CONTINUITY AND DIFFERENTIABILITY

CHAPTER 05

CONTINUOUS FUNCTION

CONTINUOUS FUNCTION:

A function f(x) is said to be continuous at a point x = a of its domain, iff $\lim_{x\to a} f(x) = f(a)$ i.e

it must satisfy these three conditions:

i. f(a) exists. (a lies in the domain of f)

ii. $\lim_{x\to a} f(x)$exists. i.e, $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$

iii. $\lim_{x\to a^+} f(x) = f(a)$

POINTS TO REMEMBER:

i. Every constant function is continuous.

ii. Every identity function is continuous.

iii. Every rational function is always continuous.

iv. Every polynomial function is continuous.

v. Modulus function f(x) =|x| is continuous.

vi. All trigonometric functions are continuous in their domain.

ALGEBRA OF CONTINUOUS FUNCTIONS:

Theorem 1: Let f and g be two real functions continuous at a real number a, then

i. (f+g) is continuous at x=a.

ii. (f-g) is continuous at x=a.

iii. fg is continuous at x=a.

iv. $\frac{f}{g}$ is continuous at x=c provided that, g(x)≠0.

Theorem 2: Composition of f and g is continuous i.e. fog and gof are continuous.

DIFFERENTIABILITY AT A POINT:

Let f(x) be a real valued function defined on an open interval (a,b) and let c belongs to (a,b). Then, f(x) is said to be differentiable or derivable at x=c, iff

i. $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ exists finitely.

ii. $\lim_{x \to c^-} \frac{f(x)-f(c)}{x-c} = \lim_{x \to c^+} \frac{f(x)-f(c)}{x-c}$

iii. $\lim_{h \to 0} \frac{f(c-h)-f(c)}{-h} = \lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$

f(x) is differentiable at x=c => f(x) is continuous at x=c.

DIFFERENTIATION:

The process of finding derivative of a function is called the differentiation.

DERIVATIVES TO REMEMBER:

i. $\frac{d}{dx}$(constant) = 0

ii. $\frac{d}{dx}(x^n)$ = nx^{n-1}

iii. $\frac{dy}{dx}$(cx^n) = cnx^{n-1} where c is a constant

iv. $\frac{d}{dx}$(sinx) = cosx

v. $\frac{d}{dx}$(cosx) = -sinx

vi. $\frac{d}{dx}$(tanx) = sec^2x

vii. $\frac{d}{dx}$(cosecx) = -cosecx cotx

viii. $\frac{d}{dx}$(secx) = secx tanx

ix. $\frac{d}{dx}$(cotx) = - $cosec^2$x

x. $\frac{d}{dx}(e^x) = e^x$

xi. $\frac{d}{dx}(a^x) = a^x \log_e a$, a>0

xii. $\frac{d}{dx}(\log_e x) = \frac{1}{x}$, x>0

xiii. $\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a}$, a>0,a≠0

ALGEBRA OF DERIVATIVES:

i. *SUM AND DIFFERENCE RULE:*

$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$

ii. *PRODUCT RULE:*

$\frac{d}{dx}(u.v) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u)$

iii. *QUOTIENT RULE:*

$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$

where u and v are functions of x.

DERIVATIVE OF COMPOSITE FUNCTIONS:

Let y be a real valued function which is composite of two functions, say y = f(u) and u = g(x).

Then, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = f'(u) \cdot g'(x)$

i.e. $\frac{d}{dx}[f\{g(x)\}] = f'[g(x)].\ g'(x)$

DERIVATIVES OF IMPLICIT FUNCTIONS:

Let f(x,y) = 0 be an implicit function of x. Then, to find $\frac{dy}{dx}$ we first differentiate both sides of equation w.r.t. x and then take all terms involving $\frac{dy}{dx}$ on LHS and remaining terms on RHS to get required value.

DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS:

i. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$, $-1<x<1$

ii. $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$, $-1<x<1$

iii. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

iv. $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

v. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$, $|x|>1$

vi. $\frac{d}{dx}(\text{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$, $|x|>1$

DERIVATIVE OF A FUNCTION W.R.T. ANOTHER FUNCTION:

Let y = f(x) and z = g(x) be two given functions, we firstly differentiate both functions with respect to x separately and then put values in the following formulae

$$\frac{dy}{dx} = \frac{dy/dx}{dz/dx} \text{ or } \frac{dz}{dy} = \frac{dz/dx}{dy/dx}$$

DIFFERENTIATION OF LOGARITHMIC FUNCTION:

Suppose, given function is of the form $u(x)^{v(x)}$.In such cases, we take logarithm on both sides and use properties of logarithm to simplify it and then differentiate it.

DIFFERENTIATION OF PARAMETRIC FUNCTIONS:

If $x=\varphi(t)$ and $y=\omega(t)$,then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

DIFFERENTIATION OF INFINITE SERIES:

When the value of y is given as an infinite series, then the process to find the derivatives of such infinite series is called differentiation of infinite series. In this case, we use the fact that if one term is deleted from an infinite series, it remains unaffected to replace all terms except first form by y. Thus, we convert it into a finite series or function. Then we differentiate it to find the required value.

SECOND ORDER DERIVATIVE:

Let y=f(x) be a given function, then $\frac{dy}{dx} = f'$(x) is called the first derivative of y or f(x) and $\frac{d}{dx}$ $(\frac{dy}{dx})$ is called the second order derivative of y w.r.t. x and it is denoted by $\frac{d^2y}{dx^2}$ or y''.

ROLLE'S THEOREM:

If a function y=f(x) is defined on [a, b] and

i. function f is continuous in [a, b].
ii. function f is differentiable in(a,b).
iii. f(a) = f(b)

Then there exists at least one value c belongs to (a,b) such that f'(c) = 0.

LAGRANGE'S MEAN VALUE THEOREM:

If a function y=f(x) is defined on [a,b] and

i. function f is continuous in[a,b].
ii. function f is differentiable in(a,b).

Then, there exists at least one value c belongs to (a,b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Continuity and Differentiability

Continuous Function

Suppose f is a real function on a subset of the real numbers and let 'c' be a point in the domain of f.

Then f is continuous at c if $\lim_{x \to c} f(x) = f(c)$

A real function f is said to be continuous if it is continuous at every point in the domain of f.

eg: The function $f(x) = \frac{1}{x}, x \neq 0$ is continuous

Let 'c' be any non-zero real number, then $\lim_{x \to c} f(x) = \lim_{x \to c} \frac{1}{x} = \frac{1}{c}$. For $c = 0, f(c) = \frac{1}{c}$ So $\lim_{x \to c} f(x) = f(c)$ and hence f is continuous at every point in the domain of f.

Algebra of continuous functions

Suppose f and g are two real functions continuous at a real number c, then, $f+g$, $f-g$, $f.g$ and $\frac{f}{g}$ are continuous at $x=c[g(c) \neq 0]$.

Differentiability

Suppose f is a real function and c is a point in its domain. The derivative of f at c is $f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$

Every differentiable function is continuous, but the converse is not true.

Chain Rule

If $y=f(u)$, where $u=g(x)$ and if both $\frac{dy}{du}, \frac{du}{dx}$ exists, then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

Logarithmic differentiation

Let $y=f(x)=[u(x)]^{v(x)}$

$\log y = v(x) \log[u(x)]$

$\frac{1}{y} \cdot \frac{dy}{dx} = v(x) \frac{1}{u(x)} u'(x) + v'(x) \log[u(x)]$

$\frac{dy}{dx} = y\left[\frac{v(x)}{u(x)} u'(x) + v'(x) \log[u(x)]\right]$

e.g. : Let $y = a^x$ Then $\log y = x \log a$

$\frac{1}{y} \cdot \frac{dy}{dx} = \log a$

$\frac{dy}{dx} = y \log a = a^x \log a$.

Derivatives of Implicit functions

If two variables are expressed by some relation then one will be the implicit function of other, is called Implicit function.

For example: Let $y = \cos x - \sin y$, then $\frac{dy}{dx} = \frac{d}{dx}\cos x - \frac{d}{dx}\sin y$

or, $\frac{dy}{dx} = -\sin x - \cos y \cdot \frac{dy}{dx}$ or, $\frac{dy}{dx} = -\sin x / (1 + \cos y)$, where $y \neq (2n+1)\pi$

Derivatives of functions in parametric form

Let $x = f(t), y = g(t)$ be two functions of parameter 't'.

Then, $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$ or $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Thus, $\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$ (provided $f'(t) \neq 0$)

eg : if $x = a\cos\theta$, $y = a\sin\theta$ then $\frac{dx}{d\theta} = -a\sin\theta$ and $\frac{dy}{d\theta} = a\cos\theta$, and so $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = -\frac{a\cos\theta}{a\sin\theta} = -\cot\theta$.

Second order derivative

Let $y = f(x)$ then $\frac{dy}{dx} = f'(x)$, if $f'(x)$ is differentiable, then $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx} f'(x)$ i.e.,

$\frac{d^2y}{dx^2} = f''(x)$ is the second order derivative of y w.r.t. x.

eg : if $y = 3x^2+2$, then $y' = 6x$ and $y'' = 6$.

Some Standard derivatives

(i) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

(ii) $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$

(iii) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

(iv) $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

(v) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

(vi) $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$

(vii) $\frac{d}{dx}(e^x) = e^x$

(viii) $\frac{d}{dx}(\log x) = \frac{1}{x}$

Trace the Mind Map

- First Level
- Second Level
- Third Level

PRACTICE QUESTIONS:

1. Let $[x]$ denotes the greatest integer less than or equal to x and $f(x) = [\tan^2 x]$. Then,
   a) $\lim_{x\to 0} f(x)$ does not exist.
   b) $f(x)$ is continuous at $x = 0$
   c) $f(x)$ is not differentiable at $x = 0$
   d) $f'(0) = 1$
2. Let $f(x)$ be defined on R such that $f(1) = 2, f(2) = 8$ and $f(u+v) = f(u) + kuv - 2\,v^2$ for all $u, v \in R$ (k is a fixed constant). Then,
   a) $f'(x) = 8x$
   b) $f(x) = 8x$
   c) $f'(x) = x$
   d) None of these
3. If $f(x) = \frac{\log_e(1+x^2 \tan x)}{\sin x^3}, x \neq 0$, is to be continuous at $x = 0$, then $f(0)$ must be defined as
   a) 1
   b) 0
   c) ½
   d) -1
4. If for a function $f(x), f(2) = 3, f'(2) = 4$, then $\lim_{x\to 2}[f(x)]$, where $[\cdot]$ denotes the greatest integer function, is
   a) 2
   b) 3
   c) 4
   d) Non-existent

5. If $f(x)$ defined by $f(x) = \begin{cases} \frac{|x^2-x|}{x^2-x}, x \neq 0, 1 \\ 1, \ x = 0 \\ -1, \ x = 1 \end{cases}$ then $f(x)$ is continuous for all,

a) x

b) x except x=0

c) x except x=1

d) x except x=0 and x=1

6. On the interval $I = [-2, 2]$, the function $f(x) = \begin{cases} (x+1)\, e^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

a) Is continuous for all $x \in I - \{0\}$

b) Assumes all intermediate values from $f(-2)$ to $f(2)$

c) Has a maximum value equal to $\frac{3}{e}$

d) All the above

7. If $f(x) = \begin{cases} \frac{36^x-9^x-4^x+1}{\sqrt{2}-\sqrt{1+\cos x}}, x \neq 0 \\ k, \ x = 0 \end{cases}$ is continuous at $x = 0$, then k equals

a) $16\sqrt{2}\log 2 \log 3$

b) $16\sqrt{2}\ln 6$

c) $16\sqrt{2}\ln 2 \ln 3$

d) None of these

8. If $f(x) = \text{Min}\{\tan x, \cot x\}$, then

a) $f(x)$ is not differentiable at $x = 0, \frac{\pi}{4}, \frac{5\pi}{4}$

b) $f(x)$ is continuous at $x = 0, \frac{\pi}{2}, \frac{3\pi}{2}$

c) $\int_0^{\frac{\pi}{2}} f(x)dx = \ln\sqrt{2}$

d) $f(x)$ is periodic with period $\frac{\pi}{2}$

9. For the function $f(x) = \frac{\log_e(1+x)+\log_e(1-x)}{x}$ to be continuous at $= 0$, the value of $f(0)$ is

a) -1

b) 0

c) -2

d) 2

10. If $f(x) = \int_{-1}^{x} |t|\, dt, x \geq -1$, then

a) f and f' are continuous for $x + 1 > 0$

b) f is continuous but f' is not so for $x + 1 > 0$

c) f and f' are continuous at $x = 0$

d) f is continuous at $x = 0$ but f' is not so

11. The function $f(x) = |x| + |x - 1|$, is

a) Continuous at $x = 1$, but not differentiable

b) Both continuous and differentiable at $x = 1$

c) Not continuous at $x = 1$

d) None of these

12. If $f(x) = |\log_e x|$, then

a) $f'(1^+) = 1, f'(1^-) = -1$

b) $f'(1^-) = -1, f'(1^+) = 0$

c) $f'(1) = 1, f'(1^-) = 0$

d) $f'(1) = -1, f'(1^+) = -1$

13. At $x = 0$, the function $f(x) = |x|$ is

a) Continuous but not differentiable

b) Discontinuous and differentiable

c) Discontinuous and not differentiable

d) Continuous and differentiable

14. If $f(x) = \begin{cases} 1 + x, & 0 \leq x \leq 2 \\ 3 - x, & 2 < x \leq 3 \end{cases}$ then the set of points of discontinuity of $g(x) = fof(x)$, is

a) $\{1, 2\}$

b) $\{0, 1, 2\}$

c) $\{0, 1\}$

d) None of these

15. Let $f(x) = \begin{cases} \frac{x^4-5x^2+4}{|(x-1)(x-2)|}, x \neq 1, 2 \\ 6, \ x = 10 \\ 12, \quad x = 2 \end{cases}$ Then, $f(x)$ is continuous on the set

a) R

b) R-{1}

c) R-{2}

d) R-{1,2}

16. If $f(x) = \begin{cases} x \sin\frac{1}{x}, & x \neq 0 \\ k, & x = 0 \end{cases}$ is continuous at $x = 0$, then the value of k is

a) 1

b) -1

c) 0

d) 2

17. If function $f(x) = \begin{cases} x, \text{if } x \text{ is rational} \\ 1 - x, \ \text{if } x \text{ is irrational} \end{cases}$, then the number of points at which $f(x)$ is continuous, is

a) ∞

b) 1

c) 0

d) None of these

18. If $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$, then

a) f and f' are continuous at $x = 0$

b) f is derivable at $x = 0$ and f' is continuous at $x = 0$

c) f is derivable at $x = 0$ and f' is not continuous at $x = 0$

d) f' is derivable at $x = 0$

19. Let $f(x)$ be an odd function. Then $f'(x)$

a) Is an even function

b) Is an odd function

c) May be even or odd

d) None of these

20. Let $f(x)$ be an odd function. Then $f'(x)$

e) Is an even function

f) Is an odd function

g) May be even or odd

h) None of these

21. $f(x) = |x-3|$ is ... at $x = 3$

a) Continuous and not differentiable

b) Continuous and differentiable

c) Discontinuous and not differentiable

d) Discontinuous and differentiable

22. Let $f(x) = [x]$ and $g(x) = \begin{cases} 0, & x \in Z \\ x^2, & x \in R - Z \end{cases}$ Then, which one of the following is incorrect?

a) $\lim_{x\to 1} g(x)$ exists, but $g(x)$ is not continuous at $x = 1$

b) $\lim_{x\to 1} f(x)$ does not exist and $f(x)$ is not continuous at $x = 1$

c) gof is continuous for all x

d) fog is continuous for all x

23. The function $f(x) = \begin{cases} \frac{e^{1/x}-1}{e^{1/x}+1}, x \neq 0 \\ 0, \ x = 0 \end{cases}$

a) Is continuous at $x = 0$

b) Is not continuous at $x = 0$

c) Is not continuous at $x = 0$, but can be made continuous $x = 0$

d) None of these

24. The function $f(x) = |\cos x|$ is

a) Everywhere continuous and differentiable

b) Everywhere continuous and but not differentiable at $(2n+1)\frac{\pi}{2}, n \in Z$

c) Neither continuous nor differentiable at $(2n+1)\frac{\pi}{2}, n \in Z$

d) None of these

25. If $f(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}(\log_e a)^n, a>0, a\neq 0$, then at $x=0, f(x)$ is

a) Everywhere continuous but not differentiable

b) Everywhere differentiable

c) Nowhere continuous

d) None of these

26. If $f(x)$ is continuous function and $g(x)$ be discontinuous, then

a) $f(x)+g(x)$ must be continuous

b) $f(x)+g(x)$ must be discontinuous

c) $f(x)+g(x)$ for all x

d) None of these

27. Let $f(x)$ be a function satisfying $f(x+y)=f(x)f(y)$ for all $x,y\in R$ and $f(x)=1+x\,g(x)$ where $\lim_{x\to 0} g(x)=1$. Then, $f'(x)$ is equal to

a) $g'(x)$

b) $g(x)$

c) $f(x)$

d) None of these

28. The points of discontinuity of $\tan x$ are

a) $n\pi, n\in I$

b) $2n\pi, n\in I$

c) $(2n+1)\frac{\pi}{2}, n\in I$

d) None of these

29. Let $f(x)=\frac{\sin 4\pi[x]}{1+[x]^2}$, where $[x]$ is the greatest integer less than or equal to x, then

a) $f(x)$ is not differentiable at some points

b) $f'(x)$ exists but is different from zero

c) $f'(x)=0$ for all x

d) $f'(x)=0$ but f is not a constant function

30. If $f(9) = 9, f'(9) = 4$, then $\lim_{x \to 9} \frac{\sqrt{f(x)} - 3}{\sqrt{x} - 3}$ equals

a) 4

b) 0

c) C

d) 9

31. $\lim_{h \to 0} \frac{f(2h+2+h^2) - f(2)}{f(h-h^2+1) - f(1)}$, given that $f'(2) = 0$ and $f'(1) = 4$

a) does not exist

b) is equal to -3/2

c) is equal to 3/2

d) is equal to 3

32. The differential coefficient off(logx) w.r.t. x, where f(x)=logx is

a) $\frac{x}{\log x}$

b) $\frac{logx}{x}$

c) $(x \log x)^{-1}$

d) None of these

33. If x = $a cos^3\theta$, y= $a Sin^3\theta$, then $\sqrt{1 + (\frac{dy}{dx})^2} =$

a) $\tan^2\theta$

b) $sec^2\theta$

c) $\sec\theta$

d) $|\sec\theta|$

34. If $f(x) = x^n$, then the value of $f(1) - \frac{f'(1)}{1!} + \frac{f''(1)}{2!} - \frac{f'''(1)}{3!} + \cdots + \frac{(-1)^n f^n(1)}{n!}$ Is

a) 2^n

b) 2^{n-1}

c) 0

d) 1

35. Derivative of $\sec^{-1}\left(\frac{1}{1-2x^2}\right)$ w. r. t. $\sin^{-1}(3x-4x^3)$ is

a) $\frac{1}{4}$

b) $\frac{3}{2}$

c) 1

d) $\frac{2}{3}$

36. If $(x+y)\sin u = x^2y^2$, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ is equal to

a) $\sin u$

b) $\operatorname{cosec} u$

c) $2\tan u$

d) $3\tan u$

37. If $u = x^2 + y^2$ and $x = s + 3t, y = 2s - t$, then $\frac{d^2u}{ds^2}$ is equal to

a) 12

b) 32

c) 36

d) 10

38. If $f: R \to R$ is an even function which is twice differentiable on R and $f''(\pi) = 1$, then $f''(-\pi)$ is equal to

a) -1

b) 0

c) 1

d) 2

39. Let a function $y = y(x)$ be defined parametrically by $x = 2t - |t|, y = t^2 + t|t|$. Then, $y'(x), x > 0$

a) 0

b) 4x

c) 2x

d) Doesn't exist

40. If y= $\sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \cdots \infty}}}}$, then $\frac{dy}{dx}$ is equal to

a) $\frac{x}{2y-1}$

b) $\frac{x}{2y+1}$

c) $\frac{1}{x(2y-1)}$

d) $\frac{1}{x(1-2y)}$

41. The expression of $\frac{dy}{dx}$ of the function $y = a^{x^{a^{x \ldots \infty}}}$, is

a) $\frac{y^2}{x(1-y\log x)}$

b) $\frac{y^2 \log y}{x(1-y\log x)}$

c) $\frac{y^2 \log y}{x(1-y\log x \log y)}$

d) $\frac{y^2 \log y}{x(1+y\log x \log y)}$

42. If $f(x) = x^n + 4$, then the value of $f(1) + \frac{f'(1)}{1!} + \frac{f''(1)}{2!} + \ldots + \frac{f^n(1)}{n!}$ Is

a) 2^{n-1}

b) $2^n + 4$

c) $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$

d) None of these

43. Let y be an implicit function of x defined by $x^{2x} - 2x^x \cot y - 1 = 0$. Then $y'(1)$ equals

a) -1

b) 1

c) log 2

d) –log 2

44. The derivative of $\left[\frac{e^x+1}{e^x}\right]$ is equal to

a) 0

b) $\frac{1}{e^x}$

c) $-\frac{1}{e^x}$

d) e^x

45. $x = \frac{1-\sqrt{y}}{1+\sqrt{y}} \Rightarrow \frac{dy}{dx}$ is equal to

a) $\frac{4}{(x+1)^2}$

b) $\frac{4(x-1)}{(1+x)^3}$

c) $\frac{x-1}{(1+x)^3}$

d) $\frac{4}{(x+1)^3}$

46. If $y = \sin(\log_e x)$, then $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx}$ is equal to

a) $\sin(\log_e x)$

b) $\cos(\log_e x)$

c) y^2

d) $-y$

47. If $2^x + 2^y = 2^{x+y}$, then $\frac{dy}{dx}$ is equal to

a) $\frac{2^x+2^y}{2^x-2^y}$

b) $\frac{2^x+2^y}{1+2^{x+y}}$

c) $2^{x-y}\left(\frac{2^y-1}{1-2^x}\right)$

d) $\frac{2^{x+y}-2^x}{2^y}$

48. If $y = \sqrt{x + \sqrt{y + \sqrt{x + \sqrt{y + \cdots \infty}}}}$, then $\frac{dy}{dx}$ is equal to

a) $\frac{y+x}{y^2-2x}$

b) $\frac{y^3-x}{2y^2-2xy-1}$

c) $\frac{y^3+x}{2y^2-x}$

d) None of these

49. Observe the following statements:

I. If $f(x) = ax^{41} + bx^{-40}$, then $\frac{f''(x)}{f(x)} = 1640\ x^{-2}$

II. $\frac{d}{dx}\left\{\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right\} = \frac{1}{1+x^2}$

Which of the following is correct?

a) I is true, but II is false

b) Both I and II true

c) Neither I nor II is true

d) I is false, but II is true

50. If $y = f(x)$ and $y \cos x + x \cos y = \pi$, then the value of $f''(0)$ is

a) Π

b) –π

c) 0

d) 2π

ANSWER KEY

1. A
2. A
3. A
4. C
5. D
6. D
7. C
8. A
9. B
10. A
11. A
12. A
13. A
14. A
15. A
16. D
17. C
18. C
19. C
20. A
21. A
22. D
23. B
24. B
25. B

26.B
27. C
28.C
29.C
30.A
31.D
32.C
33.D
34.C
35.D
36.D
37.D
38.C
39.D
40.C
41.C
42.B
43.A
44.C
45.B
46.B
47.D
48.C
49.A
50.A

HINTS AND SOLUTIONS

1. We have, $-\pi/4 < x < \pi/4$

   $\Rightarrow -1 < \tan x < 1 \Rightarrow 0 \le \tan^2 x < 1 \Rightarrow [\tan^2 x] = 0$

   $\therefore f(x) = [\tan^2 x] = 0$ for all $x \in (-\pi/4, \pi/4)$

   Thus, $f(x)$ is a constant function on $\in (-\pi/4, \pi/4)$

   So, it is continuous on $\in (-\pi/4, \pi/4)$ and $f'(x) = 0$ for all $x \in (-\pi/4, \pi/4)$

2. We have,

   $f(u+v) = f(u) + kuv - 2v^2$ for all $u, v \in R$ ...(i)

   Putting $u = v = 1$, we get

   $f(2) = f(1) + k - 2 \Rightarrow 8 = 2 + k - 2 \Rightarrow k = 8$

   Putting $u = x, v = h$ in (i), we get

   $\frac{f(x+h)-f(x)}{h} = kx - 2h \Rightarrow \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = kx \Rightarrow f'(x) = 8x \quad [\because k = 8]$

3. At x=0, f(0)=$\frac{0}{0}$ [Indeterminate form]

   So applying L Hospital Rule we get

   f(0) = $\lim_{x\to 0} \frac{\log(1+x^2\tan x)}{\sin x^3}$

   $= \lim_{x\to 0} \frac{(2x\,tanx + x^2sec^2x)}{(1+x^2tanx)(3x^2cosx^3)} = \lim_{x\to 0} \frac{(2tanx + xsec^2x)}{(1+x^2tanx)x(3cosx^3)} = \lim_{x\to 0} \frac{\frac{2tanx}{x} + sec^2x}{3cosx^3(1+x^2tanx)}$

   [Since $\lim_{x\to 0} \frac{tanx}{x} = 1$ & $\lim_{x\to 0} sec^2 x = 1$]

   Therefore substituting x=0, we get(0) =$\frac{(2\times 1)+1}{3\times 1\times 1}$ =1

   So f(0)=1

4. $f'(2)$=4

   $\lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = 4$

   If h>0, then f(2^+)>2, [f(2+h)]=4 and If h<0, then f(2^-) < 2, [f(2-h)]=4

   Hence limit is 4.

5. We have, $f(x) = \begin{cases} \frac{x^2-x}{x^2-x} = 1, \text{ if } x < 0 \text{ or } x > 1 \\ -\frac{(x^2-x)}{x^2-x} = -1, \text{ if } 0 < x < 1 \\ 1, \text{ if } x = 0 \\ -1, \text{ if } x = 1 \end{cases} \Rightarrow f(x) = \begin{cases} 1, \text{if } x \leq 0 \text{ or } x > 1 \\ -1, \text{if } 0 < x \leq 1 \end{cases}$

Now, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0} 1 = 1$ and, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0} -1 = -1$

Clearly, $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$

So, $f(x)$ is not continuous at $x = 0$. It can be easily seen that it is not continuous at $x = 1$

6. We have, $f(x) = \begin{cases} (x+1)\, e^{-\left(\frac{1}{x}+\frac{1}{x}\right)} = (x+1), \; x < 0 \\ (x+1)\, e^{-\left(\frac{1}{x}+\frac{1}{x}\right)} = (x+1)e^{-2/x}, x > 0 \end{cases}$

Clearly, $f(x)$ is continuous for all $x \neq 0$

So, we will check its continuity at $x = 0$

We have,

(LHL at $x = 0$) $= \lim_{x\to 0^-} f(x) = \lim_{x\to 0}(x+1) = 1$

(RHL $at\ x = 0$) $= \lim_{x\to 0^+} f(x) = \lim_{x\to 0}(x+1)\; e^{-2/x} = \lim_{x\to 0} \frac{x+1}{e^{2/x}} = 0$

$\therefore \lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+ f(x)}$

So, $f(x)$ is not continuous at $x = 0$ Also, $f(x)$ assumes all values from $f(-2)$ to f(2) and $f(2) = 3/e$ is the maximum value of $f(x)$

7. For $f(x)$ to be continuous at $x = 0$, we must have

$\lim_{x\to 0} f(x) = f(0)$

$\Rightarrow \lim_{x\to 0} \frac{(9^x-1)(4^x-1)}{\sqrt{2}-\sqrt{2}\cos^2 x/2} = k \Rightarrow \lim_{x\to 0} \frac{(9^x-1)(4^x-1)}{\sqrt{2}.2\sin^2 x/4} = k$

$\Rightarrow \lim_{x\to 0} \frac{16\times\left(\frac{9^x-1}{x}\right)\left(\frac{4^x-1}{x}\right)}{2\sqrt{2}\left(\frac{\sin x/2}{x/4}\right)^2} = k \Rightarrow \frac{16}{2\sqrt{2}} \log 9 . \log 4 = k = 4\sqrt{2} \log 9 . \log 4 =$

$16\sqrt{2} \log 3 \log 2$

8. We have, $f(x) = \begin{cases} \tan x, & 0 \le x \le \pi/4 \\ \cot x, & -\pi/4 \le x \le \pi/2 \\ \tan x, & \pi/2 < x \le 3\pi/4 \\ \cot x, & 3\pi/4 \le x < \pi \end{cases}$

Since $\tan x$ and $\cot x$ are periodic functions with period π. So, $f(x)$ is also periodic with period π.It is evident from the graph that $f(x)$ is not continuous at $x = \pi/2$. Since $f(x)$ is periodic with period π. So, it is not continuous at $x = 0, \pm\pi/2, \pm\pi, \neq 3\pi/2$

Also, $f(x)$ is not differentiable $x = \pi/4, 3\pi/4, 5\pi/4$ etc

9. Since, the function $f(x)$ is continuous

$\therefore\ f(0) =$RHL $f(x) =$LHL$f(x)$

Now, RHL$f(X) = \lim_{h\to 0} \frac{\log(1+0+h)+\log(1-0-h)}{0+h} = \lim_{h\to 0} \frac{\log(1+h)+\log(1-h)}{h} = \lim_{h\to 0} \frac{\frac{1}{1+h}-\frac{1}{1-h}}{1} = 0$

[by L 'Hospital's rule]

$\therefore\ f(0) =$RHL $f(x) = 0$

10. If $-1 \le x < 0$, then

$$f(x) = \int_{-1}^{x} |t|\, dt = \int_{-1}^{x} -t\, dt = -\frac{1}{2}(x^2 - 1)$$

If $x \ge 0$, then

$$f(x) = \int_{-1}^{0} -t\, dt + \int_{-1}^{x} -t\, dt = \frac{1}{2}(x^2 + 1) \therefore f(x) = \begin{cases} -\frac{1}{2}(x^2 - 2), & -1 \le x < 0 \\ \frac{1}{2}(x^2 + 1), & 0 \le x \end{cases}$$

It can be easily seen that $f(x)$ is continuous at $x = 0$. So, it is continuous for all $x > -1$

Also, $Rf'(0) = 0 = Lf'(0)$. So, $f(x)$ is differentiable at $x = 0$

$$\therefore f'(x) = \begin{cases} -x, & -1 < x = 0 \\ 0, & x = 0 \\ x, & x > 0 \end{cases}$$

Clearly, $f'(x)$ is continuous at $x = 0$.Consequently, it is continuous for all $x > -1$ i.e. for $x + 1 > 0$.Hence, f and f' are continuous for $x + 1 > 0$

11. We have, $f(x) = |x| + |x-1| = \begin{cases} -2x+1, & x < 0 \\ 1, & 0 \le x < 1 \\ 2x-1, & 1 \le x \end{cases}$

Clearly, $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1} 1 = 1$, $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1}(2x-1) = 1$

and, $f(1) = 2 \times 1 - 1 = 1 \therefore \lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} f(x) = f(1)$

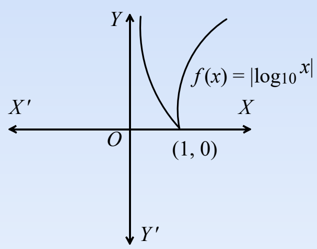
So, $f(x)$ is continuous at $x = 1$

Now, $\lim\limits_{x \to 1^-} \frac{f(x)-f(1)}{x-1} = \lim\limits_{h \to 0} \frac{f(1-h)-f(1)}{-h} = \lim\limits_{h \to 0} \frac{1-1}{-h} = 0$

and, $\lim\limits_{x \to 1^+} \frac{f(x)-f(1)}{x-1} = \lim\limits_{h \to 0} \frac{f(1+h)-f(1)}{h} \Rightarrow \lim\limits_{x \to 1^+} \frac{f(x)-f(1)}{x-1} = \lim\limits_{h \to 0} \frac{2(1+h)-1-1}{h} = 2$

$\therefore$ (LHD at $x = 1$) $\neq$ (RHD at $x = 1$)So, $f(x)$ is not differentiable at $x = 1$

12. As is evident from the graph of $f(x)$ that it is continuous but not differentiable at $x = 1$



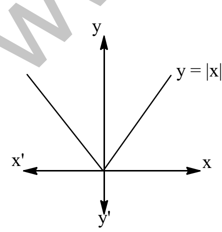
Now,$f''(1^+) = \lim\limits_{x \to 1^+} \frac{f(x)-f(1)}{x-1} \Rightarrow f''(1^+) = \lim\limits_{h \to 0} \frac{f(1+h)-f(1)}{h}$

$\Rightarrow f''(1^+) = \lim\limits_{h \to 0} \frac{\log_{10}(1+h)-0}{h} \Rightarrow f''(1^+) = \lim\limits_{h \to 0} \frac{\log(1+h)}{h.\log_e 10} = \frac{1}{\log_e 10} = \log_{10} e$

$f''(1^-) = \lim\limits_{x \to 1^+} \frac{f(x)-f(1)}{x-1} \Rightarrow f''(1^-) = \lim\limits_{h \to 0} \frac{f(1-h)-f(1)}{h}$

$\Rightarrow f''(1^-) = \lim\limits_{h \to 0} \frac{\log_{10}(1-h)}{h} = \lim\limits_{h \to 0} \frac{\log_e(1-h)}{h \log_e 10} = -\log_{10} e$

13.



It is clear from the graph that $f(x)$ is continuous everywhere and also differentiable everywhere except at $x = 0$

14. We have,

$$f(x)=\begin{cases}1+x, & 0\le x\le 2\\ 3-x, & 2<x\le 3\end{cases} \therefore g(x)=fof(x)\Rightarrow f(x)=f\big(f(x)\big)$$

$$\Rightarrow g(x)=\begin{cases}f(1+x), & 0\le x\le 2\\ f(3-x), & 2<x\le 3\end{cases}\Rightarrow g(x)=\begin{cases}1+(1+x), & 0\le x\le 1\\ 3-(1+x), & 1<x\le 2\\ 1+(3-x), & 2<x\le 3\end{cases}$$

$$\Rightarrow g(x)=\begin{cases}2+x, & 0\le x\le 1\\ 2-x, & 1<x\le 2\\ 4-x, & 2<x\le 3\end{cases}$$

Clearly, $g(x)$ is continuous in $(0,1)\cup(1,2)\cup(2,3)$ except possibly at $x=0,1,2$ and 3. We observe that $\lim\limits_{x\to 0^+} g(x)=\lim\limits_{x\to 0^+}(2+x)=2=g(0)$ and $\lim\limits_{x\to 3^-} g(x)=\lim\limits_{x\to 3^-} 4-x=1=g(3)$. Therefore, $g(x)$ is right continuous at $x=0$ and left continuous at $x=3$. At $x=1$, we have $\lim\limits_{x\to 1^-} g(x)=\lim\limits_{x\to 1^-} 2+x=3$ and, $\lim\limits_{x\to 1^+} g(x)=\lim\limits_{x\to 1^+} 2-x=1$ $\therefore$ $\lim\limits_{x\to 1^+} g(x)\ne\lim\limits_{x\to 1^-} g(x)$.So, $g(x)$ is not continuous at $x=1$At $x=2$, we have $\lim\limits_{x\to 2^-} g(x)=\lim\limits_{x\to 2^-}(2-x)=0$ and, $\lim\limits_{x\to 2^+} g(x)=\lim\limits_{x\to 2^+}(4-x)=0$ $\therefore \lim\limits_{x\to 2^-} g(x)\ne\lim\limits_{x\to 2^+} g(x)$.So, $g(x)$ is not continuous at $x=2$.Hence, the set of points of discontinuity of $g(x)$ is $\{1,2\}$

15. Since, $\lim\limits_{x\to 0} f(x)=f(0)$

$$\Rightarrow \lim_{x\to 0}\frac{\sin \pi x}{5x}=k \Rightarrow (1)\frac{\pi}{5}=k \Rightarrow k=\frac{\pi}{5} \qquad \left[\because \lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

16. For any $x\ne 1,2$, we find that $f(x)$ is the quotient of two polynomials and a polynomial is everywhere continuous. Therefore, $f(x)$ is continuous for all $x\ne 1,2$. Continuity at $x=1$: We have,

$$\lim_{x\to 1^-} f(x)=\lim_{h\to 0} f(1-h)$$

$$\Rightarrow \lim_{x\to 1^-} f(x)=\lim_{h\to 0}\frac{(1-h-2)(1-h+2)(1-h+1)(1-h-1)}{|(1-h-1)(1-h-2)|}$$

$$\Rightarrow \lim_{x\to 1^-} f(x)=\lim_{h\to 0}\frac{(3-h)(2-h)(-1-h)(-h)}{|(-h)(-1-h)|}\Rightarrow \lim_{x\to 1^-} f(x)=\lim_{h\to 0}\frac{(3-h)(2-h)\,h(h+1)}{h(h+1)}=6$$

and, $\lim\limits_{x\to 1^+} f(x)=\lim\limits_{h\to 0} f(1+h)$

$$\lim_{x\to 1^+} f(x)=\lim_{h\to 0}\frac{(1+h-2)(1+h+2)(1+h+1)(1+h-1)}{|(1+h-1)(1+h-2)|}$$

$$\lim_{x\to1^+} f(x) = \lim_{h\to0} \frac{(h-1)(3+h)(2+h)(h)}{|h(h-1)|}$$

$$\lim_{x\to1^+} f(x) = -\lim_{h\to0} \frac{(h-1)(3+h)(2+h)h}{h(1-h)} = -6$$

$$\therefore \lim_{x\to1^-} f(x) \neq \lim_{x\to1^+} f(x)$$

So, $f(x)$ is not continuous at $x = 1$

Similarly, $f(x)$ is not continuous at $x = 2$

17. If function $f(x)$ is continuous at $x = 0$, then

$$f(0) = \lim_{x\to0} f(x) \therefore \quad f(0) = k = \lim_{x\to0} x\sin\frac{1}{x}$$

$$\Rightarrow k = 0 \qquad \left[\because -1 \leq \sin\frac{1}{x} \leq 1\right]$$

18. At no point, function is continuous

19. We have, $\lim_{x\to0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to0} \frac{x^2\sin\left(\frac{1}{x}\right)}{x} = \lim_{x\to0} x\sin\frac{1}{x} = 0$

So, $f(x)$ is differentiable at $x = 0$ such that $f'(0) = 0$

For $x \neq 0$, we have

$$f'(x) = 2\,x\sin\left(\frac{1}{x}\right) + x^2\cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) \Rightarrow f'(x) = 2\,x\sin\frac{1}{x} - \cos\frac{1}{x}$$

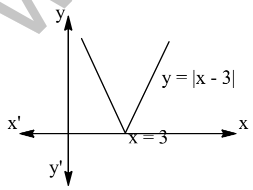
$$\Rightarrow \lim_{x\to0} f'(x) = \lim_{x\to0} 2\,x\sin\frac{1}{x} - \lim_{x\to0}\cos\left(\frac{1}{x}\right) = 0 - \lim_{x\to0}\cos\left(\frac{1}{x}\right)$$

Since $\lim_{x\to0}\cos\left(\frac{1}{x}\right)$ does not exist

$\therefore \lim_{x\to0} f'(x)$ does not exist. Hence, $f'(x)$ is not continuous at $x = 0$

20. Solve by yourself.

21. From the graph it is clear that $f(x)$ is continuous everywhere but not differentiable at $x = 3$



22. We have,

$\lim_{x\to1^-} g(x) = \lim_{x\to1^+} g(x) = 1$ and $\quad g(1) = 0$

So, $g(x)$ is not continuous at $x = 1$ but $\lim_{x\to1} g(x)$ exists

We have, $\lim_{x\to1^-} f(x) = \lim_{h\to0} f(1-h) = \lim_{h\to0}[1-h] = 0$

and, $\lim_{x\to1^+} f(x) = \lim_{h\to0} f(1+h) = \lim_{h\to0}[1+h] = 1$

So, $\lim_{x\to1} f(x)$ does not exist and so $f(x)$ is not continuous at $x = 1$

We have, $gof(x) = g\big(f(x)\big) = g([x]) = 0$, for all $x \in R$. So, gof is continuous for all x

We have, $fog(x) = f\big(g(x)\big)$

$\Rightarrow fog(x) = \begin{cases} f(0), & x \in Z \\ f(x^2), & x \in R - Z \end{cases} \Rightarrow fog(x) = \begin{cases} 0, & x \in Z \\ [x^2], & x \in R - Z \end{cases}$

Which is clearly not continuous

23. We have,

$$\lim_{x\to0^-} f(x) = \lim_{h\to0} f(-h) = \lim_{h\to0} \frac{e^{-1/h} - 1}{e^{-1/h} + 1} = -1$$

and, $\lim_{x\to0^+} f(x) = \lim_{h\to0} f(h) = \lim_{x\to0} \frac{e^{1/h}-1}{e^{1/h}+1} = \lim_{h\to0} \frac{e^{-1/h}}{e^{-1/h}} = 1$

$\therefore \lim_{x\to0^-} f(x) \neq \lim_{x\to0^+} f(x)$

Hence, $f(x)$ is not continuous at $x = 0$

24. It can be easily seen from the graph of $f(x) = |\cos x|$ that it is everywhere continuous but not differentiable at odd multiples of $\frac{\pi}{2}$

25. $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} (\log_e a)^n$

$$\Rightarrow f(x) = \sum_{n=0}^{\infty} \frac{(x \log_e a)^n}{n!} = \sum_{n=0}^{\infty} \frac{(\log_e a^x)^n}{n!}$$

$\Rightarrow f(x) = e^{\log_e a^x} = a^x$, which is everywhere continuous and differentiable

26. If possible, let $f(x)+g(x)$ be continuous. Then, $\{f(x)+g(x)\}-f(x)$ must be continuous$\Rightarrow g(x)$ must be continuous. This is a contradiction to the given fact that $g(x)$ is discontinuous. Hence, $f(x)+g(x)$ must be discontinuous

27. We have,

$$f(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\Rightarrow f'(x)=\lim_{h\to 0}\frac{f(x)f(h)-f(x)}{h}$$

$$\Rightarrow f'(x)=f(x)\lim_{h\to 0}\frac{f(h)-1}{h}\qquad [\because f(x+y)=f(x)f(y)]$$

$$\Rightarrow f'(x)=f(x)\left\{\lim_{h\to 0}\frac{1+h\,g(h)-1}{h}\right\}\qquad [\because f(x)=1+x\,g(x)]$$

$$\Rightarrow f'(x)=f(x)\lim_{h\to 0}g(h)=f(x)\cdot 1=f(x)$$

28. The point of discontinuity of $f(x)$ are those points where $\tan x$ is infinite.

$ie,\quad \tan x=\tan\infty \Rightarrow\quad x=(2n+1)\frac{\pi}{2},\ n\in I$

29. We have, $f(x)=\frac{\sin 4\pi[x]}{1+[x]^2}=0$ for all x $[\because 4\pi[x]$is a multiple of $\pi]$

$\Rightarrow f'(x)=0$ for all x

30. We have,

$$\lim_{x\to 9}\frac{\sqrt{f(x)}-3}{\sqrt{x}-3}=\lim_{x\to 9}\frac{\frac{f'(x)}{2\sqrt{f(x)}}}{\frac{1}{2\sqrt{x}}}\quad \text{[Using L' Hospital Rule]}=\lim_{x\to 9}\frac{\sqrt{x}f'(x)}{\sqrt{f(x)}}=\frac{3\times 4}{3}=4$$

31. $\lim_{h\to 0}\frac{f(2h+2+h^2)-f(2)}{f(h-h^2+1)-f(1)}=\lim_{h\to 0}\frac{\{f'(2h+2+h^2)\}.(2+2h)-0}{\{f'(h-h^2+1)\}.(1-2h)-0}$ [using L' Hospital's rule]

$=\frac{f'(2).2}{f'(1).1}=\frac{6.2}{4.1}=3$

32. Solve by yourself.

33. Solve by yourself.

34. $f(x)=x^n \quad\Rightarrow\quad f(1)=1$

$f'(x)=nx^{n-1}\quad\Rightarrow\quad f'(1)=n$

$f''(x)=n(n-1)x^{n-2}\quad\Rightarrow\quad f''(1)=n(n-1)$

… … … … … … … … … … … … … …

$f^n(x)=n(n-1)(n-2)\ldots 2.1\Rightarrow\ f^n(1)=n(n-1)(n-2)\ldots 2.1$

Now, $f(1)-\frac{f'(1)}{1!}+\frac{f''(1)}{2!}-\frac{f'''(1)}{3!}+\ldots+\frac{(-1)^n f^n(1)}{n!}$

$=1-\frac{n}{1!}+\frac{n(n-1)}{2!}-\frac{n(n-1)(n-2)}{3!}+\ldots+\frac{(-1)^n n(n-1)(n-2)\ldots 2.1}{n!}=(1-1)^n=0$

35. Let $u = \sin^{-1}\left(\frac{1}{1-2x^2}\right)$, $v = \sin^{-1}(3x - 4x^3)$

Put $x = \sin\theta$, we get

$u = \sec^{-1}(\sec 2\theta), v = \sin^{-1}(\sin 3\theta) \Rightarrow u = 2\theta, v = 3\theta$

$\Rightarrow u = 2\sin^{-1}x, \quad v = 3\sin^{-1}x \quad \Rightarrow \frac{u}{v} = \frac{2}{3} \quad \Rightarrow u = \frac{2}{3}v \Rightarrow \frac{du}{dv} = \frac{2}{3}$

36. $f(x,y) = \sin u = \frac{x^2y^2}{x+y}$

Here, $f(x)$ is a homogenous function of degree 3. By Euler theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3f \Rightarrow x\frac{\partial \sin u}{\partial x} + y\frac{\partial \sin u}{\partial y} = 3\sin u \Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3\tan u$$

37. Given, $u = x^2 + y^2, x = s + 3t, y = 2s - t$

Now ,$\frac{dx}{ds} = 1, \frac{dy}{ds} = 2$ ... (i)

$\frac{d^2x}{ds^2} = 0, \frac{d^2y}{ds^2} = 0$ ... (ii)

Now, $u = x^2 + y^2$

$\frac{du}{ds} = 2x\frac{dx}{ds} + 2y\frac{dy}{ds}$

$\frac{d^2u}{ds^2} = 2\left(\frac{dx}{ds}\right)^2 + 2x\frac{d^2x}{ds^2} + 2\left(\frac{dy}{ds}\right)^2 + 2y\left(\frac{d^2y}{ds^2}\right)$

$\Rightarrow \frac{d^2u}{ds^2} = 2(1)^2 + 2x(0) + 2(2)^2 + 2y(0) = 2 + 8 = 10$

38. Let the even function be

$f(x) = \cos x \Rightarrow f'(x) = -\sin x \quad \Rightarrow f''(x) = -\cos x$

At $x = \pi$, $f''(\pi) = -\cos\pi = 1$

$\therefore$ Our assumption is true.

$\therefore$ At $x = -\pi$

$f''(-\pi) = -\cos(-\pi) = 1$

Alternate

Since the function is twice differentiable

$\therefore f''(x) = \text{const.} \forall x f''(-\pi) = -f''(\pi) = 1$

39. Let $y = \log x$

On differentiating w.r.t. x from 1 to n times, we get

$$y_1 = \frac{1}{x}, y_2 = -\frac{1}{x^2}, y_3 = \frac{2}{x^3}, y_4 = -\frac{6}{x^4}, \ldots,$$

$\therefore$ by symmetry. $y_n = \frac{(-1)^{n-1}(n-1)!}{x^n}$

40. Given, $y = \sqrt{\log x + y} \quad \Rightarrow \quad y^2 = \log x + y$

$\Rightarrow 2y\frac{dy}{dx} = \frac{1}{x} + \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{x(2y-1)}$

41. We have,

$y = a^{x^{a^{x\ldots\infty}}} \Rightarrow y = a^{x^y}$

$\Rightarrow \log y = x^y \log a \Rightarrow \log(\log y) = y \log x + \log(\log a)$

Differentiating w.r.t. x, we get

$\frac{1}{\log y}\frac{1}{y}\frac{dy}{dx} = \frac{dy}{dx} \cdot \log x + \frac{y}{x} \Rightarrow \frac{dy}{dx} = \frac{y^2 \log y}{x(1-\log x \log y)}$

42. $f(1) = 5, f'(x) = nx^{n-1}$so $f'(1) = n$

$f''(1) = n(n-1), \ldots f^n(1) = 1.2. \ldots . n$

Thus, $f(1) + \frac{f'(1)}{1!} + \ldots + \frac{f^n(1)}{n!} = 5 + \frac{n}{1} + \frac{n(n-1)}{2!} + \ldots + \frac{n!}{n!} = (1+1)^n + 4 = 2^n + 4$

43. $x^{2x} - 2x^x \cot y - 1 = 0 \quad \ldots$(i)

At $x = 1$,

$1 - 2\cot y - 1 = 0 \Rightarrow \cot y = 0 \Rightarrow y = \frac{\pi}{2}$

On differentiating Eq. (i), w.r.t.$'x'$, we get

$2x^{2x}(1 + \log x) - 2$

$$\left[x^x(-\operatorname{cosec}^2 y)\frac{dy}{dx} + \cot y \, x^x(1 + \log x)\right] = 0$$

At $\left(1, \frac{\pi}{2}\right)$,

$2(1 + \log 1) - 2\left(1(-1)\left(\frac{dy}{dx}\right)_{\left(1,\frac{\pi}{2}\right)} + 0\right) = 0 \qquad \Rightarrow 2 + 2\left(\frac{dy}{dx}\right)_{\left(1,\frac{\pi}{2}\right)} = 0 \Rightarrow \left(\frac{dy}{dx}\right)_{\left(1,\frac{\pi}{2}\right)} = -1$

44. $\frac{d}{dx}\left[\frac{e^x+1}{e^x}\right] = \frac{d}{dx}[1+e^{-x}] = -e^{-x} = -\frac{1}{e^x}$

45. Given, $\frac{x}{1} = \frac{1-\sqrt{y}}{1+\sqrt{y}}$

Applying component and dividend, we get

$\frac{1+x}{1-x} = \frac{(1+\sqrt{y})+(1-\sqrt{y})}{(1+\sqrt{y})-(1-\sqrt{y})} \Rightarrow \frac{1+x}{1-x} = \frac{2}{2\sqrt{y}} \Rightarrow y = \left(\frac{1-x}{1+x}\right)^2$

On differentiating w.r.t. x, we get

$\frac{dy}{dx} = \frac{-2(1+x)^2(1-x)-(1-x)^2.2(1+x)}{(1+x)^4} = \frac{(1-x)(1+x)(-2-2x-2+2x)}{(1+x)^4}$

$= \frac{4(x-1)}{(x+1)^3}$

46. Given, $y = \sin\ (\log_e x)$ ...(i)

$\Rightarrow \frac{dy}{dx} = \cos(\log_e x).\frac{1}{x}$ ...(ii)

$\Rightarrow \frac{d^2y}{dx^2} = \frac{-x.\sin(\log_e x).\frac{1}{x}=\cos(\log_e x).1}{x^2} = \frac{-\sin\ (\log_e x)-\cos\ (\log_e x)}{x^2}$

$x^2\frac{d^2y}{dx^2} = -\sin(\log_e x) - x\frac{dy}{dx}$ [using Eq. (ii)]

$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -y$ [using Eq. (i)]

47. We have,

$2^x + 2^y = 2^{x+y}$

$\Rightarrow 2^x \log 2 + 2^y \log 2\frac{dy}{dx} = 2^{x+y}\log 2\left(1+\frac{dy}{dx}\right) \Rightarrow (2^y - 2^{x+y})\frac{dy}{dx} = 2^{x+y} - 2^x$

$\Rightarrow \frac{dy}{dx} = \frac{2^x(2^y-1)}{2^y(1-2^x)} = 2^{x-y}\left(\frac{2^y-1}{1-2^x}\right)$

48. We have,

$y = \sqrt{x+\sqrt{y+\sqrt{x+\sqrt{y+\cdots\infty}}}} \Rightarrow y^2 = x + \sqrt{y+\sqrt{x+\sqrt{y+\cdots\infty}}}$

$\Rightarrow y^2 = x + \sqrt{y+y} \Rightarrow (y^2-x)^2 = 2y$

Differentiating both sides w.r.t. x, we get

$2(y^2-x)\left(2y\frac{dy}{dx}-1\right) = 2\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{y^2-x}{y^3-xy-1}$

49. If $f(x) = ax^{41} + bx^{-40}$, then $f'(x) = 41ax^{40} - 40bx^{-41}$

$\Rightarrow f''(x) = 1640\, ax^{39} + 1640\, bx^{-42} \Rightarrow f''(x) = \frac{1640}{x^2}(ax^{41} + bx^{-40})$

$$\Rightarrow f''(x) = \frac{1640}{x^2} f(x) \Rightarrow \frac{f''(x)}{f(x)} = 1640x^{-2}$$

So, statement-1 is true

We have, $\tan^{-1}\frac{2x}{1-x^2} = \begin{cases} \pi + 2\tan^{-1}x & x < -1 \\ 2\tan^{-1}x\,, \text{if} & -1x \leq 1 \\ -\pi + 2\tan^{-1}x & x > 1 \end{cases}$

$\therefore \frac{d}{dx}\left(\tan^{-1}\frac{2x}{1-x^2}\right) = \frac{2}{1+x^2}$ for all x

So, statement-II is not true

50. Given, $y = \cos x + x\cos y = \pi$

On differentiating both sides w.r.t. x, we get

$\frac{dy}{dx}\cos + y(-\sin x) + x(-\sin y)\frac{dy}{dx} + \cos y = 0 \Rightarrow \frac{dy}{dx} = \frac{y\sin x - \cos y}{\cos x - x\sin y}$

Again, differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{\begin{bmatrix} (\cos x - x\sin y)\left(y\cos x + \sin x\frac{dy}{dx} + \sin y\frac{dy}{dx}\right) \\ -(y\sin x - \cos y)\left(-\sin x - \sin y - x\cos y\frac{dy}{dx}\right) \end{bmatrix}}{(\cos x - x\sin y\,)^2}$$

At $x = 0$,

$$f''(0) = \frac{1(y + \sin y) - (-1)(-\sin y)}{(1-0)^2} = y$$

As $y\cos x + x\cos y = \pi$

$\therefore$ At $x = 0 \Rightarrow y = \pi$

Hence, $f''(0) = \pi$