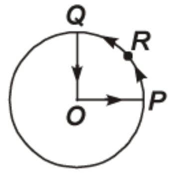
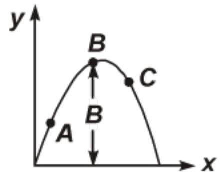
Date: 13/10/2021

Q1. A cyclist starts from centre O of a circular park of radius 1 km and moves along the path $OPRQO$ as shown figure. If he maintains constant speed of $10 \, ms^{-1}$, what is his acceleration at point R in magnitude and direction?



Q2. A particle is projected in air at some angle to the horizontal, moves along parabola as shown in figure, where x and y indicate horizontal and vertical directions, respectively. Show in the diagram, direction of velocity and acceleration at points A, B and C.



Q3. A football is kicked into the air vertically upwards. What is its (a) acceleration, and (b) velocity at the highest point?

Q4. A boy travelling in an open car moving on a levelled road with constant speed tosses a ball vertically up in the air and catches it back. Sketch the motion of the ball as observed by a boy standing on the footpath. Give explanation to support your diagram.

Q5. A ball is thrown from a roof top at an angle of 45° above the horizontal. It hits the ground a few seconds later. At what point during its motion, does the ball have

(a) greatest speed. (b) smallest speed. (c) greatest acceleration?

Explain

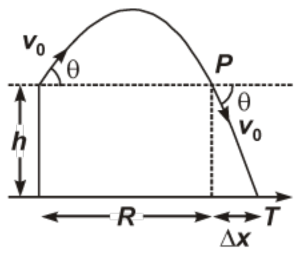
Q6. A fighter plane is flying horizontally at an altitude of 1.5 km with speed 720 km/h. At what angle of sight (w.r.t. horizontal) when the target is seen, should the pilot drop the bomb in order to attack the target?

Q7. A boy throws a ball in air at 60° to the horizontal along a road with a speed of 10 m/s (36 km/h). Another boy sitting in a passing by car observes the ball. Sketch the motion of the ball as observed by the boy in the car, if car has a speed of (18 km/h). Give explanation to support your diagram.

Q8. In dealing with motion of projectile in air, we ignore effect of air resistance on motion. This gives trajectory as a parabola as you have studied. What would the trajectory look like if air resistance is included? Sketch such a trajectory and explain why you have drawn it that way.

Q9. A gun can fire shells with maximum speed v_0 and the maximum horizontal range that can be achieved is $R = \frac{v_0^2}{g}$.

If a target farther away by distance Δx (beyond R) has to be hit with the same gun (see figure), show that it could be achieved by raising the gun to a height at least $h = \Delta x \left[1 + \frac{\Delta x}{R}\right]$.

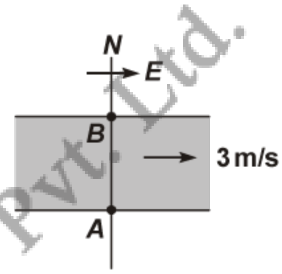


(*Hint :* This problem can be approached in two different ways:

(a) Refer to the diagram: target T is at horizontal distance $x = R + \Delta x$ and below point of projection $y = -h$.

(b) From point P in the diagram: Projection at speed v_0 at an angle θ below horizontal with height h and horizontal range Δx.)

Q10. A river is flowing due East with a speed 3 m/s. A swimmer can swim in still water at a speed of 4 m/s (see figure).



(a) If swimmer starts swimming due North, what will be his resultant velocity (magnitude and direction)?

(b) If he wants to start from point A on South bank and reach opposite point B on North bank,

(i) which direction should he swim?

(ii) what will be his resultant speed?

(c) From two different cases as mentioned in (a) and (b) above, in which case will he reach opposite bank in shorter time?

Q11. A hill is 500 m high. Supplies are to be sent across the hill using a canon that can hurl packets at a speed of 125 m/s over the hill. The canon is located at a distance of 800 m from the foot of hill and can be moved on the ground at a speed of 2 m/s; so that its distance from the hill can be adjusted. What is the shortest time in which a packet can reach on the ground across the hill? Take $g = 10\ m/s^2$.

Q12. A cricket fielder can throw the cricket ball with a speed v_0. If he throws the ball while running with speed u at an angle θ to the horizontal, find

(a) the effective angle to the horizontal at which the ball is projected in air as seen by a spectator.

(b) what will be time of flight?

(c) what is the distance (horizontal range) from the point of projection at which the ball will land?

(d) find θ at which he should throw the ball that would maximise the horizontal range as found in (iii).

(e) how does θ for maximum range change if $u > v_0$, $u = v_0$, $u < v_0$?

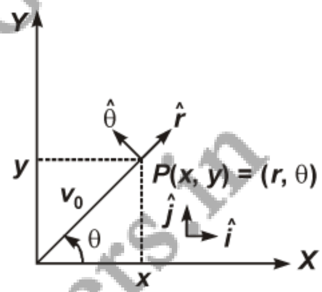
(f) how does θ in (v) compare with that for $u = 0$ (*i.e.*, 45°)?

Q13. (a) Earth can be thought of as a sphere of radius 6400 km. Any object (or a person) is performing circular motion around the axis of earth due to earth's rotation (period 1 day). What is acceleration of object on the surface of the earth (at equator) towards its centre? what is it at latitude θ? How does these accelerations compare with $g = 9.8 \text{ m/s}^2$?

(b) Earth also moves in circular orbit around sun once every year with on orbital radius of 1.5×10^{11} m. What is the acceleration of Earth (or any object on the surface of the Earth) towards the centre of the Sun? How does this acceleration compare with $g = 9.8 \text{ m/s}^2$?

Q14. A girl riding a bicycle with a speed of 5 m/s towards north direction, observes rain falling vertically down. If she increases her speed to 10 m/s, rain appears to meet her at 45° to the vertical. What is the speed of the rain? In what direction does rain fall as observed by a ground based observer?

Q15. Motion in two dimensions, in a plane can be studied by expressing position, velocity and acceleration as vectors in Cartesian co-ordinates $A = A_x\hat{i} + A_y\hat{j}$ where $\hat{i}$ and $\hat{j}$ are unit vector along x and y directions, respectively and A_x and A_y are corresponding components of A (see figure). Motion can also be studied by expressing vectors in circular polar co-ordinates as $A = A_r\hat{r} + A_\theta\hat{\theta}$ where $\hat{r} = \frac{r}{r} = \cos\theta\hat{i} + \sin\theta\hat{j}$ and $\hat{\theta} = -\sin\theta\hat{i} + \cos\theta\hat{j}$ are unit vectors along direction in which 'r' and 'θ' are increasing.



(a) Express $\hat{i}$ and $\hat{j}$ in terms of $\hat{r}$ and $\hat{\theta}$.

(b) Show that both $\hat{r}$ and $\hat{\theta}$ are unit vectors and are perpendicular to each other.

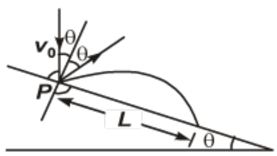
(c) Show that $\frac{d}{dt}(\hat{r}) = \omega\hat{\theta}$ where

$$\omega = \frac{d\theta}{dt} \text{ and } \frac{d}{dt}(\theta) = -\omega\hat{r}$$

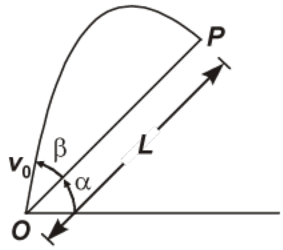
(d) For a particle moving along a spiral given by $r = a\theta\hat{r}$, where $a = 1$ (unit), find dimensions of 'a'.

(e) Find velocity and acceleration in polar vector representation for particle moving along spiral described in (d) above.

Q16. A particle falling vertically from a height hits a plane surface inclined to horizontal at an angle θ with speed v_0 and rebounds elastically (see figure). Find the distance along the plane where if will hit second time.



Q17. A particle is projected in air at an angle β to a surface which itself is inclined at an angle α to the horizontal (see figure).

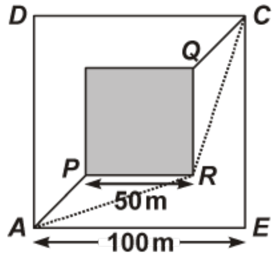


(a) Find an expression of range on the plane surface (distance on the plane from the point of projection at which particle will hit the surface).

(b) Time of flight.

(c) β at which range will be maximum.

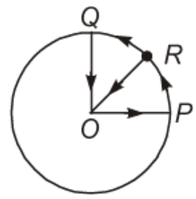
Q18. A man wants to reach from A to the opposite corner of the square C (see figure). The sides of the square are 100 m. A central square of $50\text{ m} \times 50\text{ m}$ is filled with sand. Outside this square, he can walk at a speed 1 m/s. In the central square, he can walk only at a speed of v m/s ($v < 1$). What is smallest value of v for which he can reach faster via a straight path through the sand than any path in the square outside the sand?



Date: 13/10/2021

S1. As shown in the adjacent figure. The cyclist covers the path *OPRQO*.

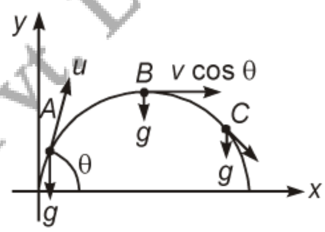
As we know whenever an object performing circular motion, acceleration is called centripetal acceleration and is always directed towards the centre.



Hence, acceleration at *R* is $a = \dfrac{v^2}{r}$

$$\Rightarrow \qquad a = \frac{(10)^2}{1\,\text{km}} = \frac{100}{10^3} = 0.1\,\text{m/s}^2 \text{ along } RO.$$

S2. Consider the adjacent diagram in which a particle is projected at an angle θ.

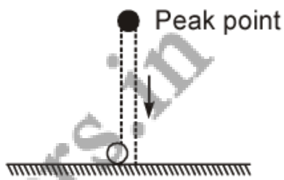


v_x = Horizontal component of velocity $v \cos\theta$ = constant.

v_y = Vertical component of velocity = $v \sin\theta$.

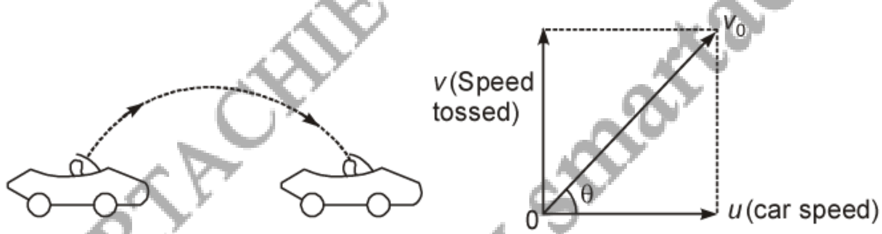
Velocity will always be tangential to the curve in the direction of motion and acceleration is always vertically downward and is equal to *g* (acceleration due to gravity).

S3. (a) Consider the adjacent diagram in which a football is kicked into the air vertically upwards. Acceleration of the football will always be vertical downward and is equal to *g*.



(b) When the football reaches the highest point velocity will be zero as it is continuously retared by acceleration due to gravity *g*.

S4. The path of the ball observed by a boy standing on the footpath is parabolic. The horizontal speed of the ball is same as that of the car, therefore, ball as well car travells equal horizontal distance. Due to its vertical speed, the ball follows a parabolic path.



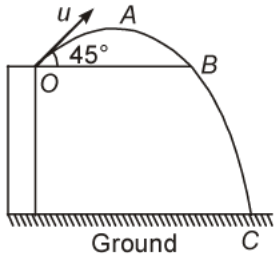
Note: We must be very clear that we are working with respect to ground. When we observe with respect to the car motion will be along vertical direction only.

S5. Consider the adjacent diagram in which a ball is projected from point *O*, and covering the path *OABC*.

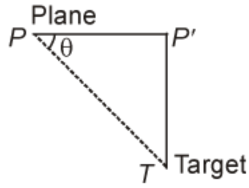
(a) At point B, it will gain the same speed u and after that speed increases and will be maximum just before reaching C.

(b) During upward journey from O to A speed decreases and will be minimum at point A.

(c) Acceleration is always constant throughout the journey and is vertically downward equal to g.



S6. Consider the adjacent diagram. Let a fighter plane, when it be at position P, drops a bomb to hit a target T.



Let $\angle P'PT = \theta$

Speed of the plane = 720 km/h

$$= 720 \times \frac{5}{18} \text{ m/s} = 200 \text{ m/s}$$

Altitude of the plane ($P'T$) = 1.5 km = 1500 m

If bomb hits the target after time t, then horizontal distance travelled by the bomb,

$$PP' = u \times t = 200t \quad \text{... (i)}$$

Vertical distance travelled by the bomb,

$$P'T = \frac{1}{2}gt^2 \Rightarrow 1500 = \frac{1}{2} \times 9.8\, t^2$$

$$\Rightarrow \quad t^2 = \frac{1500}{4.9} \Rightarrow t = \sqrt{\frac{1500}{49}} = 17.49 \text{ s}$$

Using value of t in Eq. (i),

$$PP' = 200 \times 17.49 \text{ m}$$

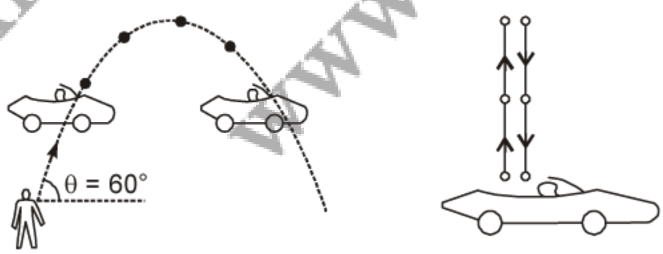
Now, $$\tan\theta = \frac{P'T}{P'P} = \frac{1500}{200 \times 17.49}$$

$$= .49287 = \tan 23°\, 12'$$

$$\theta = 23°\, 12'$$

Note: Angle is with respect to target. As seen by observer in the plane motion of the bomb will be vertically downward below the plane.

S7. Consider the diagram below



(a) (b)

The boy throws the ball at an angle of 60°

∴ Horizontal component of velocity = 4 cos θ

$$= (10 \text{ m/s}) \cos 60°$$

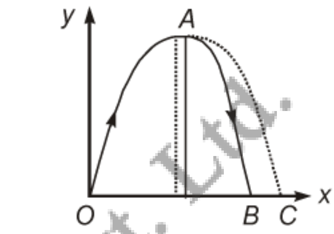
$$= 10 \times \frac{1}{2} = 5 \text{ m/s}$$

Speed of the car = 18 km/h = 5 m/s

As horizontal speed of ball and car is same, hence relative velocity of car and ball in the horizontal direction will be zero.

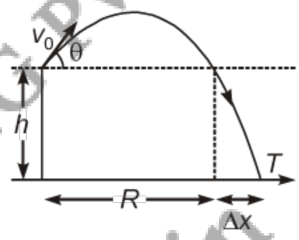
Only vertical motion of the ball be seen by the boy in the car, as shown in figure (b).

S8. Due to air resistance, particle energy as well as horizontal component of velocity keep on decreasing making the fall steeper than rise as shown in the figure.



When we are neglecting air resistance path was symmetric parabola (*OAB*). When air resistance is considered path is asymmetric parabola (*OAC*).

S9. This problem can be approached in two different ways



(a) Refer to the diagram, target *T* is at horizontal distance $x = R + \Delta x$ and between point of projection $y = -h$.

(b) From point *P* in the diagram projection at speed v_0 at an angle θ below horizontal with height *h* and horizontal range $\Delta x A$)

Applying method (i)

$$\text{Maximum horizontal range} \quad R = \frac{v_0^2}{g}, \quad \text{for} \quad \theta = 45° \qquad \text{... (i)}$$

Let the gun be raised through a height *h* from the ground so that it can hit the target. Let vertically downward direction is taken as positive.

Horizontal component of initial velocity = $v_0 \cos \theta$

Vertical component of initial velocity = $-v_0 \sin \theta$

Taking motion in vertical direction

$$h = (-v_0 \sin \theta)\, t + \frac{1}{2} g t^2 \qquad \text{... (ii)}$$

Taking motion in horizontal direction

$$(R + \Delta x) = v_0 \cos \theta \times t$$

$$\Rightarrow \qquad t = \frac{(R + \Delta x)}{v_0 \cos \theta} \qquad \text{... (iii)}$$

Substituting value of t in Eq. (ii), we get

$$h = (-v_0 \sin\theta) \times \left(\frac{R+\Delta x}{v_0 \cos\theta}\right) + \frac{1}{2} g \left(\frac{R+\Delta x}{v_0 \cos\theta}\right)^2$$

$$h = -(R+\Delta x)\tan\theta + \frac{1}{2} g \frac{(R+\Delta x)^2}{v_0^2 \cos^2\theta}$$

As angle of projection is $\theta = 45°$, therefore

$$h = -(R+\Delta x) + \tan 45° + \frac{1}{2} g \frac{(R+\Delta x)^2}{v_0^2 \cos^2 45°}$$

$$h = -(R+\Delta x) \times 1 + \frac{1}{2} g \frac{(R+\Delta x)^2}{v_0^2 (1/2)} \qquad \left(\because \ \tan 45° = 1 \text{ and } \cos 45° = \frac{1}{\sqrt{2}}\right)$$

$$h = -(R+\Delta x) + \frac{(R+\Delta x)^2}{R} \qquad \text{[Using Eq. (i), } R = v_0^2/g]$$

$$h = -(R+\Delta x) + \frac{1}{R}(R^2 + \Delta x^2 + 2R\Delta x)$$

$$h = -R - \Delta x + \left(R + \frac{\Delta x^2}{R} + 2\Delta x\right)$$

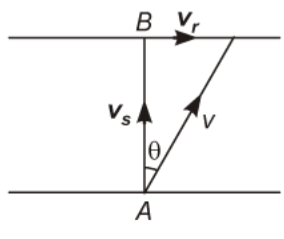
$$h = \Delta x + \frac{\Delta x^2}{R}$$

$$h = \Delta x\left(1 + \frac{\Delta x}{R}\right).$$

S10. Given, speed of the river (v_r) = 3 m/s (East)

Speed of swimmer (v_s) = 4 m/s (East)

(a) When swimmer starts swimming due north then his resultant velocity



$$v = \sqrt{v_r^2 + v_s^2} = \sqrt{(3)^2 + (4)^2}$$

$$= \sqrt{9+16} = \sqrt{25} = 5 \text{ m/s}$$

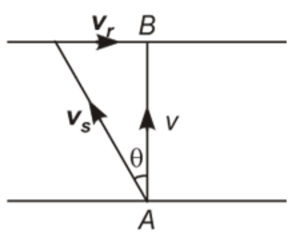
$$\tan\theta = \frac{v_r}{v_s} = \frac{3}{4}$$

$$= 0.75 = 36°54'$$

Hence, $\theta = 36°54'$.

(b) To reach opposite points B, the swimmer should swim at an angle θ of North.

Resultant speed of the swimmer



$$v = \sqrt{v_s^2 - v_r^2} = \sqrt{(4)^2 - (3)^2}$$

$$= \sqrt{16-9} = \sqrt{7} \text{ m/s}$$

$$\tan\theta = \frac{v_r}{v_s} = \frac{3}{\sqrt{7}}$$

$$\Rightarrow \qquad \theta = \tan^{-1}\left(\frac{3}{\sqrt{7}}\right) \text{ of North.}$$

(c) In case (a)

Time taken by the swimmer to cross the river,

$$t_1 = \frac{d}{v_s} = \frac{d}{4}\,s$$

In case (b)

Time taken by the swimmer to cross the river,

$$t_1 = \frac{d}{v} = \frac{d}{\sqrt{7}}$$

As $\quad \dfrac{d}{4} < \dfrac{d}{\sqrt{7}}$, therefore $t_1 < t_2$

Hence, the swimmer will cross the river in shorter time in case (a).

S11. The minimm vertical velocity required for crossing the hill is given by

$$v_\perp^2 \geq 2\,gh = 10{,}000$$

$$v_\perp > 100 \text{ m/s}$$

As canon can haul packets with a speed of 125 m/s, so the maximum value of horizontal velocity, $v_{||}$ will be

$$v_{||} = \sqrt{125^2 - 100^2} = 75\text{m/s } v$$

The time taken to reach the top of the hill with velocity $v_\perp$ is given by

$$\frac{1}{2}gT^2 = h \quad \Rightarrow \quad T = 10\,\text{s}.$$

In 10 s the horizontal distance covered = 750 m.

So cannon has to be moved through a distance of 50 m on the ground.

So total time taken (shortest) by the packet to reach ground across the hill

$$= \frac{50}{2}\,\text{s} + 10\,\text{s} + 10\,\text{s} = 45\,\text{s}.$$

S12. Consider the adjacent diagram.

(a) Initial velocity in x-direction,

u_x (Speed of cricketer + Speed of ball) = $u + v_0 \cos\theta$

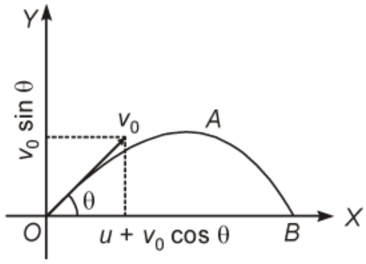
u_y = Initial velocity in y-direction

$= v_o \sin\theta$

where angle of projection is θ.

Now, we can write

$$\tan\theta = \frac{u_y}{u_x} = \frac{v_0 \sin\theta}{u + v_0\cos\theta}$$



$\Rightarrow$ $$\theta = \tan^{-1}\left(\frac{v_0 \sin\theta}{u + v_0\cos\theta}\right).$$

(b) Let T be the time of flight.

As net displacement is zero over time period T.

$$y = 0, \quad u_y = v_0 \sin\theta, \quad a_y = -g, \quad t = T$$

We know that $$y = u_y t + \frac{1}{2} a_y t^2$$

$\Rightarrow$ $$0 = v_0 \sin\theta T + \frac{1}{2}(-g)T^2$$

$\Rightarrow$ $$T\left[v_0 \sin\theta - \frac{g}{2}T\right] = 0 \quad \Rightarrow \quad T = 0, \frac{2v_0 \sin\theta}{g}$$

$T = 0$, corresponds to point O.

Hence, $$T = \frac{2v_0 \sin\theta}{g}.$$

(c) Horizontal range, $$R = (u + v_0\cos\theta)T = (u + v_0\cos\theta)\frac{2v_0\sin\theta}{g}$$

$$= \frac{v_0}{g}[2u\sin\theta + v_0 \sin 2\theta]$$

(d) For horizontal range to be maximum, $\frac{dR}{d\theta} = 0$

$\Rightarrow$ $$\frac{v_0}{g}[2u\cos\theta + v_0\cos 2\theta \times 2] = 0$$

$\Rightarrow$ $$2u\cos\theta + 2v_0[2\cos^2\theta - 1] = 0$$

$\Rightarrow$ $$4v_0\cos^2\theta + 2u\cos\theta - 2v_0 = 0$$

$\Rightarrow$ $$2v_0\cos^2\theta + u\cos\theta - v_0 = 0$$

$\Rightarrow$ $$\cos\theta = \frac{-u \pm \sqrt{u^2 + 8v_0^2}}{4v_0}$$

$$\Rightarrow \qquad \theta_{max} = \cos^{-1}\left[\frac{-u \pm \sqrt{u^2 + 8v_0^2}}{4v_0}\right]$$

$$= \cos^{-1}\left[\frac{-u + \sqrt{u^2 + 8v_0^2}}{4v_0}\right].$$

(e) If $u = v_0$,

$$\cos\theta = \frac{-v_0 \pm \sqrt{v_0^2 + 8v_0^2}}{4v_0} = \frac{-1+3}{4} = \frac{1}{2}$$

$$\Rightarrow \qquad \theta = 60°$$

If $u \ll v_0$, then $\quad 8v_0^2 + u^2 \approx 8v_0^2$

$$\theta_{max} = \cos^{-1}\left[\frac{-u \pm 2\sqrt{2}\,v_0}{4v_0}\right] = \cos^{-1}\left[\frac{1}{\sqrt{2}} - \frac{u}{4v_0}\right]$$

If $u \ll v_0$, then $\quad \theta_{max} = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

If $u > u_0$ and $u >> v_0$

$$\theta_{max} = \cos^{-1}\left[\frac{-u \pm u}{4v_0}\right] = 0 \quad \Rightarrow \theta_{max} = \frac{\pi}{2}$$

(f) If $u = 0$, $\quad \theta_{max} = \cos^{-1}\left[\frac{0 \pm \sqrt{8v_0^2}}{4v_0}\right] = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4} = 45°.$

S13. (a) Radius of the Earth (R) = 6400 km = 6.4×10^6 m

Time period (T) = 1 day = $24 \times 60 \times 60$ s = 86400 s

Centripetal acceleration (a_c) = $\omega^2 R = R\left(\frac{2\pi}{T}\right)^2 = \frac{4\pi^2 R}{T^2}$

$$= \frac{4 \times (22/7)^2 \times 6.4 \times 10^6}{(24 \times 60 \times 60)^2}$$

$$= \frac{4 \times 484 \times 64 \times 10^6}{49 \times (24 \times 3600)^2}$$

$$= 0.034 \text{ m/s}^2$$

At equator, $\quad$ latitude $\theta = 0°$

$$\therefore \qquad \frac{a_c}{g} = \frac{0.034}{9.8} = \frac{1}{288}$$

(b) Orbital radius of the Earth around the Sun (R) = 1.5×10^{11} m

Time period = 1 yr = 365 day

$= 365 \times 24 \times 50 \times 60\,\text{s} = 3.5 \times 10^{7}\,\text{s}$

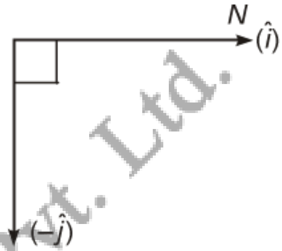
Centripetal acceleration (a_c) = $R\omega^2 = \dfrac{4\pi^2 R}{T^2}$

$$= \frac{4 \times (22/7)^2 \times 1.5 \times 10^{11}}{(3.15 \times 10^7)^2}$$

$$= 5.97 \times 10^{-3}\,\text{m/s}^2$$

$$\therefore \qquad \frac{a_c}{g} = \frac{5.97 \times 10^{-3}}{9.8} = \frac{1}{1642}.$$

S14. Assume North to be $\hat{i}$ direction and vertically downward to be $-\hat{j}$.



Let the rain velocity v_r be $a\hat{i} + b\hat{j}$.

$$\boldsymbol{v_r} = a\hat{i} + b\hat{j}$$

Case I: Given velocity of girl = $\boldsymbol{v_g}$ (5 m/s) $\hat{i}$

Let $\boldsymbol{v_{rg}}$ = Velocity of rain w.r.t. girl

$$= \boldsymbol{v_r} - \boldsymbol{v_g} = (a\hat{i} + b\hat{j}) - 5$$

$$= (a-5)\hat{i} + b\hat{j}$$

According to question rain, appears to fall vertically downward.

Hence, $\quad a - 5 = 0 \quad \Rightarrow \quad a = 5$

Case II: Given velocity of the girl,

$$\boldsymbol{v_g} = (10\,\text{m/s})\hat{i}$$

$$\therefore \qquad \boldsymbol{v_{rg}} = \boldsymbol{v_r} - \boldsymbol{v_g}$$

$$= (a\hat{i} + b\hat{j}) - 10\hat{i} = (a-10)\hat{i} + b\hat{j}$$

According to question rain appears to fall at 45 to the vertical hence

$$\tan 45 = \frac{b}{a-10} = 1$$

$$\Rightarrow \qquad b = a - 10 = 5 - 10 = -5$$

Hence, velocity of rain = $a\hat{i} + b\hat{j}$

$$\Rightarrow \qquad \boldsymbol{v_r} = 5\hat{i} - 5\hat{j}$$

Speed of rain = $|\boldsymbol{v_r}| = \sqrt{(5)^2 + (-5)^2} = \sqrt{50} = 5\sqrt{2}\ m/s$.

S15. (a) Given, unit vector $\quad \hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$ ... (i)

$\hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$ ... (ii)

Multiplying Eq. (i) by sin θ and Eq. (ii) with cos θ and adding

$$\hat{\boldsymbol{r}}\sin\theta + \hat{\theta}\cos\theta = \sin\theta\,.\,\cos\theta\,\hat{\boldsymbol{i}} + \sin^2\theta\,\hat{\boldsymbol{j}} + \cos^2\theta\,\hat{\boldsymbol{j}} - \sin\theta\,.\,\cos\theta\,\hat{\boldsymbol{i}}$$

$$= \hat{\boldsymbol{j}}(\cos^2\theta + \sin^2\theta) = \hat{\boldsymbol{j}}$$

= $\hat{\boldsymbol{r}}\sin\theta + \hat{\theta}\cos\theta = \hat{\boldsymbol{j}}$

By Eq. (i) × cos θ – Eq (ii) × sin θ

$$n(\hat{\boldsymbol{r}}\cos\theta - \hat{\theta}\sin\theta) = \hat{\boldsymbol{i}}$$

(b) $\hat{\boldsymbol{r}}\,.\,\hat{\theta} = (\cos\theta\,\hat{\boldsymbol{i}} + \sin\theta\,\hat{\boldsymbol{j}})\,.\,(-\sin\theta\,\hat{\boldsymbol{i}} + \cos\theta\,\hat{\boldsymbol{j}})$

$$= -\cos\theta\,.\,\sin\theta + \sin\theta\,.\,\cos\theta = 0$$

⇒ θ = 90° Angle between $\hat{\boldsymbol{r}}$ and $\hat{\theta}$.

(c) Given, $\hat{\boldsymbol{r}} = \cos\theta\,\hat{\boldsymbol{i}} + \sin\theta\,\hat{\boldsymbol{j}}$

$$\frac{d\hat{\boldsymbol{r}}}{dt} = \frac{d}{dt}(\cos\theta\,\hat{\boldsymbol{i}} + \sin\theta\,\hat{\boldsymbol{j}})$$

$$= -\sin\theta\,.\,\frac{d\theta}{dt}\hat{\boldsymbol{i}} + \cos\theta\,.\,\frac{d\theta}{dt}\hat{\boldsymbol{j}}$$

$$= \omega[-\sin\theta\,\hat{\boldsymbol{i}} + \cos\theta\,\hat{\boldsymbol{j}}] \qquad \left[\because\ \omega = \frac{d\theta}{dt}\right]$$

(d) Given, $\boldsymbol{r} = a\theta\hat{\boldsymbol{r}}$, here writing dimensions $[\boldsymbol{r}] = [a][\theta][\hat{\boldsymbol{r}}]$

⇒ $L = [a]\ \%\ 1 \quad \Rightarrow \quad [a] = L = [M^0 L^1 T^0]$

(e) Given, a = 1 unit $\quad \boldsymbol{r} = \theta\hat{\boldsymbol{r}} = \theta[\cos\theta\,\hat{\boldsymbol{i}} + \sin\theta\,\hat{\boldsymbol{j}}]$

Velocity $\quad v = \dfrac{d\boldsymbol{r}}{dt} = \dfrac{d\theta}{dt}\hat{\boldsymbol{r}} + \theta\dfrac{d}{dt}\hat{\boldsymbol{r}} = \dfrac{d\theta}{dt}\hat{\boldsymbol{r}} + \theta\dfrac{d}{dt}[(\cos\theta\hat{\boldsymbol{i}} + \sin\theta\hat{\boldsymbol{j}})]$

$$= \frac{d\theta}{dt}\hat{\boldsymbol{r}} + \theta\left((-\sin\theta\hat{\boldsymbol{i}} + \sin\theta\hat{\boldsymbol{j}})\frac{d\theta}{dt}\right)$$

$$= \frac{d\theta}{dt}\hat{\boldsymbol{r}} + \theta\hat{\theta}\omega = \omega\hat{\boldsymbol{r}} + \omega\theta\hat{\theta}$$

Acceleration, $\quad \boldsymbol{a} = \dfrac{d}{dt}[\omega\hat{\boldsymbol{r}} + \omega\theta\hat{\theta}] = \dfrac{d}{dt}\left[\dfrac{d\theta}{dt}\hat{\boldsymbol{r}} + \dfrac{d\theta}{dt}(\theta\hat{\theta})\right]$

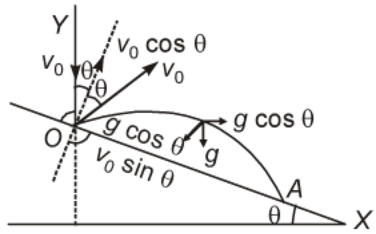
$$= \frac{d^2\theta}{dt^2}\hat{\boldsymbol{r}} + \frac{d\theta}{dt}\cdot\frac{d\hat{\boldsymbol{r}}}{dt} + \frac{d^2\theta}{dt^2}\theta\hat{\theta} + \frac{d\theta}{dt}\frac{d}{dt}(\theta\hat{\theta})$$

$$= \frac{d^2\theta}{dt^2}\hat{\boldsymbol{r}} + \omega[-\sin\theta\hat{\boldsymbol{i}} + \sin\theta\hat{\boldsymbol{j}}] + \frac{d^2\theta}{dt^2}\theta\hat{\theta} + \frac{\omega d}{dt}(\theta\hat{\theta})$$

$$= \frac{d^2\theta}{dt^2}\hat{\boldsymbol{r}} + \omega^2\hat{\theta} + \frac{d^2\theta}{dt^2}\times\theta\hat{\theta} + \omega^2\hat{\theta} + \omega^2\theta(-\hat{\boldsymbol{r}})$$

$$\left(\frac{d^2\theta}{dtv^2} - \omega^2\right)\hat{\boldsymbol{r}} + \left(2\omega^2 + \frac{d^2\theta}{dt^2}\theta\right)\theta.$$

S16. Considering x and y axes as shown in the diagram.



For the motion of the projectile from O to A.

$$y = 0, \quad u_y = v_0 \cos\theta$$
$$a_y = -g\cos\theta, \quad t = T$$

Applying equation of kinematics,

$$y = u_y t + \frac{1}{2} a_y t^2$$

$\Rightarrow$
$$0 = v_0 \cos\theta T + \frac{1}{2}(-g\cos\theta)T^2$$

$\Rightarrow$
$$T\left[v_0 \cos\theta - \frac{g\cos\theta}{2}\right] = 0$$

$$T = \frac{2v_0 \cos\theta}{g\cos\theta}$$

As $T = 0$, corresponds to point O

Hence,
$$T = \frac{2v_0}{g}$$

Now, considering motion along OX.

$$x = L, \quad u_x = v_0 \sin\theta, \quad a_x = g\sin\theta, \quad t = T = \frac{2v_0}{g}$$

Applying equation of kinematics,

$$x = u_x t + \frac{1}{2} a_x t^2$$

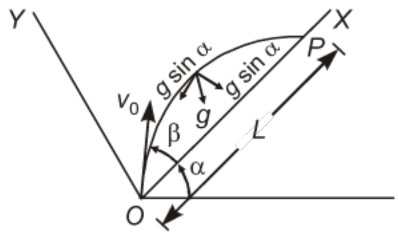
$\Rightarrow$
$$L = v_0 \sin\theta t + \frac{1}{2} g \sin\theta t^2 = (v_0 \sin\theta)(T) + \frac{1}{2} g\sin\theta T^2$$

$$= (v_0 \sin\theta)\left(\frac{2v_0}{g}\right) + \frac{1}{2} g \sin\theta \times \left(\frac{2v_0}{g}\right)^2$$

$$= \frac{2v_0^2}{g}\sin\theta + \frac{1}{2} g\sin\theta \times \frac{4v_0^2}{g^2} = \frac{2v_0^2}{g}[\sin\theta + \sin\theta]$$

$\Rightarrow$
$$L = \frac{4v_0^2}{g}\sin\theta.$$

S17. Consider the adjacent diagram.



Mutually perpendicular x and y-axes are shown in the diagram.

Particle is projected from the point O.

Let time taken in reaching from point O to point P is T.

(a) Considering motion along OX.

$$x = L, \quad u_x = v_0 \cos\beta, \quad a_x = -g\sin\alpha$$

$$t = T = \frac{2v_0 \sin\beta}{g\cos\alpha}$$

$$x = u_x t + \frac{1}{2} a_x t^2$$

$\Rightarrow$ $$L = v_0 \cos\beta T + \frac{1}{2} (-g\sin\alpha)\, T^2$$

$\Rightarrow$ $$L = v_0 \cos\beta T - \frac{1}{2} g\sin\alpha T^2$$

$$= T\left[v_0 \cos\beta - \frac{1}{2} g\cos\alpha T\right]$$

$$= T\left[v_0 \cos\beta - \frac{1}{2} g\sin\alpha \times \frac{2v_0\sin\beta}{g\cos\alpha}\right]$$

$$= \frac{2v_0\sin\beta}{g\cos^2\alpha}\left[v_0\cos\beta - \frac{v_0\sin\alpha\sin\beta}{\cos\alpha}\right]$$

$$= \frac{2v_0^2\sin\beta}{g\cos^2\alpha}[\cos\beta\cos\alpha - \sin\alpha\sin\beta]$$

$\Rightarrow$ $$= \frac{2v_0^2\sin\beta}{g\cos^2\alpha}\cos(\alpha+\beta)$$

(b) Considering motion along vertical upward direction perpendicular to OX.
For the journey O to P.

$$y = 0, \quad u_y = v_0\sin\beta, \quad a_y = -g\cos\alpha, \quad t = T$$

Applying equation,

$$y = u_y t + \frac{1}{2} + a_y t^2$$

$\Rightarrow$ $$0 = v_0\sin\beta T + \frac{1}{2}(-g\cos\alpha)\, T^2$$

$$\Rightarrow\ T\left[v_0\sin\beta - \frac{g\cos\alpha}{2}T\right] = 0$$

$\Rightarrow$ $$T = 0, \quad T = \frac{2v_0\sin\beta}{g\cos\alpha}$$

As $T = 0$, corresponding to point O

Hence, $$T = \text{time of flight} = \frac{2v_0\sin\beta}{g\cos\alpha}$$

(c) For range (L) to be maximum,
$\sin\beta \,.\, \cos(\beta+\beta)$ should be maximum.

Let, $\quad Z = \sin\beta \,.\, \cos(\alpha + \beta)$

$$= \sin\beta\,[\cos\alpha \,.\, \cos\beta - \sin\alpha \,.\, \sin\beta]$$

$$= \frac{1}{2}\,[\cos\alpha \,.\, \sin 2\beta - 2\sin\alpha \,.\, \sin^2\beta]$$

$$= \frac{1}{2}\,[\sin 2\beta \,.\, \cos\alpha - \sin\alpha\,(1 - \cos 2\beta)]$$

$$\Rightarrow \quad z = \frac{1}{2}\,[\sin 2\beta \,.\, \cos\alpha - \sin\alpha + \sin\alpha \,.\, \cos 2\beta]$$

$$= \frac{1}{2}\,[\sin 2\beta \,.\, \cos\alpha + \cos 2\beta \,.\, \sin\alpha - \sin\alpha]$$

$$= \frac{1}{2}\,[\sin(2\beta + \alpha) - \sin\alpha]$$

For z to be maximum,

$$\sin(2 = \beta + \alpha) = \text{maximum} = 1$$

$$\Rightarrow \quad 2\beta + \alpha = \frac{\pi}{2} \quad \text{or} \quad \beta = \frac{\pi}{4} - \frac{\alpha}{2}.$$

S18. Consider the straight line path $APQC$ through the sand.

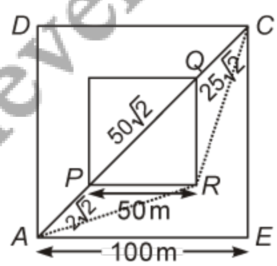
Time taken to go from A to C via this path

$$T_{sand} = \frac{AP + QC}{1} + \frac{PQ}{v}$$

$$= \frac{25\sqrt{2} + 25\sqrt{2}}{1} + \frac{50\sqrt{2}}{v} = 50\sqrt{2}\left[\frac{1}{v} + 1\right]$$

The shortest path outside the sand will be ARC.

Time taken to go from A to C via this path



$$= T_{outside} = \frac{AR + RC}{1}\ \text{s}$$

$$= \sqrt{75^2 + 25^2}\ \text{s}$$

$$= 2 \times 25\sqrt{10}\ \text{s}$$

For $\quad T_{sand} < T_{outside},$

$$50\sqrt{2}\left[\frac{1}{v} + 1\right] < 2 \times 25\sqrt{10}$$

$$\Rightarrow \quad \frac{1}{v} + 1 < \sqrt{5}$$

$$\Rightarrow \quad \frac{1}{v} < \sqrt{5} - 1 \quad \text{or} \quad v > \frac{1}{\sqrt{5} - 1} \approx \text{m/s} \quad \text{or} \quad v > 0.81\ \text{m/s}.$$