Definite, Indefinite Integration & Areas

Single Correct Answer Type

1. If a circle of radius r is touching the lines $x^2 - 4xy + y^2 = 0$ in the first quadrant at points A and B, then area of triangle OAB (O being the origin) is

(A) $\frac{3\sqrt{3}r^2}{4}$ (B) $\frac{\sqrt{3}r^2}{4}$ (C) $\frac{3r^2}{4}$ (D) r^2

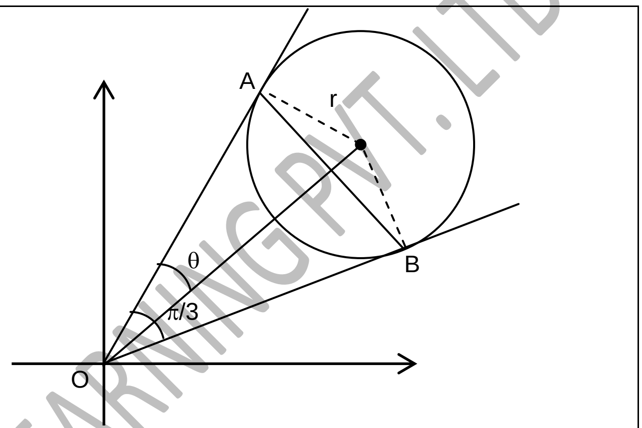
Key. A

Sol.

Here $\tan 2\theta = \frac{2\sqrt{4-1}}{2} = \sqrt{3} \Rightarrow \theta = \frac{\pi}{6}$

Area of $\Delta OAB = \frac{1}{2}(r\cot\theta)^2(\sin 2\theta)$

$= \frac{1}{2}(r\sqrt{3})^2 \frac{\sqrt{3}}{2}$.



2. ABCD is a quadrilateral with side lengths AB = 4, BC = 10, CD = 6 and AD = 6, and diagonal BD = 8 units. If the incircles of triangles ABD and BCD touch BD at P and Q respectively, then area of quadrilateral C_1PC_2Q (where C_1 and C_2 are incentres of triangle ABD and BCD respectively), is

(A) $3+\frac{\sqrt{15}}{2}$ sq. units (B) 3 sq. units

(C) $\frac{\sqrt{15}}{6}$ sq. units (D) 4 sq. units

Key. A

Sol.

In triangle ABD, we have $BP = \frac{6+4+8}{2} - 6 = 3$.

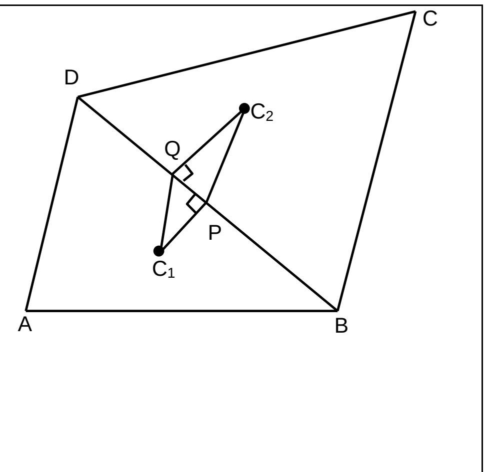
In triangle BCD, we have

$DQ = \frac{10+6+8}{2} - 10 = 2$.

$\Rightarrow PQ = 8 - (3 + 2) = 3$

$\Rightarrow$ area of trapezium $C_1PC_2Q = \frac{1}{2}(r_1 + r_2)\cdot PQ$,

where $r_1 = \frac{\sqrt{9\times3\times5\times1}}{9} = \frac{\sqrt{15}}{3}$ and $r_2 = \frac{\sqrt{12\times2\times6\times4}}{12} = 2$.



$\Rightarrow$ area of quadrilateral $C_1PC_2Q = \frac{1}{2}\left(2+\frac{\sqrt{15}}{3}\right)\times 3 = 3+\frac{\sqrt{15}}{2}$ sq. units.

3. The area of the region whose boundaries are defined by the curves y = 2 cos x, y = 3 tan x and the y-axis, is

(A) $1+3\ln\left(\frac{2}{\sqrt{3}}\right)$ (B) $1 + \frac{3}{2}\ln 3 - 3\ln 2$

(C) $1 + \frac{3}{2}\ln 3 - \ln 2$ (D) $\ln 3 - \ln 2$

Key. B

Sol. Solving 2 cos x = 3 tan x we get, $2 - 2\sin^2 x = 3\sin x$

$\Rightarrow \sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}$.

Required area = $\int_0^{\pi/6}(2\cos x - 3\tan x)dx = 2\sin x - 3\ln\sec x\Big|_0^{\pi/6} = 1-3\ln 2+\frac{3}{2}\ln 3$.

4. The area of the region bounded by the curve $y = x^2$ and $y = \sec^{-1}[-\sin^2 x]$, (where [.] denotes the greatest integer function), is

(A) $\pi\sqrt{\pi}$ (B) $\frac{4}{3}\pi\sqrt{\pi}$ (C) $\frac{2}{3}\pi\sqrt{\pi}$ (D) $\frac{1}{3}\pi\sqrt{\pi}$

Key. B

Sol. $[-\sin^2 x] = 0$ or -1 but $\sec^{-1}(0)$ is not defined.

$\Rightarrow \sec^{-1}[-\sin^2 x] = \sec^{-1}(-1) = \pi$.

The required area = $\int_{-\sqrt{\pi}}^{\sqrt{\pi}}(\pi - x^2)dx = \frac{4}{3}\pi\sqrt{\pi}$.

5. The range of the function $f(x) = \int_1^x |t|\,dt,\ x \in \left[\frac{-1}{2},\frac{1}{2}\right]$ is

(A) $\left[\frac{3}{8},\frac{5}{8}\right]$ (B) $\left[\frac{-5}{8},\frac{3}{8}\right]$ (C) $\left[\frac{-3}{8},\frac{5}{8}\right]$ (D) $\left[\frac{-5}{8},\frac{-3}{8}\right]$

Key. D

Sol. If $0 \le x \le \frac{1}{2} \Rightarrow f(x) = \int_1^x t\,dt = \frac{x^2-1}{2}$

If $-\frac{1}{2} \le x \le 0 \Rightarrow f(x) = \int_1^0 t\,dt - \int_0^x t\,dt = -\left(\frac{x^2+1}{2}\right)$

$\Rightarrow f^1(x) > 0 \forall x \in \left[\frac{-1}{2},\frac{1}{2}\right]$

$\therefore$ *Range of* $f(x)$ *is* $\left[f\left(\frac{-1}{2}\right), f\left(\frac{1}{2}\right)\right] \Rightarrow \left[\frac{-5}{8},\frac{-3}{8}\right]$

6. $\int_0^1 Tan^{-1}(1-x+x^2)\,dx =$ ________

(A) $\ln 2$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{2}$ (D) $\frac{\pi}{2}-\ln 2$

Key. A

Sol. $\int_0^1 \tan^{-1}\left(1-x+x^2\right)dx = \int_0^1\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{x+(1-x)}{1-x(1-x)}\right)\right)dx$

$$= \frac{\pi}{2}-\int_0^1\left(\tan^{-1}(1-x)+\tan^{-1}x\right)dx$$

$$\Rightarrow \frac{\pi}{2}-2\int_0^1 \tan^{-1}(x)\,dx = \ln 2$$

7. $\int_0^{\frac{\pi}{4}}\left(\pi x-4x^2\right)\ln(1+Tanx)\,dx =$ ________

(A) $\frac{\pi^3}{192}$ (B) $\frac{\pi^3}{192}\ln 2$ (C) $\frac{\pi^2}{96}$ (D) $\frac{\pi^2}{96}\ln 2$

Key. B

Sol. Let $I = \int_0^{\frac{\pi}{4}} 4x\left(\frac{\pi}{4}-x\right)\ln(1+\tan x)\,dx = \int_0^{\frac{\pi}{4}} 4\left(\frac{\pi}{4}-x\right)x\ln\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx$

$$= 4\ln 2\int_0^{\frac{\pi}{4}} x\left(\frac{\pi}{4}-x\right)dx - I$$

$$\Rightarrow I = 2\ln 2\int_0^{\frac{\pi}{4}} x\left(\frac{\pi}{4}-x\right)dx$$

$$= \frac{\pi^3}{192}\ln 2$$

8. $\int_0^{2\pi} x\ln\left(\frac{3+\cos x}{3-\cos x}\right)dx =$ _____

(A) $\frac{\pi}{2}\ln 3$ (B) $\frac{\pi}{6}\ln 3$ (C) $\frac{\pi}{12}\ln 3$ (D) 0

Key. D

Sol.

$$I = \int_0^{2\pi}(2\pi-x)\ln\left(\frac{3+\cos x}{3-\cos x}dx\right) \Rightarrow 2I = 2\pi\int_0^{2\pi}\ln\left(\frac{3+\cos x}{3-\cos x}\right)dx$$

$$\Rightarrow I = 2\pi\int_0^{\pi}\ln\left(\frac{3+\cos x}{3-\cos x}\right)dx = 2\pi\int_0^{\pi}\ln\left(\frac{3-\cos x}{3+\cos x}\right)dx = -I \Rightarrow I = 0$$

9. If a point P moves such that its distance from line y = $\sqrt{3}$ x – 7 is same as its distance from $\left(2\sqrt{3},\ -1\right)$, then area bounded by locus of P and the coordinate axes is (in sq. units)

(A) $\frac{\sqrt{3}}{2}$ (B) $2\sqrt{3}$ (C) 6 (D) $\frac{3\sqrt{3}}{2}$

Key. A

Sol. As point lise on the line. Locus of the point is straight line perpendicular to given line passing through $\left(2\sqrt{3},\ -1\right)$ i.e. $\frac{x}{\sqrt{3}}+y=1$

$\Rightarrow$ area of triangle = $\frac{\sqrt{3}\times 1}{2}$.

10. If A$\left(\frac{3}{\sqrt{2}},\ \sqrt{2}\right)$, B$\left(-\frac{3}{\sqrt{2}},\ \sqrt{2}\right)$, C$\left(-\frac{3}{\sqrt{2}},\ -\sqrt{2}\right)$ and D (3 cos θ, 2 sin θ) are four points, then the value of θ for which the area of quadrilateral ABCD is maximum, $\left(\frac{3\pi}{2}\le\theta\le 2\pi\right)$ is (in sq. units)

(A) $2\pi-\sin^{-1}\frac{1}{3}$ (B) $\frac{7\pi}{4}$ (C) $2\pi-\cos^{-1}\frac{3}{\sqrt{85}}$ (D) $\frac{\pi}{4}$

Key. B

Sol. Area of quadrilateral ABCD is maximum when area of ACD is maximum
$\Rightarrow$ distance of D from AC is maximum i.e. $(\cos\theta-\sin\theta)$ is maximum.

$\Rightarrow$ $\sqrt{2}\cos\left(\theta+\frac{\pi}{4}\right)$ is maximum

$\Rightarrow \theta = \frac{7\pi}{4}$.

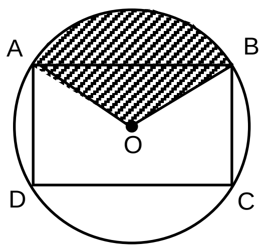
11. A square ABCD is inscribed in a circle of radius 4. A point P moves inside the circle such that d (P, AB) ≤ min (d (P, BC), d (P, CD), d (P, DA)) where d (P, AB) is the distance of a point P from line AB. The area of region covered by moving point P is (in sq. units)

(A) 4π (B) 8π (C) $8\pi - 16$ (D) $3\pi-4$

Key. A

Sol.

Shaded area is the required region = $\frac{\pi r^2}{4}=4\pi$.



12. Let $I_1 = \int_{\sec^2 z}^{2-\tan^2 z} x f\left(x(3-x)\right)dx$ and let $I_2 = \int_{\sec^2 z}^{2-\tan^2 z} f\left(x(3-x)\right) dx$

where 'f' is a continuous function and 'z' is any real number, then $\frac{I_1}{I_2} =$

(A) $\frac{3}{2}$ (B) $\frac{1}{2}$ (C) 1 (D) $\frac{2}{3}$

Key. A

Sol. Conceptual

13. If f, g, h are continuous functions on [0, a] such that

$f(a-x) = f(x), g(a-x) = -g(x), \quad 3h(x) - 4h(a-x) = 5$ then $\int_0^a f(x)g(x)h(x)dx =$

a) 0 b) a c) a/2 d) 2a

Key. A

Sol. Conceptual

14. The value of $\int_{-2}^{2} \frac{\sin^2 x}{\left[\frac{x}{\pi}\right] + \frac{1}{2}} dx,$ where [x] is the greatest integer less than or equal to x, is

a) 1 b) 0 c) $4 - \sin 4$ d) $4 + \sin 4$

Key. B

Sol. Conceptual

15. $\int_0^{4/\pi} (3x^2 . \sin\frac{1}{x} - x.\cos\frac{1}{x})dx =$

a) $\frac{8\sqrt{2}}{\pi^3}$ b) $\frac{24\sqrt{2}}{\pi^3}$ c) $\frac{32\sqrt{2}}{\pi^3}$ d) $\frac{32\sqrt{2}}{\pi}$

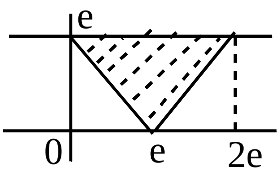
Key. C

Sol. Conceptual

16. The area bounded by the curves $f(x) = \begin{cases} x^{\frac{1}{\ln x}}, & x \neq 1 \\ e, & x = 1 \end{cases}$ and $y = |x-e|$ is

(A) $\frac{e^2}{2}$ (B) e^2 (C) $2e^2$ (D) 1

Key. B



Sol. $f(x) = \begin{cases} x^{\log_x^e} = e, & x \neq 1 \\ e, & x = 1 \end{cases}$

17. The area of the region bounded by the point $P(x, y)$ satisfying $\log_x \log_y x > 0$ and $\frac{1}{2} < x < 2$ is

(A) $\frac{3}{4}$ (B) 1 (C) 2 (D) $\frac{7}{8}$

Key. D

Sol. (i) $\frac{1}{2} < x < 1$

a) $0 < y < 1 \Rightarrow x > y \wedge x < 1$

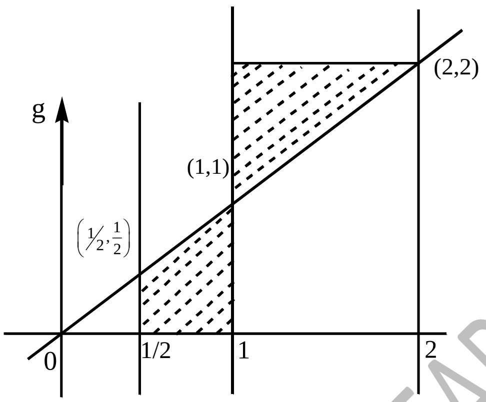
b) $y > 1 \Rightarrow x < y \wedge x > 1$

not possible

(ii) $1 < x < 2$

a) $0 < y < 1$

$x > y \wedge x < 1$



Not possible

b) y > 1

$x < y \wedge x > 1$

Area $= \frac{1}{2} - \frac{1}{2}.\frac{1}{2}.\frac{1}{2} + \frac{1}{2}$

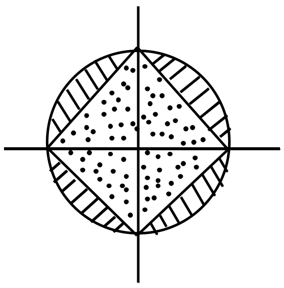
$= 1 - \frac{1}{8} = \frac{7}{8}$

18. Area of the region defined by $\left||x| - |y|\right| \geq 1$ and $x^2 + y^2 \leq 1$ is

(A) 1 (B) 2 (C) $\pi - 1$ (D) $2\pi - 1$

Key. C

Sol. $-1 \leq \left||x| - |y|\right| \leq 1$



$\left||x| - |y|\right| \leq 1 \wedge \left||x| - |y|\right| \geq -1$

Required area $= \pi(1)^2 = \pi$

19. If $I_n = \int \tan^n x dx$, then $I_0 + I_1 + 2(I_2 + ... + I_8) + I_9 + I_{10}$, is equal to

1) $\sum_{n=1}^{9} \frac{\tan^n x}{n}$ 2) $1 + \sum_{n=1}^{8} \frac{\tan^n x}{n}$ 3) $\sum_{n=1}^{9} \frac{\tan^n x}{n+1}$ 4) $\sum_{n=2}^{10} \frac{\tan^n x}{n+1}$

Key. 1

Sol. We have $I_n = \int \tan^n x\, dx = \int \tan^{n-2} x(\sec^2 x - 1) dx = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$

i.e. $I_{n-2} + I_n = \frac{\tan^{n-1} x}{n-1} (n \geq 2)$

Thus, we have

$I_0 + I_1 + 2(I_2 + ... + I_8) + I_9 + I_{10}$

$= (I_0 + I_2) + (I_1 + I_3) + (I_2 + I_4) + (I_3 + I_5) + (I_4 + I_6) + (I_5 + I_7) + (I_6 + I_8) + (I_7 + I_9) + (I_8 + I_{10})$

.

$= \sum_{n=2}^{10} \frac{\tan^{n-1}}{n-1} = \sum_{n=1}^{9} \frac{\tan^n x}{n}$

20. $\int e^{\sin x} \left(\frac{x \cos^3 x - \sin x}{\cos^2 x} \right) dx$, is equal to

1) $e^{\sin x}(\tan x + x) + C$ 2) $e^{\sin x}(x - \sec x) + C$

3) $e^{\sin x}(\sec x + \tan x) + C$ 4) none of these

Key. 2

Sol. We have

$I = \int e^{\sin x} \left(\frac{x \cos^3 x - \sin x}{\cos^2 x} \right) dx$

$= \int x e^{\sin x} \cos x dx - \int e^{\sin x} (\sec x \tan x) dx$

$= \left[x e^{\sin x} - \int e^{\sin x} dx \right] - \left[e^{\sin x} \sec x - \int e^{\sin x} dx \right]$

$= e^{\sin x}(x - \sec x) + C$

21. If $\int \frac{dx}{x\sqrt{1-x^3}} = a\ln\left(\frac{\sqrt{1-x^3}+b}{\sqrt{1-x^3}+1}\right)+k$, then

1) b=1, a =1

2) $b=-1,\ a=-\frac{1}{3}$

3) $b=1, a=-\frac{2}{3}$

4) None of these

Key. 2

Sol. $I=\int \frac{dx}{x\sqrt{1-x^3}} = \int \frac{x^2dx}{x^3\sqrt{1-x^3}}$

Put $1-x^3=t^2 \Rightarrow -3x^2dx = 2tdt$

22. $\frac{\sin^3 x}{(\cos^4 x+3\cos^2 x+1)\tan^{-1}(\sec x+\cos x)}dx =$

1) $\tan^{-1}(\sec x+\cos x)+c$

2) $\log\left|\tan^{-1}(\sec x+\cos x)\right|+c$

3) $\frac{1}{(\sec x+\cos^2 x)^2}+c$

4) $\log|\sec x+\cos x|+c$

Key. 2

Sol. Put $Tan^{-1}(\sec x+\cos x)=f(x)$

$f^1(x)=\frac{\sin^3 x}{\cos^4 x+3\cos^2 x+1}$

$\therefore \int \frac{f^1(x)}{f(x)} = \log|f(x)|+c$

23. $\int \left(\sqrt{\sin x}+\sqrt{\cos x}\right)^{-4} dx =$

1) $\frac{2}{\left(\sqrt{\tan x}+1\right)^2}\left[\frac{1}{3\left(\sqrt{\tan x}+1\right)}-\frac{1}{2}\right]+c$

2) $\frac{2}{\left(\sqrt{\tan x}+1\right)^2}\left[\frac{1}{3\left(\sqrt{\tan x}+1\right)}+\frac{1}{2}\right]+c$

3) $\frac{2}{\left(\sqrt{\tan x}+1\right)^2}\left[\frac{1}{\sqrt{3}\left(\sqrt{\tan x}+1\right)}+\frac{1}{2}\right]+c$

4) $\frac{2}{\left(\sqrt{\tan x}+1\right)^2}\left[\frac{1}{\sqrt{3}\left(\sqrt{\tan x}+1\right)}-\frac{1}{2}\right]+c$

Key. 1

Sol.

$$I=\int\left(\sqrt{\sin x}+\sqrt{\cos x}\right)^{-4}dx=\int\frac{dx}{\left[\sqrt{\cos x}\left(\sqrt{\tan x}+1\right)\right]^4}=\int\frac{1}{\cos^2 x\left(\sqrt{\tan x}+1\right)^4}$$

$$=\int\frac{\sec^2 x dx}{\left(\sqrt{\tan x}+1\right)^4}$$

$$Put\sqrt{\tan x}+1=y\Rightarrow\frac{1}{2\sqrt{\tan x}}\sec^2 x dx=dy\Rightarrow\sec^2 x dx=2\sqrt{\tan x}dy=2(y-1)dy$$

$$I=\int\frac{1}{y^4}.2(y-1)dy=2\int\left(\frac{1}{y^3}-\frac{1}{y^4}\right)dy=2\left[\frac{-1}{2y^2}+\frac{1}{3y^3}\right]+c=\frac{2}{y^2}\left[\frac{1}{3y}-\frac{1}{2}\right]$$

$$=\frac{2}{\left(\sqrt{\tan x}+1\right)^2}\left[\frac{1}{3\left(\sqrt{\tan x}+1\right)}-\frac{1}{2}\right]+c$$

24. $\int\sqrt{\tan x}dx=$

1) $\frac{1}{2\sqrt{2}}\left[2\tan^{-1}\left(\frac{\tan x-1}{\sqrt{2\tan x}}\right)+\log\left[\frac{\tan x-\sqrt{2\tan x}+1}{\tan x+\sqrt{2\tan x}+1}\right]\right]+c$

2) $\frac{1}{2\sqrt{2}}\left[2\tan^{-1}\left(\frac{\tan x-1}{2\tan x}\right)+\log\left[\frac{\tan x-\sqrt{2\tan x}+1}{\tan x+\sqrt{2\tan x}+1}\right]\right]+c$

3) $\frac{1}{2\sqrt{2}}\left[2\tan^{-1}\left(\frac{\tan x+1}{\sqrt{2}\tan x}\right)+\log\left[\frac{\tan^2 x-2\tan x+1}{\tan x+2\tan x+1}\right]\right]+c$

4) $\frac{1}{5\sqrt{2}}\left[2\tan^{-1}\left(\frac{\tan x+1}{\sqrt{2}\tan x}\right)+\log\left[\frac{\tan^2 x-2\tan x-1}{\tan x+2\tan x+1}\right]\right]+c$

Key. 1

Sol. $\frac{1}{2}\int 2\sqrt{\tan x}dx=\frac{1}{2}\int\left(\sqrt{\tan x}+\sqrt{\cot x}\right)dx+\frac{1}{2}\int\left(\sqrt{\tan x}-\sqrt{\cot x}\right)dx$

25. $\int\sqrt{x+\sqrt{x^2+2}}\,dx=$

1) $\left(\frac{x+\sqrt{x^2+2}}{3}\right)^{\frac{3}{2}}-2\left(x+\sqrt{x^2+2}\right)^{-\frac{1}{2}}+c$

2) $\frac{1}{3}\left(x+\sqrt{x^2+2}\right)^{\frac{3}{2}}-2\left(x+\sqrt{x^2+2}\right)^{\frac{-1}{2}}+c$

3) $\frac{2}{7}\left(x+\sqrt{x^2+2}\right)^{\frac{7}{2}}-2\left(x+\sqrt{x^2+2}\right)+c$

4) $\frac{2}{7}\left(x+\sqrt{x^2+2}\right)^{\frac{7}{2}}+2\left(x+\sqrt{x^2+2}\right)+c$

Key. 2

Sol. Put $x+\sqrt{x^2+2}=t$

26. If $\int \frac{2\cos x-\sin x+\lambda}{\cos x+\sin x-2}dx = A\ln\left|\cos x+\sin x-2\right|+Bx+C$ Then the ordered triplet A,B,λ is

1) $\left(\frac{1}{2},\frac{3}{2},-1\right)$ 2) $\left(\frac{3}{2},\frac{1}{2},-1\right)$ 3) $\left(\frac{1}{2},-1,-\frac{3}{2}\right)$ 4) $\left(\frac{3}{2},-1,\frac{1}{2}\right)$

Key. 2

Sol.

$$\frac{d}{dx}\left(A\,ln\left|\cos x+\sin x-2\right|+Bx+C\right)$$

$$=A\frac{\cos x-\sin x}{\cos x+\sin x-2}+B=\frac{A\cos x-A\sin x+B\cos x+B\sin x-2B}{\cos x+\sin x-2}$$

$\therefore \quad 2=A+B, -1=-A+B, \lambda=-2B$

$\therefore \quad A=3/2, B=1/2, \lambda=-1$

27. $\int \frac{x^8+4}{x^4-2x^2+2}dx=$

1) $\frac{x^5}{5}-\frac{2x^3}{3}+2x+C$ 2) $\frac{x^5}{5}-\frac{2x^3}{3}-2x+C$

3) $\frac{x^5}{5}+\frac{2x^3}{3}-2x+C$ 4) $\frac{x^5}{5}+\frac{2x^3}{3}+2x+C$

Key. 4

Sol.

$$\int \frac{\left(x^8+4+4x^4\right)-4x^4}{x^4-2x^2+2}dx=\int \frac{\left(x^4+2\right)^2-\left(2x^2\right)^2}{\left(x^4-2x^2+2\right)}dx$$

$$=\int \frac{\left(x^4+2-2x^2\right)\left(x^4+2+2x^2\right)}{\left(x^4-2x^2+2\right)}dx=\frac{x^5}{5}+\frac{2x^3}{3}+2x+C$$

28. $\int \frac{\cos^4 x\, dx}{\sin^3 x\left(\sin^5 x+\cos^5 x\right)^{3/5}} = -\frac{1}{2}\left(1+\cot^A x\right)^B + C$

1) 5 2) $\frac{2}{5}$ 3) 2 4) 1

Key. 3

Sol.

$$I = \int \frac{\cos^4 x dx}{\sin^3 x\left(\sin^5 x+\cos^5 x\right)^{3/5}}$$

$$= \int \frac{\cos^4 x dx}{\sin^6 x\left(1+\cot^5 x\right)^{3/5}}$$

$$= \int \frac{\cot^4 x \operatorname{cosec}^2 x\, dx}{\left(1+\cot^5 x\right)^{3/5}} \; put\; 1+\cot^5 x = t \qquad 5\cot^4 x \operatorname{cosec}^2 x\, dx = -dt$$

29. If $f(x)=\sqrt{x}, g(x)=e^{x-1}$, and $\int fog(x)dx = A\; fog(x)+B\tan^{-1}\left(fog(x)\right)+C$, then $A+B$ is equal to

1) 1 2) 2 3) 3 4) 0

Key. 4

Sol.

$$fog(x) = \sqrt{e^x-1}$$

$$\therefore \qquad I = \int \sqrt{e^x-1}\, dx = \int \frac{2t^2}{t^2+1}dt \quad \left\{where\; \sqrt{e^x-1}=t\right\}$$

$$= 2t-2\tan^{-1}t+C = 2\sqrt{e^x-1}-2\tan^{-1}\left(\sqrt{e^x-1}\right)+C = 2fog(x)-2\tan^{-1}\left(fog(x)\right)+C$$

$$\therefore \qquad A+B = 2+(-2) = 0$$

30. If $\int \sin^{-1}x\cos^{-1}x dx = f^{-1}\left[\frac{\pi}{2}x - xf^{-1}(x) - 2\sqrt{1-x^2}\right]+\frac{\pi}{2}\sqrt{1-x^2}+2x+C$, then f(x) is equal to

1) sin *3x* 2) sin *2x* 3) sin *x* 4) sin *4x*

Key. 3

Sol.

$$\int \sin^{-1}x\cos^{-1}x dx = \int\left[\frac{\pi}{2}\sin^{-1}x-\left(\sin^{-1}x\right)^2\right]dx$$

$$\Rightarrow \qquad \frac{\pi}{2}\left(x\sin^{-1}x+\sqrt{1-x^2}\right)-\left(x\left(\sin^{-1}x\right)^2+\sin^{-1}x\sqrt{1-x^2}-x\right)+c \; By\; parts$$

$$\Rightarrow \qquad \sin^{-1}x\left[\frac{\pi}{2}x-x\sin^{-1}x-2\sqrt{1-x^2}\right]+\frac{\pi}{2}\sqrt{1-x^2}+2x+c$$

$$\therefore \qquad f^{-1}(x)=\sin^{-1}x, f(x)=\sin x$$

31. Let $F(x)=e^{\sin^{-1}x}\left(1-\frac{x}{\sqrt{1-x^2}}\right)dx$ and $F(0)=1,$ If $F(1/2)=\frac{k\sqrt{3}e^{x/6}}{\pi}$, then k =

1) $\frac{\pi}{4}$ 2) $\frac{\pi}{6}$ 3) $\frac{\pi}{2}$ 4) $\frac{\pi}{3}$

Key. 3

Sol.

$$F(x)=\int e^{\sin^{-1}x}\left(1-\frac{x}{\sqrt{1-x^2}}\right)dx=\int e^{\sin^{-1}x}\left(\frac{1}{\sqrt{1-x^2}}\sqrt{1-x^2}-\frac{x}{\sqrt{1-x^2}}\right)dx$$

$$F(x)=e^{\sin^{-1}x}\sqrt{1-x^2}+c$$

$$F(0)=1+c \quad \Rightarrow \quad c=0 \quad (\because F(0)=1)$$

$$F(1/2)=e^{\pi/6}.\frac{\sqrt{3}}{2}=\frac{k\sqrt{3}}{\pi}e^{\pi/6}$$

$$\therefore \quad k=\frac{\pi}{2}$$

32. $\int\left(\frac{x-1}{x+1}\right)\frac{dx}{\sqrt{x^3+x^2+x}}=2\tan^{-1}\sqrt{f(x)}+C$ then find $f(x)$.

1) $x+\frac{1}{x}+1$ 2) $x+\frac{1}{x}+2$ 3) $x-\frac{1}{x}+1$ 4) $x-\frac{1}{x}-2$

Key. 1

Sol.

$$I=\int\left(\frac{x-1}{x+1}\right)\frac{dx}{x\sqrt{x+1+\frac{1}{x}}}$$

$$so,\ I=\int\frac{(x-1)dx}{(x+1)x\sqrt{x+1+\frac{1}{x}}}=\int\frac{\left(1-\frac{1}{x}\right)\left(1+\frac{1}{x}\right)dx}{(x+1)\left(1+\frac{1}{x}\right)\sqrt{x+1+\frac{1}{x}}}=\int\frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}+2\right)\sqrt{x+\frac{1}{x}+1}}$$

$$Put\ x+1+\frac{1}{x}=t^2$$

$$\left(1-\frac{1}{x^2}\right)dx=2t\,dt$$

$= \int \frac{2t\,dt}{(t^2+1)t} = 2\tan^{-1}t + c = 2\tan^{-1}\left(\sqrt{x+\frac{1}{x}+1}\right)+c$

$x+\frac{1}{x}+1$ Ans.

33. $\int 2\cos 2x$ ln $(\tan x)\,dx$, is equal to

1) $\sin 2x\ln(\tan x) - 2x + C$
2) $\sin 2x\ln(\tan x) + 2x + C$
3) $\sin x\ln(\tan x) - x + C$
4) none of these

Key. 1

Sol. We have $I = \int 2\cos 2x\ln(\tan x)\,dx = \sin 2x\ln(\tan x) - \int \sin 2x.\frac{\sec^2 x}{\tan x}dx$

$= \sin 2x\ln(\tan x) - \int 2dx = \sin 2x\ln(\tan x) - 2x + C$

34. $\int \frac{\cos x + x\sin x}{x(x+\cos x)}dx =$

1) $\log\left|\frac{x}{x+\cos x}\right| + c$
2) $-\log\left|\frac{x}{x+\cos x}\right| + c$
3) $\log\left|\frac{x+\cos x}{2x}\right| + c$
4) $-\log\left|\frac{x+\cos x}{2x}\right| + c$

Key. 1

Sol. $\int \frac{x+\cos x + x\sin x - x}{x(x+\cos x)}dx$

35. $\int [1+\tan x\tan(x+\alpha)]\,dx$, is equal to

1) $\tan\alpha\ln\left|\frac{\sin(x+\alpha)}{\sin x}\right| + C$
2) $\cot\alpha\ln\left|\frac{\sin(x+\alpha)}{\sin x}\right| + C$
3) $\cot\alpha\ln\left|\frac{\sin x}{\sin(x+\alpha)}\right| + C$
4) $\cot\alpha\ln\left|\frac{\cos x}{\cos(x+\alpha)}\right| + C$

Key. 4

Sol. We have $\tan\alpha = \tan(x+\alpha-x) = \frac{\tan(x+\alpha)-\tan x}{1+\tan x\tan(x+\alpha)}$

Then, we have

$\int [1+\tan x\tan(x+\alpha)]\,dx = \int \cot\alpha[\tan(x+\alpha)-\tan x]\,dx$

$= \cot\alpha\left[-\ln\left|\cos(x+\alpha)\right| + \ln\left|\cos x\right|\right] + C$

$= \cot\alpha \ln\left|\dfrac{\cos x}{\cos(x+\alpha)}\right| + C$

36. Let $x^2 \neq n\pi - 1, n \in N$. Then, the value of $\int x\sqrt{\dfrac{2\sin(x^2+1) - \sin 2(x^2+1)}{2\sin(x^2+1) + \sin 2(x^2+1)}}$ dx is equal to

1) $\log\left|\dfrac{1}{2}\sec(x^2+1)\right| + C$

2) $\log\left|\sec\left(\dfrac{x^2+1}{2}\right)\right| + C$

3) $\dfrac{1}{2}\log\left|\sec(x^2+1)\right| + C$

4) None of these

Key. 2

Sol. We have, $\int x\sqrt{\dfrac{2\sin(x^2+1) - \sin 2(x^2+1)}{2\sin(x^2+1) + \sin 2(x^2+1)}}dx$

$$= \int x\sqrt{\frac{2\sin(x^2+1) - 2\sin(x^2+1)\cos(x^2+1)}{2\sin(x^2+1) + 2\sin(x^2+1)\cos(x^2+1)}}dx$$

$$= \int x\sqrt{\frac{1-\cos(x^2+1)}{1+\cos(x^2+1)}}dx$$

$$= \int x\tan\left(\frac{x^2+1}{2}\right)dx$$

$$= \int \tan\left(\frac{x^2+1}{2}\right)d\left(\frac{x^2+1}{2}\right)$$

$$= \log\left|\sec\left(\frac{x^2+1}{2}\right)\right| + C$$

37. $\displaystyle\int \frac{dx}{\cos(2x)\cos(4x)}$ is equal to

1) $\dfrac{1}{2\sqrt{2}}\log\left|\dfrac{1+\sqrt{2}\sin 2x}{1-\sqrt{2}\sin 2x}\right| - \dfrac{1}{2}\left(\log\left|\sec 2x - \tan 2x\right|\right) + C$

2) $\frac{1}{2\sqrt{2}}\log\left|\frac{1+\sqrt{2}\sin 2x}{1+\sqrt{2}\sin x}\right|-\frac{1}{2}\left(\log\left|\sec 2x-\tan 2x\right|\right)+C$

3) $\frac{1}{\sqrt{2}}\log\left|\frac{1+\sqrt{2}\sin 2x}{1-\sqrt{2}\sin 2x}\right|-\frac{1}{2}\left(\log\left|\sec 2x-\tan 2x\right|\right)+C$

4) None of these

Key. 1

Sol. $\int\frac{\sin(4x-2x)\,dx}{\sin(2x)\cos(2x)\cos(4x)}$

$=\int\frac{\sin(4x)\,dx}{\sin(2x)\cos(4x)}-\int\sec 2x\,dx$

$=2\int\frac{\cos 2x\,dx}{\cos 4x}-\frac{1}{2}\left(\log|\sec 2x-\tan 2x|\right)$

38. $\int\frac{1-7\cos^2 x}{\sin^7 x\cos^2 x}dx=\frac{f(x)}{(\sin x)^7}+C$, then f(x) is equal to

1) Sin x 2) Cos x 3) Tan x 4) Cot x

Key. 3

Sol. $\int\frac{1-7\cos^2 x}{\sin^7 x\cos^2 x}dx=\int\left(\frac{\sec^2 x}{\sin^7 x}-\frac{7}{\sin^7 x}\right)dx$

$=\int\frac{\sec^2 x}{\sin^7 x}dx-\int\frac{7}{\sin^7 x}dx=I_1+I_2$

Now, $I_1=\int\frac{\sec^2 x}{\sin^7 x}dx$

$=\frac{\tan x}{\sin^7 x}+7\int\frac{\tan x.\cos x}{\sin^8 x}dx$

$=\frac{\tan x}{\sin^7 x}-I_2$

$\therefore I_1+I_2=\frac{\tan x}{\sin^7 x}+C$

$\Rightarrow f(x)=\tan x$

39. Integral of $\sqrt{1+2\cot x(\cot x+\operatorname{cosec} x)}$ with respect to x is:

1) $2\ln\cos\frac{x}{2}+c$ 2) $2\ln\sin\frac{x}{2}+c$

3) $\frac{1}{2}\ln\cos\frac{x}{2}+c$ 4) $\ln\sin x-\ln(\operatorname{cosec} x-\cot x)+c$

Key. 2

Sol. I = $\int\sqrt{1+2\operatorname{cosec} x\cot x+2\cot^2 x}$

= $\int\sqrt{\operatorname{cosec}^2 x+2\operatorname{cosec} x\cot x+\cot^2 x}\,dx$

= $\int(\operatorname{cosec} x+\cot x)\,dx$

40. Let $f(x)=\frac{1}{x}\ln\left(\frac{x}{e^x}\right)$ then its primitive with respect to x is

1) $\frac{1}{2}e^x-\ln x+C$ 2) $\frac{1}{2}\ln x-e^x+C$

3) $\frac{1}{2}\ln^2 x-x+C$ 4) $\frac{e^x}{2x}+C$

Key. 3

Sol. $\int\frac{1}{x}\ln\frac{x}{e^x}dx$ = $\int\frac{1}{x}(\ln x-\ln e^x)\,dx$

= $\int\frac{\ln x-x}{x}dx$ = $\left[\int\frac{1}{x}\ln x\,dx-\int\frac{1}{x}x\,dx\right]$ (put $\ln x = u$; $\frac{1}{x}dx=du$)

= $\int u\,du$ – $\int 1dx$ = $\frac{1}{2}\ln^2 x - x + C$]

41. Primitive of $f(x)=x.2^{\ln(x^2+1)}$ with respect to x is

1) $\frac{2^{\ln(x^2+1)}}{2(x^2+1)}+C$ 2) $\frac{(x^2+1)2^{\ln(x^2+1)}}{\ln 2+1}+C$ 3) $\frac{(x^2+1)^{\ln 2+1}}{2(\ln 2+1)}+C$ 4) $\frac{(x^2+1)^{\ln 2}}{2(\ln 2+1)}+C$

Key. 3

Sol. I = $\int x\,2^{\ln(x^2+1)}dx$ let $x^2 + 1 = t$; $x\,dx=\frac{dt}{2}$

Hence I = $\frac{1}{2}\int 2^{\ln t}dt$ = $\frac{1}{2}\int t^{\ln 2}dt$ = $\frac{1}{2}\cdot\frac{t^{\ln 2+1}}{\ln 2+1}$ + C = $\frac{1}{2}\cdot\frac{(x^2+1)^{\ln 2+1}}{\ln 2+1}$ + C $\Rightarrow$ (C)

42. Let g(x) be an antiderivative for f(x). Then $\ln\left(1+(g(x))^2\right)$ is an antiderivate for

1) $\frac{2f(x)g(x)}{1+(f(x))^2}$ 2) $\frac{2f(x)g(x)}{1+(g(x))^2}$ 3) $\frac{2f(x)}{1+(f(x))^2}$ 4) none

Key. 2

Sol. Given $\int f(x)\,dx$ = g (x) $\Rightarrow$ g ' (x) = f (x)

now $\frac{d}{dx}\left(\ln(1+g^2(x)\right) = \frac{2g(x)g'(x)}{1+g^2(x)} = \frac{2f(x)g(x)}{1+g^2(x)} \Rightarrow$ (B)

43. $\int \frac{\cos^3 x+\cos^5 x}{\sin^2 x+\sin^4 x}dx$ equals

1) $\sin x-6\tan^{-1}(\sin x)+c$ 2) $\sin x-2\sin^{-1}x+c$

3) $\sin x-2(\sin x)^{-1}-6\tan^{-1}(\sin x)+c$ 4) $\sin x-2(\sin x)^{-1}+5\tan^{-1}(\sin x)+c$

Key. 3

Sol. sin x = t ; I = $\int \frac{(1-t^2)(2-t^2)}{t^2(1+t^2)}$ dt = $\int \frac{(y-1)(y-2)}{y(1+y)}$ dy = 1 + $\frac{2(1-2y)}{y(y+1)}$; y = t^2

= 1 + 6 $\left[\frac{1}{3y}-\frac{1}{y+1}\right]$ = $\left(1+\frac{2}{t^2}-\frac{6}{1+t^2}\right)$dt

44. The evaluation of $\int \frac{px^{p+2q-1}-qx^{q-1}}{x^{2p+2q}+2x^{p+q}+1}dx$ is

1) $-\frac{x^p}{x^{p+q}+1}+C$ 2) $\frac{x^q}{x^{p+q}+1}+C$ 3) $-\frac{x^q}{x^{p+q}+1}+C$ 4) $\frac{x^p}{x^{p+q}+1}+C$

Key. 3

Sol. $\int \frac{px^{p+2q-1}-qx^{q-1}}{\left(x^{p+q}+1\right)^2}$ dx = $\int \frac{px^{p-1}-qx^{-q-1}}{\left(x^p+x^{-q}\right)^2}$ dx

taking x^q as x^{2q} common from Denominator and take it in N^r

45. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\left(x+\sqrt{x}\right)dx$ equals

1) $2e^{\sqrt{x}}\left[x-\sqrt{x}+1\right]+C$ 2) $e^{\sqrt{x}}\left[x-2\sqrt{x}+1\right]$

3) $e^{\sqrt{x}}\left[x+\sqrt{x}\right]+C$ 4) $e^{\sqrt{x}}\left[x+\sqrt{x}+1\right]+C$

Key. 1

Sol. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\left(x+\sqrt{x}\right)dx$; put x = t^2 ; dx = 2t dt

= $\int e^t(t^2+t)\,dt$ = e^t (At^2 + Bt + C) (Let)

Diffrentiate both the sides

e^t (t^2 + t) = e^t (2At + B) + (At^2 + Bt + C) e^t

On comparing coefficient we get

A = 1 ; B = – 1 ; C = 1]

46. $\int \frac{e^{\tan^{-1}x}}{\left(1+x^2\right)}\left[\left(\sec^{-1}\sqrt{1+x^2}\right)^2+\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]dx$ (x > 0)

1) $e^{\tan^{-1}x}.\tan^{-1}x+C$ 2) $\frac{e^{\tan^{-1}x}.\left(\tan^{-1}x\right)^2}{2}+C$

3) $e^{\tan^{-1}x}.\left(\sec^{-1}\left(\sqrt{1+x^2}\right)\right)^2+C$ 4) $e^{\tan^{-1}x}.\left(\cos ec^{-1}\left(\sqrt{1+x^2}\right)\right)^2+C$

Key. 3

Sol. note that $\sec^{-1}\sqrt{1+x^2} = \tan^{-1}x$; $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = 2\tan^{-1}x$ for x > 0

$I = \int \frac{e^{\tan^{-1}x}}{1+x^2}\left((\tan^{-1}x)^2+2\tan^{-1}x\right)dx$ put $\tan^{-1}x = t$

$= \int e^t(t^2+2t)\,dt = e^t.t^2 = e^{\tan^{-1}x}\left(\tan^{-1}x\right)^2 + C$

47. Let $f(x)=\frac{2\sin^2 x-1}{\cos x}+\frac{\cos x\left(2\sin x+1\right)}{1+\sin x}$ then $\int e^x\left(f(x)+f'(x)\right)dx$ equals

(where c is the constant of integeration)

1) $e^x\tan x+c$ 2) $e^x\cot x+c$ 3) $e^x\cos ec^2 x+c$ 4) None of these

Key. 1

Sol. $\frac{\cos x(1+2\sin x)}{1+\sin x}-\frac{\cos^2 x-\sin^2 x}{\cos x}$

$=\frac{\cos^2 x(1+2\sin x)-(1+\sin x)(\cos^2 x-\sin^2 x)}{\cos x(1+\sin x)}=\frac{-\sin x\cos^2 x+\sin^3 x}{\cos x(1+\sin x)}$

$=\frac{\sin x\cos^2 x+\sin^2 x(1+\sin x)}{\cos x(1+\sin x)}=\frac{\sin x(1-\sin x)+\sin^2 x}{\cos x}$ = tan x

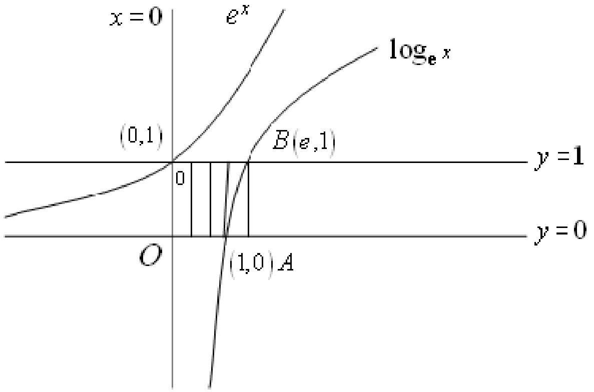
48. Area bounded by the curves $y=e^x,\ y=\log_e x$ and the lines $x=0,\ y=0,\ y=1$ is

A) $e^2+2sq.units$ B) $e+1sq.unit$

C) $e+2sq.units$ D) $e-1\ sq.unit$

Key: D

Hint:



Area = Area of rectangle OABC $-\int_0^1 \log_e x$

49. The area of the loop of the curve $y^2 = x^4(x+2)$ is [in square units]

(A) $\frac{32\sqrt{2}}{105}$ (B) $\frac{64\sqrt{2}}{105}$ (C) $\frac{128\sqrt{2}}{105}$ (D) $\frac{256\sqrt{2}}{105}$

Key: D

Hint: Area = $2\int_{-2}^{0} y\,dx = 2\int_{-2}^{0} x^2\sqrt{x+2}\,dx = 4\sqrt{2}\int_{0}^{2}\left(z^2-2\right)^2 z^2 dz (where \sqrt{x+2} = z)$

$= 4\left[\frac{z^7}{7} - \frac{4z^5}{5} + \frac{4z^3}{3}\right]_0^{\sqrt{2}}$

$= \frac{256\sqrt{2}}{105}$

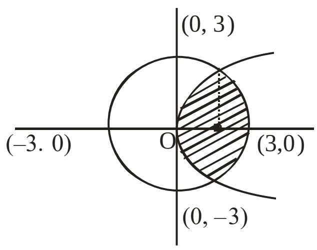
50. The area of the smaller portion enclosed by the curves $x^2+y^2 = 9$ and $y^2 = 8x$ is

A) $\frac{\sqrt{2}}{3} + \frac{9\pi}{4} - \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)$ B) $2\left(\frac{\sqrt{2}}{3} + \frac{9\pi}{4} - \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right)$

C) $2\left(\frac{\sqrt{2}}{3} + \frac{9\pi}{4} + \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right)$ D) $\frac{\sqrt{2}}{3} + \frac{9\pi}{4} + \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)$

Key: B

HINT :



$x^2 + y^2 = 9,$

$x^2 + 8x - 9 = 0$

$x = \frac{-8 \pm \sqrt{64+36}}{2}$

$x = \frac{-8 \pm 10}{2} - 9, 1$

$\boxed{x = 1}$

Area enclosed $= 2\left[\int_0^1 2\sqrt{2x}dx + \int_1^3 \sqrt{9 - x^2}\,dx\right] = 2\left[2\sqrt{2}\int_0^1 \sqrt{x}dx + \int_1^3 \sqrt{9 - x^2}\,dx\right]$

On simplifying we get

$= 2\left[\frac{\sqrt{2}}{3} + \frac{9\pi}{4} - \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right]$

51. The area of the region in the xy-plane defined by the inequalities $x - 2y^2 \geq 0$, $1 - x - |y| \geq 0$ is

A) $\frac{1}{2}$ B) $\frac{1}{3}$ C) $\frac{1}{4}$ D) $\frac{7}{12}$

Key: D

Hint: Area $= 2\int_0^{1/2} \sqrt{\frac{x}{2}}dx + \frac{1}{4} = \frac{7}{12}$

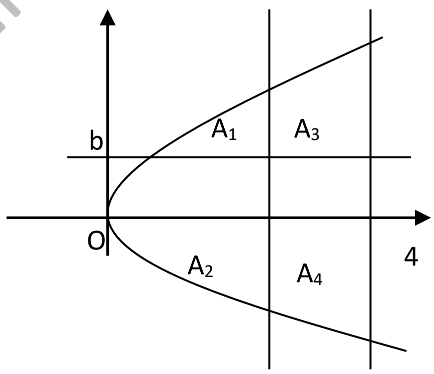
52. Area bounded by curve $y^2 = x$ and x = 4 is divided into 4 equal parts by the lines x = a and y = b then.

a) Area of each part = $\frac{8}{3}$ b) b = 0

c) $a = \sqrt{2}$ d) $a = (16)^{1/3}$

Key: D

Hint: Total area = $2\int_0^4 \sqrt{x}\,dx = \frac{32}{3}$



Area of each part = 8/3

$A_3 = A_4 \Rightarrow \int_a^4 \left(\sqrt{x} - b\right)dx =$

$\int_a^4 \left(b + \sqrt{x}\right)dx = \frac{8}{3} \Rightarrow b = 0$

$\int_a^4 \sqrt{x}\,dx = \frac{8}{3} \Rightarrow a^3 = 16$

53. Area of the region in which point $p(x, y)$, $\{x > 0\}$ lies; such that $y \leq \sqrt{16 - x^2}$ and $\left|\tan^{-1}\left(\frac{y}{x}\right)\right| \leq \frac{\pi}{3}$ is

(A) $\left(\frac{16}{3}\pi\right)$ (B) $\left(\frac{8\pi}{3} + 8\sqrt{3}\right)$ (C) $\left(4\sqrt{3} - \pi\right)$ (D) $\left(\sqrt{3} - \pi\right)$

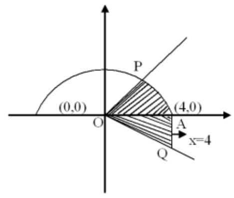
Key: B

Hint: Required area is the area of shaded region (APOQ)

= area of ΔOAQ + area of sector (OAP)

$$=\frac{1}{2}\times 4\times 4\sqrt{3}+\frac{\pi(4\times 4)}{6}$$

$$=\left(\frac{8\pi}{3}+8\sqrt{3}\right)$$



54. Area bounded between the curves $y=\sqrt{4-x^2}$ and $y^2=3|x|$ is/are

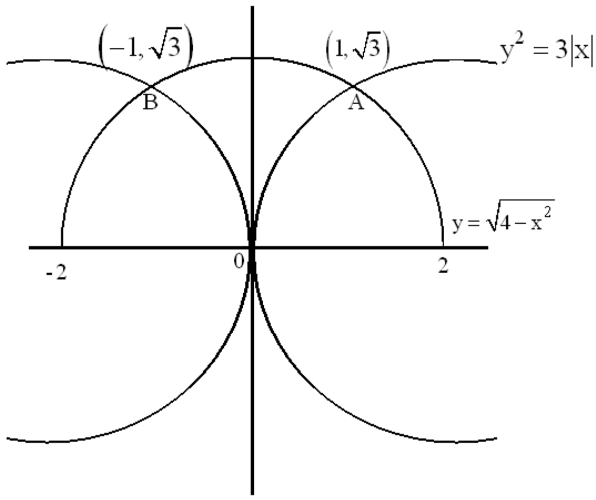
(A) $\frac{\pi-1}{\sqrt{3}}$ (B) $\frac{2\pi-1}{3\sqrt{3}}$ (C) $\frac{2\pi-\sqrt{3}}{3}$ (D) $\frac{2\pi-\sqrt{3}}{3\sqrt{3}}$

Key: C

Hint: Required area = $2\int_0^1\left(\sqrt{4-x^2}-\sqrt{3x}\right)dx$

$$=2\left(\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)-\frac{\sqrt{3}.2\,x^{3/2}}{3}\right)_0^1$$

$$=\frac{2\pi-\sqrt{3}}{3}$$



55. Let $f:[0,\infty)\to R$ be a continuous and strictly increasing function such that $f^3(x)=\int_0^x t.f^2(t)dt, \forall x>0$. The area enclosed by $y=f(x)$, the x-axis and the ordinate at x=3, is

(A) 1 (B) $\frac{3}{2}$ (C) 2 (D) 3

Key: B

Hint: $f(x)=\frac{x^2}{6}$

$A = \frac{1}{6}\int_0^3 x^2 dx = 3/2$

56. Let f(x) = x + sin x. The area bounded by $y = f^{-1}(x), y = x,\ x \in [0, \pi]$ is

(a) 1 (b) 2 (c) 3

(d) canot be found because $f^{-1}(x)$ cannot be determined

Key: B

Hint: The curves given by y = x + sin x and $y = f^{-1}(x)$ are images of each other in the line y = x.

Hence required area = $\int_0^{\pi}((x + \sin x) - x)dx = -[\cos x]_0^{\pi} = 2$

57. The area of the region bounded by the curves $|x + y| \le 2$, $|x - y| \le 2$ and $2x^2 + 6y^2 \ge 3$ is

(A) $\left(8 + \frac{\sqrt{3}}{2}\pi\right)$ sq. units (B) $\left(8 - \frac{\sqrt{3}}{2}\pi\right)$ sq. units

(C) $\left(4 - \frac{3\sqrt{3}}{2}\pi\right)$ sq. units (D) $\left(8 - \frac{3\sqrt{3}}{2}\pi\right)$ sq. units

Key : B

Sol :

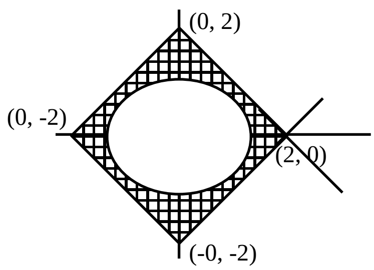
$2x^2 + 6y^2 \ge 3$ . . . . . . (1)

area of ellipse = $\pi \times \frac{\sqrt{3}}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{\pi\sqrt{3}}{2}$

$|x + y| \le 2 \Rightarrow -2 \le (x + y) \le 2$ . . . . . .(2)

$|x - y| \le 2 \Rightarrow -2 \le (x - y) \le 2$ . . . . . . (3)

Required area = $\left(8 - \frac{\pi\sqrt{3}}{2}\right)$ sq. units



58. If $A = \int_1^{\sin\theta} \frac{t\,dt}{1+t^2}dt, B = \int_1^{\operatorname{cosec}\theta} \frac{1}{t(1+t^2)}dt$ then $\begin{vmatrix} A & A^2 & B \\ e^{A+B} & B^2 & -1 \\ 1 & A^2 + B^2 & -1 \end{vmatrix} = ?$

a) $\sin\theta$ b) $\operatorname{cosec}\theta$ c) 0 d) 1

Key: C

$$B=\int_1^{\operatorname{cosec}\theta}\frac{1}{t\left(1+t^2\right)}$$

Hint:

let $\frac{1}{t}=u\Rightarrow B=\int_1^{\sin\theta}\frac{-udu}{1+u^2}$

$\Rightarrow A+B=0 \quad \Rightarrow A=-B$

$$\therefore\begin{vmatrix}A & A^2 & -A\\ e^0 & A^2 & -1\\ 1 & A^2+B^2 & -1\end{vmatrix}=0$$

59. The value of $\int_0^2\frac{2x^3-6x^2+9x-5}{x^2-2x+5}dx$ equals

(A) 4 (B) 1

C) – 1 (D) None of the above

Key: D

Hint Make the substitution x – 1= t. It turns into an odd integral and so reduces to zero.

60. Let $I_n=\int_0^\infty e^{-x}\left(\sin x\right)^n dx,\ n\in N, n>1$ then $\frac{I_{2008}}{I_{2006}}$ equals

(A) $\frac{2007\times 2006}{2008^2+1}$ (B) $\frac{2008\times 2007}{2008^2+1}$

(C) $\frac{2006\times 2004}{2008^2-1}$ (D) $\frac{2008\times 2007}{2008^2-1}$

Key: B

Hint In $=\int_0^\infty e^{-x}\left(\sin x\right)^n dx$

$$=\left[\sin^n x\left(-e^{-x}\right)\right]_0^\infty+\int_0^\infty n\sin^{n-1}x\cos x\,e^{-x}dx$$

$$=0+n\int_0^\infty\left(\sin^{n-1}x\cos x\right)e^{-x}dx$$

$$=n\left[\left(\sin^{n-1}x\cos x\right)\left(-e^{-x}\right)\right]_0^\infty-n\int_0^\infty\left\{-\sin^n x+(n-1)\sin^{n-2}x\cos^2 x\right\}\left(-e^{-x}\right)dx$$

$$=0+n\int_0^\alpha e^{-x}\left\{-\sin^n x+(n-1)\sin^{n-2}x\,(1-\sin^2 x)\right\}dx$$

$$=n\int_0^\alpha e^{-x}\left\{(n-1)\sin^{n-2}x-n\sin^n x\right\}dx$$

$$=n(n-1)I_{n-2}-n^2I_n$$

$we\ \ have\ \ \left(1+n^2\right)I_n = n(n-1)\,I_{n-2}$

$then\ \ \frac{I_n}{I_{n-2}} = \frac{n(n-1)}{n^2+1}$

61. A hyperbola passing through origin has 3x-4y-1=0 and 4x-3y-6=0 as its asymptotes. Then the equation of its transverse axis is

a) x-y-5=0 b)x+y+1=0 c) x+y-5=0 d)x-y-1=0

Key: C

Hint: Asymptotes are equally inclined to the axes of hyperbola
Find the bisector of the asymptotes which bisects the angle containing the origin.

62. If $\int_{\sin x}^{1} t^2.f(t)dt = 1-\sin x,\ \forall x \in \left(0,\frac{\pi}{2}\right)$ then the value of $f\left(\frac{1}{\sqrt{3}}\right)$ is

(A) $\frac{1}{\sqrt{3}}$ (B) $\sqrt{3}$ (C) $\frac{1}{3}$ (D) 3

Key: D

Hint: $\int_{\sin x}^{1} t^2.f(t)dt = 1-\sin x$

Differentiating both sides with respect to 'x'

$0-\sin^2 x.f(\sin x).\cos x = -\cos x \ \Rightarrow \cos x\left[1-\sin^2 x.f(\sin x)\right] = 0$

But $\cos x \neq 0$

So, $f(\sin x) = \frac{1}{\sin^2 x}$

$f\left(\frac{1}{\sqrt{3}}\right) = 3$

63. $\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}dx =$

(A) $\frac{\pi^2}{4}$ (B) π^2 (C) 0 (D) $\frac{\pi}{2}$

Key: B

Hint: $I = 4\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx = 4\int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}dx$

$\therefore 2I = 4\pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx = 8\pi\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}dx$

$= 2\pi^2$

$\therefore I = \pi^2$

64. Let $f:(0,\infty)\to R$ and F(x) = $\int_1^x f(t)dt$. If $F(x^2) = x^2(1 + x)$ then f(4) equals

(A) 5/4 (B) 7 (C) 4 (D) 2

Key: C

Hint: $F'(x)=f(x)$

$F(x)=x(1+\sqrt{x})=x+x^{3/2}$

$\therefore\ F'(x)=f(x)=1+\frac{3}{2}\sqrt{x}$

$\therefore$ f(4) = 4

65. If $f(x)=\int_0^x (1+t^3)^{-1/2}dt$ and g (x) is the inverse of f, then the value of $\frac{g''(x)}{g^2(x)}$ is

(A) 3/2 (B) 2/3 (C) 1/3 (D) 1/2

Key: A

Hint: $f(x)=\int_0^x (1+t^3)^{-1/2}dt$

i.e. $f[g(x)]=\int_0^{g(x)} (1+t^3)^{-1/2}dt$

i.e. $x=\int_0^{g(x)} (1+t^3)^{-1/2}dt$ [$\because$ g is inverse of f $\Rightarrow$ f[g(x)] = x]

Differentiating with respect to x, we have

$1 = (1 + g^3)^{-1/2} . g'$

i.e. $(g')^2 = 1 + g^3$

Differentiating again with respect to x, we have

$2g'g''=3g^2g'$

gives $\frac{g''}{g^2}=\frac{3}{2}$

66. $\int_0^a \ln(\cot a+\tan x)dx$, where $a\in\left(0,\frac{\pi}{2}\right)$ is

a) a ln (sin a) b) – a ln (sin a) c) – a ln (cos a) d) none of these

Key: B

Hint:
$$I=\int_0^a \ln\frac{\cos(a-x)}{\sin a.\cos x}dx=\int_0^a \ln\frac{\cos x}{\sin a\cos(a-x)}dx$$

adding $$2I=\int_0^a \ln\frac{1}{\sin^2 a}dx=\int_0^a -2\ln\sin a\ dx=-2a\ln\sin a$$

67. If $f(x)=\int_1^x \frac{dt}{2+t^4}$, then

(A) $f(2)<\frac{1}{3}$ (B) $f(2)>\frac{1}{3}$

(C) $f(2)=\frac{1}{3}$ (D) $f(2)>1$

Key: A

Hint: $f'(x)=\frac{1}{2+x^4}$

By LMVT $f'(C)=\frac{f(2)-f(1)}{2-1}$ for some $c\in(1,2)$

$\Rightarrow f(2)=\frac{1}{2+c^4}$ as $f(1)=0$ $\Rightarrow 1<c<2 \Rightarrow 3<2+c^4<18 \Rightarrow f(2)<\frac{1}{3}$

68. Let $I_1=\int_0^{\pi/4} x^{2008}(\tan x)^{2008}dx$, $I_2=\int_0^{\pi/4} x^{2009}(\tan x)^{2009}dx$ and $I_3=\int_0^{\pi/4} x^{2010}(\tan x)^{2010}dx$ Then the correct order sequence, is

(A) $I_2<I_3<I_1$ (B) $I_1<I_2<I_3$

(C) $I_3<I_1<I_2$ (D) $I_3<I_2<I_1$

Key: D

Hint:

69. .If f(x + y) = f(x) + f(y) + 2xy - 6 for all x, y$\in$R and f ′(0) = 2, then y = f(x) will be

(A) straight line (B) parabola

(C) ellipse (D) circle

Key: B

Hint: $f'(x) = \lim_{n\to 0}\frac{f(x+h)-f(x+0)}{h}$

$= \lim_{n\to 0}\frac{f(x)+f(h)+2xh-6-f(x)-f(0)+6}{h}$

$= 2x + \lim_{n\to 0}\frac{f(0+h)-f(0)}{h} = 2x + f'(0)$

f(x) = x^2 + 2x + C, but f(0) = 6

So, f(x) = x^2 + 2x + 6.

70. The value of $\int_3^6\left(\sqrt{x+\sqrt{12x-36}}+\sqrt{x-\sqrt{12x-36}}\right)dx$ is equal to

(A) $6\sqrt{3}$ (B) $4\sqrt{3}$ (C) $12\sqrt{3}$ (D) $2\sqrt{3}$

Key: A

Hint: $\int_0^3 \left(\sqrt{(x+3)+2\sqrt{3}\sqrt{x}}+\sqrt{(x+3)-2\sqrt{3}\sqrt{x}}\right)dx$

$\int_0^3 \left(\left(\sqrt{x}+\sqrt{3}\right)+\left(\sqrt{3}-\sqrt{x}\right)\right)dx = \int_0^3 2\sqrt{3}\,dx = 6\sqrt{3}$

71. Let $f:\left[0,\frac{\pi}{2}\right]\to[0,1]$ be a differentiable function such that $f(0)=0,\ f\left(\frac{\pi}{2}\right)=1$, then

(A) $f'(\alpha)=\sqrt{1-(f(\alpha))^2}$ for all $\alpha\in\left(0,\frac{\pi}{2}\right)$

(B) $f'(\alpha)=\frac{2}{\pi}$ for all $\alpha\in\left(0,\frac{\pi}{2}\right)$

(C) $f(\alpha)f'(\alpha)=\frac{1}{\pi}$ for at least one $\alpha\in\left(0,\frac{\pi}{2}\right)$

(D) $f'(\alpha)=\frac{8\alpha}{\pi^2}$ for at least one $\alpha\in\left(0,\frac{\pi}{2}\right)$

Key: A

Hint: Let : $\left[0,\frac{\pi}{2}\right]\to$ [0,1] be a ....

(A) Consider $g(x)=\sin^{-1}f(x)-x$

Since $g(0)=0, g\left(\frac{\pi}{2}\right)=0$

$\therefore$ There is at least one value of $\alpha\in\left(0,\frac{\pi}{2}\right)$ such that

$$g'(\alpha)=\frac{f'(\alpha)}{\sqrt{1-(f(\alpha))^2}}-1=0$$

i. e. $f'(\alpha)=\sqrt{1-(f(\alpha))^2}$ for atleast one value of α but may not be for all $\alpha\in\left(0,\frac{\pi}{2}\right)$

$\therefore$ flase

(B) Consider $g(x)=f(x)-\frac{2x}{\pi}$

Since $g(0)=0, g\left(\frac{\pi}{2}\right)=0$

$\therefore$ there is at least one value of $\alpha\in\left(0,\frac{\pi}{2}\right)$ such that

$$g'(\alpha)=f'(\alpha)-\frac{\pi}{2}=0$$

i. e $f'(\alpha)=\frac{2}{\pi}$ for atleast one value of α but may not be for all $\alpha \in \left(0, \frac{\pi}{2}\right)$

$\therefore$ false

(C) Consider $g(x)=(f(x))^2-\frac{2x}{\pi}$

Since $g(0)=0, g\left(\frac{\pi}{2}\right)=0$

$\therefore$ There is at least one value of $\alpha \in \left(0, \frac{\pi}{2}\right)$ such that

$g'(\alpha)=2f(\alpha)f'(\alpha)-\frac{\pi}{2}=0$

$\therefore f(\alpha)f'(\alpha)=\frac{1}{\pi}$

$\therefore$ True

(D) Consider $g(x)=f(x)-\frac{4x^2}{\pi^2}$

Since $g(0)=0,\ g\left(\frac{\pi}{2}\right)=0$

$\therefore$ there is at least one value of $\alpha \in \left(0, \frac{\pi}{2}\right)$ such that

$g'(\alpha)=f'(\alpha)-\frac{8\alpha}{\pi^2}=0$

$\therefore f'(\alpha)=\frac{8\alpha}{\pi^2}$

$\therefore$ True

72. If x = a cos t, y = a sin t, then $\frac{d^2y}{dx^2}$ at t = $\frac{\pi}{4}$ is

(a) $\frac{a}{2\sqrt{2}}$ (b) $-\frac{a}{2\sqrt{2}}$ (c) $\frac{2\sqrt{2}}{a}$ (d) $-\frac{2\sqrt{2}}{a}$

Key: d

Hint: Clearly $x^2 + y^2 = a^2$ and $y(\pi/4) = a/\sqrt{2}$, $x(\pi/4) = a/\sqrt{2}$. Differentiating we get,

$2x + 2yy_1 = 0 \Rightarrow y_1 = -\frac{x}{y}$, so $y_1(\pi/4)=-1$.

Now $x + yy_1 = 0 \Rightarrow 1+y_1^2+yy_2=0$

$\Rightarrow y_2(\pi/4)=-\frac{1+(y_1(\pi/4))^2}{y(\pi/4)}=\frac{-2\sqrt{2}}{a}$

73. If $z \neq 0$, then $\int_{x=0}^{100}\left[\arg|z|\right]dx$ is (where [.] denotes the greatest integer function)

(A) 0 (B) 10

(C) 100 (D) not defined

Key : A

Sol : $\because |z|$ = real and positive, imaginary part is zero

$\therefore \arg|z| = 0$

$\Rightarrow [\arg|z|] = 0$

$\therefore \int_{x=0}^{100} [\arg|z|]dx = \int_{x=0}^{100} 0.dx = 0$

74. $\lim_{x\to\frac{\pi}{4}} \frac{\int_{2}^{\sec^2 x} f(t)dt}{x^2 - \frac{\pi^2}{16}}$ equals

(a) $\frac{8}{\pi}f(2)$ (b) $\frac{2}{\pi}f(2)$ (c) $\frac{2}{\pi}f\left(\frac{1}{2}\right)$ (d) $4f(2)$

Key: A

Hint: Required limits is of the form $\frac{0}{0}$, so it is equal to

$\lim_{x\to\frac{\pi}{4}} \frac{2\sec x \sec x \tan x f(\sec^2 x)}{2x} = \lim_{x\to\frac{\pi}{4}} \frac{\sec^2 x \tan x\ f(\sec^2 x)}{x} = \frac{8}{\pi}f(2)$

75. *Let* $f(x) = \int_0^1 |t-x|t\ dt$ for all real *x*. Then the minimum value of f is

a) $\frac{1}{2}$ b) $\frac{1}{3}\left(1+\frac{1}{\sqrt{2}}\right)$ c) $\frac{1}{3}\left(1-\frac{1}{\sqrt{2}}\right)$ d) $\frac{1}{6}$

Key: C

Hint: $f(x) = \begin{cases} \frac{1}{3} - \frac{x}{2} if\ \ x \le 0 \\ \frac{1}{3} + \frac{x^3}{3} - \frac{x}{2} if\ 0 \le x \le 1 \\ \frac{x}{2} - \frac{1}{3} if\ x \ge 1 \end{cases}$

f attains its minimum at x = $\frac{1}{\sqrt{2}}$

76. $\int \frac{(1+\sqrt{\tan x})(1+\tan^2 x)}{2\tan x}dx$ equal to

A) $\log \tan^2 x + \sqrt{\tan x} + c$ B) $\log \tan^2 x + \frac{1}{2\sqrt{\tan x}} + c$

C) $\log|\tan x| + 2\sqrt{\tan x} + c$ D) $\log|\tan| + \sqrt{\tan x} + c$

Key: A

Hint: $\int \frac{1}{2\sin x \cos x} dx + \frac{1}{2}\int \frac{\sqrt{\tan x}}{\sin x \cos x} dx$

= $\frac{1}{2}\int \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} dx + \frac{1}{2}\int \frac{\sec^2 x}{\sqrt{\tan x}} dx$ = $\log(\tan^2 x) + \sqrt{\tan x} + c$

77. $\int \frac{(2+\sec x)\sec x}{(1+2\sec x)^2} dx =$

a) $\frac{1}{2\text{cosec}x + \cot x} + C$ b)2 cosec x +cot x+C c) $\frac{1}{2\text{cosec}x - \cot x} + C$ d) 2 cosec x – cot x + C

Key: A

Hint $I = \int \frac{(2\cos x + 1)}{(2+\cos x)^2} dx = \int \frac{(2+\cos x)\cos x + \sin^2 x}{(2+\cos x)^2} dx$

$= \int \frac{\cos x}{2+\cos x} dx - \int \frac{-\sin^2 x}{(2+\cos x)^2} dx = \frac{\sin x}{2+\cos x} + c$

78. $\int \frac{\mathrm{dx}}{(\mathrm{x}-3)^{4/5}(\mathrm{x}+1)^{6/5}} =$

A) $((\mathrm{x}-3)(\mathrm{x}+1))^{1/5} + \mathrm{c}$ B) $\frac{5}{4}\left(\frac{\mathrm{x}-3}{\mathrm{x}+1}\right)^{1/5} + \mathrm{c}$

C) $\left(\frac{\mathrm{x}+1}{\mathrm{x}-3}\right)^{1/5} + \mathrm{c}$ D) $(\mathrm{x}-3)^{6/5}(\mathrm{x}+1)^{4/5} + \mathrm{c}$

Key: B

Hint Put $\mathrm{t} = \frac{\mathrm{x}-3}{\mathrm{x}+1}$

$\Rightarrow \mathrm{dx} = \frac{(\mathrm{x}+1)^2 \mathrm{dt}}{4}$

79. If the system of linear equations $x+y+z=6, x+2y+3z=14$ and $2x+5y+\lambda z=\mu$, $(\lambda, \mu \in R)$ has no solution, then

a) $\lambda \neq 8$ b) $\lambda = 8, \mu \neq 36$ c) $\lambda = 8, \mu = 36$ d) None of these

Key: B

Hint $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 2 & 5 & \lambda & \mu \end{bmatrix}$ $R_2 \to R_2 - R_1$; $\mathrm{R}_3 \to R_3 - 2R_1$

- $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & \lambda-2 & \mu-12 \end{bmatrix} R_3 \to R_3 - 3R_2$

- $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & \lambda-8 & \mu-36 \end{bmatrix}$

$\lambda - 8 = 0$ & $\mu - 36 \neq 0$

80. A man starts from the point P(3,-3) and reaches the point Q(0,2) after touching the line 2x+y=7 at R. The least value of PR+RQ is

a) $5\sqrt{2}$ b) $3\sqrt{2}$ c) $7\sqrt{2}$ d) $2\sqrt{2}$

Key: A

Hint P, Q lies on same side of the line find image of P w.r.t. line

81. Let $S(x) = \int \frac{dx}{e^x + 8e^{-x} + 4e^{-3x}}$, $R(x) = \int \frac{dx}{e^{3x} + 8e^x + 4e^{-x}}$ and $M(x) = S(x) - 2R(x)$. If $M(x) = \frac{1}{2}\tan^{-1}\left(f(x)\right) + c$ where c is an arbitrary constant then $f(\log_e 2) =$

A) $\frac{3}{2}$ B) $\frac{1}{2}$ C) $\frac{5}{2}$ D) $\frac{7}{2}$

Key: A

Hint: $M(x) = \int \frac{e^x\left(e^{2x}-2\right)}{e^{4x} + 8e^{2x} + 4}dx$ $e^x = t \Rightarrow \int \frac{\left(t^2 - 2\right)dt}{t^4 + 8t^2 + 4} = \frac{1}{2}\tan^{-1}\left(\frac{t + 2/t}{2}\right) + c$

$= \frac{1}{2}\tan^{-1}\left(\frac{e^x + 2e^{-x}}{2}\right) + c$

82. If $\int\left(2 - 3\sin^2 x\right)\sqrt{\sec x}\, dx = 2f(x)\sqrt{g(x)} + c$ and f(x) is non constant function then

(A) $f^2(x) + g^2(x) = 1$ (B) $f^2(x) - g^2(x) = 1$ (C) $f(x)g(x) = 1$ (D) $f(x) = g(x)$

Key: A

Hint:

$$\int \frac{2 - 3\sin^2 x}{\sqrt{\cos x}}dx = \int \frac{2\cos^2 x - \sin^2 x}{\sqrt{\cos x}}dx = 2\int(\cos x)\sqrt{\cos x}dx - \int \frac{\sin^2 x}{\sqrt{\cos x}}dx$$

$= 2\sin x\sqrt{\cos x} \quad + c$

$\Rightarrow f(x) = \sin x, g(x) = \cos x$

83. $\int \frac{e^{\cot x}}{\sin^2 x}\left(2\ln \operatorname{cosec} x + \sin 2x\right)dx =$

a) $-2e^{\cot x}\ln(\cos ecx)+c$ b) $e^{\cot x}\ln x+c$

c) $e^{\cot x}\ln(\cos ecx)+c$ d) $e^{\cot x}\ln(\sin x)+c$

Key: A

Hint:

$$t=\cot x\Rightarrow -\cos ec^2 xdx=dt\Rightarrow dx=\frac{-1}{1+t^2}dt$$

$$I=-\int e^t\left(\ln(1+t^2)+\frac{2t}{1+t^2}\right)dt=-e^t\ln(1+t^2)+c=-2e^{\cot x}\ln\cos ecx+c$$

84. If $\int\frac{\sec^2 x-2010}{\sin^{2010}x}dx=\frac{P(x)}{\sin^{2010}x}+C$, then value of $P\left(\frac{\pi}{3}\right)$ is

(A) 0 (B) $\frac{1}{\sqrt{3}}$

(C) $\sqrt{3}$ (D) None of these

Key: C

Hint: $\int\frac{\sec^2 x-2010}{\sin^{2010}x}dx$

$$=\int\sec^2 x(\sin x)^{-2010}-2010\int\frac{1}{(\sin x)^{2010}}dx=I_1-I_2$$

Applying by parts on I_1, we get

$$I_1=\frac{\tan x}{(\sin x)^{2010}}+2010\int\frac{\tan x\cos x}{(\sin x)^{2011}}dx=\frac{\tan x}{(\sin x)^{2010}}+2010\int\frac{dx}{(\sin x)^{2010}}$$

$$\Rightarrow I=I_1-I_2=\frac{\tan x}{(\sin x)^{2010}}=\frac{P(x)}{(\sin x)^{2010}}$$

$$P\left(\frac{\pi}{3}\right)=\tan\frac{\pi}{3}=\sqrt{3}$$

85. If $I=\int\frac{dx}{\sin\left(x-\frac{\pi}{3}\right)\cos x}$, then I equals

a) 2 log $\left|\sin x+\sin\left(x-\frac{\pi}{3}\right)\right|+C$ b) $2\log\left|\sin\left(x-\frac{\pi}{3}\right)\sec x\right|+C$

c) $2\log\left|\sin x-\sin\left(x-\frac{\pi}{3}\right)\right|+C$ d) None of these

Key: B

Hint: $I=\frac{1}{\cos\left(\frac{\pi}{3}\right)}\int \frac{\cos\left(x-\left(x-\frac{\pi}{3}\right)\right)}{\sin\left(x-\frac{\pi}{3}\right)\cos x}dx$

86. $\int_0^1 \frac{x^6-x^3}{(2x^3+1)^3}dx$ is equal to

(A) 0 (B) $-\frac{1}{6}$

(C) $-\frac{1}{12}$ (D) $-\frac{1}{36}$

Key. D

Sol. $\int_0^1 \frac{\left(1-\frac{1}{x^3}\right)}{\left(2x+\frac{1}{x^2}\right)^3}dx$

and proceed

87. Suppose f, g are continuous and differentiable on [0,b], f', g' are non-negative on [0,b] and f is non constant with f(0) = 0, then the minimum value of $\int_0^a g(x)f'(x)dx+\int_0^b g'(x)f(x)dx$ on $a\in(0,b]$ is

(A) $f(a).g(a)$ (B) $f(b).g(b)$

(C) $f(a).g(b)$ (D) $f(b).g(a)$

Key. C

Sol. Integrating by parts the integral

$\int_0^a g(x)f'(x)dx+\int_0^b g'(x)f(x)dx$

$=g(x)f(x)\Big|_0^a-\int_0^a g'(x)f(x)dx \;+\int_0^b g'(x)f(x)dx$

$=f(a).g(a)+\int_a^b g'(x)f(x)dx$

$\geq f(a).g(a)+\int_a^b g'(x)f(x)dx=f(a).g(b)$

88. If f(x) be a real valued function, $f(x)+f(x+4)=f(x+2)+f(x+6)$, $g(x)=\int_x^{x+8} f(t)dt$. Then $g^1(x)$ is equal to

a) $f(x)$ b) $f(x+8)$ c) 8 d) 0

Key. D

Sol. Conceptual

89. Value of $\int_1^5\left(\sqrt{x+2\sqrt{x-1}}+\sqrt{x-2\sqrt{(x-1)}}\right)dx$ is

a) $\frac{8}{3}$ b) $\frac{16}{3}$ c) $\frac{32}{3}$ d) $\frac{34}{3}$

Key. D

Sol. $\because \sqrt{x+\left(2\sqrt{(x-1)}\right)}=\sqrt{\sqrt{(x-1)}^2+1^2+2\sqrt{(x-1)}}$

$=\sqrt{(x-1)}+1$

And $\sqrt{x-2\sqrt{(x-1)}}=\sqrt{\left(\sqrt{(x-1)}\right)^2+1^2-2\sqrt{(x-1)}}$

$=\left|\sqrt{(x-1)}-1\right|$

Then $\int_1^5 \sqrt{x+2\sqrt{(x-1)}}+\sqrt{x-2\sqrt{(x-1)}}dx$

$=\int_1^5\left(\sqrt{(x-1)}+1\right)+\int_1^5\left|\sqrt{(x-1)}-1\right|dx$

$=\int_1^5\left(\sqrt{(x-1)}+1\right)dx+\int_1^2\left(1-\sqrt{(x-1)}\right)dx+\int_2^5\left(\sqrt{(x-1)}-1\right)dx$

$=\int_0^4\left(\sqrt{x}+1\right)dx+\int_0^1\left(1-\sqrt{x}\right)dx+\int_1^4\left(\sqrt{x}-1\right)dx$

$=\left[\frac{2}{3}\left(x^{3/2}\right)+x\right]_0^4+\left[x-\frac{2}{3}x^{3/2}\right]_0^1+\left[\frac{2}{3}x^{3/2}-x\right]_1^4$

$=\left(\frac{16}{3}+4\right)+\left(1-\frac{2}{3}\right)+\left(\frac{16}{3}-4\right)+\left(\frac{2}{3}-1\right)=\frac{32}{3}$

90. $\int_{-\pi/4}^{\pi/4}\frac{e^x.\sec^2 xdx}{e^{2x}-1}$ is equal to

a) 0 b) 2 c) e d) 2e

Key. A

Sol. Let $I=\int_{-\pi/4}^{\pi/4}\frac{e^x\sec^2 xdx}{e^{2x}-1}$

If $f(x)=\frac{e^x\sec^2 x}{e^{2x}-1}$

$\therefore$ $f(-x)=\frac{e^x\sec^2 x}{e^{2x}-1}$

$=\frac{e^x\sec^2 x}{1-e^{2x}}$

$=-\frac{e^x\sec^2 x}{e^{2x}-1}$

$= -f(x)$

$\therefore$ I = 0 (f(x) is odd function)

91. Let $f(x) = \frac{1}{2}a_0 + \sum_{i=1}^{n} a_i \cos(ix) + \sum_{j=1}^{n} b_j \sin(jx)$, then $\int_{-\pi}^{\pi} f(x)\cos kx dx$ is equal to

a) a_k b) b_k c) πa_k d) πb_k

Key. C

Sol. Conceptual

92. Let $\int_0^x \left(\frac{bt\cos 4t - a\sin 4t}{t^2}\right)dt = \frac{a\sin 4x}{x}$, then a and b are given by

a) 1/4, 1 b) 2,2 c) -1, 4 d) 2, 4

Key. A

Sol. Since, $\int_0^x \left(\frac{bt\cos 4t - a\sin 4t}{t^2}\right)dt = \frac{a\sin 4x}{x}$

Differentiating both sides w.r.t. x

$\therefore \frac{bx\cos 4x - a\sin 4x}{x^2} = \frac{a\{4x\cos 4x - \sin 4x\}}{x^2}$

On comparing $b = 4a$

$a = 1/4$ and $b = 1$

93. If $f'''(x) = k$ in $[0, a]$, then $\int_0^a f(x)dx - \left\{xf(x) - \frac{x^2}{2!}f'(x) + \frac{x^3}{3!}f''(x)\right\}_0^a$ is

a) $-ka^4/12$ b) $ka^4/24$ c) $-ka^4/24$ d) $ka^4/12$

Key. C

Sol. Conceptual

94. If $f(x)$ is a differentiable function and $\int_0^{x^3} t^2 f(t)dt = \frac{3}{13}x^{13} + 5$ then $f\left(\frac{8}{27}\right) =$

A) 8 / 27 B) 16 / 27 C) 16 / 81 D) 8 / 9

Key. C

Sol. Diff. w.r.t. $x \Rightarrow f(x^3) = x^4$

95. $\int_{-2\pi}^{2\pi} \frac{\sin^6 x}{(\sin^6 x + \cos^6 x)(1+e^{-x})}dx =$

A) 2π B) π C) $\pi/2$ D) 4π

Key. B

Sol. $\int_{-a}^{a} f(x)dx = \int_{0}^{a}\left(f(x) + f(-x)\right) dx$

96. Let $f(x), g(x), h(x)$ be continous in $[0, 2a]$ and satisfies
$f(2a-x) = f(x), g(2a-x) = g(x), h(x) + h(2a-x) = 3,\ f(2a-x)g(2a-x) = f(x)g(x)$
then $\int_{0}^{2a} f(x)g(x)h(x)dx =$

A) $\int_{0}^{2a} f(x)g(x)dx$ B) $3\int_{0}^{a} f(x)g(x)dx$ C) $2\int_{0}^{a} f(x)g(x)dx$ D) $\int_{0}^{a} f(x)g(x)dx$

Key. B

Sol. $I = \int_{0}^{2a} f(x)g(x)h(x)dx = \int_{0}^{2a} f(2a-x)g(2a-x)h(2a-x)dx = \int_{0}^{2a} f(x)g(x)\left[3-h(x)\right]dx$

$I = \frac{3}{2}\int_{0}^{2a} f(x)g(x)dx = 3\int_{0}^{a} f(x)g(x)dx$

97. $\int \frac{\sqrt[3]{x^2} + \sqrt[6]{x}}{x\left(1+\sqrt[3]{x}\right)} dx =$

A) $\frac{1}{2}\tan^{-1}\left(\frac{x^6 + x^{-6}}{2}\right) + C$ B) $\frac{x^{24}}{2} - \log(1+x^{24}) + \tan^{-1}(x^3) + C$

C) $\frac{3}{2}x^{12} + 6\tan^{-1}(x^6) + C$ D) $6\tan^{-1}(x^6) + 3x^{12} - 6\log_e\sqrt{1+x^{12}} + C$

Key. D

Sol. $x = t^6$

98. $\int \frac{dx}{x^{20}(1+x^{20})^{\frac{1}{20}}} =$

A) $-\frac{1}{19}\left(1+\frac{1}{x^{20}}\right)^{\frac{19}{20}} + C$ B) $\frac{1}{21}\left(1-\frac{1}{x^{20}}\right)^{\frac{19}{20}} + C$

C) $\frac{1}{19}\left(1-\frac{1}{x^{20}}\right)^{\frac{19}{20}} + C$ D) $-\frac{1}{21}\left(1+\frac{1}{x^{20}}\right)^{\frac{19}{20}} + C$

Key. A

Sol. $\int \frac{dx}{x^{21}\left(\frac{1}{x^{20}}+1\right)^{\frac{1}{20}}}$, Put $\frac{1}{x^{20}}+1=t$

99. $\int \frac{\sin\left(\frac{\pi}{4}-x\right)dx}{2+\sin 2x}=A\tan^{-1}\left(f(x)\right)+B$, where A, B are constants then the range of $Af(x)$ is

A) $[-1,1]$ B) $[-\sqrt{2},\sqrt{2}]$ C) $[0,1]$ D) $[-1,0]$

Key. A

Sol. $\frac{1}{\sqrt{2}}\int \frac{d(\sin x+\cos x)}{(\sin x+\cos x)^2+1}$

100. The area bounded by the curves $y=2-|x-1|,\ y=\sin x;\ x=0$ and $x=2$ is

A) $1+2\cos^2 1$ B) $2+\sin^2 1$ C) $\frac{\pi}{2}$ D) $1+\log 2$

Key. A

Sol. Area $=\frac{3}{2}+(\cos 1-1)+3-\frac{3}{2}+(\cos 2-\cos 1)=2+\cos 2$

101. $\int_0^{\pi/2} \frac{\sin x}{1+\cos x+\sin x}dx=$

A) $\frac{\pi}{4}$ B) $\frac{\pi}{4}+\log\sqrt{2}$ C) $\frac{\pi}{4}-\log\sqrt{2}$ D) $\frac{\pi}{4}-\log 2$

Key. C

Sol. $I=\int_0^{\pi/2} \frac{\sin\frac{x}{2}}{\sin\frac{x}{2}+\cos\frac{x}{2}}dx=2\int_0^{\pi/4} \frac{\sin x}{\sin x+\cos x}dx$

Let $\sin x=A(\sin x+\cos x)+B(\cos x-\sin x)$ $A-B=1$ and $A+B=0,\ A=\frac{1}{2},\ B=\frac{-1}{2}$

$I=2\times\frac{1}{2}\times\frac{\pi}{4}-2\times\frac{1}{2}\left[\log(\sin x+\cos x)\right]_0^{\pi/4}=\frac{\pi}{4}-\log\sqrt{2}$

102. $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^3+6k^2+11k+5}{(k+3)!}=$

A) $\frac{2}{3}$ B) 1 C) $\frac{5}{3}$ D) $\frac{7}{3}$

Key. C

Sol. $k^3+6k^2+11k+5=(k+1)(k+2)(k+3)-1$

$\therefore \frac{k^3+6k^2+11k+5}{(k+3)!}=\frac{1}{k!}-\frac{1}{(k+3)!}$

103. $I_n=\int_0^{\pi/2}\cos^n x\cos(nx)dx,\ n\in N$ then $\sqrt{I_{2001}:I_{2002}}$ can be the eccentricity of

A) Parabola B) Ellipse C) Circle D) Hyperbola

Key. D

Sol. $I_{n+1}=\int_0^{\pi/2}\cos^{n+1}x\cos(n+1)x\,dx$

$=\int_0^{\pi/2}\cos^{n+1}(\cos nx\cos x-\sin nx\sin x)dx$

$=\int_0^{\pi/2}\cos^n x\cos nx(1-\sin^2 x)dx-\int_0^{\pi/2}\cos^{n+1}x.\sin nx.\sin x\,dx$

$\therefore\ I_{n+1}=I_n-I_{n+1}\Rightarrow 2I_{n+1}=I_n$

$\therefore\ I_n:I_{n+1}=2$

104. If $\int\frac{x\,dx}{\sqrt[2012]{(1+x^2)^{1012}(2+x^2)^{3012}}}=\frac{\alpha}{\beta}(1-f(x))^{\frac{\beta}{2\alpha}}+k$ then which is true

A) $\alpha=503;\ \beta=500,\ f(\sqrt{2})=\frac{1}{\beta-\alpha}$ B) $\alpha=503;\ \beta=250,\ f(\sqrt{2})=\frac{1}{\alpha-\beta}$

C) $\alpha=503;\ \beta=500,\ f(1)=\frac{1}{\alpha-\beta}$ D) $\alpha=503;\ \beta=225,\ f(\sqrt{3})=\frac{1}{\alpha-\beta}$

Key. C

Sol. $\int \frac{x}{\left(1+x^2\right)^2\left(\frac{2+x^2}{1+x^2}\right)^{\frac{3012}{2012}}}dx$

Let $\frac{2+x^2}{1+x^2}=t$ then $f(x)=\frac{1}{2+x^2}$ $f(1)=\frac{1}{3}=\frac{1}{\alpha-\beta}$

Where $\alpha=503:\beta=500$

105. $I_n=\int_0^1 x^n\tan^{-1}x\,dx.$ If $a_n I_{n+2}+b_n I_n=c_n\ \forall\ n\in N,\ n\geq 1$ then

A) $a_1,a_2,a_3.......$ are in A.P. B) $b_1,b_2,b_3.......$ are in G.P.

C) $c_1,c_2,c_3.......$ are in H.P. D) $a_1,a_2,a_3.......$ are in H.P.

Key. A

Sol. $I_n=\left(\frac{x^{n+1}}{n+1}\tan^{-1}x\right)_0^1-\int_0^1\frac{x^{n+1}}{n+1}\cdot\frac{1}{1+x^2}dx$

$(n+1)I_n=\frac{\pi}{4}-\int_0^1\frac{x^{n+1}}{1+x^2}dx$

$(n+3)I_{n+2}=\frac{\pi}{4}-\int_0^1\frac{x^{n+3}}{1+x^2}dx$

$\therefore\ (n+1)I_n+(n+3)I_{n+2}=\frac{\pi}{2}-\frac{1}{n+2}$

$\therefore\ a_n=(n+3)\Rightarrow a_1,a_2,a_3......$ are in A.P.

$b_n=(n+1)\Rightarrow b_1,b_2.....$ are in A.P.

$c_n=\frac{\pi}{2}-\frac{1}{n+2}$ not in any progression.

106. $\int_0^1\frac{x^6-x^3}{(2x^3+1)^3}dx$ is equal to

(A) 0 (B) $-\frac{1}{6}$

(C) $-\frac{1}{12}$ (D) $-\frac{1}{36}$

Key. D

Sol. $\int_0^1 \frac{\left(1-\frac{1}{x^3}\right)}{\left(2x+\frac{1}{x^2}\right)^3}dx$

and proceed

107. Suppose f, g are continuous and differentiable on [0,b], f', g' are non-negative on [0,b] and f is non constant with f(0) = 0, then the minimum value of $\int_0^a g(x)f'(x)dx + \int_0^b g'(x)f(x)dx$ on $a \in (0, b]$ is

(A) $f(a).g(a)$ (B) $f(b).g(b)$

(C) $f(a).g(b)$ (D) $f(b).g(a)$

Key. C

Sol. Integrating by parts the integral

$$\int_0^a g(x)f'(x)dx + \int_0^b g'(x)f(x)dx$$

$$= g(x)f(x)\Big|_0^a - \int_0^a g'(x)f(x)dx \;\; + \int_0^b g'(x)f(x)dx$$

$$= f(a).g(a) + \int_a^b g'(x)f(x)dx$$

$$\geq f(a).g(a) + \int_a^b g'(x)f(x)dx = f(a).g(b)$$

108. Let $I_1 = \int_{\sec^2 z}^{2-\tan^2 z} x\, f\left(x(3-x)\right)dx$ and let $I_2 = \int_{\sec^2 z}^{2-\tan^2 z} f\left(x(3-x)\right)dx$

where 'f' is a continuous function and 'z' is any real number, then $\frac{I_1}{I_2} =$

A) $\frac{3}{2}$ B) $\frac{1}{2}$ C) 1 D) $\frac{2}{3}$

Key. A

Sol.

$$I_1 = \int_{\sec^2 z}^{2-\tan^2 z} x\, f\left(x(3-x)\right)dx$$

$$I_2 = \int_{\sec^2 z}^{2-\tan^2 z} f\left(x(3-x)\right)dx$$

$$2I_1 = \int_{\sec^2 z}^{2-\tan^2 z} (3-x) f((3-x)x)\,dx$$

$$2I_1 = \int_{\sec^2 z}^{2-\tan^2 z} 3f(x(3-x))\,dx$$

$$= 3\int_{\sec^2 z}^{2-\tan^2 z} ((x)(3-x))\,dx$$

$$2I_1 = 3I_2$$

$$\Rightarrow \frac{I_1}{I_2} = \frac{3}{2}$$

109. If $\int \frac{\sec^2 x - 2010}{\sin^{2010} x} dx = \frac{P(x)}{\sin^{2010} x} + C,$ then value of $P\left(\frac{\pi}{3}\right)$ is

A) 0 B) $\frac{1}{\sqrt{3}}$ C) $\sqrt{3}$ D) None of these

Key. C

Sol. $\int \frac{\sec^2 x - 2010}{\sin^{2010} x} dx$

$$= \int \sec^2 x(\sin x)^{-2010} - 2010\int \frac{1}{(\sin x)^{2010}} dx = I_1 - I_2$$

Applying, by parts on I_1, we get

$$I_1 = \frac{\tan x}{(\sin x)^{2010}} + 2010\int \frac{\tan x \cos x}{(\sin x)^{2011}} dx = \frac{\tan x}{(\sin x)^{2010}} + 2010\int \frac{dx}{(\sin x)^{2010}}$$

$$\Rightarrow I = I_1 - I_2 = \frac{\tan x}{(\sin x)^{2010}} = \frac{P(x)}{(\sin x)^{2010}}$$

$$P\left(\frac{\pi}{3}\right) = \tan\frac{\pi}{3} = \sqrt{3}$$

110. If $c > 0$ and the area of the region enclosed by the parabolas $y = x^2 - c^2$ *and* $y = c^2 - x^2$ is 576, then c =

a) 6 b) 4 c) 3 d) 8

Key: A

Hint: Area between the two parabolas = $4\int_0^C (c^2 - x^2)\,dx = \frac{8c^3}{3} = 576$

only if c = 6

111. Let $f(x) = x^2 + 6x + 1$ and let R denote the set of points (x, y) in the XY-plane such that $f(x) + f(y) \le 0$ and $f(x) - f(y) \le 0$. Then the area of the region R is

A) 6π B) $3\pi + 2$ C) $2\pi + 8$ D) 8π

Key: D

Hint: $f(x) + f(y) \le 0 \Rightarrow (x+3)^2 + (y+3)^2 \le 16$

$f(x) - f(y) \le 0 \Rightarrow (x-y)(x+y+6) \le 0$

112. The quadrilateral formed by the lines $y = ax + c, y = ax + d, y = bx + c$ and $y = bx + d$ has area 18. The quadrilateral formed by the lines $y = ax + c, y = ax - d, y = bx + c$ and $y = bx - d$ has area 72. If a, b, c, d are positive integers then the least possible value of the sum $a + b + c + d$ is

A) 13 B) 14 C) 15 D) 16

Key: D

Hint: $\frac{(c-d)^2}{|a-b|} = 18$ and $\frac{(c+d)^2}{|a-b|} = 72$.

$a = 3, b = 1, d = 3, c = 9$ is a solution for which the minimum is attained.

113. Area of a square ABCD is 36 and side AB is parallel to the X-axis. Vertices A, B and C lie on the graphs of $y = \log_a x, y = 2\log_a x$ and $y = 3\log_a x$ respectively. Then $a =$

A) $3^{1/6}$ B) $\sqrt{3}$ C) $6^{1/3}$ D) $\sqrt{6}$

Key: A

Hint: Let $A = (p, \log_a p); B = (q, 2\log_a q),\ p, q > 0 \,\&\, a > 0, a \ne 1$, then $C = (q, 3\log_a q)$

$AB \text{ P } X - \text{axis} \Rightarrow p = q^2 . |AB| = 6 \Rightarrow |p - q| = 6$ also $|\log_a q| = 6$.

114. The area bounded by the curve $\left(y - \text{Sin}^{-1}x\right)^2 = x - x^2$ s

a) $\frac{\pi}{4}$ b) $\frac{\pi}{2}$ c) π d) $\frac{\pi}{3}$

Key: A

Hint: The given curves on $y = \text{Sin}^{-1}x - \left(\sqrt{x - x^2}\right)$ & $y = \text{Sin}^{-1}x + \left(\sqrt{x - x^2}\right)$

$$\text{Requred area} = \int_0^1 2\sqrt{x - x^2}\,dx$$

115. Let F(x) = $\sin x \int_0^x \cos t\,dt + 2\int_0^x t\,dt + \cos^2 x - x^2$. Then area bounded by xF(x) and ordinate x = 0 and x = 5 with x-axis is

(A) 16 (B) $\frac{25}{2}$ (C) $\frac{35}{2}$ (D) 25

Key: B

Hint: $F(x)=\sin x\int_0^x \cos t\,dt+2\int_0^x t\,dt-x^2+\cos^2 x$

$=\sin x(\sin t)_0^x+2\left(\frac{t^2}{2}\right)_0^x-x^2+\cos^2 x$

= $\sin^2 x + x^2 - x^2 + \cos^2 x = 1$

$A=\int_0^5 xF(x)dx=\int_0^5 (x)(1)dx=\left[\frac{x^2}{2}\right]_0^5=\frac{25}{2}$

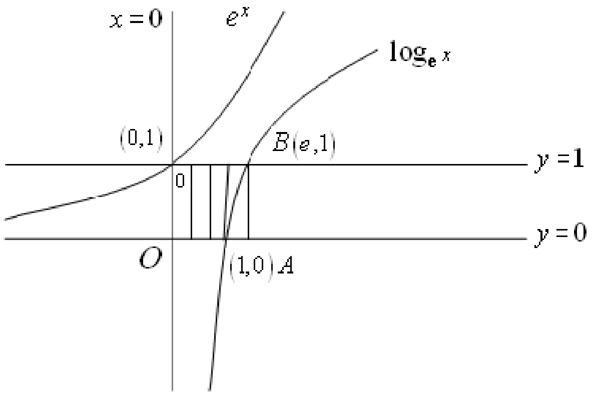
116. Area bounded by the curves $y=e^x,\ y=\log_e x$ and the lines $x=0,\ y=0,\ y=1$ is

A) $e^2+2 sq.units$ B) $e+1 sq.unit$

C) $e+2 sq.units$ D) $e-1\ sq.unit$

Key: D

Hint:



Area = Area of rectangle OABC $-\int_0^1 \log_e x$

117. The area of the loop of the curve $y^2=x^4(x+2)$ is [in square units]

(A) $\frac{32\sqrt{2}}{105}$ (B) $\frac{64\sqrt{2}}{105}$ (C) $\frac{128\sqrt{2}}{105}$ (D) $\frac{256\sqrt{2}}{105}$

Key: D

Hint: Area = $2\int_{-2}^{0} y\,dx=2\int_{-2}^{0} x^2\sqrt{x+2}\,dx=4\sqrt{2}\int_0^2 \left(z^2-2\right)^2 z^2dz(where\sqrt{x+2}=z)$

= $4\left[\frac{z^7}{7}-\frac{4z^5}{5}+\frac{4z^3}{3}\right]_0^{\sqrt{2}}$

$= \frac{256\sqrt{2}}{105}$

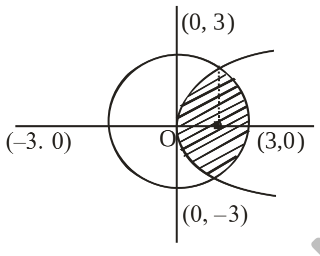
118. The area of the smaller portion enclosed by the curves $x^2+y^2=9$ and $y^2=8x$ is

A) $\frac{\sqrt{2}}{3}+\frac{9\pi}{4}-\frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)$ B) $2\left(\frac{\sqrt{2}}{3}+\frac{9\pi}{4}-\frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right)$

C) $2\left(\frac{\sqrt{2}}{3}+\frac{9\pi}{4}+\frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right)$ D) $\frac{\sqrt{2}}{3}+\frac{9\pi}{4}+\frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)$

Key: B

HINT :



$x^2+y^2=9,$

$x^2+8x-9=0$

$x=\frac{-8\pm\sqrt{64+36}}{2}$

$x=\frac{-8\pm 10}{2}-9,1$

$\boxed{x=1}$

Area enclosed $=2\left[\int_0^1 2\sqrt{2x}dx+\int_1^3\sqrt{9-x^2}\,dx\right]=2\left[2\sqrt{2}\int_0^1\sqrt{x}dx+\int_1^3\sqrt{9-x^2}\,dx\right]$

On simplifying we get

$=2\left[\frac{\sqrt{2}}{3}+\frac{9\pi}{4}-\frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right]$

119. The area of the region in the xy-plane defined by the inequalities $x-2y^2\geq 0$, $1-x-|y|\geq 0$ is

A) $\frac{1}{2}$ B) $\frac{1}{3}$ C) $\frac{1}{4}$ D) $\frac{7}{12}$

Key: D

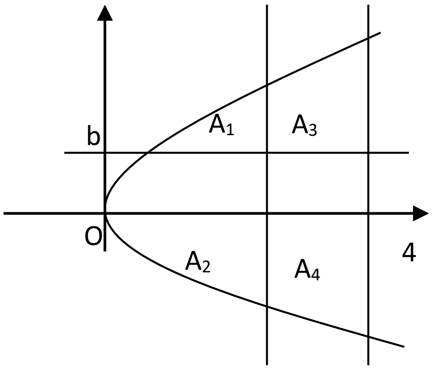
Hint: Area $=2\int_0^{1/2}\sqrt{\frac{x}{2}}dx+\frac{1}{4}=\frac{7}{12}$

120. Area bounded by curve $y^2 = x$ and $x = 4$ is divided into 4 equal parts by the lines $x = a$ and $y = b$ then.

a) Area of each part = $\frac{8}{3}$ b) b = 0

c) $a=\sqrt{2}$ d) $a=(16)^{1/3}$

Key: D



Hint: Total area = $2\int_0^4 \sqrt{x}\,dx=\frac{32}{3}$

Area of each part = 8/3

$$A_3=A_4 \Rightarrow \int_a^4\left(\sqrt{x}-b\right)dx=$$

$$\int_a^4\left(b+\sqrt{x}\right)dx=\frac{8}{3}\Rightarrow b=0$$

$$\int_a^4\sqrt{x}\,dx=\frac{8}{3}\Rightarrow a^3=16$$

121. Area of the region in which point $p(x,y)$, $\{x>0\}$ lies; such that $y\le\sqrt{16-x^2}$ and $\left|\tan^{-1}\left(\frac{y}{x}\right)\right|\le\frac{\pi}{3}$ is

(A) $\left(\frac{16}{3}\pi\right)$ (B) $\left(\frac{8\pi}{3}+8\sqrt{3}\right)$ (C) $\left(4\sqrt{3}-\pi\right)$ (D) $\left(\sqrt{3}-\pi\right)$

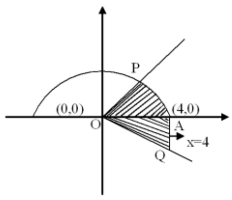
Key: B

Hint: Required area is the area of shaded region (APOQ)

= area of ΔOAQ + area of sector (OAP)

$$=\frac{1}{2}\times4\times4\sqrt{3}+\frac{\pi(4\times4)}{6}$$

$$=\left(\frac{8\pi}{3}+8\sqrt{3}\right)$$



122. Area bounded between the curves $y=\sqrt{4-x^2}$ and $y^2=3|x|$ is/are

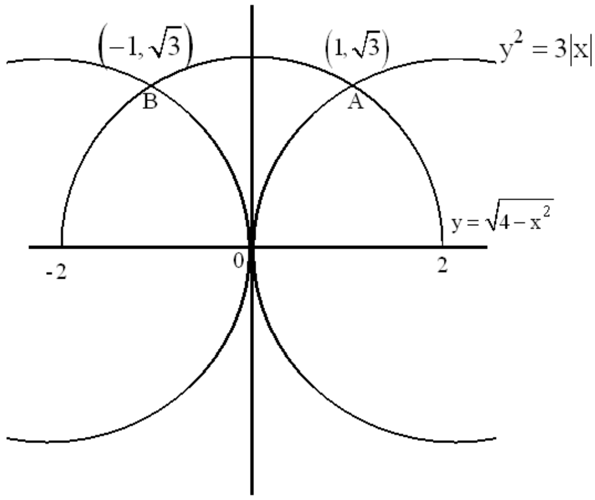
(A) $\frac{\pi-1}{\sqrt{3}}$ (B) $\frac{2\pi-1}{3\sqrt{3}}$ (C) $\frac{2\pi-\sqrt{3}}{3}$ (D) $\frac{2\pi-\sqrt{3}}{3\sqrt{3}}$

Key: C

Hint: Required area = $2\int_0^1\left(\sqrt{4-x^2}-\sqrt{3x}\right)dx$

$$=2\left(\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)-\frac{\sqrt{3}.2x^{3/2}}{3}\right)_0^1$$

$$=\frac{2\pi-\sqrt{3}}{3}$$



123. Let $f:[0,\infty)\to R$ be a continuous and strictly increasing function such that $f^3(x)=\int_0^x t.f^2(t)dt, \forall x>0$. The area enclosed by $y=f(x)$, the x-axis and the ordinate at x=3, is

(A) 1 (B) $\frac{3}{2}$ (C) 2 (D) 3

Key: B

Hint: $f(x)=\frac{x^2}{6}$

$A=\frac{1}{6}\int_0^3 x^2 dx = 3/2$

124. Let f(x) = x + sin x. The area bounded by $y=f^{-1}(x), y=x, x\in[0,\pi]$ is

(a) 1 (b) 2 (c) 3

(d) canot be found because $f^{-1}(x)$ cannot be determined

Key: B

Hint: The curves given by y = x + sin x and $y = f^{-1}(x)$ are images of each other in the line y = x.

Hence required area = $\int_0^\pi ((x+\sin x)-x)dx = -[\cos x]_0^\pi = 2$

125. The area of the region bounded by the curves $|x+y|\le 2$, $|x-y|\le 2$ and $2x^2+6y^2\ge 3$ is

(A) $\left(8+\frac{\sqrt{3}}{2}\pi\right)$ sq. units (B) $\left(8-\frac{\sqrt{3}}{2}\pi\right)$ sq. units

(C) $\left(4-\frac{3\sqrt{3}}{2}\pi\right)$ sq. units (D) $\left(8-\frac{3\sqrt{3}}{2}\pi\right)$ sq. units

Key : B

Sol :

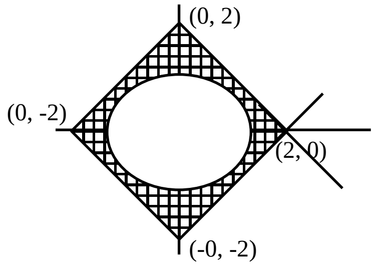
$2x^2+6y^2\ge 3$ . . . . (1)

area of ellipse = $\pi \times \frac{\sqrt{3}}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{\pi\sqrt{3}}{2}$

$|x + y| \le 2 \Rightarrow -2 \le (x + y) \le 2$ . . . .(2)

$|x - y| \le 2 \Rightarrow -2 \le (x - y) \le 2$ . . . . (3)

Required area = $\left(8 - \frac{\pi\sqrt{3}}{2}\right)$ sq. units



126. The curve $y = (|x| - 1)\text{sgn}(x - 1)$ divides $\frac{9x^2}{64} + \frac{4}{25}y^2 = \frac{1}{\pi}$ in two parts having area A_1 and A_2 (where $A_1 < A_2$), then

a) $\frac{A_1}{A_2} = \frac{7}{13}$ b) $\frac{A_1}{A_2} = \frac{3}{7}$

c) $A_1 = \frac{7}{3}$ d) $A_2 = \frac{13}{7}$

Key: A

Sol : $A_1 = \frac{10}{3} - 1, A_2 = \frac{10}{3} + 1 \Rightarrow \frac{A_1}{A_2} = \frac{7}{13}$

127. Area bounded by the circle which is concentric with the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ and which passes through $\left(4, -\frac{9}{5}\right)$, the vertical chord common to both circle and ellipse on the positive side of x-axis is

a) $\frac{481}{25}\tan^{-1}\left(\frac{9}{20}\right) - \frac{36}{5}$ b) $2\tan^{-1}\left(\frac{9}{20}\right)$

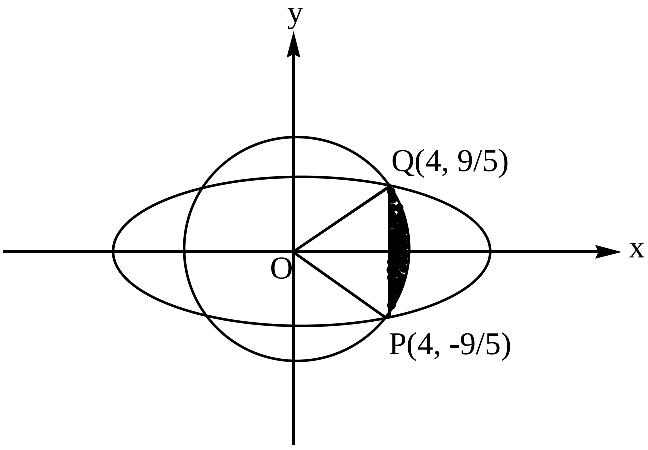
c) $\frac{481}{25}\tan^{-1}\left(\frac{9}{20}\right)$ d) none of these

Key : a

Sol : As eccentricity of ellipse = $\frac{4}{5}$

Co-ordinate of foci = (4, 0), (-4, 0)

$\Rightarrow \left(4, -\frac{9}{5}\right)$ is one of the end point of latus – rectum



$\Rightarrow$ required area is

$$\frac{1}{2\pi}\times\pi\times\left(4^2+\frac{9^2}{5^2}\right)\times 2\tan^{-1}\left(\frac{9}{20}\right)-\text{ area of }\Delta POQ$$

$$=\frac{481}{25}\tan^{-1}\left(\frac{9}{20}\right)-\frac{1}{2}\times4\times\left(\frac{18}{5}\right)=\frac{481}{25}\tan^{-1}\left(\frac{9}{20}\right)-\frac{36}{5}$$

128. Area of the region in which point p(x, y), {x > 0} lies; such that $y\le\sqrt{16-x^2}$ and $\left|\tan^{-1}\left(\frac{y}{x}\right)\right|\le\frac{\pi}{3}$ is

a) $\left(\frac{16}{3},\pi\right)$

b) $\left(\frac{8\pi}{3}+8\sqrt{3}\right)$

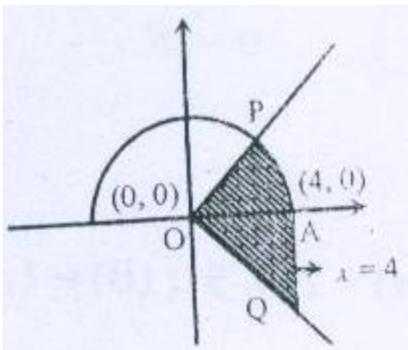
c) $\left(4\sqrt{3}-\pi\right)$

d) none of these

Key : b

Sol : Required area is the area of shaded region (APOQ)

= area of ΔOAQ + area of sector (OAP)



$$=\frac{1}{2}\times4\times4\sqrt{3}+\frac{\pi(4\times4)}{6}$$

$=\left(\frac{8\pi}{3}+8\sqrt{3}\right)$

129. If $c>0$ and the area of the region enclosed by the parabolas $y=x^2-c^2$ *and* $y=c^2-x^2$ is 576, then c =

a) 6 b) 4 c) 3 d) 8

Key: A

Hint: Area between the two parabolas = $4\int_0^C\left(c^2-x^2\right)dx=\frac{8c^3}{3}=576$

only if c = 6

130. Let $f(x)=x^2+6x+1$ and let R denote the set of points (x,y) in the XY-plane such that $f(x)+f(y)\le 0$ and $f(x)-f(y)\le 0$. Then the area of the region R is

A) 6π B) $3\pi+2$ C) $2\pi+8$ D) 8π

Key: D

Hint: $f(x)+f(y)\le 0 \Rightarrow (x+3)^2+(y+3)^2\le 16$

$f(x)-f(y)\le 0 \Rightarrow (x-y)(x+y+6)\le 0$

131. The quadrilateral formed by the lines $y=ax+c, y=ax+d, y=bx+c$ and $y=bx+d$ has area 18. The quadrilateral formed by the lines $y=ax+c, y=ax-d, y=bx+c$ and $y=bx-d$ has area 72. If a,b,c,d are positive integers then the least possible value of the sum $a+b+c+d$ is

A) 13 B) 14 C) 15 D) 16

Key: D

Hint: $\frac{(c-d)^2}{|a-b|}=18$ and $\frac{(c+d)^2}{|a-b|}=72$.

$a=3, b=1, d=3, c=9$ is a solution for which the minimum is attained.

132. Area of a square ABCD is 36 and side AB is parallel to the X-axis. Vertices A, B and C lie on the graphs of $y=\log_a x, y=2\log_a x$ and $y=3\log_a x$ respectively. Then $a=$

A) $3^{1/6}$ B) $\sqrt{3}$ C) $6^{1/3}$ D) $\sqrt{6}$

Key: A

Hint: Let $A=(p,\log_a p); B=(q,2\log_a q),\ p,q>0 \,\&\, a>0, a\ne 1$, then $C=(q,3\log_a q)$

$AB\,P\,X-\text{axis}\Rightarrow p=q^2.|AB|=6\Rightarrow|p-q|=6$ also $|\log_a q|=6$.

133. The area bounded by the curve $\left(y-\text{Sin}^{-1}x\right)^2=x-x^2$ s

a) $\frac{\pi}{4}$ b) $\frac{\pi}{2}$ c) π d) $\frac{\pi}{3}$

Key: A

Hint: The given curves on $y = \text{Sin}^{-1}x - \left(\sqrt{x - x^2}\right)$ & $y = \text{Sin}^{-1}x + \left(\sqrt{x - x^2}\right)$

$$\text{Requred area} = \int_0^1 2\sqrt{x - x^2}\,dx$$

134. Let F(x) = $\sin x \int_0^x \cos t\,dt + 2\int_0^x t\,dt + \cos^2 x - x^2$. Then area bounded by xF(x) and ordinate x = 0 and x = 5 with x-axis is

(A) 16 (B) $\frac{25}{2}$ (C) $\frac{35}{2}$ (D) 25

Key: B

Hint: $F(x) = \sin x \int_0^x \cos t\,dt + 2\int_0^x t\,dt - x^2 + \cos^2 x$

$= \sin x(\sin t)_0^x + 2\left(\frac{t^2}{2}\right)_0^x - x^2 + \cos^2 x$

= $\sin^2 x + x^2 - x^2 + \cos^2 x = 1$

$A = \int_0^5 xF(x)dx = \int_0^5 (x)(1)dx = \left[\frac{x^2}{2}\right]_0^5 = \frac{25}{2}$

135. If $\int \frac{x\,dx}{\sqrt[2012]{\left(1+x^2\right)^{1012}\left(2+x^2\right)^{3012}}} = \frac{\alpha}{\beta}\left(1 - f\left(x\right)\right)^{\frac{\beta}{2\alpha}} + k$ then which is true

a) $\alpha = 503; \beta = 500,\ f\left(\sqrt{2}\right) = \frac{1}{\beta - \alpha}$ b) $\alpha = 503; \beta = 250,\ f\left(\sqrt{2}\right) = \frac{1}{\alpha - \beta}$

c) $\alpha = 503; \beta = 500,\ f\left(1\right) = \frac{1}{\alpha - \beta}$ d) $\alpha = 503; \beta = 225,\ f\left(\sqrt{3}\right) = \frac{1}{\alpha - \beta}$

Key. C

Sol. $\frac{2 + x^2}{1 + x^2} = t$

136. The area of the region in the xy-plane defined by the inequalities $x - 2y^2 \geq 0$, $1 - x - |y| \geq 0$ is

a) $\frac{1}{2}$ b) $\frac{1}{3}$ c) $\frac{1}{4}$ d) $\frac{7}{12}$

Key. D

Sol. Area $= 2\int_0^{1/2} \sqrt{\frac{x}{2}}dx + \frac{1}{4} = \frac{7}{12}$

137. The area bounded by $y = \sin^{-1} x, y = \cos^{-1} x$ and the x-axis is

1) 2+$\sqrt{2}$ 2) 2-$\sqrt{2}$ 3) $\sqrt{2}$ +1 4) $\sqrt{2}$ -1

Key. 4

Sol. By the graph

Required area $= \int_0^{1/\sqrt{2}} \sin^{-1} x dx + \int_{1/\sqrt{2}}^{1} \cos^{-1} x dx$

138. Let f be a real valued function satisfying $f\left(\frac{x}{y}\right) = f(x) - f(y)$ and $\underset{x \to 0}{Lt} \frac{f(1+x)}{x} = 3$. The area bounded by the curve y=f(x), the y-axis and the line y=3 is

1) 9e 2) 2e 3) 3e 4) none

Key. 3

Sol. $f^1(x) = \underset{h \to 0}{Lt} \frac{f(x+h) - f(x)}{h} = \underset{h \to 0}{Lt} \frac{f\left(1+\frac{h}{x}\right)}{\frac{h}{x}.x} = \frac{3}{x}$

139. Area enclosed by the curve y = f(x) defined parametrically as $x = \frac{1-t^2}{1+t^2}, y = \frac{2t}{1+t^2}$ is equal to

1) π sq.units 2) π /2 sq.unit 3) $\frac{3\pi}{4}$ sq.units 4) $\frac{3\pi}{2}$ sq.units

Key. 1

Sol. Clearly t can be any real number

Let t= $\tan\theta$

140. The maximum area of a rectangle whose two vertices lie on the x-axis and two on the curve y=3-|x| , $-3 \le x \le 3$ is

1) 9 2) 9/4 3) 3 4) 9/2

Key. 4

Sol. Take 2a and 3-a are the length of the sides of the rectangle

141. The area of the closed figure bounded by x= -1, x=2 and $y = \begin{cases} -x^2+2 & x \le 1 \\ 2x-1, & x > 1 \end{cases}$ and the x-axis is

1) 16 /3 sq. units 2) 10/3 sq. units 3) 13/3 sq. units 4) 7/3 sq. units

Key. 1

Sol. By the graph

$A = \int_{-1}^{1} \left(-x^2+2\right) + \int_{1}^{2} (2x-1)dx$

142. The value of the parameter 'a' such that the area bounded by $y = a^2x^2 + ax + 1$, positive coordinate axes and the line x=1 attains its least value, is equal to

1) $-\frac{1}{4}$ 2) $-\frac{1}{2}$ 3) $-\frac{3}{4}$ 4) -1

Key. 3

Sol. $A = \int_0^1 \left(a^2x^2 + ax + 1\right) dx$

$= \frac{a^2}{3} + \frac{a}{2} + 1$ which is minimum for a = -3/4

143. The area enclosed by the curves y= $\sqrt{4-x^2}$, $y \geq \sqrt{2}$ sin $\left(\frac{x\pi}{2\sqrt{2}}\right)$ and x-axis is divided by the y-axis in the ratio

1) $\frac{\pi^2 - 8}{\pi^2 + 8}$ 2) $\frac{\pi^2 - 4}{\pi^2 + 4}$ 3) $\frac{\pi - 4}{\pi + 4}$ 4) $\frac{2\pi^2}{2\pi + \pi^2 - 8}$

Key. 4

Sol. By the graph

Area of the left of y-axis is π ,Area of the right of y-axis = $\int_0^{\sqrt{2}} \left(\sqrt{4-x^2} - \sqrt{2}\sin\left(\frac{x\pi}{2\sqrt{2}}\right)\right)dx$

144. The area enclosed by the curves, $xy^2 = a^2(a-x)$ and $(a-x)y^2 = a^2x$ is

1) $(\pi - 2)a^2 sq.units$ 2) $(4-\pi)a^2 sq.units$ 3) $\pi a^2 / 3 sq.units$ 4) $\frac{\pi a^2}{2}$ sq. units

Key. 1

Sol. By the graph, required area

= $2\int_0^a \left[a - \frac{a^3}{a^2+y^2} - \frac{a^3}{a^2+y^2}\right]dy$ (integrating along y-axis)

145. The area of the region bounded by the parabola $(y-2)^2 = x-1$, the tangent to it at the point with the ordinate 3 and the x-axis is

1) 7 sq. units 2) 6 sq. units 3) 9 sq. units 4) 8 sq. units

Key. 3

Sol. Given parabola is $(y-2)^2 = x-1$

Tangent at (2,3) is $y-3 = \frac{1}{2}(x-2) \Rightarrow x - 2y + 4 = 0$

By the graph

$\int_0^3 \left((y-2)^2 + 1\right)dy - \int_0^3 (2y-4)dy$

146. Consider the region formed by the lines x = 0, y = 0, x = 2, y = 2. Area enclosed by the curves $y = e^x$ and y = 1n x, within this region, is being removed. Then, the area of the remaining region is

1) 2(ln2-1) sq. units 2) 2(1n2+1) sq. units

3) 2(21n2-1) sq. units 4) 2(2ln2+1)sq. units

Key. 3

Sol. By the graph

$2\int_0^{In2} (2-e^x)dx$

147. The area bounded by the curve f(x)=x+sinx and its inverse function between the ordinates x=0 and x= 2π is

1) 4π sq. units 2) 8π sq. units 3) 4 sq. units 4) 8 sq. units

Key. 4

Sol. By the graph, required Area = 4A, where

$$A=\int_0^{\pi}(x+\sin x)dx-\int_0^{\pi}xdx$$

$$=\frac{\pi^2}{2}-\cos\pi+\cos 0-\frac{\pi^2}{2}=2 \text{ square units}$$

148. The area bounded by the curves y= $\sin^{-1}|\sin x|$ and $y=\left(\sin^{-1}|\sin x|\right)^2$, where $0\le x\le 2\pi$, is

1) $\frac{1}{3}+\frac{\pi^2}{4}$ sq.units
2) $\frac{1}{6}+\frac{\pi^3}{8}$ sq. units
3) $\frac{2}{3}+\frac{\pi^2(\pi-3)}{6}$ sq .units
4) $\frac{4}{3}+\frac{\pi^2(\pi-3)}{6}$ sq .units

Key. 4

Sol. $$y=\sin^{-1}|\sin x|=\begin{cases} x, & 0\le x<\frac{\pi}{2} \\ \pi-x, & \frac{\pi}{2}\le x<\pi \\ x-\pi, & \pi\le x<\frac{3\pi}{2} \\ 2\pi-x, & \frac{3\pi}{2}\le x<2\pi \end{cases}$$

By the graph

$$4\int_0^1(x-x^2)dx+4\int_1^{\pi/2}(x^2-x)dx$$

149. Area bounded by the curve $xy^2=a^2(a-x)$ and its asymptote is

1) $\pi a^2/2$ sq. units
2) πa^2 sq. units
3) $3\pi a^2$ sq. units
4) $4\pi a^2$ sq. units

Key. 2

Sol. By the graph, required area = $2\int_0^{\infty}\frac{a^3}{y^2+a^2}dy$

150. Consider two curves $C_1: y^2=4\left[\sqrt{y}\right]x$ and $C_2: x^2=4\left[\sqrt{x}\right]y$, where[.] denotes the greatest integer function. Then the area of the region enclosed by these two curves within the square formed by the lines x=1, y=1 ,x=4,y=4 is

1) 8/3 sq. units
2) 10/3 sq. units
3) 11/3 sq. units
4) 11/4 sq. units

Key. 2

Sol. By the graph

The required area

$$A=\int_1^2(2\sqrt{x}-1)dx+\int_2^4\left(2\sqrt{x}-\frac{x^2}{4}\right)dx$$

Definite, Indefinite Integration & Areas

Integer Answer Type

1. Let $f(x)$ be differentiable function such that $f(x)=x^2+\int_0^x e^{-t}f(x-t)\,dt$ then $6f(1)=$ __

Key. 8

Sol. $f(x)=x^2+\int_0^x e^{-t}f(x-t)\,dt=x^2+e^{-x}\int_0^x e^{t}f(t)\,dt$

$\Rightarrow f^1(x)=2x-e^{-x}\left(e^x\left(f(x)-x^2\right)\right)+e^{-x}.e^x f(x)$

$\Rightarrow f^{|}(x)=2x+x^2 \Rightarrow f(x)=\frac{x^3}{3}+x^2+K$

$But,\quad f(0)=0 \Rightarrow k=0$

$\therefore f(1)=\frac{4}{3}$

2. If the value of definite integral $\int_1^a x.a^{-\left[\log_a^x\right]}dx$ where $a>1, and\,[.]$ denotes the greatest integer, is $\frac{e-1}{2}$ then the value of $5[a]$ is ____

Key. 5

Sol. Let $\log_a^x=t \Rightarrow a^t=x \Rightarrow dx=a^t\log_e^a$

$\therefore I=\ln a\int_0^1 a^t.a^{-(t)}.a^t dt=\ln a\int_0^1 a^{2t}dt=\frac{a^2-1}{2}=\frac{e-1}{2}\Rightarrow a=\sqrt{e}$

3. If $I=\int_0^{\pi} x\left(\sin^2(\sin x)+\cos^2(\cos x)\right)dx$, then $[I]=$ ____, where $[.]$ denotes the greatest integer function

Key. 4

Sol.

$$I=\int_0^{\pi}(\pi-x)\left(\left(\sin^2(\sin x)\right)+\cos^2(\cos x)\right)dx \Rightarrow 2I=2\pi\int_0^{\frac{\pi}{2}}\left(\sin^2(\sin x)+\cos^2(\cos x)\right)dx$$

$$\Rightarrow I=\pi\int_0^{\frac{\pi}{2}}\left(\sin^2(\sin x)\right)+\cos^2(\cos x)\,dx=\pi\int_0^{\frac{\pi}{2}}\left(\sin^2(\cos x)+\cos^2(\sin x)\right)dx$$

$$\Rightarrow 2I = \pi\int_0^{\frac{\pi}{2}} 2dx \Rightarrow I = \frac{\pi^2}{2}$$

4. If $\int_{\cos x}^{1} t^2 f(t)dt = 1-\cos x \ \forall \ x \in \left(0, \frac{\pi}{2}\right)$, then $\left[f\left(\frac{\sqrt{3}}{4}\right)\right] =$ ([.] denotes the greatest integer function.)

Key. 5

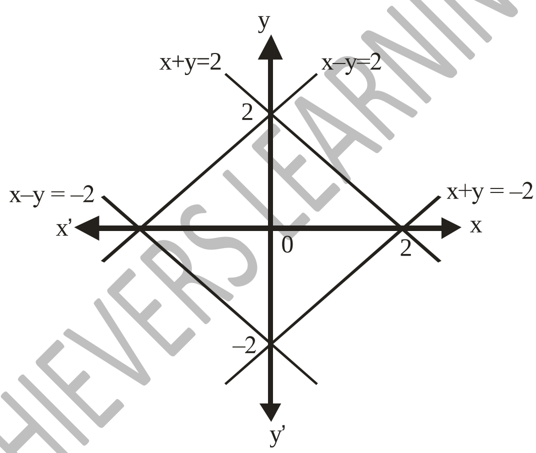
Sol. Conceptual

5. Area bounded by $2 \geq \max.\{|x-y|,|x+y|\}$ is k sq. units then k =

Key. 8

Sol. $2 \geq \max.\{|x-y|,|x+y|\}$

$\Rightarrow$ $|x-y| \leq 2$ and $|x+y| \leq 2$, which forms a square of diagonal length 4 units.



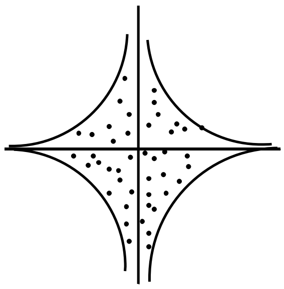
$\Rightarrow$ The area of the region is $\frac{1}{2}\times 4\times 4 = 8$ sq. units

This is equal to the area of the square of side length $2\sqrt{2}$.

6. The area bounded by the curves $y = \ln x, y = ln|x|, y = |\ln x|, y = |\ln|x||$ is

Key. 4

Sol. Area $= 4\int_0^1 |\ln x|dx$



$= -4\int_0^1 \ln x dx$

$= -4\left[x\ln x - x\right]_0^1$

= 4

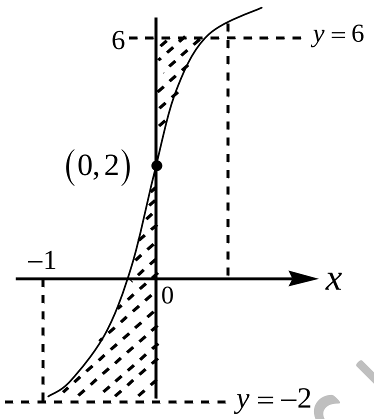
7. Let $f(x)=x^3+3x+2$ and g(x) is the inverse of it. The area bounded by g(x), the x-axis and the ordinates at x= -2 and x= 6 is $\frac{m}{n}$ where $m,n\in N$ & $G.C.D\ of\ (m,n)=1$ then $m-2=$

Key. 7

Sol. The required area will be equal to area enclosed by $y=f(x)$, the y-axis between the abscissa at $y=-2$ and $y=6$

Required area

$=\int_0^1\{6-f(x)\}dx+\int_{-1}^0[f(x)-(-2)]dx$



$\frac{9}{2}=\frac{m}{n} \Rightarrow m-n=7$

8. The integral $\int_{\pi/4}^{5\pi/4}(|\cos t|\sin t+|\sin t|\cos t)\,dt$ has the value equal to

Key. 0

Sol. I = $\int_{\pi/4}^{\pi/2} 2\sin t\cos t\,dt + \int_{\pi/2}^{\pi}\{(-\sin t\cos t)+(\sin t\cos t)\}dt + \int_{\pi}^{5\pi/4} -2\sin t\cos t\,dt$

$= \int_{\pi/4}^{\pi/2}\sin 2t\,dt - \int_{\pi}^{5\pi/4}\sin 2t\,dt$

= 0.

9. Let 'f' is a differentiable function such that $f'(x)=f(x)+\int_0^2 f(x)dx,\ f(0)=\frac{4-e^2}{3}$ then the value of $[f(2)]$ where [.] denotes the greatest integer $\le x$ is.

Key. 5

Sol. Given $f'(x)=f(x)+A$ where $A=\int_0^2 f(x)dx$

Solving – (1)

$$f(x) = \lambda\left(e^x - 1\right) + \frac{4 - e^2}{3}$$

$$\because \int_0^2 f(x)\,dx = A \Rightarrow \lambda = 1 \text{ and } A = \frac{e^2 - 1}{3}$$

$$\therefore f(x) = e^x - 1 + \frac{4 - e^2}{3} = e^x - \frac{1}{3}\left(e^2 - 1\right)$$

$$f(2) = \frac{2e^2 + 1}{3} \qquad \therefore \left[f(2)\right] = 5$$

10. If the area bounded by the curves $y = -x^2 + 6x - 5$, $y = -x^2 + 4x - 3$ and the line $y = 3x - 15$ is $\frac{73}{\lambda}$, then the value of λ is

Ans: 6

Hint: Area = $\left|\int_1^5 (6x - x^2 - 5)\,dx - \int_1^3 (4x - x^2 - 3)\,dx\right|$

$+ \left|\int_3^4 (4x - x^2 - 3)dx\right| + \left|\int_4^5 (3x - 15)dx\right| = \frac{73}{6}$

11. The minimum area bounded by the function y= f(x) and $y = \alpha x + 9$ ($\alpha \in R$) where f satisfies the relation $f(x + y) = f(x) + f(y) + y\sqrt{f(x)}$ $\forall x, y \in R$ and $f'(0) = 0$ is 9A, value of A is

Ans: 1

Hint: $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x + 0)}{h} \Rightarrow \lim_{h \to 0} \frac{f(x) + f(h) + h\sqrt{f(x)} - f(x) - f(0) - 0\sqrt{f(x)}}{h}$

$\Rightarrow \lim_{h \to 0}\left(\frac{f(h) - f(0)}{h - 0}\right) + \sqrt{f(x)}$

$f'(x) = \sqrt{f(x)}$

$\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = \int dx$

$2\sqrt{f(x)} = x + c$

$f(x) = \frac{x^2}{4}$

when $\alpha = 0$ area is minimum

required minimum area = $2\int_0^9 2\sqrt{y}\,dy$

$\Rightarrow\ 4\left(\dfrac{y^{3/2}}{3/2}\right)_0^9$ = 72 sq. unit.

12. Let R = {x, y : $x^2 + y^2 \le 144$ and sin(x + y) ≥ 0}. And S be the area of region given by R, then find S/9π.

Ans: 8

Hint: $x^2 + y^2 \le 144$ and $\sin(x + y) \ge 0 \Rightarrow 2n\pi \le x + y \le (2n + 1)\pi$; n∈I

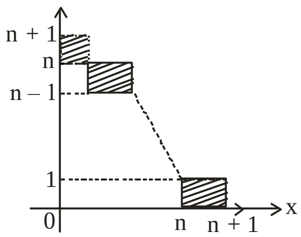
Hence we get the area

S = $\dfrac{\pi.144}{2} \Rightarrow \dfrac{S}{9\pi} = 8$

13. If the area bounded by [x] + [y] = n and y = k; n, $k \in N$ and $k \le (n+1)$ and [.] is greatest integer function, in the first quadrant, is n + r, then find r.

Ans: 1

Hint: Area = *n* + 1



14. Let α, β be roots of the quadratic equation $18x^2 - 9\pi x + \pi^2 = 0$, where $\alpha < \beta$. Also $f(x) = x^2$ and $g(x) = \cos x$. If the area bounded by the curve y = (fog) (x), the vertical lines x = α, x = β and x-axis is $\dfrac{\pi}{\lambda}$, then find the sum of the digit in λ

Ans: 3

Hint: Let α, β be roots of the ……….

$(fog)(x) = f(\cos x) = \cos^2 x$

$\alpha, \beta : 18x^2 - 9\pi x + \pi^2 = 0 \Rightarrow \alpha = \dfrac{\pi}{6}, \beta = \dfrac{\pi}{3}$

$$\text{Area} = \int_{\pi/6}^{\pi/3} \cos^2 x\ dx = \frac{\pi}{12}$$

15. The minimum area bounded by the function y= f(x) and y = αx + 9 (α∈R) where f satisfies the relation f(x + y) = f(x) + f(y) + y $\sqrt{f(x)}$ ∀x,y∈R and f′(0) = 0 is 9A, value of A is

Key: 8

Hint: $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x+0)}{h} \Rightarrow \lim_{h\to 0} \dfrac{f(x) + f(h) + h\sqrt{f(x)} - f(x) - f(0) - 0\sqrt{f(x)}}{h}$

$$\Rightarrow \lim_{h\to 0}\left(\frac{f(h) - f(0)}{h - 0}\right) + \sqrt{f(x)}$$

f′(x) = $\sqrt{f(x)}$

$\int \frac{f'(x)}{\sqrt{f(x)}}dx = \int dx$

$2\sqrt{f(x)}$ = x + c

f(x) = $\frac{x^2}{4}$

when α = 0 area is minimum

required minimum area = $2\int_0^9 2\sqrt{y}\,dy$

$\Rightarrow 4\left(\frac{y^{3/2}}{3/2}\right)_0^9$ = 72 sq. unit.

16. Given that

$\int_1^y \sec^{-1}x\,dx = \lambda$ then $\int_{-y}^{-1}\sec^{-1}x - \tan^{-1}\left(\sqrt{x^2-1}\right)dx + \int_1^y \sec^{-1}x - \tan^{-1}\left(\sqrt{x^2-1}\right)dx =$ ___$(|y| \geq 1)$

equals to $\pi(y-a) - b\lambda$ then a+b=___

Ans: 3

$\int_{-y}^{y}\sec^{-1}x - \tan^{-1}\left(\sqrt{x^2-1}\right)dx = \int_{-y}^{-1}\sec^{-1}x\left(\pi - \sec^{-1}x\right)dx$

$+\int_1^y \sec^{-1}x - \sec^{-1}x\,dx$

Hint: $= 2\int_{-y}^{-1}\sec^{-1}x\,dx - \int_{-y}^{-1}\sec^{-1}x$

$= 2\int_1^y\left(\pi - \sec^{-1}x\right)dx - \pi(y-1) = \pi(y-1) - 2\lambda$

$\therefore a + b = 3$

17. $Let\ I_n = \int_0^{\pi/2}(\sin x + \cos x)^n\,dx\ (n \geq 2)$. Then the value of $nI_n - 2(n-1)I_{n-2} is$

Ans: 2

Hint $I_n = \int_0^{\pi/2}(\sin x + \cos x)^{n-1}(\sin x - \cos x)'dx$

$= 2 + (n-1)\int_0^{\pi/2}(\sin x + \cos x)^{n-2}(\cos x - \sin x)^2\,dx$

$= 2 + (n-1)\int_0^{\pi/2}(\sin x + \cos x)^{n-2}\left[2 - (\sin x + \cos x)^2\right]dx$

$= 2 + 2(n-1)I_{n-2} - (n-1)I_n$

$\Rightarrow nI_n - 2(n-1)I_{n-2} = 2$

18. If $\lim_{x\to 0}\int_0^x \frac{t^2dt}{(x-\sin x)\sqrt{a+t}} = 1$ then the value of a is …………………..

Key: 4

Hint:

$$\underset{x\to 0}{\text{lt}}\frac{\int_0^x \frac{t^2}{\sqrt{a+t}}dt}{x-\sin x}=\underset{x\to 0}{\text{lt}}\frac{x^2}{\sqrt{a+x}\left(1-\cos x\right)}=\underset{x\to 0}{\text{lt}}\frac{x^2}{\sqrt{a+x}\;\left(2\sin^2\frac{x}{2}\right)}$$

$$=\frac{2}{\sqrt{a}}=1\Rightarrow a=4$$

19. Given $\int_0^{\pi/2}\ln\sin x dx=\frac{\pi}{2}\ln\frac{1}{2}$ and $\int_0^{\pi/2}\left(\frac{x}{\sin x}\right)^2 dx=\frac{k.\pi}{2}\ln 2$ then k = ………………

Key: 2

Hint:

$$I=\int_0^{\pi/2}x^2\cos ec^2 x dx=\left[-x^2\cot x\right]_0^{\pi/2}+\int_0^{\pi/2}2x\cot x dx$$

$$=0+\underset{x\to 0^+}{Lt}\frac{x^2}{\tan x}+2\left[x\ln\sin x\right]_0^{\pi/2}-2\int_0^{\pi/2}\ln\sin x dx$$

$$=0-2\underset{x\to 0^+}{Lt}x\ln\sin x-2\frac{\pi}{2}\ln\frac{1}{2}=\pi\ln 2$$

$$\left(\because \underset{x\to 0^+}{Lt}\frac{\ln\sin x}{1/x}=\underset{x\to 0^+}{Lt}\frac{\frac{1}{\sin x}.\cos x}{-\frac{1}{x^2}}=0\right)$$

20. Find the value of $\frac{29\int_0^1\left(1-x^4\right)^7dx}{4\int_0^1\left(1-x^4\right)^6dx}$

Key: 7

Hint: $l=\int_0^1\left(1-x_1^4\right)^7-\underset{\pi}{1}\,dx$

$$=\left[x\left(1-x^4\right)^7\right]_0^1+7\times 4\int_0^1 x\left(1-x^4\right)^6x^3dx$$

$$=-28\int_0^1\left(1-x^4\right)^6dx+28\int_0^1\left(1-x^4\right)^6dx=-28I+28\int_0^1\left(1-x^4\right)^6dx$$

$$29I-28\int_0^1\left(1-x^4\right)dx$$

$$\frac{29\int_0^1 \left(1-x^4\right)^7 dx}{4\int_0^1 \left(1-x^4\right)^7 dx} = 7$$

21. If $f(x) = a\cos(\pi x) + b, f'\left(\frac{1}{2}\right) = \pi$ and $\int_{1/2}^{3/2} f(x)dx = \frac{2}{\pi} + 1$, then find the value of, $-\frac{12}{\pi}\left(\sin^{-1} a + \cos^{-1} b\right)$

Key: 6

Hint: $f'(x) = -\alpha\pi\sin(\pi x)$

$$\Rightarrow f'\left(\frac{1}{2}\right) = -a\pi\sin\frac{\pi}{2} = -a\pi = \pi \Rightarrow a = -1$$

$$\int(a\cos\pi x + b)dx = \left(\frac{a\sin\pi x}{\pi} + bx\right)_{1/2}^{3/2} = \left(\frac{-a}{\pi} + \frac{3b}{2}\right) - \left(\frac{a}{\pi} + \frac{b}{2}\right)$$

$$\Rightarrow \frac{2a}{\pi} + b = \frac{2}{\pi} + 1 \Rightarrow b = 1$$

So, $\frac{-12}{\pi}\left(\sin^{-1}(-1) + \cos^{-1} 1\right) = \frac{-12}{\pi}\left(-\frac{\pi}{2} + 0\right) = 6$

22. Evaluate $\left[\int_0^{\pi} \frac{dx}{1+2\sin^2 x}\right]$ where [.] denotes greatest integer function

Key: 1

Hint: $\int_0^{\pi} \frac{dx}{1+2\sin^2 x} = 2\int_0^{\pi/2} \frac{dx}{1+2\sin^2 x} = 2\int_0^{\pi/2} \frac{\sec^2 x\, dx}{1+3\tan^2 x}$

Put tan x = t = $2\int_0^{\infty} \frac{dt}{1+\left(\sqrt{3}t\right)^2}$

$$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\sqrt{3}t\right]_0^{\infty} = \frac{\pi}{\sqrt{3}}$$

$$\left[\frac{\pi}{\sqrt{3}}\right] = 1$$

23. Let $F(x)$ be a non-negative continuous function defined on R such that $F(x) + F\left(x + \frac{1}{2}\right) = 3$ and the value of $\int_0^{1500} F(x)dx$ is $\frac{9000}{\lambda}$. Then the numerical value of λ is

Key: 4

Hint: We have $F(x)+F\left(x+\frac{1}{2}\right)-3$ ......(i)

Replace x by $x+\frac{1}{2}$ in (I), we get $F\left(x+\frac{1}{2}\right)+F(x+1)=3$....(2)

$\therefore$ From (1) and (2), we get $F(x)=F(x+1)$ ....(3) $\Rightarrow$ F(x) is periodic function.

Now consider $I=\int_0^{1500} F(x)dx\int_0^1 F(x)dx=1500\left|\int_0^{\frac{1}{2}} F(x)dx+\int_{\frac{1}{2}}^{1} F(x)dx\right|\begin{pmatrix}\text{Using property}\\ \text{of periodic}\\ \text{function}\end{pmatrix}$

Put $x+y+\frac{1}{2}m2^{nd}$ integral, we get

$$I=1500\left|\int_0^{\frac{1}{2}} F(x)dx+\int_0^{\frac{1}{2}} F\left(y+\frac{1}{2}\right)dy\right|-1500\int_0^{\frac{1}{2}}\left(F(x)+F\left(x+\frac{1}{2}\right)\right)dx=1500\int_0^{\frac{1}{2}} 3dx\left|\text{Using}(1)\right|$$

Hence I = $1500(3)\left(\frac{1}{2}\right)=750\times 3=2250$

24. Let $f(x)=\frac{4^x}{4^x+2}, I_1=\int_{f(1-a)}^{f(a)} xf(x(1-x))dx$ and $I_2=\int_{f(1-a)}^{f(a)} f(x(1-x))dx$ where $f(a)>f(1-a)$ then the value of $\frac{I_2}{I_1}$ is

Key: 2

Hint: $f(x)+f(1-x)=1$

$I_1=\int_{f(1-x)}^{f(x)} x+(x(-x))dx$

$x+f(1-x)+f(x)=1$

$f(1-x)$

25. If $I=\int_{-\frac{\pi}{2}}^{2\pi} \sin^{-1}(\sin x)dx$ then $-\frac{16}{\pi^2}I=$

Key: 2

Hint: $I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} xdx+\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}(\pi-x)dx+\int_{\frac{3\pi}{2}}^{2\pi}(x-2\pi)dx=\frac{-\pi^2}{8}$

26. The value of $\int_{-1}^{1}\left[x\left[1+\sin\pi x\right]+1\right]dx$ is ([.] denote the greatest integer function)

Key : 2

Sol : Let $I=\int_{-1}^{1}\left[x\left[1+\sin\pi x\right]+1\right]dx=\int_{-1}^{1}\left[x\left[1+\sin\pi x\right]+1\right]dx+\int_{1}^{0}\left[x\left[1+\sin\pi x\right]+1\right]dx$

Now, $-1<x<0\Rightarrow\left[1+\sin\pi x\right]=0$

and $0<x<1\Rightarrow\left[1+\sin\pi x\right]=1$

$\therefore\ I=\int_{-1}^{0}1.dx+\int_{0}^{1}\left[x+1\right]dx$

$=\int_{-1}^{0}1.dx+\int_{0}^{1}\left[x\right]dx+\int_{0}^{1}1.dx$

$=\left[0-(-1)\right]+0+(1-0)$ = 2

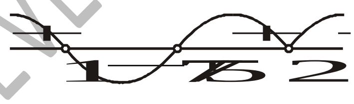
27. The function $f(x)=\int_{1}^{x}\left\{2(t-1)(t-2)^3+3(t-1)^2(t-2)^2\right\}dt$ attains its maximum at x is equal to

Key : 1

Sol : $f'(x)=2(x-1)(x-2)^3+3(x-1)^2(x-2)^2$

$=(x-1)(x-2)^2\left\{2(x-2)+3(x-1)\right\}$

$=(x-1)(x-2)^2(5x-7)$



sign change of $f'(x)$ from +ve to –ve at x = 1

$\therefore$ maximum at x = 1.

28. If $\lim_{n\to\infty}\frac{\sum_{r=1}^{n}\sqrt{r}\sum_{r=1}^{n}\frac{1}{\sqrt{r}}}{\sum_{r=1}^{n}r}=\frac{k}{3}$ then find k.

ANS: 8

Hint: $\lim_{n\to\infty}\frac{\sum_{r=1}^{n}\sqrt{r}\sum_{r=1}^{n}\frac{1}{\sqrt{r}}}{\frac{n(n+1)}{2}}\quad\lim_{n\to\infty}\frac{\sum_{r=1}^{n}\sqrt{\frac{r}{n}}\sum_{r=1}^{n}\frac{1}{\sqrt{\frac{r}{n}}}}{\frac{n^2}{2}\left(1+\frac{1}{n}\right)}$

$=2\times\int_{0}^{1}\sqrt{x}\,dx\int_{0}^{1}\frac{1}{\sqrt{x}}dx$

$$= 2\times\left[\frac{x^{3/2}}{\frac{3}{2}}\right]_0^1\left[\frac{x^{1/2}}{\frac{1}{2}}\right]_0^1$$

$$= 2\times\frac{2}{3}\times 2 = \frac{8}{3}$$

$\therefore\ k = 8$

29. $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{k^{1/a\left\{n^{a-\frac{1}{a}}+k^{a-\frac{1}{a}}\right\}}}{n^{a+1}}$ is equal to

Key : 1

Sol : $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{k^{\frac{1}{a}}\left\{n^{a-\frac{1}{a}}+k^{a-\frac{1}{a}}\right\}}{n^{a+1}}$

$= \lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{1}{n}.\left\{\left(\frac{k}{n}\right)^{1/a}+\left(\frac{k}{n}\right)^{a}\right\}$

$= \int_0^1\left(x^{1/a}+x^{a}\right)dx$

$$= \left\{\frac{x^{(1/a)+1}}{\frac{1}{a}+1}+\frac{a^{a+1}}{a+1}\right\}_0^1$$

$= \frac{a}{a+1}+\frac{1}{a+1} = 1$

30. If $\int \sin 4x\ e^{\tan^2 x}dx = c - A\cos^4 x.e^{\tan^2 x}$ then A=___

Ans: 2

Hint:

$$I = 4\int \sin x\cos x\left(\frac{1-\tan^2 x}{1+\tan^2 x}\right)e^{\tan^2 x}dx = 4\int \tan x\sec^2 x\cos^6 x\left(1-\tan^2 x\right)e^{\tan^2 x}$$

$t = \tan^2 x$

$$\Rightarrow I = 2\int\frac{(1-t)e^t}{(1+t)^3}dt = \frac{-2e^t}{(1+t)^2}+c$$

$= -2\cos^4 x e^{\tan^2 x}+c$

31. If $\int\frac{x^4+1}{x\left(x^2+1\right)^2}dx = A\,ln\,|x| + \frac{B}{1+x^2} + c$, where c is the constant of integration then A + B is :

Ans: 2

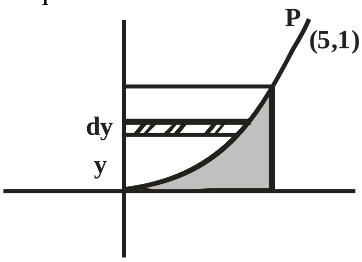
Hint: Add and subtract $2x^2$ in the numerator]

32. Let y = g(x) be the inverse of a bijective mapping f : R → R defined as f (x) = $3x^3 + 2x$. The area bounded by the graph of g(x), the x-axis and the coordinate at x = 5 is 'A' then the value of (4A –7) is

Key. 6

Sol. Inverse of $y = 3x^3 + 2x$ is $x = 3y^3 + 2y$

Required area



$$A = \int_0^1 5 - (3y^3 + 2y)dy = \left[5y - \left(\frac{3y^4}{4} + y^2\right)\right]_0^1$$

$$= 5 - \left(\frac{3}{4} + 1\right) = \frac{20-7}{4} = \frac{13}{4}$$

$4A = 13$

$\therefore \quad 4A - 7 = 6$

33. A point P(x, y) moves is such a way that [x + y + 1] = [x] (where [.] greatest integer function) and x ∈ (0, 2). Then the area representing all the possible positions of P equals

Key. 2

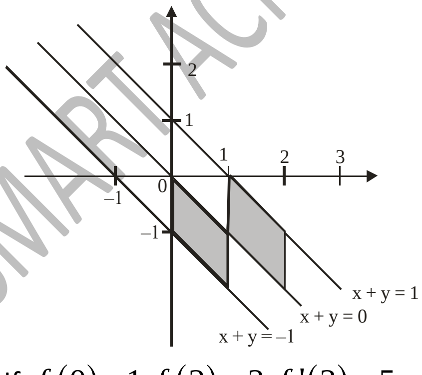
Sol. If x ∈ (0, 1)

Then $-1 \le x + y < 0$

And if x ∈ [1, 2)

$0 \le x + y < 1$

Required area $= 4\left(\frac{1}{2}.1.\sqrt{2}\sin\frac{\pi}{4}\right) = 2$ sq. units



34. If $f(0)=1, f(2)=3, f'(2)=5$

then the value of the definite integral $\int_0^1 x f''(2x)dx$ is

Key. 2

Sol. $\int \underset{I}{x}\, \underset{II}{f''(2x)}dx = x\frac{f'(2x)}{2} - \int \frac{f'(2x)}{2}dx$

$= x\frac{f'(2x)}{2} - \frac{f(2x)}{4}$

$\int_0^1 x f''(2x)dx = \left|\frac{xf'(2x)}{2} - \frac{f(2x)}{4}\right|_0^1$

$= \left(\frac{f'(2)}{2} - \frac{f(2)}{4}\right) - \left(0 - \frac{f(0)}{4}\right)$

$\left(\frac{5}{2} - \frac{3}{4}\right) - \left(0 - \frac{1}{4}\right) = 2$

35. A point P(x, y) moves is such a way that [x + y + 1] = [x] (where [.] greatest integer function) and $x \in (0, 2)$. Then the area representing all the possible positions of P equals

Key. 2

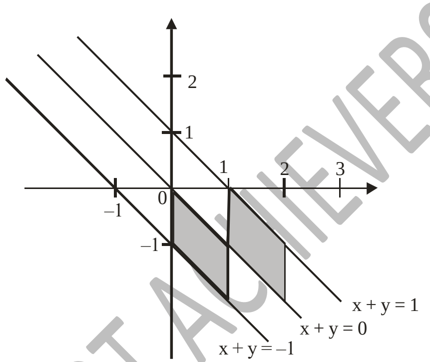
Sol. If $x \in (0, 1)$

Then $-1 \le x + y < 0$

And if $x \in [1, 2)$

$0 \le x + y < 1$

Required area $= 4\left(\frac{1}{2}.1.\sqrt{2}\sin\frac{\pi}{4}\right) = 2$ sq. units



36. If $\int_0^{x^2(1+x)} f(t)dt = x,$ then the value of $10f(2)$ must be

Key. 2

Sol. Differentiating both sides w.r.t. x, then

$f(x^2(1+x)) \times (2x + 3x^2) = 1$

At $x = 1 \Rightarrow f(2) = \frac{1}{5} \therefore\ 10f(2) = 2$

37. If $I = \int_0^2 x[2x]dx,$ where [.] denotes the greatest integer function, then the value of $\frac{4}{17}I =$

Key. 1

Sol. $I = \int_0^2 x[2x]dx$

$= \int_0^{1/2} 0.dx + \int_{1/2}^1 xdx + \int_1^{3/2} 2xdx + \int_{3/2}^2 3xdx$

$= 0 + [\frac{x^2}{2}]_{1/2}^1 + [x^2]_1^{3/2} + [\frac{3x^2}{2}]_{3/2}^2$

$= \frac{1}{2}(1 - \frac{1}{4}) + (\frac{9}{4} - 1) + \frac{3}{2}(4 - \frac{9}{4})$

$= \frac{3}{8} + \frac{5}{4} + \frac{21}{8}$

$= \frac{34}{8} = \frac{17}{4}$

$\therefore \frac{4}{17} I = 1$

38. If $\int_0^{\pi/2} \sin^8 x \cos^4 xdx = \frac{k\pi}{2048}$ then the value of k must be

Key. 7

Sol. $\int_0^{\pi/2} \sin^8 x \cos^4 xdx = \frac{(7.5.3.1)(3.1)}{12.10.8.6.4.2}.\frac{\pi}{2}$

(by Wallis' formula) $= \frac{7\pi}{2048} \Rightarrow k = 7$

39. If $I = \int_0^1 x(1-x)^{49} dx,$ then the value of $5100I =$

Key. 2

Sol. $\because I = \int_0^1 x(1-x)^{49} dx$

$= \int_0^1 (1-x)(1-(1-x))^{40} dx$

$= \int_0^1 (1-x)x^{49} dx = \int_0^1 (x^{49} - x^{50})dx$

$= [\frac{x^{50}}{50} - \frac{x^{51}}{51}]_0^1$

$$= \frac{1}{50} - \frac{1}{51} = \frac{1}{2550} \Rightarrow 5100I$$

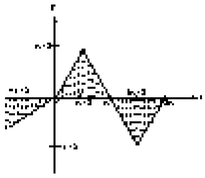
$$\Rightarrow 5100\frac{1}{2550} = 2$$

40. If $I = \int_{-\pi/2}^{2\pi} \sin^{-1}(\sin x)dx = \frac{-\pi^2}{k}$ then $k =$

Key. 8

Sol. $I = \int_{-\pi/2}^{0} \sin^{-1}(\sin x)dx + \int_{0}^{\pi} \sin^{-1}(\sin x)dx + \int_{\pi}^{2\pi} \sin^{-1}(\sin x)dx$

=Area of shaded region



$$= -\left(\frac{1}{2} \times \frac{\pi}{2} \times \frac{\pi}{2}\right) + \left(\frac{1}{2} \times \pi \times \frac{\pi}{2}\right) - \left(\frac{1}{2} \times \pi \times \frac{\pi}{2}\right)$$

$$= -\frac{\pi^2}{8}$$

Since, $k = 8$

41. If $f(x) = \int_a^x \frac{1}{f(x)} dx$ and $\int_a^1 \frac{1}{f(x)} dx = \sqrt{2}$, then the value of $f(2)$ =

Key. 2

Sol. Since, $f(x) = \int_a^x \frac{1}{f(x)} dx$

Differentiating both sides w.r.t. x, then

$$f^1(x) = \frac{1}{f(x)} \Rightarrow 2f(x)f^1(x) = 2$$

Integrating both sides, then

$$(f(x))^2 = 2x + c$$

$$\therefore f(x) = \sqrt{(2x+c)}$$

But $\int_0^1 \frac{1}{f(x)}dx = \sqrt{2}$

And $f(1) = \int_a^1 \frac{1}{f(x)}dx = \sqrt{2}$

$\Rightarrow \sqrt{(2+c)} = \sqrt{2}$

$\therefore c = 0$

Then, $f(x) = \sqrt{2x}$

42. If $\int_0^{\pi} x\sin^5 x\cos^6 xdx = \frac{k\pi}{693}$ then k =

Key. 8

Sol. Let

$$I = \int_0^{\pi} x\sin^5 x.\cos^6 xdx = \int_0^{\pi}(\pi - x).\sin^5(\pi - x).\cos^6(\pi - x)dx = \int_0^{\pi}(\pi - x)\sin^5 x.\cos^6 xdx$$

$$= \int_0^{\pi}\pi\sin^5 x.\cos^6 xdx - \int_0^{\pi}x\sin^5 x\cos^6 xdx$$

$$\Rightarrow 2I = \pi.2\int_0^{\pi/2}\sin^5 x.\cos^6 xdx \Rightarrow I = \pi\left[\frac{4}{11}.\frac{2}{9}.\frac{1}{7}\right] = \frac{8\pi}{693}.$$

43. If $\int_0^{\pi/4}\tan^5 xdx = \frac{1}{k}\log 2 - \frac{1}{4}$ then k =

Key. 2

Sol. $\int_0^{\pi/4}\tan^5 xdx = \frac{1}{4} - \frac{1}{2} + \frac{1}{2}\log 2 = \frac{1}{2}\log 2 - \frac{1}{4}$

44. If $I = \int_{-\pi/2}^{\pi/2}\sqrt{\cos x - \cos^3 x}dx = \frac{k}{3}$ then $k =$

Key. 4

Sol.

$$I = \int_{-\pi/2}^{\pi/2}\sqrt{\cos x}\,|\sin x|\,dx = 2\int_{-\pi/2}^{\pi/2}\sqrt{\cos x}\,|\sin|\,xdx = 2\int_0^{\pi/2}\sqrt{\cos x}\sin xdx = \left[-\frac{4}{3}(\cos x)^{3/2}\right]_0^{\pi/2} = \frac{4}{3}$$

45. The value of $\int_{-1}^{3}\{|x-2| + [x]\}dx,$ where $[x]$ denotes the greatest integer less than or equal to x is

Key. 7

Sol. $\int_{-1}^{3}\{|x-2| + [x]\}dx$

$$= \int_{1}^{0} \{|x-2|+[x]\}\,dx + \int_{0}^{1} \{|x-2|+[x]\}\,dx + \int_{1}^{2} \{|x-2|+[x]\}dx + \int_{2}^{3} \{|x-2|+[x]\}\,dx$$

$$= \int_{-1}^{0} (2-x-1)dx + \int_{0}^{1} (2-x+0)dx + \int_{1}^{2} (2-x+1)dx + \int_{2}^{3} (x-2+2)dx$$

$$= \left[x-\frac{x^2}{2}\right]_{-1}^{0} + \left[2x-\frac{x^2}{2}\right]_{0}^{1} + \left[3x-\frac{x^2}{2}\right]_{1}^{2} + \left[\frac{x^2}{2}\right]_{2}^{3} = -\left(-1-\frac{1}{2}\right)+\left(2-\frac{1}{2}\right)+(6-2)-\left(3-\frac{1}{2}\right)+\frac{9}{2}-2=7$$

46. $I = 3\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\cos x - \cos^3 x}\,dx$ is

KEY. 4

SOL. Given integral

$$I = \int_{-\pi/2}^{\pi/2} \sqrt{[\cos x(1-\cos^2 x)]}dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{(\cos x \sin^2 x)}dx$$

$$I = \int_{-\pi/2}^{\pi/2} \sqrt{(\cos x)}\,|\sin x|\,dx \qquad (1)$$

Now $|\sin x| = \{ \begin{matrix} -\sin x, if\ -\pi/2 \le x < 0 \\ \sin x, if\ 0 < x \le \pi/2 \end{matrix}$

From (1), we have

$$I = \int_{-\pi/2}^{0} \sqrt{(\cos x)}(-\sin x)dx + \int_{0}^{\pi/2} \sqrt{(\cos x)}\sin x\,dx$$

Putting $\cos x = t, -\sin x\,dx = dt$, we get

$$I = \int_{0}^{1} t^{1/2}dt - \int_{1}^{0} t^{1/2}dt = 2\int_{0}^{1} t^{1/2}dt$$

$$= 2\times(\frac{2}{3})[t^{3/2}]_0^1$$

$$= \frac{4}{3}$$

47. If $\int_{0}^{\pi/2} x^n \sin x\,dx = (3/4)(\pi^2-8)$, then the value of n is _____

KEY. 3

SOL. Let $I_n = \int_{0}^{\pi/2} x^n \sin x\,dx$

Integrating by parts choosing $\sin x$ as the second function, we get

$$I_n = [x^n(-\cos x)]_0^{\pi/2} - \int_0^{\pi/2} nx^{n-1}(-\cos x)dx$$

$$= 0 - n\int_0^{\pi/2} x^{n-1}\cos x\,dx$$

Again integrating by parts,

$$I_n = n[x^{n-1}\sin x]_0^{\pi/2} - n(n-1)\int_0^{\pi/2} x^{n-2}\sin x\,dx$$

$$\Rightarrow I_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}$$

R.H.S. contains π^2. So putting $n=3$, we get

$$I_3 = 3\left(\frac{\pi}{2}\right)^2 - 3\times 2I_1$$

$$= \frac{3\pi^2}{4} - 6\int_0^{\pi/2} x\sin x\,dx$$

$$= \frac{3\pi^2}{4} - 6[x(-\cos x) + \sin x]_0^{\pi/2}$$

$$= \frac{3\pi^2}{4} - 6(1) = \frac{3}{2}(\pi^2 - 8)$$, which is true.

Hence, $n=3$.

48. If $I_n = \int_0^{\pi} x^n \sin x\,dx$ and $I_5 + 20I_3 = \pi^k$, then the value of k is _____

KEY. 5

SOL.
$$I_n = \int_0^{\pi} x^n \sin x\,dx$$

$$= [-x^n\cos x]_0^{\pi} + n\int_0^{\pi} x^{n-1}\cos x\,dx$$

$$= \pi^n + n[x^{n-1}\sin x]_0^{\pi} - n(n-1)\int_0^{\pi} x^{n-2}\sin x\,dx$$

$$\Rightarrow I_n = \pi^n + n\times 0 - n(n-1)I_{n-2}$$

Putting $n=5$, we get

$$I_5 = \pi^5 - 20I_3$$

$$\Rightarrow I_5 + 20I_3 = \pi^5$$

49. If $I = \int_{-1}^{1}(1+x)^{1/2}(1-x)^{3/2}\,dx$, then the value of $\sec^3(I/2)$ is _____

KEY. 8

SOL. Using property $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

$$I=\int_{-1}^{1}(1-x)^{1/2}(1+x)^{3/2}dx$$

$$\Rightarrow 2I=\int_{-1}^{1}(1+x)^{1/2}(1-x)^{1/2}[(1-x)+(1+x)]dx$$

$$\Rightarrow 2I=2\int_{-1}^{1}\sqrt{1-x^2}\,dx$$

$$\Rightarrow I=2\int_{0}^{1}\sqrt{1-x^2}\,dx(x=\sin\theta)\Rightarrow dx=\cos\theta d\theta$$

$$\Rightarrow I=2\int_{0}^{\pi/2}\cos^2\theta\,d\theta=\frac{\pi}{2}$$

50. The value of $\int_{-\pi/4}^{\pi/4}[(x^9-3x^5+7x^3-x+1)/\cos^2 x]dx$ is ____

KEY. 2

SOL. $f(x)=\dfrac{x^9-3x^5+7x^3-x}{\cos^2 x}+\sec^2 x$

$$=\sec^2 x(x^9-3x^5+7x^3-x)+\sec^2 x$$

$$\Rightarrow \int_{-\pi/4}^{\pi/4}f(x)dx=\int_{-\pi/4}^{\pi/4}\sec^2 x\,dx$$

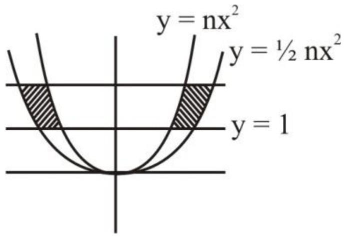
$$=2\int_{0}^{\pi/4}\sec^2 x\,dx$$

$$=2\tan x\,|_0^{\pi/4}=2$$

51. Find natural number *n* so that area bounded by y = nx^2, $y=\frac{1}{2}nx^2$ and $y^2-4y+3=0$ is greatest.

Key. 1

Sol. $\text{Area}=\int_1^3\left(\sqrt{\frac{2y}{n}}-\sqrt{\frac{y}{n}}\right)dy$



is greatest when n is least

52. $f:[0,5]\to R,\ y=f(x)$ such that $f''(x)=f''(5-x)\,\forall x\in[0,5]$ $f'(0)=1$ and $f'(5)=7$, then evaluate $\int_1^4 f'(x)dx-4$.

Key. 8

Sol. $\int_1^4 f'(x)dx = \left[xf'(x)\right]_1^4 - \int_1^4 xf''(x)dx$

$I = \int_1^4 x f''(x)dx = \int_1^4 (5-x)f''(5-x)dx$

$= 5\int_1^4 f''(x)dx - I$

$I = \frac{5}{2}\left[f'(4) - f'(1)\right]$

So, $\int_1^4 f'(x)dx = \frac{3}{2}\left[f'(4) + f'(1)\right]$

Now, $f''(x) = f''(5-x) \Rightarrow f'(x) = -f'(5-x) + c$

$f'(0) + f'(5) = c \Rightarrow c = 8$

so $f'(x) + f'(5-x) = 8 \Rightarrow f'(4) + f'(1) = 8$