APPLICATIONS OF DERIVATIVES

CHAPTER

06

APPLICATIONS

DERIVATIVE AS A RATE MEASURER:

If y=f(x) be a function of x, where y is dependent variable and x is independent variable.

Then $\frac{dy}{dx}$ or $f'(x)$ represents the rate of change of y with respect to x for a definite value of x.

REMARK:1

The value of $\frac{dy}{dx}$ at $x = x_0$ i.e. $(\frac{dy}{dx})_{x=x_0}$ represents the rate of change of y with respect to x at $x = x_0$.

REMARK:2

If x=$\varphi(t)$ and y=$\omega(t)$, then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, provided that $\frac{dx}{dt} \neq 0$.

Thus, the rate of change of y with respect to x can be calculated by using the rate of change of y and that of x each with respect to t.

IMPORTANT FORMULAS:

i. Area of a square = x^2 and Perimeter = 4x
where x is the side of the square.

ii. Area of a rectangle = xy and Perimeter=2(x+y)
where x and y are the length and breadth of rectangle.

iii. Area of a trapezium = $\frac{1}{2}$(sum of parallel sides) × (perpendicular distance between them)

iv. Area of a circle =πr^2
and Circumference of a circle = $2\pi r$
where r is the radius of circle.

v. Volume of sphere = $\frac{4}{3}\pi r^3$

and Surface area of sphere = $4\pi r^2$

Where r is the radius of sphere

vi. Total surface area of a right circular cylinder = $2\pi rh + 2\pi r^2$

Curved surface area of right circular cylinder = $2\pi rh$

and Volume = $\pi r^2 h$

where r is the radius and h is the height of the cylinder.

vii. Volume of a right circular cone = $\frac{1}{3}\pi r^2 h$

Curved surface area = πrl

Total surface area = $\pi r^2 + \pi rl$

where r is the radius, h is the height and l is the slant height of a cone.

viii. Volume of a parallelopiped = xyzzy

and Surface area = 2(xy+yz+zx)

where x, y and z are the dimensions of a parallelepiped.

ix. Volume of a cube = x^3

and Surface area = $6x^2$

where x is the side of cube.

x. Area of an equilateral triangle = $\frac{\sqrt{3}}{4}(side)^2$

MARGINAL COST:

Marginal cost represents the instantaneous rate of change of the total cost with respect to the number of items produced at an instant. If C(x) represents the cost function for x units produced, then marginal cost denoted by MC, is given by MC= $\frac{d}{dx}$\{C(x)\}.

MARGINAL REVENUE:

Marginal revenue represents the rate of change of total revenue with respect to the number of items sold at an instant. If R(x) represents the revenue function for x units sold, then marginal revenue denoted by MR, is given by MR= $\frac{d}{dx}$\{R(x)\}.

INCREASING AND DECREASING FUNCTIONS:

i. *INCREASING FUNCTIONS:* Let I be an open interval contained in the domain of a real valued function f. Then f is said to be

a) Increasing on I, if $x_1 < x_2$

$\Rightarrow f(x_1) \leq f(x_2), \forall\, x_1, x_2 \in I$

b) Strictly increasing on I, if $x_1 < x_2$

$\Rightarrow f(x_1) < f(x_2), \forall\, x_1, x_2 \in I$

ii. *DECREASING FUNCTIONS:* Let I be an open interval contained in the domain of a real valued function f. Then f is said to be

a) Decreasing on I, if $x_1 < x_2$

$\Rightarrow f(x_1) \geq f(x_2), \forall\, x_1, x_2 \in I$

b) Strictly decreasing on I, if $x_1 < x_2$

$\Rightarrow f(x_1) > f(x_2), \forall\, x_1, x_2 \in I$

MONOTONIC FUNCTIONS:

A function f(x) is said to be monotonic on an interval (a, b) if it is either increasing or decreasing on (a,b).

THEOREM: Let f be continuous on [a, b] and differentiable on (a, b).

i. If $f'(x) > 0$ for each $x \in (a, b)$, then f(x) is said to be increasing in [a, b] and strictly increasing in (a, b).

ii. If $f'(x) < 0$ for each $x \in (a, b)$, then f(x) is said to be decreasing in [a, b] and strictly increasing in (a, b).

iii. If $f'(x) = 0$ for each $x \in (a, b)$, then f is said to be a constant function in [a, b].

TANGENTS AND NORMALS:

i. *TANGENTS:* A tangent is a straight line, which touches the curve y=f(x) at a point.

ii. *NORMAL:* A normal is a straight line perpendicular to a tangent to the curve y=f(x) intersecting at the point of contact

SLOPE OF TANGENT AND NORMAL:

$\frac{dy}{dx}$ represents the gradient or slope of a curve y=f(x).

If a tangent line to the curve y=f(x) makes an angle θ

With x-axis in the positive direction, then

SLOPE OF TANGENT = $\frac{dy}{dx}$= $\tan\theta$

SLOPE OF NORMAL = $\frac{-1}{SLOPE\ OF\ TANGENT} = \frac{-1}{\frac{dy}{dx}}$

EQUATIONS OF TANGENT AND NORMAL:

Let y=f(x) be a curve and P (x_1, y_1) be a point on it. Then,

EQUATION OF TANGENT AT P (x_1, y_1) IS

$$(y-y_1) = [\frac{dy}{dx}]_{(x_1,y_1)}(x - x_1)$$

EQUATION OF NORMAL AT P (x_1, y_1) IS

$$(y-y_1) = \frac{-1}{[\frac{dy}{dx}]_{(x_1,y_1)}}(x - x_1)$$

ANGLE BETWEEN INTERSECTION OF TWO CURVES:

Let y= f_1 (x) and y= f_2 (x) be the two curves and φ be the angle between their tangents at the point of their intersection P (x_1, y_1).

Then, $tan\varphi = [\frac{m_1-m_2}{1+m_1m_2}]$

where $m_1 = [\frac{dy}{dx}]_{(x_1,y_1)}$ for y= f_1 (x)

and $m_2 = [\frac{dy}{dx}]_{(x_1,y_1)}$ for y= f_2 (x)

If $m_1m_2 = -1$, then tangents are perpendicular to each other. In this case, we say that the curves intersect each other orthogonally. This also happens, when $m_1 = 0$ and $m_2 = \infty$.

If $m_1 = m_2$, then tangents are parallel to each other.

APPROXIMATIONS:

Let y=f(x) be a function such that f:D→R, D⊂R. Here, x is an independent variable and y is the dependent variable.

Let Δx be a small change in x and Δy be the corresponding change in y and given by $\Delta y = f(x + \Delta x) - f(x)$. Then,

i. the differential of x, denoted by dx, is defined by dx = Δx.

ii. the differential of y, denoted by dy, is defined by dy $= f'(x)$dx or dy$=\left(\frac{dy}{dx}\right)\Delta x$.

iii. If $dx = \Delta x$ is relatively small, when compared with x, dy is a good approximation of Δy and we denote it by $dy \cong \Delta y$.

ABSOLUTE ERROR: The error Δx in x is called the absolute error in x.

RELATIVE ERROR: If Δx is an error in x, then $\frac{\Delta x}{x}$ is called the relative error in x.

PERCENTAGE ERROR: If Δx is an error in x, then $\frac{\Delta x}{x} \times 100$ is called the percentage error in x.

MAXIMA AND MINIMA:

Let f be a real valued function and c be an interior point in the domain of f . Then,

I. Point c is called a local maxima, if there is a h>0 such that f(c)≥ f(x), ∀ x in (c-h,c+h). Here, value f(c) is called the local maximum value of f.

II. Point c is called a local minima, if there is a h>0 such that f(c)<f(x), ∀ x in (c-h,c+h). Here, value f(c) is called the local minimum value of f.

CRITICAL POINT:

A point c in the domain of a function f at which either $f'(c) = 0$ or f is not differentiable, is called a critical point of f. If f is continuous at c and $f'(c) = 0$, then there exists h>0 such that f is differentiable in the interval (c-h, c+h).

FIRST DERIVATIVE TEST:

Let f be a function defined on an open interval I and let f be continuous at a critical point c in I. Then,

I. If $f'(x)$ change sign from positive to negative as x increases through point c, then c is a point of local maxima.

II. If $f'(x)$ change sign from negative to positive as x increases through point c, then c is a point of local minima.

III. If $f'(x)$ does not change sign as x increases through point c, then c is neither a point of local maxima nor a point of local minima. Infect, such a point is called point of inflection.

SECOND DERIVATIVE TEST:

Let f be a function defined on an interval I and $c \in I$. Let f be twice differentiable at c, then

I. x=c is a point of local maxima, if $f'(c)$=0 and $f''(c)$<0. The value f(c) is local maximum value of f.

II. x=c is a point of local minima, if $f'(c)$=0 and $f''(c)$>0. The value f(c) is local minimum value of f.

III. If $f'(c)$=0 and $f''(c)$=0, then the test fails.

ABSOLUTE MAXIMA AND ABSOLUTE MINIMA:

Let f be a differentiable function on [a, b] and c be a point in [a, b] such that $f'(c)$=0. Then f(a), f(b) and f(c), the maximum of these values gives a maxima or absolute maxima and minimum of these values gives a minima or absolute minima.

Applications of Derivatives

Maxima and Minima

A point C in the domain of 'f' at which either $f'(C)=0$ or is not differentiable is called a critical point of f.

First derivative test

Let f be continuous at a critical point C in open interval. Then

(i) If $f'(x) > 0$ at every point left of C and $f'(x) < 0$ at every point right of C, then 'C' is a point of local maxima.

(ii) If $f'(x) < 0$ at every point left of C and $f'(x) > 0$ at every point right of C, then 'C' is a point of local minima.

(iii) If $f'(x)$ does not change sign as 'x' increases through C, then 'C' is called the point of inflection.

Second derivative test

Let f be a function defined on given interval, f is twice differentiable at C. Then

(i) $x=C$ is a point of local maxima If $f'(C)=0$ and $f''(C)<0$, $f(C)$ is local maxima of f.

(ii) $x=C$ is a point of local minima if $f'(C)=0$ and $f''(C)>0$. $f(C)$ is local minima of f.

(iii) The test fails if $f'(C)=0$ and $f''(C)=0$

Rate of change of bodies

The change in quantity w.r.t. time is known as rate of change. If a quantity y varies w.r.t. another quantity x, satisfying $y = f(x)$, then $\frac{dy}{dx}$ represents rate of change of y w.r.t. 'x'

Increasing and decreasing functions

If $f'(x) \geq 0 \,\forall\, x \in (a,b)$ then f is increasing in (a,b) and if $f'(x) \leq 0 \,\forall\, x \in (a,b)$, then f is decreasing in (a,b)

eg: Let $f(x) = x^3 - 3x^2 + 4x, x \in R$, then

$f'(x) = 3x^2 - 6x + 4 = 3(x-1)^2 + 1 > 0 \,\forall\, x \in$ R.

So, the function f is strictly increasing on R.

A function f is said to be

(i) increasing on (a,b) if $x_1 < x_2$ in $(a,b) \Rightarrow f(x_1) \leq f(x_2) \,\forall\, x_1, x_2 \in (a,b)$, and

(ii) decreasing on (a,b) if $x_1 < x_2$ in $(a,b) \Rightarrow f(x_1) \geq f(x_2) \,\forall\, x_1, x_2 \in (a,b)$

Trace the Mind Map

- First Level
- Second Level
- Third Level

PRACTICE QUESTIONS

1. The function $f(x)$ given by $f(x) = \begin{vmatrix} x-1 & x+1 & 2x+1 \\ x+1 & x+3 & 2x+3 \\ 2x+1 & 2x-1 & 4x+1 \end{vmatrix}$ has

    a) One point of maximum and one point of minimum

    b) One point of maximum only

    c) One point of minimum only

    d) None of the above

2. The points of extrema of $f(x) = \int_0^x \frac{\sin t}{t} dt$ in the domain $x > 0$ are

    a) $(2n+1)\frac{\pi}{2}, n = 1, 2, \ldots$

    b) $(4n+1)\frac{\pi}{2}, n = 1, 2, \ldots$

    c) $(2n+1)\frac{\pi}{4}, n = 1, 2, \ldots$

    d) $n\pi, n = 1, 2, \ldots$

3. The critical points of the function

    $f(x) = 2\sin^2\left(\frac{x}{6}\right) + \sin\left(\frac{x}{3}\right) - \left(\frac{x}{3}\right)$

    Whose coordinates satisfy the inequality $x^2 - 10 < -19.5x$, is

    a) -6π

    b) 6π

    c) $\frac{9\pi}{2}$

    d) -4π

4. Let $g(x) = f(x) - 2\{f(x)\}^2 + 9\{f(x)\}^3$ for all $x \in R$. Then,

    a) $g(x)$ and $f(x)$ increases and decrease together

    b) $g(x)$ increases whenever $f(x)$ decreases and vice-versa

    c) $g(x)$ increases for all $x \in R$

    d) $g(x)$ decreases for all $x \in R$

5. The tangent and the normal drawn to the curve $y = x^2 - x + 4$ at $P(1, 4)$ cut the x-axis at A and B respectively. If the length of the subtangent drawn to the curve at P is equal to the length of the subnormal, then the area of the triangle PAB in sq units is
   a) 4
   b) 32
   c) 8
   d) 16
6. Consider the following statements :
   I. The function $x + \frac{1}{x} (x \neq 0)$ is a non-increasing function in the interval $[-1, 1]$
   II. The maximum and minimum values of the function $|\sin 4x + 3|$ are 2, 4
   III. The function $x^2 \log x$ in the interval has a point of maxima

   Which of the statement given above is/are correct?
   a) Only 1
   b) Only 2
   c) Only 3
   d) All 1,2,3
7. The radius and height of a cylinder are equal. If the radius of the sphere is equal to the height of the cylinder, then the ratio of the rates of increase of the volume of the sphere and the volume of the cylinder is
   a) 4:3
   b) 3:4
   c) $4 : 3\pi$
   d) $3\pi : 4$
8. Let f, g and h be real-valued functions defined on the interval [0, 1] by $f(x) = e^{x^2} + e^{-x^2}$, $g(x) = xe^{x^2} + e^{-x^2}$ and $h(x) = x^2 e^{x^2} + e^{-x^2}$. If a, b and c denote respectively, the absolute maximum of f, g and h on [0, 1], then
   a) $a = b$ and $c \neq b$
   b) $a = c$ and $a \neq b$
   c) $a \neq b$ and $c \neq b$
   d) a=b=c

9. A particle is moving in a straight line such that the distance described 's' and the time taken 't' are given by $t = as^2 + bs + c, a > 0$. If v is the velocity of the particle at any time t, then acceleration is

   a) -2av
   b) $-2av^2$
   c) $-2av^3$
   d) None of these

10. Let $f(x) = \begin{cases} |x^3 + x^2 + 3x + \sin x| \left(3 + \sin\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$, then number of points (where $f(x)$ attains its minimum value) is

   a) 1
   b) 2
   c) 3
   d) Infinite many

11. Divide 12 into two parts such that the product of the square on one part and the fourth power of the second port is maximum, are

   a) 6,6
   b) 5,7
   c) 4,8
   d) 3,9

12. If the sum of the squares of the intercepts on the axes cut off by the tangent to the curve $x^{1/3} + y^{1/3} = a^{1/3}$ (with $a > 0$) at $P(a/8, a/8)$ is 2, then $a =$

   a) 1
   b) 2
   c) 4
   d) 8

13. A variable triangle ABC is inscribed in a circle of diameter x units. If at a particular instant the rate of change of side $'a'$ is $\frac{x}{2}$ times the rate of change of the opposite angle A, then $A =$

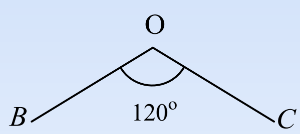
a) $\frac{\pi}{6}$

b) $\frac{\pi}{3}$

c) $\frac{\pi}{4}$

d) $\frac{\pi}{2}$

14. OB And OC are two roads enclosing an angle of $120°$. X And Y start from 'O' at the same time. X Travels along OB with a speed of 4 km/h and Y travels along OC with a speed of 3 km/h. The rate at which the shortest distance between X and Y is increasing after 1 h is



a) $\sqrt{37}$ km/hr

b) 37 km/hr

c) 13 km/hr

d) $\sqrt{13}$ km/hr

15. A particle moves along a straight line according to the law $s = 16 - 2t + 3t^3$, where s a metre is the distance of the particle from a fixed point at the end of t seconds. The acceleration of the particle at the end of 2 s is

a) $\frac{36m}{s^2}$

b) $\frac{34m}{s^2}$

c) 36m

d) None of these

16. If the function $f(x) = x^3 + 3(a-7)x^2 + 3(a^2-9)x - 1$ has positive points of extremum then

a) $a \in (3, \infty) \cup (-\infty, -3)$

b) $a \in (-\infty, -3) \cup \left(3, \frac{29}{7}\right)$

c) $(-\infty, 7)$

d) $\left(-\infty, \frac{29}{7}\right)$

17. The maximum distance from the origin of a point on the curve $x = a \sin t - \sin\left(\frac{at}{b}\right), y = a \cos t - b \cos\left(\frac{at}{b}\right)$, both $a, b > 0$, is

a) a-b

b) a+b

c) $\sqrt{a^2 + b^2}$

d) $\sqrt{a^2 - b^2}$

18. An object is moving in the clockwise direction around the unit circle $x^2 + y^2 = 1$. As it passes through the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$,its y −coordinate is decreasing at the rate of 3unit per second.The rate at which the$x$ −coordinate changes at this point is (in unit per second)

a) 2

b) $3\sqrt{3}$

c) $\sqrt{3}$

d) $2\sqrt{3}$

19. A spherical iron ball 10 cm in radius is coated with a layer of ice of uniform thickness that melts at a rate of 50 cm^2/min. When the thickness of ice is 15 cm. then the rate at which the thickness of ice decreases, is

a) $\frac{5}{6\pi}$ cm/min

b) $\frac{5}{54\pi}$ cm/min

c) $\frac{5}{18\pi}$ cm/min

d) $\frac{1}{36\pi}$ cm/min

20. If the rate of change of area of a square plate is equal to that of the rate of change of its perimeter, then length of the side is

a) 1 unit
b) 2 units
c) 3 units
d) 4 units

21. A particle moves on a line according to the law $s = at^2 + bt + c$. If the displacement after 1 sec is 16 cm, the velocity after 2 sec is 24 cm/sec and acceleration is $8\text{cm}/\sec^2$, then

a) $a = 4, b = 8, c = 4$
b) $a = 4, b = 4, c = 8$
c) $a = 8, b = 4, c = 4$
d) None of these

22. The maximum value $x^3 - 3x$ in the interval [0, 2] is

a) -2
b) 0
c) 2
d) 1

23. The function $f(x) = \cot^{-1} x + x$ increases in the interval

a) $(1, \infty)$
b) $(-1, \infty)$
c) $(-\infty, \infty)$
d) $(0, \infty)$

24. The equation of the normal to the curve $y = e^{-2|x|}$ at the point where the curve cuts the line $x = 1/2$, is

a) $2e(ex + 2y) = e^2 - 4$
b) $2e(ex - 2y) = e^2 - 4$
c) $2e(ey - 2x) = e^2 - 4$
d) None of these

25. If a particle moves according to the law $s = 6t^2 - \frac{t^3}{2}$, then the time at which it is momentarily at rest is

a) t=0 only

b) t=8 only

c) t=0,8

d) None of these

26. The length of subtangent to the curve $x^2y^2 = a^4$ at the point $(-a, a)$ is

a) 3a

b) 2a

c) A

d) 4a

27. A particle moves along a straight line with the law of motion given by $s^2 = at^2 + 2bt + c$. then the acceleration varies are

a) $\frac{1}{s^3}$

b) $\frac{1}{s}$

c) $\frac{1}{s^4}$

d) $\frac{1}{s^2}$

28. The maximum value of xy when $x + 2y = 8$ is

a) 20

b) 16

c) 24

d) 8

29. The length of the subtangent to the curve $x^2 + xy + y^2 = 7$ at $(1, -3)$ is

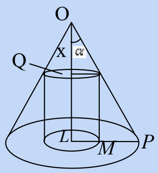
a) 3

b) 5

c) 3/5

d) 15

30. A given right circular cone has volume p and the largest right circular cylinder that can be inscribed in the cone has a volume$q$. Then, $p: q$ is



a) 9:4

b) 8:3

c) 7:2

d) None of the above

31. The angle between the curves $y = a^x$ and $y = b^x$ is equal to

a) $\tan^{-1}\left(\left|\frac{a-b}{1+ab}\right|\right)$

b) $\tan^{-1}\left(\left|\frac{a+b}{1-ab}\right|\right)$

c) $\tan^{-1}\left(\left|\frac{\log b+\log a}{1+\log a \log b}\right|\right)$

d) $\tan^{-1}\left(\left|\frac{\log a-\log b}{1+\log a \log b}\right|\right)$

32. If a and b are positive numbers such that $a > b$, then the minimum value of a sec $\theta - b\tan\theta \left(0 < \theta < \frac{\pi}{2}\right)$ is

a) $\frac{1}{\sqrt{a^2-b^2}}$

b) $\frac{1}{\sqrt{a^2+b^2}}$

c) $\sqrt{a^2+b^2}$

d) $\sqrt{a^2-b^2}$

33. Let y be the number of people in a village at time t. Assume that the rate of change of the population is proportional to the number of people in the village at any time and further assume that the population never increase in time. Then, the population of the village at any fixed t is given by

a) $y = e^{kt} + c$, for some constants $c \leq 0$ and $k \geq 0$

b) $y = ce^{kt}$, for some constants $c \geq 0$ and $k \leq 0$

c) $y = e^{ct} + k$, for some constants $c \leq 0$ and $k \geq 0$

d) $y = k\,e^{ct}$, for some constants $c \geq 0$ and $k \leq 0$

34. The function $f(x) = \cos(\pi/x)$ is increasing in the interval

a) $(2n+1, 2\,n), n \in N$

b) $\left(\frac{1}{2\,n+1}, 2\,n\right), n \in N$

c) $\left(\frac{1}{2\,n+2}, \frac{1}{2\,n+1}\right), n \in N$

d) None of these

35. If $f(x) = x^3 - 6x^2 + 9x + 3$ be a decreasing function, then x lies in

a) $(-\infty, -1) \cap (3, \infty)$

b) $(1, 3)$

c) $(3, \infty)$

d) None of these

36. If $0 < x < \frac{\pi}{2}$, then

a) $\cos(\sin x) > \cos x$

b) $\cos(\sin x) < \cos x$

c) $\cos(\sin x) = \sin(\cos x)$

d) $\cos(\sin x) < \sin(\cos x)$

37. The longest distance of the point $(a, 0)$ from the curve $2x^2 + y^2 - 2\,x = 0$, is given by

a) $\sqrt{1 - 2\,a + a^2}$

b) $\sqrt{1 + 2\,a + 2a^2}$

c) $\sqrt{1 + 2\,a - a^2}$

d) $\sqrt{1 - 2\,a + 2\,a^2}$

38. If PQ and PR are the two sides of a triangle, then the angle between them which gives maximum area of the triangle, is

a) π

b) $\frac{\pi}{3}$

c) $\frac{\pi}{4}$

d) $\frac{\pi}{2}$

39. If the curves $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $\frac{x^2}{l^2}-\frac{y^2}{m^2}=1$ cut each other orthogonally, then

a) $a^2+b^2=l^2+m^2$

b) $a^2-b^2=l^2-m^2$

c) $a^2-b^2=l^2+m^2$

d) $a^2+b^2=l^2-m^2$

40. Let the function $g:(-\infty,\infty)\to\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ be given by$g(u)=2\tan^{-1}(e^u)-\frac{\pi}{2}$. Then, g is

a) Even and is strictly increasing in $(0,\infty)$

b) Odd and is strictly decreasing in $(-\infty,\infty)$

c) Odd and is strictly increasing in $(-\infty,\infty)$

d) Neither even nor odd, but is strictly increasing in $(-\infty,\infty)$

41. Let $f(x)=1+2x^2+2^2x^4+\ldots\ldots+2^{10}x^{20}$.Then, $f(x)$ has

a) More than one minimum

b) Exactly one minimum

c) At least one maximum

d) None of these

42. Let $f(x)=\begin{cases}|x|, & for\ 0<|x|\le 2\\ 1, & for\ \ x=0\end{cases}$, then at $x=0, f$ has

a) A local maximum

b) A local minimum

c) No local extremum

d) No local maximum

43. All points on the curve $y^2=4a\left(x+a\sin\frac{x}{a}\right)$ at which the tangents are parallel to the axis of x lie on a

a) Circle

b) Parabola

c) Line

d) None of these

44. If $\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \cdots + \frac{a_{n-1}}{2} + a_n = 0$. Then the function $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$ has in $(0, 1)$

a) At least one zero

b) At most one zero

c) Only 3 zeroes

d) Only 2 zeroes

45. The length of the subtangent at any point (x_1, y_1)on the curve $y = a^x, (a > 0)$ is

a) $2 \log a$

b) $\frac{1}{\log a}$

c) $\log a$

d) $a^{2x_1} \log a$

46. Suppose the cubic $x^3 - px + q$ has three distinct real roots where $p > 0$ and $q > 0$. Then, which one of the following holds?

a) The cubic has maxima at both $\frac{p}{3}$ *and* $-\frac{p}{3}$

b) The cubic has minima at $\frac{p}{3}$ and maxima at $-\frac{p}{3}$

c) The cubic has minima at$-\frac{p}{3}$ and maxima at$\frac{p}{3}$

d) The cubic has minima at both $\frac{p}{3}$ *and* $-\frac{p}{3}$

47. ΔABC is not right angled and is inscribed in a fixed circle. If a, A, b, B be slightly varied keeping c, C fixed, then $\frac{da}{\cos A} + \frac{db}{\cos B} =$

a) 2R

b) π

c) 0

d) None of these

48. The equation(s) of the tangent(s) to the curve $y = x^4$ from the point (2, 0) not on the curve is given by

a) $y = \frac{4098}{81}$

b) $y - 1 = 5(x - 1)$

c) $y - \frac{4096}{81} = \frac{2048}{27}\left(x - \frac{8}{3}\right)$

d) $y - \frac{32}{243} = \frac{80}{81}\left(x - \frac{2}{3}\right)$

49. If $f(x) = \begin{cases} 3x^2 + 12\,x - 1, -1 \le x \le 2 \\ 37 - x, \;\; 2 < x \le 3 \end{cases}$, then

a) $f(x)$ is increasing in $[-1, 2]$

b) $f(x)$ is continuous in $[-1, 3]$

c) $f(x)$ is maximum at $x = 2$

d) All the above

50. If $a^2x^4 + b^2y^4 = c^6$, then maximum value of xy is

a) $\frac{c^2}{\sqrt{ab}}$

b) $\frac{c^3}{ab}$

c) $\frac{c^3}{\sqrt{2ab}}$

d) $\frac{c^3}{2ab}$

ANSWER KEY

1. D	26.C
2. D	27. A
3. B	28.D
4. A	29.D
5. D	30.A
6. A	31.D
7. A	32.D
8. D	33.B
9. C	34.D
10. A	35.B
11. C	36.A
12. C	37.D
13. B	38.D
14. A	39.C
15. A	40.C
16. B	41.B
17. B	42.A
18. B	43.B
19. C	44.A
20. B	45.B
21. A	46.B
22. C	47.C
23. C	48.C
24. B	49.D
25. C	50.C

HINTS AND SOLUTIONS

1. We have, $f(x)=\begin{vmatrix} x-1 & x+1 & 2x+1 \\ x+1 & x+3 & 2x+3 \\ 2x+1 & 2x-1 & 4x+1 \end{vmatrix} \Rightarrow f(x)=\begin{vmatrix} x-1 & x+1 & 2x+1 \\ 2 & 2 & 2 \\ 3 & -3 & -1 \end{vmatrix}$

   $\Rightarrow f(x)=\begin{vmatrix} x-1 & 2 & 3 \\ 2 & 0 & -2 \\ 3 & -6 & -7 \end{vmatrix}$ $\left[\begin{matrix}\text{Applying } C_2 \to C_2 - C_1 \\ C_3 \to C_3 - 2C_1\end{matrix}\right]$

   $\Rightarrow f(x) = -12(x-1) - 2(-14+6) + 3(-12) \Rightarrow f(x) = -12x - 8$

   Clearly, $f'(x) \neq 0$ for any $x \in R$.So, $f(x)$ has no point of maximum or minimum

2. $\because\ f'(x) = 2\left(\frac{1}{3}\right)\sin\left(\frac{x}{6}\right)\cos\left(\frac{x}{6}\right) + \left(\frac{1}{3}\right)\cos\frac{x}{3} - \left(\frac{1}{3}\right)$

   $= \left(\frac{1}{3}\right)\left[2\sin\left(\frac{x}{6}\right)\cos\left(\frac{x}{6}\right) - 2\sin^2\left(\frac{x}{6}\right)\right] = \left(\frac{2}{3}\right)\sin\left(\frac{x}{6}\right)\cos\left(\frac{x}{6}\right) - 2\sin\left(\frac{x}{6}\right)$

   Put $f'(x) = 0 \iff \sin\left(\frac{x}{6}\right) = 0 \Rightarrow \tan\left(\frac{x}{6}\right) = 1 \Rightarrow \frac{x}{6} = k\pi, k \in I$ or $\frac{x}{6} = n\pi + \frac{\pi}{4}$,

   $n \in I$ $\Rightarrow x^2 - 10 < -19.5\, x \iff (x+9.75)^2 < 105.0625 \iff (x-0.5)(x+20) < 0 \iff -20 < x < 0.5$

   So, the critical points satisfying the last inequality will be $0, 6\pi, -\frac{9\pi}{2}$

3. Let there be a value of k for which $x^3 - 3x + k = 0$ has two distinct roots between 0 and 1.Let a, b be two distinct roots of $x^3 - 3x + k = 0$ lying between 0 and 1 such that $a < b$.Let $f(x) = x^3 - 3x + k$. Then, $f(a) = f(b) = 0$.Since between any two roots of a polynomial $f(x)$ there exists at least one roots of its derivative $f'(x)$. Therefore, $f'(x) = 3x^2 - 3$ has at least one root between a and b. But, $f'(x) = 0$ has two roots equal to ± 1 which do not lie between a and b.Hence, $f(x) = 0$ has no real roots lying between 0 and 1 for any value of k

4. We have, $g(x) = f(x) - 2\{f(x)\}^2 + 9\{f(x)\}^3$ for all $x \in R$

   $\Rightarrow g'(x) = f'(x) - 4f(x)f'(x) + 27\{f(x)\}^2 f'(x)$ for all $x \in R$

   $\Rightarrow g'(x) = [1 - 4f(x) + 27\{f(x)\}^2]f'(x)$ for all $x \in R$

   Clearly, $27\{f(x)\}^2 - 4f(x) + 1$ is a quadratic expression with discriminant less than zero. So, its sign is same as that of the coefficient of $\{f(x)^2\}$ i.e. positive for all $x \in R$

   Thus, $g'(x)$ and $f'(x)$ have the same sign. Hence, $g(x)$ and $f(x)$ increase and decrease together

5. Given equation of curve is $y = x^2 - x + 4$

Slope of tangent at $P(1, 4)$ is

$$\left(\frac{dy}{dx}\right) = 2x - 1 \quad \Rightarrow \quad \left(\frac{dy}{dx}\right)_{(1,4)} = 1$$

$\therefore$ Equation of tangent is

$y - 4 = 1(x - 1) \Rightarrow \quad y - x = 3 \qquad$ ...(i)

And equation of normal at point $P(1, 4)$ is

$y - 4 = -1(x - 1) \;\Rightarrow \quad x + y = 5 \qquad$ ...(ii)

Since the tangent cuts x-axis at $A(-3, 0)$

And the normal cuts x-axis at $B(5, 0)$

$$\therefore \text{Area of } \Delta PAB = \frac{1}{2}\left|\begin{vmatrix} 1 & 4 & 1 \\ -3 & 0 & 1 \\ 5 & 0 & 1 \end{vmatrix}\right| = \frac{1}{2}\,|[-4(-3-5)]| = 16 \text{ sq units}$$

6. DO IT BY YOURSELF

7. Let r be the radius of the cylinder. Let V_1 and V_2 be the volumes of the sphere and cylinder respectively. Then, $V_1 = \frac{4}{3}\pi r^3$ and $V_2 = \pi r^3 \quad [\because h = r]$

$$\Rightarrow \frac{dV_1}{dt} = 4\pi r^2 \frac{dr}{dt} \text{ and } \frac{dV_2}{dt} = 3\pi r^2 \frac{dr}{dt} \Rightarrow \frac{\frac{dV_1}{dt}}{\frac{dV_2}{dt}} = 4 : 3$$

8. Given function

$f(x) = e^{x^2} + e^{-x^2}$, $g(x) = xe^{x2} + e^{-x2}$ and $h(x) = x^2e^{x^2} + e^{-x^2}$ Are strictly increasing on [0, 1]. Hence, at $x = 1$, the given function attains absolute maximum all equal to $e + \frac{1}{e}$

9. Given , $t = as^2 + bs + c$

$\Rightarrow 1 = 2as\frac{ds}{dt} + b\frac{ds}{dt}$ [differentiating]

$\Rightarrow 1 = 2asv + bv$ …(i)

$\Rightarrow 0 = 2a\frac{ds}{dt}v + 2as\frac{dv}{dt} + b\frac{dv}{dt}$ [differentiating]

$\Rightarrow \frac{dv}{dt}(2as + b) = -2av^2 \Rightarrow \frac{dv}{dt}\left(\frac{1}{v}\right) = -2av^2$ [from Eq.(i)]

$\Rightarrow \frac{dv}{dt} = -2av^3$

10. $f(x)=\begin{cases}|x^3+x^2+3x+\sin x|\left(3+\sin\left(\frac{1}{x}\right)\right), & x\neq 0\\ 0, & x=0\end{cases}$

Let $g(x)=x^3+x^2+3x+\sin x$

$g'(x)=3x^2+2x+3+\cos x$

$=3\left(x^2+\frac{2x}{3}+1\right)+\cos x=3\left\{\left(x+\frac{1}{3}\right)^2+\frac{8}{9}\right\}+\cos x>0$

And $2<3+\sin\left(\frac{1}{x}\right)<4$

Hence, minimum value of $f(x) is\ 0\ at\ x=0$. Hence, number of points=1

11. Let x and y be a two parts. Then,

$x+y=12$

Let $P=x^2y^4$

$\Rightarrow \quad \sqrt{p}=xy^2=L$ [say]

$\Rightarrow \quad L=x(12-x)^2 \Rightarrow \quad \frac{dL}{dx}=(12-x)^2-2x(12-x)$

For maxima, pot $\frac{dL}{dx}=0$

$\Rightarrow \quad (12-x)[12-3x]=0 \Rightarrow \quad x=12, x=4$

At $x=4$, it is maximum ($\because x\neq 12$)

Hence, it is maximum, when the parts are 4, 8

12. We have,

$x^{1/3}+y^{1/3}=a^{1/3}$

$\Rightarrow \frac{1}{3}x^{-2/3}+\frac{1}{3}y^{-2/3}\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\left(\frac{x}{y}\right)^{-2/3}=-\left(\frac{y}{x}\right)^{2/3} \Rightarrow \left(\frac{dy}{dx}\right)_P=-1$

The equation of the tangent at P is

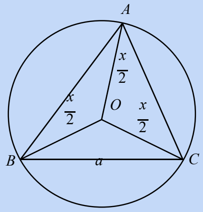
$y-\frac{a}{8}=-1\left(x-\frac{a}{8}\right) \Rightarrow x+y=\frac{a}{4}$

This cuts intercepts $\frac{a}{4}$ and $\frac{a}{4}$ with each of the coordinate axes

$\therefore \frac{a^2}{16}+\frac{a^2}{16}=2 \Rightarrow a^2=16 \Rightarrow a=4$

13. We have, $\frac{x}{2} = \frac{a}{2\sin A}$ [Using : $R = \frac{a}{2\sin A}$]

$\Rightarrow a = x\sin A$

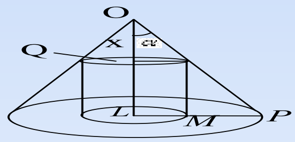


$\Rightarrow \frac{da}{dt} = x\cos A\frac{dA}{dt} \Rightarrow \frac{x}{2}\frac{dA}{dt} = x\cos A\frac{dA}{dt}$ $\left[\frac{da}{dt} = \frac{x}{2}\frac{dA}{dt}\text{(given)}\right]$

$\Rightarrow \cos A = \frac{1}{2} \Rightarrow A = \frac{\pi}{3}$

14. After time t the distance covered by X is $4t$ and Y is $3t$.

Let shortest distance between X and Y is A. Then, by cosine law



$A^2 = (4t)^2 + (3t)^2 - (4t)(3t)2\cos 120°$

$\Rightarrow \quad A^2 = 16t^2 + 9t^2 - 24t^2\left(-\frac{1}{2}\right) = 37t^2 \quad ....(i) \Rightarrow \quad A = \sqrt{37t}$

If $t = 1h$, then $A = \sqrt{37}km$.Now, differentiating Eq.(i)w. r. t. t,we get

$2AA'' = 37(2t)$. After $t = 1h$, we get $2\sqrt{37}A'' = 2(37)$

$\Rightarrow \quad A'' = \sqrt{37}km/h$

15. Given , $s = 16 - 2t + 3t^3$

$\Rightarrow \frac{ds}{dt} = -2 + 9t^2 \quad \Rightarrow \frac{d^2s}{dt^2} = 18t$

At $t = 2s$, acceleration a$= \frac{d^2s}{dt^2} = 18 \times 2 = 36m/s^2$

16. We have,$f(x) = x^3 + 3(a-7)x^2 + 3(a^2-9)x - 1 \Rightarrow f'(x) = 3x^2 + 6x(a-7) + 3(a^2-9)$

If the function $f(x)$ has a positive point of minimum, then $f'(x) = 0$ must have positive roots$\Rightarrow x = 0$ is less than the roots of $f'(x) = 0 \Rightarrow f'(0) > 0$ and Discriminant $\geq 0 \Rightarrow 4(a-7)^2 - 4(a^2-9) \geq 0$ and $3(a^2-9) > 0 \Rightarrow 7a - 29 \leq 0$ and $a^2 - 9 > 0 \Rightarrow a < \frac{29}{7}$ and $(a < -3$ or $a > 3) \Rightarrow a \in (-\infty, -3) \cup (3, 29/7)$

17. Let $P(x, y$ be a point of the curve

$$x = a\sin t - b\sin\left(\frac{at}{b}\right), y = a\cos t - b\cos\left(\frac{at}{b}\right)$$

and, O be the origin. Then, $OP^2 = \left\{a\sin t - b\sin\left(\frac{at}{b}\right)\right\}^2 + \left\{a\cos t - b\cos\left(\frac{at}{b}\right)\right\}^2$

$$\Rightarrow OP^2 = a^2 + b^2 - 2ab\cos\left(\frac{a-b}{b}\right)t$$

Clearly, OP^2 is maximum when $\cos\left(\frac{a-b}{t}\right)t$ minimum $i.e.$ equal to -1

In that case, we have $OP^2 = a^2 + b^2 + 2ab = (a+b)^2 \Rightarrow OP = a + b$

18. Let x be the length of a side of the triangle. Then, its area A is given by

$$A = \frac{\sqrt{3}}{4}x^2 \Rightarrow \frac{dA}{dx} = \frac{\sqrt{3}}{2}x \therefore \Delta A = \frac{dA}{dx}\Delta x \Rightarrow \Delta A = \frac{\sqrt{3}}{2}x\,\Delta x$$

$$\Rightarrow \frac{\Delta A}{A} \times 100 = \frac{\frac{\sqrt{3}}{2}x\,\Delta x}{\frac{\sqrt{3}}{4}x^2} \times 100 = 2\left(\frac{\Delta x}{x} \times 100\right) = 2k$$

19. Given, $\frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = -50 \Rightarrow \frac{dr}{dt} = -\frac{50}{4\pi r^2} \Rightarrow \left(\frac{dr}{dt}\right)_{r=15} = -\frac{50}{4\pi \times 225} = -\frac{1}{18\pi}$ cm/min.Hence , the thickness of ice decrease by $1/18\pi$ cm/min

20. Let at any point t, the length of a side of the square be x. Then,

$A =$ Area $= x^2$ and $P =$ Perimeter $= 4x$

$$\Rightarrow \frac{dA}{dt} = 2x\frac{dx}{dt} \text{ and } \frac{dP}{dt} = 4\frac{dx}{dt}$$

It is given that $\frac{dA}{dt} = \frac{dP}{dt} \Rightarrow 2x\frac{dx}{dt} = 4\frac{dx}{dt} \Rightarrow x = 2$

21. We have,

$s = at^2 + bt + c$ ...(i)

$\Rightarrow \frac{ds}{dt} = 2at + b$ ...(ii)

$\Rightarrow \frac{d^2s}{dt^2} = 2a$ ....(iii)

At $t = 1$, we have $s = 16$

$\therefore a + b + c = 16$

At $t = 2$, we have

$\frac{ds}{dt} = 24$ and $\frac{d^2s}{dt^2} = 8 \Rightarrow 2a + b = 24$ and $2a = 8$

Solving these equations, we get

$a = 4, b = 8$ and $c = 4$

22. Given, $f(x) = x^3 - 3x$

$\therefore \quad f'(x) = 3x^2 - 3$

For maxima, $f'(x) = 6x$

$\Rightarrow \quad 3x^2 - 3 = 0 \quad \Rightarrow \quad x = \pm 1$

$\because \quad x = 1 \in [0, 2]$

At $x = 1$, $f''(x) > 0$, minima

$f(0) = 0$, $f(1) = -2$ and $f(2) = 2$

Hence, maximum value is 2

23. We have, $f(x) = \cot^{-1} x + x \Rightarrow f'(x) = \frac{-1}{1+x^2} + 1 = \frac{x^2}{1+x^2}$

Clearly, $f'(x) > 0$ for all x.So, $f(x)$ increases in $(-\infty, \infty)$

24. We have, $y = e^{-2|x|} = \begin{cases} e^{2x}, x < 0 \\ e^{-2x}, x \geq 0 \end{cases}$

The line $x = 1/2$ cuts this curve at the point $P(1/2, e^{-1})$

Also, $\frac{dy}{dx} = \begin{cases} 2e^{2x}, x < 0 \\ -2e^{-2x}, x > 0 \end{cases} \therefore \left(\frac{dy}{dx}\right)_{x=1/2} = -\frac{2}{e}$

The equation of the normal at P is

$$y - \frac{1}{e} = \frac{e}{2}\left(x - \frac{1}{2}\right)$$

$\Rightarrow 4(ey - 1) = e^2(2x - 1) \Rightarrow 2e(ex - 2y) = e^2 - 4$

25. We have,

$s = 6t^2 - \frac{t^3}{2} \Rightarrow \frac{ds}{dt} = 12t - \frac{3t^2}{2}$ and $\frac{d^2s}{dt^2} = 12 - 3t$

When the particle is momentarily at rest, we have

$\frac{ds}{dt} = 0$ and $\frac{d^2s}{dt^2} \neq 0$

Now, $\frac{ds}{dt} = 0 \Rightarrow t = 0, t = 8$

Clearly, $\frac{d^2s}{dt^2} \neq 0$ for $t = 0, 8$

26. Equation of the curve is $x^2y^2 = a^4$

On differentiating w.r.t. x, we get

$$x^2 2y\frac{dy}{dx} + y^2 2x = 0 \Rightarrow \frac{dy}{dx} = \frac{-y}{x} \Rightarrow \left(\frac{dy}{dx}\right)_{(-a,a)} = -\left(\frac{a}{-a}\right) = 1$$

Therefore, length of subtangent at the point $(-a, a) = \frac{y}{\left(\frac{dy}{dx}\right)} = \frac{a}{1} = a$

27. Given, s=$\sqrt{at^2 + 2bt + c}$

$$\Rightarrow \frac{ds}{dt} = \frac{2at+2b}{2\sqrt{at^2+2bt+c}} = \frac{at+b}{\sqrt{at^2+2bt+c}} \Rightarrow \frac{d^2s}{dt^2} = \frac{\left[\frac{\sqrt{at^2+2bt+c+a} - (at+b)\frac{(2at+2b)}{2\sqrt{at^2+2bt+c}}}{\left(\sqrt{at^2+2bt+c}\right)^2}\right]}{\left(\sqrt{at^2+2bt+c}\right)^2}$$

$$\Rightarrow \frac{d^2}{dt^2} = \frac{ac-b^2}{s^3} \Rightarrow \text{Acceleration} \propto \frac{1}{s^3}$$

28. Given, $y = \frac{8-x}{2}$

Let $p = xy = \frac{8x-x^2}{2}$

$$\Rightarrow \quad \frac{dp}{dx} = \frac{1}{2}(8 - 2x)$$

For maxima or minima, put $\frac{dp}{dx} = 0$

$\therefore \quad 8 - 2x = 0 \Rightarrow \quad x = 4$

Again, $\frac{d^2p}{dx^2} = -1 \Rightarrow \quad \left(\frac{d^2p}{dx^2}\right) = -1 < 0$

Thus, function is maximum at $x = 4$ and $y = 2$

Therefore, maximum value of $p = 4 \times 2 = 8$

29. Given curve is $x^2 + xy + y^2 = 7$

$$\Rightarrow 2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{(2x+y)}{x+2y} \Rightarrow \left(\frac{dy}{dx}\right)_{(1,-3)} = \frac{-(2-3)}{(1-6)} = -\frac{1}{5}$$

$\therefore$ Length of subtangent= $\frac{y}{\frac{dy}{dx}} = \frac{-3}{\frac{1}{-5}} = 15$

30. Let H be the height of the cone and α be its semi-vertical angle. Suppose that x is the radius of the inscribed cylinder and h be its height

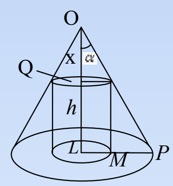
$\therefore \quad h = QL = OL - OQ = H - x\cot\alpha$

V =Volume of the cylinder= $\pi x^2(H - x\cot\alpha)$

Also, $p = \frac{1}{3}\pi(H\tan\alpha)^2 H \quad \ldots$(i)

$\frac{dV}{dx} = \pi(2Hx - 3x^2\cot\alpha) \quad \therefore \frac{dV}{dx} = 0 \quad \Rightarrow \quad x = 0,$

$x = \frac{2}{3}H\tan\alpha \Rightarrow \left.\frac{d^2V}{dx^2}\right|_{x=\frac{2}{3}H\tan\alpha} = -2\pi H < 0$



$\therefore \quad V$ is maximum when $x = \frac{2}{3}H\tan\alpha$

And $q = V_{\max} = \pi\frac{4}{9}H^2\tan^2\alpha\frac{1}{3}H = \frac{4}{9}p$ [from Eq. (i)]

Hence, $p:q = 9:4$

31. The point of intersection of given curve is (0, 1). On differentiating given curves, we get $\frac{dy}{dx} = a^x\log a, \frac{dy}{dx} = b^x\log b \Rightarrow m_1 = a^x \log a, m_2 = b^x\log b$

At (0,1) m_1 =log a, m_2 =log b

$\therefore \quad \tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} \Rightarrow \theta = \tan^{-1}\left|\frac{\log a - \log b}{1 + \log a \log b}\right|$

32. Let $y = a\sec\theta - b\tan\theta$

$\Rightarrow \quad \frac{dy}{d\theta} = a\,sec\theta tan\theta - bsec^2\theta$

Put $\frac{dy}{d\theta} = 0 \Rightarrow sec\theta(a\,tan\theta - bsec\theta) = 0$

$\Rightarrow sin\theta = \frac{b}{a} \qquad (\because sec\theta \neq 0)$

Now, $\frac{d^2y}{d\theta^2} > 0, at\ sin\theta = \frac{b}{a}$

$\therefore$minimum value is $y = a\frac{a}{\sqrt{a^2-b^2}} - b\frac{b}{\sqrt{a^2-b^2}} = \sqrt{a^2 - b^2}$

33. Given, $\frac{dy}{dt} \propto y, where$ y is the position of village $\Rightarrow \frac{1}{y}dy = k\,dt$

$\Rightarrow \log y = \log c + kt$ [on integrating]$\Rightarrow \log\frac{y}{c} = kt \Rightarrow y = ce^{kt}$

34. We have, $f(x)=\cos\left(\frac{\pi}{x}\right) \Rightarrow f'(x)=\frac{\pi}{x^2}\sin\left(\frac{\pi}{x}\right)$

For $f(x)$ to be increasing, we must have $f'(x)>0$

$\Rightarrow \frac{\pi}{x^2}\sin\left(\frac{\pi}{x}\right)>0 \Rightarrow \sin\left(\frac{\pi}{x}\right)>0 \Rightarrow 2\,n\,\pi<\frac{\pi}{x}<(2\,n+1)\,\pi$

$\Rightarrow \frac{1}{2\,n}>x>\frac{1}{2\,n+1} \Rightarrow x\in\left(\frac{1}{2\,n+1},\frac{1}{2\,n}\right)$

35. $\because\ \ f(x)=x^3-6x^2+9x+3$

On differentiating w.r.t. x, we get

$f'(x)=3x^2-12x+9 \Rightarrow f'(x)=3(x^2-4x+3)$

For decreasing, $f'(x)<0$

$\Rightarrow (x-3)(x-1)<0,$

$\therefore\ \ x\in(1,3)$

36. We know that $\cos x$ is decreasing on $(0,\pi/2)$ and $\sin x<x$ for $0<x<\frac{\pi}{2}$

$\therefore \cos(\sin x)>\cos x$ for $0<x<\frac{\pi}{2}$

Also, $0<\cos x<1<\frac{\pi}{2}$ for $0<x<\frac{\pi}{2}$ and, $\sin x<x$ for $0<x<\frac{\pi}{2}$

$\therefore \sin(\cos x)<\cos x$ for $0<x<\frac{\pi}{2}$

Hence, $\cos(\sin x)>\cos x$ and $\sin(\cos x)<\cos x$ for $0<x<\frac{\pi}{2}$

37. Let (x,y) be the point on the curve $2x^2+y^2-2x=0$. Then its distance from $(a,0)$ is given by $S=\sqrt{(x-a)^2+y^2}$

$\Rightarrow S^2=x^2-2\,ax+a^2+2\,x-2\,x^2$ [Using $2\,x^2+y^2-2\,x=0$]

$\Rightarrow S^2=-x^2+2\,x(1-a)+a^2$ ...(i) $\Rightarrow 2\,S\,\frac{dS}{dx}=-2\,x+2(1-a)$

For S to be maximum, we must have,

$\frac{dS}{dx}=0 \Rightarrow -2x+2(1-a)=0 \Rightarrow x=1-a$

It can be easily checked that $\frac{d^2S}{dx^2}<0$ for $x=1-a$

Hence, S is maximum for $x=1-a$

Putting $x=1-a$ in (i), we get $S=\sqrt{1-2\,a+2\,a^2}$

38. Let $PQ = a$ and $PR = b$, then $\Delta = \frac{1}{2}ab\sin\theta \because -1 \le \sin\theta \le 1$

$\therefore$ Area is maximum when $\sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

39. The two curves are

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \ldots(\text{i}) \quad$ and, $\frac{x^2}{l^2} - \frac{y^2}{m^2} = 1 \quad \ldots(\text{ii})$

Differentiating with respect to x, we get

$$\left(\frac{dy}{dx}\right)_{C_1} = -\frac{b^2}{a^2}\frac{x}{y}, \left(\frac{dy}{dx}\right)_{C_2} = \frac{m^2}{l^2}\frac{x}{y}$$

The two curves intersect orthogonally, iff.

$$\left(\frac{dy}{dx}\right)_{C_1}\left(\frac{dy}{dx}\right)_{C_2} = -1 \Rightarrow -\frac{b^2}{a^2}\frac{x}{y} \times \frac{m^2}{l^2}\frac{x}{y} = -1$$

$\Rightarrow m^2b^2x^2 = a^2l^2y^2 \quad \ldots(\text{iii})$

Subtracting (ii) from (i), we get

$x^2\left(\frac{1}{a^2} - \frac{1}{l^2}\right) + y^2\left(\frac{1}{b^2} + \frac{1}{m^2}\right) = 0 \quad \ldots(\text{iv})$

From (iii) and (iv), we get

$$\frac{1}{m^2b^2}\left(\frac{1}{a^2} - \frac{1}{l^2}\right) = -\frac{1}{a^2l^2}\left(\frac{1}{b^2} + \frac{1}{m^2}\right)$$

$\Rightarrow l^2 - a^2 = -b^2 - m^2 \Rightarrow a^2 - b^2 = l^2 + m^2$

40. Given, $g(u) = 2\tan^{-1}(e^u) - \frac{\pi}{2}, for\ u \in (-\infty, \infty)$

Now$g(-u) = 2\tan^{-1}(e^{-u}) - \frac{\pi}{2} = 2(\cot^{-1}(e^u)) - \frac{\pi}{2} = 2\left(\frac{\pi}{2} - \tan^{-1}(e^u)\right) - \frac{\pi}{2}$

$= -g(u) \therefore g(u)$ Is an odd function. Also, $g'(u) = 2\frac{1}{1+(e^u)^2}.e^u - 0 > 0$

Which is strictly increasing in $(-\infty, \infty)$

41. $f(x) = 1 + 2x^2 + 2^2x^4 + \ldots + 2^{10}x^{20}$

$f'(x) = 4x + 4.2^2x^3 + \ldots + 20.2^{10}x^{19} = x(4 + 4.2^2x^2 + \ldots + 20.2^{10}x^{18})$

For a maximum or minimum, put $f'(x) = 0$

$\Rightarrow \quad x = 0$

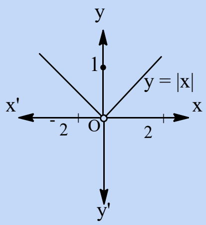
But $4 + 12.2^2 + x^2 + \cdots + 20.19.2^{10}x^{18} > 0$

For $x < 0 \Rightarrow f'(x) < 0$ and $x > 0 \Rightarrow f'(x) > 0$

$\therefore$ Exactly one minimum.

42. Given, $f(x) = \begin{cases} |x|, & \text{for } 0 < |x| \le 2 \\ 1, & \text{for } x = 0 \end{cases}$

It is clear from the graph that $f(x)$ has local maximum.



43. We have, $y^2 = 4a\left(x + a\sin\frac{x}{a}\right)$ …(i)

$$\Rightarrow 2y\frac{dy}{dx} = 4a\left(1 + \cos\frac{x}{a}\right)$$

For points at which the tangents are parallel to x-axis, we must have $\frac{dy}{dx} = 0$

$$\Rightarrow 4a\left(1 + \cos\frac{x}{a}\right) = 0 \Rightarrow \cos\frac{x}{a} = -1 \Rightarrow \frac{x}{a} = (2n+1)\pi$$

For these values of x, we have $\sin\frac{x}{a} = 0$

Putting $\sin\frac{x}{a} = 0$ in (i), we get $y^2 = 4ax$

Therefore, all these points lie on the parabola $y^2 = 4ax$

44. Consider the function $\phi(x) = a_0\frac{x^{n+1}}{n+1} + a_1\frac{x^n}{n} + a_2\frac{x^{n-1}}{n-1} + \cdots + a_{n-1}\frac{x^2}{2} + a_n x$

Since $\phi(x)$ is a polynomial. Therefore, it is continuous on $[0,1]$ and differentiable on $(0,1)$. Also, $\phi(0) = 0$ and, $\phi(1) = \frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-2} + \cdots + a_n = 0$ [Given]

$\therefore \phi(0) = \phi(1)$. Thus, $\phi(x)$ satisfies conditions of Rolle's theorem on $[0,1]$

Consequently, there exist $c \in (0,1)$ such that $\phi'(c) = 0$ i.e. $c \in (0,1)$ is a zero of $\phi'(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = f(x)$

45. Given, $y = a^x \Rightarrow \frac{dy}{dx} = a^x \log a$

Now, length of subtangent at any point $= \frac{y}{dy/dx} = \frac{a^{x_1}}{a^{x_1}\log a} = \frac{1}{\log a}$

46. Let $f(x) = x^3 - px + q$. Then $f'(x) = 3x^2 - p$

Put $f'(x) = 0 \Rightarrow x = \sqrt{\frac{p}{3}}, -\sqrt{\frac{p}{3}}$

Now, $f''(x) = 6x$

$\therefore$ At $x = \sqrt{\frac{p}{3}, \ f''(x) = 6\sqrt{\frac{p}{3}} > 0,}$ minima And at $= -\sqrt{\frac{p}{3}}, \ f''(x) < 0$, maxima

47. DO IT YOURSELF.

48. Suppose the tangent from the point (2, 0) to $y = x^4$ touches the curve at (x_1, y_1).

The equation of the tangent at (x_1, y_1) is $y - y_1 = 4x_1^3(x - x_1)$

If it passes through (2, 0), then $0 - y_1 = 4x_1^3(2 - x_1)$

$\Rightarrow y_1 = 4x_1^3(x_1 - 2) \Rightarrow x_1^4 = 4x_1^3(x_1 - 2)$ [$\because$ (x_1, y_1) lies on $y = x^4$ $\therefore y_1 = x_1^4$]

$\Rightarrow 3x_1^4 - 8x_1^3 = 0 \Rightarrow x_1^3(3x_1 - 8) = 0 \Rightarrow x_1 = 0$ or, $x_1 = 8/3$

Now, $x_1 = 0$, and $y_1 = x_1^4 \Rightarrow y_1 = 0$

$x_1 = 8/3$, and $y_1 = x_1^4 \Rightarrow y_1 = \left(\frac{8}{3}\right)^4 = \frac{4096}{81}$

Hence, the equations of the tangents are

$y = 0$ and $y = -\frac{4096}{81} = \frac{2048}{27}\left(x - \frac{8}{3}\right)$

49. We have,

$$f(x) = \begin{cases} 3x^2 + 12x - 1, & -1 \le x \le 2 \\ 37 - x, & 2 < x \le 3 \end{cases}$$

Clearly, $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$

So, $f(x)$ is continuous at $x = 2$

Hence, it is continuous on $[-1, 3]$

Thus, option (b) is correct

We find that

$f'(x) = 6x + 12 > 0$ for all $x \in [-1, 2]$

$\Rightarrow f(x)$ is increasing on $[-1, 2]$

Thus, option (a) is correct

Also,

$f'(x) < 0$ for all $x \in (2, 3]$

$\Rightarrow f(x)$ is decreasing on (2, 3]

Hence, $f(x)$ is attains the maximum value at 2

So, option (c) is correct

50. $y = \left(\frac{c^6 - a^2x^4}{b^2}\right)^{\frac{1}{4}}$

Let $f(x) = xy = \left(\frac{c^6x^4 - a^2x^8}{b^2}\right)^{\frac{1}{4}} \Rightarrow f'(x) = \frac{1}{4}\left(\frac{c^6x^4 - a^2x^8}{b^2}\right)^{-3/4}$

$\left(\frac{4x^3c^6}{b^2} - \frac{8x^7a^2}{b^2}\right)$

Put $f'(x) = 0 \Rightarrow \quad x = \pm \frac{c^{\frac{3}{2}}}{2^{\frac{1}{4}}\sqrt{a}}$

$\therefore \quad f\left(\frac{c^{3/2}}{c^{1/4}\sqrt{a}}\right) = \frac{c^3}{\sqrt{2ab}}$