Date: 13/10/2021

Q1. Molecules in air in the atmosphere are attracted by gravitational force of the Earth. Explain why all of them do not fall into the Earth just like an apple falling from a tree.

Q2. What is the angle between the equatorial plane and the orbital plane of

(a) Polar satellite? (b) Geostationary satellite?

Q3. Out of aphelion and perihelion, where is the speed of the Earth more and why?

Q4. An astronaut inside a small spaceship orbiting around the Earth cannot detect gravity. If the space station orbiting around the Earth has a large size, can he hope to detect gravity?

Q5. What is the direction of areal velocity of the Earth around the Sun?

Q6. Draw areal velocity versus time graph for Mars.

Q7. Is it possible for a body to have inertia but no weight?

Q8. Give one example each of central force and non-central force.

Q9. The gravitational force between a hollow spherical shell (of radius R and uniform density) and a point mass is F. Show the nature of F vs r graph where r is the distance of the point from the centre of the hollow spherical shell of uniform density.

Q10. How is the gravitational force between two point masses affected when they are dipped in water keeping the separation between them the same?

Q11. Six point masses of mass m each are at the vertices of a regular hexagon of side l. Calculate the force on any of the masses.

Q12. An object of mass m is raised from the surface of the Earth to a height equal to the radius of the Earth, that is, taken from a distance R to $2R$ from the centre of the Earth. What is the gain in its potential energy?

Q13. We can shield a charge from electric fields by putting it inside a hollow conductor. Can we shield a body from the gravitational influence of nearby matter by putting it inside a hollow sphere or by some other means?

Q14. Mean solar day is the time interval between two successive noon when Sun passes through Zenith point (Meridian).

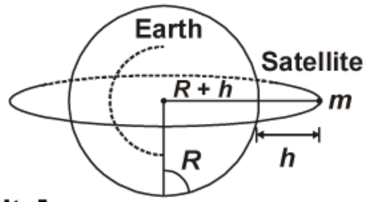
Sidereal day is the time interval between two successive transit of a distant star through the Zenith point (Meridian).

By drawing appropriate diagram showing Earth's spin and orbital motion, show that mean solar day is four minutes longer than the sidereal day. In other words, distant stars would rise 4 minutes early every successive day.

Q15. A satellite is to be placed in equatorial geostationary orbit around Earth for communication.

(a) Calculate height of such a satellite.

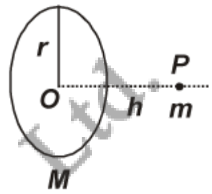
(b) Find out the minimum number of satellites that are needed to cover entire Earth so that at least one satellites is visible from any point on the equator.



[$M = 6 \times 10^{24}$ kg, $R = 6400$ km, $T = 24$ h, $G = 6.67 \times 10^{-11}$ SI units]

Q16. A Star like the Sun has several bodies moving around it at different distances. Consider that all of them are moving in circular orbits. Let r be the distance of the body from the centre of the Star and let its linear velocity be v, angular velocity ω, kinetic energy K, gravitational potential energy U, total energy E and angular momentum l. As the radius r of the orbit increases, determine which of the above quantities increase and which ones decrease.

Q17. A mass m is placed at P a distance h along the normal through the centre O of a thin circular ring of mass M and radius r (see figure).



If the mass is removed further away such that OP becomes $2h$, by what factor the force of gravitation will decrease, if $h = r$?

Q18. Show the nature of the following graph for a satellite orbiting the Earth.

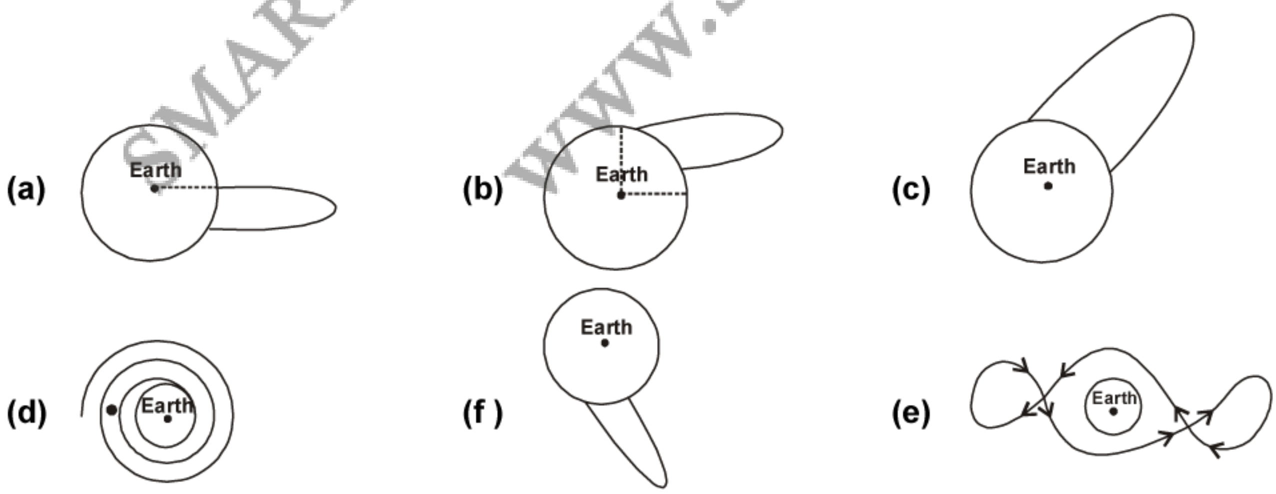
(a) K.E. *vs* orbital radius R (b) P.E. *vs* orbital radius R (c) T.E. *vs* orbital radius R.

Q19. Two identical heavy spheres are separated by a distance 10 times their radius. Will an object placed at the mid point of the line joining their centres be in stable equilibrium or unstable equilibrium? Give reason for your answer.

Q20. A satellite is in an elliptic orbit around the Earth with aphelion of $6R$ and perihelion of $2R$ where $R = 6400$ km is the radius of the Earth. Find eccentricity of the orbit. Find the velocity of the satellite at apogee and perigee. What should be done if this satellite has to be transferred to a circular orbit of radius $6R$? [$G = 6.67 \times 10^{-11}$ SI units and $M = 6 \times 10^{24}$ kg]

Q21. Earth's orbit is an ellipse with eccentricity 0.0167. Thus, Earth's distance from the Sun and speed as it moves around the Sun varies from day to day. This means that the length of the solar day is not constant through the year. Assume that Earth's spin axis is normal to its orbital plane and find out the length of the shortest and the longest day. A day should be taken from noon to noon. Does this explain variation of length of the day during the year?

Q22. Shown are several curves (see figure). Explain with reason, which ones amongst them can be possible trajectories traced by a projectile (neglect air friction).

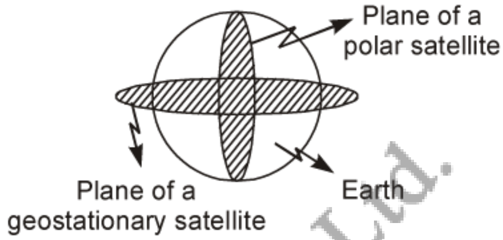


Date: 13/10/2021

S1. Molecules experience the vertically downward force due to gravity just like an apple falling from a tree. Due to thermal motion, which is random, their velocity is not in the vertical direction. The downward force of gravity causes the density of air in the atmosphere close to Earth higher than the density as we go up.

S2. Consider the diagram where plane of geostationary and polar satellite are shown.

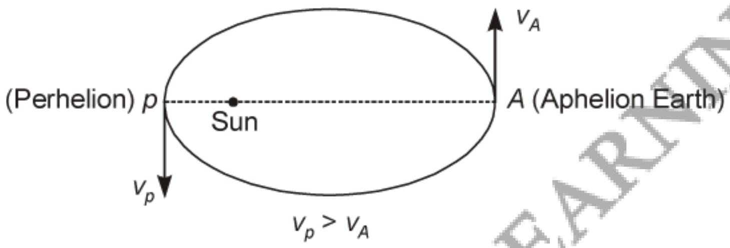
Clearly:



(a) Angle between the equatorial plane and orbital plane of a polar satellite is 90°.

(b) Angle between equatorial plane and orbital plane of a geostationary satellite is 0°

S3. Aphelion is the location of the Earth where it is at the greatest distance from the Sun and perihelion is the location of the Earth where it is at the nearest distance from the Sun.



The areal velocity $\left(\frac{1}{2}(\boldsymbol{r} \times \boldsymbol{v})\right)$ of the Earth around the Sun is constant (Kepler's law). Therefore, the speed of the Earth is more at the perihelion than at the aphelion.

S4. Inside a small spaceship orbiting around the Earth, the value of acceleration due to gravity g, can be considered as constant and hence astronaut feels weightlessness.

If the space station orbiting around the Earth has a large size, such that variation in g matters in heat case astronaut inside the spaceship will experience gracitational force and hence can detect gravity. *e.g.*, On the Moon, due to large size gravity can be detected.

S5. Areal velocity of the Earth around the Sun is given by

$$\frac{d\boldsymbol{A}}{dt} = \frac{L}{2m}.$$

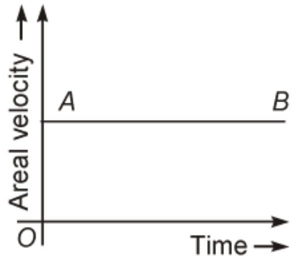
where, $\boldsymbol{L}$ is the angular momentum and m is the mass of the Earth.

But angular momentum $L = r \times p = r \times m\boldsymbol{v}$

$\therefore$ Areal velocity $\left(\frac{d\boldsymbol{A}}{dt}\right) = \frac{1}{2m}(\boldsymbol{r} \times m\boldsymbol{v}) = \frac{1}{2}(\boldsymbol{r} \times \boldsymbol{v})$

Therefore, the direction of areal velocity $\left(\frac{d\mathbf{A}}{dt}\right)$ is in the direction of $(\mathbf{r} \times \mathbf{v})$, *i.e.*, perpendicular to the plane containing r and v and directed as given by right hand rule.

S6. Areal velocity of a planet revolving around the Sun is constant with time. Therefore, graph between areal velocity and time is a straight line (*AB*), parallel to time axis. (Kapler's second law).

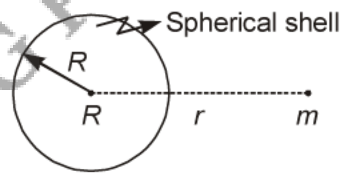


S7. Yes, a body will always have mass but the gravitational force on it can be zero; for example, when it is kept at the centre of the Earth.

S8. Central force; gravitational force of a point mass, electrostatic force due to a point charge. Non-central force: spin-dependent nuclear forces, magnetic force between two current carrying loops.

S9. Consider the diagram, density of the shell is constant.

Let it is ρ.



$$\text{Mass of the shell} = (\text{Density}) \times (\text{Volume})$$

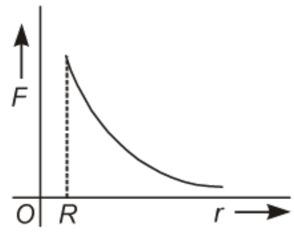
$$= (\rho) \times \frac{4}{3}\pi R^3 = M$$

As the density of the shell is uniform, it can be treated as a point mass placed at its centre.

Therefore, F = gravitational force between M and $m = \frac{GMm}{r^2}$

$$F = 0 \quad \text{for} \quad r < R$$ (i.e., force inside the shell is xero)

$$= \frac{GM}{r^2} \quad \text{for} \quad r \geq R$$

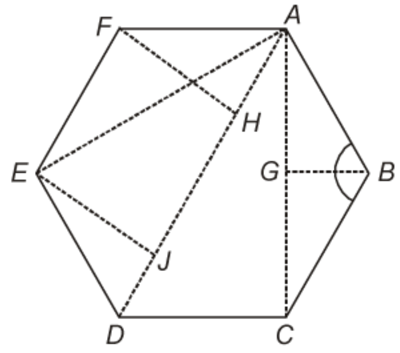


The variation of F versus r is shown in the diagram.

Note: When r tends to infinity, force tends to zero, also force is maximum on the surface of the hollow spherical shell.

S10. Gravitational force acting between two point masses m_1 and m_2, $F = \frac{Gm_1m_2}{r^2}$, is independent of the nature of medium between them. Therefore, gravitational force acting between two point masses will remain unaffected when they are dipped in water.

S11. Consider the diagram below, in which six point masses are placed at six vertices *A*, *B*, *C*, *D*, *E* and *F*.



$$AB = BC$$

$$(AC) = 2\,AG = 2.l.\frac{\sqrt{3}}{2} = \sqrt{3}l$$

$$AD = AH + HJ + JD$$

$$= \frac{l}{2} + l + \frac{l}{2}$$

$$= 2l.$$

$$AE = AC = \sqrt{3}l, \quad AF = l.$$

Force along *AD* due to *m* at *F* and *B*

$$F_1 = Gm^2\left[\frac{1}{l^2}\right]\frac{1}{2} + Gm^2\left[\frac{1}{l^2}\right]\frac{1}{2} = \frac{Gm^2}{l^2}$$

Force along *AD* due to masses at *E* and *C*

$$F_2 = Gm^2\,\frac{1}{3l^2}\cos(30°) + \frac{Gm^2}{3l^2}\cos(30°)$$

$$= \frac{Gm^2}{3l^2}\sqrt{3} = \frac{Gm^2}{\sqrt{3}l^2}$$

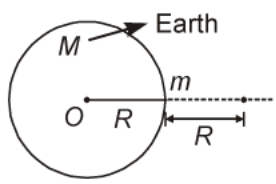
Force due to mass *M* at *D*

$$F_3 = \frac{Gm^2}{4l^2}$$

$\therefore$ Total Force = $F_1 + F_2 + F_3$

$$= \frac{Gm^2}{l^2}\left[1 + \frac{1}{\sqrt{3}} + \frac{1}{4}\right].$$

S12. Consider the diagram where an object of mass *m* is raised from the surface of the Earth to a distance (height) equal to the radius of the Earth (*R*).



Potential energy of the object at the surface of the Earth = $-\dfrac{GMm}{R}$

P.E. of the object at a height equal to the radius of the Earth = $-\dfrac{GMm}{2R}$

$\therefore$ Gain in P.E. of the object = $-\dfrac{GMm}{2R} - \left(-\dfrac{GMm}{R}\right)$

$$= \frac{-GMm + 2GMm}{2R} = +\frac{GMm}{2R}$$

$$= \frac{gR^2 \times m}{2R} = \frac{1}{2}mgR \qquad [\because \; GM = gR^2]$$

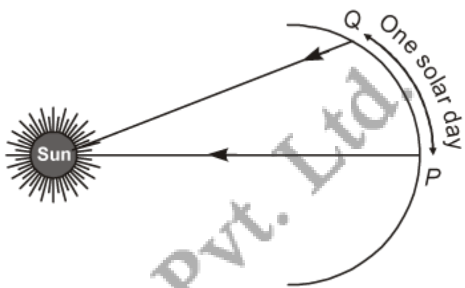
Inside a small spaceship orbiting around the Earth, the value of acceleration due to gravity g, can be considered as zero and hence astronaut feels weightlessness.

If the space station orbiting around the Earth has a large size, such that variation in g, matters in that case astronaut inside the spaceship will experience gravitational force and hence can detect gravity, *e.g.*, On the Moon, due to larger size gravity can be detected.

S13. A body cannot be shielded from the gravitational influence of nearby matter, because gravitational force between two point mass bodies is independent of the intervening medium between them.

It is due to the above reason, we cannot shield a body from the gravitational influence of nearby matter by putting it either inside a hollow sphere or by some other means.

S14. Consider the diagram below, the Earth moves from the point P to Q in one solar day.



Every day the Earth advances in the orbit by approximately 1°. Then, it will have to rotate by 361° (which we define as 1 day) to have sun at zenith point again.

$\because$ 361° corresponds to 24 hours.

$\therefore$ 1° corresponds to $\frac{24}{361} \times 1 = 0.066 \text{ h} = 3.99 \text{ min} \approx 4 \text{ min}$.

Hence, distance stars would rise 4 min early every successive day.

S15. Consider the adjacent diagram

Given, Mass of the Earth, $M = 6 \times 10^{24}$ kg

Radius of the Earth, $R = 6400 \text{ km} = 6.4 \times 10^6$ m

Time period $T = 24$ h

$= 24 \times 60 \times 60 = 86400$ s

$G = 6.67 \times 10^{-11}$ N-m^2/kg^2

(a) Time period $T = 2\pi\sqrt{\frac{(R+h)^3}{GM}}$ $\left[\because v_0 = \sqrt{\frac{GM}{R+h}} \text{ and } T = \frac{2\pi(R+h)}{v_0}\right]$

$\Rightarrow$ $T^2 = 4\pi^2 \frac{(R+h)^3}{GM}$ $\Rightarrow$ $(R+h)^3 = \frac{T^2 GM}{4\pi^2}$

$\Rightarrow$ $R + h = \left(\frac{T^2 GM}{4\pi^2}\right)^{1/3}$ $\Rightarrow$ $h = \left(\frac{T^2 GM}{4\pi^2}\right)^{1/3} - R$

$\Rightarrow$ $h = \left[\frac{(24 \times 60 \times 60)^2 \times 6.67 \times 10^{-11} \times 6 \times 10^{24}}{4 \times (3.14)^2}\right]^{1/3} - 6.4 \times 10^6$

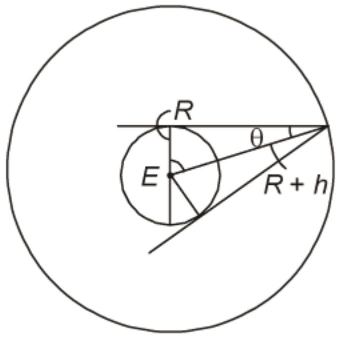
$= 4.23 \times 10^7 - 6.4 \times 10^6$

$= (42.3 - 6.4) \times 10^6$

$= 35.9 \times 10^6$ m

$= 3.59 \times 10^7$ m.

(b) If satellite is at height h from the Earth's surface, then according to the diagram.



$$\cos\theta = \frac{R}{R+h} = \frac{1}{\left(1+\frac{h}{R}\right)} = \frac{1}{\left(1+\frac{3.59\times10^7}{6.4\times10^6}\right)}$$

$$= \frac{1}{1+5.61} = \frac{1}{6.61} = 0.1513 = \cos 81^\circ 18'$$

$$\theta = 81^\circ 18'$$

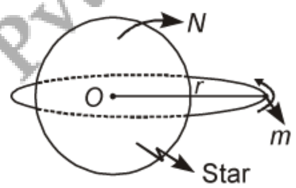
$$\therefore \quad 2\theta = 2\times(81^\circ 18') = 162^\circ 36'$$

If n is the number of satellites needed to cover entire the Earth, then

$$n = \frac{360^\circ}{2\theta} = \frac{360^\circ}{162^\circ 36'} = 2.31$$

$\therefore$ Minimum 3 satellites are required to cover entire the Earth.

S16. The situation is shown in the diagram, where a body of mass m is revolving around a Star of mass M.



Linear velocity of the body $\quad v = \sqrt{\dfrac{GM}{r}}$

$\Rightarrow \quad v \propto \dfrac{1}{\sqrt{r}}$

Therefore, when r increases, v decreased,

Angular velocity of the body $\quad \omega = \dfrac{2\pi}{T}$

According to Kepler's law of period,

$$T^2 \propto r^3 \quad \Rightarrow \quad T = kr^{3/2} \qquad [k = \text{constant}]$$

where k is a constant

$$\therefore \quad \omega = \frac{2\pi}{kr^{3/2}} \quad \Rightarrow \quad \omega \propto \frac{1}{r^{3/2}} \qquad \left(\because \quad \omega = \frac{2\pi}{T}\right)$$

Therefore, when r increases, ω decreases.

Kinetic energy of the body $\quad K = \dfrac{1}{2}mv^2 = \dfrac{1}{2}m\times\dfrac{GM}{r} = \dfrac{GMm}{2r}$

$\therefore \quad K \propto \dfrac{1}{r}$

Therefore, when r increases, K.E. decreases.

Gravitational potential energy of the body.

$$U = -\frac{GMm}{r} \quad \Rightarrow \quad U \propto -\frac{1}{r}$$

Therefore, when r increases, P.E. becomes less negative, *i.e.*, increases.

Total energy of the body $\quad E = \text{K.E.} + \text{P.E.} = \dfrac{GMm}{2r} + \left(-\dfrac{GMm}{r}\right) = -\dfrac{GMm}{2r}$

Therefore, when r increased, total energy becomes less negative, *i.e.*, increases.

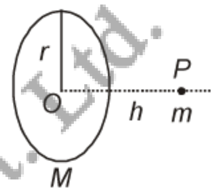
Angular momentum of the body $L = mvr = mr\sqrt{\dfrac{GM}{r}} = m\sqrt{GMr}$

$\therefore \quad L = \sqrt{r}$

Therefore, when r increases, angular momentum L increases.

Note: In this case, we have not considered the Sun-object system as isolated and the force on the system is not zero. So, angular momentum is not conserved.

S17. Consider the diagram, in which a system consisting of a ring and a point mass is shown.



Gravitational force acting on an object of mass m, placed at point P at a distance h along the normal through the centre of a circular ring of mass M and radius r is given by

$$F = \frac{GMmh}{(r^2 + h^2)^{3/2}} \qquad \text{[Along } PQ\text{]} \quad \text{... (i)}$$

When mass is displaced upto distance $2h$, then

$$F' = \frac{GMm}{[r^2 + (2h)^2]^{3/2}} \qquad [\because\ h = 2r]$$

$$= \frac{2GMmh}{(r^2 + 4h^2)^{3/2}} \qquad \text{... (ii)}$$

When $h = r$, then from Eq. (i)

$$F = \frac{GMm \times r}{(r^2 + h^2)^{3/2}} \quad \Rightarrow \quad F = \frac{GMm}{2\sqrt{2}r^2}$$

and $\quad F' = \dfrac{2GMmr}{(r^2 + 4r^2)^{3/2}} = \dfrac{2GMm}{5\sqrt{5}r^2} \qquad$ [From Eq. (ii) substituting $h = r$]

$$\therefore \quad \frac{F'}{F} = \frac{4\sqrt{2}}{5\sqrt{5}}$$

$$\Rightarrow \quad F' = \frac{4\sqrt{2}}{5\sqrt{5}}F.$$

S18. Consider the diagram, where a satellite of mass *m*, moving around the Earth in a circular orbit of radius R.

Orbital speed of the satellite orbiting the Earth is given by

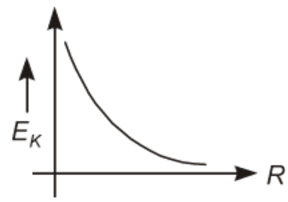
$$v_0 = \sqrt{\frac{GM}{R}}$$

where *M* and *R* are the mass and radius of the Earth.

(a) $\therefore$ K.E. of a satellite of mass *m*,

$$E_K = \frac{1}{2}mv_0^2 = \frac{1}{2}m \times \frac{GM}{R}$$

$\therefore$ $$E_K \propto \frac{1}{R}$$



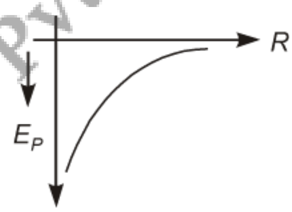
It means the K.E. decreases exponentially with radius.

The graph for K.E. versus orbital radius *R* is shown in figure.

(b) Potential energy of a satellite

$$E_P = -\frac{GMm}{R}$$

$$E_P \propto -\frac{1}{R}$$

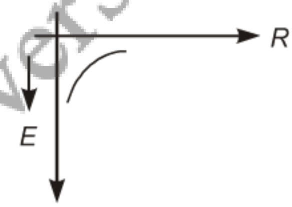


The graph for P.E. versus orbital radius *R* is shown in figure

(c) Total energy of the satellite

$$E = E_K + E_P = \frac{GMm}{2R} - \frac{GMm}{R}$$

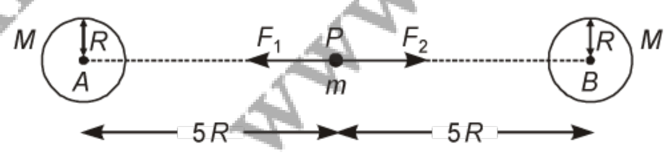
$$= -\frac{GMm}{2R}$$



The graph for tatol energy versus orbital radius *R* is shown in the figure.

Note: We should keep in mind that Potential Energy (P.E.) and Kinetic Energy (K.E.) of the satellite-Earth system is always negative.

S19. Let the mass and radius of each identical heavy sphere be *M* and *R* respectively. An object of mass *m* be placed at the mid-point *P* of the line joining their centres.



Force acting on the object placed at the mid-point,

$$F_1 = F_2 = \frac{GMm}{(5R)^2}$$

The direction of forces are opposite, therefore net force acting on the object is zero.

To check the stability of the equilibrium, we displace the object through a small distance x towards sphere A and it will try to come back to original position. So, it is in stable equilibrium.

S20.

$$r_a = a(1 + e) = 6R$$

$$r_p = a(1 - e) = 2R \quad \Rightarrow \quad e = \frac{1}{2}$$

Conservation of angular momentum:

Angular momentum at perigee = Angular momentum at apogee

$$\therefore \quad mv_p r_p = mv_a r_a$$

$$\therefore \quad \frac{v_a}{v_p} = \frac{1}{3}.$$

Conservation of Energy:

Energy at perigee = Energy at apogee

$$\frac{1}{2}mv_p^2 - \frac{GMm}{r_p} = \frac{1}{2}mv_a^2 - \frac{GMm}{r_a}$$

$$\therefore \quad v_p^2\left(1 - \frac{1}{9}\right) = -2GM\left[\frac{1}{r_a} - \frac{1}{r_p}\right] = 2GM\left[\frac{1}{r_a} - \frac{1}{r_p}\right]$$

$$v_p = \frac{2GM\left[\frac{1}{r_p} - \frac{1}{r_a}\right]^{1/2}}{\left[1 - \left(v_a/v_p\right)\right]^2} = \left[\frac{\frac{2GM}{R}\left[\frac{1}{2} - \frac{1}{6}\right]}{\left(1 - \frac{1}{9}\right)}\right]^{1/2}$$

$$= \left(\frac{2/3}{8/9}\frac{GM}{R}\right)^{1/2} = \sqrt{\frac{3}{4}\frac{GM}{R}} = 6.85 \text{ km/s.}$$

$$v_p = 6.85 \text{ km/s}, \qquad v_a = 2.28 \text{ km/s.}$$

For $r = 6r$, $v_c = \sqrt{\frac{GM}{6R}} = 3.23$ km/s.

Hence to transfer to a circular orbit at apogee, we have to boost the velocity by

$$\Delta = (3.23 - 2.28) = 0.95 \text{ km/s.}$$

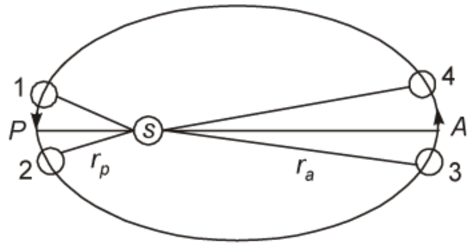
This can be done by suitably firing rockets from the satellite.

S21. Angular momentum and areal velocity are constant as Earth orbits the Sun.

At perigee $\quad r_p^2\omega_p = r_a^2\omega_a \quad$ at apogee.

If 'a' is the semi-major axis of Earth's orbit, then $r_p = a(1 - e)$ and $r_a = a(1 + e)$.

$$r_a = a(1 + e)$$



$\therefore$ $$\frac{\omega_p}{\omega_a} = \left(\frac{1+e}{1-e}\right)^2 \cdot e = 0.0167$$

$\therefore$ $$\frac{\omega_p}{\omega_a} = 1.0691$$

Let ω be angular speed which is geometric mean of ω_p and ω_a and corresponds to mean solar day,

$\therefore$ $$\left(\frac{\omega_p}{\omega}\right)\left(\frac{\omega}{\omega_a}\right) = 1.0691$$

$\therefore$ $$\frac{\omega_p}{\omega} = \frac{\omega}{\omega_a} = 1.034.$$

If ω corresponds to 1° per day (mean angular speed), then ω_p = 1.034 per day and ω_a = 0.967 per day. Since 361° = 14 hrs: mean solar day, we get 361.034° which corresponds to 24 hrs 8.14″ (8.1″ longer) and 360.967° corresponds to 23 hrs 59 min 52″ (7.9″ smaller).

This does not explain the actual variation of the length of the day during the year.

S22. The trajectory of a particle under gravitational force of the Earth will be a conic section (for motion outside the Earth) with the centre of the Earth as a focus. Only (c) meets this requirement.