Differential Calculus

Single Correct Answer Type

1. Let $f(x)=4x+8\cos x-\ln\{\cos x(1+\sin x)\}+\tan x-2\sec-6$. If $f(x)>0\,\forall\, x\in(0,a)$ then

a) $a=\frac{\pi}{6}$ b) $a=\frac{\pi}{3}$ c) $a=\frac{\pi}{2}$ d) none of these

Ans. a

Sol. $f'(x)=4-8\sin x-\frac{(-\sin x+\cos^2 x-\sin^2 x)}{\cos x(1+\sin x)}+\sec^2 x-\sec x\tan x$

$=4(1-2\sin x)+\sec^2 x(1-2\sin x)-4\sec(1-2\sin x)$

$=f(x)=(\sec x-2)^2(1-2\sin x)$

If $f(x)>0\,\forall\, x\in(0,a)$, then f(x) is increasing in (0, a) $\Rightarrow a=\frac{\pi}{6}$

2. If f(x) is continuous for all real values of x, then $\sum_{r=1}^{n}\int_0^1 f(r-1+x)\,dx=$

a) $\int_0^n f(x)\,dx$ b) $\int_0^1 f(x)\,dx$ c) $n\int_0^1 f(x)\,dx$ d) $(n-1)\int_0^1 f(x)\,dx$

Ans. a

Sol. $\sum_{r=1}^{n}\int_0^1 f(r-1+x)\,dx=\int_0^1 f(x)\,dx+\int_0^1 f(1+x)\,dx+\int_0^1 f(2+x)\,dx+......+\int_0^1 f(n-1+x)\,dx$

= $\int_0^1 f(x)\,dx+\int_1^2 f(x)\,dx+\int_2^3 f(x)\,dx+......+\int_{n-1}^{n} f(x)\,dx=\int_0^n f(x)\,dx$

3. The coordinates of the point on the curve $x^3 = y(x-a)^2$, a > 0 where the ordinate is minimum

a) (2a, 8a) b) $\left(-2a,\frac{-8a}{9}\right)$ c) $\left(3a,\frac{27a}{4}\right)$ d) $\left(-3a,\frac{-27a}{16}\right)$

Ans. c

The ordinates of any point on the curve is given by $y=\frac{x^3}{(x-a)^2}$

Sol. $\frac{dy}{dx}=\frac{x^2(x-3a)}{(x-a)^3}$

Now, $\frac{dy}{dx}=0\Rightarrow x=0\ or\ x=3a$

$\left.\frac{d^2y}{dx^2}\right|_{x=0}=0\ and\ \left.\frac{d^2y}{dx^2}\right|_{x=3a}=\frac{72a^5}{(2a)^6}>0$

Hence y is minimum at x = 3a and is equal to $\frac{27a}{4}$

4. Let $I_n = \int_0^{\frac{\pi}{2}} \frac{\sin(2n-1)x}{\sin x} dx$, $J_n = \int_0^{\frac{\pi}{2}} \frac{\sin^2 nx}{\sin^2 x} dx$, $n \in N$, then

a) $J_{(n+1)} - J_n = I_n$ b) $J_{(n+1)} - J_n = I_{(n+1)}$ c) $J_{n+1} + J_n = J_n$ d) $J_{n+1} + J_{n+1} = J_n$

Ans. b

Sol. $J_n - J_{n-1} = \int_0^{\frac{\pi}{2}} \frac{\sin^2 nx - \sin^2(n-1)x}{\sin^2 x} dx - \int_0^{\frac{\pi}{2}} \frac{\sin(2n-1)x - \sin x}{\sin^2 x} dx = I_n$

i.e $J_n - J_{n-1} = I_n \Rightarrow J_{n+1} - J_n = I_{n+1}$

5. A curve whose concavity is directly proportional to the logarithm of its x-coordinates at any of the curve, is given by

a) $c_1 x^2 (2\log x - 3) + c_2 x + c_3$ b) $c_1 x^2 (2\log x + 3) + c_2 x + c_3$

c) $c_1 x^2 (2\log x) + c_2$ d) none of these

Ans. a

Sol. $\frac{d^2 y}{dx^2} = k\log x \Rightarrow \frac{dy}{dx} = k(x\log x - x) + A$

$\Rightarrow y = k\left[\frac{1}{2}x^2 \log x - \int \frac{1}{2}x^2 \cdot \frac{1}{x} - \frac{x^2}{2} dx\right] + Ax + B$

$\Rightarrow y = \frac{k}{4}\{2x^2 \log x - x^2 - 2x^2] + Ax + B$

$\Rightarrow y = c_1(2\log x - 3)x^2 + c_2 x + c_3$

6. The domain of the function $f(x) = \sqrt{3 - 2^x - 2^{1-x}} + \sqrt{\sin^{-1} x}$ is

a) $[-1, 0]$ b) $[0, 1]$ c) $\left[\frac{1}{2}, 1\right]$ d) [1, 2]

Ans. b

Sol. $\sin^{-1} x \geq 0 \Rightarrow 0 \leq x \leq 1$

and $2^x + 2^{1-x} \leq 3 \Rightarrow 2^x + 2.2^{-x} - 3 \leq 0$

Put $2^x = t$, then $t^2 - 3t + 2 \leq 0 \Rightarrow (t-2)(t-1) \leq 0$

$\Rightarrow 1 \leq t \leq 2$ *i.e* $1 \leq 2^x \leq 2$

$\Rightarrow 0 \leq x \leq 1$

7. Area bounded by the curve y = sinx, y = cosx, $x = -\frac{\pi}{3}$, $x = 2\pi$

a) $4\sqrt{2} - \left(\frac{\sqrt{3}+1}{2}\right)$ b) $\sqrt{2} + \left(\frac{\sqrt{3}+1}{2}\right)$ c) $\sqrt{2} - \left(\frac{\sqrt{3}+2}{2}\right)$ d) $4\sqrt{2} + \left(\frac{\sqrt{3}+1}{2}\right)$

Ans. d

Sol. $A = \int_{-\pi/3}^{2\pi} |\sin x - \cos x| dx \Rightarrow 4\sqrt{2} + \frac{\sqrt{3}+1}{2}$

8. Let f(1) = 1 and $f(n)=2\sum_{r=1}^{n-1} f(r)$, then $\sum_{n=1}^{m} f(n)$ is equal to

a) $3^{m-1}-1$ b) 3^{m-1} c) $3^{m}-1$ d) none of these

Ans. b

Sol. $f(n)=2\left(f(1)+f(2)+....+f(n-1)\right)$

$\therefore$ f(n + 1) = 2(f(1) + f(2) + ...+ f(n))

$\Rightarrow$ f(n + 1) = 3f(n) for $n\geq 2$

Also f(2) = 2f(1) = 2

f(3) = 3f(2) = $2\cdot 3$

$\sum_{n=1}^{m} f(n)=f(1)+f(2)+...+f(m)$

= 1 + 2 + 2.3 + 2.3^2 + ... + 2.3^{m-2} = 1 + 2(1 + 3 + 3^2 + ... + 3^{m-2})

9. $I=\int \frac{2+3\cos\theta}{\sin\theta+2\cos\theta+3}\,d\theta$, then

a) $I=\frac{6\theta}{5}+\frac{3}{5}\log\left|\sin\theta+2\cos\theta+3\right|-\frac{8}{5}\tan^{-1}\left(\frac{\tan\left(\frac{\theta}{2}\right)+1}{2}\right)+c$

b) $I=\frac{6\theta}{5}-\frac{3}{5}\log\left|\sin\theta+2\cos\theta+3\right|-\frac{8}{5}\tan^{-1}\left(\frac{\tan\left(\frac{\theta}{2}\right)+1}{2}\right)+c$

c) $I=\frac{6\theta}{5}-\frac{3}{5}\log\left|\sin\theta+2\cos\theta+3\right|$

d) none of these

Ans. a

Sol. $2+3\cos\theta=I\left(\sin\theta+2\cos\theta+3\right)+m\left(\cos\theta-2\sin\theta\right)+n$, then integrate

10. The value of $\lim_{n\to\infty}\left(\frac{\sqrt{n}}{\left(3+4\sqrt{n}\right)^2}+\frac{\sqrt{n}}{\sqrt{2}\left(3\sqrt{2}+4\sqrt{n}\right)^2}+.....+\frac{1}{49n}\right)$ is equal to

a) $\frac{1}{14}$ b) $\frac{2}{7}$ c) $\frac{3}{7}$ d) none of these

Ans. a

Sol. $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{\sqrt{n}}{\sqrt{r}\left(3\sqrt{r}+4\sqrt{n}\right)^2}$

Put $\frac{r}{n}=x\Rightarrow\frac{1}{n}=dx$

$=\int_0^1\frac{dx}{\sqrt{x}\left(3\sqrt{x}+4\right)^2}=\frac{1}{14}$

11. Solution of differential equation $xdy-\left(y+xy^3\left(1+\log x\right)\right)dx=0$

a) $\frac{-x^2}{y^2}=\frac{2x}{3}\left(\frac{2}{3}+\log x\right)+c$ b) $\frac{x^2}{y^2}=\frac{2x^2}{3}\left(\frac{2}{3}+\log x\right)+c$

c) $\frac{-x^2}{y^2}=\frac{2x^3}{3}\left(\frac{2}{3}+\log x\right)+c$ d) none of these

Ans. c

Sol. $-d\left(\frac{x}{y}\right)=xy\left(1+\log x\right)dx$

$\int -\frac{x}{y}d\left(\frac{x}{y}\right)=\int x^2\left(1+\log x\right)dx$ gives solution

12. Let $f:R\to R$ and $g:R\to R$ be twice differentiable function satisfying $f''(x)=g''(x)$, $2f'(1)=g'(1)=4$ and 3f(2) = g(2) = 9. The value of f(4) – g(4) is equal to

a) – 6 b) – 16 c) – 10 d) – 8

Ans. c

Sol. $f'(x)=g(x)-2$

f(x) = g(x) – 2x – 2

f(u) – g(u) = - 10

13. Let a, b, c be three real numbers such that a < b < c. Let f(x) be continuous $\forall\ x\in[a,c]$ and differentiable $\forall\ x\in(a,c)$. If $f''(x)>0\ \forall\ x\in(a,c)$ then

a) (c – b) f(a) + (b – a) f(c) > (c – a) f(b) b) (c – b) f(a) + (a – c) f(b) < (a – b) f(c)

c) f(a) < f(b) < f(c) d) none of these

Ans. a

Sol. By LMVT

$$\frac{f(b)-f(a)}{b-a}>\frac{f(c)-f(b)}{c-b}$$

14. The solution of $y^5x+y-x\frac{dy}{dx}=0$ is

a) $\frac{x^4}{4}+\frac{1}{5}\left(\frac{x}{y}\right)^5=c$ b) $\frac{x^5}{5}+\frac{1}{4}\left(\frac{x}{y}\right)^4=c$ c) $\left(\frac{x}{y}\right)^5+\frac{x^4}{4}=c$ d) $(xy)^4+\frac{x^5}{5}=c$

Ans. b

Sol. $y^5xdx+ydx-xdy=0$, multiply by x^3/y^5

$\Rightarrow x^4dx+\frac{x^3}{y^3}\left(d\left(x/y\right)\right)=0$

$\Rightarrow \frac{x^5}{5}+\frac{1}{4}\left(\frac{x}{y}\right)^4=c$

15. A point P lying inside the curve $y=\sqrt{2ax-x^2}$ is moving such that its shortest distance from the curve at any position is greater than its distance from x-axis. The point P enclose a region whose area is equal to

a) $\frac{\pi a^2}{2}$ b) $\frac{a^2}{3}$ c) $\frac{2a^2}{3}$ d) $\left(\frac{3\pi-4}{6}\right)a^2$

Ans. c

Sol.

$y=\sqrt{2ax-x^2} \Rightarrow (x-a)^2+y^2=a^2$

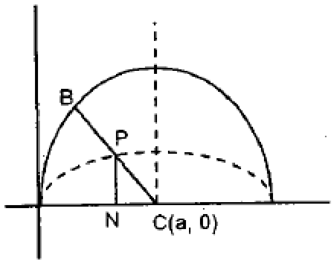
Let P(h, k) be a point then BP > PN

For the boundary condition BP = PN = k

Now $AP=a-k=\sqrt{(h-a)^2+k^2} \Rightarrow k=h-\frac{h^2}{2a}$

$\therefore$ boundary of the region is $y=x-\frac{x^2}{2a}$

Required area = $2\int_0^a\left(x-\frac{x^2}{2a}\right)dx=\frac{2a^2}{3}$



16. If $\log_x\left(\log_y k\right)>0$ where $x,k\in(0,1)$ then $y\in$

a) (0, x) b) (0, k) c) (k, 1) d) R^+

Ans. c

Sol. $\log_y k<1$

case 1 : if y > 1 $\Rightarrow$ k < y

for $\log_y k>0 \Rightarrow$ k > 1 which is not possible

case 2 : if y < 1 $\Rightarrow$ k > y

and for $\log_y k>0 \Rightarrow$ k < 1 which is true

17. Period of $f(x)=x-[x+\lambda]-\mu$ where $\lambda,\ \mu\in R$ and $[\cdot]$ denotes the g.i.f is

a) λ b) μ c) $|\lambda-\mu|$ d) 1

Ans. d

Sol. $f(x)=x-[x+\lambda]-\mu=x+\lambda-[x+\lambda]-(\lambda+\mu)$

$=\{x+\lambda\}-(\lambda+\mu)$

$\therefore$ Period of f(x) = 1

18. If $f(x)=2\sin^3 x-3\sin^2 x+12\sin x+5\ \forall\ x\in\left(0,\frac{\pi}{2}\right)$, then

a) f is increasing in $\left(0,\frac{\pi}{2}\right)$ b) f is decreasing in $\left(0,\frac{\pi}{2}\right)$

c) f is increasing $\left(0,\frac{\pi}{4}\right)$ and decreasing in $\left(\frac{\pi}{4},\frac{\pi}{2}\right)$

d) f is decreasing is $\left(0,\frac{\pi}{4}\right)$ and increasing in $\left(\frac{\pi}{4},\frac{\pi}{2}\right)$

Ans. a

Sol. $f'(x)=6\cos x\left(\sin^2 x-\sin x+2\right)>0\ \forall \in\left(0,\dfrac{\pi}{2}\right)$

Thus f(x) is increasing in $\left(0,\dfrac{\pi}{2}\right)$

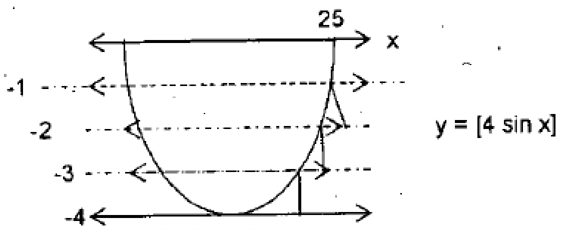
19. Total number of points of non-differentiability of $f(x)=[3+4\sin x]$ in $[\pi,2\pi]$ where $[\cdot]$ denote the g.i.f are

a) 5 b) 6 c) 8 d) 9

Ans. c

Sol.

f(x) = 3 + [4 sinx]
f(x) is non-differentiable where g(x) =[4sinx] is non differentiable
In [π, 2 π], g(x) is clearly non-differentiable at 8 points.



20. If $f(x)+2f(1-x)=x^2+1\ \forall x\in R$ and $\int_0^k f(x)dx=0$, then k equals to

a) 3 b) 2 c) 4 d) none of these

Ans. a

Sol. Putting (1 – x) for x and subtracting we get $f(x)=\dfrac{x^2-4x+3}{3}$

Now $\int_0^k \dfrac{x^2-4x+3}{3}dx=0 \Rightarrow \dfrac{k^3}{3}-2k^2+3k=0$

$\Rightarrow k=3$

21. A point p(x, y) moves is such a way that [x + y + 1] = [x] (where [ ] denotes g.i.f) and $x\in(0,2)$. Then the area representing all the possible positions of P equals

a) $\sqrt{2}$ sq. units b) $2\sqrt{2}$ sq. units c) $4\sqrt{2}$ sq. units d) none of these

Ans. d

Sol.

If $x\in(0,1)$

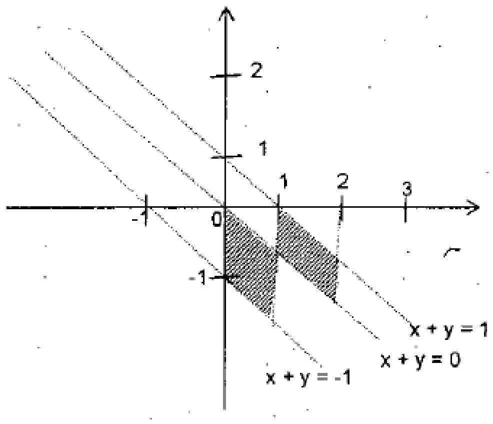
Then $-1\le x+y<0$

and if $x\in(1,2)$

$0\le x+y<1$

Required area =

$4\left(\dfrac{1}{2}\cdot 1\cdot\sqrt{2}\sin\dfrac{\pi}{4}\right)=2\,sq\,units$



22. Let f(x) be a polynomial with real coefficients satisfies $f(x) = f'(x) \times f'''(x)$. If f(x)=0 satisfies x = 1, 2, 3 only then the value of $f'(1) \times f'(2) \times f'(3)$ =

a) positive b) negative c) 0 d) inadequate data

Ans. c

Sol. $f(x) = f'(x) \times f'''(x)$ is satisfied by only the polynomial of degree 4.

Since f(x) = 0 satisfies x = 1, 2, 3 only. It is clear one of the root is twice repeated.

$\Rightarrow f'(1)\, f'(2)\, f'(3) = 0$

23. The value of $\lim_{n\to\infty}\left(\frac{n!}{(mn)^n}\right)^{1/n}$ is

a) em b) $\frac{e}{m}$ c) $\frac{1}{em}$ d) none of these

Ans. c

Sol. $L = \lim_{n\to\infty}\frac{1}{m}\left(\frac{1}{n}\cdot\frac{2}{n}\cdot\frac{3}{n}.....\frac{n}{n}\right)^{1/n}$

$In\ L = \lim_{n\to\infty}\left[\ln\left(\frac{1}{m}\right)+\frac{1}{n}\left(\ln\frac{1}{n}+\ln\frac{2}{n}+....+\ln\frac{n}{n}\right)\right]$

$= \ln m + \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\ln\left(\frac{r}{n}\right) = -\ln m + \int_0^1 \ln x\, dx = -\ln m - 1 = \ln\left(\frac{1}{em}\right)$

$\therefore L = \frac{1}{em}$

24. Let A = {1, 2, 3, 4, 5} and f : A $\to$ A be an into function such that $f(i) \neq i \ \forall \ i \in A$, then number of such functions f are

a) 1024 b) 904 c) 984 d) none of these

Ans. d

Sol. Total number of functions for which $f(i) \neq i = 4^5$

and number of onto functions in which $f(i) \neq i = 44$

$\Rightarrow$ required numbers of functions = 980

25. The area of the region bounded between the curves $y = e\|x|\ln|x\|$, $x^2 + y^2 - 2(|x|+|y|) + 1 \geq 0$ and x-axis where $|x| \leq 1$, if α is the x-coordinate of the point of intersection of curves is 1st quadrant, is

a) $4\left[\int_0^\alpha ex\ln x dx + \int_\alpha^1\left(1-\sqrt{1-(x-1)^2}\right)dx\right]$

b) $\left[\int_0^\alpha ex\ln x dx - \int_1^\alpha\left(1-\sqrt{1-(x-1)^2}\right)dx\right]$

c) $2\left[-\int_0^\alpha ex\ln x dx + \int_\alpha^1\left(1-\sqrt{1-(x-1)^2}\right)dx\right]$

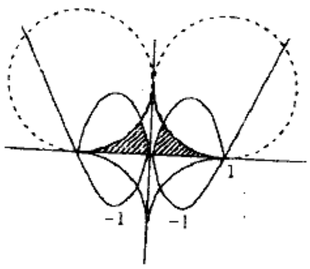
d) $2\left[\int_0^\alpha ex\ln x dx + \int_\alpha^1\left(1-\sqrt{1-(x-1)^2}\right)dx\right]$

Ans. c

Sol.

Required area is

$$2\left[\int_0^{\alpha} ex\ln x dx+\int_1^{\alpha}\left(1-\sqrt{1-(x-1)^2}\right)dx\right]$$



26. The value of $\lim_{n\to\infty} n\left[\frac{1}{3n^2+8n+4}+\frac{1}{3n^2+16n+16}+.....n\ terms\right]$ is

a) $\frac{1}{4}\ln\left(\frac{9}{5}\right)$ b) $\frac{1}{5}\ln\left(\frac{9}{5}\right)$ c) $\frac{1}{4}\ln\left(\frac{8}{5}\right)$ d) $\frac{1}{4}\ln\left(\frac{9}{7}\right)$

Ans. a

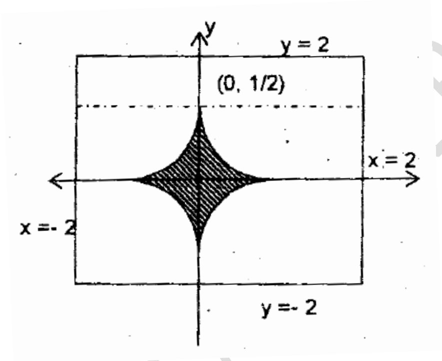
Sol. Use definite integral of first principal as a limit of sum

$$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{4\left(1+\frac{r}{n}\right)^2-1}\cdot\frac{1}{n}$$

27. The area of the region containing the points satisfying $|y|+\frac{1}{2}\le e^{-|x|}$, $\max(|x|,|y|)\le 2$ is

a) $2\log\left(\frac{e}{2}\right)$ b) $2\log\left(\frac{2e}{3}\right)$ c) $3\log\left(\frac{e}{2}\right)$ d) $3\log\left(\frac{2e}{3}\right)$

Ans. a



28. If $y=2^{\frac{1}{-2^{1-x}}}$; then $\lim_{x\to 1^+} y$ is

a) – 1 b) 1 c) 0 d) $\frac{1}{2}$

Ans. b

Sol. $\lim_{h\to 0} 2^{-2^{\frac{1}{1-(1+h)}}}=2^{-0}=1$

29. If $y=\frac{2x+5}{3x+10}$, then $2\left(\frac{dy}{dx}\right)\left(\frac{d^3y}{dx^3}\right)$ is equal to

a) $\left(\frac{d^2y}{dx^2}\right)^2$ b) $3\frac{d^2y}{dx^2}$ c) $3\left(\frac{d^2y}{dx^2}\right)^2$ d) $3\frac{d^2x}{dy^2}$

Ans. c

Sol. 3xy + 10y = 2x + 5, now differentiate 3 times.

30. If the number of solutions of $\ln|\sin x| = -x^2 + 2x$ when $x \in (0, \pi)$ is m and when $x \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$ is n, then (m + n) is equal to

a) 2 b) 4 c) 6 d) 1

Ans. a

Sol. m = 0, n = 2

31. If x, {x} and 2[x] represent the segments of a focal chord and length of latus rectum of an ellipse respectively, then length of major axis of ellipse is always greater than (where $x \notin Z$)

a) 7 b) 5 c) 8 d) 2

Ans. d

Sol. Clearly, x, [x] and {x} are in H.P = $[x] = \frac{2x\{x\}}{x+\{x\}} \Rightarrow [x] = 1$

$\Rightarrow \frac{b^2}{a} = 1 \Rightarrow a(1-e^2) = 1 \Rightarrow 2a > 2$ [since 0 < e < 1]

32. The value of $\int_3^6 \left(\sqrt{x+\sqrt{12x-36}} + \sqrt{x-\sqrt{12x-36}}\right)dx$ is equal to

a) $6\sqrt{3}$ b) $4\sqrt{3}$ c) $12\sqrt{3}$ d) none of these

Ans. a

Sol. $I = \int_3^6 \left(\left(\sqrt{x-3}+\sqrt{3}\right)+\left(\sqrt{3}-\sqrt{x-3}\right)\right)dx = 6\sqrt{3}$

33. If integral $\int \frac{dx}{(\sec x + \operatorname{cosec} x + \tan x + \cot x)^2} = \frac{x}{a} + \frac{\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)}{b} + \frac{\cos 2x}{c} + d$, then a + b + c is equal to

a) – 2 b) – 4 c) 2 d) none of these

Ans. b

Sol. Clearly,

$$I = \int \frac{\sin^2 x \cos^2 x}{(\sin x + \cos x + 1)^2}dx = \frac{1}{4}\int \frac{\left((\sin x + \cos x)^2 - 1\right)^2}{(\sin x + \cos x + 1)}dx = \frac{1}{4}\int (\sin x + \cos x - 1)^2 dx$$

On simplifying a + b + c = - 4

34. If $I_n = \int_{-n}^{n} \left(\{x+1\}\{x^2+2\} + \{x^2+3\}\{x^2+4\}\right)dx$, (where {.} denotes the fractional part) then I_1 is equal to

a) $-\frac{1}{3}$ b) $-\frac{2}{3}$ c) $\frac{1}{3}$ d) none of these

Ans. b

Sol. $I_1 = \int_{-1}^{1}\left(\{x\}+\{x^3\}\right)\{x^2\}\,dx = -2\int_{0}^{1}\{x^2\}\,dx = -2\times\left.\frac{x^3}{3}\right|_0^1 = -\frac{2}{3}$

35. Area bounded by y = $f^{-1}(x)$ and tangent and normal drawn to it at the points with abscissae π and 2π, where f(x) = sinx – x is

a) $\frac{p^2}{2} - 1$ b) $\frac{p^2}{2} - 2$ c) $\frac{p^2}{2} - 4$ d) $\frac{p^2}{2}$

Ans. b

Sol. Required area A = $\int_{\pi}^{2\pi}\left((\sin x - x) + 2\pi\right)dx = \frac{\pi^2}{2} - 2\ sq.units$

36. Let a curve y = f(x), $f(x)^3\ 0\ "\ x\hat{I}\ R$ has property that for every point P on the curve length of subnormal is equal to abscissa of P. If f(1) = 3, then f(4) is equal to

a) $-2\sqrt{6}$ b) $2\sqrt{6}$ c) $3\sqrt{5}$ d) none of these

Ans. b

Sol. Given $y\frac{dy}{dx} = x$

$y\,dy = x\,dx$

$y^2 = x^2 + c$

$f(1) = 3 \Rightarrow 9 - 1 + c \Rightarrow c = 8$

$\Rightarrow y^2 = x^2 + 8$

$f(x) = \sqrt{x^2 + 8}$

$f(4) = \sqrt{16 + 8} = 2\sqrt{6}$

37. Range of $f(x) = \cos^{-1}\left(\frac{x^2 + x + 1}{x^4 + 1}\right)$ is

a) $\left[0, \frac{\pi}{2}\right]$ b) $\left[0, \frac{\pi}{2}\right)$ c) $\left(0, \frac{\pi}{2}\right]$ d) [0, π]

Ans. b

Sol. Let $g(x) = \frac{x^2 + x + 1}{x^4 + 1}$

$\Rightarrow 0 < g(x) \le 1$

So range of f(x) is $\left[0, \frac{\pi}{2}\right)$

38. If f(x) = 0 is a cubic equation with positive and distinct roots α, β, γ such that β is H.M of the roots of $f'(x) = 0$, then α, β and γ are in

a) A.P b) G.P c) H.P d) none of these

Ans. b

Sol. $f(x) = (x - \alpha)(x - \beta)(x - \gamma)$

$\Rightarrow f'(x) = 3x^2 - 2x(\alpha + \beta + \gamma) + \alpha\beta + \beta\gamma + \gamma\alpha$

$\Rightarrow \beta = \frac{2\alpha_1\beta_1}{\alpha_1 + \beta_1}$ (where α_1, β_1 are the roots of $f'(x)$ = 0)

$\Rightarrow \beta^2 = \gamma\alpha$

39. Let a curve y = f(x), $f(x) \geq 0 \ \forall \ x \in R$ has property that for every point P on the curve, the length of subnormal is equal to abscissa of P. If f(1) = 3, then f(4) is equal to

a) $-2\sqrt{6}$ b) $2\sqrt{6}$ c) $3\sqrt{5}$ d) none of these

Ans. b

Sol. $y\frac{dy}{dx} = x \Rightarrow y^2 = x^2 + c$

$f(x) = \sqrt{x^2+8} \Rightarrow f(4) = 2\sqrt{6}$

40. If $\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{\pi}{2ab}$, then the value of $\int_0^{\pi/2} \frac{dx}{\left(4\cos^2 x + 9\sin^2 x\right)^2}$ is equal to

a) $\frac{11\pi}{864}$ b) $\frac{13\pi}{864}$ c) $\frac{17\pi}{864}$ d) none of these

Ans. b

Sol. $I = \int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{\pi}{2ab}$

$\frac{dI}{da} = \frac{-\pi}{2a^2b}$

$\Rightarrow \int_0^{\pi/2} \frac{\cos^2 x dx}{\left(a^2\cos^2 x + b^2\sin^2 x\right)} = \frac{\pi}{4a^3b}$

differentiating with respect to b

$\int_0^{\pi/2} \frac{\sin^2 x dx}{\left(a^2\cos^2 x + b^2\sin^2 x\right)} = \frac{\pi}{4a^3b}$

$\Rightarrow \int_0^{\pi/2} \frac{dx}{\left(a^2\cos^2 x + b^2\sin^2 x\right)} = \frac{\pi}{2ab}\left[\frac{1}{a^2} + \frac{1}{b^2}\right] = \frac{\pi}{24}\left[\frac{1}{4} + \frac{1}{9}\right] = \frac{13\pi}{864}$

41. If $\int \frac{dx}{\cos^3 x - \sin^3 x} = A\tan^{-1}(\sin x + \cos x) + B$ ln f(x) + C, then A is equal to

a) $\frac{2}{3}$ b) $\frac{2}{5}$ c) $-\frac{2}{3}$ d) none of these

Ans. a

Sol. $I = \int \frac{dx}{(\cos x - \sin x)\left(1 + \frac{\sin 2x}{2}\right)} = \int \frac{\cos x - \sin x}{(\cos x - \sin x)^2\left(1 + \frac{\sin 2x}{2}\right)} dx$

Put cosx + sinx = t

$I = \frac{2}{3}\tan^{-1}(\sin x + \cos x) - \frac{2}{3\sqrt{2}}\ln f(x) + c$

42. Solution of the differential equation $y\left(2x^4 + y\right)\frac{dy}{dx} = \left(1 - 4xy^2\right)x^2$ is given by

a) $3\left(x^2y\right)^2 + y^3 - x^3 = c$ b) $xy^2 + \frac{y^3}{3} - \frac{x^3}{3} + c = 0$

c) $\frac{2}{5}yx^5+\frac{y^3}{3}=\frac{x^3}{3}-\frac{4xy^3}{3}+c$ d) none of these

Ans. a

Sol. Given equation can be written as

$2x^2y(x^2dy + 2xy\,dx) + y^2\,dy - x^2\,dx = 0$

or $2x^2\,y\,d(x^2y) + y^2dy - x^2dx = 0$

Integrating, we get

$3(x^2y)^2 + y^3 - x^3 = c$

43. If $I=\int_0^{\pi}\frac{\cos x}{(x+2)^2}dx,$ then $\int_0^{\pi}\frac{\sin x}{x+1}dx$ is equal to

a) 21 b) $\frac{1}{\pi+2}-\frac{1}{2}-1$ c) 0 d) $\frac{1}{\pi+2}+\frac{1}{2}-1$

Ans. d

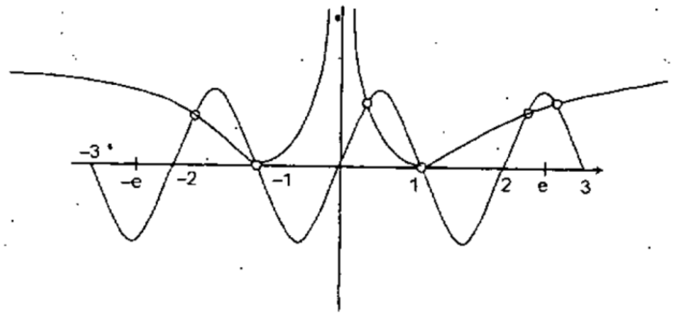
Sol. $I=\int_0^{\pi}\cos x\,d\left(-\frac{1}{x+2}\right)=\left[\frac{-\cos x}{x+2}\right]_0^{\pi}-\int_0^{\pi}\frac{\sin x}{x+2}dx$

$=\frac{1}{\pi+2}+\frac{1}{2}-\int_0^{\pi/2}\frac{\sin 2x}{x+1}dx$

44. The number of solutions of $\sin\pi x=\left|\log|x|\right|$ is

a) infinite b) 8 c) 6 d) 0

Ans. c



45. If $f(x)=\left|x^2+(k-1)|x|-k\right|$ is non differentiable at five real points, then k will lie in

a) $(-\infty,0)$ b) $(0,\infty)$ c) $(-\infty,0)-\{-1\}$ d) $(0,\infty)-\{1\}$

Ans. c

Sol. $f(x)=\left|x^2+(k-1)|x|-k\right|=\left|(|x|-1)(|x|+k)\right|$

Both roots of (x – 1) (x + k) = 0 should be positive and distinct

$\Rightarrow k\in(-\infty,0)-\{-1\}$

46. Let $g(x)=\int_a^x f(t)dt$ and $f(x)$ satisfies the following condition

$f(x+y)=f(x)+f(y)+2xy-1,\forall x,y\in R$ and $f'(0)=\sqrt{3+a-a^2}$, then the exhaustive set of values of x where $g(x)$ increases is

a) $\left(-\infty,-\frac{3}{2}\right)$ b) $\left(-\frac{3}{2},0\right)$ c) $(0,\infty)$ d) $(-\infty,\infty)$

Ans. d

Sol. $f(x) = x^2 + \left(\sqrt{3+a-a^2}\right)x + 1$

$g'(x) = f(x) > 0,\ \forall x \in R$

47. Number of positive continuous function $f(x)$ defined in $[0,1]$ for which $\int_0^1 f(x)dx = 1,\ \int_0^1 xf(x)dx = 2,\ \int_0^1 x^2 f(x)dx = 4,$ is

a) 1 b) 4 c) infinite d) none of these

Ans. d

Sol. Multiplying these three integral by 4, - 4, 1 and adding we get $\int_0^1 f(x)(x-2)^2\,dx = 0$.

Hence there does not exist any function satisfying these conditions.

48. Tangents are drawn at the point of intersection P of ellipse $x^2 + 2y^2 = 50$ and hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1,$ in the first quadrant. The area of the circle passing through the point P which cuts the intercept of 2 unit length each from these tangents, is

a) 2π b) $\sqrt{2}\pi$ c) 4π d) 6π

Ans. a

Sol. Given conic are confocal so they cut orthogonally.

49. Let $f(x) = x^3 + \frac{1}{x^3},\ x \neq 0$. If the intervals in which f(x) increases are $(-\infty, a]$ and $[b, \infty)$ then min(b – a) is equal to

a) 0 b) 2 c) 3 d) 4

Ans. b

Sol. Here $f'(x) = 3x^2 - \frac{3}{x^4} \geq 0 \Rightarrow x^6 - 1 \geq 0 \Rightarrow x \in (-\infty.1] \cup [1, \infty)$

$\therefore$ min (b – a) = min(b) – max(a) = 1 – (- 1) = 2

50. Let y = f(x), f : R $\rightarrow$ R be an odd differentiable function such that $f''(x) > 0$ and $g(\alpha, \beta) = \sin^8\alpha + \cos^8\beta + 2 - 4\sin^2\alpha\cos^2\beta$. If $f''(g(a,b)) = 0$, then $\sin^2 a + \sin^2 b$ is equal to

a) 0 b) 1 c) 2 d) 3

Ans. b

Sol. $f''(x)$ is odd function $\Rightarrow g(\alpha, \beta) = 0$

$\Rightarrow \left(\sin^4\alpha - 1\right)^2 + \left(\cos^4\beta - 1\right)^2 + 2\left(\sin^2\alpha - \cos^2\beta\right)^2 = 0$

$\Rightarrow \sin^2\alpha + \sin^2\beta = 1$

Differential Calculus

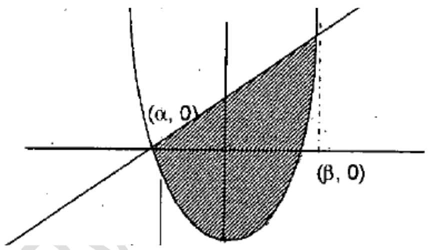
Integer Answer Type

1. If the least value of the area bounded by the line y = mx + 1 and the parabola y = x^2 + 2x – 3 is α where m is a parameter then the value of $\frac{6\alpha}{32}$ is

Ans. 2

$A=\int_{\alpha}^{\beta}\left(y_1-y_2\right)dx$ where α, β are the roots of x^2 + 2x – 3 = mx + 1, on solving we will get $\frac{1}{6}\left(m^2-5m+20\right)^{3/2}$. Hence $\alpha=\frac{32}{3}$

$\Rightarrow\frac{6\alpha}{32}=2$



2. The value of constant c such that the straight line joining the points (0, 3) and (5, -2) is tangent to the curve $y=\frac{c}{x+1}$

Ans. 4

Equation of line joining (0, 3) and (5, -2) is x + y = 3

Now it touches the curve $y=\frac{c}{x+1}$ at (x_1, y_1)

Hence $\left(\frac{dy}{dx}\right)_{(x_1,\, y_1)}=1\Rightarrow\left(x_1+1\right)^2=c,\ \left(x_1y_1\right)$ lie on the line. Substituting we get

$\pm\sqrt{c}=2\Rightarrow c=4$

3. Let $f\left(x\right)=x^2+3x-3,\ x\geq 0$, if n points x_1, x_2 …. X_n are so chosen on the x-axis such that

i) $\frac{1}{n}\sum_{i=1}^{n}f^{-1}\left(x_i\right)=f\left(\frac{1}{n}\sum_{i=1}^{n}\left(x_i\right)\right)$ ii) $\sum_{i=1}^{n}f^{-1}\left(x_i\right)=\sum_{i=1}^{n}\left(x_i\right)$

where f^{-1} denote inverse of f. Find A.M. of x_i is

Ans. 1

$f\left(x\right)=x$

$x^2+3x-3=x\Rightarrow x=1$

4. At how many points in the interval (0, 2), $f\left(x\right)=x^2\left[2x\right]-x\left[x^2\right]$ is discontinuous (where $\left[\cdot\right]$ denotes the greatest integer function)

Ans. 4

Sol. Conceptual

5. $f\left(x\right)=\lim\limits_{n\to\infty}\lim\limits_{\alpha\to1^+}\dfrac{\alpha^n\left|\sin x\right|+\left|\cos x\right|\alpha^{-n}}{\alpha^n+\alpha^{-n}}$ then $f\left(\dfrac{\pi}{2}\right)$ is

Ans. 1

$f\left(x\right)=\left|\sin x\right|$

$f\left(\dfrac{\pi}{2}\right)=1$

6. If $\lim\limits_{x\to0}\dfrac{x^n-\sin x^n}{x-\sin^n x}$ exists and has a non-zero value, then n =

Ans. 1

By putting n = 1, the result can easily be obtained.

7. If $\int\dfrac{1-x^7}{x\left(1+x^7\right)}dx=a\ln\left|x\right|+b\ln\left|x^7+1\right|+c$, then $\left|a+7b\right|$ =

Ans. 1

Differentiating both sides, we get

$$\frac{1-x^7}{x\left(1+x^7\right)}=\frac{a}{x}+b\cdot\frac{7x^6}{1+x^7}\Rightarrow a=1,\ a+7b=-1$$

8. If $\int_0^\infty\left[2e^{-x}\right]dx=\ln k\left(\left[\cdot\right]\ dentoe\ the\ g.i.f\right)$ then k =

Ans. 2

$$\int_0^\infty\left[2e^{-x}\right]dx=\int_0^{\ln2}\left[2e^{-x}\right]dx+\int_{\ln2}^\infty\left[2e^{-x}\right]dx=\int_0^{\ln2}\left[2e^{-x}\right]dx+0=\ln2$$

9. The shortest distance between $\left(1-x\right)^2+\left(x-y\right)^2+\left(y-z\right)^2+z^2=\dfrac{1}{4}$ and 4x + 2y + 4z + 7 = 0 in 3 – dimensional coordinate system is equal to

Ans. 2

Let a = 1 – x

b = x – y

c = y – z

d = z

then a + b + c + d = 1 and $a^2+b^2+c^2+d^2=\dfrac{1}{4}$

$$\Rightarrow\left(a-b\right)^2+\left(a-c\right)^2+\left(a-d\right)^2+\left(b-c\right)^2+\left(b-d\right)^2+\left(c-d\right)^2=0$$

$$\Rightarrow a=b=c=d$$

$$\therefore x=\frac{3}{4},y=\frac{1}{2},z=\frac{1}{4}$$

So the distance from the point $\left(\dfrac{3}{4},\dfrac{1}{2},\dfrac{1}{4}\right)$ from the plane 4x + 2y + 4z + 7 = 0 is

$$\frac{3+1+1+7}{6}=2$$