Differentiation

Single Correct Answer Type

1. The second derivative of a single valued function parametrically represented by $x=\phi(t)$ and $y=\psi(t)$, (where $\phi(t)$ and $\psi(t)$ are different functions and $\phi'(t)\neq 0$) is given by

A) $\frac{d^2y}{dx^2}=\frac{\left(\frac{dx}{dt}\right)\left(\frac{d^2y}{dt^2}\right)-\left(\frac{d^2x}{dt^2}\right)\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)^3}$

B) $\frac{d^2y}{dx^2}=\frac{\left(\frac{dx}{dt}\right)\left(\frac{d^2y}{dt^2}\right)-\left(\frac{d^2x}{dt^2}\right)\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)^2}$

C) $\frac{d^2y}{dx^2}=\frac{\left(\frac{d^2x}{dt^2}\right)\frac{dy}{dt}-\frac{dx}{dt}\left(\frac{d^2y}{dt^2}\right)}{\left(\frac{dx}{dt}\right)^3}$

D) $\frac{d^2y}{dx^2}=\frac{\left(\frac{d^2x}{dt^2}\right)\left(\frac{dy}{dt}\right)-\left(\frac{d^2y}{dt^2}\right)\frac{dx}{dt}}{\left(\frac{dy}{dt}\right)^3}$

Key. A

Sol. $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy/dt}{dx/dt}\right)$$

2. $y=f(x)$ be a real valued twice differentiable function defined on R, then $\frac{d^2y}{dx^2}\left(\frac{dx}{dy}\right)^3+\frac{d^2x}{dy^2}=$

(A) $\frac{dy}{dx}$ (B) $\frac{dx}{dy}$ (C) $\frac{d^2y}{dx^2}$ (D) 0

Key. D

Sol. $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{1}{\frac{dx}{dy}}\right)$

$$=\frac{d}{dy}\left(\frac{1}{\frac{dx}{dy}}\right)\frac{dy}{dx}=\frac{-\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^2}.\frac{dy}{dx}=-\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3}$$

$$\Rightarrow\frac{d^2y}{dx^2}.\left(\frac{dx}{dy}\right)^3+\frac{d^2x}{dy^2}=0$$

3. A function $f:R\rightarrow[1,\infty)$ satisfies the equation

$f(xy) = f(x)f(y) - f(x) - f(y) + 2$. If f is differentiable on $R - \{0\}$ and $f(2) = 5$,

$f'(x) = \frac{f(x)-1}{x}.\lambda$ then $\lambda =$_____

(A) $2f'(1)$ (B) $3f'(1)$ (C) $\frac{1}{2}f'(1)$ (D) $f'(1)$

Key. D

Sol. $f'(x) = \lim\limits_{x\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim\limits_{x\to 0} \frac{f\left\{x\left(1+\frac{h}{x}\right)\right\} - f(x)}{h} (x \neq 0 \text{ given})$

$= \lim\limits_{x\to 0} \frac{f\left(1+\frac{h}{x}\right)-2}{\frac{h}{x}} \cdot \frac{f(x)-1}{x}$

Putting $x = 1, y = 2$ in the given functional equation, f(1)=2

$\therefore f'(x) = \frac{f(x)-1}{x} \cdot \lim\limits_{h\to 0} \frac{f\left(1+\frac{h}{x}\right) - f(1)}{h/x}$

$= \frac{f(x)-1}{x} \cdot f'(1)$

4. If $F(x) = f(x).g(x)$ and $f'(x)g'(x) = c$, where 'c' is a constant then $\frac{f''}{f} + \frac{g''}{g} + \frac{2C}{fg} =$

(A) $\frac{f''}{F}$ (B) $\frac{f''}{F'}$ (C) $\frac{F''}{F}$ (D) none of these

Key. C

Sol. Given $F = fg \Rightarrow F' = f'g + fg'$

$F'' = f''g + 2f'g' + fg''$

$\Rightarrow F'' = f''g + 2C + fg''$

$\Rightarrow \frac{f''}{F} = \frac{f''}{f} + \frac{g''}{g} + \frac{2C}{fg}$

5. Let $g(x) = \log f(x)$ where $f(x)$ is a twice differentiable positive function on $(0, \infty)$ such that $f(x+1) = xf(x)$. Then, for $N = 1, 2, 3.....,$ $g''\left(N + \frac{1}{2}\right) - g''\left(\frac{1}{2}\right) =$

A) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+...+\frac{1}{(2N-1)^2}\right\}$ B) $4\left\{1+\frac{1}{9}+\frac{1}{25}+...+\frac{1}{(2N-1)^2}\right\}$

C) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+...+\frac{1}{(2N+1)^2}\right\}$ D) $4\left\{1+\frac{1}{9}+\frac{1}{25}+...+\frac{1}{(2N+1)^2}\right\}$

Key. 1

Sol. $g''(x+1)-g''(x)=-\frac{1}{x^2}$

$x\to x-\frac{1}{2}$, we get $g''\left(x+\frac{1}{2}\right)-g''\left(x-\frac{1}{2}\right)=-\frac{4}{(2x-1)^2}$

Put x=1,2,.....N adding we get result

6. $\frac{d^2x}{dy^2}$ equals :

A) $\left(\frac{d^2y}{dx^2}\right)^{-1}$ B) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$

C) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$ D) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$

Key. 4

Sol. $\frac{dx}{dy}=\left(\frac{dy}{dx}\right)^{-1}\Rightarrow\frac{d^2x}{dy^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)^{-1}.\frac{dx}{dy}=-\left(\frac{dy}{dx}\right)^{-2}\frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^{-1}=-\left(\frac{dy}{dx}\right)^{-3}\left(\frac{d^2y}{dx^2}\right)$

7. If $x^2+y^2=1$, then

A) $yy''-2(y')^2+1=0$ B) $yy''+(y')^2+1=0$

C) $yy''-(y')^2+1=0$ D) $yy''+2(y')^2+1=0$

Key. 2

Sol. Differentiate successively

8. If $y=x\log\left(\frac{x}{a+bx}\right)$, then $x^n\frac{d^2y}{dx^2}=\left(x\frac{dy}{dx}-y\right)^m$, where :

A) n=3,m=2 B) $n=2,m=3$

C) $m=n=2$ D) $m=n=3$

Key. 1

Sol. Conceptual

9. If $y=\left(\sin^{-1}x\right)^2+\left(\cos^{-1}x\right)^2$, then $\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=$

A) -4 B) 4 C) 2 D) -2

Key. 2

Sol. $y_1=\frac{2\sin^{-1}x}{\sqrt{1-x^2}}-\frac{2\cos^{-1}x}{\sqrt{1-x^2}}\Rightarrow\sqrt{1-x^2}\,y_1=2\left(\sin^{-1}x-\cos^{-1}x\right)$

$\Rightarrow\sqrt{1-x^2}\,y_2-\frac{2x}{2\sqrt{1-x^2}}y_1=2\left(\frac{1}{\sqrt{1-x^2}}+\frac{1}{\sqrt{1-x^2}}\right)$

10. If $y = x\sin(\log x) + x\log x,$ then $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + 2y =$

A) $\ln(x)$ B) $x\ln(x)$ C) $-x\ln(x)$ D) $\dfrac{\ln x}{x}$

Key. 2

Sol. $y_1 = \sin(\log x) + x\cos(\log x) + \log x + 1$

$$y_2 = \frac{\cos(\log x)}{x} - \frac{\sin(\log x)}{x} + \frac{1}{x}$$

11. By introducing a new variable t, putting $x = \cos t$, the expression $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y$ is transformed into :

A) $\dfrac{d^2y}{dt^2} + y$ B) $\dfrac{d^2y}{dt^2} - t\dfrac{dy}{dt} + y$ C) $\dfrac{d^2y}{dt^2} - y$ D) $\dfrac{d^2y}{dt^2}$

Key. 1

Sol. $t = \cos^{-1} x \Rightarrow \dfrac{dt}{dx} = \dfrac{-1}{\sqrt{1-x^2}}$

$$\frac{dy}{dx} = \frac{dy}{dx}.\frac{dt}{dx} = -\frac{1}{\sqrt{1-x^2}}.\frac{dy}{dt} \Rightarrow \sqrt{1-x^2}\frac{dy}{dx} = -\frac{dy}{dt}$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = \frac{d^2y}{dt^2}$$

12. Let $f(x)$ and $g(x)$ be two functions having finite non-zero third order derivatives $f'''(x)$ and $g'''(x)$ for all $x \in R$. If $f(x)g(x) = 1$ for all $x \in R,$ then $\dfrac{f'''}{f'} - \dfrac{g'''}{g'}$ is equal to :

A) $3\left(\dfrac{f''}{g} - \dfrac{g''}{f}\right)$ B) $3\left(\dfrac{f''}{f} - \dfrac{g''}{g}\right)$

C) $3\left(\dfrac{g''}{g} - \dfrac{f''}{g}\right)$ D) $3\left(\dfrac{f''}{g} - \dfrac{g''}{f}\right)$

Key. 2

Sol. $fg = 1 \Rightarrow fg_1 + f_1 g = 0$

$$\Rightarrow fg_2 + gf_2 + 2f_1g_1 = 0$$

$$\Rightarrow fg_3 + gf_3 = -3[f_1g_2 + f_2g_1]$$

$$\Rightarrow \frac{f_3}{f_1} - \frac{g_3}{g_1} = 3\left[\frac{f_2}{f} - \frac{g_2}{g}\right]$$

(using $fg_1 = -f_1 g$)

13. If $y^{1/m} = \left[x + \sqrt{1+x^2}\right]$, then $(1+x^2)y_2 + xy_1$ is equal to :

A) m^2y B) my^2 C) m^2y^2 D) my

Key. 1

Sol. $y = \left(x + \sqrt{1+x^2}\right)^m \Rightarrow y_1 = m\left(x + \sqrt{1+x^2}\right)^{m-1}.\left\{1 + \dfrac{2x}{2\sqrt{1+x^2}}\right\}$

$$\Rightarrow \sqrt{1+x^2}\,y_1 = m\left(x+\sqrt{1+x^2}\right)^m \Rightarrow \sqrt{1+x^2}\,y_1 = my$$

$$\Rightarrow \left(1+x^2\right)y_2 + xy_1 = m^2 y$$

14. If $x+\cos\theta=\sec\theta,\ y+\cos^8\theta=\sec^8\theta$ then $\left(\frac{x^2+4}{y^2+4}\right)\left(\frac{dy}{dx}\right)^2=$

a) 8 b) 16 c) 64 d) 49

Key. C

Sol. $$\frac{dy}{dx}=\frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}=\frac{8\sec^8\theta\tan\theta+8\cos^7\theta\sin\theta}{\sec\theta\tan\theta+\sin\theta}=\frac{8\tan\theta\left(\sec^8\theta+\cos^8\theta\right)}{\tan\theta\left(\sec\theta+\cos\theta\right)}$$

$$\left(\frac{dy}{dx}\right)^2=\frac{64\left(\sec^8\theta+\cos^8\theta\right)^2}{\left(\sec\theta+\cos\theta\right)^2}=\frac{64\left[\left(\sec^8\theta-\cos^8\theta\right)^2+4\right]}{\left[\left(\sec\theta-\cos\theta\right)^2+4\right]}=\frac{64\left(y^2+4\right)}{\left(x^2+4\right)}$$

$$\left(\frac{x^2+4}{y^2+4}\right)\left(\frac{dy}{dx}\right)^2=64$$

15. Let $y=f(x)$ and $f:R\to R$ be an odd function which is differentiable such that $f'''(x)>0$ and $f(\alpha,\beta)=\sin^8\alpha+\cos^8\beta+2-4\sin^2\alpha\cos^2\beta$.

If $f''(f(\alpha,\beta))=0$ then $\sin^2\alpha+\sin^2\beta=$

a) 0 b) 1 c) 2 d) 3

Key. B

Sol. $f''(x)$ is an odd function

$$\therefore f(\alpha,\beta)=0 \Rightarrow \left(\sin^4\alpha-1\right)^2+\left(\cos^4\beta-1\right)^2+2\left(\sin^2\alpha-\cos^2\beta\right)^2=0$$

$$\Rightarrow \sin^2\alpha+\sin^2\beta=1$$

16. If $y=\sin(\sin x)$ and $\frac{d^2y}{dx^2}+\frac{dy}{dx}\tan x+f(x)=0$, then $f(x)$ equals.

A) $\sin^2 x\sin(\cos x)$ B) $\sin^2 x\cos(\sin x)$ C) $\cos^2 x\sin(\cos x)$ D) $\cos^2 x\sin(\sin x)$

Key. D

Sol. $$\frac{dy}{dx}=\cos(\sin x)\cos x$$

$$\frac{d^2y}{dx^2}=-\cos(\sin x)\sin x+\cos x\left[-\sin(\sin x)\right]\cos x$$

$$\therefore \frac{d^2y}{dx^2}+\frac{dy}{dx}\tan x=-\cos^2 x\sin(\sin x)$$

$$\therefore \frac{d^2y}{dx^2}+\frac{dy}{dx}\tan x+\cos^2 x\sin(\sin x)=0$$

$\therefore f(x)=\cos^2 x\sin(\sin x)$

17. $\frac{d^2x}{dy^2}$ equals

A) $\left(\frac{d^2y}{dx^2}\right)^{-1}$ B) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$ C) $\frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^{-2}$ D) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$

Key. D

Sol. $\frac{d^2x}{dy^2}=\frac{d}{dy}\left(\frac{dx}{dy}\right)=\frac{d}{dx}\left(\frac{dx}{dy}\right)\times\frac{dx}{dy}$

$$=\frac{d}{dx}\left[\frac{1}{\frac{dy}{dx}}\right]\times\frac{1}{\left(\frac{dy}{dx}\right)}$$

$$=\frac{-1}{\left(\frac{dy}{dx}\right)^2}\times\frac{d^2y}{dx^2}\times\frac{1}{\frac{dy}{dx}}$$

$$=-\left(\frac{dy}{dx}\right)^{-3}\cdot\frac{d^2y}{dx^2}$$

18. If $(a+bx)e^{\frac{y}{x}}=x$ then $x^3\frac{d^2y}{dx^2}$ equals

A) $\left(\frac{dy}{dx}+x\right)^2$ B) $\left(x\frac{dy}{dx}-y\right)^2$ C) $\left(\frac{dy}{dx}-y\right)^2$ D) $\left(x\frac{dy}{dx}+y\right)^2$

Key. B

Sol. $\frac{y}{x}=\log x-\log(a+bx)$

Differentiable

$$\frac{x\frac{dy}{dx}-y}{x^2}=\frac{1}{x}-\frac{1}{a+bx}b=\frac{a}{x(a+bx)}$$

$$\therefore x\frac{dy}{dx}-y=\frac{ax}{a+bx}\rightarrow ....(1)$$

Differentiable again w.r.t x

$$x\frac{d^2y}{dx^2}=\frac{a^2}{(a+bx)^2}$$

$$\therefore x^3 \frac{d^2y}{dx^2} = \frac{a^2x^2}{(a+bx)^2} = \left(\frac{xdy}{dx} - y\right)^2$$

19. If $y = Tan^{-1}\left(\frac{3+2\log_e x}{1-6\log_e x}\right) + Tan^{-1}\left(\frac{\log_e\left(\frac{e}{x^2}\right)}{\log_e\left(ex^2\right)}\right)$, then $\frac{d^2y}{dx^2}$ is

A) 0 B) 1 C) -1 D) 2

Key. A

Sol. $y = Tan^{-1}\left(\frac{1-2\log_e^{x}}{1+2\log_e x}\right) + Tan^{-1}\left(\frac{3+2\log_e x}{1-6\log_e x}\right)$

$= Tan^{-1}(1) - Tan^{-1}(2\log_e x) + Tan^{-1}(3) + Tan^{-1}(2\log_e x)$

$= Tan^{-1}(1) + Tan^{-1}(3)$

$\therefore \frac{dy}{dx} = 0$

20. Let f be a function such that $f(x+y) = f(x) + f(y)$ for all x and y and $f(x) = (2x^2+3x)\,g(x)$ for all x where $g(x)$ is continuous and $g(0) = 9$ then $f'(x)$ is equals to

A) 9 B) 3 C) 27 D) 6

Key. C

Sol. $f'(x) = \underset{h\to 0}{Lt} \frac{f(x+h) - f(x)}{h}$

$= \underset{h\to 0}{Lt} \frac{f(x) + f(h) - f(x)}{h}$

$= \underset{h\to 0}{Lt} \frac{(2h^2+3h)\,g(h)}{h}$

$= \underset{h\to 0}{Lt} (2h+3)\,g(h)$

$= 3\,g(0) = 3.9 = 27$

21. If $y = (Tax)^{(Tanx)^{Tanx}}$ then $\frac{dy}{dx}$ $at\ x = \frac{\pi}{4}$ is

A) 1 B) 2 C) 3 D) 4

Key. B

Sol. $\log y = (\tan x)^{\tan x} \log(\tan x) \quad \to (1)$

Taking $\log$ again, we get from (1)

$\log(\log y) = \tan x \log(\tan x) + \log(\log(\tan x))$

Differentiate with respect to x

$$\frac{1}{\log y}.\frac{1}{y}\frac{dy}{dx}=\sec^2 x\log(Tanx)+Tanx\frac{1}{Tanx}Sec^2x+\frac{1}{\log(Tanx)}\frac{1}{Tanx}Sec^2x$$

$$\therefore \frac{dy}{dx}=y\log y\ \sec^2 x\left(\log(Tanx)+1+\frac{1}{Tanx\log(Tanx)}\right)$$

$$=y(Tanx)^{Tanx}\log Tan.Sec^2x\left[\left(\log(Tanx)+1\right)+\frac{1}{Tanx\log(Tanx)}\right]$$

$$=y(Tanx)^{Tanx}Sec^2x\left[\log(Tanx)(\log Tanx+1)+Cotx\right]$$

When at $x=\frac{\pi}{4},\ y=1$

$$\therefore \frac{dy}{dx}=1.1.2(0+1)=2$$

22. The triangle formed by the tangent to the curve $f(x)=x^2+bx-b$ at the point (1, 1) and the co-ordinate axes, lies in the first quadrant. If its area is 2 then the value of b is

A) -1 B) 3 C) -3 D) 1

Key. C

Sol. $\frac{dy}{dx}=2x+b=2+b\ at\ (1,1)$

Equation of tangent is $y-1=(2+b)(x-1)$ is intercepts A and B on the axis are obtained by putting y = 0 and then x = 0

$$\therefore A=\frac{b+1}{b+2},\ B=-(b+1)$$

$$\Delta=\frac{1}{2}AB=2$$

$$\therefore AB=4\Rightarrow -(b+1)(b+1)=4(b+2)$$

$$\therefore b=-3$$

23. If $t(1+x^2)=x$ and $x^2+t^2=y$ then at x = 2, the value of $\frac{dy}{dx}$ is

a) $\frac{24}{5}$ b) $\frac{101}{125}$ c) $\frac{488}{125}$ d) $\frac{358}{125}$

Key. 3

Sol. $\frac{dy}{dx}=2x+2t.\frac{dt}{dx}$

$t=\frac{x}{1+x^2}\quad \frac{dt}{dx}=\frac{1-x^2}{(1+x^2)^2}$ put x =2

24. Suppose f and g are functions having second derivatives f'' and g'' every where, if $f(x).g(x)=1$ for all x and f' and g' are never zero, then $\frac{f''(x)}{f'(x)}-\frac{g''(x)}{g'(x)}$ equals

a) $2\frac{f'(x)}{f(x)}$ b) $\frac{2g'(x)}{g(x)}$ c) $-\frac{f'(x)}{f(x)}$ d) $\frac{g'(x)}{g(x)}$

Key. 1

Sol. We have $f'(x)g(x)+g'(x)f(x)=0 \Rightarrow \frac{g(x)}{g'(x)}+\frac{f(x)}{f'(x)}=0$ --- (1)

Further $f''(x)g(x)+2f'(x)g'(x)+g''(x)f(x)=0$

Divide throughout by $f'(x)g'(x)$ and use (1)

25. If $x=\varphi(t)$ and $y=\psi(t)$, then $\frac{d^2y}{dx^2}$ is equal to (dashes denote the derivative w.r.t 't')

a) $\frac{\varphi'\psi''-\psi'\varphi''}{\varphi'}$ b) $\frac{\varphi'\psi''-\psi'\varphi''}{\varphi'^2}$ c) $\frac{\varphi''}{\psi''}$ d) $\frac{\psi''}{\varphi'^2}-\frac{\psi'.\varphi''}{\varphi'^3}$

Key. 4

Sol. $\frac{dy}{dx}=\frac{\psi'}{\varphi'}$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}=\frac{\frac{\varphi'\psi''-\psi'\varphi''}{(\varphi')^2}}{\varphi'}$$

26. If $x=a\sin 2\theta(1+\cos 2\theta)$ and $y=b\cos 2\theta(1-\cos 2\theta)$, then $\left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{3}}=$

a) 0 b) $\frac{ab}{\sqrt{3}}$ c) $\frac{b}{a}\sqrt{3}$ d) 1

Key. 3

Sol. $x=a\sin 2\theta\left(2\cos^2\theta\right)=2a\sin 2\theta\cos^2\theta$

$y=b\cos 2\theta\left(2\sin^2\theta\right)=2b\cos 2\theta\sin^2\theta$

$$\frac{dy}{dx}=\frac{2b\left(-2\sin 2\theta\sin^2\theta+\cos 2\theta\sin 2\theta\right)}{2a\left(2\cos 2\theta\cos^2\theta+\sin 2\theta(-\sin 2\theta)\right)}$$

$$\Rightarrow\left(\frac{dy}{dx}\right)_{\theta=\pi/3}=\frac{b}{a}.\sqrt{3}$$

27. If $x=\frac{1+t}{t^3}, y=\frac{3}{2t^2}+\frac{2}{t}$ satisfies $f(x)\left(\frac{dy}{dx}\right)^3=1+\frac{dy}{dx}$, then $f(x)$

a) x b) $\frac{x^2}{1+x^2}$ c) $x+\frac{1}{x}$ d) $x-\frac{1}{x}$

Key. 1

Sol. $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{-\frac{(3+2t)}{t^3}}{-\frac{(3+2t)}{t^4}}=t$

Since

$f(x)\left(\frac{dy}{dx}\right)^3=1+\frac{dy}{dx}\Rightarrow f(x)t^3=1+t$ $\Rightarrow f(x)=\frac{1+t}{t^3}=x$

28. If $y=(1+x)(1+x^2)(1+x^4)$, then $\frac{dy}{dx}$ $at\ x=0$ is

a) 1 b) – 1 c) 0 d) 2

Key. 1

Sol. Multiplying numerator & denominator by (1 - x)

$y=\frac{(1-x)(1+x)(1+x^2)(1+x^4)}{1-x}$ $\Rightarrow y=\frac{1-x^8}{1-x}$ $\left(\frac{dy}{dx}\right)_{x=0}=1$

29. $y=f(x)$ be a real valued twice differentiable function defined on R, then

$\frac{d^2y}{dx^2}\left(\frac{dx}{dy}\right)^3+\frac{d^2x}{dy^2}=$

(A) $\frac{dy}{dx}$ (B) $\frac{dx}{dy}$

(C) $\frac{d^2y}{dx^2}$ (D) 0

Key. D

Sol. $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{1}{\frac{dx}{dy}}\right)$

$=\frac{d}{dy}\left(\frac{1}{\frac{dx}{dy}}\right)\frac{dy}{dx}=\frac{-\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^2}\cdot\frac{dy}{dx}=-\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3}$

$\Rightarrow\frac{d^2y}{dx^2}\cdot\left(\frac{dx}{dy}\right)^3+\frac{d^2x}{dy^2}=0$

30. If cos y = x cos(a+y) and $\frac{dy}{dx}=\frac{k}{1+x^2-2x\cos a}$ then the value of k is

a) sin a b) cos a c) 1 d) – sin a

Key. A

Sol. $x=\frac{\cos y}{\cos(a+y)} \Rightarrow \frac{dx}{dy}=\frac{\sin a}{\cos^2(a+y)} \Rightarrow \frac{dy}{dx}=\frac{\cos^2 y}{x^2 \sin a}=\frac{\sin a}{1+x^2-2x\cos a} \Rightarrow K=\sin a$

31. Given that f(x) = $\begin{cases} \frac{x.g(x)}{|x|}, & x \neq 0 \\ 0, & x=0 \end{cases}$ g(0) = 0 = g′(0), then f′(0) equals

(A) 1 (B) –1

(C) 2 (D) 0

Key. D

Sol. f′(0) = $\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$

= $\lim_{h\to 0}\frac{\frac{hg(h)}{|h|}-0}{h} = \lim_{h\to 0}\frac{g(h)}{|h|}=0 = \lim_{h\to 0}\frac{g'(h)}{1}=0$

32. If $f(x)=x+\tan x$, and f is inverse of g, then $g'(x)$ equal to

a) $\frac{1}{1+[g(x)-x]^2}$ b) $\frac{1}{2-[g(x)-x]^2}$ c) $\frac{1}{2+[g(x)-x]^2}$ d) $\frac{1}{1-[g(x)-x]^2}$

Key. C

Sol. $f(x)=x+\tan x$

$f(f^{-1}(y))=f^{-1}(y)+\tan(f^{-1}(y))$

$y=g(y)+\tan(g(y))$

$x=g(x)+\tan(g(x))$

diff

$1=g'(x)+\sec^2(g(x)).g'(x)$

$g'(x)=\frac{1}{2+[g(x)-x]^2}$

33. If 'f' is an increasing function from R $\rightarrow$ R such that f''(x) > 0 and f^{-1} exists then $\frac{d^2(f^{-1}(x))}{dx^2}$ is

a) < 0 b) > 0

c) = 0 d) cannot be determined

Key. A

Sol. f'(x) > 0 and f'' (x) > 0

Let g(x) = $f^{-1}(x)$

f (g(x)) = x

f' (g(x)) . g'(x) = 1

$$g'(x) = \frac{1}{f'(g(x))}$$

$$g''(x) = -\frac{1}{(f'(g(x)))^2} f''(g(x)).\ g'(x)$$

$\therefore g''(x). < 0.$

$(\because$ f''(x) & f'(x) > 0)

$$\frac{d^2}{dx^2}\left(f^{-1}(x) = gw''(x)\right).<0$$

34. If a function $f:[-2a, 2a] \to R$ is an odd function such that $f(2a-x) = f(x), \forall x \in [a, 2a]$ and left hand derivative at x = a is 0 then find left hand derivative at x = -a

(a) 0 (b) -1 (c) $\frac{1}{2}$ (d) a

Key. A

Sol. $f'(-a^-) = \underset{h\to 0}{Lt} \frac{f(-a) - f(-a-h)}{h}$

$\underset{h\to 0}{Lt} \frac{-f(a) + f(a+h)}{h} = \underset{h\to 0}{Lt} \frac{-f(a) + f(a+h)}{h} = 0$

35. $\frac{d^2x}{dy^2} =$

(a) $\left(\frac{d^2y}{dx^2}\right)^{-1}$ (b) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$

(c) $\frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^{-2}$ (d) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$

Key. D

Sol. $\frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1}$

Again differentiating with respective y

36. If $\sqrt{x^2+y^2} = a\,e^{Tan^{-1}\left(\frac{y}{x}\right)}(a>0),(y(0)>0)$ then $y^{11}(0) =$

(a) $\frac{a}{2}e^{\pi/2}$ (b) $ae^{-\pi/2}$ (c) $\frac{-2}{a}e^{-\pi/2}$ (d) $\frac{a}{2}e^{-\pi/2}$

Key. C

Sol. Differentiating with respect to x two times

$$\frac{d^2y}{dx^2} = \frac{2(x^2+y^2)}{(x-y)^3}$$

$$Put\ x = 0 \Rightarrow \frac{2y(0)^2}{(-y(0))^3}$$

37. If $y = e^{-x}\cos x$ and $y_4 + ky = 0$, where $y_4 = \frac{d^4y}{dx^4}$, then k =

A) 4 B) – 4 C) 2 D) – 2

Key. A

Sol. $y = e^{-x}\cos x$

$y_1 = -e^{-x}\sin x - e^{-x}\cos x = -e^{-x}\sin x - y$

$y_2 = -e^{-x}\cos x + e^{-x}\sin x - y_1$

$\Rightarrow \quad y_2 = -y - y_1 + e^{-x}\sin x$

$\Rightarrow \quad y_2 = -2(y + y_1)$

$\Rightarrow \quad y_3 = -2(y_1 + y_2)$

$\Rightarrow \quad y_3 = -2(e^{-x}\sin x - y)$

$\Rightarrow \quad y_4 = 2y_1 - 2y - 2y - 2y_1$

or $\quad y_4 + 4y = 0 \quad \Rightarrow \quad k = 4$

38. Let $y = e^{2x}$. Then $\left(\frac{d^2y}{dx^2}\right)\left(\frac{d^2x}{dy^2}\right)$ is:

A) 1 B) e^{-2x} C) $2e^{-2x}$ D) $-2e^{-2x}$

Key. D

Sol. $y = e^{2x}$

$\therefore \quad \frac{dy}{dx} = 2e^{2x}$ and $\frac{d^2y}{dx^2} = 4e^{2x}$

$\frac{dx}{dy} = \frac{1}{2e^{2x}} = \frac{1}{2y}$

$\therefore \quad \frac{d^2x}{dy^2} = -\frac{1}{2y^2} = -\frac{1}{2}e^{-4x}$

$\therefore \quad \frac{d^2y}{dx^2}.\frac{dx^2}{dy^2} = 4.\, e^{2x}.\left(\frac{-e^{-2x}}{2e^{2x}}\right) = -2e^{-2x}$

39. If $y^2 = P(x)$, is a polynomial of degree 3, then $\left(\frac{d}{dx}\right)\left(y^3.\frac{d^2y}{dx^2}\right)$ equals:

A) $P'''(x) + P'(x)$ B) $P''(x).P'''(x)$ C) $P(x).P'''(x)$ D) a constant

Key. C

Sol. $y^2 = P(x) \Rightarrow 2y\frac{dy}{dx} = P'(x)$

Or $\quad 2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = P''(x)$

Or $\quad 2y\frac{d^2y}{dx^2} = P'' - 2\left(\frac{dy}{dx}\right)^2 = P'' - \frac{P'^2}{2y^2}$

$\because \quad 2y^3\frac{d^2y}{dx^2} = y^2P'' - \frac{1}{2}P'^2 = PP'' - \frac{1}{2}P'^2$

$\therefore \quad 2\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right) = P'P'' + PP''' - P'P'' = PP'''$

40. If $f'(x) = -f(x)$ and g(x) = f(x) and $F(x) = \left(f\left(\frac{x}{2}\right)\right)^2 + \left(g\left(\frac{x}{2}\right)\right)^2$ and given that F(5) = 5, then F(10) is equal to

A) 5 B) 10 C) 0 D) 15

Key. A

Sol. $F'(x)\left[f\left(\frac{x}{2}\right).f'\left(\frac{x}{2}\right) + g\left(\frac{x}{2}\right)g'\left(\frac{x}{2}\right)\right]$

Here $g(x) = f'(x)$ & $g'(x) = f''(x) = -f(x)$

So $F'(x) = f\left(\frac{x}{2}\right)g\left(\frac{x}{2}\right) - f\left(\frac{x}{2}\right)g\left(\frac{x}{2}\right) = 0$

$\Rightarrow$ F(x) is constant function

So F(10) = 5

41. If $f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$ (where p is constant), then at x = 0, $\frac{d^3 f(x)}{dx^3} = 0$

A) p B) p^2 C) – p D) 0

Key. D

Sol. $f'(x) = \begin{vmatrix} 3x^2 & \cos x & -\sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$

$f''(x) = \begin{vmatrix} 6x & -\sin x & -\cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$

$f'''(x) = \begin{vmatrix} 6 & -1 & 0 \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix} = 0$

42. If $x^y = e^{x-y}$, then $\frac{dy}{dx} =$

A) $(1+\log x)^{-1}$ B) $(1+\log x)^{-2}$

C) $\log x(1+\log x)^{-2}$ D) $\log x(1+\log x)^{-1}$

Key. C

Sol. $x^y = e^{x-y}$ i.e. $y\, ln\, x = x - y$ i.e. $y = \frac{x}{1 + ln\, x}$

$\therefore \frac{dy}{dx} = \frac{ln\, x}{(1 + ln\, x)^2}$

43. If $y = \tan^{-1}\left(\frac{2^x}{1+2^{2x+1}}\right)$, then $\frac{dy}{dx}$ at x = 0 is

A) 1 B) 2 C) ln 2 D) none of these

Key. D

Sol. $y = \tan^{-1}\left(\frac{2^{x+1} - 2^x}{1 + 2^x.2^{x+1}}\right) = \tan^{-1} 2^{(x+1)} - \tan^{-1} 2^x \Rightarrow y' = \frac{2^{x+1}\ln 2}{1+\left(2^{x+1}\right)^2} - \frac{2^x \ln 2}{1+\left(2^x\right)^2} \Rightarrow y'(0) = -\frac{1}{10}\ln 2$

44. A function g defined for all real x > 0 satisfies g(1) = 1, $g'(x^2) = x^3$ for all x > 0, then g(4) equals

A) $\frac{13}{3}$ B) 3 C) $\frac{67}{5}$ D) none of these

Key. C

Sol. $g'(x^2) = x^3$

$\Rightarrow \quad g'(t) = t^2, \quad$ where $\quad x^2 = t$

$\Rightarrow \quad g(t) = \frac{t^{5/2}}{5/2} + c$

$\therefore \quad g(1) = \frac{(1)^{5/2}}{5/2} + c$

$\therefore \quad c = \frac{3}{5}$

$\therefore \quad g(t) = \frac{t^{5/2}}{5/2} + \frac{3}{5}$

$g(4) = \frac{(4)^{5/2}}{5/2} + \frac{3}{5} = \frac{67}{5}$

45. If f(x) = log x, then $f'(\log x)$ equals

A) $\frac{x}{\log x}$ B) $\frac{\log x}{x}$ C) $\frac{1}{x \log x}$ D) $\frac{1}{\log x}$

Key. D

Sol. $f(x) = \log x$

$f'(x) = \frac{1}{x}$

$f'(\log x) = \frac{1}{\log x}$

23. f is a strictly monotonic differentiable function with $f^1(x) = \frac{1}{\sqrt{1+x^3}}$. If g is the inverse of f then $g^{11}(x) =$

A) $\frac{3x^2}{2\sqrt{1+x^3}}$ B) $\frac{3g^2(x)}{2\sqrt{1+g^2(x)}}$ C) $\frac{3}{2}g^2(x)$ D) $\frac{x^2}{\sqrt{1+x^3}}$

Key. C

Sol. $y = f(x) \Leftrightarrow x = f^{-1}(y) \Leftrightarrow x = g(y)$ since $g = f^{-1}$

$\frac{dx}{dy} = g^1(y) = \frac{1}{f^1(x)} = \sqrt{1+x^3}$

$\frac{d^2x}{dy^2} = g^{11}(y) = \frac{d}{dy}^{\left(\frac{dx}{dy}\right)} = \frac{d}{dy}^{\sqrt{1+x^3}}$

$= \frac{d}{dx}^{\left(\sqrt{1+x^3}\right)} \times \frac{dx}{dy} = \frac{3x^2}{2\sqrt{1+x^3}} \times g^1(y) = \frac{3x^2}{2}$

$g^{11}(y) = \frac{3}{2} g^2(y)$

24. Let $f(x) = \frac{g(x)}{x}$ when $x \neq 0$ and $f(0) = 0$. If $g(0) = g^1(0) = 0$ and $g^{11}(0) = 17$ then $f^1(0) =$

A) 3/4 B) –1/2 C) 17/3 D) 17/2

Key. D

Sol. $f^1(0) = \underset{x \to 0}{Lt} \frac{f(x) - f(0)}{x - 0} = \underset{x \to 0}{Lt} \frac{f(x)}{x}$

$= \underset{x \to 0}{Lt} \frac{g(x)}{x^2} = \underset{x \to 0}{Lt} \frac{g^1(x)}{2x} = \underset{x \to 0}{Lt} \frac{g^{11}(x)}{2} = \frac{g^{11}(0)}{2} = \frac{17}{2}$