

Definite, Indefinite Integration & Areas

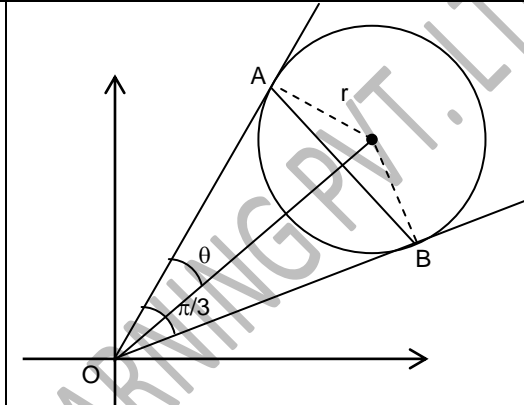
Single Correct Answer Type

1. If a circle of radius r is touching the lines $x^2 - 4xy + y^2 = 0$ in the first quadrant at points A and B, then area of triangle OAB (O being the origin) is

- (A) $\frac{3\sqrt{3}r^2}{4}$ (B) $\frac{\sqrt{3}r^2}{4}$ (C) $\frac{3r^2}{4}$ (D) r^2

Key. A

Sol.

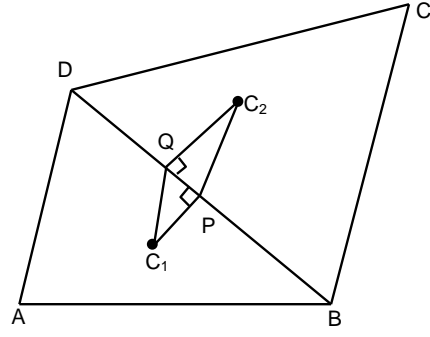
<p>Here $\tan 2\theta = \frac{2\sqrt{4-1}}{2} = \sqrt{3} \Rightarrow \theta = \frac{\pi}{6}$</p> <p>Area of $\Delta OAB = \frac{1}{2} (r \cot \theta)^2 (\sin 2\theta)$</p> <p>$= \frac{1}{2} (r\sqrt{3})^2 \frac{\sqrt{3}}{2}$.</p>	
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2. ABCD is a quadrilateral with side lengths AB = 4, BC = 10, CD = 6 and AD = 6, and diagonal BD = 8 units. If the incircles of triangles ABD and BCD touch BD at P and Q respectively, then area of quadrilateral C_1PC_2Q (where C_1 and C_2 are incentres of triangle ABD and BCD respectively), is

- (A) $3 + \frac{\sqrt{15}}{2}$ sq. units (B) 3 sq. units
- (C) $\frac{\sqrt{15}}{6}$ sq. units (D) 4 sq. units

Key. A

Sol.

<p>In triangle ABD, we have $BP = \frac{6+4+8}{2} - 6 = 3$.</p> <p>In triangle BCD, we have</p> <p>$DQ = \frac{10+6+8}{2} - 10 = 2$.</p> <p>$\Rightarrow PQ = 8 - (3+2) = 3$</p> <p>$\Rightarrow$ area of trapezium $C_1PC_2Q = \frac{1}{2} (r_1 + r_2) \cdot PQ$,</p> <p>where $r_1 = \frac{\sqrt{9 \times 3 \times 5 \times 1}}{9} = \frac{\sqrt{15}}{3}$ and $r_2 = \frac{\sqrt{12 \times 2 \times 6 \times 4}}{12} = 2$.</p>	
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$$\Rightarrow \text{area of quadrilateral } C_1PC_2Q = \frac{1}{2} \left(2 + \frac{\sqrt{15}}{3} \right) \times 3 = 3 + \frac{\sqrt{15}}{2} \text{ sq. units.}$$

3. The area of the region whose boundaries are defined by the curves $y = 2 \cos x$, $y = 3 \tan x$ and the y -axis, is

(A) $1 + 3 \ln \left(\frac{2}{\sqrt{3}} \right)$

(B) $1 + \frac{3}{2} \ln 3 - 3 \ln 2$

(C) $1 + \frac{3}{2} \ln 3 - \ln 2$

(D) $\ln 3 - \ln 2$

Key. B

Sol. Solving $2 \cos x = 3 \tan x$ we get, $2 - 2 \sin^2 x = 3 \sin x$

$$\Rightarrow \sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}$$

$$\text{Required area} = \int_0^{\pi/6} (2 \cos x - 3 \tan x) dx = 2 \sin x - 3 \ln \sec x \Big|_0^{\pi/6} = 1 - 3 \ln 2 + \frac{3}{2} \ln 3.$$

4. The area of the region bounded by the curve $y = x^2$ and $y = \sec^{-1} [-\sin^2 x]$, (where $[\cdot]$ denotes the greatest integer function), is

(A) $\pi\sqrt{\pi}$

(B) $\frac{4}{3} \pi\sqrt{\pi}$

(C) $\frac{2}{3} \pi\sqrt{\pi}$

(D) $\frac{1}{3} \pi\sqrt{\pi}$

Key. B

Sol. $[-\sin^2 x] = 0$ or -1 but $\sec^{-1}(0)$ is not defined.

$$\Rightarrow \sec^{-1} [-\sin^2 x] = \sec^{-1} (-1) = \pi.$$

$$\text{The required area} = \int_{-\sqrt{\pi}}^{\sqrt{\pi}} (\pi - x^2) dx = \frac{4}{3} \pi\sqrt{\pi}.$$

5. The range of the function $f(x) = \int_1^x |t| dt$, $x \in \left[\frac{-1}{2}, \frac{1}{2} \right]$ is

(A) $\left[\frac{3}{8}, \frac{5}{8} \right]$

(B) $\left[\frac{-5}{8}, \frac{3}{8} \right]$

(C) $\left[\frac{-3}{8}, \frac{5}{8} \right]$

(D) $\left[\frac{-5}{8}, \frac{-3}{8} \right]$

Key. D

Sol. If $0 \leq x \leq \frac{1}{2} \Rightarrow f(x) = \int_1^x t dt = \frac{x^2 - 1}{2}$

$$\text{If } -\frac{1}{2} \leq x \leq 0 \Rightarrow f(x) = \int_1^0 t dt - \int_0^x t dt = -\left(\frac{x^2 + 1}{2} \right)$$

$$\Rightarrow f'(x) > 0 \forall x \in \left[\frac{-1}{2}, \frac{1}{2} \right]$$

$$\therefore \text{Range of } f(x) \text{ is } \left[f\left(\frac{-1}{2}\right), f\left(\frac{1}{2}\right) \right] \Rightarrow \left[\frac{-5}{8}, \frac{-3}{8} \right]$$

6. $\int_0^1 \tan^{-1}(1 - x + x^2) dx = \underline{\hspace{2cm}}$

- (A) $\ln 2$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{2}$ (D) $\frac{\pi}{2} - \ln 2$

Key. A

Sol.
$$\int_0^1 \tan^{-1}(1-x+x^2) dx = \int_0^1 \left(\frac{\pi}{2} - \tan^{-1} \left(\frac{x+(1-x)}{1-x(1-x)} \right) \right) dx$$

$$= \frac{\pi}{2} - \int_0^1 (\tan^{-1}(1-x) + \tan^{-1} x) dx$$

$$\Rightarrow \frac{\pi}{2} - 2 \int_0^1 \tan^{-1}(x) dx = \ln 2$$

7.
$$\int_0^{\frac{\pi}{4}} (\pi x - 4x^2) \ln(1 + \tan x) dx = \underline{\hspace{2cm}}$$

- (A) $\frac{\pi^3}{192}$ (B) $\frac{\pi^3}{192} \ln 2$ (C) $\frac{\pi^2}{96}$ (D) $\frac{\pi^2}{96} \ln 2$

Key. B

Sol. Let
$$I = \int_0^{\frac{\pi}{4}} 4x \left(\frac{\pi}{4} - x \right) \ln(1 + \tan x) dx = \int_0^{\frac{\pi}{4}} 4 \left(\frac{\pi}{4} - x \right) x \ln \left(1 + \tan \left(\frac{\pi}{4} - x \right) \right) dx$$

$$= 4 \ln 2 \int_0^{\frac{\pi}{4}} x \left(\frac{\pi}{4} - x \right) dx - I$$

$$\Rightarrow I = 2 \ln 2 \int_0^{\frac{\pi}{4}} x \left(\frac{\pi}{4} - x \right) dx$$

$$= \frac{\pi^3}{192} \ln 2$$

8.
$$\int_0^{2\pi} x \ln \left(\frac{3 + \cos x}{3 - \cos x} \right) dx = \underline{\hspace{2cm}}$$

- (A) $\frac{\pi}{2} \ln 3$ (B) $\frac{\pi}{6} \ln 3$ (C) $\frac{\pi}{12} \ln 3$ (D) 0

Key. D

Sol.
$$I = \int_0^{2\pi} (2\pi - x) \ln \left(\frac{3 + \cos x}{3 - \cos x} \right) dx \Rightarrow 2I = 2\pi \int_0^{2\pi} \ln \left(\frac{3 + \cos x}{3 - \cos x} \right) dx$$

$$\Rightarrow I = 2\pi \int_0^{\pi} \ln \left(\frac{3 + \cos x}{3 - \cos x} \right) dx = 2\pi \int_0^{\pi} \ln \left(\frac{3 + \cos x}{3 - \cos x} \right) dx = -I \Rightarrow I = 0$$

9. If a point P moves such that its distance from line $y = \sqrt{3}x - 7$ is same as its distance from $(2\sqrt{3}, -1)$, then area bounded by locus of P and the coordinate axes is (in sq. units)

- (A) $\frac{\sqrt{3}}{2}$ (B) $2\sqrt{3}$ (C) 6 (D) $\frac{3\sqrt{3}}{2}$

Key. A

Sol. As point lie on the line. Locus of the point is straight line perpendicular to given line passing through $(2\sqrt{3}, -1)$ i.e. $\frac{x}{\sqrt{3}} + y = 1$

$$\Rightarrow \text{area of triangle} = \frac{\sqrt{3} \times 1}{2}.$$

10. If $A\left(\frac{3}{\sqrt{2}}, \sqrt{2}\right)$, $B\left(-\frac{3}{\sqrt{2}}, \sqrt{2}\right)$, $C\left(-\frac{3}{\sqrt{2}}, -\sqrt{2}\right)$ and $D(3 \cos \theta, 2 \sin \theta)$ are four points,

then the value of θ for which the area of quadrilateral ABCD is maximum, $\left(\frac{3\pi}{2} \leq \theta \leq 2\pi\right)$ is

(in sq. units)

- (A) $2\pi - \sin^{-1} \frac{1}{3}$ (B) $\frac{7\pi}{4}$ (C) $2\pi - \cos^{-1} \frac{3}{\sqrt{85}}$ (D) $\frac{\pi}{4}$

Key. B

Sol. Area of quadrilateral ABCD is maximum when area of ACD is maximum \Rightarrow distance of D from AC is maximum i.e. $(\cos \theta - \sin \theta)$ is maximum.

$$\Rightarrow \sqrt{2} \cos\left(\theta + \frac{\pi}{4}\right) \text{ is maximum}$$

$$\Rightarrow \theta = \frac{7\pi}{4}.$$

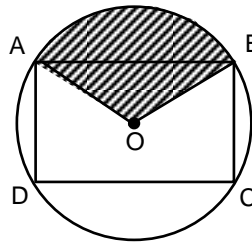
11. A square ABCD is inscribed in a circle of radius 4. A point P moves inside the circle such that $d(P, AB) \leq \min(d(P, BC), d(P, CD), d(P, DA))$ where $d(P, AB)$ is the distance of a point P from line AB. The area of region covered by moving point P is (in sq. units)

- (A) 4π (B) 8π (C) $8\pi - 16$ (D) $3\pi - 4$

Key. A

Sol. Shaded area is the required

$$\text{region} = \frac{\pi r^2}{4} = 4\pi.$$



12. Let $I_1 = \int_{\sec^2 z}^{2-\tan^2 z} x f(x(3-x)) dx$ and let $I_2 = \int_{\sec^2 z}^{2-\tan^2 z} f(x(3-x)) dx$

where 'f' is a continuous function and 'z' is any real number, then $\frac{I_1}{I_2} =$

- (A) $\frac{3}{2}$ (B) $\frac{1}{2}$ (C) 1 (D) $\frac{2}{3}$

Key. A

Sol. Conceptual

13. If f, g, h are continuous functions on [0, a] such that

$f(a-x) = f(x), g(a-x) = -g(x), 3h(x) - 4h(a-x) = 5$ then $\int_0^a f(x)g(x)h(x)dx =$

- a) 0 b) a c) a/2 d) 2a

Key. A

Sol. Conceptual

14. The value of $\int_{-2}^2 \frac{\sin^2 x}{\left[\frac{x}{\pi}\right] + \frac{1}{2}} dx$, where [x] is the greatest integer less than or equal to x, is

- a) 1 b) 0 c) $4 - \sin 4$ d) $4 + \sin 4$

Key. B

Sol. Conceptual

15. $\int_0^{4/\pi} (3x^2 \cdot \sin \frac{1}{x} - x \cdot \cos \frac{1}{x}) dx =$

- a) $\frac{8\sqrt{2}}{\pi^3}$ b) $\frac{24\sqrt{2}}{\pi^3}$ c) $\frac{32\sqrt{2}}{\pi^3}$ d) $\frac{32\sqrt{2}}{\pi}$

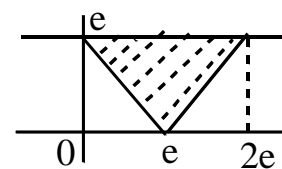
Key. C

Sol. Conceptual

16. The area bounded by the curves $f(x) = \begin{cases} x^{\frac{1}{\ln x}}, & x \neq 1 \\ e, & x = 1 \end{cases}$ and $y = |x - e|$ is

- (A) $\frac{e^2}{2}$ (B) e^2 (C) $2e^2$ (D) 1

Key. B



Sol. $f(x) = \begin{cases} x^{\log_x e} = e, & x \neq 1 \\ e, & x = 1 \end{cases}$

17. The area of the region bounded by the point P(x, y) satisfying $\log_x \log_y x > 0$ and

$\frac{1}{2} < x < 2$ is

- (A) $\frac{3}{4}$ (B) 1 (C) 2 (D) $\frac{7}{8}$

Key. D

Sol. (i) $\frac{1}{2} < x < 1$

a) $0 < y < 1 \Rightarrow x > y \wedge x < 1$

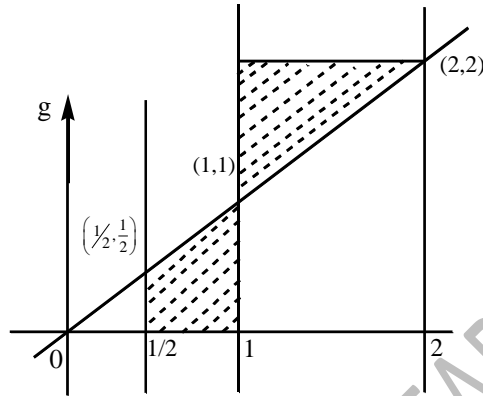
b) $y > 1 \Rightarrow x < y \wedge x > 1$

not possible

(ii) $1 < x < 2$

a) $0 < y < 1$

$x > y \wedge x < 1$



Not possible

b) $y > 1$

$x < y \wedge x > 1$

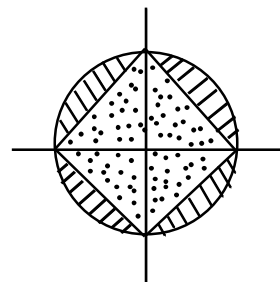
$$\begin{aligned} \text{Area} &= \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \\ &= 1 - \frac{1}{8} = \frac{7}{8} \end{aligned}$$

18. Area of the region defined by $\|x|-|y|\| \geq 1$ and $x^2 + y^2 \leq 1$ is

- (A) 1 (B) 2 (C) $\pi - 1$ (D) $2\pi - 1$

Key. C

Sol. $-1 \leq \|x|-|y|\| \leq 1$



$\|x|-|y|\| \leq 1 \wedge \|x|-|y|\| \geq -1$

Required area = $\pi(1)^2 = \pi$

21. If $\int \frac{dx}{x\sqrt{1-x^3}} = a \ln \left(\frac{\sqrt{1-x^3+b}}{\sqrt{1-x^3+1}} \right) + k$, then

1) $b=1, a=1$

2) $b=-1, a=-\frac{1}{3}$

3) $b=1, a=-\frac{2}{3}$

4) None of these

Key. 2

Sol. $I = \int \frac{dx}{x\sqrt{1-x^3}} = \int \frac{x^2 dx}{x^3 \sqrt{1-x^3}}$

Put $1-x^3 = t^2 \Rightarrow -3x^2 dx = 2t dt$

22. $\frac{\sin^3 x}{(\cos^4 x + 3\cos^2 x + 1) \tan^{-1}(\sec x + \cos x)} dx =$

1) $\tan^{-1}(\sec x + \cos x) + c$

2) $\log |\tan^{-1}(\sec x + \cos x)| + c$

3) $\frac{1}{(\sec x + \cos^2 x)^2} + c$

4) $\log |\sec x + \cos x| + c$

Key. 2

Sol. Put $\tan^{-1}(\sec x + \cos x) = f(x)$

$f'(x) = \frac{\sin^3 x}{\cos^4 x + 3\cos^2 x + 1}$

$\therefore \int \frac{f'(x)}{f(x)} = \log |f(x)| + c$

23. $\int (\sqrt{\sin x} + \sqrt{\cos x})^{-4} dx =$

1) $\frac{2}{(\sqrt{\tan x} + 1)^2} \left[\frac{1}{3(\sqrt{\tan x} + 1)} - \frac{1}{2} \right] + c$

2) $\frac{2}{(\sqrt{\tan x} + 1)^2} \left[\frac{1}{3(\sqrt{\tan x} + 1)} + \frac{1}{2} \right] + c$

$$3) \frac{2}{(\sqrt{\tan x} + 1)^2} \left[\frac{1}{\sqrt{3}(\sqrt{\tan x} + 1)} + \frac{1}{2} \right] + c$$

$$4) \frac{2}{(\sqrt{\tan x} + 1)^2} \left[\frac{1}{\sqrt{3}(\sqrt{\tan x} + 1)} - \frac{1}{2} \right] + c$$

Key. 1

$$I = \int (\sqrt{\sin x} + \sqrt{\cos x})^{-4} dx = \int \frac{dx}{[\sqrt{\cos x}(\sqrt{\tan x} + 1)]^4} = \int \frac{1}{\cos^2 x (\sqrt{\tan x} + 1)^4} dx$$

$$= \int \frac{\sec^2 x dx}{(\sqrt{\tan x} + 1)^4}$$

Put $\sqrt{\tan x} + 1 = y \Rightarrow \frac{1}{2\sqrt{\tan x}} \sec^2 x dx = dy \Rightarrow \sec^2 x dx = 2\sqrt{\tan x} dy = 2(y-1) dy$

$$I = \int \frac{1}{y^4} \cdot 2(y-1) dy = 2 \int \left(\frac{1}{y^3} - \frac{1}{y^4} \right) dy = 2 \left[\frac{-1}{2y^2} + \frac{1}{3y^3} \right] + c = \frac{2}{y^2} \left[\frac{1}{3y} - \frac{1}{2} \right]$$

$$= \frac{2}{(\sqrt{\tan x} + 1)^2} \left[\frac{1}{3(\sqrt{\tan x} + 1)} - \frac{1}{2} \right] + c$$

Sol.

24. $\int \sqrt{\tan x} dx =$

$$1) \frac{1}{2\sqrt{2}} \left[2 \tan^{-1} \left(\frac{\tan x - 1}{\sqrt{2 \tan x}} \right) + \log \left[\frac{\tan x - \sqrt{2 \tan x} + 1}{\tan x + \sqrt{2 \tan x} + 1} \right] \right] + c$$

$$2) \frac{1}{2\sqrt{2}} \left[2 \tan^{-1} \left(\frac{\tan x - 1}{2 \tan x} \right) + \log \left[\frac{\tan x - \sqrt{2 \tan x} + 1}{\tan x + \sqrt{2 \tan x} + 1} \right] \right] + c$$

$$3) \frac{1}{2\sqrt{2}} \left[2 \tan^{-1} \left(\frac{\tan x + 1}{\sqrt{2 \tan x}} \right) + \log \left[\frac{\tan^2 x - 2 \tan x + 1}{\tan x + 2 \tan x + 1} \right] \right] + c$$

$$4) \frac{1}{5\sqrt{2}} \left[2 \tan^{-1} \left(\frac{\tan x + 1}{\sqrt{2 \tan x}} \right) + \log \left[\frac{\tan^2 x - 2 \tan x - 1}{\tan x + 2 \tan x + 1} \right] \right] + c$$

Key. 1

Sol. $\frac{1}{2} \int 2\sqrt{\tan x} dx = \frac{1}{2} \int (\sqrt{\tan x} + \sqrt{\cot x}) dx + \frac{1}{2} \int (\sqrt{\tan x} - \sqrt{\cot x}) dx$

25. $\int \sqrt{x + \sqrt{x^2 + 2}} dx =$

$$1) \left(\frac{x + \sqrt{x^2 + 2}}{3} \right)^{\frac{3}{2}} - 2(x + \sqrt{x^2 + 2})^{-\frac{1}{2}} + c$$

$$2) \frac{1}{3} (x + \sqrt{x^2 + 2})^{\frac{3}{2}} - 2(x + \sqrt{x^2 + 2})^{-\frac{1}{2}} + c$$

$$3) \frac{2}{7} (x + \sqrt{x^2 + 2})^{\frac{7}{2}} - 2(x + \sqrt{x^2 + 2}) + c$$

$$4) \frac{2}{7} (x + \sqrt{x^2 + 2})^{\frac{7}{2}} + 2(x + \sqrt{x^2 + 2}) + c$$

Key. 2

Sol. Put $x + \sqrt{x^2 + 2} = t$

26. If $\int \frac{2 \cos x - \sin x + \lambda}{\cos x + \sin x - 2} dx = A \ln |\cos x + \sin x - 2| + Bx + C$ Then the ordered triplet

A, B, λ is

$$1) \left(\frac{1}{2}, \frac{3}{2}, -1 \right) \quad 2) \left(\frac{3}{2}, \frac{1}{2}, -1 \right) \quad 3) \left(\frac{1}{2}, -1, -\frac{3}{2} \right) \quad 4) \left(\frac{3}{2}, -1, \frac{1}{2} \right)$$

Key. 2

Sol.

$$\begin{aligned} & \frac{d}{dx} (A \ln |\cos x + \sin x - 2| + Bx + C) \\ &= A \frac{\cos x - \sin x}{\cos x + \sin x - 2} + B = \frac{A \cos x - A \sin x + B \cos x + B \sin x - 2B}{\cos x + \sin x - 2} \\ \therefore & \quad 2 = A + B, -1 = -A + B, \lambda = -2B \\ \therefore & \quad A = 3/2, B = 1/2, \lambda = -1 \end{aligned}$$

$$27. \int \frac{x^8 + 4}{x^4 - 2x^2 + 2} dx =$$

$$1) \frac{x^5}{5} - \frac{2x^3}{3} + 2x + C$$

$$2) \frac{x^5}{5} - \frac{2x^3}{3} - 2x + C$$

$$3) \frac{x^5}{5} + \frac{2x^3}{3} - 2x + C$$

$$4) \frac{x^5}{5} + \frac{2x^3}{3} + 2x + C$$

Key. 4

Sol.

$$\begin{aligned} & \int \frac{(x^8 + 4 + 4x^4) - 4x^4}{x^4 - 2x^2 + 2} dx = \int \frac{(x^4 + 2)^2 - (2x^2)^2}{(x^4 - 2x^2 + 2)} dx \\ &= \int \frac{(x^4 + 2 - 2x^2)(x^4 + 2 + 2x^2)}{(x^4 - 2x^2 + 2)} dx = \frac{x^5}{5} + \frac{2x^3}{3} + 2x + C \end{aligned}$$

28.
$$\int \frac{\cos^4 x \, dx}{\sin^3 x (\sin^5 x + \cos^5 x)^{3/5}} = -\frac{1}{2} (1 + \cot^5 x)^B + C$$

1) 5 2) $\frac{2}{5}$ 3) 2 4) 1

Key. 3

Sol.

$$\begin{aligned} I &= \int \frac{\cos^4 x \, dx}{\sin^3 x (\sin^5 x + \cos^5 x)^{3/5}} \\ &= \int \frac{\cos^4 x \, dx}{\sin^6 x (1 + \cot^5 x)^{3/5}} \\ &= \int \frac{\cot^4 x \operatorname{cosec}^2 x \, dx}{(1 + \cot^5 x)^{3/5}} \text{ put } 1 + \cot^5 x = t \quad 5 \cot^4 x \operatorname{cosec}^2 x \, dx = -dt \end{aligned}$$

29. If $f(x) = \sqrt{x}$, $g(x) = e^{x-1}$, and $\int fog(x) \, dx = A fog(x) + B \tan^{-1}(fog(x)) + C$, then $A+B$ is equal to

1) 1 2) 2 3) 3 4) 0

Key. 4

Sol.

$$\begin{aligned} fog(x) &= \sqrt{e^x - 1} \\ \therefore I &= \int \sqrt{e^x - 1} \, dx = \int \frac{2t^2}{t^2 + 1} \, dt \quad \left\{ \text{where } \sqrt{e^x - 1} = t \right\} \\ &= 2t - 2 \tan^{-1} t + C = 2\sqrt{e^x - 1} - 2 \tan^{-1}(\sqrt{e^x - 1}) + C = 2 fog(x) - 2 \tan^{-1}(fog(x)) + C \\ \therefore A+B &= 2 + (-2) = 0 \end{aligned}$$

30. If $\int \sin^{-1} x \cos^{-1} x \, dx = f^{-1} \left[\frac{\pi}{2} x - x f^{-1}(x) - 2\sqrt{1-x^2} \right] + \frac{\pi}{2} \sqrt{1-x^2} + 2x + C$, then $f(x)$ is equal to

1) $\sin 3x$ 2) $\sin 2x$ 3) $\sin x$ 4) $\sin 4x$

Key. 3

Sol.

$$\begin{aligned} \int \sin^{-1} x \cos^{-1} x \, dx &= \int \left[\frac{\pi}{2} \sin^{-1} x - (\sin^{-1} x)^2 \right] dx \\ \Rightarrow \frac{\pi}{2} (x \sin^{-1} x + \sqrt{1-x^2}) - (x (\sin^{-1} x)^2 + \sin^{-1} x \sqrt{1-x^2} - x) + c \text{ By parts} \\ \Rightarrow \sin^{-1} x \left[\frac{\pi}{2} x - x \sin^{-1} x - 2\sqrt{1-x^2} \right] + \frac{\pi}{2} \sqrt{1-x^2} + 2x + c \\ \therefore f^{-1}(x) &= \sin^{-1} x, f(x) = \sin x \end{aligned}$$

31. Let $F(x) = e^{\sin^{-1}x} \left(1 - \frac{x}{\sqrt{1-x^2}} \right) dx$ and $F(0) = 1$, If $F(1/2) = \frac{k\sqrt{3}e^{x/6}}{\pi}$, then $k =$

1) $\frac{\pi}{4}$

2) $\frac{\pi}{6}$

3) $\frac{\pi}{2}$

4) $\frac{\pi}{3}$

Key. 3

Sol.

$$F(x) = \int e^{\sin^{-1}x} \left(1 - \frac{x}{\sqrt{1-x^2}} \right) dx = \int e^{\sin^{-1}x} \left(\frac{1}{\sqrt{1-x^2}} \sqrt{1-x^2} - \frac{x}{\sqrt{1-x^2}} \right) dx$$

$$F(x) = e^{\sin^{-1}x} \sqrt{1-x^2} + c$$

$$F(0) = 1 + c \quad \Rightarrow \quad c = 0 \quad (\because F(0) = 1)$$

$$F(1/2) = e^{x/6} \cdot \frac{\sqrt{3}}{2} = \frac{k\sqrt{3}}{\pi} e^{x/6}$$

$$\therefore k = \frac{\pi}{2}$$

32. $\int \left(\frac{x-1}{x+1} \right) \frac{dx}{\sqrt{x^3+x^2+x}} = 2 \tan^{-1} \sqrt{f(x)} + C$ then find $f(x)$.

1) $x + \frac{1}{x} + 1$

2) $x + \frac{1}{x} + 2$

3) $x - \frac{1}{x} + 1$

4) $x - \frac{1}{x} - 2$

Key. 1

Sol.

$$I = \int \left(\frac{x-1}{x+1} \right) \frac{dx}{x \sqrt{x+1+\frac{1}{x}}}$$

$$\text{so, } I = \int \frac{(x-1) dx}{(x+1) x \sqrt{x+1+\frac{1}{x}}} = \int \frac{\left(1-\frac{1}{x}\right) \left(1+\frac{1}{x}\right) dx}{(x+1) \left(1+\frac{1}{x}\right) \sqrt{x+1+\frac{1}{x}}} = \int \frac{\left(1-\frac{1}{x^2}\right) dx}{\left(x+\frac{1}{x}+2\right) \sqrt{x+\frac{1}{x}+1}}$$

Put $x + \frac{1}{x} + 1 = t^2$

$$\left(1 - \frac{1}{x^2} \right) dx = 2t dt$$

$$= \int \frac{2t \, dt}{(t^2 + 1)t} = 2 \tan^{-1} t + c = 2 \tan^{-1} \left(\sqrt{x + \frac{1}{x} + 1} \right) + c$$

$$x + \frac{1}{x} + 1 \quad \text{Ans.}$$

33. $\int 2 \cos 2x \ln(\tan x) \, dx$, is equal to

1) $\sin 2x \ln(\tan x) - 2x + C$

2) $\sin 2x \ln(\tan x) + 2x + C$

3) $\sin x \ln(\tan x) - x + C$

4) none of these

Key. 1

Sol. We have $I = \int 2 \cos 2x \ln(\tan x) \, dx = \sin 2x \ln(\tan x) - \int \sin 2x \cdot \frac{\sec^2 x}{\tan x} \, dx$

$$= \sin 2x \ln(\tan x) - \int 2 \, dx = \sin 2x \ln(\tan x) - 2x + C$$

34. $\int \frac{\cos x + x \sin x}{x(x + \cos x)} \, dx =$

1) $\log \left| \frac{x}{x + \cos x} \right| + c$

2) $-\log \left| \frac{x}{x + \cos x} \right| + c$

3) $\log \left| \frac{x + \cos x}{2x} \right| + c$

4) $-\log \left| \frac{x + \cos x}{2x} \right| + c$

Key. 1

Sol. $\int \frac{x + \cos x + x \sin x - x}{x(x + \cos x)} \, dx$

35. $\int [1 + \tan x \tan(x + \alpha)] \, dx$, is equal to

1) $\tan \alpha \ln \left| \frac{\sin(x + \alpha)}{\sin x} \right| + C$

2) $\cot \alpha \ln \left| \frac{\sin(x + \alpha)}{\sin x} \right| + C$

3) $\cot \alpha \ln \left| \frac{\sin x}{\sin(x + \alpha)} \right| + C$

4) $\cot \alpha \ln \left| \frac{\cos x}{\cos(x + \alpha)} \right| + C$

Key. 4

Sol. We have $\tan \alpha = \tan(x + \alpha - x) = \frac{\tan(x + \alpha) - \tan x}{1 + \tan x \tan(x + \alpha)}$

Then, we have

$$\int [1 + \tan x \tan(x + \alpha)] \, dx = \int \cot \alpha [\tan(x + \alpha) - \tan x] \, dx$$

$$= \cot \alpha \left[-\ln |\cos(x + \alpha)| + \ln |\cos x| \right] + C$$

$$= \cot \alpha \ln \left| \frac{\cos x}{\cos(x + \alpha)} \right| + C$$

36. Let $x^2 \neq n\pi - 1, n \in N$. Then, the value of $\int x \sqrt{\frac{2 \sin(x^2 + 1) - \sin 2(x^2 + 1)}{2 \sin(x^2 + 1) + \sin 2(x^2 + 1)}} dx$ is equal to

1) $\log \left| \frac{1}{2} \sec(x^2 + 1) \right| + C$

2) $\log \left| \sec \left(\frac{x^2 + 1}{2} \right) \right| + C$

3) $\frac{1}{2} \log |\sec(x^2 + 1)| + C$

4) None of these

Key. 2

Sol. We have, $\int x \sqrt{\frac{2 \sin(x^2 + 1) - \sin 2(x^2 + 1)}{2 \sin(x^2 + 1) + \sin 2(x^2 + 1)}} dx$

$$= \int x \sqrt{\frac{2 \sin(x^2 + 1) - 2 \sin(x^2 + 1) \cos(x^2 + 1)}{2 \sin(x^2 + 1) + 2 \sin(x^2 + 1) \cos(x^2 + 1)}} dx$$

$$= \int x \sqrt{\frac{1 - \cos(x^2 + 1)}{1 + \cos(x^2 + 1)}} dx$$

$$= \int x \tan \left(\frac{x^2 + 1}{2} \right) dx$$

$$= \int \tan \left(\frac{x^2 + 1}{2} \right) d \left(\frac{x^2 + 1}{2} \right)$$

$$= \log \left| \sec \left(\frac{x^2 + 1}{2} \right) \right| + C$$

37. $\int \frac{dx}{\cos(2x) \cos(4x)}$ is equal to

1) $\frac{1}{2\sqrt{2}} \log \left| \frac{1 + \sqrt{2} \sin 2x}{1 - \sqrt{2} \sin 2x} \right| - \frac{1}{2} (\log |\sec 2x - \tan 2x|) + C$

$$2) \frac{1}{2\sqrt{2}} \log \left| \frac{1 + \sqrt{2} \sin 2x}{1 + \sqrt{2} \sin x} \right| - \frac{1}{2} (\log |\sec 2x - \tan 2x|) + C$$

$$3) \frac{1}{\sqrt{2}} \log \left| \frac{1 + \sqrt{2} \sin 2x}{1 - \sqrt{2} \sin 2x} \right| - \frac{1}{2} (\log |\sec 2x - \tan 2x|) + C$$

4) None of these

Key. 1

Sol.
$$\int \frac{\sin(4x - 2x) dx}{\sin(2x) \cos(2x) \cos(4x)}$$

$$= \int \frac{\sin(4x) dx}{\sin(2x) \cos(4x)} - \int \sec 2x dx$$

$$= 2 \int \frac{\cos 2x dx}{\cos 4x} - \frac{1}{2} (\log |\sec 2x - \tan 2x|)$$

38.
$$\int \frac{1 - 7 \cos^2 x}{\sin^7 x \cos^2 x} dx = \frac{f(x)}{(\sin x)^7} + C$$
, then f(x) is equal to

1) Sin x

2) Cos x

3) Tan x

4) Cot x

Key. 3

Sol.
$$\int \frac{1 - 7 \cos^2 x}{\sin^7 x \cos^2 x} dx = \int \left(\frac{\sec^2 x}{\sin^7 x} - \frac{7}{\sin^7 x} \right) dx$$

$$= \int \frac{\sec^2 x}{\sin^7 x} dx - \int \frac{7}{\sin^7 x} dx = I_1 + I_2$$

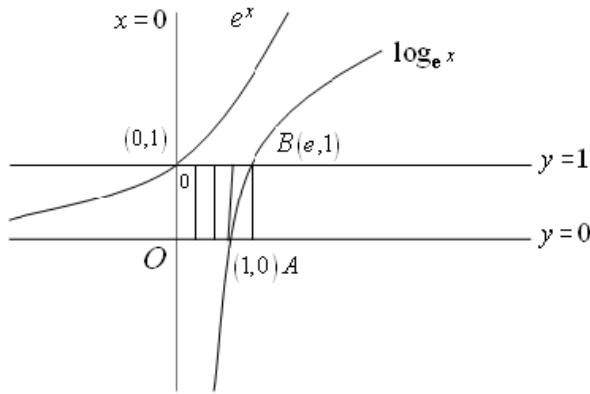
Now,
$$I_1 = \int \frac{\sec^2 x}{\sin^7 x} dx$$

$$= \frac{\tan x}{\sin^7 x} + 7 \int \frac{\tan x \cdot \cos x}{\sin^8 x} dx$$

$$= \frac{\tan x}{\sin^7 x} - I_2$$

$$\therefore I_1 + I_2 = \frac{\tan x}{\sin^7 x} + C$$

$$\Rightarrow f(x) = \tan x$$



$$\text{Area} = \text{Area of rectangle OABC} - \int_0^1 \log_e x$$

49. The area of the loop of the curve $y^2 = x^4(x+2)$ is [in square units]

- (A) $\frac{32\sqrt{2}}{105}$ (B) $\frac{64\sqrt{2}}{105}$ (C) $\frac{128\sqrt{2}}{105}$ (D) $\frac{256\sqrt{2}}{105}$

Key: D

Hint: $\text{Area} = 2 \int_{-2}^0 y dx = 2 \int_{-2}^0 x^2 \sqrt{x+2} dx = 4\sqrt{2} \int_0^2 (z^2 - 2)^2 z^2 dz$ (where $\sqrt{x+2} = z$)

$$= 4 \left[\frac{z^7}{7} - \frac{4z^5}{5} + \frac{4z^3}{3} \right]_0^{\sqrt{2}}$$

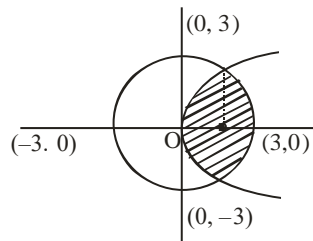
$$= \frac{256\sqrt{2}}{105}$$

50. The area of the smaller portion enclosed by the curves $x^2 + y^2 = 9$ and $y^2 = 8x$ is

- A) $\frac{\sqrt{2}}{3} + \frac{9\pi}{4} - \frac{9}{2} \sin^{-1}\left(\frac{1}{3}\right)$ B) $2 \left(\frac{\sqrt{2}}{3} + \frac{9\pi}{4} - \frac{9}{2} \sin^{-1}\left(\frac{1}{3}\right) \right)$
 C) $2 \left(\frac{\sqrt{2}}{3} + \frac{9\pi}{4} + \frac{9}{2} \sin^{-1}\left(\frac{1}{3}\right) \right)$ D) $\frac{\sqrt{2}}{3} + \frac{9\pi}{4} + \frac{9}{2} \sin^{-1}\left(\frac{1}{3}\right)$

Key: B

HINT:



$$x^2 + y^2 = 9,$$

$$x^2 + 8x - 9 = 0$$

$$x = \frac{-8 \pm \sqrt{64 + 36}}{2}$$

$$x = \frac{-8 \pm 10}{2} = -9, 1$$

$$\boxed{x=1}$$

$$\text{Area enclosed} = 2 \left[\int_0^1 2\sqrt{2x} dx + \int_1^3 \sqrt{9-x^2} dx \right] = 2 \left[2\sqrt{2} \int_0^1 \sqrt{x} dx + \int_1^3 \sqrt{9-x^2} dx \right]$$

On simplifying we get

$$= 2 \left[\frac{\sqrt{2}}{3} + \frac{9\pi}{4} - \frac{9}{2} \sin^{-1} \left(\frac{1}{3} \right) \right]$$

51. The area of the region in the xy-plane defined by the inequalities $x - 2y^2 \geq 0$, $1 - x - |y| \geq 0$ is

- A) $\frac{1}{2}$ B) $\frac{1}{3}$ C) $\frac{1}{4}$ D) $\frac{7}{12}$

Key: D

Hint: Area = $2 \int_0^{1/2} \sqrt{\frac{x}{2}} dx + \frac{1}{4} = \frac{7}{12}$

52. Area bounded by curve $y^2 = x$ and $x = 4$ is divided into 4 equal parts by the lines $x = a$ and $y = b$ then.

- a) Area of each part = $\frac{8}{3}$ b) $b = 0$
 c) $a = \sqrt{2}$ d) $a = (16)^{1/3}$

Key: D

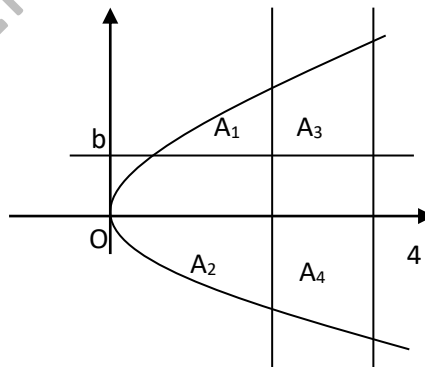
Hint: Total area = $2 \int_0^4 \sqrt{x} dx = \frac{32}{3}$

Area of each part = $\frac{8}{3}$

$$A_3 = A_4 \Rightarrow \int_a^4 (\sqrt{x} - b) dx =$$

$$\int_a^4 (b + \sqrt{x}) dx = \frac{8}{3} \Rightarrow b = 0$$

$$\int_a^4 \sqrt{x} dx = \frac{8}{3} \Rightarrow a^3 = 16$$



53. Area of the region in which point $p(x, y)$, $\{x > 0\}$ lies; such that $y \leq \sqrt{16 - x^2}$ and $\left| \tan^{-1} \left(\frac{y}{x} \right) \right| \leq \frac{\pi}{3}$ is

- (A) $\left(\frac{16}{3} \pi \right)$ (B) $\left(\frac{8\pi}{3} + 8\sqrt{3} \right)$ (C) $(4\sqrt{3} - \pi)$ (D) $(\sqrt{3} - \pi)$

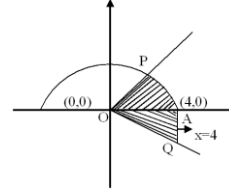
Key: B

Hint: Required area is the area of shaded region (APOQ)

= area of ΔOAQ + area of sector (OAP)

$$= \frac{1}{2} \times 4 \times 4\sqrt{3} + \frac{\pi(4 \times 4)}{6}$$

$$= \left(\frac{8\pi}{3} + 8\sqrt{3} \right)$$



54. Area bounded between the curves $y = \sqrt{4-x^2}$ and $y^2 = 3|x|$ is/are

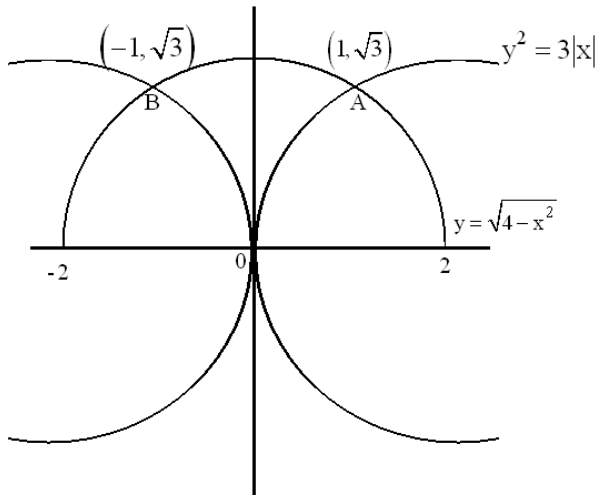
- (A) $\frac{\pi-1}{\sqrt{3}}$ (B) $\frac{2\pi-1}{3\sqrt{3}}$ (C) $\frac{2\pi-\sqrt{3}}{3}$ (D) $\frac{2\pi-\sqrt{3}}{3\sqrt{3}}$

Key: C

Hint: Required area = $2 \int_0^1 (\sqrt{4-x^2} - \sqrt{3x}) dx$

$$= 2 \left(\frac{x}{2} \sqrt{4-x^2} + \frac{4}{2} \sin^{-1} \left(\frac{x}{2} \right) - \frac{\sqrt{3} \cdot 2x^{3/2}}{3} \right)_0^1$$

$$= \frac{2\pi - \sqrt{3}}{3}$$



55. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a continuous and strictly increasing function such that

$f^3(x) = \int_0^x t \cdot f^2(t) dt, \forall x > 0$. The area enclosed by $y = f(x)$, the x-axis and the ordinate at $x=3$, is

- (A) 1 (B) $\frac{3}{2}$ (C) 2 (D) 3

Key: B

Hint: $f(x) = \frac{x^2}{6}$

$$A = \frac{1}{6} \int_0^3 x^2 dx = 3/2$$

56. Let $f(x) = x + \sin x$. The area bounded by $y = f^{-1}(x)$, $y = x$, $x \in [0, \pi]$ is

- (a) 1 (b) 2 (c) 3
 (d) cannot be found because $f^{-1}(x)$ cannot be determined

Key: B

Hint: The curves given by $y = x + \sin x$ and $y = f^{-1}(x)$ are images of each other in the line $y = x$.

Hence required area = $\int_0^\pi ((x + \sin x) - x) dx = -[\cos x]_0^\pi = 2$

57. The area of the region bounded by the curves $|x + y| \leq 2$, $|x - y| \leq 2$ and $2x^2 + 6y^2 \geq 3$ is

- (A) $\left(8 + \frac{\sqrt{3}}{2} \pi\right)$ sq. units (B) $\left(8 - \frac{\sqrt{3}}{2} \pi\right)$ sq. units
 (C) $\left(4 - \frac{3\sqrt{3}}{2} \pi\right)$ sq. units (D) $\left(8 - \frac{3\sqrt{3}}{2} \pi\right)$ sq. units

Key: B

Sol:

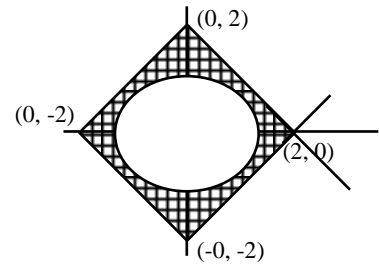
$$2x^2 + 6y^2 \geq 3 \quad \dots \dots \dots (1)$$

area of ellipse = $\pi \times \frac{\sqrt{3}}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{\pi\sqrt{3}}{2}$

$$|x + y| \leq 2 \Rightarrow -2 \leq (x + y) \leq 2 \quad \dots \dots \dots (2)$$

$$|x - y| \leq 2 \Rightarrow -2 \leq (x - y) \leq 2 \quad \dots \dots \dots (3)$$

Required area = $\left(8 - \frac{\pi\sqrt{3}}{2}\right)$ sq. units



58. If $A = \int_1^{\sin \theta} \frac{t dt}{1+t^2}$, $B = \int_1^{\operatorname{cosec} \theta} \frac{1}{t(1+t^2)} dt$ then $\begin{vmatrix} A & A^2 & B \\ e^{A+B} & B^2 & -1 \\ 1 & A^2 + B^2 & -1 \end{vmatrix} = ?$

- a) $\sin \theta$ (b) $\operatorname{cosec} \theta$ (c) 0 (d) 1

Key: C

$$B = \int_1^{\operatorname{cosec}\theta} \frac{1}{t(1+t^2)}$$

$$\text{let } \frac{1}{t} = u \Rightarrow B = \int_1^{\sin\theta} \frac{-udu}{1+u^2}$$

Hint: $\Rightarrow A + B = 0 \Rightarrow A = -B$

$$\therefore \begin{vmatrix} A & A^2 & -A \\ e^0 & A^2 & -1 \\ 1 & A^2 + B^2 & -1 \end{vmatrix} = 0$$

59. The value of $\int_0^2 \frac{2x^3 - 6x^2 + 9x - 5}{x^2 - 2x + 5} dx$ equals

(A) 4

(B) 1

(C) -1

(D) None of the above

Key: D

Hint Make the substitution $x - 1 = t$. It turns into an odd integral and so reduces to zero.

60. Let $I_n = \int_0^\infty e^{-x} (\sin x)^n dx$, $n \in \mathbb{N}$, $n > 1$ then $\frac{I_{2008}}{I_{2006}}$ equals

(A) $\frac{2007 \times 2006}{2008^2 + 1}$

(B) $\frac{2008 \times 2007}{2008^2 + 1}$

(C) $\frac{2006 \times 2004}{2008^2 - 1}$

(D) $\frac{2008 \times 2007}{2008^2 - 1}$

Key: B

Hint $I_n = \int_0^\infty e^{-x} (\sin x)^n dx$

$$= \left[\sin^n x (-e^{-x}) \right]_0^\infty + \int_0^\infty n \sin^{n-1} x \cos x e^{-x} dx$$

$$= 0 + n \int_0^\infty (\sin^{n-1} x \cos x) e^{-x} dx$$

$$= n \left[(\sin^{n-1} x \cos x) (-e^{-x}) \right]_0^\infty - n \int_0^\infty \{ -\sin^n x + (n-1) \sin^{n-2} x \cos^2 x \} (-e^{-x}) dx$$

$$= 0 + n \int_0^\infty e^{-x} \{ -\sin^n x + (n-1) \sin^{n-2} x (1 - \sin^2 x) \} dx$$

$$= n \int_0^\infty e^{-x} \{ (n-1) \sin^{n-2} x - n \sin^n x \} dx$$

$$= n(n-1) I_{n-2} - n^2 I_n$$

we have $(1+n^2)I_n = n(n-1)I_{n-2}$

then $\frac{I_n}{I_{n-2}} = \frac{n(n-1)}{n^2+1}$

61. A hyperbola passing through origin has $3x-4y-1=0$ and $4x-3y-6=0$ as its asymptotes. Then the equation of its transverse axis is
- a) $x-y-5=0$ b) $x+y+1=0$ c) $x+y-5=0$ d) $x-y-1=0$

Key: C

Hint: Asymptotes are equally inclined to the axes of hyperbola

Find the bisector of the asymptotes which bisects the angle containing the origin.

62. If $\int_{\sin x}^1 t^2 \cdot f(t) dt = 1 - \sin x, \forall x \in \left(0, \frac{\pi}{2}\right)$ then the value of $f\left(\frac{1}{\sqrt{3}}\right)$ is
- (A) $\frac{1}{\sqrt{3}}$ (B) $\sqrt{3}$ (C) $\frac{1}{3}$ (D) 3

Key: D

Hint: $\int_{\sin x}^1 t^2 \cdot f(t) dt = 1 - \sin x$

Differentiating both sides with respect to 'x'

$$0 - \sin^2 x \cdot f(\sin x) \cdot \cos x = -\cos x \Rightarrow \cos x [1 - \sin^2 x \cdot f(\sin x)] = 0$$

But $\cos x \neq 0$

$$\text{So, } f(\sin x) = \frac{1}{\sin^2 x}$$

$$f\left(\frac{1}{\sqrt{3}}\right) = 3$$

63. $\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x} dx =$
- (A) $\frac{\pi^2}{4}$ (B) π^2 (C) 0 (D) $\frac{\pi}{2}$

Key: B

Hint: $I = 4 \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx = 4 \int_0^{\pi} \frac{(\pi-x) \sin x}{1+\cos^2 x} dx$

$$\begin{aligned} \therefore 2I &= 4\pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx = 8\pi \int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x} dx \\ &= 2\pi^2 \end{aligned}$$

$$\therefore I = \pi^2$$

64. Let $f : (0, \infty) \rightarrow \mathbb{R}$ and $F(x) = \int_1^x f(t) dt$. If $F(x^2) = x^2(1+x)$ then $f(4)$ equals

- (A) 5/4 (B) 7 (C) 4 (D) 2

Key: C

Hint: $F'(x) = f(x)$

$$F(x) = x(1 + \sqrt{x}) = x + x^{3/2}$$

$$\therefore F'(x) = f(x) = 1 + \frac{3}{2}\sqrt{x}$$

$$\therefore f(4) = 4$$

65. If $f(x) = \int_0^x (1+t^3)^{-1/2} dt$ and $g(x)$ is the inverse of f , then the value of $\frac{g''(x)}{g^2(x)}$ is

- (A) 3/2 (B) 2/3 (C) 1/3 (D) 1/2

Key: A

Hint: $f(x) = \int_0^x (1+t^3)^{-1/2} dt$

$$\text{i.e. } f[g(x)] = \int_0^{g(x)} (1+t^3)^{-1/2} dt$$

$$\text{i.e. } x = \int_0^{g(x)} (1+t^3)^{-1/2} dt \quad [\text{Q } g \text{ is inverse of } f \Rightarrow f[g(x)] = x]$$

Differentiating with respect to x , we have

$$1 = (1+g^3)^{-1/2} \cdot g'$$

$$\text{i.e. } (g')^2 = 1 + g^3$$

Differentiating again with respect to x , we have

$$2g'g'' = 3g^2g'$$

gives
$$\frac{g''}{g^2} = \frac{3}{2}$$

66. $\int_0^a \ln(\cot a + \tan x) dx$, where $a \in \left(0, \frac{\pi}{2}\right)$ is

- a) $a \ln(\sin a)$ b) $-a \ln(\sin a)$ c) $-a \ln(\cos a)$
 a) none of these d) none of these

Key: B

$$I = \int_0^a \ln \frac{\cos(a-x)}{\sin a \cdot \cos x} dx = \int_0^a \ln \frac{\cos x}{\sin a \cos(a-x)} dx$$

Hint:

$$\text{adding } 2I = \int_0^a \ln \frac{1}{\sin^2 a} dx = \int_0^a -2 \ln \sin a dx = -2a \ln \sin a$$

67. If $f(x) = \int_1^x \frac{dt}{2+t^4}$, then

- (A) $f(2) < \frac{1}{3}$ (B) $f(2) > \frac{1}{3}$
 (C) $f(2) = \frac{1}{3}$ (D) $f(2) > 1$

Key: A

Hint: $f'(x) = \frac{1}{2+x^4}$

By LMVT $f'(c) = \frac{f(2)-f(1)}{2-1}$ for some $c \in (1,2)$

$\Rightarrow f(2) = \frac{1}{2+c^4}$ as $f(1) = 0 \Rightarrow 1 < c < 2 \Rightarrow 3 < 2+c^4 < 18 \Rightarrow f(2) < \frac{1}{3}$

68. Let $I_1 = \int_0^{\pi/4} x^{2008} (\tan x)^{2008} dx$, $I_2 = \int_0^{\pi/4} x^{2009} (\tan x)^{2009} dx$ and

$I_3 = \int_0^{\pi/4} x^{2010} (\tan x)^{2010} dx$ Then the correct order sequence, is

- (A) $I_2 < I_3 < I_1$ (B) $I_1 < I_2 < I_3$
 (C) $I_3 < I_1 < I_2$ (D) $I_3 < I_2 < I_1$

Key: D

Hint:

69. .If $f(x+y) = f(x) + f(y) + 2xy - 6$ for all $x, y \in \mathbb{R}$ and $f'(0) = 2$, then $y = f(x)$ will be

- (A) straight line (B) parabola
 (C) ellipse (D) circle

Key: B

Hint: $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x+0)}{h}$

$= \lim_{h \rightarrow 0} \frac{f(x) + f(h) + 2xh - 6 - f(x) - f(0) + 6}{h}$

$= 2x + \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = 2x + f'(0)$

$f(x) = x^2 + 2x + C$, but $f(0) = 6$

So, $f(x) = x^2 + 2x + 6$.

70. The value of $\int_3^6 \left(\sqrt{x + \sqrt{12x - 36}} + \sqrt{x - \sqrt{12x - 36}} \right) dx$ is equal to

- (A) $6\sqrt{3}$ (B) $4\sqrt{3}$ (C) $12\sqrt{3}$ (D) $2\sqrt{3}$

Key: A

Hint: $\int_0^3 \left(\sqrt{(x+3)+2\sqrt{3}\sqrt{x}} + \sqrt{(x+3)-2\sqrt{3}\sqrt{x}} \right) dx$
 $\int_0^3 \left((\sqrt{x} + \sqrt{3}) + (\sqrt{3} - \sqrt{x}) \right) dx = \int_0^3 2\sqrt{3} dx = 6\sqrt{3}$

71. Let $f : \left[0, \frac{\pi}{2}\right] \rightarrow [0, 1]$ be a differentiable function such that $f(0) = 0$, $f\left(\frac{\pi}{2}\right) = 1$, then

- (A) $f'(\alpha) = \sqrt{1 - (f(\alpha))^2}$ for all $\alpha \in \left(0, \frac{\pi}{2}\right)$
- (B) $f'(\alpha) = \frac{2}{\pi}$ for all $\alpha \in \left(0, \frac{\pi}{2}\right)$
- (C) $f(\alpha)f'(\alpha) = \frac{1}{\pi}$ for at least one $\alpha \in \left(0, \frac{\pi}{2}\right)$
- (D) $f'(\alpha) = \frac{8\alpha}{\pi^2}$ for at least one $\alpha \in \left(0, \frac{\pi}{2}\right)$

Key: A

Hint: Let $g : \left[0, \frac{\pi}{2}\right] \rightarrow [0, 1]$ be a

(A) Consider $g(x) = \sin^{-1} f(x) - x$

Since $g(0) = 0, g\left(\frac{\pi}{2}\right) = 0$

\therefore There is at least one value of $\alpha \in \left(0, \frac{\pi}{2}\right)$ such that

$$g'(\alpha) = \frac{f'(\alpha)}{\sqrt{1 - (f(\alpha))^2}} - 1 = 0$$

i. e. $f'(\alpha) = \sqrt{1 - (f(\alpha))^2}$ for atleast one value of α but may not be for all $\alpha \in \left(0, \frac{\pi}{2}\right)$

\therefore false

(B) Consider $g(x) = f(x) - \frac{2x}{\pi}$

Since $g(0) = 0, g\left(\frac{\pi}{2}\right) = 0$

\therefore there is at least one value of $\alpha \in \left(0, \frac{\pi}{2}\right)$ such that

$$g'(\alpha) = f'(\alpha) - \frac{2}{\pi} = 0$$

i. e $f'(\alpha) = \frac{2}{\pi}$ for atleast one value of α but may not be for all $\alpha \in \left(0, \frac{\pi}{2}\right)$

\therefore false

(C) Consider $g(x) = (f(x))^2 - \frac{2x}{\pi}$

Since $g(0) = 0, g\left(\frac{\pi}{2}\right) = 0$

\therefore There is at least one value of $\alpha \in \left(0, \frac{\pi}{2}\right)$ such that

$$g'(\alpha) = 2f(\alpha)f'(\alpha) - \frac{\pi}{2} = 0$$

$$\therefore f(\alpha)f'(\alpha) = \frac{1}{\pi}$$

\therefore True

(D) Consider $g(x) = f(x) - \frac{4x^2}{\pi^2}$

Since $g(0) = 0, g\left(\frac{\pi}{2}\right) = 0$

\therefore there is at least one value of $\alpha \in \left(0, \frac{\pi}{2}\right)$ such that

$$g'(\alpha) = f'(\alpha) - \frac{8\alpha}{\pi^2} = 0$$

$$\therefore f'(\alpha) = \frac{8\alpha}{\pi^2}$$

\therefore True

72. If $x = a \cos t, y = a \sin t$, then $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$ is

(a) $\frac{a}{2\sqrt{2}}$

(b) $-\frac{a}{2\sqrt{2}}$

(c) $\frac{2\sqrt{2}}{a}$

(d) $-\frac{2\sqrt{2}}{a}$

Key: d

Hint: Clearly $x^2 + y^2 = a^2$ and $y(\pi/4) = a/\sqrt{2}, x(\pi/4) = a/\sqrt{2}$. Differentiating we get,

$$2x + 2yy_1 = 0 \Rightarrow y_1 = -\frac{x}{y}, \text{ so } y_1(\pi/4) = -1.$$

$$\text{Now } x + yy_1 = 0 \Rightarrow 1 + y_1^2 + yy_2 = 0$$

$$\Rightarrow y_2(\pi/4) = -\frac{1 + (y_1(\pi/4))^2}{y(\pi/4)} = \frac{-2\sqrt{2}}{a}$$

73. If $z \neq 0$, then $\int_{x=0}^{100} [\arg|z|] dx$ is (where $[.]$ denotes the greatest integer function)

- (A) 0
(C) 100

- (B) 10
(D) not defined

Key: A

Sol: $Q |z| = \text{real and positive, imaginary part is zero}$

$$\therefore \arg |z| = 0$$

$$\Rightarrow [\arg |z|] = 0$$

$$\therefore \int_{x=0}^{100} [\arg |z|] dx = \int_{x=0}^{100} 0 \cdot dx = 0$$

74. $\lim_{x \rightarrow \frac{\pi}{4}} \frac{\int_{\frac{\pi}{4}}^{\sec^2 x} f(t) dt}{x^2 - \frac{\pi^2}{16}}$ equals

- (a) $\frac{8}{\pi} f(2)$ (b) $\frac{2}{\pi} f(2)$ (c) $\frac{2}{\pi} f\left(\frac{1}{2}\right)$ (d) $4f(2)$

Key: A

Hint: Required limits is of the form $\frac{0}{0}$, so it is equal to

$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{2 \sec x \sec x \tan x f(\sec^2 x)}{2x} = \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sec^2 x \tan x f(\sec^2 x)}{x} = \frac{8}{\pi} f(2)$$

75. Let $f(x) = \int_0^1 |t-x|t dt$ for all real x . Then the minimum value of f is

- a) $\frac{1}{2}$ b) $\frac{1}{3} \left(1 + \frac{1}{\sqrt{2}}\right)$ c) $\frac{1}{3} \left(1 - \frac{1}{\sqrt{2}}\right)$ d) $\frac{1}{6}$

Key: C

Hint: $f(x) = \begin{cases} \frac{1}{3} - \frac{x}{2} & \text{if } x \leq 0 \\ \frac{1}{3} + \frac{x^3}{3} - \frac{x}{2} & \text{if } 0 \leq x \leq 1 \\ \frac{x}{2} - \frac{1}{3} & \text{if } x \geq 1 \end{cases}$

f attains its minimum at $x = \frac{1}{\sqrt{2}}$

76. $\int \frac{(1 + \sqrt{\tan x})(1 + \tan^2 x)}{2 \tan x} dx$ equal to

- A) $\log \tan^2 x + \sqrt{\tan x} + c$ B) $\log \tan^2 x + \frac{1}{2\sqrt{\tan x}} + c$

C) $\log |\tan x| + 2\sqrt{\tan x} + c$

D) $\log |\tan x| + \sqrt{\tan x} + c$

Key: A

Hint:
$$\int \frac{1}{2 \sin x \cos x} dx + \frac{1}{2} \int \frac{\sqrt{\tan x}}{\sin x \cos x} dx$$

$$= \frac{1}{2} \int \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} dx + \frac{1}{2} \int \frac{\sec^2 x}{\sqrt{\tan x}} dx = \log(\tan^2 x) + \sqrt{\tan x} + c$$

77.
$$\int \frac{(2 + \sec x) \sec x}{(1 + 2 \sec x)^2} dx =$$

a) $\frac{1}{2 \operatorname{cosec} x + \cot x} + C$ b) $2 \operatorname{cosec} x + \cot x + C$ c) $\frac{1}{2 \operatorname{cosec} x - \cot x} + C$ d) $2 \operatorname{cosec} x - \cot x + C$

Key: A

Hint
$$I = \int \frac{(2 \cos x + 1)}{(2 + \cos x)^2} dx = \int \frac{(2 + \cos x) \cos x + \sin^2 x}{(2 + \cos x)^2} dx$$

$$= \int \frac{\cos x}{2 + \cos x} dx - \int \frac{-\sin^2 x}{(2 + \cos x)^2} dx = \frac{\sin x}{2 + \cos x} + c$$

78.
$$\int \frac{dx}{(x-3)^{4/5} (x+1)^{6/5}} =$$

A) $((x-3)(x+1))^{1/5} + c$ B) $\frac{5}{4} \left(\frac{x-3}{x+1}\right)^{1/5} + c$
 C) $\left(\frac{x+1}{x-3}\right)^{1/5} + c$ D) $(x-3)^{6/5} (x+1)^{4/5} + c$

Key: B

Hint Put $t = \frac{x-3}{x+1}$

$$\Rightarrow dx = \frac{(x+1)^2 dt}{4}$$

79. If the system of linear equations $x + y + z = 6, x + 2y + 3z = 14$ and

$2x + 5y + \lambda z = \mu, (\lambda, \mu \in R)$ has no solution, then

- a) $\lambda \neq 8$ b) $\lambda = 8, \mu \neq 36$ c) $\lambda = 8, \mu = 36$ d) None of these

Key: B

Hint
$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 2 & 5 & \lambda & \mu \end{bmatrix} \quad R_2 \rightarrow R_2 - R_1 \quad ; \quad R_3 \rightarrow R_3 - 2R_1$$

$$\therefore \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & \lambda-2 & \mu-12 \end{bmatrix} R_3 \rightarrow R_3 - 3R_2$$

$$\therefore \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & \lambda-8 & \mu-36 \end{bmatrix}$$

$$\lambda-8=0 \text{ \& } \mu-36 \neq 0$$

80. A man starts from the point P(3,-3) and reaches the point Q(0,2) after touching the line $2x+y=7$ at R. The least value of PR+RQ is

- a) $5\sqrt{2}$ b) $3\sqrt{2}$ c) $7\sqrt{2}$ d) $2\sqrt{2}$

Key: A

Hint P, Q lies on same side of the line find image of P w.r.t. line

81. Let $S(x) = \int \frac{dx}{e^x + 8e^{-x} + 4e^{-3x}}$, $R(x) = \int \frac{dx}{e^{3x} + 8e^x + 4e^{-x}}$ and $M(x) = S(x) - 2R(x)$. If $M(x) = \frac{1}{2} \tan^{-1}(f(x)) + c$ where c is an arbitrary constant then $f(\log_e^2) =$

- A) $\frac{3}{2}$ B) $\frac{1}{2}$ C) $\frac{5}{2}$ D) $\frac{7}{2}$

Key: A

Hint: $M(x) = \int \frac{e^x(e^{2x}-2)}{e^{4x}+8e^{2x}+4} dx \quad e^x = t \Rightarrow \int \frac{(t^2-2) dt}{t^4+8t^2+4} = \frac{1}{2} \tan^{-1}\left(\frac{t+2/t}{2}\right) + c$
 $= \frac{1}{2} \tan^{-1}\left(\frac{e^x+2e^{-x}}{2}\right) + c$

82. If $\int (2-3\sin^2 x)\sqrt{\sec x} dx = 2f(x)\sqrt{g(x)} + c$ and f(x) is non constant function then

- (A) $f^2(x) + g^2(x) = 1$ (B) $f^2(x) - g^2(x) = 1$ (C) $f(x)g(x) = 1$ (D) $f(x) = g(x)$

Key: A

Hint:

$$\int \frac{2-3\sin^2 x}{\sqrt{\cos x}} dx = \int \frac{2\cos^2 x - \sin^2 x}{\sqrt{\cos x}} dx = 2\int (\cos x)\sqrt{\cos x} dx - \int \frac{\sin^2 x}{\sqrt{\cos x}} dx$$

$$= 2\sin x\sqrt{\cos x} + c$$

$$\Rightarrow f(x) = \sin x, g(x) = \cos x$$

83. $\int \frac{e^{\cot x}}{\sin^2 x} (2\ln \cos ecx + \sin 2x) dx =$

a) $-2e^{\cot x} \ln(\cos ecx) + c$

b) $e^{\cot x} \ln x + c$

c) $e^{\cot x} \ln(\cos ecx) + c$

d) $e^{\cot x} \ln(\sin x) + c$

Key: A

$$t = \cot x \Rightarrow -\cos ec^2 x dx = dt \Rightarrow dx = \frac{-1}{1+t^2} dt$$

Hint:

$$I = -\int e^t \left(\ln(1+t^2) + \frac{2t}{1+t^2} \right) dt = -e^t \ln(1+t^2) + c = -2e^{\cot x} \ln \cos ecx + c$$

84. If $\int \frac{\sec^2 x - 2010}{\sin^{2010} x} dx = \frac{P(x)}{\sin^{2010} x} + C$, then value of $P\left(\frac{\pi}{3}\right)$ is

(A) 0

(B) $\frac{1}{\sqrt{3}}$

(C) $\sqrt{3}$

(D) None of these

Key: C

Hint: $\int \frac{\sec^2 x - 2010}{\sin^{2010} x} dx$

$$= \int \sec^2 x (\sin x)^{-2010} - 2010 \int \frac{1}{(\sin x)^{2010}} dx = I_1 - I_2$$

Applying by parts on I_1 , we get

$$I_1 = \frac{\tan x}{(\sin x)^{2010}} + 2010 \int \frac{\tan x \cos x}{(\sin x)^{2011}} dx = \frac{\tan x}{(\sin x)^{2010}} + 2010 \int \frac{dx}{(\sin x)^{2010}}$$

$$\Rightarrow I = I_1 - I_2 = \frac{\tan x}{(\sin x)^{2010}} = \frac{P(x)}{(\sin x)^{2010}}$$

$$P\left(\frac{\pi}{3}\right) = \tan \frac{\pi}{3} = \sqrt{3}$$

85. If $I = \int \frac{dx}{\sin\left(x - \frac{\pi}{3}\right) \cos x}$, then I equals

a) $2 \log \left| \sin x + \sin\left(x - \frac{\pi}{3}\right) \right| + C$

b) $2 \log \left| \sin\left(x - \frac{\pi}{3}\right) \sec x \right| + C$

c) $2 \log \left| \sin x - \sin\left(x - \frac{\pi}{3}\right) \right| + C$

d) None of these

Key: B

Hint:
$$I = \frac{1}{\cos\left(\frac{\pi}{3}\right)} \int \frac{\cos\left(x - \left(x - \frac{\pi}{3}\right)\right)}{\sin\left(x - \frac{\pi}{3}\right)\cos x} dx$$

86.
$$\int_0^1 \frac{x^6 - x^3}{(2x^3 + 1)^3} dx$$
 is equal to

- (A) 0 (B) $-\frac{1}{6}$
 (C) $-\frac{1}{12}$ (D) $-\frac{1}{36}$

Key. D

Sol.
$$\int_0^1 \frac{\left(1 - \frac{1}{x^3}\right)}{\left(2x + \frac{1}{x^2}\right)^3} dx$$

and proceed

87. Suppose f, g are continuous and differentiable on [0, b], f', g' are non-negative on [0, b] and f is non constant with f(0) = 0, then the minimum value of $\int_0^a g(x)f'(x)dx + \int_0^b g'(x)f(x)dx$ on

$a \in (0, b]$ is

- (A) f(a).g(a) (B) f(b).g(b)
 (C) f(a).g(b) (D) f(b).g(a)

Key. C

Sol. Integrating by parts the integral

$$\begin{aligned} & \int_0^a g(x)f'(x)dx + \int_0^b g'(x)f(x)dx \\ &= g(x)f(x)\Big|_0^a - \int_0^a g'(x)f(x)dx + \int_0^b g'(x)f(x)dx \\ &= f(a).g(a) + \int_a^b g'(x)f(x)dx \\ &\geq f(a).g(a) + \int_a^b g'(x)f(x)dx = f(a).g(b) \end{aligned}$$

88. If f(x) be a real valued function, $f(x) + f(x + 4) = f(x + 2) + f(x + 6)$,

$g(x) = \int_x^{x+8} f(t)dt$. Then $g'(x)$ is equal to

- a) f(x) b) f(x + 8) c) 8 d) 0

Key. D

Sol. Conceptual

89. Value of $\int_1^5 \left(\sqrt{x + 2\sqrt{x-1}} + \sqrt{x - 2\sqrt{x-1}} \right) dx$ is

a) $\frac{8}{3}$

b) $\frac{16}{3}$

c) $\frac{32}{3}$

d) $\frac{34}{3}$

Key. D

Sol. Q $\sqrt{x + (2\sqrt{x-1})} = \sqrt{\sqrt{(x-1)}^2 + 1^2 + 2\sqrt{(x-1)}}$
 $= \sqrt{(x-1)} + 1$

And $\sqrt{x - 2\sqrt{x-1}} = \sqrt{(\sqrt{(x-1)})^2 + 1^2 - 2\sqrt{(x-1)}}$
 $= |\sqrt{(x-1)} - 1|$

Then $\int_1^5 \sqrt{x + 2\sqrt{x-1}} + \sqrt{x - 2\sqrt{x-1}} dx$
 $= \int_1^5 (\sqrt{(x-1)} + 1) + \int_1^5 |\sqrt{(x-1)} - 1| dx$
 $= \int_1^5 (\sqrt{(x-1)} + 1) dx + \int_1^2 (1 - \sqrt{(x-1)}) dx + \int_2^5 (\sqrt{(x-1)} - 1) dx$
 $= \int_0^4 (\sqrt{x} + 1) dx + \int_0^1 (1 - \sqrt{x}) dx + \int_1^4 (\sqrt{x} - 1) dx$
 $= \left[\frac{2}{3} (x^{3/2}) + x \right]_0^4 + \left[x - \frac{2}{3} x^{3/2} \right]_0^1 + \left[\frac{2}{3} x^{3/2} - x \right]_1^4$
 $= \left(\frac{16}{3} + 4 \right) + \left(1 - \frac{2}{3} \right) + \left(\frac{16}{3} - 4 \right) + \left(\frac{2}{3} - 1 \right) = \frac{32}{3}$

90. $\int_{-\pi/4}^{\pi/4} \frac{e^x \cdot \sec^2 x dx}{e^{2x} - 1}$ is equal to

a) 0

b) 2

c) e

d) 2e

Key. A

Sol. Let $I = \int_{-\pi/4}^{\pi/4} \frac{e^x \sec^2 x dx}{e^{2x} - 1}$

If $f(x) = \frac{e^x \sec^2 x}{e^{2x} - 1}$

$\therefore f(-x) = \frac{e^x \sec^2 x}{e^{2x} - 1}$

$= \frac{e^x \sec^2 x}{1 - e^{2x}}$

$= -\frac{e^x \sec^2 x}{e^{2x} - 1}$

$$= -f(x)$$

∴ I = 0 (f(x) is odd function)

91. Let $f(x) = \frac{1}{2}a_0 + \sum_{i=1}^n a_i \cos(ix) + \sum_{j=1}^n b_j \sin(jx)$, then $\int_{-\pi}^{\pi} f(x) \cos kx dx$ is equal to

- a) a_k b) b_k c) πa_k d) πb_k

Key. C

Sol. Conceptual

92. Let $\int_0^x \left(\frac{bt \cos 4t - a \sin 4t}{t^2} \right) dt = \frac{a \sin 4x}{x}$, then a and b are given by

- a) 1/4, 1 b) 2, 2 c) -1, 4 d) 2, 4

Key. A

Sol. Since, $\int_0^x \left(\frac{bt \cos 4t - a \sin 4t}{t^2} \right) dt = \frac{a \sin 4x}{x}$

Differentiating both sides w.r.t. x

$$\therefore \frac{bx \cos 4x - a \sin 4x}{x^2} = \frac{a \{4x \cos 4x - \sin 4x\}}{x^2}$$

On comparing $b = 4a$
 $a = 1/4$ and $b = 1$

93. If $f'''(x) = k$ in $[0, a]$, then $\int_0^a f(x) dx - \left\{ xf(x) - \frac{x^2}{2!} f'(x) + \frac{x^3}{3!} f''(x) \right\}_0^a$ is

- a) $-ka^4 / 12$ b) $ka^4 / 24$ c) $-ka^4 / 24$ d) $ka^4 / 12$

Key. C

Sol. Conceptual

94. If $f(x)$ is a differentiable function and $\int_0^{x^3} t^2 f(t) dt = \frac{3}{13} x^{13} + 5$ then $f\left(\frac{8}{27}\right) =$

- A) 8 / 27 B) 16 / 27 C) 16 / 81 D) 8 / 9

Key. C

Sol. Diff. w.r.t. $x \Rightarrow f(x^3) = x^4$

95. $\int_{-2\pi}^{2\pi} \frac{\sin^6 x}{(\sin^6 x + \cos^6 x)(1 + e^{-x})} dx =$

A) 2π

B) π

C) $\pi/2$

D) 4π

Key. B

Sol. $\int_{-a}^a f(x)dx = \int_0^a (f(x) + f(-x))dx$

96. Let $f(x), g(x), h(x)$ be continuous in $[0, 2a]$ and satisfies
 $f(2a-x) = f(x), g(2a-x) = g(x), h(x) + h(2a-x) = 3, f(2a-x)g(2a-x) = f(x)g(x)$
 then $\int_0^{2a} f(x)g(x)h(x)dx =$

A) $\int_0^{2a} f(x)g(x)dx$

B) $3\int_0^a f(x)g(x)dx$

C) $2\int_0^a f(x)g(x)dx$

D) $\int_0^a f(x)g(x)dx$

Key. B

Sol. $I = \int_0^{2a} f(x)g(x)h(x)dx = \int_0^{2a} f(2a-x)g(2a-x)h(2a-x)dx = \int_0^{2a} f(x)g(x)[3-h(x)]dx$
 $I = \frac{3}{2} \int_0^{2a} f(x)g(x)dx = 3\int_0^a f(x)g(x)dx$

97. $\int \frac{\sqrt[3]{x^2} + \sqrt{x}}{x(1 + \sqrt[3]{x})} dx =$

A) $\frac{1}{2} \tan^{-1}\left(\frac{x^6 + x^{-6}}{2}\right) + C$

B) $\frac{x^{24}}{2} - \log(1 + x^{24}) + \tan^{-1}(x^3) + C$

C) $\frac{3}{2} x^{12} + 6 \tan^{-1}(x^6) + C$

D) $6 \tan^{-1}(x^6) + 3x^{12} - 6 \log_e \sqrt{1 + x^{12}} + C$

Key. D

Sol. $x = t^6$

98. $\int \frac{dx}{x^{20}(1+x^{20})^{\frac{1}{20}}} =$

A) $-\frac{1}{19} \left(1 + \frac{1}{x^{20}}\right)^{\frac{19}{20}} + C$

B) $\frac{1}{21} \left(1 - \frac{1}{x^{20}}\right)^{\frac{19}{20}} + C$

C) $\frac{1}{19} \left(1 - \frac{1}{x^{20}}\right)^{\frac{19}{20}} + C$

D) $-\frac{1}{21} \left(1 + \frac{1}{x^{20}}\right)^{\frac{19}{20}} + C$

Key. A

Sol. $\int \frac{dx}{x^{21} \left(\frac{1}{x^{20}} + 1\right)^{\frac{1}{20}}}$, Put $\frac{1}{x^{20}} + 1 = t$

99. $\int \frac{\sin\left(\frac{\pi}{4} - x\right) dx}{2 + \sin 2x} = A \tan^{-1}(f(x)) + B$, where A, B are constants then the range of $Af(x)$ is

- A) $[-1, 1]$ B) $[-\sqrt{2}, \sqrt{2}]$ C) $[0, 1]$ D) $[-1, 0]$

Key. A

Sol. $\frac{1}{\sqrt{2}} \int \frac{d(\sin x + \cos x)}{(\sin x + \cos x)^2 + 1}$

100. The area bounded by the curves $y = 2 - |x - 1|$, $y = \sin x$; $x = 0$ and $x = 2$ is

- A) $1 + 2 \cos^2 1$ B) $2 + \sin^2 1$ C) $\frac{\pi}{2}$ D) $1 + \log 2$

Key. A

Sol. Area $= \frac{3}{2} + (\cos 1 - 1) + 3 - \frac{3}{2} + (\cos 2 - \cos 1) = 2 + \cos 2$

101. $\int_0^{\pi/2} \frac{\sin x}{1 + \cos x + \sin x} dx =$

- A) $\frac{\pi}{4}$ B) $\frac{\pi}{4} + \log \sqrt{2}$ C) $\frac{\pi}{4} - \log \sqrt{2}$ D) $\frac{\pi}{4} - \log 2$

Key. C

Sol. $I = \int_0^{\pi/2} \frac{\sin \frac{x}{2}}{\sin \frac{x}{2} + \cos \frac{x}{2}} dx = 2 \int_0^{\pi/4} \frac{\sin x}{\sin x + \cos x} dx$

Let $\sin x = A(\sin x + \cos x) + B(\cos x - \sin x)$ $A - B = 1$ and $A + B = 0$, $A = \frac{1}{2}$, $B = \frac{-1}{2}$

$I = 2 \times \frac{1}{2} \times \frac{\pi}{4} - 2 \times \frac{1}{2} [\log(\sin x + \cos x)]_0^{\pi/4} = \frac{\pi}{4} - \log \sqrt{2}$

102.
$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{k^3 + 6k^2 + 11k + 5}{(k+3)!} =$$

- A) $\frac{2}{3}$ B) 1 C) $\frac{5}{3}$ D) $\frac{7}{3}$

Key. C

Sol. $k^3 + 6k^2 + 11k + 5 = (k+1)(k+2)(k+3) - 1$

$$\therefore \frac{k^3 + 6k^2 + 11k + 5}{(k+3)!} = \frac{1}{k!} - \frac{1}{(k+3)!}$$

103. $I_n = \int_0^{\pi/2} \cos^n x \cos(nx) dx, n \in N$ then $\sqrt{I_{2001} : I_{2002}}$ can be the eccentricity of

- A) Parabola B) Ellipse C) Circle D) Hyperbola

Key. D

$$I_{n+1} = \int_0^{\pi/2} \cos^{n+1} x \cos(n+1)x dx$$

Sol.

$$= \int_0^{\pi/2} \cos^{n+1} x (\cos nx \cos x - \sin nx \sin x) dx$$

$$= \int_0^{\pi/2} \cos^n x \cos nx (1 - \sin^2 x) dx - \int_0^{\pi/2} \cos^{n+1} x \sin nx \sin x dx$$

$$\therefore I_{n+1} = I_n - I_{n+1} \Rightarrow 2I_{n+1} = I_n$$

$$\therefore I_n : I_{n+1} = 2$$

104. If $\int \frac{x dx}{\sqrt{(1+x^2)^{2012} (2+x^2)^{3012}}} = \frac{\alpha}{\beta} (1-f(x))^{\frac{\beta}{2\alpha} + k}$ then which is true

- A) $\alpha = 503; \beta = 500, f(\sqrt{2}) = \frac{1}{\beta - \alpha}$ B) $\alpha = 503; \beta = 250, f(\sqrt{2}) = \frac{1}{\alpha - \beta}$
 C) $\alpha = 503; \beta = 500, f(1) = \frac{1}{\alpha - \beta}$ D) $\alpha = 503; \beta = 225, f(\sqrt{3}) = \frac{1}{\alpha - \beta}$

Key. C

Key. D

Sol.
$$\int_0^1 \frac{\left(1 - \frac{1}{x^3}\right)}{\left(2x + \frac{1}{x^2}\right)^3} dx$$

and proceed

107. Suppose f, g are continuous and differentiable on $[0, b]$, f', g' are non-negative on $[0, b]$ and f is non constant with $f(0) = 0$, then the minimum value of $\int_0^a g(x)f'(x)dx + \int_0^b g'(x)f(x)dx$ on

$a \in (0, b]$ is

- (A) $f(a).g(a)$ (B) $f(b).g(b)$
 (C) $f(a).g(b)$ (D) $f(b).g(a)$

Key. C

Sol. Integrating by parts the integral

$$\begin{aligned} & \int_0^a g(x)f'(x)dx + \int_0^b g'(x)f(x)dx \\ &= g(x)f(x)|_0^a - \int_0^a g'(x)f(x)dx + \int_0^b g'(x)f(x)dx \\ &= f(a).g(a) + \int_a^b g'(x)f(x)dx \\ &\geq f(a).g(a) + \int_a^b g'(x)f(x)dx = f(a).g(b) \end{aligned}$$

108. Let $I_1 = \int_{\sec^2 z}^{2-\tan^2 z} x f(x(3-x)) dx$ and let $I_2 = \int_{\sec^2 z}^{2-\tan^2 z} f(x(3-x)) dx$

where ' f ' is a continuous function and ' z ' is any real number, then $\frac{I_1}{I_2} =$

- A) $\frac{3}{2}$ B) $\frac{1}{2}$ C) 1 D) $\frac{2}{3}$

Key. A

Sol.
$$I_1 = \int_{\sec^2 z}^{2-\tan^2 z} x f(x(3-x)) dx$$

$$I_2 = \int_{\sec^2 z}^{2-\tan^2 z} f(x(3-x)) dx$$

$$\begin{aligned}
 2I_1 &= \int_{\sec^2 z}^{2-\tan^2 z} (3-x)f((3-x)x)dx \\
 2I_1 &= \int_{\sec^2 z}^{2-\tan^2 z} 3f(x(3-x))dx \\
 &= 3 \int_{\sec^2 z}^{2-\tan^2 z} ((x)(3-x))dx \\
 2I_1 &= 3I_2 \\
 \Rightarrow \frac{I_1}{I_2} &= \frac{3}{2}
 \end{aligned}$$

109. If $\int \frac{\sec^2 x - 2010}{\sin^{2010} x} dx = \frac{P(x)}{\sin^{2010} x} + C$, then value of $P\left(\frac{\pi}{3}\right)$ is

- A) 0 B) $\frac{1}{\sqrt{3}}$ C) $\sqrt{3}$ D) None of these

Key: C

Sol.
$$\begin{aligned}
 &\int \frac{\sec^2 x - 2010}{\sin^{2010} x} dx \\
 &= \int \sec^2 x (\sin x)^{-2010} - 2010 \int \frac{1}{(\sin x)^{2010}} dx = I_1 - I_2
 \end{aligned}$$

Applying, by parts on I_1 , we get

$$I_1 = \frac{\tan x}{(\sin x)^{2010}} + 2010 \int \frac{\tan x \cos x}{(\sin x)^{2011}} dx = \frac{\tan x}{(\sin x)^{2010}} + 2010 \int \frac{dx}{(\sin x)^{2010}}$$

$$\Rightarrow I = I_1 - I_2 = \frac{\tan x}{(\sin x)^{2010}} = \frac{P(x)}{(\sin x)^{2010}}$$

$$P\left(\frac{\pi}{3}\right) = \tan \frac{\pi}{3} = \sqrt{3}$$

110. If $c > 0$ and the area of the region enclosed by the parabolas $y = x^2 - c^2$ and $y = c^2 - x^2$ is 576, then c =

- a) 6 b) 4 c) 3 d) 8

Key: A

Hint: Area between the two parabolas = $4 \int_0^c (c^2 - x^2) dx = \frac{8c^3}{3} = 576$

only if $c = 6$

111. Let $f(x) = x^2 + 6x + 1$ and let R denote the set of points (x, y) in the XY-plane such that $f(x) + f(y) \leq 0$ and $f(x) - f(y) \leq 0$. Then the area of the region R is

A) 6π B) $3\pi + 2$ C) $2\pi + 8$ D) 8π

Key: D

Hint: $f(x) + f(y) \leq 0 \Rightarrow (x+3)^2 + (y+3)^2 \leq 16$
 $f(x) - f(y) \leq 0 \Rightarrow (x-y)(x+y+6) \leq 0$

112. The quadrilateral formed by the lines $y = ax + c, y = ax + d, y = bx + c$ and $y = bx + d$ has area 18. The quadrilateral formed by the lines $y = ax + c, y = ax - d, y = bx + c$ and $y = bx - d$ has area 72. If a, b, c, d are positive integers then the least possible value of the sum $a + b + c + d$ is

A) 13 B) 14 C) 15 D) 16

Key: D

Hint: $\frac{(c-d)^2}{|a-b|} = 18$ and $\frac{(c+d)^2}{|a-b|} = 72$.

$a = 3, b = 1, d = 3, c = 9$ is a solution for which the minimum is attained.

113. Area of a square ABCD is 36 and side AB is parallel to the X-axis. Vertices A, B and C lie on the graphs of $y = \log_a x, y = 2\log_a x$ and $y = 3\log_a x$ respectively. Then $a =$

A) $3^{1/6}$ B) $\sqrt{3}$ C) $6^{1/3}$ D) $\sqrt{6}$

Key: A

Hint: Let $A = (p, \log_a p); B = (q, 2\log_a q), p, q > 0 \& a > 0, a \neq 1$, then $C = (q, 3\log_a q)$
 $AB \parallel X\text{-axis} \Rightarrow p = q^2. |AB| = 6 \Rightarrow |p - q| = 6$ also $|\log_a q| = 6$.

114. The area bounded by the curve $(y - \sin^{-1}x)^2 = x - x^2$ is

a) $\frac{\pi}{4}$ b) $\frac{\pi}{2}$ c) π d) $\frac{\pi}{3}$

Key: A

Hint: The given curves on $y = \sin^{-1}x - (\sqrt{x - x^2})$ & $y = \sin^{-1}x + (\sqrt{x - x^2})$

$$\text{Required area} = \int_0^1 2\sqrt{x - x^2} dx$$

115. Let $F(x) = \sin x \int_0^x \cos t dt + 2 \int_0^x t dt + \cos^2 x - x^2$. Then area bounded by $y = F(x)$ and ordinate $x = 0$ and $x = 5$ with x-axis is

- (A) 16 (B) $\frac{25}{2}$ (C) $\frac{35}{2}$ (D) 25

Key: B

Hint: $F(x) = \sin x \int_0^x \cos t \, dt + 2 \int_0^x t \, dt - x^2 + \cos^2 x$

$$= \sin x (\sin t)_0^x + 2 \left(\frac{t^2}{2} \right)_0^x - x^2 + \cos^2 x$$

$$= \sin^2 x + x^2 - x^2 + \cos^2 x = 1$$

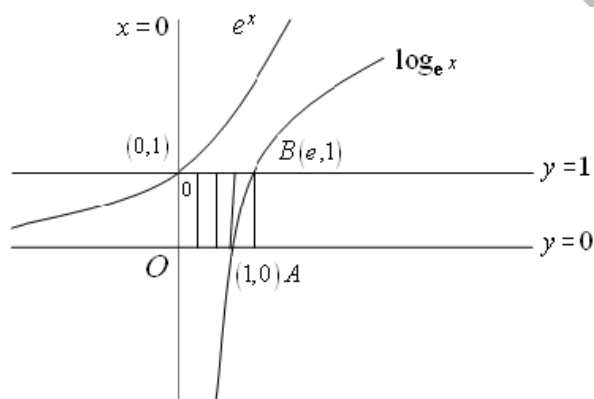
$$A = \int_0^5 xF(x)dx = \int_0^5 (x)(1)dx = \left[\frac{x^2}{2} \right]_0^5 = \frac{25}{2}$$

116. Area bounded by the curves $y = e^x$, $y = \log_e x$ and the lines $x = 0$, $y = 0$, $y = 1$ is

- A) $e^2 + 2sq.units$ B) $e + 1sq.unit$
 C) $e + 2sq.units$ D) $e - 1sq.unit$

Key: D

Hint:



$$\text{Area} = \text{Area of rectangle OABC} - \int_0^1 \log_e x$$

117. The area of the loop of the curve $y^2 = x^4(x+2)$ is [in square units]

- (A) $\frac{32\sqrt{2}}{105}$ (B) $\frac{64\sqrt{2}}{105}$ (C) $\frac{128\sqrt{2}}{105}$ (D) $\frac{256\sqrt{2}}{105}$

Key: D

Hint: $\text{Area} = 2 \int_{-2}^0 y \, dx = 2 \int_{-2}^0 x^2 \sqrt{x+2} \, dx = 4\sqrt{2} \int_0^2 (z^2 - 2)^2 z^2 \, dz$ (where $\sqrt{x+2} = z$)

$$= 4 \left[\frac{z^7}{7} - \frac{4z^5}{5} + \frac{4z^3}{3} \right]_0^{\sqrt{2}}$$

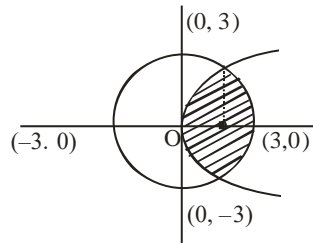
$$= \frac{256\sqrt{2}}{105}$$

118. The area of the smaller portion enclosed by the curves $x^2+y^2=9$ and $y^2=8x$ is

- A) $\frac{\sqrt{2}}{3} + \frac{9\pi}{4} - \frac{9}{2} \sin^{-1}\left(\frac{1}{3}\right)$ B) $2\left(\frac{\sqrt{2}}{3} + \frac{9\pi}{4} - \frac{9}{2} \sin^{-1}\left(\frac{1}{3}\right)\right)$
 C) $2\left(\frac{\sqrt{2}}{3} + \frac{9\pi}{4} + \frac{9}{2} \sin^{-1}\left(\frac{1}{3}\right)\right)$ D) $\frac{\sqrt{2}}{3} + \frac{9\pi}{4} + \frac{9}{2} \sin^{-1}\left(\frac{1}{3}\right)$

Key: B

HINT :



$$x^2 + y^2 = 9,$$

$$x^2 + 8x - 9 = 0$$

$$x = \frac{-8 \pm \sqrt{64 + 36}}{2}$$

$$x = \frac{-8 \pm 10}{2} = -9, 1$$

$$\boxed{x=1}$$

$$\text{Area enclosed} = 2 \left[\int_0^1 2\sqrt{2x} dx + \int_1^3 \sqrt{9-x^2} dx \right] = 2 \left[2\sqrt{2} \int_0^1 \sqrt{x} dx + \int_1^3 \sqrt{9-x^2} dx \right]$$

On simplifying we get

$$= 2 \left[\frac{\sqrt{2}}{3} + \frac{9\pi}{4} - \frac{9}{2} \sin^{-1}\left(\frac{1}{3}\right) \right]$$

119. The area of the region in the xy -plane defined by the inequalities $x-2y^2 \geq 0$, $1-x-|y| \geq 0$ is

- A) $\frac{1}{2}$ B) $\frac{1}{3}$ C) $\frac{1}{4}$ D) $\frac{7}{12}$

Key: D

Hint: Area = $2 \int_0^{1/2} \sqrt{\frac{x}{2}} dx + \frac{1}{4} = \frac{7}{12}$

120. Area bounded by curve $y^2 = x$ and $x = 4$ is divided into 4 equal parts by the lines $x = a$ and $y = b$ then.

- a) Area of each part = $\frac{8}{3}$ b) $b = 0$

c) $a = \sqrt{2}$

d) $a = (16)^{1/3}$

Key: D

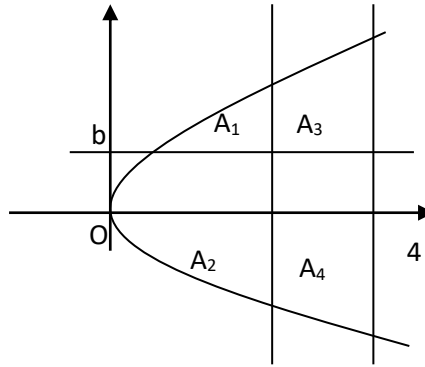
Hint: Total area = $2 \int_0^4 \sqrt{x} dx = \frac{32}{3}$

Area of each part = $8/3$

$A_3 = A_4 \Rightarrow \int_a^4 (\sqrt{x} - b) dx =$

$\int_a^4 (b + \sqrt{x}) dx = \frac{8}{3} \Rightarrow b = 0$

$\int_a^4 \sqrt{x} dx = \frac{8}{3} \Rightarrow a^3 = 16$



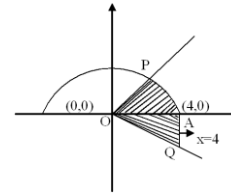
121. Area of the region in which point $p(x, y), \{x > 0\}$ lies; such that $y \leq \sqrt{16 - x^2}$ and $\left| \tan^{-1} \left(\frac{y}{x} \right) \right| \leq \frac{\pi}{3}$ is

- (A) $\left(\frac{16}{3} \pi \right)$ (B) $\left(\frac{8\pi}{3} + 8\sqrt{3} \right)$ (C) $(4\sqrt{3} - \pi)$ (D) $(\sqrt{3} - \pi)$

Key: B

Hint: Required area is the area of shaded region (APOQ) = area of ΔOAQ + area of sector (OAP)

$= \frac{1}{2} \times 4 \times 4\sqrt{3} + \frac{\pi(4 \times 4)}{6}$
 $= \left(\frac{8\pi}{3} + 8\sqrt{3} \right)$



122. Area bounded between the curves $y = \sqrt{4 - x^2}$ and $y^2 = 3|x|$ is/are

- (A) $\frac{\pi - 1}{\sqrt{3}}$ (B) $\frac{2\pi - 1}{3\sqrt{3}}$ (C) $\frac{2\pi - \sqrt{3}}{3}$ (D) $\frac{2\pi - \sqrt{3}}{3\sqrt{3}}$

Key: C

Hint: Required area = $2 \int_0^1 \left(\sqrt{4 - x^2} - \sqrt{3x} \right) dx$

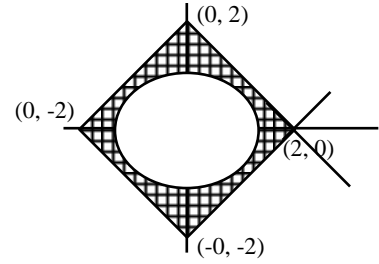
$= 2 \left(\frac{x}{2} \sqrt{4 - x^2} + \frac{4}{2} \sin^{-1} \left(\frac{x}{2} \right) - \frac{\sqrt{3} \cdot 2x^{3/2}}{3} \right)_0^1$
 $= \frac{2\pi - \sqrt{3}}{3}$

area of ellipse = $\pi \times \frac{\sqrt{3}}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{\pi\sqrt{3}}{2}$

$|x + y| \leq 2 \Rightarrow -2 \leq (x + y) \leq 2 \dots \dots \dots (2)$

$|x - y| \leq 2 \Rightarrow -2 \leq (x - y) \leq 2 \dots \dots \dots (3)$

Required area = $\left(8 - \frac{\pi\sqrt{3}}{2}\right)$ sq. units



126. The curve $y = (|x| - 1)\text{sgn}(x - 1)$ divides $\frac{9x^2}{64} + \frac{4}{25}y^2 = \frac{1}{\pi}$ in two parts having area A_1 and

A_2 (where $A_1 < A_2$), then

a) $\frac{A_1}{A_2} = \frac{7}{13}$

b) $\frac{A_1}{A_2} = \frac{3}{7}$

c) $A_1 = \frac{7}{3}$

d) $A_2 = \frac{13}{7}$

Key: A

Sol : $A_1 = \frac{10}{3} - 1, A_2 = \frac{10}{3} + 1 \Rightarrow \frac{A_1}{A_2} = \frac{7}{13}$

127. Area bounded by the circle which is concentric with the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ and which

passes through $\left(4, -\frac{9}{5}\right)$, the vertical chord common to both circle and ellipse on the positive side of x-axis is

a) $\frac{481}{25} \tan^{-1}\left(\frac{9}{20}\right) - \frac{36}{5}$

b) $2 \tan^{-1}\left(\frac{9}{20}\right)$

c) $\frac{481}{25} \tan^{-1}\left(\frac{9}{20}\right)$

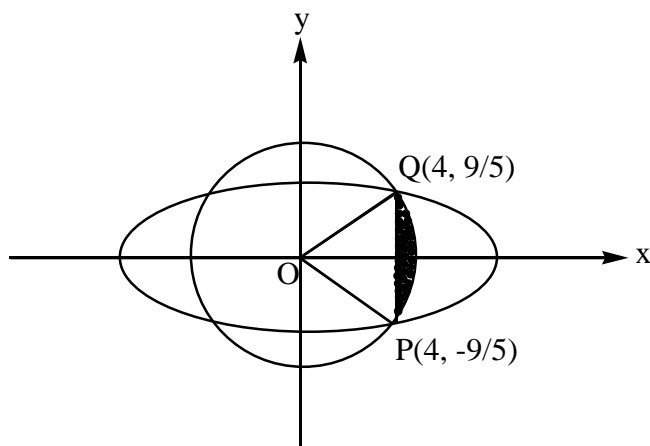
d) none of these

Key: a

Sol : As eccentricity of ellipse = $\frac{4}{5}$

Co-ordinate of foci = $(4, 0), (-4, 0)$

$\Rightarrow \left(4, -\frac{9}{5}\right)$ is one of the end point of latus – rectum



⇒ required area is

$$\begin{aligned} & \frac{1}{2\pi} \times \pi \times \left(4^2 + \frac{9^2}{5^2}\right) \times 2 \tan^{-1}\left(\frac{9}{20}\right) - \text{area of } \Delta POQ \\ &= \frac{481}{25} \tan^{-1}\left(\frac{9}{20}\right) - \frac{1}{2} \times 4 \times \left(\frac{18}{5}\right) = \frac{481}{25} \tan^{-1}\left(\frac{9}{20}\right) - \frac{36}{5} \end{aligned}$$

128. Area of the region in which point $p(x, y)$, $\{x > 0\}$ lies; such that $y \leq \sqrt{16-x^2}$ and

$$\left| \tan^{-1}\left(\frac{y}{x}\right) \right| \leq \frac{\pi}{3} \text{ is}$$

a) $\left(\frac{16}{3}, \pi\right)$

b) $\left(\frac{8\pi}{3} + 8\sqrt{3}\right)$

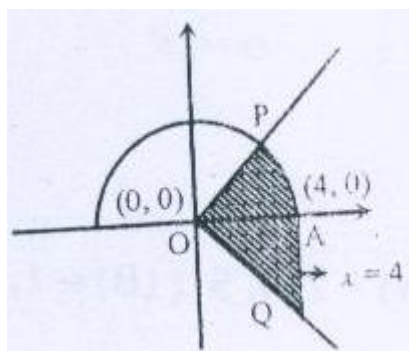
c) $(4\sqrt{3} - \pi)$

d) none of these

Key: b

Sol: Required area is the area of shaded region (APOQ)

= area of ΔOAQ + area of sector (OAP)



$$= \frac{1}{2} \times 4 \times 4\sqrt{3} + \frac{\pi(4 \times 4)}{6}$$

Hint: The given curves on $y = \sin^{-1}x - (\sqrt{x-x^2})$ & $y = \sin^{-1}x + (\sqrt{x-x^2})$

$$\text{Required area} = \int_0^1 2\sqrt{x-x^2} dx$$

134. Let $F(x) = \sin x \int_0^x \cos t dt + 2 \int_0^x t dt + \cos^2 x - x^2$. Then area bounded by $y = F(x)$ and ordinate $x = 0$ and $x = 5$ with x -axis is

- (A) 16 (B) $\frac{25}{2}$ (C) $\frac{35}{2}$ (D) 25

Key: B

Hint: $F(x) = \sin x \int_0^x \cos t dt + 2 \int_0^x t dt - x^2 + \cos^2 x$

$$= \sin x (\sin t)_0^x + 2 \left(\frac{t^2}{2} \right)_0^x - x^2 + \cos^2 x$$

$$= \sin^2 x + x^2 - x^2 + \cos^2 x = 1$$

$$A = \int_0^5 xF(x)dx = \int_0^5 (x)(1)dx = \left[\frac{x^2}{2} \right]_0^5 = \frac{25}{2}$$

135. If $\int \frac{x dx}{\sqrt[2012]{(1+x^2)^{1012} (2+x^2)^{3012}}} = \frac{\alpha}{\beta} (1-f(x))^{\frac{\beta}{2\alpha}} + k$ then which is true

- a) $\alpha = 503; \beta = 500, f(\sqrt{2}) = \frac{1}{\beta-\alpha}$ b) $\alpha = 503; \beta = 250, f(\sqrt{2}) = \frac{1}{\alpha-\beta}$
 c) $\alpha = 503; \beta = 500, f(1) = \frac{1}{\alpha-\beta}$ d) $\alpha = 503; \beta = 225, f(\sqrt{3}) = \frac{1}{\alpha-\beta}$

Key: C

Sol. $\frac{2+x^2}{1+x^2} = t$

136. The area of the region in the xy -plane defined by the inequalities $x-2y^2 \geq 0$, $1-x-|y| \geq 0$ is

- a) $\frac{1}{2}$ b) $\frac{1}{3}$ c) $\frac{1}{4}$ d) $\frac{7}{12}$

Key: D

Sol. Area = $2 \int_0^{1/2} \sqrt{\frac{x}{2}} dx + \frac{1}{4} = \frac{7}{12}$

137. The area bounded by $y = \sin^{-1} x, y = \cos^{-1} x$ and the x -axis is

- 1) $2+\sqrt{2}$ 2) $2-\sqrt{2}$ 3) $\sqrt{2}+1$ 4) $\sqrt{2}-1$

Key. 4

Sol. By the graph

$$\text{Required area} = \int_0^{1/\sqrt{2}} \sin^{-1} x dx + \int_{1/\sqrt{2}}^1 \cos^{-1} x dx$$

138. Let f be a real valued function satisfying $f\left(\frac{x}{y}\right) = f(x) - f(y)$ and $\lim_{x \rightarrow 0} \frac{f(1+x)}{x} = 3$. The area bounded by the curve $y=f(x)$, the y -axis and the line $y=3$ is

- 1) $9e$ 2) $2e$ 3) $3e$ 4) none

Key. 3

Sol.
$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{f\left(1 + \frac{h}{x}\right) - f\left(\frac{h}{x}\right)}{\frac{h}{x}} = \frac{3}{x}$$

139. Area enclosed by the curve $y = f(x)$ defined parametrically as $x = \frac{1-t^2}{1+t^2}$, $y = \frac{2t}{1+t^2}$ is equal to

- 1) π sq.units 2) $\pi/2$ sq.unit 3) $\frac{3\pi}{4}$ sq.units 4) $\frac{3\pi}{2}$ sq.units

Key. 1

Sol. Clearly t can be any real number
Let $t = \tan \theta$

140. The maximum area of a rectangle whose two vertices lie on the x -axis and two on the curve $y=3-|x|$, $-3 \leq x \leq 3$ is

- 1) 9 2) $9/4$ 3) 3 4) $9/2$

Key. 4

Sol. Take $2a$ and $3-a$ are the length of the sides of the rectangle

141. The area of the closed figure bounded by $x=-1$, $x=2$ and $y = \begin{cases} -x^2 + 2 & x \leq 1 \\ 2x - 1, & x > 1 \end{cases}$ and the x -axis is

- 1) $16/3$ sq. units 2) $10/3$ sq. units 3) $13/3$ sq. units 4) $7/3$ sq. units

Key. 1

Sol. By the graph

$$A = \int_{-1}^1 (-x^2 + 2) dx + \int_1^2 (2x - 1) dx$$

142. The value of the parameter 'a' such that the area bounded by $y = a^2x^2 + ax + 1$, positive coordinate axes and the line $x=1$ attains its least value, is equal to

- 1) $-\frac{1}{4}$ 2) $-\frac{1}{2}$ 3) $-\frac{3}{4}$ 4) -1

Key. 3

Sol.
$$A = \int_0^1 (a^2x^2 + ax + 1) dx$$

$$= \frac{a^2}{3} + \frac{a}{2} + 1 \text{ which is minimum for } a = -3/4$$

143. The area enclosed by the curves $y = \sqrt{4 - x^2}$, $y \geq \sqrt{2} \sin\left(\frac{x\pi}{2\sqrt{2}}\right)$ and x-axis is divided by the y-axis in the ratio

1) $\frac{\pi^2 - 8}{\pi^2 + 8}$ 2) $\frac{\pi^2 - 4}{\pi^2 + 4}$ 3) $\frac{\pi - 4}{\pi + 4}$ 4) $\frac{2\pi^2}{2\pi + \pi^2 - 8}$

Key. 4

Sol. By the graph

$$\text{Area of the left of y-axis is } \pi, \text{ Area of the right of y-axis} = \int_0^{\sqrt{2}} \left(\sqrt{4 - x^2} - \sqrt{2} \sin\left(\frac{x\pi}{2\sqrt{2}}\right) \right) dx$$

144. The area enclosed by the curves, $xy^2 = a^2(a - x)$ and $(a - x)y^2 = a^2x$ is

1) $(\pi - 2)a^2 \text{ sq. units}$ 2) $(4 - \pi)a^2 \text{ sq. units}$ 3) $\pi a^2 / 3 \text{ sq. units}$ 4) $\frac{\pi a^2}{2} \text{ sq. units}$

Key. 1

Sol. By the graph, required area

$$= 2 \int_0^a \left[a - \frac{a^3}{a^2 + y^2} - \frac{a^3}{a^2 + y^2} \right] dy \text{ (integrating along y-axis)}$$

145. The area of the region bounded by the parabola $(y - 2)^2 = x - 1$, the tangent to it at the point with the ordinate 3 and the x-axis is

1) 7 sq. units 2) 6 sq. units 3) 9 sq. units 4) 8 sq. units

Key. 3

Sol. Given parabola is $(y - 2)^2 = x - 1$

$$\text{Tangent at } (2, 3) \text{ is } y - 3 = \frac{1}{2}(x - 2) \Rightarrow x - 2y + 4 = 0$$

By the graph

$$\int_0^3 \left((y - 2)^2 + 1 \right) dy - \int_0^3 (2y - 4) dy$$

146. Consider the region formed by the lines $x = 0$, $y = 0$, $x = 2$, $y = 2$. Area enclosed by the curves $y = e^x$ and $y = \ln x$, within this region, is being removed. Then, the area of the remaining region is

1) $2(\ln 2 - 1) \text{ sq. units}$ 2) $2(1 - \ln 2 + 1) \text{ sq. units}$
 3) $2(2 - \ln 2 - 1) \text{ sq. units}$ 4) $2(2 - \ln 2 + 1) \text{ sq. units}$

Key. 3

Sol. By the graph

$$2 \int_0^{\ln 2} (2 - e^x) dx$$

147. The area bounded by the curve $f(x) = x + \sin x$ and its inverse function between the ordinates $x = 0$ and $x = 2\pi$ is

1) $4\pi \text{ sq. units}$ 2) $8\pi \text{ sq. units}$ 3) 4 sq. units 4) 8 sq. units

Key. 4

Sol. By the graph, required Area = 4A, where

$$A = \int_0^{\pi} (x + \sin x) dx - \int_0^{\pi} x dx$$

$$= \frac{\pi^2}{2} - \cos \pi + \cos 0 - \frac{\pi^2}{2} = 2 \text{ square units}$$

148. The area bounded by the curves $y = \sin^{-1} |\sin x|$ and $y = (\sin^{-1} |\sin x|)^2$, where

$$0 \leq x \leq 2\pi, \text{ is}$$

1) $\frac{1}{3} + \frac{\pi^2}{4}$ sq.units

2) $\frac{1}{6} + \frac{\pi^3}{8}$ sq. units

3) $\frac{2}{3} + \frac{\pi^2(\pi-3)}{6}$ sq .units

4) $\frac{4}{3} + \frac{\pi^2(\pi-3)}{6}$ sq .units

Key. 4

Sol. $y = \sin^{-1} |\sin x| = \begin{cases} x, & 0 \leq x < \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} \leq x < \pi \\ x - \pi, & \pi \leq x < \frac{3\pi}{2} \\ 2\pi - x, & \frac{3\pi}{2} \leq x < 2\pi \end{cases}$

By the graph

$$4 \int_0^1 (x - x^2) dx + 4 \int_1^{\pi/2} (x^2 - x) dx$$

149. Area bounded by the curve $xy^2 = a^2(a - x)$ and its asymptote is

1) $\pi a^2 / 2$ sq. units 2) πa^2 sq. units 3) $3\pi a^2$ sq. units 4) $4\pi a^2$ sq. units

Key. 2

Sol. By the graph, required area = $2 \int_0^{\infty} \frac{a^3}{y^2 + a^2} dy$

150. Consider two curves $C_1 : y^2 = 4[\sqrt{y}]x$ and $C_2 : x^2 = 4[\sqrt{x}]y$, where [.] denotes the greatest integer function. Then the area of the region enclosed by these two curves within the square formed by the lines $x=1, y=1, x=4, y=4$ is

1) $8/3$ sq. units 2) $10/3$ sq. units 3) $11/3$ sq. units 4) $11/4$ sq. units

Key. 2

Sol. By the graph

The required area

$$A = \int_1^2 (2\sqrt{x} - 1) dx + \int_2^4 \left(2\sqrt{x} - \frac{x^2}{4} \right) dx$$

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