# Maxima & Minima

## *Single Correct Answer Type*

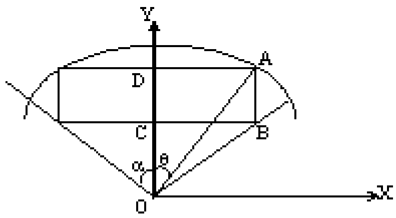
1. A sector subtends an angle $2\alpha$ at the centre then the greatest area of the rectangle inscribed in the sector is (R is radius of the circle)

A) $R^2 \tan\frac{\alpha}{2}$ B) $\frac{R^2}{2}\tan\frac{\alpha}{2}$ C) $R^2\tan\alpha$ D) $\frac{R^2}{2}\tan\alpha$

Key. A

Sol. Let A be any point on the arc such that $\angle YOA = \theta$

Where $0 \le \theta \le \alpha$




DA = CB = R sin $\theta$, OD = R cos $\theta$

$\Rightarrow$ CO = CB cot $\alpha = R\sin\theta\cot\alpha$

Now, CD = OD - OC = R cos $\theta - R\sin\theta\cot\alpha$

= R (cos $\theta$ – sin $\theta$ cot $\alpha$ )

Area of rectangle ABCD , S = 2.CD.CB

= 2R (cos$\theta$ - sin $\theta$ cot $\alpha$) R sin $\theta$ = 2R$^2$(sin$\theta$ cos$\theta$ - sin$^2$ $\theta$ cot$\alpha$ )

$R^2\left(\sin 2\theta - (1-\cos 2\theta)\cot\alpha\right) = \frac{R^2}{\sin\alpha}\left[\cos(2\theta-\alpha)-\cos\alpha\right]$

$S_{max} = \frac{R^2}{\sin\alpha}(1-\cos\alpha) \quad \left(\text{for } \theta = \alpha/2\right)$

Hence, greatest area of the rectangle = $R^2\tan\frac{\alpha}{2}$

2. Let f(x) = x$^2$ – bx + c , b is a odd positive integer, f(x) = 0 have two prime numbers as roots and
b + c = 35. Then the global minimum value of f(x) is

A) $-\frac{183}{4}$ B) $\frac{173}{16}$ C) $-\frac{81}{4}$ D) data not sufficient

Key. C

Sol. Let $\alpha$, $\beta$ be roots of x$^2$ – bx + c = 0,

Then $\alpha + \beta$ = b

$\Rightarrow$ one of the roots is ‘2’ (Since $\alpha$, $\beta$ are primes and b is odd positive integer)

$\therefore$ f(2) = 0 $\Rightarrow$ 2b - c = 4 and b + c = 35

$\therefore$ b = 13, c = 22

Minimum value = $f\left(\frac{13}{2}\right)=-\frac{81}{4}$.

3. Let $f(x)$ be a positive differentiable function on $[0,a]$ such that $f(0)=1$ and $f(a)=3^{1/4}$ If $f^1(x)\ge(f(x))^3+(f(x))^{-1}$, then, maximum value of a is

a) $\frac{\pi}{12}$ b) $\frac{\pi}{24}$ c) $\frac{\pi}{36}$ d) $\frac{\pi}{48}$

Key. B

Sol. $f^1(x)f(x)\ge(f(x))^4+1$

$\Rightarrow \frac{2f^1(x)f(x)}{\left\{(f(x))^2\right\}^2+1}\ge 2$

$\Rightarrow \int_0^a \frac{2f^1(x)f(x)}{\left\{(f(x))^2\right\}^2+1}\ge 2\int_0^a 1dx$

$\Rightarrow \left|\tan^{-1}(f(x))^2\right|_0^a \ge 2a \Rightarrow \frac{\pi}{3}-\frac{\pi}{4}\ge 2a$

4. The least value of 'a' for which the equation $\frac{4}{\sin x}+\frac{1}{1-\sin x}=a$ for atleast one solution on the interval $\left(0,\frac{\pi}{2}\right)$ is,

a) 1 b) 4 c) 8 d) 9

Key. D

Sol. Q $a=\frac{4}{\sin x}+\frac{1}{1-\sin x}$, where a is least

$\Rightarrow \frac{da}{dx}=\left(\frac{-4}{\sin^2 x}+\frac{1}{(1-\sin x)^2}\right)\cos x=0$

Q $\cos x\ne 0\Rightarrow \sin x=2/3$

$\frac{d^2a}{dx^2}=45>0$ for $\sin x=2/3\Rightarrow \frac{4}{2/3}+\frac{1}{1-2/3}=6+3=9$

5. Let domain and range of $f(x)$ and $g(x)$ are respectively $[0,\infty)$. If $f(x)$ be an increasing function and $g(x)$ be an decreasing function. Also, $h(x)=f(g(x)), h(0)=0$ and $p(x)=h\left(x^3-2x^2+2x\right)-h(4)$ then for every $x\in(0,2]$

a) $p(x)\in(0,-h(4))$ b) $p(x)\in[-h(4),0]$

c) $p(x)\in(-h(4),h(4))$ d) $p(x)\in(h(4),h(4)]$

Key. A

Sol. $h(x)=f(g(x))$

$h^1(x)=f^1(g(x))g^1(x)<0\forall x\in[0,\infty)$

Q $g^1(x)<0\forall x\in[0,\infty)$ and $f^1(g(x))>0\forall x\in[0,\infty)$

Also, $h(0)=0$ and hence, $h(x)<0\forall x\in[0,\infty)$

$p(x)=h(x^3-2x^2+2x)-h(4)$

$p^1(x)=h^1(x^3-2x^2+2x).(3x^2-4x+2)<0\forall x\in(0,2)$

$\because h^1(x^3-2x^2+2x)<0\forall x\in(0,\infty)$ and $3x^2-4x+2>0\forall x\in R$

$\Rightarrow p(x)$ is an decreasing function

$\Rightarrow p(2)<p(x)<p(0)\forall x\in(0,2)$

$\Rightarrow h(4)-h(4)<p(x)<h(0)-h(4)$

$\Rightarrow 0<p(x)<-h(4)$

6. If $f(x)=\begin{cases}3-x^2, x\le 2\\ \sqrt{a+14}-|x-48|, x>2\end{cases}$ and if $f(x)$ has a local maxima at x = 2, then, greatest value of a is

a) 2013 b) 2012 c) 2011 d) 2010

Key. C

Sol. Local maximum at $x=2\Rightarrow$

$$\Rightarrow \lim_{h\to 0} f(2+h)\le f(2)$$

$$\Rightarrow \lim_{h\to 0}\left(\sqrt{a+14}-|2+h-48|\right)\le 3-2^2$$

$$\Rightarrow \sqrt{a+14}\le 45\Rightarrow a\le 2011$$

7. Two runners A and B start at the origin and run along positive x-axis, with B running three times as fast as A. An observer, standing one unit above the origin, keeps A and B in view. Then the maximum angle of sight '$\theta$' between the observes view of A and B is

a) $\pi/8$ b) $\pi/6$ c) $\pi/3$ d) $\pi/4$

Key. B

Sol. $$\tan\theta=\tan(\theta_2-\theta_1)\Rightarrow \tan\theta=\frac{3x-x}{1+3x.x}=\frac{2x}{1+3x^2}$$

$$\text{let } y=\frac{2x}{1+3x^2}\quad \frac{dy}{dx}=\frac{2(1-3x^2)}{(1+3x^2)^2}$$

$$\frac{dy}{dx}=0\Rightarrow x=\frac{1}{\sqrt{3}} \text{ and } \frac{d^2y}{dx^2}=\frac{-24x}{(1+3x^2)^3}<0 \text{ for } x=1/\sqrt{3}$$

8. If the function $f(x)=ax^3+bx^2+11x-6$ satisfies conditions of Rolle's theorem in [1, 3] and $f'\left(2+\frac{1}{\sqrt{3}}\right)=0$, then value of a and b are respectively

(A) 1, –6 (B) –1, 6 (C) –2, 1 (D) –1, 1/2

Key. A

Sol. $\because$ f (1) = f(3)

$\Rightarrow$ a + b + 11 – 6 = 27a + 9b + 33 – 6

$\Rightarrow$ 13a + 4b = –11

and $f'(x) = 3ax^2 + 2bx + 11$ ... (i)

$\Rightarrow$ $f'\left(2+\frac{1}{\sqrt{3}}\right) = 3a\left(2+\frac{1}{\sqrt{3}}\right)^2 + 2b\left(2+\frac{1}{\sqrt{3}}\right) + 11 = 0$

$\Rightarrow$ $3a\left(4+\frac{1}{3}+\frac{4}{\sqrt{3}}\right) + 2b\left(2+\frac{1}{\sqrt{3}}\right) + 11 = 0$ ... (ii)

From eqs. (i) and (ii), we get a = 1, b = –6.

9. Let $f(x)$ be a positive differentiable function on $[0,a]$ such that $f(0)=1$ and $f(a)=3^{1/4}$ If $f^1(x) \geq (f(x))^3 + (f(x))^{-1}$, then, maximum value of a is

a) $\frac{\pi}{12}$ b) $\frac{\pi}{36}$ c) $\frac{\pi}{24}$ d) $\frac{\pi}{48}$

Key. C

Sol. $f^1(x)f(x) \geq (f(x))^4 + 1$

$\Rightarrow \frac{2f^1(x)f(x)}{\left\{(f(x))^2\right\}^2+1} \geq 2$

$\Rightarrow \int_0^a \frac{2f^1(x)f(x)}{\left\{(f(x))^2\right\}^2+1} \geq 2\int_0^a 1dx$

$\Rightarrow \left|\tan^{-1}(f(x))^2\right|_0^a \geq 2a \Rightarrow \frac{\pi}{3}-\frac{\pi}{4} \geq 2a$

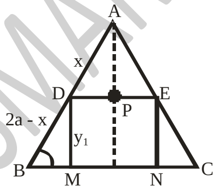
Given expansion = $\{x-(1+\cos t)\}^2 + \left\{\frac{K}{x}-(1+\sin t)\right\}^2$

10. A rectangle is inscribed in an equilateral $\Delta$ of side length 2a units. Maximum area of this rectangle is

(A) $\sqrt{3}a^2$ (B) $\frac{\sqrt{3}a^2}{4}$ (C) $a^2$ (D) $\frac{\sqrt{3}a^2}{2}$

Key. D

Sol.




Let AD = x Let DM = $y_1$

BD = (2a – x) DE = $2x_1$

In $\Delta$DBM

$\angle B = \frac{\pi}{3}$ $\sin 60° = \frac{y_1}{2a-x}$

$y_1 = (2a-x)\times\frac{\sqrt{3}}{2}$

In $\Delta ADP$

$\angle D = \frac{\pi}{3}$

$\cos 60° = \frac{x_1}{x}$

$x_1 = x \times \frac{1}{2}$

$2x_1 = x$

$\Delta(x)$ = Area of rectangle = $2x_1 y$

$\Delta(x) = x \times (2a - x)\frac{\sqrt{3}}{2}$

$\Delta'(x) = \frac{\sqrt{3}}{2}(2a - 2x) = 0 \Rightarrow x = a$

$\Delta''(a) = -ve$

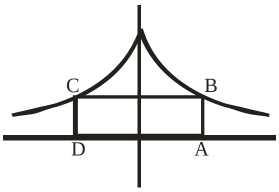
x = a point of maxima

maximum area = $a \times \frac{a\sqrt{3}}{2} = \frac{\sqrt{3}a^2}{2}$

11. The maximum area of a rectangle whose two consecutive vertices lie on the x-axis and another two lie on the curve $y = e^{-|x|}$ is equal to

(A) 2e sq. Units (B) $\frac{2}{e}$ sq. Units (C) e sq. units (D) $\frac{1}{e}$ sq. units

Key. B




Sol.

Let the rectangle is (ABCD)

$A = (t, 0), B = (t, e^{-t}), C = (-t, e^{-t}), D = (-t, 0)$

ABCD = $2te^{-t} = f(t)$

$\frac{df}{dt} = 2(t(-e^{-t}) + e^{-t}) = 2e^{-t}(1 - t)$

$\frac{df}{dt} > 0 \Rightarrow t \in (0,1)$

$\frac{df}{dt} < 0 \Rightarrow t \in (1,\infty)$

t = 1 is point of maxima

Maximum area = $f(1) = \frac{2}{e}$

12. Let $f:[0,4] \to R$, be a differentiable function. Then, there exists real numbers $a, b \in (0,4)$ such that, $(f(4))^2 - (f(0))^2 = Kf^1(a)f(b)$ Where K, is

a) $\frac{1}{4}$ b) 8 c) $\frac{1}{12}$ d) 4

Key. B

Sol. By LMVT, $\exists a \in (0,4) \ni \frac{f(4)-f(0)}{4-0} = f^1(a) \Rightarrow f(4)-f(0) = 4f^1(a)$

$\because \frac{f(4)+f(0)}{2}$ lies between $f(0)$ and $f(4)$, by Intermediate value theorem $\exists b \in (0,4) \ni \frac{f(4)+f(0)}{2} = f(b)$ hence, $\left(f(4)^2\right)-\left(f(0)\right)^2 = 8 \quad f^1(a)f(b)$

13. A window is in the shape of a rectangle surmounted by a semi circle .If the perimeter of the window is of fixed length '$l$' then the maximum area of the window is

1) $\frac{l^2}{2\pi+4}$ 2) $\frac{l^2}{\pi+8}$ 3) $\frac{l^2}{2\pi+8}$ 4) $\frac{l^2}{8\pi+4}$

Key. 3

$l = 2x + 2r + \pi r$

Sol. $A = 2rx + \frac{1}{2}\pi r^2$

$\frac{dA}{dV} = 0 \Rightarrow r = \frac{l}{4+\pi}$

14. If the petrol burnt per hour in driving a motor boat varies as the cube of its velocity when going against a current of 'C' kmph , the most economical speed
Is (in kmph)

1) $\frac{C}{2}$ 2) $\frac{3C}{2}$ 3) $\frac{\sqrt{3}C}{2}$ 4) C

Key. 2

Sol. y be the petrol burnt hour $y = kv^3$ 'S' be the distance traveled by boat the petrol burnt = $\frac{S}{V-C} \times kv^3$

$f'(v) = 0 \Rightarrow v = \frac{3c}{2}$

15. A point 'P' is given on the circumference of a Circle of radius 'r' .The chord 'QR' is parallel to the tangent line at 'P' the maximum area of the triangle PQR is

1) $\frac{3\sqrt{2}}{4}r^2$ 2) $\frac{3\sqrt{3}}{4}r^2$ 3) $\frac{3}{8}r^2$ 4) $\frac{3\sqrt{2}}{4}r$

Key. 2

Sol. The area maximum when the triangle is equilateral

16. The minimum value of $f(x) = x^2 + \frac{250}{x}$ is

1) 15 2) 25 3) 45 4) 75

Key. 4

Sol. $f'(x)=0$ *and* $f''(5)>0$ minimum value $= f(5)$

17. The sum of two numbers is '6'.The minimum value of the sum of their reciprocals is

1) $\frac{3}{4}$ 2) $\frac{6}{5}$ 3) $\frac{2}{3}$ 4) $\frac{2}{5}$

Key. 3

Sol. $x=y=\frac{6}{2}=3,\ \frac{1}{x}+\frac{1}{y}=\frac{2}{3}$

18. Minimum value of $\frac{(6+x)(11+x)}{2+x}$ is

1) 5 2) 15 3) 45 4) 25

Key. 4

Sol. $f'(x)=0$ *when put* $x=4$

19. The maximum area of a rectangle inscribed in a circle of radius 5 cm is

1) $25\ sq.cm$ 2) $50\ sq.cm$ 3) $100\ sq.cm$ 4) $\frac{25}{2}sq.cm$

Key. 2

Sol. $Area=2r^2=50\ sq.cm$

20. The diagonal of the rectangle of maximum area having perimeter 100 cm is

1) $10\sqrt{2}$ 2) 10 3) $25\sqrt{2}$ 4) 15

Key. 3

Sol. The maximum perimeter of the rectangle that can be inscribed in a circle is a square .Here the lengths are $x=\sqrt{2}\ r,\ y=\sqrt{2}\ r$

21. The maximum value of $x^{-x},(x>0)$ is

1) $e^e$ 2) $e^{1\backslash e}$ 3) $e^{-e}$ 4) $1\backslash e$

Key. 2

Sol. $f(x)=x^{-x},\ f'(x)=0\Rightarrow x=e^{-1}$
$f''(e-1)<0$

22. Which fraction exceeds its $p^{th}$ power by the greatest number possible is?

1) $p^p$ 2) $\left(\frac{1}{P}\right)^{P-1}$ 3) $p^{\frac{1}{1-p}}$ 4) $\frac{1}{p^p}$

Key. 3

$y = x - x^p$

Sol. $\frac{dy}{dx} = 0 \Rightarrow x = \left(\frac{1}{p}\right)^{\frac{1}{p-1}}$

23. In $(0, 2\pi)$, $f(x) = x + \sin 2x$ is

1) Minimum at $x = \frac{2\pi}{3}$ 2) Maximum at $x = \frac{2\pi}{3}$

3) Maximum at $x = \frac{\pi}{4}$ 4) Minimum at $x = \frac{\pi}{6}$

Key. 1

Sol. $f'(x) = 0 \Rightarrow f''(x) > 0 \text{ when } x = \frac{2\pi}{3}$

24. The Value of 'a' for which $f(x) = a\sin x + \frac{1}{3}\sin 3x$ has an extremum at $x = \frac{\pi}{3}$ is

1) 1 2) -1 3) 0 4) 2

Key. 4

Sol. $\frac{d^2y}{dx^2} = 0$ then find '$x$' and substitute in $\frac{dy}{dx}$.

25. A person wishes to lay a straight fence across a triangular field ABC, with $\underline{|A} < \underline{|B} < \underline{|C}$ so as to divide it into two equal areas. The length of the fence with minimum expense, is

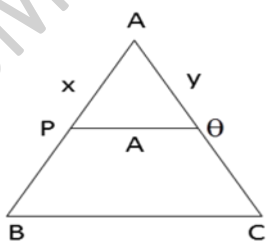
a) $\sqrt{2\Delta \cot \frac{B}{2}}$ b) $\sqrt{2\Delta \tan \frac{C}{3}}$

c) $\sqrt{\tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}}$ d) $\sqrt{2\Delta \tan \frac{A}{2}}$

(where '$\Delta$' represents, area of triangle ABC)

Key. D

Sol.



$$\frac{1}{2}xy\sin A = \frac{1}{2}\left(\frac{1}{2}bc\sin A\right)$$

$$\Rightarrow xy = \frac{1}{2}bc$$

$$z_A^2 = (PQ)^2 = x^2 + y^2 - 2xy\cos A$$

$$= x^2 + \frac{b^2c^2}{4x^2} - bc\cos A$$

$$\Rightarrow 2Z_A\left(\frac{dZ_A}{dx}\right) = 2x - \frac{b^2c^2}{2x^3}$$

$$\frac{dZ_A}{dx} = 0 \Rightarrow x = \sqrt{\frac{bc}{2}},\text{ and } \frac{d^2Z_A}{dx^2} > 0$$

Hence $Z_A$ is minimum if $x = \sqrt{\frac{bc}{2}}$ and the minimum value of $Z_A$,is

$$\sqrt{\frac{bc}{2} + \frac{bc}{2} - bc\cos A} = \sqrt{2\Delta\tan\frac{A}{2}}$$

26. The number of critical point of $f(x) = \frac{|x-1|}{x^2}$ is

1) 1 2) 2 3) 3 4) 0

Key. 2

Sol.
$$f(x) = \left|\frac{x-1}{x^2}\right|, f(x) = 0 \ for \ x = \pm 2$$
$$f(x) = \pm\left(x - \frac{1}{x}\right) \Rightarrow f'(x) = \pm\left(1 + \frac{1}{x^2}\right) \neq 0$$

27. The total cost of producing '$x$' pocket radio sets per day is Rs.$\left(\frac{1}{4}x^2 + 35x + 25\right)$ and the price per set at which they may be sold is Rs.$\left(50 - x/2\right)$ to obtain maximum profit the daily out put should be--------- radio sets.

1) 10 2) 5 3) 15 4) 20

Key. 1

Sol. If daily out put is x sets and p be the total point then

$$p = x\left(50 - 1/2\,x\right) - \left(\frac{1}{4}x^2 + 35x - 25\right)$$

$$\frac{dp}{dx} = 0 \Rightarrow x = 10 \ and \ \left(\frac{d^2p}{dx^2}\right)_{(x=10)} = -3/2 < 0$$

28. If $f(x) = a\log|x| + bx^2 + x$ has extreme values at $x = -1,\ x = 2$ then a = ---- b = --

1) $2, \frac{-1}{2}$ 2) $\frac{-1}{2}, 2$ 3) $\frac{1}{2}, 2$ 4) $2, \frac{1}{2}$

Key. 1

Sol.
$$f'(-1)=0 \Rightarrow -a-2b+1=0$$
$$f'(2)=0 \Rightarrow -\frac{a}{2}+4b+1=0$$

29. A quadratic function in ' $x$ ' has the values '10' when $x=1$ and has minimum value '1' when $x=-2$ the function is

1) $2x^2+3x+5$ 2) $3x^2+2x+5$ 3) $x^2+3x+6$ 4) $x^2+4x+5$

Key. 4

Sol.
$$f(x)=ax^2+bx+c$$
$$a+b+c=10,\ f'(-2)=0,\ f(-2)=1$$

30. The equation of a line passing through the point (3,4) and which forms a triangle of minimum area with the coordinate axes in the first quadrant

1) $4x+3y-24=0$ 2) $3x+4y-12=0$ 3) $2x+3y-12=0$ 4) $3x+2y-24=0$

Key. 1

Sol. (3 ,4 ) is the mid point of the line segment

31. The maximum of $f(x)=2x^3-9x^2+12x+4$ occurs at $x=$

1) 1 2) 2 3) -1 4) -2

Key. 1

Sol.
$$f'(x)=0 \Rightarrow 6x^2-18x+12=0$$
$$f''(x)=12x-18$$

32. $f(x)=4+5x^2+6x^4$ has

1) Only one minimum 2) Neither maximum n or minimum
3) Only one maximum 4) No minimum.

Key. 1

Sol. $f(x)$ is minimum at $x=0$

33. At $x=0,\ f(x)=(3-x)e^{2x}-4xe^x-x$

1) Has a minimum 2) Has a maximum
3) Has no extremum 4) Is not defined

Key. 3

Sol.
$At\ x=0,\ f'(x)=0$
$At\ x=0,\ f''(x)=0$
$At\ x=0,\ f'''(x)\neq 0$
$\therefore f(x)\, is\ neither\ \max imum\ nor\ \min imum$

34. The number of critical points of $f(x)=\frac{|x-1|}{x^2}$ is

(A) 1 (B) 2 (C) 3 (D) None of these

Key. C

Sol. f(x) is not differentiable at x = 0 and x = 1.
$f'(x)=0$ at $x=2$

35. A differentiable function f(x) has a relative minimum at x =0, then the function y = f(x) + ax + b has a relative minimum at x = 0 for

(A) all a and all b (B) all b > 0 (C) all b, if a = 0 (D) all a > 0

Key. C

Sol. $f'(0)=0$ and $f''(0)>0$
y = f(x) + ax + b has a relative minimum at x = 0.
Then $\frac{dy}{dx}=0$ at x = 0
$f'(x)+a=0 \Rightarrow\ a=0$
$f''(x)>0 \Rightarrow\quad f''(0)>0$
Hence y has relative minimum at x = 0 if a = 0 and b $\in$ R.

36. Let $f:[0,4]\rightarrow R$, be a differentiable function. Then, there exists real numbers $a,b\in(0,4)$ such that, $(f(4))^2-(f(0))^2=Kf^1(a)f(b)$ Where K, is

a) $\frac{1}{4}$ b) 8 c) $\frac{1}{12}$ d) 4

Key. B

Sol. By LMVT, $\exists a\in(0,4) \ni \frac{f(4)-f(0)}{4-0}=f^1(a)\Rightarrow f(4)-f(0)=4f^1(a)$

$\because \frac{f(4)+f(0)}{2}$ lies between $f(0)$ and $f(4)$, by Intermediate value theorem

$\exists b\in(0,4)\ni \frac{f(4)+f(0)}{2}=f(b)$ hence, $\left(f(4)^2\right)-(f(0))^2=8\quad f^1(a)f(b)$

37. If $f(x)=(1-x)^{5/2}$ satisfies the relation, $f(x)=f(o)+xf^{1}(o)+\frac{x^2}{2}f^{11}(\theta x)$ then, as $x\to 1$, the value of $\theta$, is

a) $\frac{4}{25}$ b) $\frac{25}{4}$ c) $\frac{25}{9}$ d) $\frac{9}{25}$

Key. D

Sol. $f^{1}(x)=\frac{-5}{2}(1-x)^{3/2}$ and $f^{11}(x)=\frac{15}{4}(1-x)^{1/2}$ and $f(0)=1, f^{1}(0)=\frac{-5}{2}$,

$f^{11}(\theta x)=\frac{15}{4}(1-\theta x)^{1/2}$

Hence, $(1-x)^{5/2}=\frac{2-5x}{2}+\frac{x^2}{2}(1-\theta x)^{1/2}\times\frac{15}{4}$ as

$x\to 1,\ 0=1-\frac{5}{2}+\frac{15}{8}(1-\theta)^{1/2}\Rightarrow\theta=9/25$

38. $A(1,0), B(e,1)$ are two points on the curve $y=\log_e x$. If P is a point on the curve at which the tangent to the curve is parallel to the chord AB, then, abscissa of P, is

a) $\frac{e-1}{2}$ b) $\frac{e+1}{2}$ c) $e-1$ d) $e+1$

Key. C

Sol. By LMVT, applied to $f(x)=\log_e x$ on $[1,e], \exists$ an $x_0\in(1,e) \ni f^{1}(x_0)=\frac{f(e)-f(1)}{e-1}$

$\Rightarrow x_0=e-1$

39. Consider the following statements

Statement – I: If f and g are continuous and monotonic on R, then, f + g is also a monotonic function.

Statement- II: If $f(x)$ is a continuous decreasing function $\forall x>0$, and $f(1)$ is positive, then, $f(x)=0$ happens exactly at one value of x. Then,

a) Both I and II are true b) I is true, II is false

c) I is false, II is true d) both I and II are false

Key. D

Sol. I : $f(x)=x$ and $g(x)=-x^2$ on R

II : $f(x)=\frac{1}{x}, x>0$

40. The number of values of x at which the function, $f(x)=(x-1)x^{2/3}$ has extreme values, is

a) 4 b) 3 c) 2 d) 1

Key. C

Sol. $f^1(x)=\dfrac{5x-2}{3x^{1/3}}$

Let $x<0, f^1(x)>0$ and for $x>0, f^1(x)<0$ $\Rightarrow f$ has maximum at x = 0

$x<\dfrac{2}{5}, f^1(x)<0$ and $x>\dfrac{2}{5}, f^1(x)>0 \Rightarrow f$ has minimum at $X=\dfrac{2}{5}$

41. A person wishes to lay a straight fence across a triangular field ABC, with $\angle A<\angle B<\angle C$ so as to divide it into two equal areas. The length of the fence with minimum expense, is

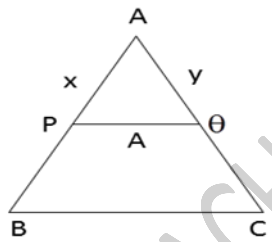
a) $\sqrt{2\Delta\cot\dfrac{B}{2}}$ b) $\sqrt{2\Delta\tan\dfrac{C}{3}}$

c) $\sqrt{\tan\dfrac{A}{2}\tan\dfrac{B}{2}\tan\dfrac{C}{2}}$ d) $\sqrt{2\Delta\tan\dfrac{A}{2}}$

(where '$\Delta$' represents, area of triangle ABC)

Key. D

Sol.




$$\frac{1}{2}xy\sin A=\frac{1}{2}\left(\frac{1}{2}bc\sin A\right)$$

$$\Rightarrow xy=\frac{1}{2}bc$$

$$z_A^2=(PQ)^2=x^2+y^2-2xy\cos A$$

$$=x^2+\frac{b^2c^2}{4x^2}-bc\cos A$$

$$\Rightarrow 2Z_A\left(\frac{dZ_A}{dx}\right)=2x-\frac{b^2c^2}{2x^3}$$

$$\frac{dZ_A}{dx}=0 \Rightarrow x=\sqrt{\frac{bc}{2}},\ \text{and}\ \frac{d^2Z_A}{dx^2}>0$$

Hence $Z_A$ is minimum if $x=\sqrt{\frac{bc}{2}}$ and the minimum value of $Z_A$,is

$$\sqrt{\frac{bc}{2}+\frac{bc}{2}-bc\cos A}=\sqrt{2\Delta\tan\frac{A}{2}}$$

42. If the function $f(x)=ax^3+bx^2+11x-6$ satisfies conditions of Rolle's theorem in [1, 3] and $f'\left(2+\frac{1}{\sqrt{3}}\right)=0$, then value of a and b are respectively

(A) 1, –6 (B) –1, 6 (C) –2, 1 (D) –1, 1/2

Key. A

Sol. $\because$ f (1) = f(3)

$\Rightarrow$ a + b + 11 – 6 = 27a + 9b + 33 – 6

$\Rightarrow$ 13a + 4b = –11

and $f'(x)=3ax^2+2bx+11$ ... (i)

$\Rightarrow$ $f'\left(2+\frac{1}{\sqrt{3}}\right)=3a\left(2+\frac{1}{\sqrt{3}}\right)^2+2b\left(2+\frac{1}{\sqrt{3}}\right)+11=0$

$\Rightarrow$ $3a\left(4+\frac{1}{3}+\frac{4}{\sqrt{3}}\right)+2b\left(2+\frac{1}{\sqrt{3}}\right)+11=0$ ... (ii)

From eqs. (i) and (ii), we get a = 1, b = –6.

43. Let $f(x)$ be a positive differentiable function on $[0,a]$ such that $f(0)=1$ and $f(a)=3^{1/4}$ If $f^1(x)\ge(f(x))^3+(f(x))^{-1}$, then, maximum value of a is

a) $\frac{\pi}{12}$ b) $\frac{\pi}{36}$ c) $\frac{\pi}{24}$ d) $\frac{\pi}{48}$

Key. C

Sol. $f^1(x)f(x)\ge(f(x))^4+1$

$$\Rightarrow \frac{2f^1(x)f(x)}{\left\{(f(x))^2\right\}^2+1}\ge 2$$

$$\Rightarrow \int_0^a \frac{2f^1(x)f(x)}{\left\{(f(x))^2\right\}^2+1}\ge 2\int_0^a 1dx$$

$$\Rightarrow \left|\tan^{-1}(f(x))^2\right|_0^a \ge 2a \Rightarrow \frac{\pi}{3}-\frac{\pi}{4}\ge 2a$$

Given expansion = $\{x-(1+\cos t)\}^2+\left\{\frac{K}{x}-(1+\sin t)\right\}^2$

44. For $x > 0, 0 \le t \le 2\pi, K > \frac{3}{2} + \sqrt{2}$, K being a fixed real number the minimum value of $x^2 + \frac{K^2}{x^2} - 2\left\{(1+\cos t)x + \frac{K(1+\sin t)}{x}\right\} + 3 + 2\cos t + 2\sin t$ is

a) $\left\{\sqrt{K} - \left(1 + \frac{1}{\sqrt{2}}\right)\right\}^2$  b) $\frac{1}{2}\left\{\sqrt{K} - \left(1 + \frac{1}{\sqrt{2}}\right)\right\}^2$

c) $3\left\{\sqrt{K} - \left(1 + \frac{1}{\sqrt{2}}\right)\right\}^2$  d) $2\left\{\sqrt{K} - \left(1 + \frac{1}{\sqrt{2}}\right)\right\}^2$

Key. D

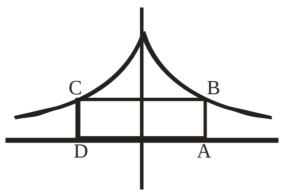
Sol. Given expansion = $\{x - (1+\cos t)\}^2 + \left\{\frac{K}{x} - (1+\sin t)\right\}^2$

45. The maximum area of a rectangle whose two consecutive vertices lie on the x-axis and another two lie on the curve $y = e^{-|x|}$ is equal to

(A) 2e sq. Units  (B) $\frac{2}{e}$ sq. Units (C) e sq. units  (D) $\frac{1}{e}$ sq. units

Key. B

Sol.




Let the rectangle is (ABCD)

$A = (t, 0), B = (t, e^{-t}), C = (-t, e^{-t}), D = (-t, 0)$

ABCD = $2te^{-t}$ = f(t)

$\frac{df}{dt} = 2(t(-e^{-t}) + e^{-t}) = 2e^{-t}(1-t)$

$\frac{df}{dt} > 0 \Rightarrow t \in (0,1)$

$\frac{df}{dt} < 0 \Rightarrow t \in (1,\infty)$

t = 1 is point of maxima

Maximum area = $f(1) = \frac{2}{e}$

46. The number of critical points of $f(x) = \frac{|x-1|}{x^2}$ is

(A) 1  (B) 2  (C) 3  (D) None of these

Key. C

Sol. f(x) is not differentiable at x = 0 and x = 1.

$f'(x) = 0$ at $x = 2$

47. A differentiable function f(x) has a relative minimum at x =0, then the function y = f(x) + ax + b has a relative minimum at x = 0 for

(A) all a and all b (B) all b > 0 (C) all b, if a = 0 (D) all a > 0

Key. C

Sol. $f'(0)=0$ and $f''(0)>0$

y = f(x) + ax + b has a relative minimum at x = 0.

Then $\frac{dy}{dx}=0$ at x = 0

$f'(x)+a=0 \Rightarrow a=0$

$f''(x)>0 \Rightarrow f''(0)>0$

Hence y has relative minimum at x = 0 if a = 0 and b $\in$ R.

48. Let A(1, 2), B(3, 4) be two points and C(x, y) be a point such that area of $\Delta ABC$ is 3 sq.units and $(x-1)(x-3)+(y-2)(y-4)=0$. Then maximum number of positions of C, in the xy plane is

a) 2 b) 4 c)8 d) none of these

Key: D

Hint: (x,y) lies on the circle ,with AB as a diameter . Area

$(\Delta ABC)=3$

$\Rightarrow \left(\frac{1}{2}\right)(AB)(\text{altitude})=3.$

$\Rightarrow \text{altitude} = \frac{3}{\sqrt{2}} \Rightarrow \text{no such "C" exists}$

49. If $y,z>0$ and $y+z=C$, then minimum value of $\sqrt{\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right)}$ is equal to

A) $\frac{C}{2}+1$ B) $\frac{2}{C}+3$ C) $1+\frac{2}{C}$ D) $\frac{C}{2}$

Key: C

Hint: $\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right)=1+\frac{1}{y}+\frac{1}{z}+\frac{1}{yz}$

$=1+\frac{1}{y}+\frac{1}{z}+\frac{1}{yz}\geq 1+\frac{2}{\sqrt{yz}}+\frac{1}{yz}=\left(1+\frac{1}{\sqrt{yz}}\right)^2$ = $\frac{1}{\sqrt{yz}}\geq\frac{2}{y+z}\geq\frac{2}{C}$ = $\left(1+\frac{1}{\sqrt{yz}}\right)^2\geq\left(1+\frac{2}{C}\right)^2$

50. Let a, b, c, d, e, f, g, h be distinct elements in the set {–7, –5, –3, –2, 2, 4, 6, 13}. The minimum value of $(a + b + c + d)^2 + (e + f + g + h)^2$ is

a) 30 b) 32 c) 34 d) 40

Key: B

Hint: Note that sum of the elements is 8

Let a + b + c + d = x

$\therefore$ e + f + g + h =8 – x

Again, let y = $x^2$ + (8 - x)$^2$

$\therefore$ y = 2$x^2$ - 16 x + 64

= 2[$x^2$-8x + 32]

=2(x-4)$^2$+16]

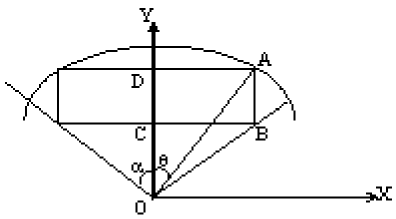
$\therefore$ $_{min}$ = 32 when x = 4

51. A sector subtends an angle $2\alpha$ at the centre then the greatest area of the rectangle inscribed in the sector is (R is radius of the circle)

a) $R^2 \tan\frac{\alpha}{2}$ b) $\frac{R^2}{2}\tan\frac{\alpha}{2}$ c) $R^2\tan\alpha$ d) $\frac{R^2}{2}\tan\alpha$

Key: A

Hint: Let A be any point on the arc such that $\angle YOA = \theta$
Where $0 \le \theta \le \alpha$




DA = CB = R sin $\theta$, OD = R cos $\theta$

$\Rightarrow$ CO = CB cot $\alpha = R\sin\theta\cot\alpha$

Now, CD = OD - OC = R cos $\theta - R\sin\theta\cot\alpha$

= R (cos $\theta$ – sin $\theta$ cot $\alpha$ )

Area of rectangle ABCD , S = CD.CB

R = (cos$\theta$ - sin $\theta$ cot $\alpha$) R sin $\theta$ = $R^2$(sin$\theta$ cos$\theta$ - sin$^2$ $\theta$ cot$\alpha$ )

$\frac{R^2}{2}\left(\sin2\theta-(1-\cos2\theta)\cot\alpha\right)$ $\frac{R^2}{2\sin\alpha}\left[\cos(2\theta-\alpha)\right]$

$S_{man} = \frac{R^2}{\sin\alpha}(1-\cos\alpha)$ $\left(\text{for } \theta = \alpha/2\right)$

Hence, greatest area of the rectangle = $R^2\tan\frac{\alpha}{2}$

52. Let $f:(0,\infty)\to R$ be a (strictly) decreasing function. If $f\left(2a^2+a+1\right) < f\left(3a^2-4a+1\right)$, then the range of $a \in R$ is

(A) $\left(-\infty,\frac{1}{3}\right)\cup(1,\infty)$ (B) (0, 5) (C) $\left(0,\frac{1}{3}\right)\cup(1,5)$ (D) [0, 5]

Key: C

Hint: we have $2a^2+a+1>3a^2-4a+1 \Rightarrow a^2-5a<0 \Rightarrow 0<a<5$ …….(A)

ALSO $3a^2-4a+1>(3a-1)(a-1)>0 \Rightarrow a\in(-\infty,1/3)\cup(1,\infty)$.........(B)

INTERSECTION OF (A) AND (B) YIELDS $a \in (0, 1/3) \cup (1, 5)$

53. The greatest possible value of the expression $\tan\left(x+\frac{2\pi}{3}\right)-\tan\left(x+\frac{\pi}{6}\right)+\cos\left(x+\frac{\pi}{6}\right)$ on the interval $\left[-5\pi/12, -\pi/3\right]$ is

(A) $\frac{12}{5}\sqrt{2}$ (B) $\frac{11}{6}\sqrt{2}$ (C) $\frac{12}{5}\sqrt{3}$ (D) $\frac{11}{6}\sqrt{3}$

Key: D

Hint: Let $u=-x-\pi/6$ then $u \in [\pi/6, \pi/4]$ and then $2u \in [\pi/3, \pi/2]$

$\tan(x+2\pi/3) = -\cot(x+\pi/6) = \cot u$

NOW $\tan(x+2\pi/3) - \tan(x+\pi/6) + \cos(x+\pi/6)$

$= \cot u + \tan u + \cos u$

$= \frac{2}{\sin 2u} + \cos u$

BOTH $\frac{2}{\sin 2u}$ AND $\cos u$ MONOTONIC DECREASING ON $[\pi/6, \pi/4]$ AND THUS THE GREATEST VALUE OCCURS AT $u = \pi/6$

I.E $\frac{2}{\sin \pi/3} + \cos \pi/6 = \frac{4}{\sqrt{3}} + \frac{\sqrt{3}}{2} = \frac{11}{2\sqrt{3}} = \frac{11\sqrt{3}}{6}$

54. Let the smallest positive value of x for which the function $f(x) = \sin\frac{x}{3} + \sin\frac{x}{11}$, $(x \in R)$ achieves its maximum value be $x_0$. Express $x_0$ in degrees i.e, $x_0 = \alpha^0$. Then the sum of the digits in $\alpha$ is

(A) 15 (B) 17 (C) 16 (D) 18

Key: D

Hint The maximum possible values is 2

$\sin(x/3)$ TAKES THE VALUES 1 WHEN

$x/3 = 2n\pi + \pi/2$

I.E $x/3 = 90 + 360m$

$\sin(x/11)$ TAKES THE VALUE 1

WHEN $x/11 = 2n\pi + \pi/2$

I.E $x/11 = 90 + 360n$

WE ARE LOOKING FOR A COMMON SOLUTION

WE HAVE $3m - 11n = 2$. THEN SMALLEST POSITIVE SOLUTION TO THIS IS $m = 8, n = 2$, THUS $x_0 = 8910^0$, GIVING $\alpha = 8910$

55. Let $f(x) = \begin{cases} (x+1)^3 & -2 < x \le -1 \\ x^{2/3} - 1 & -1 < x \le 1 \\ -(x-1)^2 & 1 < x < 2 \end{cases}$

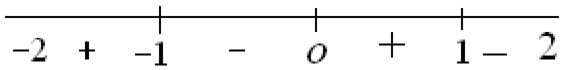
The total number of maxima and minima of $f(x)$ is

(A) 4 (B) 3 (C) 2 (D) 1

KEY : B

HINT : $f'(x)=\begin{cases} 3(x+1)^2 & -2<x<-1 \\ \frac{2}{3}\times x^{-1/3} & -1<x<1-\{0\} \\ -2(x-1) & 1<x<2 \end{cases}$

$f'(x)\,DNE\ \ at\ \ x=-1,0,1$




Sign of $f'(x)$

56. Let f(x) = $x^2$ – bx + c , b is a odd positive integer, f(x) = 0 have two prime numbers as roots and b + c = 35. Then the global minimum value of f(x) is

(A) $-\frac{183}{4}$ (B) $\frac{173}{16}$

(C) $-\frac{81}{4}$ (D) data not sufficient

KEY : C

SOL : Let $\alpha$, $\beta$ be roots of $x^2$ – bx + c = 0,

Then $\alpha + \beta$ = b

$\Rightarrow$ one of the roots is '2' (Since $\alpha$, $\beta$ are primes and b is odd positive integer)

$\therefore$ f(2) = 0 $\Rightarrow$ 2b - c = 4 and b + c = 35

$\therefore$ b = 13, c = 22

Minimum value = $f\left(\frac{13}{2}\right)=-\frac{81}{4}$.

57. Maximum value of $\log_5(3x+4y)$ , if $x^2+y^2=25$ is

(A) 2 (B) 3 (C) 4 (D) 5

Key : A

Hint : Since $x^2+y^2=25 \Rightarrow$ x = 5 cos $\theta$ and y = 5 sin $\theta$

So, therefore, $\log_5(3x+4y)=\log_5(15\cos\theta+20\sin\theta)$

$\Rightarrow \{\log_5(3x+4y)\}_{max}=2$

58. The greatest area of the rectangular plot which can be laid out within a triangle of base 36 ft. & altitude 12ft equals (Assume that one side of the rectangle lies on the base of the triangle)

(A) 90 (B) 108

(C) 72 (D) 126

Key: B

Hint: Area of rectangle = A = xy ........(i)

Also $\frac{36}{x}=\frac{12}{12-y}\Rightarrow 3y=(36-x)$ ......(ii)

$\therefore A=\frac{A}{3}(36-x)=\frac{1}{3}\left(36x-x^2\right)$

Now $A'(x)=0\Rightarrow 36-2x=0\Rightarrow x=18$

$A''(x)=\frac{1}{3}(-2)<0$

Also $y=\frac{36-x}{3}-\frac{36-18}{3}=6$

$\therefore A_{mas}18\times 6=108\text{sq.feet}$

59. Let f(x) = $\begin{cases} 3x+|a^2-4|, & x<1 \\ -x^2+2x+7, & x\ge 1\end{cases}$. Then set of values of a for which f(x) has maximum value at x = 1 is

(A) $(3, \infty)$ (B) [-3, 3]

(C) $(-\infty, 3)$ (D) none of these

Key: B

Hint: Since $-x^2 + 2x + 7$ takes maximum value 8 at x = 1, so f(x) take maximum value at x = 1,

if $\lim_{x\to 1} f(x)\le f(1)$

$\Rightarrow |a^2 - 4| \le 5 \Rightarrow a\in [-3, 3]$

60. Let $f(x)=(\sin\theta)\left(x^2-2\right)((\sin\theta)x+\cos\theta),(\theta\ne m\pi, m\in I)$ Then f(x) has

(A) local maxima at certain $x\in R^+$ (B) a local maxima at certain $x\in R^-$

(C) a local minima at certain x = 0 (D) a local minima at certain $x\in R^-$

Key: B

Hint: $f(x) = (\sin^2\theta)x^3 + \frac{1}{2}\sin 2\theta\, x^2 - 2\sin^2\theta.\, x - \sin 2\theta$

$f'(x) = (3\sin^2\theta)x^2 + \sin 2\theta x - 2\sin^2\theta$

Then D > 0 and product of roots < 0

So f(x) has local maxima at some $x\in R^-$

and local minima at some $x\in R^+$

61. Let $g(x)=\frac{1}{4}f\left(2x^2-1\right)+\frac{1}{2}f\left(1-x^2\right)\forall x\in R,$ where $f''(x)>0\forall x\in R$, g(x) is necessarily increasing in the interval

(A) $\left(-\sqrt{\frac{2}{3}},\sqrt{\frac{2}{3}}\right)$ (B) $\left(-\sqrt{\frac{2}{3}},0\right)\cup\left(\sqrt{\frac{2}{3}},\infty\right)$

(C) $(-1,1)$ (D) None of these

Key: B

Hint: $f''(x)>0$

$\Rightarrow f'$ is inc. fn

To find : where g is nec. Inc

g is inc $\Rightarrow g'>0$

$$\Rightarrow \frac{1}{4}.f'\left(2x^2-1\right)(4x)+\frac{1}{2}P\left(1-x^2\right)(-2x)>0$$

$$\Rightarrow x\left\{f'\left(2x^2-1\right)-f'\left(1-x^2\right)\right\}>0$$

Case 1 : $x>0\rightarrow(1)f'\left(2x^2-1\right)>f'\left(1-x^2\right)$

$$\Rightarrow 2x^2-1>1-x^2$$

$$\Rightarrow x\in\left(-\infty,\sqrt{\frac{2}{3}}\right)\cup\left(\sqrt{\frac{2}{3}},\infty\right)\rightarrow(2)$$

$(1)\cap(2)\Rightarrow x\in\left(\sqrt{\frac{2}{3}},\infty\right)$ ..........(3)

Case II : $x<0\rightarrow(3)\ f'\left(2x^2-1\right)<f'\left(1-x^2\right)$

$$\Rightarrow 2x^2-1<1-x^2$$

$$\Rightarrow x\in\left(-\sqrt{\frac{2}{3}},\sqrt{\frac{2}{3}}\right)\rightarrow(4)$$

$$(3)\cap(4)\Rightarrow\in\left(-\sqrt{\frac{2}{3}},0\right)\rightarrow(6)$$

$\therefore$ g is inc in $x\in(5)\cup(6)$

$$\Rightarrow x\in\left(-\sqrt{\frac{2}{3}},0\right)\cup\left(\sqrt{\frac{2}{3}},\infty\right)$$

62. A variable line through A(6,8) meets the curve $x^2+y^2=2$ at B and C. P is a point on BC such that AB, AP, AC are in HP. The minimum distance of the origin from the locus of P is

a) 1 b) $\frac{1}{2}$ c) $\frac{1}{3}$ d) $\frac{1}{5}$

Key: D

Hint: Locus of P is the chord of contact of tangent, from A is $3x+4y-1=0$

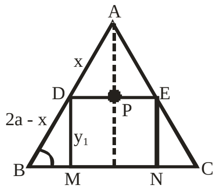
Distance of (0,0) is $\frac{1}{5}$

63. A rectangle is inscribed in an equilateral $\Delta$ of side length 2a units. Maximum area of this rectangle is

(A) $\sqrt{3}a^2$ (B) $\frac{\sqrt{3}a^2}{4}$ (C) $a^2$ (D) $\frac{\sqrt{3}a^2}{2}$

Key. D

Sol.




Let AD = x

BD = (2a – x)

In $\Delta DBM$

$\angle B = \frac{\pi}{3}$

Let DM = $y_1$

DE = $2x_1$

$\sin 60° = \frac{y_1}{2a - x}$

$y_1 = (2a - x) \times \frac{\sqrt{3}}{2}$

In $\Delta ADP$

$\angle D = \frac{\pi}{3}$

$\cos 60° = \frac{x_1}{x}$

$x_1 = x \times \frac{1}{2}$

$2x_1 = x$

$\Delta(x)$ = Area of rectangle = $2x_1 y$

$\Delta(x) = x \times (2a - x)\frac{\sqrt{3}}{2}$

$\Delta'(x) = \frac{\sqrt{3}}{2}(2a - 2x) = 0 \Rightarrow x = a$

$\Delta''(a) = -ve$

x = a point of maxima

maximum area = $a \times \frac{a\sqrt{3}}{2} = \frac{\sqrt{3}a^2}{2}$

64 If the equation $a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x = 0\,(a_1 \neq 0, n \geq 2)$ has a+ve root $x = \alpha$, then the equation $na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + .... + a_1 = 0$ has a positive root, which is :

1. equal to $\alpha$ 2. $\geq \alpha$ 3. $< \alpha$ 4. $> \alpha$

Key. 3

Sol. $f(x) = a_n x^n + a_{n-1}x^{n-1} + .... + a_1 x = 0$ has a+ve root $x = \alpha$ ; by observation x = 0 is also a root

$f(\alpha) = f(0) = 0$

f(x) is continuous on $[0, \alpha]$ and differentiable on $(0, \alpha)$ by Rolle's Theorem

$\Rightarrow \exists$ at least one root $c \in (0, \alpha)$

Such that $f'(c) = 0$

$\therefore \; 0 < c < \alpha$

65 The minimum & maximum value of $f(x) = \sin(\cos x) + \cos(\sin x) \forall -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ are respectively.

1. cos 1 and 1+ sin 1
2. sin 1 and 1+ cos 1
3. cos 1 & $\cos\left(\frac{1}{\sqrt{2}}\right)+\sin\left(\frac{1}{\sqrt{2}}\right)$
4. 2

Key. 1

Sol. Given $f(x)=\sin(\cos x)+\cos(\sin x)$

Fact when a function is even & defined in negative as well as positive interval for maxima & minima, we check the maxima/minimum in the positive internal only so it suffices to find the maximum & minimum values of $f$ in $0\le x\le\frac{\pi}{2}$.

Now $x\in[0,\frac{\pi}{2}]$, $\sin(\cos x)\,\&\cos(\sin x)$ are decreasing functions so maximum of f(x) is f(0) & minimum of f(x) is $f(\pi/2)$

$\therefore f(\pi/2)=\sin(\cos\pi/2)+\cos(\sin\pi/2)=\cos 1$

And $f(0)=\sin(\cos 0^0)+\cos(\sin 0^0)=\sin 1+\cos 0^0=1+\sin 1$

66 Let $f(x)=\begin{cases}\dfrac{\cos(\pi x)}{2} & \forall\, 0\le x<1\\ 3+5x & \forall\, x\ge 1\end{cases}$

1. f(x) has local minimum at x= 1
2. f(x) has local maximum at x= 1
3. f(x) does not have any local maximum or local minimum at x = 1
4. f(x) has a global minimum at x = 1

Key. 1

Sol. $f(x)=\begin{cases}\cos\dfrac{\pi}{2}x & \forall\, 0\le x<1\\ 5x+3 & \forall\, x\ge 1\end{cases}$

$f'(x)=\begin{cases}-\dfrac{\pi}{2}\sin\dfrac{\pi}{2}x & \forall\, 0\le x<1\\ 5 & \forall\, x\ge 1\end{cases}$

$\Rightarrow f'(x)$ changes its sign from –ve to +ve in the immediate neighbourhood of

x= 1

$\Rightarrow f(x)$ changes from decreasing function to increasing function

$\Rightarrow f(x)$ has a local minimum value at x= 1

67 The minimum value of $x^2-x+1+\sin x$ is given by

1. $\frac{1}{4}$
2. $\frac{3}{4}$
3. $-\frac{1}{4}$
4. $-\frac{7}{4}$

Key. 3

Sol. Let $f(x)=x^2-x+1+\sin x$

$=(x-1/2)^2+(\frac{3}{4}+\sin x)$

$\geq \frac{3}{4}+\sin x$ $\quad (\because (x-\frac{1}{2})^2 \geq 0)$

$\geq \frac{3}{4}-1=-1/4$ ($\because$ minimum value of sinx = -1)

68. If $f(x)$ is a differentiable function $\forall x \in R$ so that, $f(2)=4, f^1(x) \geq 5\ \forall x \in [2,6]$, then, $f(6)$ is

a) $\geq 24$ b) $\leq 24$ c) $\geq 9$ d) $\leq 9$

Key. A

Sol. By mean value theorem, $f(6)-f(2)=(6-2)f^1(c)$ where $c \in (2,6)$

$\Rightarrow f(6)=f(2)+4f^1(c)=4+4f^1(1)>4+4(5)$

$(\therefore f^1(x) \geq 5)$ $f(6) \geq 24$

69. The values of parameter 'a' for which the point of minimum of the function $f(x)=1+a^2x-x^3$ satisfies the inequality $\frac{x^2+x+2}{x^2+5x+6}<0$ are,

a) $(-3\sqrt{3},-2\sqrt{3}) \cup (2\sqrt{3},3\sqrt{3})$ b) $(-5\sqrt{3},-3\sqrt{3}) \cup (3\sqrt{3},5\sqrt{3})$

c) $(-7\sqrt{3},-5\sqrt{3}) \cup (5\sqrt{3},7\sqrt{3})$ d) $(-9\sqrt{3},-6\sqrt{3}) \cup (6\sqrt{3},9\sqrt{3})$

Key. A

Sol. $\frac{x^2+x+2}{x^2+5x+6}<0 \Rightarrow x \in (-3,-2)$

Let $f(x)=1+a^2x-x^3$ for maximum (or) minimum,

$f^1(x)=0 \Rightarrow a^2-3x^2=0 \Rightarrow x=\pm\frac{a}{\sqrt{3}}$

And $f^1(x)=-6x$ is positive when x is negative if a > 0 then point of minimum is $\frac{-a}{\sqrt{3}}$

$\Rightarrow -3<\frac{-a}{\sqrt{3}}<-2$

$\Rightarrow 2\sqrt{3}<a<3\sqrt{3}$

If $a<0,$ the point of minimum is $a|\sqrt{3}$

$\Rightarrow -3<\frac{a}{\sqrt{3}}<-2 \Rightarrow -3\sqrt{3}<a<-2\sqrt{3}$

$\Rightarrow a \in (-3\sqrt{3},-2\sqrt{3}) \cup (2\sqrt{3},3\sqrt{3})$

70. Let $\phi(x)=\frac{(x-b)(x-c)}{(a-b)(a-c)}f(a)+\frac{(x-c)(x-a)}{(b-c)(b-a)}f(b)+\frac{(x-a)(x-b)}{(c-a)(c-b)}f(c)-f(x)$ Where $a<c<b$ and $f^{11}(x)$ exists at all points in $(a,b)$. Then, there exists a real number $\mu, a<\mu<b$ such that $\frac{f(a)}{(a-b)(a-c)}+\frac{f(b)}{(b-c)(b-a)}+\frac{f(c)}{(c-a)(c-b)}=$

a) $f^{11}(\mu)$ b) $2f^{11}(\mu)$ c) $\frac{1}{2}f^{11}(\mu)$ d) $\frac{1}{3}f^{111}(\mu)$

Key. C

Sol. Apply RT's, twice

71. If $f(x)$ is a differentiable function $\forall x\in R$ so that, $f(2)=4, f^1(x)\geq 5\ \forall x\in[2,6]$, then, $f(6)$ is

a) $\geq 24$ b) $\leq 24$ c) $\geq 9$ d) $\leq 9$

Key. A

Sol. By mean value theorem, $f(6)-f(2)=(6-2)f^1(c)$ where $c\in(2,6)$

$\Rightarrow f(6)=f(2)+4f^1(c)=4+4f^1(1)>4+4(5)$

$(\therefore f^1(x)\geq 5)\ f(6)\geq 24$

72. The values of parameter 'a' for which the point of minimum of the function $f(x)=1+a^2x-x^3$ satisfies the inequality $\frac{x^2+x+2}{x^2+5x+6}<0$ are,

a) $(-3\sqrt{3},-2\sqrt{3})\cup(2\sqrt{3},3\sqrt{3})$ b) $(-5\sqrt{3},-3\sqrt{3})\cup(3\sqrt{3},5\sqrt{3})$

c) $(-7\sqrt{3},-5\sqrt{3})\cup(5\sqrt{3},7\sqrt{3})$ d) $(-9\sqrt{3},-6\sqrt{3})\cup(6\sqrt{3},9\sqrt{3})$

Key. A

Sol. $\frac{x^2+x+2}{x^2+5x+6}<0\Rightarrow x\in(-3,-2)$

Let $f(x)=1+a^2x-x^3$ for maximum (or) minimum,

$f^1(x)=0\Rightarrow a^2-3x^2=0\Rightarrow x=\pm\frac{a}{\sqrt{3}}$

And $f^1(x)=-6x$ is positive when x is negative if a > 0 then point of minimum is $\frac{-a}{\sqrt{3}}$

$\Rightarrow -3<\frac{-a}{\sqrt{3}}<-2$

$\Rightarrow 2\sqrt{3}<a<3\sqrt{3}$

If $a<0$, the point of minimum is $a|\sqrt{3}$

$\Rightarrow -3<\frac{a}{\sqrt{3}}<-2\Rightarrow -3\sqrt{3}<a<-2\sqrt{3}$

$\Rightarrow a\in(-3\sqrt{3},-2\sqrt{3})\cup(2\sqrt{3},3\sqrt{3})$

73. Let domain and range of $f(x)$ and $g(x)$ are respectively $[0,\infty)$. If $f(x)$ be an increasing function and $g(x)$ be an decreasing function. Also, $h(x)=f(g(x)),h(0)=0$ and $p(x)=h(x^3-2x^2+2x)-h(4)$ then for every $x\in(0,2]$

a) $p(x)\in(0,-h(4))$ b) $p(x)\in[-h(4),0]$

c) $p(x)\in(-h(4),h(4))$ d) $p(x)\in(h(4),h(4)]$

Key. A

Sol. $h(x)=f(g(x))$

$h^1(x)=f^1(g(x))g^1(x)<0\forall x\in[0,\infty)$

$\because g^1(x)<0\forall x\in[0,\infty)$ and $f^1(g(x))>0\forall x\in[0,\infty)$

Also, $h(0)=0$ and hence, $h(x)<0\forall x\in[0,\infty)$

$p(x)=h(x^3-2x^2+2x)-h(4)$

$p^1(x)=h^1(x^3-2x^2+2x).(3x^2-4x+2)<0\forall x\in(0,2)$

$\because h^1(x^3-2x^2+2x)<0\forall x\in(0,\infty)$ and $3x^2-4x+2>0\forall x\in R$

$\Rightarrow p(x)$ is an decreasing function

$\Rightarrow p(2)<p(x)<p(0)\forall x\in(0,2)$

$\Rightarrow h(4)-h(4)<p(x)<h(0)-h(4)$

$\Rightarrow 0<p(x)<-h(4)$

74. Let $f(x)$ be a positive differentiable function on $[0,a]$ such that $f(0)=1$ and $f(a)=3^{1/4}$ If $f^1(x)\ge(f(x))^3+(f(x))^{-1}$, then, maximum value of a is

a) $\frac{\pi}{12}$ b) $\frac{\pi}{24}$ c) $\frac{\pi}{36}$ d) $\frac{\pi}{48}$

Key. B

Sol. $f^1(x)f(x)\ge(f(x))^4+1$

$$\Rightarrow \frac{2f^1(x)f(x)}{\left\{(f(x))^2\right\}^2+1}\ge 2$$

$$\Rightarrow \int_0^a \frac{2f^1(x)f(x)}{\left\{(f(x))^2\right\}^2+1}\ge 2\int_0^a 1dx$$

$$\Rightarrow \left|\tan^{-1}(f(x))^2\right|_0^a\ge 2a\Rightarrow \frac{\pi}{3}-\frac{\pi}{4}\ge 2a$$

75. The least value of 'a' for which the equation $\frac{4}{\sin x}+\frac{1}{1-\sin x}=a$ for atleast one solution on the interval $\left(0,\frac{\pi}{2}\right)$ is,

a) 1 b) 4 c) 8 d) 9

Key. D

Sol. $\because a = \frac{4}{\sin x} + \frac{1}{1-\sin x}$, where a is least

$\Rightarrow \frac{da}{dx} = \left( \frac{-4}{\sin^2 x} + \frac{1}{(1-\sin x)^2} \right) \cos x = 0$

$\because \cos x \neq 0 \Rightarrow \sin x = 2/3$

$\frac{d^2a}{dx^2} = 45 > 0$ for $\sin x = 2/3 \Rightarrow \frac{4}{2/3} + \frac{1}{1-2/3} = 6+3 = 9$

76. $f(x) = x^4 - 10x^3 + 35x^2 - 50x + c.$ where c is a constant. the number of real roots of $f'(x) = 0$ and $f''(x) = 0$ are respectively

(1) 1, 0 (2) 3, 2 (3) 1, 2 (4) 3, 0

Key. 2

Sol. $g(x) = (x-1)(x-2)(x-3)(x-4)$

$f(x) = g(x) + c_0 : c_0 = c - 24$

$g(x) = 0$ has 4 roots viz. $x = 1, 2, 3, 4$

$f'(x) = g'(x)$ and $f''(x) = g''(x)$

By Rolle's theorem $g'(x) = 0$ has min. one root in each of the intervals (1, 2); (2, 3); (3, 4)

By Rolle's theorem, between two roots of $f'(x) = 0$, $f''(x) = 0$ has minimum one root.

77. Let $h(x) = f(x) - (f(x))^2 + (f(x))^3$ for every real number x. Then

(1) h is increasing whenever f is increasing

(2) h is increasing whenever f is decreasing

(3) h is decreasing whenever f is increasing

(4) nothing can be said in general

Key. 1

Sol. $h'(x) = f'(x) - 2f(x)f'(x) + 3(f(x))^2 f'(x)$

$= f'(x)\left[1 - 2f(x) + 3(f(x))^2\right]$

Since, $1 - 2f(x) + 3(f(x))^2 > 0$ for all $f(x)$

$\Rightarrow h'(x) > 0$ if $f'(x) > 0$

$\Rightarrow$ h is increasing when ever f is increasing and $h'(x) < 0$ if $f'(x) < 0$

$\Rightarrow$ h is decreasing when ever f is decreasing.

78. The set of critical points of the function $f(x) = (x-2)^{\frac{2}{3}}.(2x+1)$ is

(1) $\{1, 2\}$ (2) $\left\{-\frac{1}{2}, 1\right\}$ (3) $\{-1, 2\}$ (4) $\{1\}$

Key. 1

Sol. $f'(x)=(x-2)^{\frac{2}{3}}.2+(2x+1).\frac{2}{3}\frac{1}{(x-2)^{\frac{1}{3}}}$

$=2\left[\frac{3(x-2)+2x+1}{3(x-2)^{\frac{1}{3}}}\right]$

$=\frac{2}{3}\frac{(5x-5)}{(x-2)^{\frac{1}{3}}}=\frac{10}{3}\frac{(x-1)}{(x-2)^{\frac{1}{3}}}$

Critical points are $x=1$ and $x=2$

79. For $x\in(0,1)$ which of the following is true?

(1) $e^x<1+x$ (2) $\log_e(1+X)<X$ (3) $\sin x>x$ (4) $\log_e x>x$

Key. 2

Sol. Let $f(x)=e^x-1-x, g(x)=\log(1+x)-x$

h(x) = sin x – x, p(x) =log x – x

for g(x) = log(1 + x) – x

$g'(x)=\frac{1}{1+x}-1=\frac{-x}{1+x}<0 \quad \forall x\in(0, 1)$

g(x) is decreasing when 0 < x < 1.

g(0) > g(x)_ $\Rightarrow$ log (1 + x) < x

Similarly it can be done for other functions.

80. $f(x)=|x\ln x|:\ x\in(0,1)$, then $f(x)$ has maximum value=

(1) e (2) $\frac{1}{e}$ (3) 1 (4) None of these

Key. 2

Sol. $f(x)=-x\ln x$

$\lim_{x\to 0+} f(x)=0$

$$f'(x)=-(1+\ln x)\begin{cases}>0 & \text{if} \quad 0<x<\frac{1}{e}\\ =0 & \text{if} \quad x=\frac{1}{e}\\ <0 & \text{if} \quad \frac{1}{e}<x<1\end{cases}$$

f has maximum value at $x=\frac{1}{e}$ and $f\left(\frac{1}{e}\right)=\frac{1}{e}$

81. Let $f(x)=\begin{cases}(x+1)^3 & -2<x\le -1\\ x^{2/3}-1 & -1<x\le 1\\ -(x-1)^2 & 1<x>2\end{cases}$

The total number of maxima and minima of $f(x)$ is

(1) 4 (2) 3 (3) 2 (4) 1

Key. 2

Sol.

$$f'(x)=\begin{cases}3(x+1)^2 & -2<x<-1\\ \frac{2}{3}\times x^{-1/3} & -1<x<1-\{0\}\\ -2(x-1) & 1<x<2\end{cases}$$

$f'(x)\,DNE\ \ at\ \ x=-1,0,1$

$-2\ \ +\ \ -1\ \ -\ \ o\ \ +\ \ 1\ -\ \ 2$

Sign of $f'(x)$

82. Given $f(x)=\begin{cases}x^2e^{2(x-1)} & 0\le x\le 1\\ a\cos(2x-2)+bx^2 & 1<x\le 2\end{cases}$ $f(x)$ is differentiable at $x=1$ provided

(1) $a=-1,b=2$ (2) $a=1,b=-2$

(3) $a=-3,b=4$ (4) $a=3,b=-4$

Key. 1

Sol. $f(1+0)=f(1-0)\Rightarrow a+b=1$

$f^1(1-0)=f^1(1+0)\Rightarrow 4=2b$

$\Rightarrow b=2,a=-1$

83. Define $f:[0,\pi]\to R$ by is continuous at $x=\frac{\pi}{2}$, then k=

(1) $\frac{1}{12}$ (2) $\frac{1}{6}$ (3) $\frac{1}{24}$ (4) $\frac{1}{32}$

Key. 1

Sol. Let $\sin x=t$ and evaluate $\lim_{t\to 1}\frac{t^2}{1-t^2}\left[\sqrt{2t^2+3t+4}-\sqrt{t^2+6t+2}\right]$ by rationalization

84. If $f(x)=\frac{1}{(x-1)(x-2)}$ and $g(x)=\frac{1}{x^2}$, then the number of discontinuities of the composite function $f(g(x))$ is

(1) 2 (2) 3 (3) 4 (4) $\ge 5$

Key. 4

Sol. Conceptual

85. Find which function does not obey lagrange's mean value theorem in [0, 1]

(1) $f(x)=\begin{cases}\frac{1}{2}-x: & x<\frac{1}{2}\\ \left(\frac{1}{2}-x\right)^2: & x\geq\frac{1}{2}\end{cases}$ (2) $f(x)=\begin{cases}\frac{\sin x}{x}: & x\neq 0\\ 1 \quad if & x=0\end{cases}$

(3) $f(x)=x|x|$ (4) $f(x)=|x|$

Key. 1

Sol. In (a), $f'\left(\frac{1}{2}-\right)=-1$ while $f'\left(\frac{1}{2}+\right)=0$

f is not differentiable at $x=\frac{1}{2}$.

86. Rolle's theorem holds in [1, 2] for the function $f(x)=x^3+bx^2+cx$ at the point $\frac{4}{3}$. The values of b, c are respectively

(1) 8, –5 (2) –5, 8 (3) 5, –8 (4) –5, –8

Key. 2

Sol. $f(1)=f(2)$ and $f'(4/3)=0$

$3b+c=-7$ and $8b+3c=-16$

$b=-5; c=8$

87. If $f(x)=\begin{cases}x^{\alpha}\log x, & x>0\\ 0, & x=0\end{cases}$ and Rolle's theorem is applicable to f(x) for $x\in[0, 1]$ then $\alpha$ is equal to

1. -2 2. -1 3. 0 4. 1/2

Key. 4

Sol. for Rolle's theorem in [a, b]

$f(a)=f(b)\Rightarrow f(0)=f(1)=0$

Since the function has to be continuous in [0, 1]

$\underset{x\to 0^+}{Lt} f(x)=f(0)$

$\Rightarrow \underset{x\to 0^+}{Lt} x^{\alpha}\log x=0$

$\Rightarrow \underset{x\to 0}{Lt} \frac{\log x}{x^{-\alpha}}=0$

Applying L – H rule

$$\underset{x\to 0}{Lt}\frac{1/x}{-\alpha x^{-\alpha-1}}=0$$

$$\Rightarrow \underset{x\to 0}{Lt}\frac{-x^{\alpha}}{\alpha}=0$$

This is true for $\alpha > 0$

88. Let $f:(0,\infty)\to R$ be a (strictly) decreasing function.

If $f\left(2a^2+a+1\right)< f\left(3a^2-4a+1\right)$, then the range of $a\in R$ is

a) $\left(-\infty,\frac{1}{3}\right)\cup(1,\infty)$ b) $(0,\ 5)$ c) $\left(0,\frac{1}{3}\right)\cup(1,5)$ d) $[0,\ 5]$

Key. 3

Sol. we have $2a^2+a+1>3a^2-4a+1\Rightarrow a^2-5a<0\ \Rightarrow 0<a<5$ .......(A)

Also $3a^2-4a+1>(3a\ -1)(a-1)>0\ \Rightarrow a\in(-\infty,1/3)\cup(1,\infty)$.........$(B)$

Intersection of (A) and (B) yields $a\in(0,\ 1/3)\cup(1,5)$

89. Suppose $f:[1,2]\to R$ is such that $f(x)=x^3+bx^2+cx$. If $f$ satisfies the hypothesis of Rolle's theorem on $[1,2]$ and the conclusion of Rolle's theorem holds for $f$ on $[1,2]$ at the point $\frac{4}{3}$, then

a) $b=-5$ b) $b=5$ c) $c=-8$ d) $c=9$

Key. 1

Sol. $f(1)=f(2)\Rightarrow 1+b+c=8+4b+2c\Rightarrow 3b+c=-7\to(1)$.

Now, $f'(x)=3x^2+2bx+c$; $\therefore f'\left(\frac{4}{3}\right)=0$ (given) $\Rightarrow 3.\frac{16}{9}+2b.\frac{4}{3}+c=0\ \Rightarrow 8b+3c=-16\to$ (2). From $(1),(2)$ we get $b=-5$ and $c=8$.

90. Given a function $f:[0,4]\to R$ is differentiable, then for some $a,b\in(0,4)$ $[f(4)]^2-[f(0)]^2$ =

a) $8f'(b)f(a)$ b) $4f'(b)f(a)$ c) $2f'(b)f(a)$ d) $f'(b)f(a)$

Key. 1

Sol. Since $f(x)$ is differentiable in [0, 4], using Lagrange's Mean Value Theorem.

$$f'(b)=\frac{f(4)-f(0)}{4},\ b\in(0,4) \qquad \text{.......... (1)}$$

Now, $\{f(4)\}^2-\{f(0)\}^2=\frac{4\{f(4)-f(0)\}}{4}\{f(4)+f(0)\}$ $=4f'(b)\{f(4)+f(0)\}$ ........... (2)

Also, from Intermediate Mean Value Theorem,

$$\frac{f(4)+f(0)}{2}=f(a) \text{ for } a\in(0,4)$$

Hence, from (2) $[f(4)]^2-[f(0)]^2=8f'(b)f(a)$

91. Suppose $\alpha$, $\beta$ and $\theta$ are angles satisfying $0<\alpha<\theta<\beta<\frac{\pi}{2}$, then $\frac{\sin\alpha-\sin\beta}{\cos\beta-\cos\alpha}$ =

a) $\tan\theta$ b) $-\tan\theta$ c) $\cot\theta$ d) $-\cot\theta$

Key. 3

Sol. Let $f(x)=\sin x$ and $g(x)=\cos x$, then f and g are continuous and derivable. Also, $\sin x\neq 0$ for any $x\in\left(0,\frac{\pi}{2}\right)$ so by Cauchy's MVT, $\frac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)}=\frac{f'(\theta)}{g'(\theta)} \Rightarrow \frac{\sin\beta-\sin\alpha}{\cos\beta-\cos\alpha}=\frac{\cos\theta}{-\sin\theta}$

92. If $f''(x)>0$, $\forall\ x\in R$, $f'(3)=0$ and $g(x)=f(\tan^2 x-2\tan x+4)$, $0<x<\frac{\pi}{2}$, then $g(x)$ is increasing in

a) $\left(0,\frac{\pi}{4}\right)$ b) $\left(\frac{\pi}{6},\frac{\pi}{3}\right)$ c) $\left(0,\frac{\pi}{3}\right)$ d) $\left(\frac{\pi}{4},\frac{\pi}{2}\right)$

Key. 4

Sol. $g'(x)=(f'((\tan x-1)^2+3))2(\tan x-1)\sec^2 x$ since $f''(x)>0 \Rightarrow f'(x)$ is increasing

So, $f'((\tan x-1)^2+3)>f'(3)=0\ \forall x\in\left(0,\frac{\pi}{4}\right)\cup\left(\frac{\pi}{4},\frac{\pi}{2}\right)$

Also, $(\tan x-1)>0$ for $x\in\left(\frac{\pi}{4},\frac{\pi}{2}\right)$. So, $g(x)$ in increasing in $\left(\frac{\pi}{4},\frac{\pi}{2}\right)$

93. Let $f(x)=2x^3+ax^2+bx-3\cos^2 x$ is an increasing function for all $a,b,x\in R$. Then

a) $a^2-6b-18>0$ b) $a^2-6b+18<0$ c) $a^2-3b-6<0$ d) $a>0$, $b>0$

Key. 2

Sol. $f(x)=2x^3+ax^2+bx-3\cos^2 x$

$\therefore\ f'(x)=6x^2+2ax+b+3\sin 2x$

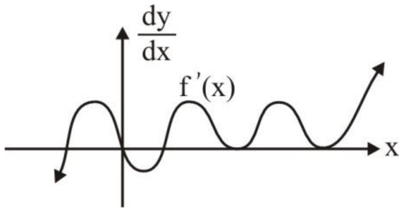
$\therefore\ f(x)$ is increasing for all $x \Rightarrow 6x^2+2ax+b+3\sin 2x>0$

Also, $6x^2+2ax+b+3\sin 2x\geq 6x^2+2ax+b-3$ as $\sin 2x\geq -1$

Hence $6x^2+2ax+b-3>0$

$\therefore\ 4a^2-4\,.\,6(b-3)<0 \Rightarrow a^2-6b+18<0$

94. f : R $\to$ R be differentiable function. Study following graph of $f'(x)=\frac{dy}{dx}$. Find sum of total no. of points of inflexion and extrema of y = f(x).




Key. 9

Sol. No. of points of inflexion = 6, no. of extrema = 3

95. The minimum value of $(8x^2 + y^2 + z^2)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2$, (x, y, z > 0), is

(A) 8 (B) 27

(C) 64 (D) 125

Key. C

Sol. $\frac{2(2x)^2+y^2+z^2}{2+1+1} \ge \left(\frac{2(2x)+y+z}{2+1+1}\right)^2 \ge \left(\frac{2+1+1}{\frac{2}{2x}+\frac{1}{y}+\frac{1}{z}}\right)^2 \Rightarrow (8x^2 + y^2 + z^2)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2 \ge 64.$

96. Let $f(x) = \begin{cases} (3-\sin(1/x))\,|x|, & x \ne 0 \\ 0 \quad\quad , & x = 0 \end{cases}$. Then at $x = 0$ $f$ has a

(A) maxima (B) minima

(C) neither maxima nor minima (D) point of discontinuity

Key. B

Sol. f is continuous at x = 0

Further f(0 + h) > f(0) and f(0 – h) > f(0), for positive ‘h’. Hence f has minimum value at x = 0.

97. A car is to be driven 200kms on a highway at an uniform speed of x km/hrs (speed Rules of the high way require $40 \le x \le 70$) . The cost of diesel is Rs 30/litre and is consumed at the rate of $100+\frac{x^2}{60}$ litres per hour. If the wage of the driver is Rs 200 per hour then the most economical speed to drive the car is

a) 55.5 b) 70 c) 40 d) 80

Key. B

Sol. Let cost incurred to travel 200 kms be

$C(x)$.*Then*

$$C(x) = \left(100+\frac{x^2}{60}\right)\frac{200}{x}\times 30 + 200\times\frac{200}{x}$$

$$= \frac{640000}{x} + 100x$$

$$\Rightarrow C'(x) < 0 \;\; for \;\; x \in [40,70]$$

$\Rightarrow C(x)$ is minimum for x = 70 in $x \in [40,70]$.

98. Let a, n $\in$ N such that a $\geq n^3$ then $\sqrt[3]{a+1}-\sqrt[3]{a}$ is always

(A) less than $\frac{1}{3n^2}$ (B) less than $\frac{1}{2n^3}$

(C) more than $\frac{1}{n^3}$ (D) more than $\frac{1}{4n^2}$

Key. A

Sol. Let $f(x)=x^{1/3} \Rightarrow f'(x)=\frac{1}{3x^{2/3}}$, applying LMVT in [a, a + 1], we get one c$\in$(a, a + 1)

$$f'(c)=\frac{f(a+1)-f(a)}{a+1-a} \Rightarrow \sqrt[3]{a+1}-\sqrt[3]{a}=\frac{1}{3c^{2/3}}<\frac{1}{3a^{2/3}}\leq\frac{1}{3n^2} \Rightarrow \sqrt[3]{a+1}-\sqrt[3]{a}<\frac{1}{3n^2}\ \forall a\geq n^3$$

99. If $x^2 + 9y^2 = 1$, then minimum and maximum value of $3x^2 - 27y^2 + 24xy$ is

(A) 0, 5 (B) – 5, 5

(C) – 5, 10 (D) 0, 10

Key. B

Sol. Put x = cos $\theta$, $y=\frac{1}{3}\sin\theta$

Let $u = 3x^2 - 27y^2 + 24xy$

$u = 3\cos 2\theta + 4\sin 2\theta$

$-5 \leq u \leq 5$.

100. Let the function g : $(-\infty, \infty) \to \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ be given by $g(u) = 2\tan^{-1}(e^u)-\frac{\pi}{2}$. Then g is

(A) even and is strictly increasing in $(0, \infty)$

(B) odd and is strictly decreasing in $(-\infty, \infty)$

(C) odd and is strictly increasing in $(-\infty, \infty)$

(D) neither even nor odd but is strictly increasing in $(-\infty, \infty)$

Key. C

Sol. $g(-u)=2\tan^{-1}e^{-u}-\frac{\pi}{2}=2\cot^{-1}e^u-\frac{\pi}{2}=2\left(\frac{\pi}{2}-\tan^{-1}e^u\right)-\frac{\pi}{2}$

$=-\left(2\tan^{-1}e^u-\frac{\pi}{2}\right)=-g(u)$

$g'(u)=2.\frac{1}{1+e^{2u}}.e^u>0.$

So, g(u) is odd and strictly increasing.

101. Let $f(x)$ be a differentiable function in the interval $(0,2)$, then the value of $\int_0^2 f(x)dx$ is___

a) $f(c)$ where $c\in(0,2)$ b) $2f(c)$ where $c\in(0,2)$

c) $f'(c)$ where $c \in (0,2)$ d) $f''(0)$

Key. B

Sol. Consider $g(t) = \int_0^t f(x)dx$ Applying LMVT in (0,2)

$\frac{g(2)-g(0)}{2-0} = g'(c);\ c \in (0,2)$ $\Rightarrow \int_0^2 f(x)dx = 2f(c)$ for $c \in (0,2)$

102. Let $g(x) = \int_{1-x}^{1+x} t\,|f'(t)|\,dt$, where f(x) does not behave like a constant function in any interval (a, b) and the graph of y = f ′(x) is symmetric about the line x = 1. Then

(A) g(x) is increasing $\forall$ x $\in$ R (B) g(x) is increasing only if x < 1

(C) g(x) is increasing if f is increasing (D) g(x) is decreasing $\forall$ x $\in$ R

Key. A

Sol. $g'(x) = (1+x)|f'(x+1)| + (1-x)|f'(1-x)|$

$= |f'(1+x)|(1+x+1-x) > 0 \ \ \forall x \in R$

103. The equation $2x^3 - 3x^2 - 12x + 1 = 0$ has in the interval $(-2,1)$

A) no real root B) exactly one real root

C) exactly two real roots D) all three real roots

Key. C

Sol. Let $f(x) = 2x^3 - 3x^2 - 12x + 1$

$f(-2) < 0;\ f(0) > 0;\ f(1) < 0$

$\therefore f(x) = 0$ has atleast two roots in the interval $(-2,1)$.

Suppose all the real roots of $f(x) \in (-2,1)$.

Then by Rolle's theorm, both the roots of the equation $f^1(x) = 0$ should belong to $(-2,1)$

$f^1(x) = 6x^2 - 6x - 12 = 0 \Rightarrow x^2 - x - 2 = 0$

$\Rightarrow (x-2)(x+1) = 0 \Rightarrow x = 2, -1$

104. If f: [1, 5] $\rightarrow$ R is defined by $f(x) = (x-1)^{10} + (5-x)^{10}$ then the range of f is

A) $\left[0, 2^{20}\right]$ B) $\left[0, 2^{11}\right]$ C) $\left[2^{11}, 2^{20}\right]$ D) $R^+$

Key. C

Sol. Conceptual

105. If $3(a+2c) = 4(b+3d) \neq 0$ then the equation $ax^3 + bx^2 + cx + d = 0$ will have

(A) no real solution (B) at least one real root in $(-1,0)$

(C) at least one real root in $(0,1)$ (D) none of these

Key. B

Sol. Consider $f(x) = \frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx$ and apply Rolle's theorem

106. The function in which Rolle's theorem is verified is

(A) $f(x) = \log\left(\frac{x^2+ab}{(a+b)x}\right)$ in $[a,b]$ (where $0 < a < b$) (B) $f(x) = (x-1)(2x-3)$ in $[1, 3]$

(C) $f(x)=2+(x-1)^{2/3}$ in $[0\ ,2]$ (D) $f(x)=\cos\left(1/x\right)$ in $[-1,1]$

Key. A

Sol. $f(x)=\log\left(\frac{x^2+ab}{(a+b)x}\right)$ is continuous in $[a,b]$ and differentiable in $(a,b)$ and $f(a)=f(b)$

107. If $f(x)=x^{\alpha}\log x$ and $f(0)=0$ then the value of $\alpha$ for which Rolle's theorem can be applied in $[0,1]$ is

(A) $-2$ (B) $-1$ (C) 0 (D) $\frac{1}{2}$

Key. D

Sol. for the function $f(x)=x^{\alpha}\log x$ Rolle's theorem is applicable for $\alpha>0$ in $[0,1]$

108. Let $f(x)=2x^2-ln|x|, x\neq 0$, then $f(x)$ is

a) monotonically increasing in $\left(-\frac{1}{2},0\right)\cup\left(\frac{1}{2},\infty\right)$

b) monotonically decreasing in $\left(-\frac{1}{2},0\right)\cup\left(\frac{1}{2},\infty\right)$

c) monotonically increasing in $\left(-\infty,-\frac{1}{2}\right)\cup\left(0,\frac{1}{2}\right)$

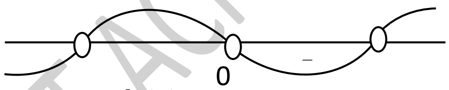
d) monotonically decreasing in $\left(-\infty,-\frac{1}{2}\right)\cup\left(0,\frac{1}{2}\right)$

Key. A,D

Sol. $\because f(x)=2x^2-\ln|x|$

$\therefore \quad f'(x)=4x-\frac{1}{x}$

$=\frac{(2x+1)(2x-1)}{x}$




For increasing, $f'(x)>0$

$\therefore \quad x\in\left(-\frac{1}{2},0\right)\cup\left(\frac{1}{2},\infty\right)$

And for decreasing, $f'(x)<0$

$\therefore \quad x\in\left(-\infty,-\frac{1}{2}\right)\cup\left(0,\frac{1}{2}\right)$

109. For $x>1, y=\log_e x$ satisfies the inequality

a) $x-1>y$ b) $x^2-1>y$ c) $y>x-1$ d) $\frac{x-1}{x}<y$

Key. A,B,D

Sol. Let $f(x)=\log_e x-(x-1)$

$\Rightarrow f'(x)=\frac{1}{x}-1=\frac{1-x}{x}<0$

$\because$ $f(x)$ is decreasing function $(\because x > 1)$

$$x > 1 \Rightarrow f(x) < f(1)$$

$\Rightarrow$ $\log_e x - (x-1) < 0$

$\Rightarrow$ $(x-1) > \log_e x$

$= y$

Or $(x-1) > y$

Now, let $g\left(x\right) = \log_e x - \left(x^2 - 1\right)$.

$\Rightarrow g'(x) = \dfrac{1}{x} - 2x = \left(\dfrac{1-2x^2}{x}\right) < 0$ (for $x > 1$)

$\therefore g(x)$ is decreasing function

$\because$ $x > 1 \Rightarrow g(x) < g(1)$

$\Rightarrow$ $\log_e x - (x^2 - 1) < 0$

$\therefore$

Or $(x^2 - 1) > y$

Again, let $h(x) = \dfrac{x-1}{x} - \log_e x$

$\therefore$ $h'(x) = 0 + \dfrac{1}{x^2} - \dfrac{1}{x} = \dfrac{1-x}{x^2} < 0$ (for $x > 1$)

$\therefore h(x)$ is decreasing function

$\because$ $x > 1 \Rightarrow h(x) < h(1)$

$\Rightarrow$ $\dfrac{x-1}{x} - \log_e x < 0$

$\Rightarrow$ $\dfrac{x-1}{x} < y$.

110. Let 'a'(a < 0, a $\notin$ I) be a fixed constant and 't' be a parameter then the set of values of 't' for the function f(x) = $\left(\dfrac{|[t]+1|+a}{|[t]+1|+1-a}\right)$x to be a non increasing function of x,
([·] denotes the greatest integer function) is

a) [[a], [–a + 1]) b) [[a], [–a]) c) [[a + 1], [–a + 1] ) d) [[a – 1], [–a + 1])

Key. B

Sol. f′(x) ≤ 0 $\Rightarrow$ $\dfrac{|[t]+1|+a}{|[t]+1|+1-a}$ ≤ 0, but as a < 0, 1 – a > 0.

So |[t] + 1| ≤ -a $\Rightarrow$ a ≤ [t] + 1 ≤ -a $\Rightarrow$ a – 1 ≤ [t] ≤ -a – 1

$\Rightarrow$ [a] ≤ [t] ≤ [-a] – 1 (as a$\notin$I) $\Rightarrow$ [a] ≤ t < [-a]

111. The number of critical values of $f(x) = \dfrac{|x-1|}{x^2}$ is

a) 0 b) 1 c) 2 d) 3

Key. D

Sol. $f'(x)=\dfrac{|x-1|\left\{\dfrac{x^2}{x-1}-2x\right\}}{x^4} \Rightarrow f'(x)=0 \quad at\, x=2$

$\Rightarrow f'(x)$ does not exist at x = 0,1

112. The absolute minimum value of $x^2-4x-10|x-2|+29$ occurs at

a) one value of $x\in R$ b) at two values of $x\in R$ c) *x=7,3* d) no value of $x\in R$

Key. B

Sol. Given function is $(|x-2|-5)^2$ which has global minimum value equal to 0 ,when $|x-2|=5$

113. The function $f(x)=x(x-1)(x-2)(x-3)-----(x-50)$ in (0,50) has *m* local maxima and *n* local minimum then

a) *m=25 , n=26* b) *m=26 , n=25* c) *m=n=26* d) *m=n=25*

Key. D

Sol. From the given conditions, it follows that $f(x)=x^3+1 \Rightarrow f^1(2)=3(2)^2=12$

114. The value of *c* in the Lagrange's mean value theorem applied to the function $f(x)=x(x+1)(x+2)$ for $0\le x\le 1$ is

a) $\dfrac{\sqrt{21}}{4}$ b) $\dfrac{\sqrt{21}-3}{3}$ c) $\dfrac{1}{5}$ d) $\dfrac{\sqrt{21}+3}{8}$

Key. B

Sol. $f^1(c)=3c^2+6c+2=\dfrac{f(1)-f(0)}{1}=6 \Rightarrow 3c^2+6c-4=0 \Rightarrow c=-1+\dfrac{\sqrt{21}}{3}\in(0,1)$

115. A twice differentiable function $f(x)$ on *(a,b)* and continuous on [a,b] is such that $f^{11}(x)<0$ for all $x\in(a,b)$ then for any $c\in(a,b), \dfrac{f(c)-f(a)}{f(b)-f(c)}>$

a) $\dfrac{b-c}{c-a}$ b) $\dfrac{c-a}{b-c}$ c) $(b-c)(c-a)$ d) $\dfrac{1}{(b-c)(c-a)}$

Key. B

Sol. Let $u\in(a,c), v\in(c,b)$ then by LMVT on (a,c),(c,b) it follows

$$f^1(u)=\frac{f(c)-f(a)}{c-a}, f^1(v)=\frac{f(b)-f(c)}{b-c}.$$

But u<v and $f^{11}(x)<0$ for all $x\in(a,b) \Rightarrow f^1(x)\downarrow \Rightarrow f^1(u)>f^1(v) \Rightarrow \dfrac{f(c)-f(a)}{f(b)-f(c)}>\dfrac{c-a}{b-c}$.

116. The number of roots of $x^5-5x+1=0$ in (-1,1) is

a) 0 b) 1 c) 2 d) 3

Key. B

Sol. Let $f(x)=x^5-5x+1$. Q $f(1)f(-1)<0$ $\exists$ atleast one root say $\alpha$ of $f(x)=0$ in (-1,1). If $\exists$ another root $\beta$ $(\alpha<\beta)$ in (-1,1) then by RT applied to $[\alpha,\beta]$, it follows that there exist $\gamma\in(\alpha,\beta)$ such that $f^1(\gamma)=5\gamma^4-5=0$ *i.e* $\gamma=1,-1$ but $\gamma\in(\alpha,\beta)\subset(-1,1) \therefore \gamma\ne 1,-1$ , *a contradiction*. Hence number of roots of $f(x)=0$ *in* $(-1,1)$ is 1.

117. If $\dfrac{a_0}{5}+\dfrac{a_1}{4}+\dfrac{a_2}{3}+\dfrac{a_3}{2}+a_4=0$ then the equation $a_0x^4+a_1x^3+a_2x^2+a_3x+a_4=0$

A) does not have root between 0 and 1
B) possesses at least one root between 0 and 1
C) has exactly one root between 0 and 1
D) has a root between 1 and 2

Key. B

Sol. Consider the function $f(x)=\dfrac{a_0x^5}{5}+\dfrac{a_1x^4}{4}+\dfrac{a_2x^3}{3}+\dfrac{a_3x^2}{2}+a_4x$

$f(0)=0$ *and* $f(1)=0$ by hypothesis

$\therefore$ f satisfies all conditions of Rolle's theorem

$\therefore$ $f^1(x)=0$ has at least one root in (0,1)

118. The largest area of the rectangle which has one side on the X-axis and two vertices on the curve $y=e^{-x^2}$ is

A) $\dfrac{1}{\sqrt{2e}}$
B) $\dfrac{1}{2e^2}$
C) $\sqrt{\dfrac{2}{e}}$
D) $\dfrac{\sqrt{2}}{e^2}$

Key. C

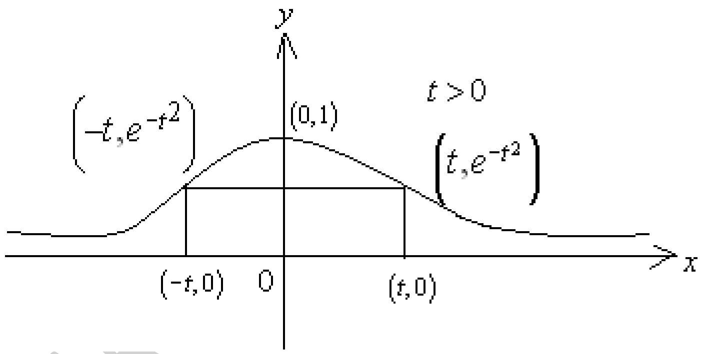
Sol. Let $f(t)=t\ e^{-t^2}$

$f^1(t)=-2t^2\ e^{-t^2}+e^{-t^2}$

$=e^{-t^2}\left(1-2t^2\right)$

$f^1(t)=0\Rightarrow t=\dfrac{1}{\sqrt{2}}$

Max area $=2\times\dfrac{1}{\sqrt{2}}\times e^{\frac{-1}{2}}=\dfrac{\sqrt{2}}{\sqrt{e}}$




119. $f(x)=\dfrac{x}{\sin x}$ and $g(x)=\dfrac{x}{\tan x}$ where $0<x\le 1$. Then in this interval

(a) f(x) and g(x) both are increasing
(b) f(x) is decreasing and g(x) is increasing
(c) f(x) is increasing and g(x) is decreasing
(d) none of the above

Key. C

Sol. $f'(x)=\dfrac{\sin x-x\cos x}{\sin^2 x}$

Now $h(x)=\sin x-x\cos x$

$h'(x)=x\sin x>0 \quad \forall\, 0<x\le 1$

h(x) is increasing in (0, 1]

h(0) < h(x) $\Rightarrow$ sin x – x cos x > 0 for 0 < x $\leq 1$

$\Rightarrow \quad f'(x) > 0$

Hence f(x) is increasing. Similarly it can be done for g(x).

120. For $x \in (0,1)$, which of the following is true?

(a) $e^x < 1 + x$ (b) $\log_e(1+x) < x$ (c) $\sin x > x$ (d) $\log_e x > x$

Key. B

Sol. Let $f(x) = e^x - 1 - x$, $g(x) = \log(1+x) - x$

h(x) = sin x – x, p(x) =log x – x

for g(x) = log(1 + x) – x

$$g'(x) = \frac{1}{1+x} - 1 = \frac{-x}{1+x} < 0 \quad \forall x \in (0, 1)$$

g(x) is decreasing when 0 < x < 1.

g(0) > g(x)_ $\Rightarrow$ log (1 + x) < x

Similarly it can be done for other functions.

121. $f(x) = |x \ln x| : x \in (0,1)$ has maximum value

(A) $e$ (B) $\frac{1}{e}$

(C) 1 (D) None of these

Key. B

Sol. $f(x) = -x \ln x$

$$\lim_{x \to 0+} f(x) = 0$$

$$f'(x) = -(1 + \ln x) \begin{cases} > 0 & \text{if} \quad 0 < x < \frac{1}{e} \\ = 0 & \text{if} \quad x = \frac{1}{e} \\ < 0 & \text{if} \quad \frac{1}{e} < x < 1 \end{cases}$$

$f$ has maximum value at $x = \frac{1}{e}$ and $f\left(\frac{1}{e}\right) = \frac{1}{e}$

122. $f(x) = \begin{cases} x^a \ln x & : \quad x > 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$.

If Lagrange's theorem applies to $f$ on [0, 1] then '$a$' can be

(A) –2 (B) –1

(C) 0 (D) $\frac{1}{2}$

Key. D

Sol. $f$ is continuous at $x = 0$

$\therefore 0 = \lim_{x \to 0+} f(x) = \lim_{x \to 0+} x^a \ln x$ forces "$a > 0$" is necessary.

123. Rolle's theorem holds in [1, 2] for the function $f(x)=x^3+bx^2+cx$ at the point "$\frac{4}{3}$". The values of $b$, $c$ are respectively

(A) 8, –5 (B) –5, 8
(C) 5, –8 (D) –5, –8

Key. B

Sol. $f(1)=f(2)$ and $f'(4/3)=0$

$3b+c=-7$ and $8b+3c=-16$

$b=-5; c=8$

124. Point on the curve $y^2=4(x-10)$ which is nearest to the line $x + y = 4$ may be

(A) (11, 2) (B) (10, 0)
(C) (11, –2) (D) None of these

Key. C

Sol. $P(x_{0,}, y_0): pt$ on curve nearest to line.

Normal at $P$ is perpendicular to the line

Normal at $P$ has slope "$-\frac{y_0}{2}$"

$\therefore y_0=2$ and $x_0=11$; $P(11,-2)$

125. $f(x)=\left(\sin^2 x\right)e^{-2\sin^2 x}$; $\max f(x)-\min f(x)=$

(A) $\frac{1}{e^2}$ (B) $\frac{1}{2e}-\frac{1}{e^2}$
(C) 1 (D) None of these

Key. D

Sol. Let $t=\sin^2 x; t\in[0,1]$

$f(x)=g(t)=te^{-2t}$

$$g'(t)=(1-2t)\,e^{-2t}\begin{cases}>0 & if \quad t\in[0,\frac{1}{2}) \\ <0 & if \quad t\in(\frac{1}{2},1]\end{cases}$$

$\max f=\max g=g\left(\frac{1}{2}\right)=\frac{1}{2e}$

$\min f=\min g=\min\{g(0),g(1)\}=0$

$\max f-\min f=\frac{1}{2e}.$

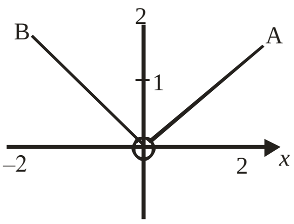
126. $f(x)=\begin{cases}|x| & if \quad 0<|x|\le 2 \\ 1 & if \quad x=0\end{cases}$ HAS AT $X = 0$

(A) LOCAL MAXIMA (B) LOCAL MINIMA

(C) TANGENT (D) NONE OF THESE

KEY. A

SOL.




A(2,0), B(–2, 0)

O(0, 0) is not a point on the graph

127. $f(x) = x^4 - 10x^3 + 35x^2 - 50x + c.$ WHERE C IS A CONSTANT. THE NUMBER OF REAL ROOTS OF $f'(x) = 0$ AND $f''(x) = 0$ ARE RESPECTIVELY

(A) 1, 0 (B) 3, 2 (C) 1, 2 (D) 3, 0

KEY. B

Sol. $g(x) = (x-1)(x-2)(x-3)(x-4)$

$f(x) = g(x) + c_0 : c_0 = c - 24$

$g(x) = 0$ has 4 roots viz. $x = 1, 2, 3, 4$

$f'(x) = g'(x)$ and $f''(x) = g''(x)$

By Rolle's theorem $g'(x) = 0$ has min. one root in each of the intervals (1, 2); (2, 3); (3, 4)

BY ROLLE'S THEOREM, BETWEEN TWO ROOTS OF $f'(x) = 0, f''(x) = 0$ HAS MINIMUM ONE ROOT.

128. THE DIFFERENCE BETWEEN THE GREATEST AND LEAST VALUE OF $f(x) = \sin 2x - x : x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(A) $\frac{\sqrt{3}+\sqrt{2}}{2}$ (B) $\frac{\sqrt{3}+\sqrt{2}}{2} + \frac{\pi}{6}$

(C) $\frac{\sqrt{3}}{2} - \frac{\pi}{3}$ (D) NONE OF THESE

KEY. D

Sol. $f'(x) = 2\cos 2x - 1;$ $f'(x) = 0$ if $x = -\frac{\pi}{6}, \frac{\pi}{6}$

$f'(x) > 0$ if $x \in \left(-\frac{\pi}{6}, \frac{\pi}{6}\right)$

$f'(x) < 0$ if $x \in [-\frac{\pi}{2}, -\frac{\pi}{6})$ or $x \in (\frac{\pi}{6}, \frac{\pi}{2}]$

Max $f = \max\{f\left(-\frac{\pi}{2}\right), f\left(\frac{\pi}{6}\right)\} = \max\left\{\frac{\pi}{2}, \frac{\sqrt{3}}{2} - \frac{\pi}{6}\right\} = \frac{\pi}{2}$

MIN $f = -\frac{\pi}{2}$ IS $F$ IS AN ODD FUNCTION.

129. $f : R \to R$ IS A FUNCTION SUCH THAT $f(x) = 2x + \sin x$; THEN, $F$ IS

(A) ONE-ONE AND ONTO (B) ONE-ONE BUT NOT ONTO

(C) ONTO BUT NOT ONE-ONE (D) NEITHER ONE-ONE NOR ONTO

KEY. A

Sol. $f'(x) = 2 + \cos x > 0$; $\therefore f$ is one-one

$f$ is continuous; $\lim\limits_{x\to\infty} f(x) \equiv \infty$; $\lim\limits_{x\to-\infty} f(x) \equiv -\infty$

$\therefore f$ IS ONE-ONE AND ONTO

130. FIND WHICH FUNCTION DOES NOT OBEY LAGRANGE'S MEAN VALUE THEOREM IN [0, 1]

(A) $f(x) = \begin{cases} \frac{1}{2} - x & : \quad x < \frac{1}{2} \\ \left(\frac{1}{2} - x\right)^2 & : \quad x \geq \frac{1}{2} \end{cases}$ (B) $f(x) = \begin{cases} \frac{\sin x}{x} & : \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases}$

(C) $f(x) = x|x|$ (D) $f(x) = |x|$

KEY. A

Sol. In (a), $f'\left(\frac{1}{2}-\right) = -1$ while $f'\left(\frac{1}{2}+\right) = 0$

$F$ IS NOT DIFFERENTIABLE AT $x = \frac{1}{2}$.

131. IF $A > 0$, $B < 0$ AND $A = \frac{\pi}{3} + B$ THEN MINIMUM VALUE OF TAN$A$ TAN$B$ IS

(A) $-\frac{1}{2}$ (B) $-1$

(C) $-\frac{1}{3}$ (D) NONE OF THESE

KEY. C

Sol. $B_0 = -B > 0$; $A + B_0 = \frac{\pi}{3}$.

By $A.M.-G.M.$, max $\tan A \tan B_0$ happens when

$A = B_0 = \frac{\pi}{6}$

$\therefore$ MIN $\tan A \tan B = -\frac{1}{3}$.

132. The point on the curve $x^2 = 2y$ which is nearest to a (0, 3) may be

(A) (2, 2) (B) $\left(1, \frac{1}{2}\right)$

(C) (0, 0) (D) $\left(-3, \frac{9}{2}\right)$

KEY. A

Sol. Let $P(x_0, y_0)$ be the nearest point

$$PA^2 = (y_0 - 3)^2 + (x_0 - 0)^2$$

$$= y_0^2 - 4y_0 + 9 \text{ as } x_0^2 = 2y_0$$

$$= (y_0 - 2)^2 + 5$$

$PA^2$ is minimum if $y_0 = 2; x_0 = \pm 2$

$P(\pm 2, 2)$.

Aliter : A lies on normal to curve at *P.*

133. POINT ON THE LINE $x - y = 3$ WHICH IS NEAREST TO THE CURVE $x^2 = 4y$ IS

(A) (0, –3) (B) $(3, 0)$

(C) $(2, -1)$ (D) NONE OF THESE

KEY. B

Sol. $P(x_0, y_0)$ is the nearest point; $y_0 = x_0 - 3$

Line through *P,* perpendicular to $x - y = 3$ is normal to given curve at, say, $Q(x_1, y_1)$

$$\therefore -\frac{2}{x_1} = -1; \;\; x_1 = 2; y_1 = 1.$$

Normal is $y - 1 = -(x - 2)$; This cuts $x - y = 3$ at *P.*

$\therefore P(3, 0)$.

134. $f(x) = \begin{cases} \dfrac{|x-1|}{x^2} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$ INCREASES IN

(A) (0, 2) (B) [0, 2]

(C) $[0, \infty)$ (D) NONE OF THESE

KEY. D

Sol. $$f(x) = \begin{cases} \dfrac{x-1}{x^2} & \text{if} \quad x > 1 \\ \dfrac{1-x}{x^2} & \text{if} \quad x < 1 : x \neq 0 \\ 0 & \text{if} \quad x = 0, 1 \end{cases}$$

$$f'(x)=\begin{cases}\dfrac{2-x}{x^3} & \text{if} \qquad x>1\\ \dfrac{x-2}{x^3} & \text{if} \quad x\in(0,1) \text{ or } x\in(-\infty,0)\end{cases}$$

*f* is not differentiable at *x* = 0, 1

$f'(x)>0$ IF $x\in(1,2)$ OR $x\in(-\infty,0)$

# Maxima & Minima

## *Integer Answer Type*

1. From a point perpendicular tangents are drawn to ellipse $x^2 + 2y^2 = 2$. The chord of contact touches a circle which is concentric with given ellipse. Then find the ratio of maximum and minimum area of circle.

Key. 4

Sol. The director circle of ellipse $\frac{x^2}{2}+\frac{y^2}{1}=1$ is $x^2 + y^2 = 3$

Let a point $P(\sqrt{3}\cos\theta, \sqrt{3}\sin\theta)$

Equation of chord of contact is

$x.\sqrt{3}\cos\theta + 2y\sqrt{3}\sin\theta - 2 = 0$

It touches $x^2 + y^2 = r^2$

$r = \frac{2}{\sqrt{3\cos^2\theta+12\sin^2\theta}}=\frac{2}{\sqrt{3+9\sin^2\theta}}$

$r_{max} = \frac{2}{\sqrt{3}}$ & $r_{min} = \frac{2}{\sqrt{12}} \Rightarrow \frac{A_{max}}{A_{min}} = 4.$

2. The maximum value of the function $f(x)=2x^3-15x^2+36x-48$ on the set $A=\{x \mid x^2+20\le 9x\}$ is

Key. 7

Sol. The given function is $f(x) = 2x^3 - 15x^2 + 36x - 48$ and $A = \{x \mid x^2 + 20 \le 9x\}$

$\Rightarrow \quad A = \{x \mid x^2 - 9x + 20 \le 0\}$

$\Rightarrow \quad A = \{x \mid (x-4)(x-5) \le 0\}$

$\Rightarrow \quad A = [4, 5]$

Also

$f'(x) = 6x^2 - 30x + 36 = 6(x^2 - 5x + 6) = 6(x-2)(x-3)$

Clearly $\forall x \in A,\ f'(x) > 0$

$\therefore$ $f$ is strictly increasing function on

$\therefore$ Maximum value of $f$ on $A$

$= f(5) = 2\times 5^3 - 15\times 5^2 + 36\times 5 - 48 = 250 - 375 + 180 - 48 = 7$

3. If $a,b,c \in N$, and if $\frac{ax^4 - bx^3 + cx^2 - bx + a}{\left(x^2+1\right)^2}$ attains minimum value at x = 2 or 1/2 then the A.M of the least possible values of a, b and c is __________

Key. 4

Sol. Put $x+\frac{1}{x}=t$ $a=1, b=4, c=7, \Rightarrow$ AM is $\frac{1+4+7}{3}=4$

4. If the greatest value of $\left(3-\sqrt{4-x^2}\right)^2+\left(1+\sqrt{4-x^2}\right)^3$ is $\alpha$, then the numerical value of $\left(\frac{\alpha}{7}\right)$, is

Key. 4

Sol. Let $t=\sqrt{4-x^2}, 0 \le t \le 2$

$\therefore F(t)=(3-t)^2+(1+t)^3$ and maximum of $f(x)$ is 10

5. If the graph of $f(x)=2x^3+ax^2+bx, a, b \in N$ cuts the x-axis at three real and distinct points, then the minimum value of $\left(a^2+b^2-4\right)$, is

Key. 6

Sol. $f^1(x)=6x^2+2ax+b \Rightarrow 4a^2-24b \ge 0$

$$\Rightarrow a^2 \ge 6b$$

$\Rightarrow a \ge 3, b \ge 1, \Rightarrow a=3, b=1$

6. If $a, b, c \in N$, and if $\frac{ax^4-bx^3+cx^2-bx+a}{\left(x^2+1\right)^2}$ attains minimum value at x = 2 or $1/2$ then the A.M of the least possible values of a, b and c is __________

Key. 4

Sol. Put $x+\frac{1}{x}=t$ $a=1, b=4, c=7, \Rightarrow$ AM is $\frac{1+4+7}{3}=4$

7. The maximum value of the function $f(x)=2x^3-15x^2+36x-48$ on the set $A=\left\{x \middle| x^2+20 \le 9x\right\}$ is

Key. 7

Sol. The given function is $f(x)=2x^3-15x^2+36x-48$ and $A=\{x \mid x^2+20 \le 9x\}$

$\Rightarrow$ $A=\{x \mid x^2-9x+20 \le 0\}$ $\Rightarrow$ $A=\{x \mid (x-4)(x-5) \le 0\}$ $\Rightarrow$ $A=[4, 5]$

Also $f'(x)=6x^2-30x+36=6(x^2-5x+6)$ $=6(x-2)(x-3)$

Clearly $\forall x \in A,\ f'(x)>0$

$\therefore$ $f$ is strictly increasing function on $A$.

$\therefore$ Maximum value of $f$ on $A$

$= f(5)=2 \times 5^3-15 \times 5^2+36 \times 5-48$ $=250-375+180-48=7$

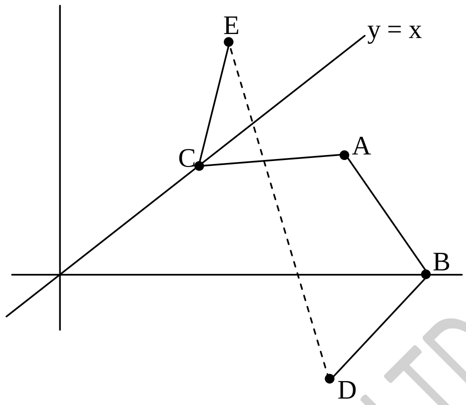
8. Given a point (2, 1). If the minimum perimeter of a triangle with one vertex at (2, 1), one on the x-axis, and one on the line y = x, is k, then [k] is equal to (where [ ] denotes the greatest integer function)

Key. 3

Sol.

Let, $D=(2,-1)$ be the reflection of A in x-axis, and let $E=(1,2)$ be the reflection in the line y = x. Then AB = BD and AC = CE, so the perimeter of ABC is

DB + BC + CE ≥ DE = $\sqrt{1+9}=\sqrt{10}$




9. The minimum value of, $\frac{\sec^4\alpha}{\tan^2\beta}+\frac{\sec^4\beta}{\tan^2\alpha},\alpha,\beta\neq\frac{K\pi}{2},K\in I,$ is

Key. 8

Sol.

$$\frac{(a+1)^2}{b}+\frac{(b+1)^2}{a}=\frac{a^2}{b}+\frac{1}{b}+\frac{b^2}{a}+\frac{1}{a}+2\left(\frac{a}{b}+\frac{b}{a}\right)\geq 4\left[\frac{a^2}{b}\cdot\frac{1}{b}\cdot\frac{b^2}{a}\cdot\frac{1}{a}\right]^{\frac{1}{4}}+4\left(\frac{a}{b}\cdot\frac{b}{a}\right)^{\frac{1}{2}}\geq 8$$

Where a = $\tan^2\alpha, b=\tan^2\beta$

10. If one root of $x^2-4ax+a+f(a)=0$ is three times the other and if minimum value of $f(a)$ is $\alpha$ , then $|12\alpha|$ has a value

Key. 1

Sol. $\theta$ and $3\theta\Rightarrow 4\theta=4a\Rightarrow\theta=a$ and $a-4a^2+f(a)=0$

$\Rightarrow f(a)=3a^2-a\Rightarrow f_{min}$ is $\frac{-1}{12}$

11. For a twice differentiable function $f(x)$, a function $g(x)$ is defined as $g(x)=(f^1(x))^2+f(x)f^{11}(x)$ on $[a,e]$. If $a<b<c<d<e$ and $f(a)=0,f(b)=2,f(c)=-1,f(d)=2,f(e)=0,$ then, the minimum number of roots of the equation $g(x)=0$ , is/are

Key. 6

Sol. $\because f(b)f(c)<0$ and $f(c)f(d)<0$

$\Rightarrow f(x)=0$ has at least four roots,

$a,c_1,c_2,e,$ Where $c_1\in(b,c)$ and $c_2\in(c,d)$. Then, by RT, $f^1(x)=0$ has at least three roots in, $(a,c_1),(c_1,c_2),(c_2,e)$

$\therefore f(x)f^1(x)=0$ has at least 7 roots, by RT and hence,

$g(x)=\frac{d}{dx}\{f(x)f^1(x)\}=0$ has at least 6 roots

12. Let $P(x)$ be a polynomial of degree 4 having extremum at $x=1,2$ and $\underset{x\to0}{\text{Let}}\left(1+\frac{P(x)}{x^2}\right)=2$, then, the value of $P(2)$ , is

Key. 0

Sol. Let $P(x)=a_0x^4+.....+a_4$ by hypothesis, $P^1(1)=0$ and $P^1(2)=0$

$\Rightarrow 4a_0+3a_1+2a_2+a_3=0$ and $32a_0+12a_1+4a_2+a_3=0$

Also, $\underset{x\to0}{\text{Lt}}\frac{P(x)}{x^2}=1\Rightarrow a_4=0$ and $a_3=0$ hence $\underset{x\to0}{\text{Lt}}\left(a_0x^3+a_1x+a_2\right)=1\Rightarrow a_2=1$

Solving, we get, $a_0=\frac{1}{4},a_1=-1,a_2=1,a_3=0,a_4=0$

$\therefore P(x)=\frac{1}{4}x^4-x^3+x^2\Rightarrow P(2)=0$

13. In the coordinate plane, the region M consists of all points $(x,y)$ satisfying the inequalities $y\ge0,y\le x,$ and $y\le2-x$ simultaneously. The region N which varies with parameter t, consists of all the points $(x,y)$ satisfying the inequalities $t\le x\le t+1$ and $0\le t\le1$ simultaneously. If the area of the region $M\cap N$ is a function of t, i,e., $M\cap N=f(t)$ and if $\alpha$ is the value of t for which this area is maximum, then the numerical value of $2\alpha$ is

Key. 1

Sol. $M\cap N=f(t)=-t^2+t+1/2$

$=\frac{3}{4}-\left(t-\frac{1}{2}\right)^2$ f(t) is maximum for t = 1/2 i.e. $\alpha=\frac{1}{2}\Rightarrow 2\alpha=1$

14. Let $M(-1,2)$ and $N(1,4)$ be two points in a plane rectangular coordinate system XOY. P is a moving point on the x-axis. When $\angle$MPN takes its maximum value, the x-coordinate of point P is

Key. 1

Sol. The centre of a circle passing through points M and N lies on the perpendicular bisector y = 3 – x of MN. Denote the centre by C(a, 3 – a), the equation of the circle is

$(x-a)^2+(y-3+a)^2=2(1+a^2)$

Since for a chord with a fixed length the angle at the circumference subtended by the corresponding arc will become larger as the radius of the circle becomes smaller. When $\angle$MPN reaches its maximum value the circle through the three points M, N and P will be tangent to the x-axis at P, which means

$2(1+a^2)=(a-3)^2\Rightarrow$ a = 1 or a = –7

Thus the point of contact are P(1, 0) or $P'(-7,0)$ respectively.

But the radius of circle through the points M, N and $P'$ is larger than that of circle through points M, N and P.

Therefore, $\angle MPN > \angle MP'N$. Thus P = (1, 0)

$\therefore$ x-coordinate of P = 1.

15. Put numbers 1, 2, 3, 4, 5, 6, 7, 8 at the vertices of a cube, such that the sum of any three numbers on any face is not less than 10. The minimum sum of the four number on a face is k, then k/2 is equal to

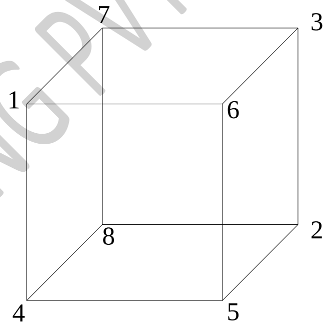
Key. 8

Sol. Suppose that the four numbers on face of the cube is $a_1$, $a_2$, $a_3$, $a_4$ such that their sum reaches the minimum and $a_1 < a_2 < a_3 < a_4$.

Since the maximum sum of any three numbers less than 5 is 9, we have $a_4 \geq 6$ and $a_1 + a_2 + a_3 + a_4 \geq 16$.

As seen in figure, we have

2 + 3 + 5 + 6 = 16




and that means minimum sum of four numbers on a face is 16.

16. Rolle's theorem holds for the function $f(x) = x^3 + mx^2 + nx$ on the interval $[1, 2]$ and the value of c is $\frac{4}{3}$. Then $m + n =$

Key. 3

Sol. $f(1) = f(2) \Rightarrow 1 + m + n = 8 + 4m + 2n \Rightarrow 3m + n + 7 = 0.$

$$f^1(C) = 0 \Rightarrow 3C^2 + 2mC + n = 0 \Rightarrow \frac{16}{3} + \frac{8m}{3} + n = 0 (C = \frac{4}{3})$$

$\Rightarrow 8m + 3n + 16 = 0$ on solving we get $m = -5, n = 8$ Hence m + n = 3

17. If the greatest value of $\left(3 - \sqrt{4 - x^2}\right)^2 + \left(1 + \sqrt{4 - x^2}\right)^3$ is $\alpha$, then the numerical value of $\left(\frac{\alpha}{7}\right)$, is

Key. 4

Sol. Let $t = \sqrt{4 - x^2}, 0 \leq t \leq 2$

$\therefore F(t) = (3 - t)^2 + (1 + t)^3$ and maximum of $f(x)$ is 10

18. If the graph of $f(x)=2x^3+ax^2+bx, a,b\in N$ cuts the x-axis at three real and distinct points, then the minimum value of $\left(a^2+b^2-4\right)$ , is

Key. 6

Sol. $f^1(x)=6x^2+2ax+b \Rightarrow 4a^2-24b\ge 0$

$$\Rightarrow a^2\ge 6b$$

$\Rightarrow a\ge 3, b\ge 1, \Rightarrow a=3, b=1$

19. The minimum value of, $\dfrac{\sec^4\alpha}{\tan^2\beta}+\dfrac{\sec^4\beta}{\tan^2\alpha}, \alpha,\beta\ne\dfrac{K\pi}{2}, K\in I,$ is

Key. 8

Sol.

$$\frac{(a+1)^2}{b}+\frac{(b+1)^2}{a}=\frac{a^2}{b}+\frac{1}{b}+\frac{b^2}{a}+\frac{1}{a}+2\left(\frac{a}{b}+\frac{b}{a}\right)\ge 4\left[\frac{a^2}{b}\cdot\frac{1}{b}\cdot\frac{b^2}{a}\cdot\frac{1}{a}\right]^{\frac{1}{4}}+4\left(\frac{a}{b}\cdot\frac{b}{a}\right)^{\frac{1}{2}}\ge 8$$

Where $a=\tan^2\alpha, b=\tan^2\beta$

20. If one root of $x^2-4ax+a+f(a)=0$ is three times the other and if minimum value of $f(a)$ is $\alpha$ , then $|12\alpha|$ has a value

Key. 1

Sol. $\theta$ and $3\theta \Rightarrow 4\theta=4a \Rightarrow \theta=a$ and $a-4a^2+f(a)=0$

$\Rightarrow f(a)=3a^2-a \Rightarrow f_{min}$ is $\dfrac{-1}{12}$

21. The sum of greatest and least values of $f(x)=\left|x^2-5x+6\right|$ in $\left[0,\dfrac{5}{2}\right]$ , is

Key. 6

Sol. Sketch its graph

22. If $A=(0,2), B=(5,10)$ are two points. If P is a Point on x-axis, then, the sum of the digits in the minimum value of AP+PB ,is

Key. 4

Sol. If $P=(x,0)$, then $AP+PB=f(x)=\sqrt{x^2+2^2}+\sqrt{(x-5)^2+10^2}$

$\Rightarrow x=\dfrac{5}{6}$ is a point of minima

$\therefore$ minimum value of $f(x)=\sqrt{\dfrac{169}{36}}+\sqrt{\dfrac{625+3600}{36}}=\dfrac{13}{6}+\dfrac{65}{6}=\dfrac{78}{6}=13$

23. For a twice differentiable function $f(x)$, a function $g(x)$ is defined as $g(x)=\left(f^{1}(x)\right)^{2}+f(x)f^{11}(x)$ on $[a,e]$. If $a<b<c<d<e$ and $f(a)=0, f(b)=2, f(c)=-1, f(d)=2, f(e)=0,$ then, the minimum number of roots of the equation $g(x)=0$ , is/are

Key. 6

Sol. $\because f(b)f(c)<0$ and $f(c)f(d)<0$

$\Rightarrow f(x)=0$ has at least four roots,

$a,c_1,c_2,e,$ Where $c_1\in(b,c)$ and $c_2\in(c,d)$. Then, by RT, $f^{1}(x)=0$ has at least three roots in, $(a,c_1),(c_1,c_2),(c_2,e)$

$\therefore f(x)f^{1}(x)=0$ has at least 7 roots, by RT and hence,

$g(x)=\dfrac{d}{dx}\left\{f(x)f^{1}(x)\right\}=0$ has at least 6 roots

24. Let $f(x)=0$ be an equation of degree six, having integer coefficients and whose one root is $2\cos\dfrac{\pi}{18}$. Then, the sum of all the roots of $f^{1}(x)=0$ , is

Key. 0

Sol. Let $\theta=\dfrac{\pi}{18}\Rightarrow 6\theta=\dfrac{\pi}{3}\Rightarrow \cos 6\theta=\dfrac{1}{2}$

$\Rightarrow 4\cos^{3}2\theta-3\cos 2\theta=\dfrac{1}{2}\Rightarrow 8\left(2\cos^{2}\theta-1\right)^{3}-6\left(2\cos^{2}\theta-1\right)=1$ let $2\cos\theta=x$

$\Rightarrow 8\left(2.\dfrac{x^{2}}{4}-1\right)^{3}-6\left(2.\dfrac{x^{2}}{4}-1\right)=1$

$\Rightarrow \left(x^{2}-2\right)^{3}-3\left(x^{2}-2\right)=1$

$\Rightarrow x^{6}-6x^{4}+9x^{2}-3=0$

$f^{1}(x)=6x\left(x^{4}-4x^{2}+3\right)$

$f^{1}(x)=0\Rightarrow x=0,\pm1,\pm\sqrt{3}$

25. Let $\alpha$ and $\beta$ respectively be the number of solutions of $e^x = x^2$ and $e^x = x^3$. Then, the numerical value of $2\alpha + 3\beta$, is

Key. 8

Sol. Sketch the graphs

26. Let $P(x)$ be a polynomial of degree 4 having extremum at $x = 1,2$ and $\underset{x\to 0}{\text{Let}}\left(1+\frac{P(x)}{x^2}\right) = 2$, then, the value of $P(2)$, is

Key. 0

Sol. Let $P(x) = a_0x^4 + ..... + a_4$ by hypothesis, $P^1(1) = 0$ and $P^1(2) = 0$

$\Rightarrow 4a_0 + 3a_1 + 2a_2 + a_3 = 0$ and $32a_0 + 12a_1 + 4a_2 + a_3 = 0$

Also, $\underset{x\to 0}{\text{Lt}} \frac{P(x)}{x^2} = 1 \Rightarrow a_4 = 0$ and $a_3 = 0$ hence $\underset{x\to 0}{\text{Lt}}\left(a_0x^3 + a_1x + a_2\right) = 1 \Rightarrow a_2 = 1$

Solving, we get, $a_0 = \frac{1}{4}, a_1 = -1, a_2 = 1, a_3 = 0, a_4 = 0$

$\therefore P(x) = \frac{1}{4}x^4 - x^3 + x^2 \Rightarrow P(2) = 0$

27. Let $f(x) = \begin{cases} \left|x^2 - 3x\right| + a, & 0 \le x < \frac{3}{2} \\ -2x + 3, & x \ge \frac{3}{2} \end{cases}$. If f(x) has a local maxima at x = $\frac{3}{2}$, and the greatest value of 'a' is k, then $|4k|$ is...........

Key. 9

Sol. $f\left(\frac{3}{2}\right) = 0 \Rightarrow \lim\limits_{x\to\frac{3}{2}}\left|x^2 - 3x\right| + a \le 0$

$a \le -\frac{9}{4}$

Hence, $|4k| = 9$

28. If $a, b, c \in N$, and if $\frac{ax^4 - bx^3 + cx^2 - bx + a}{\left(x^2+1\right)^2}$ attains minimum value at x = 2 or $1/2$ then the A.M of the least possible values of a, b and c is __________

Key. 4

Sol. Put $x + \frac{1}{x} = t$ $a = 1, b = 4, c = 7, \Rightarrow$ AM is $\frac{1+4+7}{3} = 4$

29. The maximum value of the function $f(x)=2x^3-15x^2+36x-48$ on the set $A=\left\{x\middle|x^2+20\le 9x\right\}$ is

Key. 7

Sol. The given function is $f(x)=2x^3-15x^2+36x-48$ and $A=\{x\,|\,x^2+20\le 9x\}$

$\Rightarrow \quad A=\{x\,|\,x^2-9x+20\le 0\} \Rightarrow \quad A=\{x\,|\,(x-4)(x-5)\le 0\} \Rightarrow \quad A=[4,5]$

Also $f'(x)=6x^2-30x+36=6(x^2-5x+6)=6(x-2)(x-3)$

Clearly $\forall x\in A,\ f'(x)>0$

$\therefore$ $f$ is strictly increasing function on $A$.

$\therefore$ Maximum value of $f$ on $A$

$= f(5)=2\times5^3-15\times5^2+36\times5-48=250-375+180-48=7$

30. In the coordinate plane, the region M consists of all points $(x,y)$ satisfying the inequalities $y\ge 0, y\le x,$ and $y\le 2-x$ simultaneously. The region N which varies with parameter t, consists of all the points $(x,y)$ satisfying the inequalities $t\le x\le t+1$ and $0\le t\le 1$ simultaneously. If the area of the region $M\cap N$ is a function of t, i,e., $M\cap N=f(t)$ and if $\alpha$ is the value of t for which this area is maximum, then the numerical value of $2\alpha$ is

Key. 1

Sol. $M\cap N=f(t)=-t^2+t+1/2$

$=\frac{3}{4}-\left(t-\frac{1}{2}\right)^2$ f(t) is maximum for t = 1/2 i.e. $\alpha=\frac{1}{2}\Rightarrow 2\alpha=1$

31. Let $M(-1,2)$ and $N(1,4)$ be two points in a plane rectangular coordinate system XOY. P is a moving point on the x-axis. When $\angle$MPN takes its maximum value, the x-coordinate of point P is

Key. 1

Sol. The centre of a circle passing through points M and N lies on the perpendicular bisector y = 3 – x of MN. Denote the centre by C(a, 3 – a), the equation of the circle is

$(x-a)^2+(y-3+a)^2=2(1+a^2)$

Since for a chord with a fixed length the angle at the circumference subtended by the corresponding arc will become larger as the radius of the circle becomes smaller. When $\angle$MPN reaches its maximum value the circle through the three points M, N and P will be tangent to the x-axis at P, which means

$2(1+a^2)=(a-3)^2\Rightarrow$ a = 1 or a = –7

Thus the point of contact are P(1, 0) or $P'(-7,0)$ respectively.

But the radius of circle through the points M, N and $P'$ is larger than that of circle through points M, N and P.

Therefore, $\angle MPN > \angle MP'N$. Thus P = (1, 0)

$\therefore$ x-coordinate of P = 1.

32. $f(x)=\frac{1}{1+|x|}+\frac{1}{1+|x-1|}$. Let $x_1, x_2$ are points where $f(x)$ attains local minimum and global maximum respectively. Let $k=f(x_1)+f(x_2)$ then $6k-9$

Key. 8

Sol. Local minimum $=f\left(\frac{1}{2}\right)=\frac{4}{3}$

Global maximum $=f(0)=f(1)=\frac{3}{2}\ k=\frac{4}{3}+\frac{3}{2}=\frac{17}{6}$

33. $$f(x)=\begin{cases}\left(\sqrt{2}+\sin\frac{1}{x}\right)e^{\frac{-1}{|x|}} & if \quad x\neq 0\\ 0 & if \quad x=0\end{cases}$$

Number of points where $f(x)$ has local extrema when $x\neq 0$ be $n_1$. $n_2$ be the value of global minimum of $f(x)$ then $n_1+n_2=$

Key. 0

Sol. Local extremum does not occur at any value of $x\neq 0$. But global minimum $=f(0)=0$

$\therefore\ n_1=0, n_2=0$ then $n_1+n_2=0$

34. A = (- 3,0) and B = (3 ,0) are the extremities of the base AB of triangle PAB. If the vertex P varies such that the internal bisector of angle APB of the triangle divides the opposite side AB into two segments AD and BD such that AD : BD = 2 : 1 , then the maximum value of the length of the altitude of the triangle drawn through the vertex P is

Ans: 4

Hint: The point E dividing $\overline{AB}$ externally in the ratio 2 : 1 is (9 , 0) . Since P lies on the circle described on $\overline{DE}$ as diameter, coordinates of P are of the form $(5+4\cos\theta, 4\sin\theta)$

$\therefore$ maximum length of the altitude drawn from P to the base $AB=|4\sin\theta|_{\max}=4$

35. Find the maximum value of ( $\log_{2^{1/5}}$ a).($\log_{2^{1/2}}$ b ). It is given that coefficient of 2$^{nd}$, 3$^{rd}$ and 4$^{th}$ term in expansion of $(a + b)^n$ are in A.P and the value of 3$^{rd}$ term is equal to 84 (a, b > 1).

Key: 1

Hint: In expansion of $(a + b)^n$ the coefficient of 2$^{nd}$, 3$^{rd}$ and 4$^{th}$ term are in A.P. which gives n = 7 also ${}^7C_2\, a^5 b^2 = 84 \Rightarrow a^5b^2 = 4$

Now $\frac{\log_2 a^5 + \log_2 b^2}{2} \geq (\log_2 a^5 . \log_2 b^2)^{1/2} \Rightarrow k \leq \left(\frac{\log_2 a^5 b^2}{2}\right)^2$

$k \leq 1 \Rightarrow$ maximum value of k is 1.

36. From a point perpendicular tangents are drawn to ellipse $x^2 + 2y^2 = 2$. The chord of contact touches a circle which is concentric with given ellipse. Then find the ratio of maximum and minimum area of circle.

Ans: 4

Hint: The director circle of ellipse $\frac{x^2}{2} + \frac{y^2}{1} = 1$ is $x^2 + y^2 = 3$

Let a point P($\sqrt{3}\cos\theta$, $\sqrt{3}\sin\theta$)

Equation of chord of contact is

$x.\sqrt{3}\cos\theta + 2y\sqrt{3}\sin\theta - 2 = 0$

It touches $x^2 + y^2 = r^2$

$r = \frac{2}{\sqrt{3\cos^2\theta + 12\sin^2\theta}} = \frac{2}{\sqrt{3 + 9\sin^2\theta}}$

$r_{max} = \frac{2}{\sqrt{3}}$

$r_{min} = \frac{2}{\sqrt{12}} \Rightarrow \frac{A_{max}}{A_{min}} = 4.$

37. Let $f(x) = 30 - 2x - x^3$, then find the number of positive integral values of x which satisfies $f(f(f(x))) > f(f(-x))$

Key: 2

Hint: $f(x) = 30 - 2x - x^3$

$f(x) = -2 - 3x^2 < 0 \Rightarrow f(x)$ is decreasing function

Hence $f(f(f(x))) > f(f(-x)) \Rightarrow f(f(x)) < f(-x)$

$\Rightarrow f(x) > -x$

$\Rightarrow 30 - 2x - x^3 > -x \Rightarrow x^3 + x - 30 < 0 \Rightarrow (x-3)(x^2 + 3x + 10) < 0$

$\Rightarrow x < 3$

38. The sum of greatest and least values of $f(x) = |x^2 - 5x + 6|$ in $\left[0, \frac{5}{2}\right]$, is

Key. 6

Sol. Sketch its graph

39. If $A = (0,2), B = (5,10)$ are two points. If P is a Point on x-axis, then, the sum of the digits in the minimum value of AP+PB ,is

Key. 4

Sol. If $P=(x,0)$, then $AP+PB=f(x)=\sqrt{x^2+2^2}+\sqrt{(x-5)^2+10^2}$

$\Rightarrow x=\frac{5}{6}$ is a point of minima

$\therefore$ minimum value of $f(x)=\sqrt{\frac{169}{36}}+\sqrt{\frac{625+3600}{36}}=\frac{13}{6}+\frac{65}{6}=\frac{78}{6}=13$

40. If $a,b,c\in N$, and if $\frac{ax^4-bx^3+cx^2-bx+a}{(x^2+1)^2}$ attains minimum value at x = 2 or $1/2$ then the A.M of the least possible values of a, b and c is __________

Key. 4

Sol. Put $x+\frac{1}{x}=t$ $a=1, b=4, c=7, \Rightarrow$ AM is $\frac{1+4+7}{3}=4$

41. In the coordinate plane, the region M consists of all points $(x,y)$ satisfying the inequalities $y\ge 0, y\le x$, and $y\le 2-x$ simultaneously. The region N which varies with parameter t, consists of all the points $(x,y)$ satisfying the inequalities $t\le x\le t+1$ and $0\le t\le 1$ simultaneously. If the area of the region $M\cap N$ is a function of t, i,e., $M\cap N=f(t)$ and if $\alpha$ is the value of t for which this area is maximum, then the numerical value of $2\alpha$ is

Key. 1

Sol. $M\cap N=f(t)=-t^2+t+1/2$

$=\frac{3}{4}-\left(t-\frac{1}{2}\right)^2$ f(t) is maximum for t = 1/2 i.e. $\alpha=\frac{1}{2}\Rightarrow 2\alpha=1$

42. Let $P=x^3-\frac{1}{x^3}$ , $Q=x-\frac{1}{x}$ and a is the minimum value of $P/Q^2$. Then the value of [a] is where [x] = the greatest integer $\le x$ .

Key. 3

Sol. $Q^3 = P - 3Q$

$\Rightarrow \frac{P}{Q^2}=Q+\frac{3}{Q}$

$f(Q)=Q+\frac{3}{Q}$

$f'(Q)=1-\frac{3}{Q^2}\Rightarrow Q=\pm\sqrt{3}$

f(Q)f will be minimum at Q = $\sqrt{3}$

So minimum value of f(Q) is $2\sqrt{3}$

i.e. minimum of $\left[\frac{P}{Q^2}\right]=\left[2\sqrt{3}\right]=3$

43. Let f(x) = (x – a)(x – b)(x – c)(x – d); a < b < c < d. Then minimum number of roots of the equation $f''(x)=0$ is

Key. 2

Sol. f(a) = f(b) = f(c) = f(d) = 0
f(x) = 0 (4 times). Graph of f(x) will intersect 4 times the x-axis. So there will be minimum three turnings.
and $f'(x)=0$ minimum (3 times). So $f''(x)=0$ will be minimum (2 times).

44. If $f(x)=x^3+x^2f'(1)+xf''(2)+f'''(3)\ \forall x\in R$. Then the value of $f'(1)+f''(2)+f'''(3)$ is

Key. 3

Sol. Let $f'(1)=a,\ f''(2)=b,\ f'''(3)=c$
so $f'(x)=3x^2+2ax+b,\quad f''(x)=6x+2a$
a = 3 + 2a + b
b = 12 + 2a and c = 6.
$\Rightarrow$ a = –5, b = 2 and c = 6. so a + b + c = 3

45. Let f be twice differentiable such that $f''(x)=-f(x)$ and $f'(x)=g(x)$. If h(x) = $(f(x))^2+(g(x))^2$, where h(5) = 9. Then the value of h(10) is

Key. 9

Sol. $h'(x)=2f(x)f'(x)+2g(x)g'(x)$
$f'(x)=g(x)\Rightarrow f''(x)=g'(x)$
$\Rightarrow\ g'(x)=-f(x)$
$\therefore\ h'(x)=0$ h(x) = constant
h(5) = 9 $\Rightarrow$ h(10) is also 9.