2}, -1)$ is,

$y + 1 = -2\sqrt{2}(x - \sqrt{2})$

22. Let $f(x)$ be a polynomial of degree 5. When f(x) is divided by $(x-1)^3$, the remainder 33, and when f(x) is divided by $(x+1)^3$, the remainder is –3. Then, equation to the tangent drawn to $y = f(x)$ at x = 0 is

a) $135x + 4y + 60 = 0$ b) $135x - 4y - 60 = 0$

c) $135x - 4y + 60 = 0$ d) $135x - 4y + 75 = 0$

Key. C

Sol. $f(x) = \frac{27x^5}{4} - \frac{45x^3}{2} + \frac{135x}{4} + 15$ at $x = 0, y = 15 \Rightarrow f^1(0) = \frac{135}{4}$

$\Rightarrow$ tangent equation is $y - 15 = \frac{135}{4}(x) \Rightarrow 135x - 4y + 60 = 0$

23. If the equation $x^{5/3}-5x^{2/3}=K$ has exactly one positive root, then, the complete solution set of K is,

a) $(-\infty,\infty)$ b) $(-\infty,0)$ c) $(3,\infty)$ d) $(0,\infty)$

Key. D

Sol. Sketch $y=x^{5/3}-5x^{2/3}$ and $y=K$

24. The equation of the straight lines which are both tangent and normal to the curve $27x^2=4y^3$ are

a) $x=\pm\sqrt{2}(y-2)$ b) $x=\pm\sqrt{3}(y-2)$

c) $x=\pm\sqrt{2}(y-3)$ d) $x=\pm\sqrt{3}(y-3)$

Key. A

Sol. $x=2t^3, y=3t^2 \Rightarrow$ tangent at t is $x-yt=-t^3$ Normal at t_1 is, $xt_1+y=2t_1^4+3t_1^2$

$\Rightarrow \frac{1}{t_1}=-t=\frac{-t^3}{2t_1^4+3t_1^2} \Rightarrow t^6-3t^2-2=0 \Rightarrow t^2=2 \Rightarrow t=\pm\sqrt{2}$

$\therefore$ lines are $x=\pm\sqrt{2}(y-2)$

25. If $f(x)+f(y)=f(x)f(y)+f(xy), f(1)=0, f^1(1)=-2$ then, equation to the tangent, drawn to the curve $y=f(x)$ at $x=\sqrt{2}$ is,

a) $2\sqrt{2}x-y-3=0$ b) $2\sqrt{2}x+y-3=0$

c) $2\sqrt{2}x+y+\sqrt{3}=0$ d) $2\sqrt{2}x+2y-3=0$

Key. B

Sol. Clearly $f(x)=1-x^2$ at $x=\sqrt{2}, y=-1 \Rightarrow$ tangent at $(\sqrt{2},-1)$ is, $y+1=-2\sqrt{2}(x-\sqrt{2})$

26. Let $f(x)$ be a polynomial of degree 5. When f(x) is divided by $(x-1)^3$, the remainder 33, and when f(x) is divided by $(x+1)^3$, the remainder is –3. Then, equation to the tangent drawn to $y=f(x)$ at x = 0 is

a) $135x+4y+60=0$ b) $135x-4y-60=0$

c) $135x-4y+60=0$ d) $135x-4y+75=0$

Key. C

Sol. $f(x)=\frac{27x^5}{4}-\frac{45x^3}{2}+\frac{135x}{4}+15$ at $x=0, y=15 \Rightarrow f^1(0)=\frac{135}{4}$

$\Rightarrow$ tangent equation is $y-15=\frac{135}{4}(x) \Rightarrow 135x-4y+60=0$

27. Two runners A and B start at the origin and run along positive x-axis, with B running three times as fast as A. An observer, standing one unit above the origin, keeps A and B in view. Then the maximum angle of sight 'θ' between the observes view of A and B is

a) $\pi/8$ b) $\pi/6$ c) $\pi/3$ d) $\pi/4$

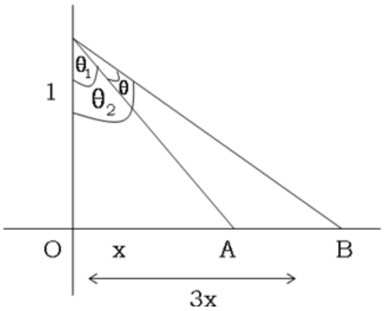
Key. B

Sol. $\tan\theta = \tan(\theta_2 - \theta_1) \Rightarrow \tan\theta = \frac{3x - x}{1 + 3x.x} = \frac{2x}{1+3x^2}$

$$\text{let}\, y = \frac{2x}{1+3x^2}\ \frac{dy}{dx} = \frac{2(1-3x^2)}{(1+3x^2)^2}$$

$$\frac{dy}{dx} = 0 \Rightarrow x = \frac{1}{\sqrt{3}} \text{and} \frac{d^2y}{dx^2} = \frac{-24x}{(1+3x^2)^3} < 0 \text{ for } x = 1/\sqrt{3}$$

$$\Rightarrow \theta = \pi \backslash 6$$



28. If the line joining the points $(0,3)$ and $(5,-2)$ is a tangent to the curve $y = \frac{c}{x+1}$, then value of c is

A) 1 B) -2 C) 4 D) -4.

Key. 3

Sol. Eqn. of the line joining given points is $(y+2) = \frac{-2-3}{5-0}(x-5)$.

$\Rightarrow y + x = 3.$

29. The number of points on the curve $y^3 - 3xy + 2 = 0$ where the tangent is either horizontal or vertical is

A) 0 B) 1 C) 2 D) > 2.

Key. 2

Sol. $3yy^1 - 3y - 3xy^1 = 0 \Rightarrow y^1 = \frac{y}{y^2 - x}$.

$y^1 = 0 \Rightarrow y = 0$, no real x

$y^1 = \infty \Rightarrow y^2 = x \Rightarrow y^3 = 1 \Rightarrow y = 1.$

The point is $(1,1)$.

30. The tangent to the curve $y = \frac{1+3x^2}{3+x^2}$ drawn at the points for which $y = 1$, intersect at

A) $(0,0)$ B) $(0,1)$ C) $(1,0)$ D) $(1,1)$

Key. 1

Sol. $y = 1 \Rightarrow x = \pm 1 \quad points\, are (1,1), (-1,1) \Rightarrow \frac{dy}{dx} = \frac{16x}{(3+x^2)^2}, \left(\frac{dy}{dx}\right)_{(1,1)} = 1, \left(\frac{dy}{dx}\right)_{(-1,1)} = -1$

Eq. of tangent at (1,1) is y – 1 = (x – 1) => x – y = 0

Eq. of tangent at (-1, 1) y – 1 = -1 (x + 1) => x + y = 0
Both tangents pass through origin.

31. The acute angle between the curves $y = |x^2 - 1|$ and $y = |x^2 - 3|$ at their points of intersection is

a) $\pi/4$ b) $\tan^{-1}(4\sqrt{2}/7)$
c) $\tan^{-1}(4\sqrt{7})$ d) $\tan^{-1}(2\sqrt{2}/7)$

Key. B

Sol. The point of intersection is $x^2 = 2, y = 1$. The given equations represent four parabolas.
$y = \pm(x^2 - 1)$ and $y = \pm(x^2 - 3)$
The curves intersect when $1 < x^2 < 3$ or $1 < x < \sqrt{3}$ or $-\sqrt{3} < x < -1$
$\therefore \quad y = x^2 - 1$ and $y = -(x^2 - 3)$
The points of intersection are $\left(\pm\sqrt{2}, 1\right)$
At $\left(\sqrt{2}, 1\right)$, $m_1 = 2x = 2\sqrt{2}, m_2 = -2x = -2\sqrt{2}$
$$\therefore \ \tan\theta = \left|\frac{4\sqrt{2}}{1-8}\right| = \frac{4\sqrt{2}}{7} \Rightarrow \theta = \tan^{-1}\left(\frac{4\sqrt{2}}{7}\right).$$

32. The angle between tangents at the point of intersection of two curves $x^3 - 3xy^2 + 2 = 0, 3x^2y - y^3 = 2$ is

a) $\frac{\pi}{6}$ b) $\frac{\pi}{4}$ c) $\frac{\pi}{3}$ d) $\frac{\pi}{2}$

Key. D

Sol. Let the point of intersection is (x, y)

33. The number of tangents to the curve $y = \cos(x + y), |x| \le 2\pi$, that are parallel to the line $x + 2y = 0$ is

A) 0 B) 1 C) 2 D) > 2

Key. 3

Sol. $y' = -\sin(x + y)(1 + y')$
Slope of tangent is $-\frac{1}{2} = y'$
$$\frac{1}{2} = \sin(x + y)\frac{1}{2} \ \Rightarrow \ \sin(x + y) = 1, \ \cos(x + y) = 0$$
$$\rightarrow y = 0 \rightarrow 0 = \cos x \rightarrow x = \frac{\pi}{2}, \frac{-3\pi}{2}$$
which satisfies the above equation.

34. The slope of the straight line which is both tangent and normal to the curve $4x^3 = 27y^2$ is

A) ± 1 B) $\pm\frac{1}{2}$ C) $\pm\frac{1}{\sqrt{2}}$ D) $\pm\sqrt{2}$.

Key. 4

Sol. $x = 3t^2, y = 2t^3, \dfrac{dy}{dx} = t$.

The tangent at t, $y - 2t^3 = t\left(x - 3t^2\right)$

The normal at t_1, $t_1 y + x = 2t_1^4 + 3t_1^2$.

(1) , (2) are identical ,

Comparing we get, $-t^3 = 2t_1^3 + 3t_1, t_1 = \dfrac{1}{t}$. Eliminating t_1, we get $t^6 = 2 + 3t^2$.

® $t^2 = 2, t = \pm\sqrt{2}$

35. The tangent at any point P on the curve $x^{2/3} + y^{2/3} = 4$ meets the coordinate axes at A and B Then $AB =$

A) 2 B) 4 C) 8 D) 16

Key. 3

Sol. $x = 8\cos^3 q$, $y = 8\sin^3 q$, $\dfrac{dy}{dx} = -\dfrac{\sin q}{\cos q}$.

Tangent at q, $y - 8\sin^2 q = -\dfrac{\sin q}{\cos q}\left(x - 8\cos^3 q\right)$

$x\sin q + y\cos q = 8\sin q\cos q$

$OA = 8\cos q$, $OB = 8\sin q$

$AB = \sqrt{OA^2 + OB^2} = 8$.

36. The rate of change of $\sqrt{x^2+16}$ with respect to $\dfrac{x}{x-1}$ at x = 3 is

a) 1 b) $\dfrac{11}{5}$ c) $-\dfrac{12}{5}$ d) -3

Key. C

Sol.
$u = \sqrt{x^2+16}\ \dfrac{du}{dx} = \dfrac{2x}{2\sqrt{x^2+16}} = \dfrac{x}{\sqrt{x^2+16}}, V = \dfrac{x}{x-1} \Rightarrow \dfrac{dv}{dx} = \dfrac{-1}{(x-1)^2}$

$\dfrac{du}{dv} = \dfrac{du/dx}{dv/dx} = \dfrac{-12}{5}$

37. The curves $x^3 - 3xy^2 = a$ and $3x^2y - y^3 = b$ intersect at an angle of

A) $\dfrac{p}{4}$ B) $\dfrac{p}{3}$ C) $\dfrac{p}{2}$ D) $\dfrac{p}{6}$.

Key. 3

Sol. Clearly $m_1 m_2 = -1$.

38. The cosine of the angle of intersection of curves $f(x) = 2^x \log_e x$ and $g(x) = x^{2x} - 1$ is

A) 1 B) 0 C) $\dfrac{1}{2}$ D) $\dfrac{\sqrt{3}}{2}$.

Key. 1

Sol. Clearly, $(1,0)$ is the point of intersection of the given curves.

Now, $f'(x) = \frac{2^x}{x} + 2^x(\log_e 2)(\log_e x)$

$\therefore$ Slope of tangent to the curve $f(x)$ at $(1,0)$, $m_1 = 2$.

$g'(x) = \frac{d}{dx}\left(e^{2x\log x} - 1\right) = x^{2x}\left(2x \times \frac{1}{x} + 2\log_e x\right)$

$\therefore$ Slope of tangent to the curve $g(x)$ at $(1,0)$, $m_2 = 2$.

Since $m_1 = m_2 = 2$.

$\therefore$ Two curves touch each other, so the angle between them is 0.
Hence, $\cos\theta = \cos 0 = 1$.

39. Let the equation of a curve be $\frac{x^2}{4} + \frac{y^2}{3} = 1$ where $\left(2\cos\theta, \sqrt{3}\sin\theta\right)$ is a general point on the curve. If the tangent to the given curve intersects the co-ordinate axes at points A, B, then the locus of midpoint of AB is

a) $2x^2 + \sqrt{3}y^2 = 4$ b) $3x^2 + 4y^2 = 4x^2y^2$

c) $3x^2 + 4y^2 = x^2y^2$ d) $4x^2 + 3y^2 = 4x^2y^2$

Key. B

Sol. Equation of tangent is

$$y - \sqrt{3}\sin\theta = \frac{-\sqrt{3}}{2}.\cot\theta\left(x - 2\cos\theta\right) \Rightarrow x\ intercept\left(x_0\right) = \frac{2}{\cos\theta} \Rightarrow \cos\theta = \frac{2}{x_0},$$

$$y\ intercept\left(y_0\right) = \frac{\sqrt{3}}{\sin\theta} \Rightarrow \sin\theta = \frac{\sqrt{3}}{y_0},\ if\ mid\ point\ is\left(h,k\right)$$

$$h = \frac{x_0}{2},\ k = \frac{y_0}{2},\ \cos\theta = \frac{1}{h}, \sin\theta = \frac{\sqrt{3}}{2k} \Rightarrow \frac{1}{h^2} + \frac{3}{4k^2} = 1$$

40. The value of n in the equation of curve $y = a^{1-n}x^n$, so that the sub-normal may be of constant length is

A) 2 B) $\frac{3}{2}$ C) $\frac{1}{2}$ D) 1

Key. 3

Sol. Taking log and differentiating both sides, we get $\frac{dy}{dx} = \frac{ny}{x}$ . . . (1)

Length of sub-normal $= na^{2-2n}.x^{2n-1}$

$n = \frac{1}{2}$.

41. Let $f(x)=x^2+xg'(1)+g''(2)$ and $g(x)=f(1)x^2+xf'(x)+f''(x)$, then $f(3)+g(3)=$

A) 7 B) -7 C) 0 D) 6

Key. 2

Sol. Let $g'(1)=a, g^{11}(a)=b$ then $f(x)=x^2+ax+b$ then $f(1)=1+a+b$

$$g(x)=(1+a+b)x^2+x(2x+a)+b$$

$$g'(x)=2x(3+a+b)+a$$

$$g'(1)=a \Rightarrow a+b+3=0\ ,\quad g''(2)=b \Rightarrow 2a+b=-6$$

42. Let the equation of a curve in parametric form be $x=a(\theta+\sin\theta),\ y=a(1-\cos\theta)$. The angle between the tangent drawn at the point $\theta=\frac{\pi}{3}$ and normal drawn at the point $\theta=\frac{2\pi}{3}$ is

a) $\frac{\pi}{6}$ b) $\frac{\pi}{4}$ c) $\frac{\pi}{3}$ d) $\frac{\pi}{2}$

Key. C

Sol. $\frac{dy}{dx}=\frac{a\sin\theta}{a(1+\cos\theta)}=\tan\frac{\theta}{2}$

$$m_1=\tan\frac{\frac{\pi}{3}}{2}=\frac{1}{\sqrt{3}},\ m_2=-\frac{1}{\tan\frac{\theta}{2}}=-\frac{1}{\tan\frac{\pi}{3}}=\frac{-1}{\sqrt{3}},\ \tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|=\sqrt{3}\Rightarrow\theta=60^0$$

43. The abscissa of two points on $y=(2010)x^2+(2011)x-2011$ are 2010 and 2012. if the chord joining those two points is parallel to tangent at P on the curve then the ordinate of P is equal to

a) $(2009)(2010)(2011)$ b) $(2010)(2011)(2012)$

c) $(2011)(2012)(2013)$ d) none

Key. B

Sol. Apply LMVT with a = 2010, b = 2012

$$f(x)=2010x^2+2011x-2011.$$

$$\frac{f(b)-f(a)}{b-a}=f'(c)\Rightarrow c=2011, f(c)=(2010)(2011)(2012)$$

44. Tangent at P_1 other than origin on the curve $y = x^3$ meets the curve again at P_2. The tangent at P_2 meets the curve again at P_3 and so on .... then $\frac{\text{area of } DP_1P_2P_3}{\text{area of } DP_2P_3P_4}$ equals

a) 1 : 20 b) 1 : 16 c) 1 : 8 d) 1 : 2

Key. B

Sol. Let $P_1 = (t_1, t_1^3) P_2 = (t_2, t_2^3), P_3(t_3, t_3^3)$.......

Solving tangent equation at P_1 with the curve again we get $t_2 = -2t_1$. Repeating the process we have $t_3 = 4t_1$ $t_4 = -8t_1$.............

$$\therefore \frac{\Delta P_1P_2P_3}{\Delta P_2P_3P_4} = \begin{vmatrix} t_1 & t_1^3 & 1 \\ t_2 & t_2^3 & 1 \\ t_3 & t_3^3 & 1 \end{vmatrix} \div \begin{vmatrix} t_2 & t_2^3 & 1 \\ t_3 & t_3^3 & 1 \\ t_4 & t_4^3 & 1 \end{vmatrix} = \frac{1}{16}$$

45. The value of parameter t so that the line $(4-t)x + ty + (a^3 - 1) = 0$ is normal to the curve $xy = 1$ may lie in the interval

A) (1, 4) B) $(-\alpha, 0) \cup (4, \alpha)$ C) $(-4, 4)$ D) $[3, 4]$

Key. B

Sol. Slope of line $(4-t)x + ty + (a^3 - 1) = 0$

is $\frac{-(4-t)}{t}$ (or) $\frac{t-4}{t}$

$\because xy = 1$

$\therefore \frac{dy}{dx} = \frac{-y}{x} = \frac{-1}{x^2}$

$\therefore$ slope of normal = $x^2 = \frac{t-4}{t}$

$\therefore x^2 > 0$

$\frac{t-4}{t} > 0$

$t \in (-\infty, 0) \cup (4, \infty)$

46. The angle of intersection of curves $y = [\,|\sin x| + |\cos x|\,]$ and $x^2 + y^2 = 5$, where $[\cdot]$ denotes greatest integral function is

A) $Tan^{-1}(2)$ B) $Tan^{-1}(\sqrt{2})$ C) $Tan^{-1}(\sqrt{3})$ D) $Tan^{-1}(3)$

Key. A

Sol. We know that $1 \le |\sin x| + |\cos x| \le \sqrt{2}$

$\therefore y = [|\sin x| + \cos x] = 1$

Let P and Q be the points of intersection of given curves clearly the given curves meet at points where $y=1$, so we get

$x^2+1=5$

$\Rightarrow x=\pm 2$

$\therefore P(2,1)\ and\ Q(-2,1)$

Now $x^2+y^2=5$

$\Rightarrow x=\pm 2$

$\therefore P(2,1)\ and\ Q(-2,1)$

Now $x^2+y^2=5$

$$\Rightarrow \frac{dy}{dx}=\frac{-x}{y},\left(\frac{dy}{dx}\right)_{(2,1)}=-2,\left(\frac{dy}{dx}\right)_{(-2,1)}=2$$

Clearly the slope of a line $y=1,$ is 0 and the slope of tangent at P and Q are -2 and 2 respectively.

$\therefore$ The angle of intersection is $\tan^{-1}(2)$

47. If the tangent at (1, 1) on $y^2=x(2-x)^2$ meets the curve again at P, then P is

a) (4, 4) b) (-1, 2) c) (9/4, 3/8) d) (9/5, 3/8)

Key. C

Sol. $2y\frac{dy}{dx}=(2-x)^2-2x(2-x)$, so $\left.\frac{dy}{dx}\right|_{(1,1)}=-\frac{1}{2}$ Therefore, the equation of tangent at (1, 1) is

$$y-1=-\frac{1}{2}(x-1)$$

$$\Rightarrow y=\frac{-x+3}{2}$$

The intersection of the tangent and the curve is given by $(1/4)(-x+3)^2=x(4+x^2-4x)$

$$\Rightarrow x^2-6x+9=16x+4x^3-16x^2$$

$$\Rightarrow 4x^3-17x^2+22x-9=0$$

$$\Rightarrow (x-1)(4x^2-13x+9)=0 \quad \Rightarrow (x-1)^2(4x-9)=0$$

Since x = 1 is already the point of tangency, x = 9/4 and $y^2 = \frac{9}{4}\left(2-\frac{9}{4}\right)^2 = \frac{9}{24}$. Thus the required point is $(9/4, 3/8)$.

48. The equation of the normal to the curve parametrically represented by $x = t^2 + 3t - 8$ and $y = 2t^2 - 2t - 5$ at the point P(2, -1) is

a) $2x+3y-1=0$ b) $6x-7y-11=0$

c) $7x+6y-8=0$ d) $3x+y-1=0$

Key. C

Sol.
$$\left.\begin{array}{l} t^2+3t-8=2 \Rightarrow t=2,-5 \\ 2t^2-2t-5=-1 \Rightarrow t=2,-1 \end{array}\right\} \Rightarrow t=2,\ \frac{dy}{dx}=\frac{4t-2}{2t+3} \Rightarrow \left(\frac{dy}{dx}\right)_{t=2}=\frac{6}{7}$$

$$equaiton\,of\ normal\ \ y+1=\frac{-7}{6}(x-2)$$

49. Tangents are drawn from origin to the curve y = cos x, their points of contact lie on the curve

a) $x^2+y^2=x^2y^2$ b) $y^2-x^2=x^2y^2$

c) $x^2+y^2=1$ d) $x^2-y^2=x^2y^2$

Key. D

Sol. Let point of contact is (h, k)

$$\Rightarrow k=\cos h--(1)\quad eq.of\ \tan gent\, at\,(h,k)\quad y-k=-\sin h(x-h), it\ passes\, through$$
$$origin \Rightarrow -k=h.\sin h---(2)$$
$$\cos^2 h+\sin^2 h=k^2+\frac{k^2}{h^2} \Rightarrow 1=y^2+\frac{y^2}{x^2}\, is\, the\, locus\, of\ \ po\,int\, of\ contact$$

50. If the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $\frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2}=1$ cut each other orthogonally, then

a) $a^2+b^2=\alpha^2+\beta^2$ b) $a^2-b^2=\alpha^2-\beta^2$

c) $a^2-b^2=\alpha^2+\beta^2$ d) $a^2+b^2=\alpha^2-\beta^2$

Key. C

$Slope\,of\ \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1\,at\,P(x_0,y_0)\,is\ -\dfrac{b^2x_0}{a^2y_0},\ Slope\,of\ \dfrac{x^2}{\alpha^2}-\dfrac{y^2}{\beta^2}=1\,at\,P(x_0,y_0)\ is\ \dfrac{\beta^2x_0}{\alpha^2y_0}$

Sol. $\because M_1M_2=-1\Rightarrow b^2\beta^2x_0^2=a^2\alpha^2y_0^2\ ---(1)$

now solving the curves

$x_0^2\left(\dfrac{1}{a^2}-\dfrac{1}{\alpha^2}\right)=-y_0^2\left(\dfrac{1}{b^2}+\dfrac{1}{\beta^2}\right)\ ----(2)$

from $(1)\,\&\,(2)$

$\dfrac{\dfrac{1}{a^2}-\dfrac{1}{\alpha^2}}{\dfrac{1}{b^2}+\dfrac{1}{\beta^2}}=\dfrac{b^2\beta^2}{a^2\alpha^2}\Rightarrow a^2-b^2=\alpha^2+\beta^2$ 51.

Tangent & Normals

Integer Answer Type

1. The parametric equations of a curve are $x = \sec^2 t \quad y = \cot t$. If the tangent, drawn to the curve at $P\left(t = \frac{\pi}{4}\right)$ meets the curve again at Q, and if $PQ = \frac{\alpha}{\beta}\sqrt{5}, (\alpha, \beta) = 1$ then, the numerical value of $\alpha + \beta$ is

Key. 5

Sol. $P = (2,1), Q = (5, -1/2) \Rightarrow PQ = \frac{3\sqrt{5}}{2}$

2. The number of points lying in $\{(x,y)/|x| \le 10, |y| \le 3\}$, on the curve $y^2 = x - \sin x$, at which the tangents to the curve are parallel to x – axis, is\are

Key. 2

Sol. $\frac{dy}{dx} = \frac{1-\cos x}{2\sqrt{x+\sin x}} = 0 \Rightarrow \cos x = 1 \Rightarrow x = 0, \pm 2\pi, \pm 4\pi....$

$\therefore x \in [-10, 10] \Rightarrow x = 0, \pm 2\pi$

For x = 0, y = 0

$\left.\begin{array}{l} x = 0, y = 0 \\ x = 2\pi, y = \pm\sqrt{2\pi} \end{array}\right\}$ they satisfy $-3 \le y \le 3$

X $x = -2\pi, y^2 = -2\pi$

Also, slope at $(0,0)$ is undefined hence points are $(2\pi, 2\pi)$ and $\left(2\pi, -\sqrt{2\pi}\right)$

3. The sum of all the integral values of K such that the variable point $\left(K, \frac{1}{K}\right)$ remains on or inside the triangle formed by the x-axis and the tangents drawn at $(2,1)$ on the curve $y = e^{-|x-2|}$ is,

Key. 2

Sol. $\frac{1+\sqrt{5}}{2} \le K \le \frac{3+\sqrt{5}}{2} \Rightarrow$ hence K = 2

4. Let $f(x)$ be a continuous function which satisfies,

$f^3(x) - 5f^2(x) + 10f(x) - 12 \le 0$,

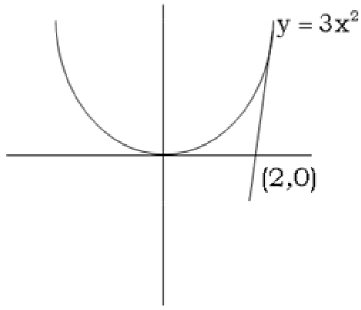
$f^2(x) - 4f(x) + 3 \ge 0$

$f^2(x) - 6f(x) + 8 \le 0$

If the equation to the tangent drawn from $(2,0)$ to the curve, $y = x^2 f(\sin x)$ is of the form, $y = \alpha x - 2\alpha,\ \alpha \in R^+$, then the numerical value of $\frac{\alpha}{6}$ is,

Key. 4

Sol.



$y = 24x - 48$

$\alpha = 24 \Rightarrow \frac{\alpha}{6} = 4$

5. Let y = f(x) be drawn with f(0) = 2 and for each real number 'a', the tangent to y = f(x) at (a, f(a)), has x intercept (a – 2). If f(x) is of the form $k\,e^{px}$, then $\left(\frac{k}{p}\right)$ has the value equal to

Key. 4

Sol. We have f(0) = 2

Now $y - f(a) = f'(a)[x - a]$

For x intercept y = 0, so

$$x = a - \frac{f(a)}{f'(a)} = a - 2 \Rightarrow \frac{f(a)}{f'(a)} = 2$$

$$\Rightarrow \frac{f'(a)}{f(a)} = \frac{1}{2}$$

$\therefore$ On integrating both sides w.r.t. a, we get

$$\ln f(a) = \frac{a}{2} + C$$

$$f(a) = Ce^{a/2}$$

$$f(x) = Ce^{x/2}$$

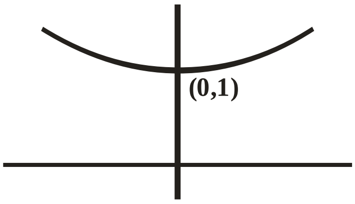
$$f(0) = C \qquad \Rightarrow C = 2$$

$$\therefore f(x) = 2e^{x/2}$$

Hence $k = 2,\ p = \frac{1}{2} \Rightarrow \frac{k}{p} = 4$

6. The shortest distance of the point (0,0) from the curve $y = \frac{1}{2}\left(e^{x} + e^{-x}\right)$ is

Key. 1

Sol. 

7. The segment of the tangent to the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=16$, contained between $'x'$ and $'y'$ axes, has length equal to λ^2 then the value of λ is

Key. 8

Sol. Let $x=64\cos^2 t$

$y=64\sin^{t}$

$\frac{dx}{dt}=-192\cos^2 t(-\sin t)$

Equation of tangent at $'t'$

$\Rightarrow y-64\sin^2 t=\frac{-\sin t}{\cos t}\left(x-64\cos^2 t\right)$

$\Rightarrow \frac{y}{\sin t}-64\sin^2 t=\frac{-\sin t}{\cos t}\left(x-64\cos^2 t\right)$

$\Rightarrow \frac{x}{64\cos t}+\frac{y}{64\sin t}=1$

$\Rightarrow$Segment of tangent between α and β

$=\sqrt{(64)^2\cos^2 t+64^2\sin^2 t}=64$

Length $=64=x^2 \therefore \lambda=8$

8. Find the value of a for which the area of the triangle included between the axes and any tangent to the curve $x^a y = k^a$ is constant

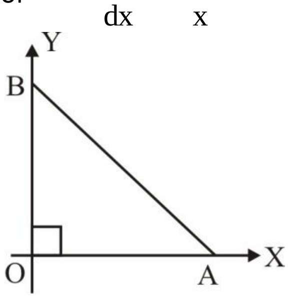
Key. 1

Sol. $\because$ $x^a y=k^a$

$\Rightarrow$ a ln x + ln y = a ln k

$\therefore$ $\frac{a}{x}+\frac{1}{y}\frac{dy}{dx}=0$

or $\frac{dy}{dx}=-\frac{ay}{x}$



Equation of tangent at (x, y)

$$Y - y = -\frac{ay}{x}(X - x)$$

or $$\frac{Y}{\left(\frac{ay}{x}\right)} - \frac{x}{a} = -X + x$$

or $$\frac{X}{1} + \frac{Y}{\left(\frac{ay}{x}\right)} = x\left(1 + \frac{1}{a}\right)$$

or $$\frac{X}{x\left(1+\frac{1}{a}\right)} + \frac{Y}{ay\left(1+\frac{1}{a}\right)} = 1$$

$\therefore$ Area $= \frac{1}{2}.x\left(1+\frac{1}{a}\right)ay\left(1+\frac{1}{a}\right)$

$$= \frac{ax}{2}.\frac{k^a}{x^a}.\left(1+\frac{1}{a}\right)^2$$

$$= \frac{ak^a}{2x^{a-1}}\left(1+\frac{1}{a}\right)^2$$

$\because$ Area is constant

$\therefore$ a – 1 = 0

$\therefore$ a = 1

9. The sum of all the integral values of K such that the variable point $\left(K, \frac{1}{K}\right)$ remains on or inside the triangle formed by the x-axis and the tangents drawn at $(2,1)$ on the curve $y = e^{-|x-2|}$ is,

Key. 2

Sol. $\frac{1+\sqrt{5}}{2} \le K \le \frac{3+\sqrt{5}}{2} \Rightarrow$ hence K = 2

10. The number of points lying in $\{(x,y)/|x| \le 10, |y| \le 3\}$, on the curve $y^2 = x - \sin x$, at which the tangents to the curve are parallel to x – axis, is\are

Key. 2

Sol. $\frac{dy}{dx} = \frac{1-\cos x}{2\sqrt{x+\sin x}} = 0 \Rightarrow \cos x = 1 \Rightarrow x = 0, \pm 2\pi, \pm 4\pi....$

$\therefore x \in [-10, 10] \Rightarrow x = 0, \pm 2\pi$

For x = 0, y = 0

$\left.\begin{array}{l} x = 0, y = 0 \\ x = 2\pi, y = \pm\sqrt{2\pi} \end{array}\right\}$ they satisfy $-3 \le y \le 3$

X $x = -2\pi, y^2 = -2\pi$

Also, slope at $(0,0)$ is undefined hence points are $(2\pi, 2\pi)$ and $\left(2\pi, -\sqrt{2\pi}\right)$

11. Let $f(x)$ be a continuous function which satisfies,

$f^3(x)-5f^2(x)+10f(x)-12\le 0$,

$f^2(x)-4f(x)+3\ge 0$

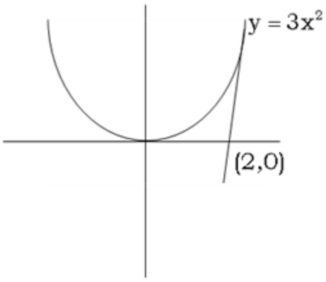
$f^2(x)-6f(x)+8\le 0$

If the equation to the tangent drawn from $(2,0)$ to the curve, $y=x^2f(\sin x)$ is of the form, $y=\alpha x-2\alpha,\ \alpha\in R^+$, then the numerical value of $\frac{\alpha}{6}$ is,

Key. 4

Sol. $y=24x-48$

$\alpha=24\Rightarrow\frac{\alpha}{6}=4$



12. The parametric equations of a curve are $x=\sec^2 t\ \ y=\cot t$. If the tangent, drawn to the curve at $P\left(t=\frac{\pi}{4}\right)$ meets the curve again at Q, and if $PQ=\frac{\alpha}{\beta}\sqrt{5},(\alpha,\beta)=1$ then, the numerical value of $\alpha+\beta$ is

Key. 5

Sol. $P=(2,1),Q=(5,-1/2)\Rightarrow PQ=\frac{3\sqrt{5}}{2}$

13. The curve $y=ax^3+bx^2+cx+5$, touches the x-axis at P(–2, 0) and cuts the y-axis at a point Q, where its gradient is 3, then find the value of 4b – 2a + c.

Key. 1

Sol. Let y = f(x), $f'(-2)=0$, $f(-2)=0$

$f'(0)=3$, $f'(x)=3ax^2+2bx+c$

Solving $a=-\frac{1}{2},b=-\frac{3}{4}$ and $c=3$

$\Rightarrow$ 4b – 2a + c = 1

14. Three normals are drawn from the point (c, 0) to the curve $y^2 = x$, c > $\frac{1}{2}$. One normal is always the x-axis. Then the value of 4c for which other two normals are perpendicular to each other is

Key. 3

Sol. Equation of normal is y = mx – 2am – am^3 to the curve $y^2 = 4ax$.

$a = \frac{1}{4}$

$\Rightarrow$ $y = mx - \frac{m}{2} - \frac{m^3}{4}$ passes through (c, 0) then $-m\left(\frac{m^2}{4} + \frac{1}{2} - c\right) = 0$

$\Rightarrow$ Remaining normals are perpendicular.

$\Rightarrow$ Product of the roots of equation $\frac{m^2}{4} + \frac{1}{2} - c = 0$ will be -1.

$\Rightarrow$ $\frac{\frac{1}{2} - c}{\frac{1}{4}} = -1 \Rightarrow c = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

so 4c = 3