Properties of Triangles

Single Correct Answer Type

1. If a,b,c be the sides of a triangle ABC and the roots of the equation $a(b-c)x^2 + b(c-a)x + c(a-b) = 0$ are equal, then $\sin^2\left(\frac{A}{2}\right), \sin^2\left(\frac{B}{2}\right), \sin^2\left(\frac{C}{2}\right)$ are in

(A)AP (B) GP (C) AGP (D) HP

Key. D

Sol. $\because$ $a(b-c) + b(c-a) + c(a-b) = 0$

$\therefore$ x = 1 is a root of the equation

$a(b-c)x^2 + b(c-a)x + c(a-b) = 0$

Then, other root = 1 ($\because$ roots are equal)

$\therefore$ $\alpha \times \beta = \frac{c(a-b)}{a(b-c)}$

$\Rightarrow$ ab – ac = ca – bc

$\therefore$ $b = \frac{2ac}{a+c}$

$\therefore$ a, b, c are in HP

Then, $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP.

$\Rightarrow \frac{s}{a}, \frac{s}{b}, \frac{s}{c}$ are in AP

$\Rightarrow \frac{s}{a}-1, \frac{s}{b}-1, \frac{s}{c}-1$ are in AP.

$\Rightarrow \frac{(s-a)}{a}, \frac{(s-b)}{b}, \frac{(s-c)}{c}$ are in AP.

Multiplying in each by $\frac{abc}{(s-a)(s-b)(s-c)}$

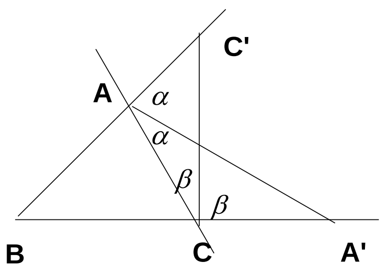
Then $\frac{bc}{(s-b)(s-c)}, \frac{ca}{(s-c)(s-a)}, \frac{ab}{(s-a)(s-b)}$ are in AP.

$\Rightarrow \frac{(s-b)(s-c)}{bc}, \frac{(s-c)(s-a)}{ca}, \frac{(s-a)(s-b)}{ab}$ are in HP.

Or $\sin^2\left(\frac{A}{2}\right), \sin^2\left(\frac{B}{2}\right), \sin^2\left(\frac{C}{2}\right)$ are in HP

2. Given in $\Delta ABC : AB = 1cm : AC = 2\,cm$ The lengths of external angular bisectors of angles A & C are equal. ie., $AA' = CC'$. If $BC \neq 1$ then BC = ________________

In the given figure $\alpha = 90^0 - \frac{A}{2}$ and $\beta = 90^0 - \frac{C}{2}$



(a) $\frac{1+\sqrt{15}}{2}$ (b) $\frac{1+\sqrt{13}}{2}$ (c) $\frac{1+\sqrt{17}}{2}$ (d) $\frac{1+\sqrt{19}}{2}$

Key. C

Sol. Length of external angular bisector of angle A is $\frac{2bc}{|b-c|}\sin\frac{A}{2}$. Length of external angular bisector of angle C is $\frac{2ab}{|a-b|}\sin\frac{C}{2}$

3. In Δ ABC, the bisector of the angle A meets the side BC at D and the circumscribed circle at E, then DE equals

(A) $\frac{a^2 \sec\frac{A}{2}}{2(b+c)}$ (B) $\frac{a^2 \sin\frac{A}{2}}{2(b+c)}$

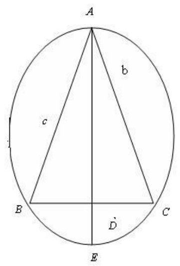
(C) $\frac{a^2 \cos\frac{A}{2}}{2(b+c)}$ (D) $\frac{a^2 \operatorname{cosec}\frac{A}{2}}{2(b+c)}$

Key. A

Sol. AD.DE = BD.DC

$$DE = \frac{BD.DC}{AD} = \frac{\left(\frac{ac}{b+c}\right)\left(\frac{ab}{b+c}\right)}{\frac{2bc}{b+c}.\cos\frac{A}{2}}$$

$$= \frac{a^2}{2(b+c)}\sec\frac{A}{2}$$



4. In Δ ABC, If A – B = 120° and R = 8r, then the value of $\frac{1+\cos C}{1-\cos C}$ equals

(All symbols used have their usual meaning in a triangle)

(A) 12 (B) 15 (C) 21 (D) 31

Key. B

Sol. $\frac{r}{R} = \cos A + \cos B + \cos C - 1$

$$\frac{1}{8} = 2\cos\frac{A+B}{2} + \cos\frac{A-B}{2} - 1 + \cos C$$

$$\Rightarrow \frac{1}{8} = \sin\frac{C}{2} - 2\sin^2\frac{C}{2}$$

$$\Rightarrow \sin\frac{C}{2} = \frac{1}{4} \quad \therefore \cos C = 1 - \frac{1}{8} = \frac{7}{8}$$

5. In a $\Delta ABC, if\ A = 30^0\ and\ \frac{b}{c} = \frac{2+\sqrt{3}+\sqrt{2}-1}{2+\sqrt{3}-\sqrt{2}+1}$, then the measure of $\angle C, is$

A) $67\frac{1^0}{2}$ B) $22\frac{1^0}{2}$ C) $52\frac{1^0}{2}$ D) $97\frac{1^0}{2}$

Key. C

Sol. use $\frac{b-c}{b+c}\cot\frac{A}{2} = \tan\left(\frac{B-C}{2}\right)$; and $B + C = 150^0$

$$\frac{b}{1+\sqrt{3}+\sqrt{2}} = \frac{c}{3+\sqrt{3}-\sqrt{2}} \Rightarrow \frac{b+c}{4+2\sqrt{3}} = \frac{b-c}{2\sqrt{2}-2} \Rightarrow \frac{b+c}{b-c} = \frac{\sqrt{3}+2}{\sqrt{2}-1}$$

$\therefore \frac{b-c}{b+c} = \frac{\sqrt{2}-1}{2+\sqrt{3}}$ which gives $\frac{b-c}{b+c}\cot 15° = \tan 22\frac{1°}{2}$

$$B - C = 45^0; B + C = 150^0$$

6. In $\Delta ABC, if \cos A+\sin A-\frac{2}{\cos B+\sin B}=0, then \frac{a+b}{c}$ is equal to

A) $\sqrt{2}$ B) 1 C) $\frac{1}{\sqrt{2}}$ D) $2\sqrt{2}$

Key. A

Sol. given $(\cos A+\sin A)(\cos B+\sin B)=2$

$\cos(A-B)+\sin(A+B)=2$

$\Rightarrow \cos(\mathrm{A-B})=1; \sin(\mathrm{A+B})=1$

$A=B;\ A+B=\frac{\pi}{2}\ \Rightarrow C=\frac{\pi}{2}$

$\therefore \frac{a+b}{c}=\sqrt{2}$

7. In a ΔABC , $\sum \sin\frac{A}{2}=\frac{6}{5}$ and $\sum II_1=9$ where I_1, I_2, I_3 are external and I is incentre, then circum radius R=

A) $\frac{15}{2}$ B) $\frac{15}{4}$ C) $\frac{15}{8}$ D) $\frac{1}{3}$

Key. C

Sol. $\sum II_1=\sum 4R\sin\frac{A}{2}\Rightarrow 9=4R\times\frac{6}{15}\Rightarrow R=\frac{45}{24}=\frac{15}{8}$

8. Let there exist a unique point P inside a ΔABC such that $\angle PAB=\angle PBC=\angle PCA=\alpha$

If PA=x, PB=y, PC=z, Δ=area of ΔABC and a,b,c are the sides opposite to the angles A,B,C respectively, then tan α is equal to

A) $\frac{a^2+b^2+c^2}{4\Delta}$ B) $\frac{a^2+b^2+c^2}{2\Delta}$ C) $\frac{2\Delta}{a^2+b^2+c^2}$ D) $\frac{4\Delta}{a^2+b^2+c^2}$

Key. D

Sol. $\cot A+\cot B+\cot c=\cot\alpha\Rightarrow \tan\alpha=\frac{4\Delta}{a^2+b^2+c^2}$

9. In a triangle ABC with usual notations, if $r=1, r_1=7$ and R = 3,then the triangleABC is

A) equilateral B) acute angled which is not equilateral

C) obtuse angled D) right angled

Key. D

Sol. $r_1 - r = 4R\sin^2\frac{A}{2} \Rightarrow \sin^2\frac{A}{2} = \frac{1}{2} \Rightarrow A = \frac{\pi}{2}$

10. In a triangle ABC, $a:b:c = 4:5:6$. The ratio of the radius of the circumcircle to that of the incircle is

A) 15/4 B) 11/5 C) 16/7 D) 16/3.

Key. C

Sol. $\frac{a}{4} = \frac{b}{5} = \frac{c}{6}$ use $\Delta rs = \frac{abc}{4R}$

11. In triangle ABC , $\frac{s-a}{\Delta} = \frac{1}{8}, \frac{s-b}{\Delta} = \frac{1}{12}, \frac{s-c}{\Delta} = \frac{1}{24}$ then b=

1) 16 2) 20 3) 24 4) 28

Key. 1

Sol.
$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$$
$$b = \sqrt{(r_2 - r)(r_1 + r_3)}$$

12. If in a triangle ABC, $\frac{s - r_2}{r_2} = \sqrt{2}$ then $\frac{a^2 + c^2 - b^2}{2ac} =$

1) $\frac{1}{\sqrt{2}}$ 2) $-\frac{1}{\sqrt{2}}$ 3) $\frac{\sqrt{3}}{2}$ 4) $-\frac{\sqrt{3}}{2}$

Key. 1

Sol. $\Rightarrow r_2(\sqrt{2}+1) = s \Rightarrow \tan\frac{B}{2} = \sqrt{2} - 1$

13. ABCD is a quadrilateral, AB=a, BC=b , CD=c , DA=d , is inscribed to a circle and circumscribed to another circle. Then the value $\tan^2\frac{A}{2}$ =

1) $\frac{ad}{bc}$ 2) $\frac{ab}{cd}$ 3) $\frac{bc}{ad}$ 4) $\frac{ac}{bd}$

Key. 3

Sol. $\cos A = \frac{ad - bc}{ad + bc} = \frac{1 - \frac{bc}{ad}}{1 + \frac{bc}{ad}}$

14. In a triangle ABC, $C = 60^0$ and R=16 then $I I_3$=

1) 30 2) 31 3) 32 4) 34

Key. 3

Sol. $I I_3 = 4R \sin \frac{C}{2}$

15. In a triangle ABC, r = 2, $\angle B = 60^0$ and $\angle C = 90^0$ then r_1 =

1) $\sqrt{3}$ 2) $2\sqrt{3}$ 3) $3\sqrt{3}$ 4) $4\sqrt{3}$

Key. 2

Sol. $r_1 = r.\cot\frac{B}{2}.\cot\frac{C}{2}$

16. If a,b,c are the sides of a triangle, then the minimum value of $\frac{2a}{b+c-a} + \frac{2b}{c+a-b} + \frac{2c}{a+b-c}$ is

1) 3 2) 6 3) 8 4) 1/8

Key. 1

Sol. $\frac{a}{s-a} + \frac{b}{s-b} + \frac{c}{s-c} = -3 + \left(\frac{s}{s-a} + \frac{s}{s-b} + \frac{s}{s-c}\right) \geq -3 + 9 = 6$

$$\left(\text{Q}\left(x_1 + x_2 + x_3\right)\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}\right) \geq 9\right)$$

17. If x,y,z are the distances of the vertices of triangle ABC from its orthocenter then x+y+z=

1) 2 (R + r) 2) 2 (R-r) 3) 2R-r 4) 2R + r

Key. 1

Sol. X =2 R cos A , y=2R cos B, z=2R cos C

18. If in a triangle the ex-radii r_1, r_2, r_3 are in the ratio 1:2:3, then their sides are in the ratio :

1) 5:8:9 2) 1:2:3 3) 3:5:7 4) 1:5:9

Key. 1

Sol. $r_1 : r_2 : r_3 = 1 : 2 : 3$, $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{11}{6}$

$a : b : c = \sqrt{(r_1 - r)(r_2 + r_3)} : \sqrt{(r_2 - r)(r_1 + r_3)} : \sqrt{(r_3 - r)(r_1 + r_2)}$

19. If length of the sides of a triangle ABC are 3,4 and 5 cm, then distance between its orthocentre and circumcentre is

1) 2.5 c.m. 2) 2 c.m. 3) 1.5 c.m. 4) 8

Key. 1

Sol. $O^1 = R\sqrt{1 - 8\cos A\cos B\cos C} = R = 2.5$

20. If length of the sides of a triangle ABC are 3,4 and 5 cm, then distance between its incentre and circumcentre is

1) $\frac{\sqrt{3}}{2}$ 2) $\frac{\sqrt{5}}{2}$ 3) $\frac{1}{2}$ 4) $\frac{1}{\sqrt{2}}$

Key. 2

$$OI = \sqrt{R^2 - 2Rr}, R = 5/2, r = \frac{\frac{1}{2} \times 3 \times 4}{6} = 1$$

Sol.

21. If P is a point on the altitude AD of the triangle ABC such that $\angle DBP = \frac{B}{3}$, then AP is equal to

A) $2a\sin\frac{C}{3}$ B) $2b\sin\frac{C}{3}$ C) $2c\sin\frac{B}{3}$ D) $2c\sin\frac{C}{3}$

Key. C

Sol. $\angle DBP = \frac{B}{3}$

$\angle DBP = \frac{B}{3}$

$\angle ABP = \frac{2B}{3}$

$\frac{AP}{\sin\frac{2B}{3}} = \frac{c}{\sin\left(90+\frac{B}{3}\right)} \Rightarrow AP = C\left(2\sin\frac{B}{3}\right)$

22. In triangle ABC, if B = 90^0 then $\cos^{-1}\left(\frac{R}{r_1+r_3}\right) =$

1) $\frac{\pi}{6}$ 2) $\frac{\pi}{4}$ 3) $\frac{\pi}{3}$ 4) $\frac{2\pi}{3}$

Key. 3

Sol. $r_1 + r_3 = 4R\cos^2\frac{B}{2}$

23. A circle is inscribed in an equilateral triangle of side 6 units. The area of any square inscribed in this circle is

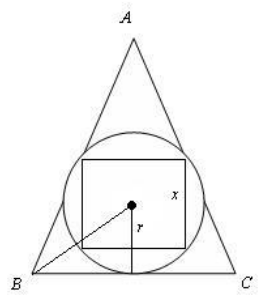
1) 6 2) 36 3) 9 4) 72

Key. 1

Sol. Let r be radius of in circle and x be side of the square

$r = \sqrt{3}$

$\sqrt{2}x = 2\sqrt{3} \Rightarrow x^2 \frac{4\times 3}{2} = 6$



24. If the area of triangle ABC is $b^2-(c-a)^2$, then $\tan B =$

1) $\frac{3}{4}$ 2) $\frac{1}{4}$ 3) $\frac{8}{15}$ 4) $\frac{15}{8}$

Key. 3

Sol. $\Delta = b^2 - (c-a)^2 = b^2 - c^2 - a^2 + 2ac$

$= 2ac\left(1 - \frac{a^1 + c^2 - b^2}{2ac}\right) = 2ac(1-\cos B)$

$\frac{abc}{4R} = 2ac.2\sin^2\frac{B}{2} \Rightarrow \tan\frac{B}{2} = \frac{1}{4}$

$\therefore \tan B = \frac{2/4}{1-1/16} = \frac{8}{15}$

25. If in a triangle ABC, $(r_2-r_1)(r_3-r_1) = 2r_2r_3$, then the triangle is :

1) Right angled 2) Isosceles
3) Equilateral 4) Right angled Isosceles

Key. 1

Sol. $\left(\frac{\Delta}{s-b} - \frac{\Delta}{s-a}\right)\left(\frac{\Delta}{s-c} - \frac{\Delta}{s-a}\right) = 2\frac{\Delta}{s-b}\frac{\Delta}{s-c}$

$(b-a)(c-a) = 2(s-a)^2$

$\Rightarrow 2(b-a)(c-a) = (b+c-a)^2$

$\Rightarrow b^2 + c^2 = a^2$

26. If r_1, r_2, r_3 are exradii of any triangle then $r_1r_2 + r_2r_3 + r_3r_1$ is equal to :

1) Δ/r 2) Δ^2/r^2 3) r/Δ 4) r^2/Δ^2

Key. 2

Sol. $r_1 = \frac{\Delta}{s-a}, r_2 = \frac{\Delta}{s-b}, r_3 = \frac{\Delta}{s-c}$

27. If in a triangle ABC, $2a = p\left(\frac{1}{r_2} + \frac{1}{r_3}\right) + q\left(\frac{1}{r} - \frac{1}{r_1}\right)$, then p+q=

1) Δ 2) 2Δ 3) 3Δ 4) 4Δ

Key. 2

Sol. $r_1r_2 = r_3 = \frac{\sqrt{3}}{2}, r = \frac{1}{2\sqrt{3}}$

28. In a triangle, if $= r_1 = 2r_2 = 3r_3$, $then\ \frac{a}{b} + \frac{b}{c} + \frac{c}{a} =$

1) $\frac{75}{100}$ 2) $\frac{155}{60}$ 3) $\frac{176}{60}$ 4) $\frac{191}{60}$

Key. 4

Sol. $\frac{\Delta}{s-a} = 2, \frac{\Delta}{s-b} = 3, \frac{\Delta}{s-c} = k$

$\Rightarrow a = \frac{5}{k}, b = \frac{4}{k}, c = \frac{3}{k}$

29. In a triangle ABC, medians AD and CE are drawn . If AD=5, $\angle DAC = \frac{\pi}{8}$ and $\angle ACE = \frac{\pi}{4}$ then the area of triangle ABC is equal to

1) $\frac{25}{9}$ 2) $\frac{25}{3}$ 3) $\frac{25}{18}$ 4) $\frac{10}{3}$

Key. 2

Sol. $AG = \frac{2}{3}, AD = \frac{10}{3}$

$$\frac{GC}{\sin\frac{\pi}{8}} = \frac{AG}{\sin\frac{\pi}{4}} \Rightarrow GC = \frac{10}{3} \times \frac{\sin\frac{\pi}{8}}{\sin\frac{\pi}{4}}$$

$\therefore$ Area fo $\Delta ABC = 3$ Area of ΔAGC

$$3\left(\frac{1}{2}\frac{10}{3} \times \left(\frac{10}{3} \times \frac{\sin\frac{\pi}{8}}{\sin\frac{\pi}{4}}\right)\right) \times \sin\left(\frac{\pi}{2} + \frac{\pi}{8}\right) = \frac{25}{3}$$

30. In a triangle ABC, $r = 1, R = 4, \Delta = 8$ then the value of $ab + bc + ca =$

1) 18 2) 81 3) 72 4) 27

Key. 2

Sol. $r_1 + r_2 + r_3 - r = 4R$

$r\left(r_1 + r_2 + r_3\right) = ab + bc + ca - S^2$

31. If in a triangle ABC $r_1=3$, $r_2=10$, $r_3=15$ then the value of R equals

1) $\frac{15}{2}$ 2) $\frac{11}{2}$ 3) $\frac{9}{2}$ 4) $\frac{13}{2}$

Key. 4

Sol. $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$

$r_1 + r_2 + r_3 - r = 4R$

32. In a triangle ABC, the maximum value of $\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$ is

1) $\frac{s}{2R}$ 2) $\frac{R}{2s}$ 3) $\frac{s}{2r}$ 4) $\frac{r}{2s}$

Key. 2

Sol. $\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} = \frac{\Delta}{s(s-a)}\cdot\frac{\Delta}{s(s-b)}\cdot\frac{\Delta}{s(s-c)}$

$= \frac{\Delta}{s^2} = \frac{r}{s} \le \ (\because 2r \le R)$

33. In triangle ABC, $\frac{r_1 + r_3}{1 + \cos B} =$

1) $\frac{abc}{4\Delta}$ 2) $\frac{abc}{2\Delta}$ 3) $\frac{2ab}{c\Delta}$ 4) $\frac{2(a+b)}{c\Delta}$

Key. 2

Sol. $\dfrac{4R\cos^2\frac{B}{2}}{2\cos^2\frac{B}{2}} = \dfrac{abc}{2\Delta}$

34. If in a triangle ABC, $r_1 = 8,\ r_2 = 12,\ r_3 = 24$ then C =

1) $\frac{\pi}{4}$ 2) $\frac{\pi}{6}$ 3) $\frac{\pi}{3}$ 4) $\frac{\pi}{2}$

Key. 4

Sol. $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$, $\tan^2\frac{C}{2} = \frac{rr_3}{r_1r_2}$

35. If H is the orthocenter of a acuteangled triangle ABC whose circumcircle is $x^2 + y^2 = 16$ then curcumdiametre of the triangle HBC is

1) 1 2) 2 3) 4 4) 8

Key. 4

Sol. since $\angle HBC = 90 - C$

$$\frac{HC}{\sin(90-c)} = 2R^1$$

$$\therefore 2R^1 = \frac{2R\cos c}{\cos c} = 2R$$

36. In triangle ABC , I is the incentre of the triangle . Then IA.IB.IC =

1) $4r^2R$ 2) $4R^2r$ 3) r^2R 4) R^2r

Key. 1

Sol. $I_A.I_B.I_C$ = r cosec A/2. r cosec B/2. r cosec C/2

$$\frac{r^3}{\sin A/2 \sin B/2 \sin C/2}.\frac{4R}{4R} = \frac{4Rr^3}{r} = 4Rr^2$$

37. In a right angled triangle ABC with $A = \frac{\pi}{2}$, a circle is drawn touching the side AB,AC and incircle of the triangle. It's radius is equal to

1) $\left(2-\sqrt{2}\right)r$ 2) $\left(3-\sqrt{2}\right)r$ 3) $\left(3+\sqrt{2}\right)r$ 4) $\left(3-2\sqrt{2}\right)r$

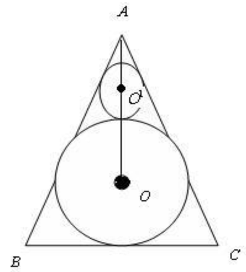
Key. 4

Sol. let r_1 be radius of required circle

$A0^1 = r_1 \operatorname{cosec}\frac{A}{2} = \sqrt{2}r_1$ $A0 = r \operatorname{cosec}\frac{A}{2} = \sqrt{2}r$

$00^1 = \sqrt{2}r\left(r - r_1\right)$ But $00^1 = r_1 + r$

$\therefore r_1 + r = \sqrt{2}\left(r - r_1\right) \Rightarrow r_1 \frac{\left(\sqrt{2}-1\right)}{\left(\sqrt{2+1}\right)} r = \left(3-2\sqrt{2}\right)r$

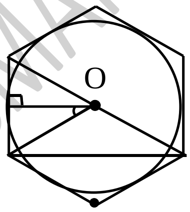


38. Let S_1 and S_2 be the areas of inscribed and circumscribed polygons of n sides respectively and S_3 is the area of regular polygon of 2n sides inscribed in a circle, then

A) $2S_3 = S_1 + S_2$ B) $S_3^2 = S_1S_2$

C) $\frac{1}{S_3} = \frac{1}{S_1} + \frac{1}{S_2}$ D) $\frac{2}{S_3} = \frac{1}{S_1} + \frac{1}{S_2}$

Key. B



Sol.

$$\tan\frac{\pi}{n} = \frac{x}{r}$$

$$x = r\tan\frac{\pi}{n}$$

$$S_1 = n \times \frac{1}{2} \times r^2 \times \sin \frac{2\pi}{n}$$

$$S_2 = n.r^2 \tan \frac{\pi}{n}$$

$$S_3 = \frac{2n}{2} \times r^2 \sin \frac{\pi}{n} \qquad S_3^2 = n^2 r^4 \sin^2 \frac{\pi}{n}$$

$$S_1 S_2 = n^2 r^4 \frac{1}{2} \times 2\sin \frac{\pi}{n} \cos \frac{\pi}{n} . \frac{\sin \frac{\pi}{n}}{\cos \frac{\pi}{n}}$$

$$= n^2 r^4 \sin^2 \frac{\pi}{n} = S_3^2$$

39. In ΔABC if $\frac{\sin A}{\sin B} + \frac{\sin B}{c} + \frac{\sin C}{b} = \frac{c}{ab} + \frac{b}{ac} + \frac{a}{bc}$ then angle A is

A) 120^0 B) 90^0 C) 60^0 D) 30^0

Key. B

Sol. $\frac{a}{bc} + \frac{b}{2Rc} + \frac{c}{2Rb} = \frac{c}{ab} + \frac{b}{ac} + \frac{a}{bc}$

$\Rightarrow 2R = a \Rightarrow A = 90^0$

40. In ΔABC, $A = \frac{2\pi}{3}, b - c = 3\sqrt{3}$ cm and area of ΔABC = $\frac{9\sqrt{3}}{2} cm^2$, then BC =

A) $6\sqrt{3}$ cm B) 9cm C) 18cm D) 27cm

Key. B

Sol. $\frac{1}{2} bc \sin \frac{2\pi}{3} = \frac{9\sqrt{3}}{2} \Rightarrow bc = 18 \Rightarrow b^2 + c^2 - 36 = 27 \Rightarrow b^2 + c^2 = 63$

$a^2 = 63 - 2 \times 18 \times \frac{-1}{2} = 81 \Rightarrow a = 9$

41. In ΔABC, if $\cot A = \sqrt{ac}, \cot B = \sqrt{\frac{c}{a}}, \cot C = \sqrt{\frac{a^3}{c}}$ then which of the following can be true?

A) $a + a^2 = 1 - c$ B) $a + a^2 = 1 + c$

C) $a+a^2=2-c$ D) $a+a^2=2+c$

Key. A

Sol. $\cot A \cot B = C,\ \cot B \cot C = a,\ \cot C \cot A = a^2$

But $\sum \cot A \cot B = 1 \Rightarrow c+a+a^2 = 1 \Rightarrow a+a^2 = 1-c$

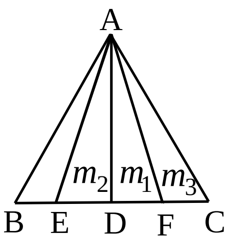
42. Let AD be a median of ΔABC. If AE and AF are medians of ΔABD and ΔADC respectively and AD = m_1, AE = m_2, AF = m_3, BC = a, then $\frac{a^2}{8} =$

A) $m_2^2+m_3^2-2m_1^2$ B) $m_1^2+m_2^2-2m_3^2$

C) $m_1^2+m_3^2-2m_2^2$ D) $m_1^2+m_2^2+m_3^2$

Key. A

Sol. $m_2^2+m_3^2 = 2\left(m_1^2+ED^2\right) \Rightarrow m_2^2+m_3^2-2m_1^2 = \frac{a^2}{8}$

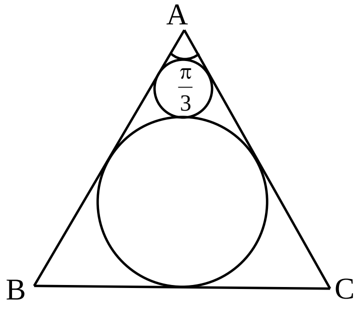


43. In ΔABC, $\angle A = \frac{\pi}{3}$ and its inradius is 6 units. The radius of the circle touching the sides AB, AC internally and the incircle of ΔABC externally is

A) 3units B) 3/2 units C) 2 units D) 4 units

Key. C

Sol. Angle between the direct common tangents is $\frac{\pi}{3}$



$$\therefore\ 2\sin^{-1}\left(\frac{6-r}{6+r}\right)=\frac{\pi}{3}\Rightarrow\frac{6-r}{6+r}=\frac{1}{2}$$

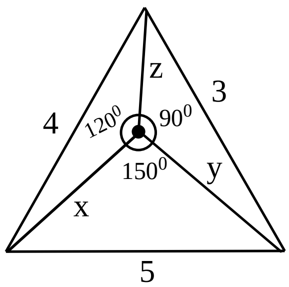
$$\Rightarrow 12-2r=6+r\Rightarrow 6=3r\Rightarrow r=2.$$

44. Three positive real numbers x, y, z satisfy the equations $x^2+\sqrt{3}xy+y^2=25$, $y^2+z^2=9$ and $x^2+xz+z^2=16$ then the value of $xy+2yz+\sqrt{3}xz$ is

A) 18 B) 24 C) 30 D) 36

Key. B

Sol.



Area of triangle = $\frac{1}{2}\times 3\times 4=\frac{1}{2}xz\frac{\sqrt{3}}{2}+\frac{1}{2}xy\times\frac{1}{2}+\frac{1}{2}yz$

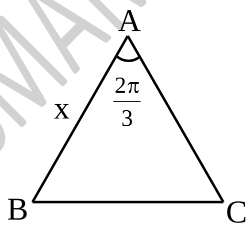
$$\Rightarrow 24=\sqrt{3}xz+xy+2yz$$

45. Let ABC be a triangle with $\angle BAC=\frac{2\pi}{3}$ and AB = x such that AB.AC=1. If x varies then the largest possible length of internal angular bisector AD is

A) 1 B) 2 C) $\frac{1}{2}$ D) $\frac{1}{4}$

Key. C

Sol.



Angular bisector $AD=\frac{2bc}{b+c}\cos\frac{A}{2}$

$$=\frac{2\times x\times\frac{1}{x}}{x+\frac{1}{x}}\times\frac{1}{2}$$

46. The sides of a triangle inscribed in a given circle subtend angles α,β,γ at the centre. Then, the minimum value of the A.M of $\cos\left(\alpha+\frac{\pi}{2}\right),\cos\left(\beta+\frac{\pi}{2}\right),\ \cos\left(\gamma+\frac{\pi}{2}\right)$ is

(A) $-\frac{\sqrt{3}}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) $\frac{1}{\sqrt{2}}$ (D) none of these

Key. A

Sol. Clearly, $\angle A=\frac{\alpha}{2},\ \angle B=\frac{\beta}{2},\ \angle C=\frac{\gamma}{2}$

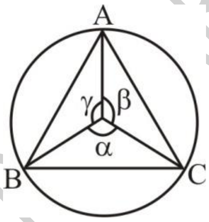
$\therefore \alpha+\beta+\gamma=2\pi$

$$\text{A.M.}=\frac{1}{3}\left[\cos\left(\alpha+\frac{\pi}{2}\right)+\cos\left(\beta+\frac{\pi}{2}\right)+\cos\left(\frac{\gamma}{2}\right)\right]$$

$$=-\frac{1}{3}\left[\sin\alpha+\sin\beta+\sin\gamma\right]$$

$$=-\frac{4}{3}\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right)$$

$$=-\frac{4}{3}\sin A\sin B\sin C$$



A.M. will be least if $\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right)$ is greatest i.e. sinA sinB sinC is greatest, we know that in a ΔABC, sin A sin B sin C is greatest if $A=B=C=\frac{\pi}{3}$

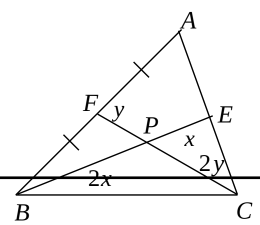
$$\therefore \text{Least A.M.}=-\frac{4}{3}\left(\frac{\sqrt{3}}{2}\right)^3=-\frac{\sqrt{3}}{2}$$

47. In the triangle ABC the medians from B and C are perpendicular. The value of $\cot B+\cot C$ cannot be

A) $\frac{1}{3}$ B) $\frac{2}{3}$ C) $\frac{4}{3}$ D) $\frac{5}{3}$

Key : A

Sol. $\tan B=\dfrac{\frac{y}{2x}+\frac{y}{x}}{1-\frac{y^2}{2x^2}}=\dfrac{3xy}{2x^2-y^2}$



$\tan\lfloor PBF=\frac{y}{2x}$

$\tan\lfloor PBC=\frac{y}{x}$

$$\cot B = \frac{2x^2 - y^2}{3xy},\ \cot C = \frac{2y^2 - x^2}{3xy}$$

$$\cot B + \cot C = \frac{x^2 + y^2}{3xy} \geq \frac{2}{3}$$

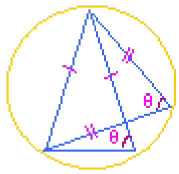
48. T_1 is an isosceles triangle with circumcircle K. Let T_2 be another isosceles triangle inscribed in K whose base is one of the equal side of T_1 and which overlaps the interior of T_1. Similarly create isosceles triangles T_3 from T_2, T_4 from T_3 and so on to the triangle T_n. Then the base angle of the triangle T_n as $n \to \infty$ is

a) 30^0 b) 60^0 c) 90^0 d) 120^0

Key : B

Sol : T_1 is an isosceles triangle with circumcricle K. Let T_2 be another isosceles triangle inscribed in K whose base is one of the equal sides of T_1 and which overlaps the interior of T_1. Similarly create isosceles triangles T_3 from T_2, T_4 from T_3 and so on, do the triangles T_n approach an equilateral triangle as $n \to \infty$?

Note that the base angle of T_n is equal to the angle opposite the base of T_{n+1} (as the figure indicates). Therefore, if θ is the base angle for T_n, then the base of angle for the next triangle (T_{n+1}) is $\frac{180^0 - \theta}{2} = 90^0 - \frac{\theta}{2}$.



Suppose, now that θ is the base angle for T_1, then the base angle for T_n is

$$90 - \frac{90}{2} + \frac{90}{4} - \frac{90}{8} + ... + (-1)^{n-2} + \frac{90}{2^{n-2}} + (-1)^{n-1}\frac{\theta}{2^{n-1}}.$$

Note that the limit as $n \to \infty$ of the above is $\frac{90}{1+1/2} = 60^0$ by formula for the sum of an infinite

49. R is the circum radius of $\Delta le\ ABC$ whose circum centre is 'S'. is the circum radius of $\Delta le\ SBC$. Then the ratio $R : R'$ is

a) 1 b) depends upon side BC

c) independent of d) depends on

c) A is true , R is false

d) A is false, R is true

KEY : D

HINT. $R = \frac{a}{\operatorname{Sin}A}, R' = \frac{a}{\operatorname{Sin}2A}$

$$\therefore \frac{R}{R'} = 2\text{CosA}$$

50. In a triangle ABC, $A - B = 120^0$ and $R = 8r$ then the value of $\cos C$ is

(A) $\frac{1}{4}$ (B) $\frac{\sqrt{15}}{4}$

(C) $\frac{7}{8}$ (D) $\frac{\sqrt{3}}{2}$

KEY : C

HINT : $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$$\Rightarrow 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{1}{16}$$

$$\Rightarrow \left[\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right]\sin\frac{C}{2} = \frac{1}{16}$$

$$\Rightarrow \left(\frac{1}{2} - \sin\frac{C}{2}\right)\sin\frac{C}{2} = \frac{1}{16}$$

$$\Rightarrow \left(\frac{1}{4} - \sin\frac{C}{2}\right)^2 = 0 \Rightarrow \sin\frac{C}{2} = \frac{1}{4}$$

Hence $\cos C = 1 - 2\sin^2\frac{C}{2} = 1 - 2\times\frac{1}{16} = \frac{7}{8}$

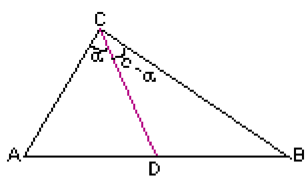
51. In a scalene $D ABC$, D is a point on the side AB such that $CD^2 = AD \cdot DB$, if $\text{SinA}\cdot\text{SinB} = \text{Sin}^2\frac{C}{2}$ then CD is

a) Median through C b) Internal bisector of

c) Altitude through C d) Divides AB in the ratio 1 : 2

Key : B

Sol : Let $\angle ACD = \alpha \Rightarrow \angle DCB = (C - \alpha)$



Applying the sine rule in ΔACD and in ΔDCB respectively, we get

$$\frac{AD}{\sin\alpha} = \frac{CD}{\sin A} \text{ and } \frac{BD}{\sin(C-\alpha)} = \frac{CD}{\sin B}$$

$$\Rightarrow \quad \frac{AD\cdot BD}{\sin\alpha\cdot\sin(C-\alpha)}=\frac{CD^2}{\sin A\cdot\sin B}$$

$$\Rightarrow \quad \frac{1}{2}\left[\cos(2\alpha-C)-\cos C\right]=\frac{1}{2}\left[\cos(2\alpha-c)-1+2\sin^2\frac{C}{2}\right]=\sin^2\frac{C}{2}-\frac{1}{2}\left(1-\cos(2\alpha-C)\right)$$

since, $1-\cos(2\alpha-C)\geq 0$

$\Rightarrow \sin A\cdot\sin B\leq\sin^2\frac{C}{2}$

and equality sign holds, if $1-\cos(2\alpha-C)=0$

$\Rightarrow \alpha=\frac{C}{2}$

That means equality sign holds, if CD is the internal angle bisector of angle C.

52. The perimeter of a triangle ABC is 6 times the arithmetic mean of the sines of its angles. If the side a is 1,then $\lfloor A$ is

a) $\frac{\pi}{6}$ b) $\frac{\pi}{3}$

c) $\frac{\pi}{2}$ d) $\frac{2\pi}{3}$

Key: A

Hint $2s=6\left(\frac{\sin A+\sin B+\sin C}{3}\right)$

53. The radii of the escribed circles of ΔABC are r_a, r_b and r_c respectively. If $r_a + r_b = 3R$ and $r_b + r_c = 2R$, then the smallest angle of triangle is

a) $\tan^{-1}(\sqrt{2}-1)$ b) $\frac{1}{2}\tan^{-1}(\sqrt{3})$

c) $\frac{1}{2}\tan^{-1}(\sqrt{2}+1)$ d) $\tan^{-1}(2-\sqrt{3})$

sol : We have $r_a + r_b = 3R \Rightarrow \frac{\Delta}{s-a}+\frac{\Delta}{s-b}=3r=\frac{3abc}{4\Delta}\left(R=\frac{abc}{4\Delta}\right)$

$$\Rightarrow \frac{\Delta(s-b+s-a)}{(s-a)(s-b)}=\frac{3abc}{4\Delta}\Rightarrow\frac{c\Delta}{(s-a)(s-b)}=\frac{3abc}{4\Delta}\Rightarrow\frac{\Delta^2}{(s-a)(s-b)}=\frac{3ab}{4}$$

$\Rightarrow 4s(s-c) = 3ab \Rightarrow (a+b+c)(a+b-c) = 3ab$

$\Rightarrow (a+b)^2 - c^2 = 3ab$ $\Rightarrow a^2 + b^2 - c^2 = ab$

$\Rightarrow c^2 = a^2 + b^2 - ab$

$\Rightarrow a^2 + b^2 - 2ab\cos C = a^2 + b^2 - ab$ (As $c^2 = a^2 + b^2 - 2ab\cos C$)

$\Rightarrow \cos C = \frac{1}{2} \Rightarrow \angle C = 60^o$ …………….. (1)

Clearly from $r_b + r_c = 2R$

$\Rightarrow \frac{\Delta}{s-b} + \frac{\Delta}{s-c} = 2R \quad \Rightarrow \frac{\Delta(2s-b-c)}{(s-b)(s-c)} = \frac{2abc}{4\Delta} \Rightarrow \frac{2\Delta^2}{(s-b)(s-c)} = bc$

$\Rightarrow 2s(s-a) = bc \Rightarrow (b+c+a)(b+c-a) = 2bc \Rightarrow (b+c)^2 - a^2 = 2bc$

Note : Angles A, C, B are in AP can be converted into more than one

54. With usual notations, in a triangle ABC, a cos(B – C) + b cos(C – A) + c cos(A – B) is equal to

(A) $\frac{abc}{R^2}$ (B) $\frac{abc}{4R^2}$ (C) $\frac{4abc}{R^2}$ (D) $\frac{abc}{2R^2}$

Key. A

Sol. Here a(cosB cosC + sinB sinC) + ...

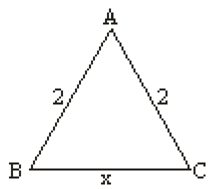
using $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

a (cosB cosC + $\frac{bc}{4R^2}$) + ...

= $\frac{3abc}{4R^2}$ + a cosB cosC + b cosC cosA + c cosA cosB = $\frac{3abc}{4R^2}$ + c cosC + c cosA cosB

= $\frac{3abc}{4R^2}$ + c [cosA cosB – cos(A + B)] = $\frac{3abc}{4R^2}$ + c sinA sinB = $\frac{3abc}{4R^2} + \frac{abc}{4R^2} = \frac{abc}{R^2}$

55. An isosceles triangle has sides of length 2, 2, and x. The value of x for which the area of the triangle is maximum, is



(A) 1 (B) $\sqrt{2}$

(C) 2 (D) $2\sqrt{2}$

Key. D

Sol. $\frac{1}{2} \times 2 \times 2\sin A$ which is maximum if A = 90° $\Rightarrow$ x = $2\sqrt{2}$]

56, In a Δ ABC if b + c = 3a then $\cot\frac{B}{2} \cdot \cot\frac{C}{2}$ has the value equal to :

(A) 4 (B) 3 (C) 2 (D) 1

Key. C

Sol. $\cot\frac{B}{2} \cdot \cot\frac{C}{2} = \frac{s(s-b)}{\Delta} \cdot \frac{s(s-c)}{\Delta} \cdot \frac{(s-a)}{s-a} = \frac{s}{s-a} = \frac{2s}{2s-2a}$

but given that a + b + c = 4a $\Rightarrow$ 2s = 4a Hence $\cot\frac{B}{2}\cdot\cot\frac{C}{2} = \frac{4a}{2a} = 2$

57. Let f, g, h be the lengths of the perpendiculars from the circumcentre of the Δ ABC on the sides a, b and c respectively . If $\frac{a}{f}+\frac{b}{g}+\frac{c}{h} = \lambda\frac{a\,b\,c}{f\,g\,h}$ then the value of λ is :

(A) 1/4 (B) 1/2 (C) 1 (D) 2

Key. A

Sol. $\tan A = \frac{a}{2f} \Rightarrow \frac{1}{2}\,\Sigma\tan A = \frac{1}{2}\,\Pi\tan A$

$= \frac{1}{4}\left(\frac{a}{f}\cdot\frac{b}{g}\cdot\frac{c}{h}\right) \Rightarrow$ A]

58. In a triangle ABC, R(b + c) = $a\sqrt{bc}$ where R is the circumradius of the triangle. Then the triangle is

(A) Isosceles but not right (B) right but not isosceles

(C) right isosceles (D) equilateral

Key. C

Sol. R(b + c) = $a\sqrt{bc}$

R(b + c) = 2RsinA $\sqrt{bc}$

$\therefore$ $\sin A = \frac{b+c}{2\sqrt{bc}}$

now applying AM $\geq$ GM for b and c

$\frac{b+c}{2bc} \geq \sqrt{bc}$; $\therefore \frac{b+c}{2bc} \geq 1$

hence sin A $\geq$ 1 which is not possible.

hence sin A = 1 $\Rightarrow$ A = 90°

$\therefore$ A = 90° and b = c $\Rightarrow$ (C)

59. A triangle with integral sides has perimeter 8 cm. Then the area of the triangle, is

(A) $2\sqrt{2}\,\text{cm}^2$ (B) $\frac{16}{9}\sqrt{3}\,\text{cm}^2$ (C) $2\sqrt{3}\,\text{cm}^2$ (D) $4\sqrt{2}\,\text{cm}^2$

Key. A

Sol. Only possibility for the sides can be 3, 3, 2 (think !)

$A = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{4\times1\times1\times2} = 2\sqrt{2}\,\text{cm}^2$

60. In triangle ABC, $a^2+c^2 = 2002b^2$ then $\frac{\cot A+\cot C}{\cot B} =$

A) $\frac{1}{2001}$ B) $\frac{2}{2001}$ C) $\frac{3}{2001}$ D) $\frac{4}{2001}$

Key. B

Sol. $\frac{\cot A+\cot C}{\cot B} = \frac{\sin(A+C)\sin B}{\sin A\sin C\sin B} = \frac{\sin^2 B}{\sin A\cos B\sin C}$

$$= \frac{4R^2b^2}{4R^2ac\cos B} = \frac{2b^2}{2ac\cos B} = \frac{2b^2}{a^2+c^2-b^2}$$

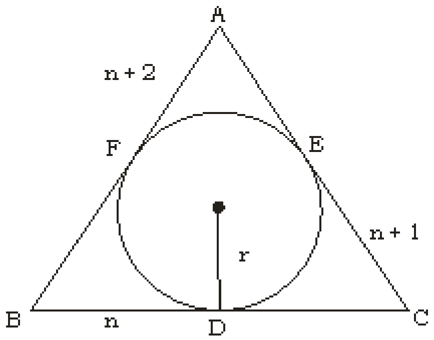
$$= \frac{2b^2}{2002b^2-b^2} = \frac{2}{2001}$$

61. The circle touches the sides BC, CA and AB of respectively at D, E and F. If the lengths BD, CE and AF are consecutive integers then the largest side of the triangle is equal to

a) 13
b) 14
c) 15
d) cannot be determined

Sol: Let BD = n, CE = n + 1, AF = n + 2.

Then BD = BF = n, CE = CD = n + 1, AF = AE = n + 2



$\therefore$ a = BC = 2n +1, b =2n +3, c =2n+2, s = 3n+3

$$r = \frac{\Delta}{s} = \frac{\sqrt{(3n+3)(n+2)n(n+1)}}{3n+3}$$

$$\therefore 4 = \sqrt{\frac{(n+2)n}{3}} \Rightarrow n(n+2) = 48 \Rightarrow n = 6$$

$\therefore$ the largest side of the triangle is 2n + 3 = 15.

62. In a ΔABC, medians AD and BE are drawn. If AD = 4, $\angle DAB = \frac{\pi}{6}$ and $\angle ABE = \frac{\pi}{3}$ then the area of ΔABC is

(A) $\frac{64}{3}$
(B) $\frac{8}{3\sqrt{3}}$
(C) $\frac{16}{3}$
(D) $\frac{32}{3\sqrt{3}}$

Key. D

Sol. The medians intersect at centroid G with AG = $\frac{8}{3}$ ($\because$ AG : GD = 2 : 1)

$$\angle AGB = \frac{\pi}{2} \Rightarrow BG = \frac{8}{3}\cot\frac{\pi}{3} = \frac{8}{3\sqrt{3}}$$

Area of $\Delta AGB = \frac{1}{2} \times \frac{8}{3\sqrt{3}} \times \frac{8}{3} = \frac{32}{9\sqrt{3}}$ $\therefore$ Area of $\Delta ABC = \frac{32}{3\sqrt{3}}$

63. In a triangle ABC, $A - B = 120^0$ and $R = 8r$ then the value of $\cos C$ is

(A) $\frac{1}{4}$ (B) $\frac{\sqrt{15}}{4}$

(C) $\frac{7}{8}$ (D) $\frac{\sqrt{3}}{2}$

Key. (c)

Sol. $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$\Rightarrow 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{1}{16}$

$\Rightarrow \left[\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right]\sin\frac{C}{2} = \frac{1}{16}$

$\Rightarrow \left(\frac{1}{2} - \sin\frac{C}{2}\right)\sin\frac{C}{2} = \frac{1}{16}$

$\Rightarrow \left(\frac{1}{4} - \sin\frac{C}{2}\right)^2 = 0 \Rightarrow \sin\frac{C}{2} = \frac{1}{4}$

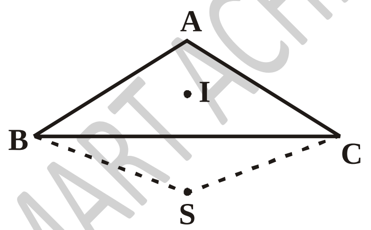
Hence $\cos C = 1 - 2\sin^2\frac{C}{2} = 1 - 2\times\frac{1}{16} = \frac{7}{8}$

64. In a Δ ABC the incentre and circumcentre are *reflections* of each other in side BC. Hence the measure of ∠ BAC (in degrees) is

(a) 120 (b) 108 (c) 135 (d) 105

Key. (b)

Sol.



I : the incentre

S : the circumcentre

$\angle BIC = 90^0 + \frac{A}{2}$ (standard result)

and reflex $\angle BSC = 2A \Rightarrow \angle BSC = 360^0 - 2A$

Hence $90^0 + \frac{A}{2} = 360^0 - 2A$

65. ABC is a triangle. Put $x = a\cos A$, $y = b\cos B$, $z = c\cos C$.

x, y, z are the side lengths of a triangle

(a) only if Δ ABC is equilateral (b) only if Δ ABC is obtuse

(c) only if Δ ABC is a right triangle (d) for any acute Δ ABC

Key. (d)

Sol. For any acute triangle ABC, x, y and z are the side lengths of the triangle formed by the feet of the altitudes of Δ ABC.

66. If ABC is a triangle in which $\frac{\pi}{2} < C < \pi$, then the quantity $\frac{a^2+b^2}{c^2}$ lies in the interval

(a) $(0, \frac{1}{2})$ (b) $(1, \frac{3}{2})$ (c) $(\frac{3}{2}, 2)$ (d) $(\frac{1}{2}, 1)$

Key. (d)

Sol. $\frac{\pi}{2} < C < \pi \Rightarrow \frac{a^2+b^2-c^2}{2ab} = \cos C < 0$

$\Rightarrow a^2 + b^2 < c^2$

$\Rightarrow \frac{a^2+b^2}{c^2} < 1.$

Further $\frac{a^2+b^2}{2} \geq \left(\frac{a+b}{2}\right)^2 > \left(\frac{c}{2}\right)^2 \Rightarrow \frac{a^2+b^2}{c^2} > \frac{1}{2}$

67. If cos A + cosB + 2cosC = 2 then the sides of the Δ ABC are in

(A) A.P. (B) G.P (C) H.P. (D) none

Key. A

Sol. $\cos A + \cos B = 2(1-\cos C) = 4\sin^2\frac{C}{2}$ or $2\cos\frac{A+B}{2}\cos\frac{A-B}{2} = 4\sin^2\frac{C}{2}$

or $\cos\frac{A-B}{2} = 2\sin\frac{C}{2}$ or $2\cos\frac{C}{2}\cos\frac{A-B}{2} = 4\sin\frac{C}{2}\cos\frac{C}{2} = 2\sin C$

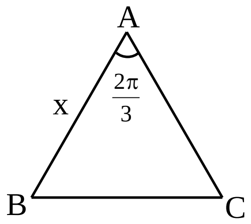
$2\sin\frac{A+B}{2}\cos\frac{A-B}{2} = 2\sin C$ or $\sin A + \sin B = 2\sin C \Rightarrow$ a, c, b are in A.P.

68. Let ABC be a triangle with $\angle BAC = \frac{2\pi}{3}$ and AB = x such that AB.AC=1. If x varies then the largest possible length of internal angular bisector AD is

A) 1 B) 2 C) $\frac{1}{2}$ D) $\frac{1}{4}$

Key. C

Sol.



Angular bisector $AD = \frac{2bc}{b+c}\cos\frac{A}{2}$

$$= \frac{2 \times x \times \frac{1}{x}}{x + \frac{1}{x}} \times \frac{1}{2}$$

69. Let I be the incentre of the triangle ABC, where $\frac{\overrightarrow{BC}}{|\overrightarrow{BC}|} + \frac{\overrightarrow{BA}}{|\overrightarrow{BA}|} = \frac{\overrightarrow{BI}}{k}$ then the diameter of the circumcircle of the triangle is

(A) k(cos A/2 + cos C/2) (B) k(sin A/2 + sin C/2)
(C) k(cot A/2 + cot C/2) (D) k (tan A/2 + tan C/2)

Key. C

Sol. Taking modulus both sides

2cos B/2 = $\frac{1}{k}|\overrightarrow{BI}| = \frac{1}{k}\frac{r}{\sin B/2} = \frac{kR\sin A/2 \sin C/2}{k}$

$\Rightarrow$ 2R = $\frac{k\sin\left(\frac{A+C}{2}\right)}{\sin A/2 \sin C/2}$ = k (cotA/2 + cotC/2)

70. Let in a triangle ABC, $\frac{\overrightarrow{BC}}{|\overrightarrow{BC}|} + \frac{\overrightarrow{BA}}{|\overrightarrow{BA}|} = \frac{1}{k}\overrightarrow{BI}$ then the diameter of the circumcircle of the ΔABC is

(A) k (cosA/2 + cosC/2) (B) k (sinA/2 + sinC/2)
(C) k (cot A/2 + cot C/2) (D) k (tan A/2 + tan C/2)

Key. C

Sol. Taking modulus both sides

2cosB/2 = $\frac{1}{k}\left|\overrightarrow{BI}\right| = \frac{1}{k}.\frac{r}{\sin B/2} = \frac{4R\sin\frac{A}{2}\sin\frac{C}{2}}{k}$

$\Rightarrow$ 2R = $\frac{k\sin\left(\frac{A+C}{2}\right)}{\sin\frac{A}{2}\sin\frac{C}{2}}$ = k (cot A/2 + cot C/2)

71. In ΔABC, $A=\frac{2\pi}{3}, b-c=3\sqrt{3}$ cm and area of ΔABC = $\frac{9\sqrt{3}}{2}cm^2$, then BC =

A) $6\sqrt{3}$ cm B) 9cm C) 18cm D) 27cm

Key. B

Sol. $\frac{1}{2}bc\sin\frac{2\pi}{3}=\frac{9\sqrt{3}}{2}\Rightarrow bc=18\Rightarrow b^2+c^2-36=27\Rightarrow b^2+c^2=63$

$a^2=63-2\times18\times\frac{-1}{2}=81\Rightarrow a=9$

72. In ΔABC, if $\cot A=\sqrt{ac}, \cot B=\sqrt{\frac{c}{a}}, \cot C=\sqrt{\frac{a^3}{c}}$ then which of the following can be true?

A) $a+a^2=1-c$ B) $a+a^2=1+c$

C) $a+a^2=2-c$ D) $a+a^2=2+c$

Key. A

Sol. $\cot A\cot B=C,\ \cot B\cot C=a,\ \cot C\cot A=a^2$

But $\sum\cot A\cot B=1\Rightarrow c+a+a^2=1\Rightarrow a+a^2=1-c$

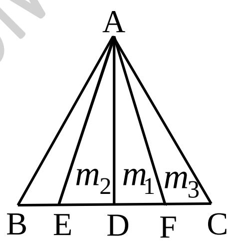
73. Let AD be a median of ΔABC. If AE and AF are medians of ΔABD and ΔADC respectively and AD = m_1, AE = m_2, AF = m_3, BC = a, then $\frac{a^2}{8}=$

A) $m_2^2+m_3^2-2m_1^2$ B) $m_1^2+m_2^2-2m_3^2$

C) $m_1^2+m_3^2-2m_2^2$ D) $m_1^2+m_2^2+m_3^2$

Key. A

Sol. $m_2^2+m_3^2=2\left(m_1^2+ED^2\right)\Rightarrow m_2^2+m_3^2-2m_1^2=\frac{a^2}{8}$

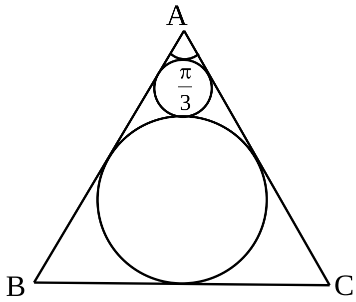


74. In ΔABC, $\angle A=\frac{\pi}{3}$ and its inradius is 6 units. The radius of the circle touching the sides AB, AC internally and the incircle of ΔABC externally is

A) 3units B) 3/2 units C) 2 units D) 4 units

Key. C

Sol. Angle between the direct common tangents is $\frac{\pi}{3}$



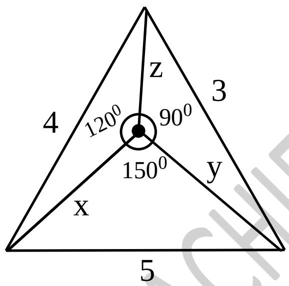
$\therefore\ 2\sin^{-1}\left(\frac{6-r}{6+r}\right)=\frac{\pi}{3}\Rightarrow\frac{6-r}{6+r}=\frac{1}{2}$

$\Rightarrow 12-2r=6+r\Rightarrow 6=3r\Rightarrow r=2$.

75. Three positive real numbers x, y, z satisfy the equations $x^2+\sqrt{3}xy+y^2=25$, $y^2+z^2=9$ and $x^2+xz+z^2=16$ then the value of $xy+2yz+\sqrt{3}xz$ is

A) 18 B) 24 C) 30 D) 36

Key. B



Sol.

Area of triangle = $\frac{1}{2}\times 3\times 4=\frac{1}{2}xz\frac{\sqrt{3}}{2}+\frac{1}{2}xy\times\frac{1}{2}+\frac{1}{2}yz$

$$\Rightarrow 24=\sqrt{3}xz+xy+2yz$$

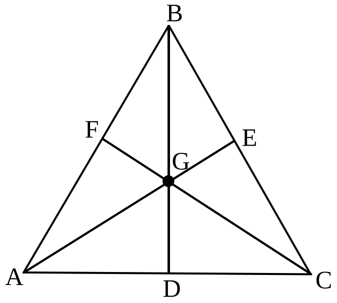
76. If m_a, m_b, m_c are lengths of medians through the vertices A, B, C of triangle ABC respectively, then length of side c =

A) $\frac{1}{3}\sqrt{2m_a^2+2m_c^2-m_b^2}$ B) $\frac{2}{3}\sqrt{2m_a^2+2m_c^2-m_b^2}$

C) $\frac{1}{3}\sqrt{2m_a^2+2m_b^2-m_c^2}$ D) $\frac{2}{3}\sqrt{2m_a^2+2m_b^2-m_c^2}$

Key. D

Sol. $AG=\frac{2}{3}ma,\ CG=\frac{2}{3}mc$



$$c^2+\frac{4}{9}mc^2=2\left(\frac{4}{9}ma^2+\frac{4}{9}mb^2\right)$$

77. If the bisector of angle 'A' of triangle ABC makes an angle 'θ' with $\overline{BC}$, then $\sin\theta$ is equal to

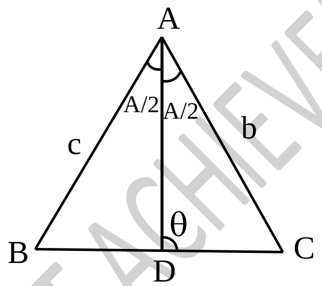
A) $\cos\left(\frac{B-C}{2}\right)$ B) $\sin\left(\frac{B-C}{2}\right)$

C) $\sin\left(B-\frac{A}{2}\right)$ D) $\sin\left(C-\frac{A}{2}\right)$

Key. A

Sol. $\theta=B+\frac{A}{2}=B+\frac{180^0-(B+C)}{2}$

$$=90^0+\left(\frac{B-C}{2}\right)$$

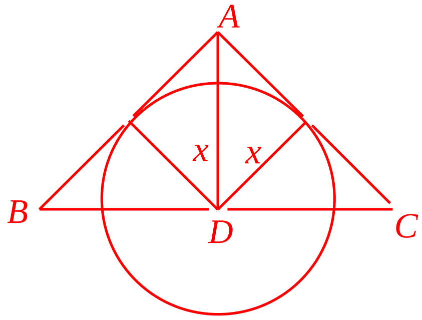


$\sin\theta=\cos\left(\frac{B-C}{2}\right)$

78. A circle of diameter '$2x$' is drawn on the side BC of triangle ABC such that it touches the sides, AB and AC. Then x =

A) $\frac{\Delta}{2(b+c)}$ B) $\frac{2\Delta}{b+c}$ C) $\frac{bc}{2\Delta}$ D) $\frac{b+c}{2\Delta}$

Key. B



Sol.

$$\Delta = \frac{1}{2}x(AB + AC) \Rightarrow x = \frac{2\Delta}{b+c}$$

79. If in a triangle ABC, $b\cos^2\frac{A}{2} + a\cos^2\frac{B}{2} = \frac{3c}{2}$ then minimum value of $\frac{a+c}{2c-a} + \frac{b+c}{2c-b}$ is equal to

A) 2 B) 4 C) 6 D) 8

Key. B

Sol. L.H.S. = $\frac{1}{2}(b + b\cos A + a + a\cos B)$

$$\Rightarrow \frac{1}{2}(a+b+c) = \frac{3}{2}c \Rightarrow 2c = a+b$$

$$\frac{a+c}{2c-a} + \frac{b+c}{2c-b} = \frac{a+c}{b} + \frac{b+c}{a} = \frac{a}{b} + \frac{b}{a} + \frac{c}{a} + \frac{c}{b}$$

$$\geq 4\left(\frac{c^2}{ab}\right)^{1/4} \geq 4$$

80. A right angled triangle ABC of maximum area is inscribed in a circle of radius R, then (Here Δ is area and s is semi perimeter, r_1, r_2, r_3 exradi of ΔABC)

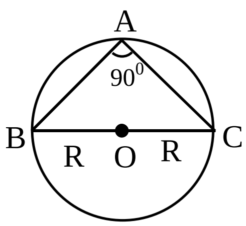
A) $\Delta = 2R^2$ B) $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{\sqrt{2}+1}{R}$

C) $r = (\sqrt{2}-1)R$ D) $s = (2+\sqrt{2})R$

Key. B

Sol. In ΔABC, $AB = AC = \sqrt{2}R$

$$S = R(\sqrt{2}+1),\ \Delta = R^2$$



$$r=\frac{\Delta}{s}=\frac{R}{\sqrt{2}+1}\Rightarrow\frac{1}{r}=\frac{\sqrt{2}+1}{R}$$

81 If an acute angled triangle ABC, if H is the orthocenter $AH = x$, $BH = y$, $CH = z$ then $x^2+y^2+z^2=$

A. $16R^2-(a^2+b^2+c^2)$ B. $12R^2-(a^2+b^2+c^2)$

C. $9R^2-(a^2+b^2+c^2)$ D. $8R^2-(a^2+b^2+c^2)$

KEY. B

SOL. $AH=2R\cos A, BH=2R\cos B, CH=2R\cos C$

$x^2+y^2+z^2=4R^2(\cos^2 A+cos^2 B+\cos^2 C)$

$=4R^2\{3-\sin^2 A-\sin^2 B-\sin^2 C\}$

$=12R^2-(a^2+b^2+c^2)$

82 Let ABC be a triangle such that $\angle ACB=\frac{\pi}{6}$ and let a,b,c denote the length of the sides opposite to A,B and C respectively . The value of x for which $a=x^2+x+1, b=x^2-1, c=2x+1$ is

A. $2+\sqrt{3}$ B. $2-\sqrt{3}$ C. $1+\sqrt{3}$ D. $4\sqrt{3}$

KEY. C

SOL. $A=120^0, C=30^0, B=30^0$

$b=c\Rightarrow x^2-1=2x+1$

$x^2-2x-2=0$

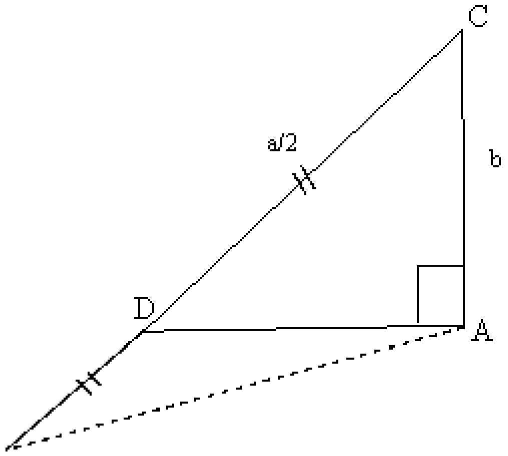
$x=\frac{2\pm\sqrt{4-4(1)(-2)}}{2.1}=\frac{2\pm\sqrt{12}}{2}=\frac{2\pm2\sqrt{3}}{2}=1\pm\sqrt{3}$

$\Rightarrow x=1+\sqrt{3}$

83 In ΔABC , D is the midpoint of BC. If AD is perpendicular to AC. Then cosA.cosC=

A. $\frac{c^2-a^2}{3ac}$ B. $\frac{3(c^2-a^2)}{2ac}$ C. $\frac{2(c^2-a^2)}{3ac}$ D. $\frac{2(a^2-c^2)}{3ac}$

KEY. C



SOL.

$$\cos C = \frac{b}{\left(\frac{a}{2}\right)} = \frac{2b}{a}$$

$$\frac{a^2+b^2-c^2}{2ab} = \frac{2b}{a} \Rightarrow a^2+b^2+c^2 = 4b^2$$

$$a^2-c^2 = 3b^2$$

$$\cos A.\cos c = \frac{b^2+c^2-a^2}{2bc}\left(\frac{2b}{a}\right) = \frac{2(c^2-a^2)}{3ac}$$

84 In ΔABC if r =1, R =5 , $\Delta = 10$ then ab + bc + ca =

A. 81 B. 121 C. 141 D. 111

KEY. B

SOL. $r(r_1+r_2+r_3)+s^2 = ab+bc+ca$

$$1(r+4r)+s^2 = ab+bc+ca$$

$$1(1+4.5)+10^2 = ab+bc+ca \Rightarrow 100+21=121$$

$$r = \frac{\Delta}{s}$$

$$1 = \frac{10}{s}$$

$$s = 10$$

85. If in an equilateral triangle, inradius is a rational number then which of the following is not true?

(A) circum-radius is always rational (B) area is always irrational

(C) ex-radii are always rational (D) perimeter is always rational

Key. D

Sol. Clearly $r = \frac{R}{2} \Rightarrow R \in Q$, now $r_1 = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = 4R\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)^2 \in Q$.

Similarly $r_2, r_3 \in Q$. Now $\Delta = \frac{abc}{4R} = 2R^2 \sin A \sin B \sin C = 2R^2\left(\frac{\sqrt{3}}{2}\right)^3 \notin Q$

Also s = a + b + c = 2R(sinA + sinB + sinC) = $= 3\sqrt{3}R \notin Q$

86. If in equilateral triangle, in-radius is a rational number then which of the following is not true?

(A) circum-radius is always rational (B) area is always irrational

(C) ex-radii are always rational (D) perimeter is always rational

Key. D

Sol. Clearly $r = \frac{R}{2} \Rightarrow R \in Q$, now $r_1 = 4R \sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = 4R\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)^2 \in Q$.

Similarly $r_2, r_3 \in Q$. Now $\Delta = \frac{abc}{4R} = 2R^2 \sin A \sin B \sin C = 2R^2\left(\frac{\sqrt{3}}{2}\right)^3 \notin Q$

Also s = a + b + c = 2R(sinA + sinB + sinC) = $= 3\sqrt{3}R \notin Q$

87. In an isosceles triangle ABC, AB=AC. If vertical angle A is 20^0, then a^3+b^3 is equal to

a) $3a^2b$ b) $3b^2c$ c) $3c^2a$ d) abc

Key. C

Sol. $\because \angle A = 20^o$

$\therefore \angle B = \angle C = 80^o$

Then, $b = c$

$\therefore \frac{a}{\sin 20^o} = \frac{b}{\sin 80^o} = \frac{c}{\sin 80^o}$

Or $\frac{a}{\sin 20^o} = \frac{b}{\cos 10^o}$

$\Rightarrow a = 2b \sin 10^o$

$\therefore a^3 + b^3 = 8b^3 \sin^3 10^o + b^3 = b^3\left\{2\left(4\sin^3 10^o\right)+1\right\} = b^3\left\{6\sin 10^o\right\} = 3ac^2$

88. Which of the following pieces of data does not uniquely determine acute angled ΔABC (R = circum radius)

a) a, sin A ,sinB b) a,b,c c) a, sinB, R d) a, sinA , R

Key. D

Sol. $\because$ In a ΔABC, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin\left\{\pi - (A+B)\right\}} = 2R$

$$\Rightarrow \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin(A+B)} = 2R$$

Alternate. (a) : If we know a, sinA, sinB then we can find b, c, A, B and C.

Alternate. (b) : We can find A, B, C by using cosine rule.

Alternate. (c) : $\because$ a, sinB, R are given then we can find sinA, b and hence.

$$\sin C = \sin\{\pi - (A+B)\} = \sin C$$

Alternate. (d) : a, sinA, R are given then we know only the ratio $\frac{b}{\sin B}$ or $\frac{c}{\sin(A+B)}$; we cannot determine the values of b, c, sinB, sinC separately.

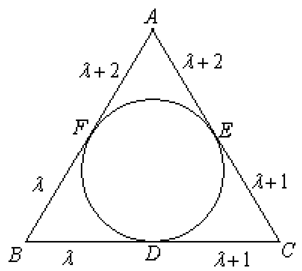
$\therefore$ Triangle ABC cannot be determined in this case.

89. The incircle of a ΔABC touches the sides BC, CA, AB at the points D, E, F respectively. If the lengths of BD, CE, AF respectively are consecutive positive integers and the inradius of the triangle is 4 units, then the perimeter of the triangle is

A) 42 B) 35 C) 84 D) 57

Key. A

Sol.



Now applying $\Delta = rs$, we get λ

90. Tangents at P,Q,R on a circle of radius r form a triangle whose sides are 3r, 4r, 5r then $PR^2 + RQ^2 + QP^2 =$

A) $\frac{84}{5}r^2$ B) $\frac{184}{5}r^2$ C) $\frac{176}{5}r^2$ D) None of these

Key. C

Sol. In $\Delta AIQ \because AI = \frac{r}{\sin A/2}$

AQ = r cot A/2 In ΔARQ

$$RQ = \sqrt{(AR)^2 + (AQ)^2} - 2(AR)(AQ)A$$

= 2(AR) sin A/2

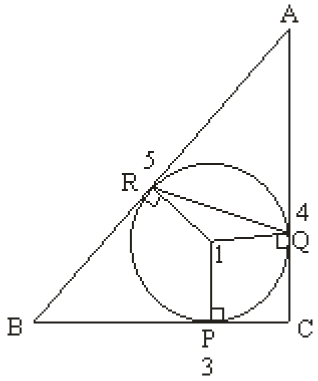
RQ = 2r cos A/2

RP = 4rcos $\left(\frac{B}{2}\right)$, $PQ = 4r\cos\left(\frac{C}{2}\right)$

$$PR^2 + RQ^2 + QP^2 = 16r^2\left[\cos^2\left(\frac{A}{2}\right) + \cos^2\left(\frac{B}{2}\right) + \cos^2\left(\frac{C}{2}\right)\right]$$

$$=16r^2\left[\frac{1+\cos A}{2}+\frac{1+\cos B}{2}+\frac{1}{2}\right]$$

$$=8r^2\left[3+\frac{3}{5}+\frac{4}{5}\right]=8r^2\left[\frac{15+7}{5}\right]=\frac{176r^2}{5}$$



91. In a triangle ABC, if a : b : c = 7 : 8 : 9 then cos A : cos B =

A) $\frac{11}{63}$ B) $\frac{22}{63}$ C) $\frac{2}{9}$ D) none of these

Key. D

Sol. $\cos A=\frac{b^2+c^2-a^2}{2bc}=\frac{64+81-49}{2.\,8.\,9}=\frac{145-49}{144}=\frac{96}{144}$

$\cos B=\frac{a^2+c^2-b^2}{2bc}=\frac{49+81-64}{2.\,7.\,9}=\frac{66}{126}=\frac{11}{21}$

92. In a triangle ABC, if cos A + cos B + cos C = $\frac{7}{4}$ then $\frac{R}{r}$ is equal to

A) $\frac{3}{4}$ B) $\frac{4}{3}$ C) $\frac{2}{3}$ D) $\frac{3}{2}$

Key. A

Sol. $\cos A+\cos B+\cos C=\frac{7}{4}$

$1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{7}{4}$

$4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{3}{4}$ (Q R = 4r sin A/2 sin B/2 sin C/2)

$\frac{R}{r}=\frac{3}{4}$

93. In $\Delta ABC\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}$ is equal to

A) $\frac{\Delta}{r^2}$ B) $\frac{(a+b+c)^2}{abc}$, 2R C) $\frac{\Delta}{r}$ D) $\frac{\Delta}{Rr}$

Key. A

Sol. $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{s(s-a)}{\Delta}=\frac{s(s-b)}{\Delta}+\frac{s(s-c)}{\Delta}$

$=\frac{s}{\Delta}\left[3s-(a+b+c)\right]$

$$=\frac{s[3s-2s]}{\Delta}=\frac{s^2}{\Delta}$$

$$=\left(\frac{a+b+c}{2}\right)^2\times\frac{4R}{abc}=\frac{(a+b+c)^2 R}{abc} \qquad \left[\because\ \Delta=\frac{abc}{4R}\right]$$

also $\frac{s^2}{\Delta}=\frac{\Delta^2}{r^2\Delta} \qquad =\frac{\Delta}{r^2}$

94. In acute angled triangle ABC, $r+r_1=r_2+r_3$ and $\angle B>\frac{\pi}{3}$ then

A) b + 2c < 2a < 2b + 2c
B) b + 4c < 4a < 2b + 4c
C) b + 4c < 4a < 4b + 4c
D) b + 3c < 3a < 3b + 3c

Key. D

Sol. $r-r_2=r_3-r_1$

$$\frac{\Delta}{s}-\frac{\Delta}{s-b}=\frac{\Delta}{s-c}-\frac{\Delta}{s-a}$$

$$\frac{-b}{s(s-b)}=\frac{-a+c}{(s-a)(s-c)}$$

$$\frac{(s-a)(s-c)}{s(s-b)}=\frac{a-c}{b}$$

$$\tan^2(B/2)=\frac{a-c}{b}$$

But $\frac{B}{2}\in\left(\frac{\pi}{6},\frac{\pi}{4}\right) \Rightarrow \tan^2\frac{B}{2}\in\left(\frac{1}{3},1\right)$

$\Rightarrow \frac{1}{3}<\frac{a-c}{b}<1$

b < 3a – 3c < 3b

b + 3c < 3a < 3b + 3c

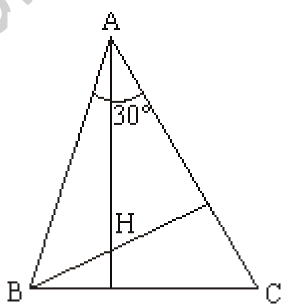
95. In a triangle ABC, $\angle A=30°$, $BC=2+\sqrt{5}$, then the distance of the vertex A from the orthocenter of the triangle is

A) 1
B) $\left(2+\sqrt{5}\right)\sqrt{3}$
C) $\frac{\sqrt{3}+1}{2\sqrt{2}}$
D) $\frac{1}{2}$

Key. B

Sol. $R=\frac{a}{2\sin A}=\frac{2+\sqrt{5}}{2\sin 30°}=\frac{2+\sqrt{5}}{2\times\frac{1}{2}}=\left(2+\sqrt{5}\right)$

Now, AH = 2R cos A = 2 $\left(2+\sqrt{5}\right)$ cos 30° = $\left(2+\sqrt{5}\right)\sqrt{3}$



96. If $c^2=a^2+b^2$, $2s=a+b+c$, then 4s(s – a) (s – b)(s – c) =

A) s^4 B) b^2c^2 C) c^2a^2 D) a^2b^2

Key. D

Sol. $c^2 = a^2 + b^2 \Rightarrow \angle C = \frac{\pi}{2}$

$\therefore \quad \Delta = \frac{1}{2}ab\sin C = \frac{1}{2}ab \Rightarrow \sqrt{s(s-a)(s-b)(s-c)} = \frac{1}{2}ab$

$\Rightarrow \quad 4s(s-a)(s-b)(s-c) = a^2b^2$.

97. If $\cot\frac{A}{2} = \frac{b+c}{a}$, then the ΔABC is

A) isosceles B) equilateral C) right angled D) none of these

Key. C

Sol. $\cot\frac{A}{2} = \frac{b+c}{a} \Rightarrow \frac{\cos A/2}{\sin A/2} = \frac{\sin B + \sin C}{\sin A}$

$\Rightarrow \quad \frac{\cos A/2}{\sin A/2} = \frac{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}{2\sin\frac{A}{2}\cos\frac{A}{2}}$

$\Rightarrow \quad \cos\frac{A}{2} = \cos\left(\frac{B-C}{2}\right) \Rightarrow \frac{A}{2} = \frac{B-C}{2}$

$\Rightarrow \quad A = B - C \quad \Rightarrow \quad A + C = B$

But A + B + C = π. Therefore, $B = \frac{\pi}{2}$

98. In a triangle ABC, (a + b + c) (b + c – a) = λ bc if

A) $\lambda < 0$ B) $\lambda > 6$ C) $0 < \lambda < 4$ D) $\lambda > 4$

Key. C

Sol. 2s(2s – 2a) = λbc

i.e., $4\frac{s(s-a)}{bc} = \lambda$ i.e., $\sin^2\frac{A}{2} = \frac{\lambda}{4}$

$\therefore \quad 0 < \frac{\lambda}{4} < 1$ i.e. $0 < \lambda < 4$

Alternative solution

$(b+c)^2 - a^2 = \lambda\, bc$

$b^2 + c^2 - a^2 = (\lambda - 2)bc$

$\frac{b^2+c^2-a^2}{2bc} = \frac{\lambda-2}{2}$ i.e. $\cos A = \frac{\lambda-2}{2}$

$\therefore \quad -1 < \frac{\lambda-2}{2} < 1$ i.e. $-2 < \lambda - 2 < 2$

i.e. $0 < \lambda < 4$

99. If 'a', 'b', 'c' are the sides of a triangle than the minimum value of $\frac{2a}{b+c-a} + \frac{2b}{c+a-b} + \frac{2c}{a+b-c}$ is

A) 3 B) 9 C) 6 D) 1

Key. C

Sol. Let a + b + c = 2S

Than we have to find minimum value of

$$\frac{a}{S-a}+\frac{b}{S-b}+\frac{c}{S-c}=-3+\frac{S}{S-a}+\frac{S}{S-b}+\frac{S}{S-c}$$

Also, $\dfrac{\frac{S}{S-a}+\frac{S}{S-b}+\frac{S}{S-c}}{3}\geq\dfrac{3}{\frac{S-a}{S}+\frac{S-b}{S}+\frac{S-c}{S}}\ \because\ \frac{S-a}{S}+\frac{S-b}{S}+\frac{S-c}{S}=1$

$\Rightarrow\quad \frac{S}{S-a}+\frac{S}{S-b}+\frac{S}{S-c}\geq 9.$

Thus minimum value of the expression is 6.

100. In triangle ABC, medians AD and BE are mutually perpendicular, then such a triangle would exist if

A) $\frac{1}{4}<\frac{a}{b}<\frac{1}{2}$ B) $\frac{1}{4}<\frac{b}{a}<\frac{3}{4}$ C) $\frac{1}{4}<\frac{a}{b}<\frac{3}{4}$ D) $\frac{1}{2}<\frac{b}{a}<2$

Key. D

Sol. AD and BE are perpendicular thus $b^2+a^2=5c^2$

Since $|a-b|<c \quad\Rightarrow\quad a^2+b^2>5(a-b)^2$

$\Rightarrow\quad 4a^2-10ab+4b^2<0 \quad\Rightarrow\quad \frac{1}{2}<\frac{a}{b}<2$

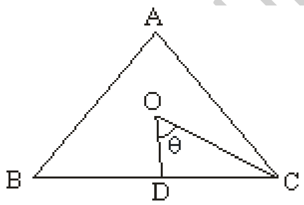
101. Consider a given acute angled triangle ABC having O as its circumcentre. Let D be a variable interior point of the side BC. The limiting value of the circumradius of the ΔOCD as point D approaches towards vertex C is equal to

A) $\frac{R}{2\cos A}$ B) $\frac{R}{\cos A}$ C) $\frac{R}{\sin A}$ D) $\frac{R}{2\sin A}$

Key. B

Sol. In the adjacent figure we have $<OCB=\frac{\pi}{2}-A$

Let $\quad <ODC=\pi-\left(\frac{\pi}{2}-A+\theta\right)=\frac{\pi}{2}+(A-\theta)$



If R, be the circumradius of ΔOCD then

$\frac{OC}{\sin\left(\frac{\pi}{2}+(A-\theta)\right)}=2R, \quad\Rightarrow\quad 2R_1=\frac{R}{\cos(A-\theta)}$

As $\quad D\to C\ \ \theta\to 0 \quad\Rightarrow\quad 2R_1\to\frac{R}{\cos A}$

102. If circumradius and inradius of a triangle be 8 and 3, then value of $\frac{a}{\tan A}+\frac{b}{\tan B}+\frac{c}{\tan C}$ equals

A) 11 B) 33 C) 44 D) 55

Key. D

Sol. $\frac{a}{\tan A}+\frac{b}{\tan B}+\frac{c}{\tan C}=a\cot A+b\cot B+c\cot C$

= 2 (R + r) = 2 (8 + 3) = 22 Ans.

103. ABCD is a quadrilateral circumscribed about a circle of unit radius then

A) $AB\sin\frac{C}{2}.\sin\frac{A}{2}=CD\sin\frac{B}{2}\sin\frac{D}{2}$
B) $AB\sin\frac{A}{2}.\sin\frac{B}{2}=CD\sin\frac{C}{2}\sin\frac{D}{2}$
C) $AB\sin\frac{A}{2}.\sin\frac{A}{2}=CD\sin\frac{C}{2}\sin\frac{B}{2}$
D) $AB\sin\frac{A}{2}.\cos\frac{B}{2}=CD\sin\frac{C}{2}\cos\frac{D}{2}$

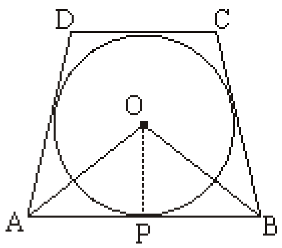
Key. B

Sol. Let 'O' be the centre of circle and 'P' be its point of contact with side AB. We have

$AP = OP.\cot\frac{A}{2}=\cot\frac{A}{2}$ and

$PB = OP.\cot\frac{B}{2}=\cot\frac{B}{2}$

$\Rightarrow \quad AP+PB=\cot\frac{A}{2}+\cot\frac{B}{2}$



$$=\frac{\sin\left(\frac{A+B}{2}\right)}{\sin\frac{A}{2}.\sin\frac{B}{2}}=AB$$

Since $A+B+C=2\pi \quad \Rightarrow \frac{A+B}{2}=\pi-\frac{C+D}{2}$

$\Rightarrow \quad \sin\left(\frac{A+B}{2}\right)=\sin\left(\frac{C+D}{2}\right)$

$\Rightarrow \quad AB.\sin\frac{A}{2}.\sin\frac{B}{2}=\sin\frac{C}{2}.\sin\frac{D}{2}.CD$

104. In triangle ABC, a : b : c = (1 + x): 1: (1 – x) where $x\in(0,1)$. If $\angle A=\frac{\pi}{2}+\angle C$, then x is equal to

A) $\frac{1}{\sqrt{6}}$
B) $\frac{1}{2\sqrt{6}}$
C) $\frac{1}{\sqrt{7}}$
D) $\frac{1}{2\sqrt{7}}$

Key. C

Sol. $a=(1+x)h,\ b=h,\ c=(1-x)h,\frac{A}{2}-\frac{C}{2}=\frac{\pi}{4}$

$\Rightarrow \quad \cos\frac{A}{2}.\cos\frac{C}{2}+\sin\frac{A}{2}\sin\frac{C}{2}=\frac{1}{\sqrt{2}}$

$\Rightarrow \quad \sqrt{\frac{S^2(S-a)(S-c)}{bc.ab}}+\sqrt{\frac{(S-b)(S-c)(S-a)(S-b)}{bc.ab}}=\frac{1}{\sqrt{2}}$

$\Rightarrow \quad \frac{S}{b}\sqrt{\frac{(S-a)(S-c)}{ac}}+\frac{(S-b)}{b}\sqrt{\frac{(S-a)(S-c)}{ac}}=\frac{1}{\sqrt{2}}$

$$\Rightarrow \quad \left(\frac{2S-b}{b}\right)\sqrt{\frac{(S-a)(S-b)}{ac}} = \frac{1}{\sqrt{2}} \Rightarrow \quad \frac{a+c}{b}\sqrt{\frac{(S-a)(S-b)}{ac}} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad 2\left(\frac{a+c}{b}\right)^2 = \frac{ac}{(s-a)(s-c)}$$

Now a + c = 2h, b = h

$$\Rightarrow \quad \frac{a+c}{b} = 2,\ s = \frac{a+b+c}{2} = \frac{3h}{2}$$

$$\Rightarrow \quad S-a = \frac{(1-2x)h}{2}, (S-c) = \frac{(1-2x)h}{2}$$

$$\Rightarrow \quad 8 = \frac{(1+x^2)4}{(1-4x^2)} \quad \Rightarrow \quad x = \frac{1}{\sqrt{7}}$$

Progressions of Triangles

Integer Answer Type

1. If a, b, c are sides of a triangle satisfying $a^2 + b^2 + c^2 = 6$, then the AM of all the integral values which lie in the interval of ab + bc + ca is

Key. 5

Sol. $\frac{1}{2}\left\{(a-b)^2 + (b-c)^2 + (c-a)^2\right\} \ge 0 \Rightarrow a^2 + b^2 + c^2 \ge ab + bc + ca \Rightarrow a^2 + b^2 + c^2 \ge 6$

$a^2 = b^2 + c^2 - 2b\cos A > b^2 + c^2 - 2bc \; Q \cos A < 1$

$b^2 > a^2 + c^2 - 2ac$ and $c^2 > a^2 + b^2 - 2ab$

$\therefore a^2 + b^2 + c^2 > 2\left(a^2 + b^2 + c^2 - ab - bc - ca\right)$

$\Rightarrow ab + bc + ca > \frac{1}{2}\left(a^2 + b^2 + c^2\right) = 3.$ hence $ab + bc + ca \in (3, 6]$

2. Let the lengths of the altitudes drawn from the veritices of a triangle ABC to the opposite sides are 2,2 and 3. if the area of ΔABC is Δ, then find the value of $2\sqrt{2}\Delta$

Key. 9

Sol. $\frac{2\Delta}{a} = 2, \frac{2\Delta}{b} = 2, \frac{2\Delta}{C} = 3$

$\Rightarrow a = \Delta, \; b = \Delta, \; c = \frac{2\Delta}{3}$

$\therefore \Delta^2 = \left(\frac{4\Delta}{3}\right)\left(\frac{\Delta}{3}\right)\left(\frac{\Delta}{3}\right)\left(\frac{2\Delta}{3}\right)$

$\Rightarrow 8\Delta^2 = 81$

$\Rightarrow 2\sqrt{2}\Delta = 9$

3. If r and R are respectively the radii of the inscribed and circumscribed circles of a regular polygon of n sides such that $\frac{R}{r} = \sqrt{5} - 1$, then n is equal to

Key. 5

Sol. $r = \frac{a}{2}\cot\frac{\pi}{n}; \; R = \frac{a}{2}\operatorname{cosec}\frac{\pi}{n}$

$\therefore \frac{R}{r} = \sec\frac{\pi}{n} = \sqrt{5} - 1 = \sec 36^0$

$\therefore \frac{\pi}{n} = \frac{\pi}{5} \Rightarrow n = 5$

4. In a $\Delta ABC, a = 5, b = 4 \; and \; \tan\frac{C}{2} = \sqrt{\frac{7}{9}}$, then measure of side 'c' is

Key. 6

Sol. $\cos C = \dfrac{1-\frac{7}{9}}{1+\frac{7}{9}} = \dfrac{1}{8}$

$\therefore c^2 = 25+16-2.5.4.\dfrac{1}{8} = 36$

5. In triangle ABC $a=\sqrt{5};\ b=2; \angle A = \pi/6$ and $c_1 and\, c_2$ are the two possible values of third side then $|c_1 - c_2|$ is

Key. 4

Sol. $a^2 = b^2 + c^2 - 2ba\cos A$

$c^2 - 2.2\dfrac{\sqrt{3}}{2}.c + 4 - 5 = 0$

$c^2 - 2\sqrt{3}.C + -1 = 0$

$c_1 + c_2 = 2\sqrt{3}; c_1c_2 = -1$

$(c_1 - c_2)^2 = 16$

6. The ratios of the lengths of the sides BC and AC of a triangle ABC to the radius of a circumscribed circle are equal to 2 and 3/2 respectively. If the ratio of the lengths of the bisectors of the interior angles B and C is $\alpha\left(\dfrac{\sqrt{\alpha}-1}{9\sqrt{\beta}}\right)$ *then* $\alpha+\beta$ *is*

Key. 9

Sol. Given $\dfrac{a}{R} = 2$; $\dfrac{b}{R} = \dfrac{3}{2}$

$\dfrac{a}{2} = \dfrac{b}{3/2} = R$

$\dfrac{BE}{CF} = \dfrac{\frac{2ac}{a+c}\cos\frac{B}{2}}{\frac{2ab}{a+b}\cos\frac{c}{2}}$

$= \dfrac{a+b}{a+c}.\dfrac{\cos\frac{B}{2}}{\cos\frac{C}{2}}.\dfrac{c}{b}$ ---------------------(1)

$\because a = 2R$ use here

$a^2 = b^2 + c^2 \Rightarrow c = \dfrac{\sqrt{7}R}{2}$

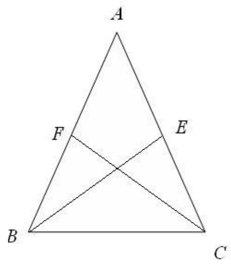
$\sin B = \dfrac{3}{4}; \sin = \dfrac{\sqrt{7}}{4}$

$\cos^2\dfrac{B}{2} = \dfrac{4+\sqrt{7}}{2}$;

$$\cos^2\frac{B}{2}=\frac{4+\sqrt{7}}{2};\cos^2\frac{c}{2}=\frac{7}{8}$$

Now from -----------------------------(1)

$$\frac{BE}{CF}=\frac{7\left(\sqrt{7}-1\right)}{9\sqrt{2}}$$



7. In $\Delta ABC.\ 3a=b+c$ then cot B/2 cot C/2 is

Key. 2

Sol. $3a=b+c$

$4a=2s$

$s=2a$

$$\cot\frac{B}{2}\cot\frac{C}{2}=\frac{s}{s-a}=\frac{2a}{2a-a}=2$$

8. In Δ ABC, if $R\left(a+b\right)=c\sqrt{ab}$ and a = 2+$\sqrt{2}$, then in radius r is

Key. 1

Sol. $\frac{a+b}{\sqrt{ab}}=\frac{C}{R}\Rightarrow\frac{a+b}{2\sqrt{ab}}=\frac{C}{2R}=\sin C$

$$\cos^2 C=1-\frac{\left(a+b\right)^2}{4ab}=\frac{-\left(a-b\right)^2}{4ab}$$

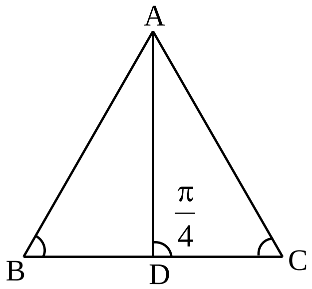
But $\cos^2 C\ge 0\Rightarrow a=b\Rightarrow c=\frac{\pi}{2}$

$$r=\frac{\Delta}{S}=\frac{\frac{1}{2}ab}{\frac{a+b+c}{2}}=\frac{a^2}{2a+\sqrt{2}a}=\frac{a}{2+\sqrt{2}}=\frac{2+\sqrt{2}}{2+\sqrt{2}}=1$$

9. If the median AD of triangle ABC makes an angle $\frac{\pi}{4}$ with the side BC, then the value of $\left|\cot B-\cot C\right|$ is

Key. 2

Sol.



From m – n theorem

$$2\cot\frac{\pi}{4} = \left|\cot C\right|\cot B \Rightarrow \left|\cot B - \cot C\right| = 2$$

10. If $A = 30^0, a = 7$ and b = 8 in ΔABC, then the number of triangles that can be constructed is

Key. 2

Sol. $\frac{7}{1/2} = \frac{8}{\sin B} \Rightarrow \sin B = \frac{8}{14} = \frac{4}{7}$

$\Rightarrow$ B can take two different values.

$\therefore$ No.of triangles = 2

11. In ΔABC if cosA + 2 cosB + cosC = 2, then the value of $\frac{2s}{b}$ (where 's' is the semi peri meter) is

Key. 3

Sol. $\cos A + \cos C = 2\left(1 - \cos B\right)$

$$\Rightarrow 2\cos\frac{A+C}{2}\cos\frac{A-C}{2} = 2 \times 2\sin^2\frac{B}{2}$$

$$\Rightarrow \frac{\cos\left(\frac{A-C}{2}\right)}{\sin\frac{B}{2}} = 2 \Rightarrow \frac{a+c}{b} = 2 \Rightarrow a + c = 2b$$

$$\therefore \frac{2S}{b} = \frac{3b}{b} = 3$$

12. In ΔABC, if r_1 = 6, R = 5, r = 2, then the value of 3tanA is

Key. 4

Sol. $r_1 - r = 4R\sin^2\frac{A}{2} \Rightarrow 4 = 20\sin^2\frac{A}{2} \Rightarrow \sin\frac{A}{2} = \frac{1}{\sqrt{5}}$

$$\tan A = \frac{2\times\frac{1}{2}}{1-\frac{1}{4}} = \frac{4}{3} \Rightarrow 3\tan A = 4$$

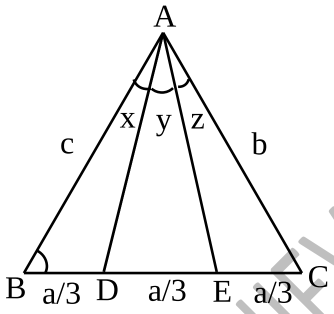
13. In ΔABC if $\angle C = 90^0$, then the value of $\frac{a+b}{r+R}$ is

Key. 2

Sol. $\angle c = 90^0 \Rightarrow 2(r+R) = a+b \Rightarrow \frac{a+b}{r+R} = 2$

14. Points D, E are taken on the side BC of an acute angled ΔABC, such that BD=DE=EC. If $\angle BAD = x,\ \angle DAE = y,\ \angle EAC = z$, then the value of $\frac{\sin(x+y).\sin(y+z)}{\sin x.\sin z}$ is

Key. 4



Sol.

$$\frac{a}{3\sin x} = \frac{AD}{\sin B} \quad \text{--------- (1)}$$

$$\frac{2a}{3\sin(x+y)} = \frac{AE}{\sin B} \quad \text{--------- (2)}$$

$$\frac{(1)}{(2)} \Rightarrow \frac{\sin(x+y)}{2\sin x} = \frac{AD}{AE} \quad \text{--------------- (3)}$$

$$\frac{2a}{3\sin(y+z)} = \frac{AD}{\sin C} \quad \text{----------- (4)}$$

$$\frac{a}{3\sin z} = \frac{AE}{\sin C} \quad \text{----------- (5)}$$

$$\frac{(5)}{(4)} \Rightarrow \frac{\sin(y+z)}{2\sin z} = \frac{AE}{AD} \quad \text{--------------- (6)}$$

$$\therefore \quad (3)\times(6) \Rightarrow \frac{\sin(x+y)\sin(y+z)}{\sin x.\sin z} = 4$$

15. If the circumcentre of triangle ABC lies on its incircle, then $\left[4\left(\cos A+\cos B+\cos C\right)\right]$

(Where $\left[x\right]$ is greatest integer less than or equal to x is)

Key. 5

Sol. $SI = r \Rightarrow R^2 - 2Rr - r^2 = 0$

$\Rightarrow \left(\frac{R}{r}\right)^2 - 2\left(\frac{R}{r}\right)^2 - 1 = 0$

$\Rightarrow \frac{R}{r} = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$

$\therefore \frac{R}{r} = \sqrt{2} + 1$

$\therefore \cos A + \cos B + \cos C = 1 + \frac{r}{R} = 1 + \sqrt{2} - 1 = \sqrt{2}$

$\therefore \left[4\left(\cos A + \cos B + \cos C\right)\right] = \left[4\sqrt{2}\right] = 5$

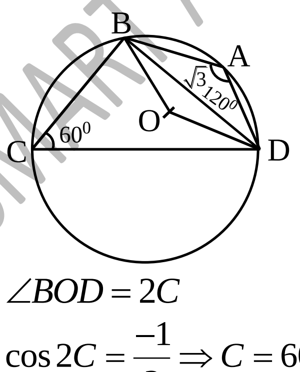
16. The area of a cyclic Quadrilateral ABCD is $\frac{3\sqrt{3}}{4}$. The radius of the circle circumscribing the Quadrilateral is 1. If AB = 1, BD = $\sqrt{3}$ then the value of 3.BC.CD is

Key. 6

Sol. $3 = 1 + AD^2 - 2 \times 1 \times AD \times \frac{-1}{2}$

$AD^2 + AD - 2 = 0$

$\left(AD + 2\right)\left(AD - 1\right) = 0 \Rightarrow AD = 1$



$\angle BOD = 2C$

$\cos 2C = \frac{-1}{2} \Rightarrow C = 60^0,\ A = 120^0$

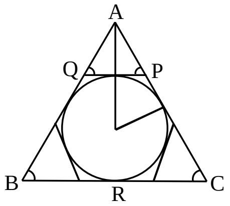
$\therefore \frac{1}{2} \times 1 \times 1 \times \frac{\sqrt{3}}{2} + \frac{1}{2} \times BC.CD \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2} B$

$BC.CD = 2$

17. The lengths of the tangents drawn from the vertices A, B, C to the incircle of ΔABC are 5, 3, 2 respectively. If the lengths of the parts of tangents with in the triangle which are drawn parallel to the sides BC, CA, AB of the triangle to the incircle be α,β,γ respectively, then $[\alpha+\beta+\gamma]$ where ($[g]$ is G.I.F. is)

Key. 6

Sol.



$$r=(S-a)\tan\frac{A}{2}$$

$$\frac{r}{S-a}=\frac{r}{AP}\Rightarrow S-a=5,\ S-b=3,\ S-c=2$$

$$\Rightarrow S=10\Rightarrow a=5,\ b=3,\ c=8$$

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1\Rightarrow\frac{\alpha}{5}+\frac{\beta}{7}+\frac{\gamma}{8}=1$$

$$\alpha=\frac{ra}{S\tan\frac{A}{2}},\ \beta=\frac{rb}{S\tan\frac{B}{2}},\ \gamma=\frac{rc}{S\tan\frac{C}{2}}$$

$$\alpha=\frac{(S-a)a}{S},\ \beta=\frac{(S-b)b}{S},\ \gamma=\frac{(S-c)c}{S}$$

$$\alpha=\frac{5\times5}{10},\ \beta=\frac{3\times7}{10},\ \gamma=\frac{2\times8}{10}$$

$$\alpha+\beta+\gamma=\frac{25+21+16}{10}=\frac{62}{10}\Rightarrow[\alpha+\beta+\gamma]=6$$

18. In ΔABC, $\frac{r}{r_1}=\frac{1}{2}$, then the value of $4\tan\left(\frac{A}{2}\right)\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)$ must be

Key. 2

Sol. $\frac{r}{r_1}=\tan\frac{B}{2}\tan\frac{C}{2}=\frac{1}{2}$

$$\tan\frac{A}{2}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)=1-\tan\frac{B}{2}\tan\frac{C}{2}=\frac{1}{2}$$

$$\therefore\ 4\ \tan\frac{A}{2}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)=2$$

19. If a, b, c are sides of a triangle satisfying $a^2+b^2+c^2=6$, then the AM of all the integral values which lie in the interval of ab + bc + ca is

Key. 5

Sol.

$$\frac{1}{2}\left\{(a-b)^2+(b-c)^2+(c-a)^2\right\}\ge 0 \Rightarrow a^2+b^2+c^2\ge ab+bc+ca \Rightarrow a^2+b^2+c^2\ge 6$$

$$a^2=b^2+c^2-2b\cos A>b^2+c^2-2bc\ \because \cos A<1$$

$$b^2>a^2+c^2-2ac \text{ and } c^2>a^2+b^2-2ab$$

$$\therefore a^2+b^2+c^2>2\left(a^2+b^2+c^2-ab-bc-ca\right)$$

$$\Rightarrow ab+bc+ca>\frac{1}{2}\left(a^2+b^2+c^2\right)=3.\ \text{hence } ab+bc+ca\in(3,6]$$

20. In ΔABC, $\frac{r}{r_1}=\frac{1}{2}$, then the value of $4\tan\left(\frac{A}{2}\right)\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)$ must be

Key. 2

Sol. $\frac{r}{r_1}=\tan\frac{B}{2}\tan\frac{C}{2}=\frac{1}{2}$

$$\tan\frac{A}{2}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)=1-\tan\frac{B}{2}\tan\frac{C}{2}=\frac{1}{2}$$

$$\therefore\ 4\tan\frac{A}{2}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)=2$$

21. With usual notation in triangle ABC, the numerical value of $\left(\frac{a+b+c}{r_1+r_2+r_3}\right)\left(\frac{a}{r_1}+\frac{b}{r_2}+\frac{c}{r_3}\right)$ is

ANS : 4

Sol. $$\sum\frac{a}{r_1}=2R\sum\frac{2\sin A/2\cos A/2}{4R\sin A/2\cos B/2\cos C/2}=\sum\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)$$

$$=2\sum\tan\frac{A}{2}=2\sum\frac{r_1}{s}=4\left(\frac{r_1+r_2+r_3}{a+b+c}\right)$$

22. In $\Delta ABC\ \frac{(r_1+r_2)(r_2+r_3)(r_3+r_1)}{Rs^2}=$ ______(where r_1,r_2,r_3 are exradii & R is circum radius and
s is Semiperimeter of triangle ABC)

KEY : 4

23. If in a triangle ABC, $b\cos^2\frac{A}{2}+a\cos^2\frac{B}{2}=\frac{3c}{2}$ *then* minimum value of $\frac{a+c}{2c-a}+\frac{b+c}{2c-b}$ is equal to …………

Key: 4

Hint:

$$LHS=\frac{1}{2}\left[b+b\cos A+a+a\cos B\right]=\frac{1}{2}\left(a+b+c\right)=\frac{3c}{2}\Rightarrow 2c=a+b$$

$$\frac{a+c}{2c-a}+\frac{b+c}{2c-b}=\frac{a+c}{b}+\frac{b+c}{a}=\frac{a}{b}+\frac{b}{a}+\frac{c}{b}+\frac{c}{a}\ge 4\left(\frac{c^2}{ab}\right)^{1/4}\ge 4$$

$$\left(\because \frac{a+b}{2}\ge\sqrt{ab}\Rightarrow c^2\ge ab\Rightarrow\frac{c^2}{ab}\ge 1\right)$$

24. If length of the side BC of a ΔABC is 4cm and $\angle BAC=120^0$, then the distance between incentre & excentre of the circle touching the side BC internally is

Key: 8

Hint: $II_1=AI_1-AI$

$=\left(r_1-r\right)\cos cA/2$

$=a\tan A/2\cos cA/2$

$=\dfrac{a}{\cos A/2}=8$

25. In ΔABC, $\frac{r}{r_1}=\frac{1}{2}$, then the value of $4\tan\left(\frac{A}{2}\right)\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)$ must be

Key. 2

Sol. $\frac{r}{r_1}=\tan\frac{B}{2}\tan\frac{C}{2}=\frac{1}{2}$

$\tan\frac{A}{2}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)=1-\tan\frac{B}{2}\tan\frac{C}{2}=\frac{1}{2}$

$\therefore\ 4\ \tan\frac{A}{2}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)=2$

26. Given a parallelogram whose acute angle is , if the squares of length of the diagonals are in ratio 1 : 3 tnen $\frac{a}{b}$ is (where a,b are the length of the sides) ________________

Sol. 1

Applying cosines rule in ΔABD and ΔBCD, we get $2a^2+2b^2=d_1^2+d_2^2$, where d_1, d_2 are length of diagonals $\left(d_1<d_2\right)$ given $d_2^2=3d_1^2$

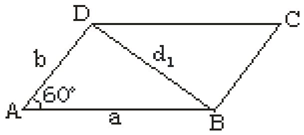
$\Rightarrow a^2+b^2=2d_1^2$

also ab = d_1^2

$\Rightarrow a^2+b^2+2ab=3ab+d_1^2=4d_1^2$ ...(1)

$\Rightarrow\left(a+b\right)^2=4d_1^2\Rightarrow a+b=2d_1$ ...(2)

using (1) and (2), we get a = b = d_1 i.e. $\frac{a}{b}=1$.



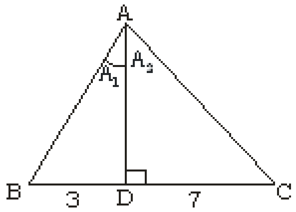
27. In a triangle ABC, the foot of the perpendicular from A divides the opposite side into parts of lengths 3 and 17 and $\tan A = \frac{22}{7}$. Let a ΔPQR is a right angle triangle (right angle at Q) such that $\angle A = \angle P$ and PQ = 7 units, the $\left[\frac{\text{Area }(ABC)}{\text{Area }(PQR)}\right]$ is __________ (where [.] denotes the greatest integer function)

Sol. 1

$\tan A = \tan(A_1 + A_2)$

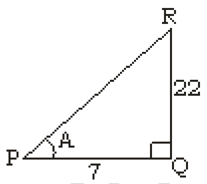
$\Rightarrow AD = 11$

hence the area of $\Delta ABC = \frac{1}{2} \times 11 \times 20 = 110$



$\Rightarrow$ Area of PQR $= \frac{1}{2} \times 22 \times 7 = 77$.

$\left[\frac{\text{Area }(ABC)}{\text{Area }(PQR)}\right] = 1$.



28. ABC is an acute angle triangle, $\angle A = 30°$.H is the orthocentre and M is the midpoint of BC. On the line HM a point T is taken such that HM = MT. If BC = 4cm, then the length of AT is ___________

Key. 8

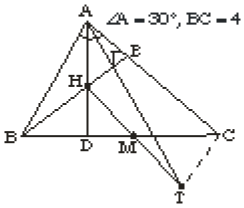
Sol. ΔBMH and ΔCMT are similar, $\angle MBH = \angle MCT$,

$\Rightarrow$ BH CT, CT $\perp$ AC and CT = BH

Now, $AT^2 = TC^2 + AC^2 = BH^2 + AC^2$

$= BD^2 \operatorname{cosec}^2 C + AD^2 \operatorname{cosec}^2 C$

$= \frac{AB^2}{\sin^2 C} = \frac{a^2}{\sin^2 A} = 16 \times 4 = 64 \Rightarrow AT = 8$



29. If I be the incentre of triangle ABC and R_1, R_2, R_3 be the circum radius of the triangle BIC, CIA, AIB respectively, then maximum value of $\frac{a^2}{R_1^2}+\frac{b^2}{R_2^2}+\frac{c^2}{R_3^2}$ is .....

Key. 9

30. The radii r_1, r_2, r_3 of inscribed circles of the triangle ABC are in H.P. If its area is 24 sq. cm and its perimeter is 24 cm, then the length of its largest side is

Key. 10

31. If in ΔABC, circle with altitude AD as diameter intersect AB at P and AC at Q such that $PQ = \lambda \frac{\Delta}{R}$, when Δ, R are area and circumradius of triangle ABC respectively, then λ is equal to

Key. 1

32. In a triangle ABC, CH and CM are the lengths of the altitude and median to the base AB. If a = 10, b = 26, c = 32 then length (HM) ?

Key. 9

33. In a ΔABC, perpendiculars are drawn from the angles A,B,C of an acute angled triangle on the opposite sides and produced to meet the circumscribing circle at D, E and F respectively. If these produced parts be α, β, γ respectively.

Then the value of $\frac{\left(\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}\right)}{\tan A+\tan B+\tan C}$ is ....

Key. 2

34. In Δ ABC, if $R(a+b)=c\sqrt{ab}$ and a = 2+$\sqrt{2}$, then in radius r is

Key. 1

Sol. $\frac{a+b}{\sqrt{ab}}=\frac{C}{R} \Rightarrow \frac{a+b}{2\sqrt{ab}}=\frac{C}{2R}=\sin C$

$$\cos^2 C = 1-\frac{(a+b)^2}{4ab}=\frac{-(a-b)^2}{4ab}$$

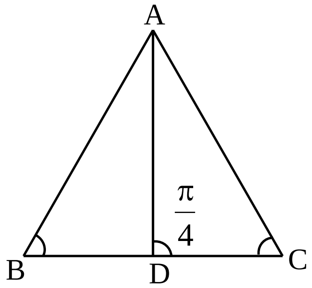
But $\cos^2 C \ge 0 \Rightarrow a=b \Rightarrow c=\frac{\pi}{2}$

$$r=\frac{\Delta}{S}=\frac{\frac{1}{2}ab}{\frac{a+b+c}{2}}=\frac{a^2}{2a+\sqrt{2}a}=\frac{a}{2+\sqrt{2}}=\frac{2+\sqrt{2}}{2+\sqrt{2}}=1$$

35. If the median AD of triangle ABC makes an angle $\frac{\pi}{4}$ with the side BC, then the value of $\left|\cot B - \cot C\right|$ is

Key. 2

Sol.



From m – n theorem

$$2\cot\frac{\pi}{4} = \left|\cot C\right|\cot B \Rightarrow \left|\cot B - \cot C\right| = 2$$

36. If $A = 30^0, a = 7$ and b = 8 in ΔABC, then the number of triangles that can be constructed is

Key. 2

Sol. $\frac{7}{1/2} = \frac{8}{\sin B} \Rightarrow \sin B = \frac{8}{14} = \frac{4}{7}$

$\Rightarrow$ B can take two different values.

$\therefore$ No.of triangles = 2

37. In ΔABC if cosA + 2 cosB + cosC = 2, then the value of $\frac{2s}{b}$ (where 's' is the semi peri meter) is

Key. 3

Sol. $\cos A + \cos C = 2\left(1 - \cos B\right)$

$$\Rightarrow 2\cos\frac{A+C}{2}\cos\frac{A-C}{2} = 2\times 2\sin^2\frac{B}{2}$$

$$\Rightarrow \frac{\cos\left(\frac{A-C}{2}\right)}{\sin\frac{B}{2}} = 2 \Rightarrow \frac{a+c}{b} = 2 \Rightarrow a + c = 2b$$

$$\therefore \frac{2S}{b} = \frac{3b}{b} = 3$$

38. In ΔABC, if $r_1 = 6$, R = 5, r = 2, then the value of 3tanA is

Key. 4

Sol. $r_1 - r = 4R\sin^2\frac{A}{2} \Rightarrow 4 = 20\sin^2\frac{A}{2} \Rightarrow \sin\frac{A}{2} = \frac{1}{\sqrt{5}}$

$$\tan A = \frac{2\times\frac{1}{2}}{1-\frac{1}{4}} = \frac{4}{3} \Rightarrow 3\tan A = 4$$

39. In ΔABC if $\angle C = 90^0$, then the value of $\dfrac{a+b}{r+R}$ is

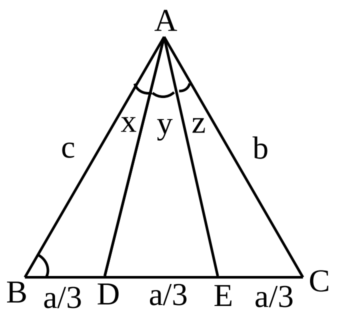
Key. 2

Sol. $\angle c = 90^0 \Rightarrow 2(r+R) = a+b \Rightarrow \dfrac{a+b}{r+R} = 2$

40. Points D, E are taken on the side BC of an acute angled ΔABC, such that BD=DE=EC. If $\angle BAD = x,\ \angle DAE = y,\ \angle EAC = z$, then the value of $\dfrac{\sin(x+y).\sin(y+z)}{\sin x.\sin z}$ is

Key. 4

Sol.



$\dfrac{a}{3\sin x} = \dfrac{AD}{\sin B}$ --------- (1)

$\dfrac{2a}{3\sin(x+y)} = \dfrac{AE}{\sin B}$ --------- (2)

$\dfrac{(1)}{(2)} \Rightarrow \dfrac{\sin(x+y)}{2\sin x} = \dfrac{AD}{AE}$ --------------- (3)

$\dfrac{2a}{3\sin(y+z)} = \dfrac{AD}{\sin C}$ ----------- (4)

$\dfrac{a}{3\sin z} = \dfrac{AE}{\sin C}$ ----------- (5)

$\dfrac{(5)}{(4)} \Rightarrow \dfrac{\sin(y+z)}{2\sin z} = \dfrac{AE}{AD}$ --------------- (6)

$\therefore\ (3)\times(6) \Rightarrow \dfrac{\sin(x+y)\sin(y+z)}{\sin x.\sin z} = 4$

41. If the circumcentre of triangle ABC lies on its incircle, then $\left[4(\cos A + \cos B + \cos C)\right]$

(Where $[x]$ is greatest integer less than or equal to x is)

Key. 5

Sol. $SI = r \Rightarrow R^2 - 2Rr - r^2 = 0$

$$\Rightarrow\left(\frac{R}{r}\right)^2-2\left(\frac{R}{r}\right)^2-1=0$$

$$\Rightarrow\frac{R}{r}=\frac{2\pm2\sqrt{2}}{2}=1\pm\sqrt{2}$$

$$\therefore\frac{R}{r}=\sqrt{2}+1$$

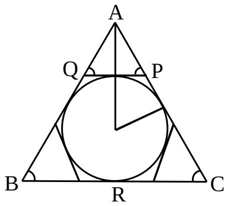
$$\therefore\cos A+\cos B+\cos C=1+\frac{r}{R}=1+\sqrt{2}-1=\sqrt{2}$$

$$\therefore\left[4\left(\cos A+\cos B+\cos C\right)\right]=\left[4\sqrt{2}\right]=5$$

42. The lengths of the tangents drawn from the vertices A, B, C to the incircle of ΔABC are 5, 3, 2 respectively. If the lengths of the parts of tangents with in the triangle which are drawn parallel to the sides BC, CA, AB of the triangle to the incircle be α,β,γ respectively, then $[\alpha+\beta+\gamma]$ where ($[g]$ is G.I.F. is)

Key. 6

Sol.



$$r=\left(S-a\right)\tan\frac{A}{2}$$

$$\frac{r}{S-a}=\frac{r}{AP}\Rightarrow S-a=5,\ S-b=3,\ S-c=2$$

$$\Rightarrow S=10\Rightarrow a=5,\ b=3,\ c=8$$

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1\Rightarrow\frac{\alpha}{5}+\frac{\beta}{7}+\frac{\gamma}{8}=1$$

$$\alpha=\frac{ra}{S\tan\frac{A}{2}},\ \beta=\frac{rb}{S\tan\frac{B}{2}},\ \gamma=\frac{rc}{S\tan\frac{C}{2}}$$

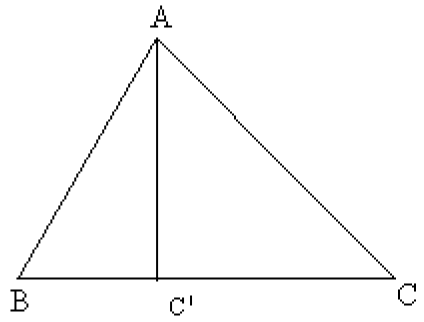
$$\alpha=\frac{(S-a)a}{S},\ \beta=\frac{(S-b)b}{S},\ \gamma=\frac{(S-c)c}{S}$$

$$\alpha=\frac{5\times5}{10},\ \beta=\frac{3\times7}{10},\ \gamma=\frac{2\times8}{10}$$

$$\alpha+\beta+\gamma=\frac{25+21+16}{10}=\frac{62}{10}\Rightarrow[\alpha+\beta+\gamma]=6$$

43. Let ABC and ABC^1 be two non congruent triangles with sides $AB = 4, AC = AC^1 = 2\sqrt{2}$ and angle $B = 30^0$, the absolute value of the difference between the area of these triangles is

KEY. 4



SOL.

$AB = 4, AC = AC^1 = 2\sqrt{2}, B = 30^0$

$C = 4, b = b^1 = 2\sqrt{2}, B = 30^0$

$\frac{b}{\sin 30} = \frac{c}{\sin C} \Rightarrow C = 45^0$

$\underline{|CAC^1} = 90^0$

$Area\ of\ \Delta ABC - \Delta ABC^1 = \Delta ACC^1$

$= \frac{1}{2} AC.AC^1$

$= \frac{1}{2} 2\sqrt{2}.2\sqrt{2} = 4$

44. Consider a triangle ABC and let a,b and c denote the lengths of the sides opposite to the vertices A,B and C respectively .Suppose a = 6, b = 10 and the area of the triangle is $15\sqrt{3}$. If $\angle ACB$ is obtuse and if 'r' denote the radius of the incirlce of the triangle then $r^2 =$

KEY. 3

SOL. $\Delta = \frac{1}{2} ab \sin C \Rightarrow \sin C = \frac{\sqrt{3}}{2} \Rightarrow C = 120^0$

$c^2 = a^2 + b^2 - 2ab\cos C \Rightarrow c = 14$

$r = \frac{\Delta}{s} = \sqrt{3} \Rightarrow r^2 = 3$

45. If $B=60^0$, $C=45^0$ and D divides BC internally in the ratio 1 : 3 and $\frac{\sin\lfloor CAD}{\sin\lfloor BAD}=\lambda$, then

$\lambda^3+\lambda^2-6\lambda+3=$

KEY. 9

SOL. By sine Rule $\frac{\sin\lfloor CAD}{\sin\lfloor BAD}=\sqrt{6}=\lambda$

$$\lambda^3+\lambda^2-6\lambda+3=6\sqrt{6}+6-6\sqrt{6}+3=9$$

46. If p_1, p_2 and p_3 are the altitudes of a triangle from vertices A,B and C respectively, and Δ is the area of the triangle, prove that $\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_3}=\frac{2ab}{(a+b+c)\Delta}\cos^2\frac{C}{2}$

Ans. $\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_3}=\frac{a+b-c}{2\Delta}=\frac{2s-2c}{2\Delta}=\frac{s-c}{\Delta}$

Sol. We have $\cos^2\frac{C}{2}=\frac{s(s-c)}{ab}$

so that $\frac{2ab}{(a+b+c)\Delta}\cos^2\frac{C}{2}=\frac{2ab}{2s\Delta}.\frac{s(s-c)}{ab}=\frac{s-c}{\Delta}$

Now, the area of triangle ABC is $\Delta=\frac{1}{2}ap_1$, i.e., $p_1=2\Delta/a$. Similarly, $p_2=2\Delta/b$ and $p_3=2\Delta/c$.

$$\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_3}=\frac{a+b-c}{2\Delta}=\frac{2s-2c}{2\Delta}=\frac{s-c}{\Delta}$$

47. In a ΔABC, the angles A,B,C are in A.P. Show that $2\cos\frac{A-C}{2}=\frac{a+c}{\sqrt{a^2-ac+c^2}}$

Ans. $2\cos\frac{A-C}{2}$

Sol.

$$\frac{a+c}{\sqrt{a^2-ac+c^2}}=\frac{\sin A+\sin C}{\sqrt{\sin^2 A-\sin A\sin C+\sin^2 C}}=\frac{2\sin\frac{A+C}{2}\cos\frac{A-C}{2}}{\sqrt{\frac{1-\cos 2A}{2}-\frac{\cos(A-C)-\cos(A+C)}{2}+\frac{1-\cos 2C}{2}}}$$

$$=\frac{2\sqrt{2},\frac{\sqrt{3}}{2}\cos\frac{A-C}{2}}{\sqrt{2-(\cos 2A+\cos 2C)-\cos(A-C)+\cos(A+C)}}$$

$$=\frac{\sqrt{6}\cos\frac{A-C}{2}}{\sqrt{\frac{3}{2}-2\cos(A+C)\cos(A-C)-\cos(A-C)}}=\frac{\sqrt{6}\cos\frac{A-C}{2}}{\sqrt{\frac{3}{2}+\cos(A-C)-\cos(A-C)}}$$

$$=2\cos\frac{A-C}{2}$$

48. Let AD, BE, CF be the length of internal bisectors of angles A,B,C of triangle ABC. Show that the harmonic mean of AD $\sec\frac{A}{2}$. BE $\sec\frac{B}{2}, CF\sec\frac{C}{2}$ is the harmonic mean of the sides of the triangle

Key. $\Sigma\frac{1}{AD\sec\frac{A}{2}}=\Sigma\frac{1}{a}$

Sol. $AD=\frac{2bc}{b+c}\cos\frac{A}{2} \quad \Rightarrow \quad \frac{1}{AD\sec\frac{A}{2}}=\frac{1}{2}\left(\frac{1}{b}+\frac{1}{c}\right)$

$\therefore \quad \Sigma\frac{1}{AD\sec\frac{A}{2}}=\Sigma\frac{1}{a}$

49. Let ABC be a triangle with altitudes h_1,h_2,h_3 and inradius r. Prove that $\frac{h_1+r}{h_1-r}+\frac{h_2+r}{h_2-r}+\frac{h_3+r}{h_3-r}\ge 6$

Ans.

Sol. $\frac{h_1+r}{h_1-r}+\frac{h_2+r}{h_2-r}+\frac{h_3+r}{h_3-r}\ge 6$

$\Delta=\frac{1}{2}ah_1 \quad \Rightarrow \quad h_1=\frac{2\Delta}{a}$

Similarly $h_1=\frac{2\Delta}{b},\ h_3=\frac{2\Delta}{c}$

So $\frac{h_1+r}{h_1-r}+\frac{h_2+r}{h_2-r}=\frac{h_3+r}{h_3-r}=\frac{2\Delta/a+\Delta/s}{2\Delta/a-\Delta/s}+\frac{2\Delta/b+\Delta/s}{2\Delta/c-\Delta/s}+\frac{2\Delta/c+\Delta/s}{2\Delta/c-\Delta/s}$

$=\frac{2s+a}{2s-a}=\frac{2s+b}{2s-b}+\frac{2s+c}{2s-c}=\frac{4s}{2s-a}+\frac{4s}{2s-b}+\frac{4s}{2s-c}-3$

$=3\left[\frac{1}{3}\left\{\frac{4s}{2s-a}+\frac{4s}{2s-b}+\frac{4s}{2s-c}\right\}\right]-3\ge 3\left(\frac{3}{\frac{2s-a}{4s}+\frac{2s-b}{4s}+\frac{2s-c}{4s}}\right]=3$ Since $(AM\ge HM)\ge 6$

50. Find the point inside a Δ from which the sum of the squares of distances to the three sides is minimum. Also find the minimum value of the sum of squares of distances.

Ans. $\frac{4(s-a)(s-b)(s-c)s}{a^2+b^2+c^2}$

Sol. If a,b, c are the lengths of the sides of the Δ and x,y,z are the length of perpendicular from the point on the sides BC, CA, AB respectively we have to minimize $x^2+y^2+z^2=t$

we have

$\frac{1}{2}ax+\frac{1}{2}by+\frac{1}{2}cz=\Delta$

$\Rightarrow \quad ax + by + cz = 2\Delta$

Where Δ is the area of the ΔABC we have the identity;

$\Rightarrow \quad (x^2+y^2+z^2)(a^2+b^2+c^2)-(ax+bx+cz)^2=(ax-by)^2+(by-cz)^2+(cz+ax)^2$

$\Rightarrow \quad (x^2+y^2+z^2)(a^2+b^2+c^2)\ge(ax+by+cz)^2$

$\Rightarrow \quad (x^2+y^2+z^2)(a^2+b^2+c^2)\ge 4\Delta^2$

$\Rightarrow \qquad x^2 + y^2 + z^2 \geq \frac{4\Delta^2}{a^2 + b^2 + c^2}$ and equality only when

$$\frac{x}{a} - \frac{y}{b} - \frac{z}{c} = \frac{ax + by + cz}{a^2 + b^2 + c^2} = \frac{2\Delta}{a^2 + b^2 + c^2}$$

The minimum value of t is $\frac{4\Delta^2}{a^2 + b^2 + c^2}$

$t\min = \frac{4(s-a)(s-b)(s-c)s}{a^2 + b^2 + c^2}$ Ans.