CHAPTER 04

DETERMINANTS

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DETERMINANT:

Every square matrix A of order n is associated with a number, called its determinant and it is denoted by det(A) or |A|.

DETERMINANT OF MATRIX OF ORDER 1:

Let A=[a] be a square matrix of order 1, then |A|=|a|=a, i.e element itself is determinant.

DETERMINANT OF MATRIX OF ORDER 2:

$$\det(A) \text{ or } |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

DETERMINANT OF MATRIX OF ORDER 3:

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$ [Expanding along R1]

PROPERTIES OF DETERMINANTS:

i) If the rows and columns of a determinant are interchanged, then the value of the determinant remains unchanged.

ii) If any two rows (columns) of a determinant are interchanged, then sign of such determinant becomes change.

iii) If any two rows(columns) of a determinant are identical, then the value of such determinant is zero.

iv) If each element of a row(column) is multiplied by a constant k, then the value of the determinant is multiplied by constant k.

v) If the element of any row(column) of a determinant is added k times, the corresponding element of any other row(column) of the determinant, then the value of the determinant remains same.

vi) If some or all elements of a row(column) of a determinant are expressed as sum of two(more) terms, then such determinant can be expressed as sum of two(more) determinants of the same order.

vii) If all elements of any two rows(column) of determinant are proportional, then the value of such determinant becomes zero.

viii) If all the elements of any row(column) of a determinant are zero, then the value of such determinant becomes zero.

AREA OF TRIANGLE:

Let $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a ΔABC. Then its area is given by

$$\Delta = \frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$$

$$= \frac{1}{2}.\left|[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]\right|$$

CONDITION OF COLLINEARITY FOR THREE POINTS:

Three points $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear if and only if the area of triangle formed by these three points is zero.

i.e. $\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right| = 0$

MINORS:

Minor of an element a_{ij} of a determinant is the determinant obtained by deleting ith row and jth column in which element a_{ij} lies. It is denoted by M_{ij}

e.g. If A= $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then

Minors of elements of A are

$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$, etc.

COFACTORS:

If M_{ij} is the minor of an element a_{ij}, then the cofactor of a_{ij} is denoted by C_{ij} or A_{ij} and defined as follows

C_{ij} or $A_{ij} = (-1)^{i+j} M_{ij}$

SINGULAR AND NON-SINGULAR MATRICES:

A square matrix A is said to be a singular matrix, if |A|=0 and if |A|≠0, then matrix A is said to be non-singular matrix.

ADJOINT OF A MATRIX:

The adjoint of a square matrix A is defined as the transpose of the matrix formed by cofactors of elements of A. Let $A = [a_{ij}]_{n\times n}$ be a square matrix, then adjoint of A, i.e. $adj(A) = C^T$, where C=$[c_{ij}]$ is the cofactor matrix of A.

PROPERTIES OF ADJOINT OF SQUARE MATRIX

If A and B are two square matrices of order n, then

i) $adj(A^T) = (adj\ A)^T$

ii) $adj(kA) = k^{n-1}(adj\ A), k \in R$

iii) $adj(AB) = (adjB)(adjA)$

iv) $|adjA| = |A|^{n-1}, if\ |A| \neq 0$

v) $|adj[adj(A)]| = |A|^{(n-1)^2}, if\ |A| \neq 0$

vi) $adj(adjA) = |A|^{n-2}.A$

INVERSE OF A MATRIX:

Suppose A is a non-zero square matrix of order n and there exists matrix B of same order n such that $AB = BA = I_n$, then such matrix B is called an inverse of matrix A. It is denoted by A^{-1} and is given by $A^{-1} = \frac{1}{|A|}[adj(A)]$

PROPERTIES OF INVERSE OF A MATRIX:

i) $(A^{-1})^{-1} = A$

ii) $(AB)^{-1} = B^{-1}A^{-1}$

iii) $(A')^{-1} = (A^{-1})'$ where A' is transpose of a matrix A.

iv) $AA^{-1} = A^{-1}A = I$

v) $|A^{-1}| = |A|^{-1}$

vi) $(kA)^{-1} = \frac{1}{k}A^{-1}, where\ k \neq 0$

CONSISTENT AND NON CONSISTENT SYSTEM:

A system of equations is consistent or inconsistent according as its solution exist or not.

SOLUTION OF SYSTEM OF LINEAR EQUATIONS:

Let the system of linear equations be

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

Then, in matrix form, this system of equations can be written as AX=B.

where, $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

SOLUTION OF SYSTEM OF LINEAR EQUATIONS:

i) When B≠O

a) If |A|≠0, then system of equations is consistent and has unique solution given by $X = A^{-1}B$.

b) If |A|=0 and (adj A)B≠0, then there exists no solution, i.e. system of equations is inconsistent.

c) If |A|=0 and (adj A)B=0, then system may be either consistent or inconsistent according as the system have either infinitely many solution or no solution.

ii) When $B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ in such cases, we have

a) |A|≠0⇒System has only trivial solution i.e. x=0, y=0 and z=0

|A|=0 ⇒System has infinitely many solutions

Determinants

Applications of determinants & matrices

- If $a_1x+b_1y+c_1z=d_1, a_2x+b_2y+c_2z=d_2, a_3x+b_3y+c_3z=d_3$ then we can write $AX=B$,

  where $A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B=\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$
- Unique solution of $AX=B$ is $X=A^{-1}B$, $|A|\neq 0$.
- $AX=B$ is consistent or inconsistent according as the solution exists or not.
- For a square matrix A in $AX=B$, if
  - (i) $|A|\neq 0$ then there exists unique solution.
  - (ii) $|A|=0$ and (adj. A) $B\neq 0$, then no solution.
  - (iii) if $|A|=0$ and (adj.A).$B=0$ then system may or may not be consistent.

Area of a triangle

If (x_1, y_1), (x_2, y_2) and (x_3, y_3) are the vertices of triangle, Area of $\Delta=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

e.g., if (1, 2), (3, 4) and (–2, 5) are the vertices, then area of the triangle is

$\Delta=\frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \\ -2 & 5 & 1 \end{vmatrix}=\frac{1}{2}|1(4-5)-2(3+2)+1(15+8)|=6$ sq.units.

we take positive value of the determinant because area is considered positive.

Minors and cofactors of a matrix

Minor of an element a_{ij} in a determinant of matrix A is the determinant obtained by deleting i^{th} row and j^{th} column and is denoted by M_{ij}. If M_{ij} is the minor of a_{ij} and cofactor of a_{ij} is A_{ij} given by $A_{ij}=(-1)^{i+j}M_{ij}$.

- If $A_{3\times 3}$ is a matrix, then $|A|=a_{11}.A_{11}+a_{12}.A_{12}+a_{13}.A_{13}$.
- If elements of one row (or column) are multiplied with cofactors of elements of any other row (or column), then their sum is zero. For e.g., $a_{11}A_{21}+a_{12}A_{22}+a_{13}A_{23}=0$.

e.g., if $A=\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$, then $M_{11}=4$ and $A_{11}=(-1)^{1+1}4=4$.

$M_{12}=-3$ and $A_{12}=(-1)^{1+2}(-3)=4$.

Determinant of a square matrix 'A', |A| is given by

(i) if $A=[a_{11}]_{1\times 1}$, then $|A|=a_{11}$

(ii) if $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2}$ then $|A|=a_{11}a_{22}-a_{12}a_{21}$

(iii) if $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$, then $|A|=a_{11}(a_{22}.a_{33}-a_{23}.a_{32})-a_{12}(a_{21}.a_{33}-a_{23}a_{31})+a_{13}(a_{21}a_{32}-a_{22}a_{31})$

e.g., If $A=\begin{bmatrix} 2 & 3 \\ 2 & 4 \end{bmatrix}$, then $|A|=2\times 4-3\times 2=2$

Adjoint and inverse of a matrix

If $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then $Adj\,A=\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$,

where A_{ij} is the cofactor of a_{ij}.

- A(adj.A)=(adj.A).$A=|A|I$, A – square matrix of order 'n'
- If $|A|=0$, then A is singular. Otherwise, A is non-singular.
- If $AB=BA=I$, where B is a square matrix, then B is called the inverse of A, $A^{-1}=B$ or $B^{-1}=A$, $(A^{-1})^{-1}=A$.

Inverse of a square matrix exists if A is non-singular i.e. $|A|\neq 0$, and is given by

$A^{-1}=\frac{1}{|A|}$(adj.A)

Trace the Mind Map
- First Level
- Second Level
- Third Level

PRACTICE QUESTIONS

1. If $f(x) = \begin{vmatrix} 1 & 1 & 1 \\ 2x & (x-1) & x \\ 3x(x-1) & (x-1)(x-2) & x(x-1) \end{vmatrix}$, then $f(50)$ is equal to

   a) 0

   b) 1

   c) 100

   d) -100

2. If $A = \begin{bmatrix} 3 & 2 & 4 \\ 1 & 2 & 1 \\ 3 & 2 & 6 \end{bmatrix}$ and A_{ij} are the cofactors of a_{ij}, then $a_{11}A_{11} + a_{12}A + a_{13}A_{13}$ is equal to

   a) 8

   b) 6

   c) 4

   d) 0

3. If $D_r = \begin{vmatrix} r & 1 & \frac{n(n+1)}{2} \\ 2r-1 & 4 & n^2 \\ 2^{r-1} & 5 & 2^n - 1 \end{vmatrix}$, then the value of $\sum_{r=0}^{n} D_r$ is

   a) 0

   b) 1

   c) $\frac{a^2}{(1+a)^2} - \frac{b^2}{(1+b)^2}$

   d) None of these

4. $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = 0$, then a, b, c are

   a) In GP

   b) InHP

   c) Equal

   d) In AP

5. The roots of the equation $\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$ are

a) -1,-2

b) -1,2

c) 1,-2

d) 1,2

6. If $A_i = \begin{bmatrix} a^i & b^i \\ b^i & a^i \end{bmatrix}$ and if $|a| < 1, |b| < 1$,then $\sum_{i=1}^{\infty} \det(A_i)$ is equal to

a) $\frac{a^2}{(1-a)^2} - \frac{b^2}{(1-b)^2}$

b) $\frac{a^2-b^2}{(1-a)^2(1-b^2)}$

c) $\frac{a^2}{(1-a)^2} + \frac{b^2}{(1-b)^2}$

d) $\frac{a^2}{(1+a)^2} - \frac{b^2}{(1+b)^2}$

7. The value of the determinant $\begin{vmatrix} 10! & 11! & 12! \\ 11! & 12! & 13! \\ 12! & 13! & 14! \end{vmatrix}$ is

a) $2\,(10!\,11!)$

b) $2\,(10!\,13!)$

c) $2\,(10!\,11!\,12!)$

d) $2\,(11!\ 12!\ 13!)$

8. If$p\lambda^4 + q\lambda^3 + q\lambda^2 + s\lambda + t = \begin{vmatrix} b^2+c^2 & a^2+\lambda & a^2+\lambda \\ b^2+\lambda & c^2+a^2 & b^2+\lambda \\ c^2+\lambda & c^2+\lambda & a^2+b^2 \end{vmatrix}$ is an identity in λ, where p, q, r, s, t are constants, then the value of t is

a) 1

b) 2

c) 0

d) None of these

9. The coefficient of x in $f(x) = \begin{vmatrix} x & 1+\sin x & \cos x \\ 1 & \log(1+x) & 2 \\ x^2 & 1+x^2 & 0 \end{vmatrix}, -1 < x \le 1$, is

a) 1

b) -2

c) -1

d) 0

10. If $\Delta_a = \begin{vmatrix} a-1 & n & 6 \\ (a-1)^2 & 2n^2 & 4n-2 \\ (a-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$, then $\sum_{a=1}^{n} \Delta_a$ is equal to

a) 0

b) 1

c) $\left\{\frac{n(n+1)}{2}\right\}\left\{\frac{a(a+1)}{2}\right\}$

d) None of these

11. If $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 5$, then the value of $\begin{vmatrix} b_2c_3 - b_3c_2 & c_2a_3 - c_3a_2 & a_2b_3 - c_3b_2 \\ b_3c_1 - b_1c_3 & c_3a_1 - c_1a_3 & a_3b_1 - a_1b_3 \\ b_1c_2 - b_2c_1 & c_1a_2 - c_2a_1 & a_1b_2 - a_2b_1 \end{vmatrix}$ is

a) 5

b) 25

c) 125

d) 0

12. The value of $\begin{vmatrix} 1990 & 1991 & 1992 \\ 1991 & 1992 & 1993 \\ 1992 & 1993 & 1994 \end{vmatrix}$ is

a) 1992

b) 1993

c) 1994

d) 0

13. $\begin{vmatrix} 1+ax & 1+bx & 1+cx \\ 1+a_1x & 1+b_1x & 1+c_1x \\ 1+a_2x & 1+b_2x & 1+c_2x \end{vmatrix} = A_0 + A_1x + A_2x^2 + A_3x^3$, then A_1 is equal to

a) abc

b) 0

c) 1

d) None of these

14. If $\begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix} = Ax-12$, then the value of A is

a) 12

b) 23

c) -12

d) 24

15. If $f(x) = \begin{vmatrix} 1+a & 1+ax & 1+ax^2 \\ 1+b & 1+bx & 1+bx^2 \\ 1+c & 1+cx & 1+cx^2 \end{vmatrix}$, where a, b, c are non-zero constants, then value of $f(10)$is

a) $10\,(b-a)(c-a)$

b) $100\,(b-a)(c-b)(a-c)$

c) $100\,abc$

d) 0

16. If $f(x)$, g(x) and $h(x)$ are three polynomials of degree 2 and $\Delta(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}$, then $\Delta(x)$ is polynomial of degree

a) 2

b) 3

c) Atmost 2

d) Atmost 3

17. The value of determinant $\begin{vmatrix} (a^x+a^{-x})^2 & (a^x-a^{-x})^2 & 1 \\ (b^x+b^{-x})^2 & (b^x-b^{-x})^2 & 1 \\ (c^x+c^{-x})^2 & (c^x-c^{-x})^2 & 1 \end{vmatrix}$ is

a) 0

b) 2 abc

c) $a^2b^2c^2$

d) None of these

18. If $x^a y^b = e^m, x^c y^d = e^n, \Delta_1 = \begin{vmatrix} m & b \\ n & d \end{vmatrix}, \Delta_2 = \begin{vmatrix} a & m \\ c & n \end{vmatrix}$ and $\Delta_3 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$, then the values of x and y are respectively

a) $\frac{\Delta_1}{\Delta_3}$ and $\frac{\Delta_2}{\Delta_3}$

b) $\frac{\Delta_2}{\Delta_1}$ and $\frac{\Delta_3}{\Delta_1}$

c) $\log\left(\frac{\Delta_1}{\Delta_3}\right)$ and $\log\left(\frac{\Delta_2}{\Delta_3}\right)$

d) $e^{\frac{\Delta_1}{\Delta_3}}$ and $e^{\frac{\Delta_2}{\Delta_3}}$

19. If $\alpha + \beta + \gamma = \pi$, then the value of the determinant $\begin{vmatrix} e^{2i\alpha} & e^{-i\gamma} & e^{-i\beta} \\ e^{-i\gamma} & e^{2i\beta} & e^{-i\alpha} \\ e^{-i\beta} & e^{-i\alpha} & e^{2i\gamma} \end{vmatrix}$, is

a) 4

b) -4

c) 0

d) None of these

20. Let a, b, c be positive real numbers. The following system of equations in x, y and z

$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, -\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ has

a) No solution

b) Unique solution

c) Infinitely many solutions

d) Finitely many solutions

21. The value of $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$ is equal to

a) 0

b) 1

c) xyz

d) $\log xyz$

22. The sum of the products of the elements of any row of a determinant A with the cofactors of the corresponding elements is equal to

a) 1

b) 0

c) $|A|$

d) $\frac{1}{2}|A|$

23. If $a > 0, b > 0, c > 0$ are respectively the p^{th}, q^{th}, r^{th} terms of a GP, then the value of the determinant $\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix}$, is

a) 1

b) 0

c) -1

d) None of these

24. If $\alpha, \beta, \gamma \in R$, then the determinant $\Delta = \begin{vmatrix} (e^{i\alpha} + e^{-i\alpha})^2 & (e^{i\alpha} - e^{-i\alpha})^2 & 4 \\ (e^{i\beta} + e^{-i\beta})^2 & (e^{i\beta} - e^{-i\beta})^2 & 4 \\ (e^{i\gamma} + e^{-i\gamma})^2 & (e^{i\gamma} - e^{-i\gamma})^2 & 4 \end{vmatrix}$ is

a) Independent of α, β and γ

b) Dependent of α, β and γ

c) Independent of α, β only

d) Independent of α, β only

25. If $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} 0$ and vectors $(1, a, a^2), (1, b, b^2)$and $(1, c, c^2)$are non-coplanar, then the product abc equals

a) 2

b) -1

c) 1

d) 0

26. The value of the determinant $\begin{vmatrix} 1 & \omega^3 & \omega^5 \\ \omega^3 & 1 & \omega^4 \\ \omega^5 & \omega^4 & 1 \end{vmatrix}$, where ω is an imaginary cube root of unity, is

a) $(1-\omega)^2$

b) 3

c) -3

d) None of these

27. Let $\Delta = \begin{vmatrix} 1+x_1y_1 & 1+x_1y_2 & 1+x_1y_3 \\ 1+x_2y_1 & 1+x_2y_2 & 1+x_2y_3 \\ 1+x_3y_1 & 1+x_3y_2 & 1+x_3y_3 \end{vmatrix}$, then value of Δ is

a) $x_1x_2x_3 + y_1y_2y_3$

b) $x_1x_2x_3y_1y_2y_3$

c) $x_2x_3y_2y_3 + x_3y_1y_3y_1 + x_1x_2y_1y_2$

d) 0

28. If $\Delta_1 = \begin{vmatrix} 10 & 4 & 3 \\ 17 & 7 & 4 \\ 4 & -5 & 7 \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} 4 & x+5 & 3 \\ 7 & x+12 & 4 \\ -5 & x-1 & 7 \end{vmatrix}$ such that $\Delta_1 + \Delta_2 = 0$, is

a) $x = 5$

b) $x = 0$

c) x has no real value

d) None of these

29. If $a_1, a_2, \ldots, a_n, \ldots$, are in GP, then the determinant

$$\Delta = \begin{vmatrix} \log a_n & \log a_{n+1} & \log a_{n+2} \\ \log a_{n+3} & \log a_{n+4} & \log a_{n+5} \\ \log a_{n+6} & \log a_{n+7} & \log a_{n+8} \end{vmatrix}$$ is equal to

a) 2

b) 4

c) 0

d) 1

30. In a third order determinant, each element of the first column consists of sum of two terms, each element of the second column consists of sum of three terms and each element of the third column consists of sum of four terms. Then, it can be decomposed into n determinant, where n has the value

a) 1
b) 9
c) 16
d) 24

31. The integer represented by the determinant

$\begin{bmatrix} 215 & 342 & 511 \\ 6 & 7 & 8 \\ 36 & 49 & 54 \end{bmatrix}$ is exactly divisible by

a) 146
b) 21
c) 20
d) 335

32. The value of $\sum_{n=1}^{N} U_n$ if $U_n = \begin{vmatrix} n & 1 & 5 \\ n^2 & 2N+1 & 2N+1 \\ n^3 & 3N^2 & 3N \end{vmatrix}$, is

a) 0
b) 1
c) -1
d) None of these

33. If $a+b+c=0$, then the solution of the equation $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$ is

a) 0
b) $\pm\frac{3}{2}(a^2+b^2+c^2)$
c) $0, \pm\sqrt{\frac{3}{2}(a^2+b^2+c^2)}$
d) $0, \pm\sqrt{(a^2+b^2+c^2)}$

34. If $1, \omega, \omega^2$ are the cube roots of unity, then

$$\Delta = \begin{vmatrix} 1 & \omega^n & \omega^{2n} \\ \omega^n & \omega^{2n} & 1 \\ \omega^{2n} & 1 & \omega^n \end{vmatrix}$$ is equal to

a) 0

b) 1

c) ω

d) ω^2

35. If $\Delta_r = \begin{vmatrix} 1 & n & n \\ 2r & n^2+n+1 & n^2+n \\ 2r-1 & n^2 & n^2+n+1 \end{vmatrix}$ and $\sum_{r=1}^{n} \Delta_r = 56$, then n equals

a) 4

b) 6

c) 7

d) 8

36. If B is a non-singular matrix and A is a square matrix such that $B^{-1}AB$ exists, then det $(B^{-1}AB)$ is equal to

a) $\det(A^{-1})$

b) $\det(B^{-1})$

c) $\det(B)$

d) $\det(A)$

37. Which one of the following is correct?

If A non-singular matrix, then

a) $\det(A^{-1}) = \det(A)$

b) $\det(A^{-1}) = \frac{1}{\det(A)}$

c) $\det(A^{-1}) = 1$

d) None of these

38. Coefficient of x in

$$f(x) = \begin{vmatrix} x & (1+\sin x)^2 & \cos x \\ 1 & \log(1+x) & 2 \\ x^2 & (1+x)^2 & 0 \end{vmatrix}, \text{ is}$$

a) 0

b) 1

c) -2

d) Cannot be determined

39. The value of $\Delta = \begin{vmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{vmatrix}$, is

a) 8

b) -8

c) 400

d) 1

40. If $a = 1 + 2 + 4 + \cdots$to n terms, $b = 1 + 3 + 9 + \cdots$to n terms and $c = 1 + 5 + 25 + \cdots$ to n terms, then $\begin{vmatrix} a & 2b & 4c \\ 2 & 2 & 2 \\ 2^n & 3^n & 5^n \end{vmatrix}$ equals

a) $(30)^n$

b) $(10)^n$

c) 0

d) $2^n + 3^n + 5^n$

41. If A, B and C are the angles of a triangle and

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A + \sin^2 A & \sin B + \sin^2 B & \sin C + \sin^2 C \end{vmatrix} = 0 \text{ then the triangle } ABC \text{ is}$$

a) Isosceles

b) Equilateral

c) Right angled isosceles

d) None of these

42. Let $D_r = \begin{vmatrix} 2^{r-1} & 2.3^{r-1} & 4.5^{r-1} \\ \alpha & \beta & \gamma \\ 2^n - 1 & 3^n - 1 & 5^n - 1 \end{vmatrix}$. Then, the value of $\sum_{r=1}^{n} D_r$ is

a) $\alpha \beta \gamma$

b) $2^n \alpha + 2^n \beta + 4^n \gamma$

c) $2 \alpha + 3 \beta + 4 \gamma$

d) None of these

43. If $\Delta(x) = \begin{vmatrix} 1 & \cos x & 1 - \cos x \\ 1 + \sin x & \cos x & 1 + \sin x - \cos x \\ \sin x & \sin x & 1 \end{vmatrix}$, then $\int_0^{\pi/2} \Delta(x) dx$ is equal to

a) $\frac{1}{4}$

b) $\frac{1}{2}$

c) 0

d) $-\frac{1}{2}$

44. The arbitrary constant on which the value of the determinant

$$\begin{vmatrix} 1 & a & a^2 \\ \cos(p-d)\, a & \cos pa & \cos(p-d)\, a \\ \sin(p-d)\, a & \sin pa & \sin(p-d)\, a \end{vmatrix}$$

does not depend, is

a) α

b) p

c) d

d) a

45. Let $[x]$ represent the greatest integer less than or equal to x, then the value of the determinant $\begin{vmatrix} [e] & [\pi] & [\pi^2 - 6] \\ [\pi] & [\pi^2 - 6] & [e] \\ [\pi^2 - 6] & [e] & [\pi] \end{vmatrix}$ is

a) -8

b) 8

c) 10

d) None of these

46. If $f(x)=\begin{vmatrix} x-3 & 2x^2-18 & 3x^3-81 \\ x-5 & 2x^2-50 & 4x^3-500 \\ 1 & 2 & 3 \end{vmatrix}$,then

$f(1).f(3)+f(3).f(5)+f(5).f(1)$ is equal to

a) $f(1)$

b) $f(3)$

c) $f(1)+f(3)$

d) $f(1)+f(5)$

47. The value of the determinant $\Delta=\begin{vmatrix} \frac{1-a_1^3 b_1^3}{1-a_1 b_1} & \frac{1-a_1^3 b_2^3}{1-a_1 b_2} & \frac{1-a_1^3 b_3^3}{1-a_1 b_3} \\ \frac{1-a_2^3 b_1^3}{1-a_2 b_1} & \frac{1-a_2^3 b_2^3}{1-a_2 b_2} & \frac{1-a_2^3 b_3^3}{1-a_2 b_3} \\ \frac{1-a_3^3 b_1^3}{1-a_3 b_1} & \frac{1-a_3^3 b_2^3}{1-a_3 b_2} & \frac{1-a_3^3 b_3^3}{1-a_3 b_3} \end{vmatrix}$, is

a) 0

b) Dependent only on a_1, a_2, a_3

c) Dependent only on b_1, b_2, b_3

d) Dependent on $a_1, a_2, a_3\ b_1, b_2, b_3$

48. Let a, b, c be such that $(b+c) \neq 0$ and

$$\begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ (-1)^{n+2}a & (-1)^{n+1}b & (-1)^n c \end{vmatrix}^1 = 0$$

Then the value of n is

a) Zero

b) Any even integer

c) Any odd integer

d) Any integer

49. If [ ] denotes the greatest integer less than or equal to the real number under consideration and $-1 \leq x < 0; 0 \leq y < 1; 1 \leq z < 2,$ then the value of the determinant $\begin{vmatrix} [x]+1 & [y] & [z] \\ [x] & [y]+1 & [z] \\ [x] & [y] & [z]+1 \end{vmatrix}$ is

a) $[x]$

b) $[y]$

c) $[z]$

d) None of these

50. If $a \neq p, b \neq q, c \neq r$ and $\begin{vmatrix} p & b & c \\ p+a & q+b & 2c \\ a & b & r \end{vmatrix} = 0$,then

$\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c}$is equal to

a) 0

b) 1

c) 2

d) 3

ANSWER KEY

1. A	26.B
2. A	27.D
3. A	28.A
4. D	29.C
5. B	30.D
6. B	31.C
7. C	32.A
8. D	33.C
9. B	34.A
10. A	35.C
11. B	36.D
12. D	37.B
13. B	38.C
14. D	39.B
15. D	40.C
16. C	41.A
17. A	42.D
18. D	43.D
19. B	44.B
20. B	45.A
21. A	46.B
22. C	47.D
23. B	48.C
24. A	49.C
25. B	50.C

HINTS AND SOLUTIONS:

1. Let $f(x)=\begin{vmatrix}1 & 1 & 1\\ 2x & (x-1) & x\\ 3x(x-1) & (x-1)(x-2) & x(x-1)\end{vmatrix}=(x-1)\begin{vmatrix}1 & 1 & 1\\ 2x & x-1 & x\\ 3x & x-2 & x\end{vmatrix}$

   Applying $C_1\to C_1-C_3$ and $C_2\to C_2-C_3$

   $=(x-1)\begin{vmatrix}0 & 0 & 1\\ x & -1 & x\\ 2x & -2 & x\end{vmatrix}=(x-1)[-2x+2x]=0$

   $\therefore f(x)=0\Rightarrow f(50)=0$

2. $a_{11}A_{11}+a_{12}A_{12}+a_{13}A_{13}$

   $=3\begin{vmatrix}2 & 1\\ 2 & 6\end{vmatrix}-2\begin{vmatrix}1 & 1\\ 3 & 6\end{vmatrix}+4\begin{vmatrix}1 & 2\\ 3 & 2\end{vmatrix}$

   $=3(12-2)-2(6-3)+4(2-6)=30-6-16=8$

3. $\sum_{r=0}^{n} D_r=\begin{vmatrix}\sum r & 1 & \frac{n(n+1)}{2}\\ 2\sum r-\sum 1 & 4 & n^2\\ \sum 2^{r-1} & 5 & 2^n-1\end{vmatrix}$

   $=\begin{vmatrix}\frac{n(n+1)}{2} & 1 & \frac{n(n+1)}{2}\\ n^2 & 4 & n^2\\ 2^n-1 & 5 & 2^n-1\end{vmatrix}=0$ [$\because$ two columns are identical]

4. Given $\begin{vmatrix}x+1 & x+2 & x+a\\ x+2 & x+3 & x+b\\ x+3 & x+4 & x+c\end{vmatrix}=0$

   Applying $R_1\to +R_1+R_3-2R_2$, we get

   $\begin{vmatrix}0 & 0 & a+c-2b\\ x+2 & x+3 & x+b\\ x+3 & x+4 & x+c\end{vmatrix}=0$

   $\Rightarrow (a+c-2b)[x^2+6x+8-(x^2+6x+9)]=0$

   $\Rightarrow (a+c-2b)(-1)=0\Rightarrow 2b=a+c$

   $\Rightarrow a,b,c$ are in AP

5. We have, $\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} 0 & 6 & 15 \\ 0 & -2-2x & 5(1-x^2) \\ 1 & 2x & 5x^2 \end{vmatrix} = 0 \quad \left(\substack{R_1 \to R_1 - R_2 \\ \text{and } R_2 \to R_2 - R_3} \right)$

$\Rightarrow 3 \cdot 2 \cdot 5 \begin{vmatrix} 0 & 1 & 1 \\ 0 & -(1+x) & 1-x^2 \\ 1 & x & x^2 \end{vmatrix} = 0$

(Taking common, 3 from R_1, 2 from C_2 , 5 from C_3)

$\Rightarrow (1+x) \begin{vmatrix} 0 & 1 & 1 \\ 0 & -1 & 1-x \\ 1 & x & x^2 \end{vmatrix} = 0 \Rightarrow (1+x)(2-x) = 0$

$\Rightarrow x + 1 = 0$ or $x - 2 = 0 \Rightarrow x = -1, 2$

6. $\because \det(A_1) = \begin{vmatrix} a & b \\ b & a \end{vmatrix} = a^2 - b^2$

$\det(A_2) = \begin{vmatrix} a^2 & b^2 \\ b^2 & a^2 \end{vmatrix} = a^4 - b^4$

$\therefore \sum_{i=1}^{\infty} \det(A_i) = \det(A_1) + \det(A_2) + \ldots$

$= a^2 - b^2 + a^4 - b^4 + \ldots = \frac{a^2}{1-a^2} - \frac{b^2}{1-b^2} = \frac{a^2-b^2}{(1-a^2)(1-b^2)}$

7. Let $\Delta = \begin{vmatrix} 10! & 11! & 12! \\ 11! & 12! & 13! \\ 12! & 13! & 14! \end{vmatrix}$

$= (10!)(11!)(12!) \begin{vmatrix} 1 & 11 & 11 \times 12 \\ 1 & 12 & 12 \times 13 \\ 1 & 13 & 13 \times 14 \end{vmatrix} = (10!)(11!)(12!) \begin{vmatrix} 1 & 11 & 11 \times 12 \\ 0 & 1 & 24 \\ 0 & 2 & 50 \end{vmatrix}$

$= 2(10!)(11!)(12!)$

8. On putting $\lambda = 0$, we get

$t = \begin{vmatrix} b^2 + c^2 & a^2 & a^2 \\ b^2 & c^2 + a^2 & b^2 \\ c^2 & c^2 & a^2 + b^2 \end{vmatrix} = 4a^2b^2c^2$

Clearly, it depends on a, b, c.

9. Given, $f(x)=\begin{vmatrix} x & 1+\sin x & \cos x \\ 1 & \log(1+x) & 2 \\ x^2 & 1+x^2 & 0 \end{vmatrix}$

$=x\{-2(1+x^2)\}-(1+\sin x)(-2x^2)+\cos x\{1+x^2-x^2\log(1+x)\}$

$=-2x-2x^3+2x^2+2x^2\sin x+\cos x\{1+x^2-x^2\log(1+x)\}$

$\therefore$ Coefficient of x in $f(x)=-2$

10. We have,

$$\Delta_a=\begin{vmatrix} a-1 & 2 & 6 \\ (a-1)^2 & 2n^2 & 4n-2 \\ (a-1)^3 & 3n^3 & 2n^2-3n \end{vmatrix}$$

$\therefore \sum_{a=1}^{n}\Delta_a \begin{vmatrix} \sum_{a=1}^{n}(a-1) & n & 6 \\ \sum_{a=1}^{n}(a-1)^2 & 2n^2 & 4n-2 \\ \sum_{a=1}^{n}(a-1)^3 & 3n^2 & 3n^2-3n \end{vmatrix}$ $\Rightarrow \sum_{a=1}^{n}\Delta_a=$

$$\begin{vmatrix} \frac{n(n-1)}{2} & n & 6 \\ \frac{n(n-1)(2n-1)}{6} & 2n^2 & 4n-2 \\ \left(\frac{n(n-1)}{2}\right)^2 & 3n^3 & 3n^2-3n \end{vmatrix}$$

$\Rightarrow \sum_{a=1}^{n}\Delta_a=\frac{n(n-1)}{12}\begin{vmatrix} 6 & n & 6 \\ 4n-2 & 2n^2 & 4n-2 \\ 3n^2-3n & 3n^3 & 3n^2-3n \end{vmatrix}=0$

11. Let $A\equiv\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=5$ ...(i)

$\therefore \begin{vmatrix} b_2c_3-b_3c_2 & c_2a_3-c_3a_2 & c_2b_3-c_3b_2 \\ b_3c_1-b_1c_3 & c_3a_1-c_1a_3 & a_3b_1-a_1b_3 \\ b_1c_2-b_2c_1 & c_1a_2-c_2a_1 & a_1b_2-a_2b_1 \end{vmatrix}$

$|\text{adj } A|=(5)^{3-1}$ [from Eq. (i)]

$=5^2=25$ $(\because |\text{adj } A|=|A|^{n-1})$

12. Applying $C_2\to C_2-C_1, C_3\to C_3-C_2$, we get $\begin{vmatrix} 1990 & 1 & 1 \\ 1991 & 1 & 1 \\ 1992 & 1 & 1 \end{vmatrix}=0$

13. We have,

$$\Delta=\begin{vmatrix} 1/a & 1 & bc \\ 1/b & 1 & ca \\ 1/c & 1 & ab \end{vmatrix}$$

$\Rightarrow \Delta=\frac{1}{abc}\begin{vmatrix} 1 & a & abc \\ 1 & b & abc \\ 1 & c & abc \end{vmatrix}$ Applying $R_1\to R_1(a)$, $R_2\to R_2(b)$ and $R_3\to R_3(c)$

14. We have, $\begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix} = Ax-12$

On putting $x = 1$ on both sides, we get

$$\begin{vmatrix} 2 & 2 & -1 \\ 4 & 3 & 0 \\ 6 & 1 & 1 \end{vmatrix} = A-12$$

Applying $C_1 \to C_1 - C_2$

$$\Rightarrow \begin{vmatrix} 0 & 2 & -1 \\ 1 & 3 & 0 \\ 5 & 1 & 1 \end{vmatrix} = A-12 \Rightarrow -2(1)+(-1)(-14) = A-12 \Rightarrow A = 24$$

15. Given, $f(x) = \begin{vmatrix} 1+a & 1+ax & 1+ax^2 \\ 1+b & 1+bx & 1+bx^2 \\ 1+b & 1+cx & 1+cx^2 \end{vmatrix}$

$\Rightarrow f(x) = \begin{vmatrix} 1+a & a(x-1) & ax(x-1) \\ 1+b & b(x-1) & bx(x-1) \\ 1+b & c(x-1) & cx(x-1) \end{vmatrix} = (x-1)x(x-1)\begin{vmatrix} 1+a & a & a \\ 1+b & b & b \\ 1+c & c & c \end{vmatrix}=0$ ($\because$ two columns are same)

16. Let $f(x) = a_0x^2 + a_1x + a_2$

and $g(x) = b_2x^2 + b_1x + b_2$

Also, $h(x) = c_0x^2 + c_1x + c_2$

Then, $\Delta(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ 2a_0x+a_1 & 2b_0x+b_1 & 2c_0x+c_1 \\ 2a_0 & 2b_0 & 2c_0 \end{vmatrix}$

$$= x\begin{vmatrix} f(x) & g(x) & h(x) \\ 2a_0 & 2b_0 & 2c_0 \\ 2a_0 & 2b_0 & 2c_0 \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ a_1 & b_1 & c_1 \\ 2a_0 & 2b_0 & 2c_0 \end{vmatrix} = 0 + 2\begin{vmatrix} f(x) & g(x) & h(x) \\ a_1 & b_1 & c_1 \\ a_0 & b_0 & c_0 \end{vmatrix}$$

$$= 2[(b_1c_0 - b_0c_1)f(x) - (a_1c_0 - a_0c_1)g(x) + (a_1b_0 - a_0b_1)h(x)]$$

Hence, degree of $\Delta(x) \leq 2$

17. $\begin{vmatrix} (a^x+a^{-x})^2 & (a^x-a^{-x})^2 & 1 \\ (b^x+b^{-x})^2 & (b^x-b^{-x})^2 & 1 \\ (c^x+c^{-x})^2 & (c^x-c^{-x})^2 & 1 \end{vmatrix}$

Applying $C_1 \to C_1 - C_2$

$$= \begin{vmatrix} 4 & (a^x-a^{-x})^2 & 1 \\ 4 & (b^x-b^{-x})^2 & 1 \\ 4 & (c^x-c^{-x})^2 & 1 \end{vmatrix} = 4\begin{vmatrix} 1 & (a^x-a^{-x})^2 & 1 \\ 1 & (b^x-b^{-x})^2 & 1 \\ 1 & (c^x-c^{-x})^2 & 1 \end{vmatrix} = 0$$

($\because$two columns are identical)

18. Given that, $x^a y^b = e^m, x^c y^d = e^n$

and $\Delta_1 = \begin{vmatrix} m & b \\ n & d \end{vmatrix}, \Delta_2 = \begin{vmatrix} a & m \\ c & n \end{vmatrix}, \Delta_3 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$\Rightarrow a\log x + b\log y = m \Rightarrow c\log x + d\log y = n$

By Cramer's rule

$\log x = \dfrac{\Delta_1}{\Delta_3}$ and $\log y = \dfrac{\Delta_2}{\Delta_3}$

$\Rightarrow x = e^{\Delta_1/\Delta_3}$ and $y = e^{\Delta_2/\Delta_3}$

19. We have,

$$\begin{vmatrix} a & a^2-bc & 1 \\ b & b^2-ca & 1 \\ c & c^2-ab & 1 \end{vmatrix} = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & -bc & 1 \\ b & -ca & 1 \\ c & -ab & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \frac{1}{abc}\begin{vmatrix} a^2 & -abc & a \\ b^2 & -abc & b \\ c^2 & -abc & c \end{vmatrix}$$ Applying $R_1 \to R_1(a)$ $R_2 \to R_2(b), R_3 \to R_3(c)$ in the IInd determinant

$$= \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} - \begin{vmatrix} a^2 & 1 & a \\ b^2 & 1 & b \\ c^2 & 1 & c \end{vmatrix} = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} - \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = 0$$

20. Let $\frac{x^2}{a^2} = X, \frac{y^2}{b^2} = Y$ and $\frac{z^2}{c^2} = Z$

Then the given system of equations becomes

$X + Y - Z = 1, X - Y + Z = 1, -X + Y + Z = 1$

The coefficient matrix is $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

Clearly, $|A| \neq 0$. So, the given system of equations has a unique solution

21. $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$

$= 1(1 - \log_y z \log_z y) - \log_x y(\log_y x - \log_z x \log_y z) + \log_x z\ (\log_z y\ \log_y x - \log\ _z x)$

$= (1 - \log_y y) - \log_x y\ (\log_y x - \log_y x) + \log_x z(\log_z x - \log_z x)$

$= (1-1) - 0 + 0 = 0$

22. We know that the sum of the products of the elements of a row with the cofactors of the corresponding elements is always equal to the value of the determinant $.ie, |A|$.

23. Let A be the first term and R be the common ratio of the GP. Then,

$a = A\,R^{p-1} \Rightarrow \log a = \log A + (p-1)\log R$

$b = A\,R^{q-1} \Rightarrow \log b = \log A + (q-1)\log R$

$c = A\,R^{r-1} \Rightarrow \log c = \log A + (r-1)\log R$

Now,

$$\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = \begin{vmatrix} (p-1) & \log R & p & 1 \\ (q-1) & \log R & q & 1 \\ (r-1) & \log R & r & 1 \end{vmatrix}$$

$$= \log R = \begin{vmatrix} p-1 & p & 1 \\ q-1 & q & 1 \\ r-1 & r & 1 \end{vmatrix} \text{ [Applying } C_1 \to C_1 - (\log A)\, C_3]$$

$$= \log R \begin{vmatrix} 0 & p & 1 \\ 0 & q & 1 \\ 0 & r & 1 \end{vmatrix} = 0 \text{ [Applying } C_1 \to C_1 - C_2 + C_3]$$

24. Given, $\Delta = \begin{vmatrix} (e^{i\alpha} + e^{-i\alpha})^2 & (e^{i\alpha} - e^{-i\alpha})^2 & 4 \\ (e^{i\beta} + e^{-i\beta})^2 & (e^{i\beta} - e^{-i\beta})^2 & 4 \\ (e^{i\gamma} + e^{-i\gamma})^2 & (e^{i\gamma} - e^{-i\gamma})^2 & 4 \end{vmatrix}$

Applying $C_1 \to C_1 - C_2$

$$= \begin{vmatrix} 4 & (e^{i\alpha} - e^{-i\alpha})^2 & 4 \\ 4 & (e^{i\beta} - e^{-i\beta})^2 & 4 \\ 4 & (e^{i\gamma} - e^{-i\gamma})^2 & 4 \end{vmatrix} = 0 \text{ } (\because \text{ two columns are same})$$

Hence, it is independent of α, β and γ.

25. $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + abc \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = 0 \Rightarrow (1+abc) \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = 0$$

$$\left[\therefore \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} \neq 0\right]$$

$\Rightarrow\ 1 + abc = 0 \Rightarrow abc = -1$

26. We have,

$$\begin{vmatrix} 1 & \omega^2 & \omega^5 \\ \omega^3 & 1 & \omega^4 \\ \omega^5 & \omega^4 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & \omega^2 \\ 1 & 1 & \omega \\ \omega^2 & \omega & 1 \end{vmatrix}$$

$= 2 - (\omega^2 - \omega) = 2 - (-1) = 3$

27. We can write $\Delta = \Delta_1 + y_1\Delta_2$, where

$$\Delta_1 = \begin{vmatrix} 1 & 1 + x_1y_2 & 1 + x_1y_3 \\ 1 & 1 + x_2y_2 & 1 + x_2y_3 \\ 1 & 1 + x_3y_2 & 1 + x_3y_3 \end{vmatrix} \text{ and } \Delta_2 = \begin{vmatrix} x_1 & 1 + x_1y_2 & 1 + x_1y_3 \\ x_2 & 1 + x_2y_2 & 1 + x_2y_3 \\ x_3 & 1 + x_3y_2 & 1 + x_3y_3 \end{vmatrix}$$

In Δ_1, use $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$ so that,

$$\Delta_1 = \begin{vmatrix} 1 & x_1y_2 & x_1y_3 \\ 1 & x_2y_2 & x_2y_3 \\ 1 & x_3y_2 & x_3y_3 \end{vmatrix} = 0 \quad (\because C_2 \text{ and } C_3 \text{ are proportional})$$

In Δ_2, $C_2 \to C_2 - y_2C_1$ and $C_3 \to C_3 - y_3C_1$ to get

$$\Delta_2 = \begin{vmatrix} x_1 & 1 & 1 \\ x_2 & 1 & 1 \\ x_3 & 1 & 1 \end{vmatrix} = 0 \quad (\because C_2 \text{ and } C_3 \text{ are identical})$$

$\therefore\ \Delta = 0$

28. Solve on your own.

29. Given $, a_1, a_2, a_3, \ldots \in$ GP

$\Rightarrow \log a_1, \log a_2, \ldots \in$ AP $\Rightarrow \log a_n, \log a_{n+1}, \log a_{n+2}, \ldots \in$ AP

$\Rightarrow \log a_{n+1} = \frac{\log a_n + \log a_{n+2}}{2}$ …(i)

Similarly, $\log a_{n+4} = \frac{\log a_{n+3} + \log a_{n+5}}{2}$ …(ii)

and $\log a_{n+7} = \frac{\log a_{n+6} + \log a_{n+8}}{2}$ …(ii)

Given, $\Delta = \begin{vmatrix} \log a_n & \log a_{n+1} & \log a_{n+2} \\ \log a_{n+3} & \log a_{n+4} & \log a_{n+5} \\ \log a_{n+6} & \log a_{n+7} & \log a_{n+8} \end{vmatrix}$

Applying $C_2 \to C_2 - \frac{C_1 + C_3}{2}$

$$\Delta = \begin{vmatrix} \log a_n & 0 & \log a_{n+2} \\ \log a_{n+3} & 0 & \log a_{n+5} \\ \log a_{n+6} & 0 & \log a_{n+8} \end{vmatrix} = 0$$

30. Since, the first column consists of sum of two terms, second column consists of sum of three terms and third column consists of sum four terms. $\therefore n = 2 \times 3 \times 4 = 24$

31. $\begin{bmatrix} 215 & 342 & 511 \\ 6 & 7 & 8 \\ 36 & 49 & 54 \end{bmatrix}$

$= 215(378 - 392) - 342(324 - 288) + 511(294 - 252) = -3010 - 12312 + 21462 = 6140$ (Which is exactly divisible by 20)

32. $\sum_{n=1}^{N} U_n = \begin{vmatrix} \sum n & 1 & 5 \\ \sum n^2 & 2N+1 & 2N+1 \\ \sum n^3 & 3N^2 & 3N \end{vmatrix}$

$$= \begin{vmatrix} \frac{N(N+1)}{2} & 1 & 5 \\ \frac{N(N+1)(2N+1)}{6} & 2N+1 & 2N+1 \\ \left(\frac{N(N+1)}{2}\right)^2 & 3N^2 & 3N \end{vmatrix} = \frac{N(N+1)}{12} \begin{vmatrix} 6 & 1 & 5 \\ 4N+2 & 2N+1 & 2N+1 \\ 3N(N+1) & 3N^2 & 3N \end{vmatrix}$$

Applying $C_3 \to C_3 + C_2$

$$= \frac{N(N+1)}{12} \begin{vmatrix} 6 & 1 & 6 \\ 4N+2 & 2N+1 & 4N+2 \\ 3N(N+1) & 3N^2 & 3N(N+1) \end{vmatrix} = 0 \quad (\because \text{two columns are identical})$$

33. Given, $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$

Applying $R_1 \to R_1 + R_2 + R_3$

$\Rightarrow (a+b+c-x) \begin{vmatrix} 1 & 1 & 1 \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0 \quad \Rightarrow (a+b+c-x) \begin{vmatrix} 1 & 0 & 0 \\ c & b-x-c & a-c \\ b & a-b & c-x-b \end{vmatrix} = 0$

$\Rightarrow (a+b+c-x)[1(b-x-c)(c-x-b) - (a-c)(a-b)] = 0$

$\Rightarrow (a+b+c-x)[bc - xb - b^2 - xc + x^2 + bx - c^2 + cx + bc - (a^2 - ab - ac + bc)] = 0 \Rightarrow (a+b+c-x)[x^2 - a^2 - b^2 - c^2 + ab + bc + ca] = 0$

$\Rightarrow x = a+b+c$ or $x^2 = a^2 + b^2 + c^2 + ab + bc + ca \Rightarrow x = 0$ or $x^2 = a^2 + b^2 + c^2 + \frac{1}{2}(a^2 + b^2 + c^2) \Rightarrow x = 0$ or $x = \pm\sqrt{\frac{3}{2}(a^2 + b^2 + c^2)}$

34. Given, $\Delta=\begin{vmatrix} 1 & \omega^n & \omega^{2n} \\ \omega^n & \omega^{2n} & 1 \\ \omega^{2n} & 1 & \omega^n \end{vmatrix}$

$= 1(\omega^{3n} - 1) - \omega^n(\omega^{2n} - \omega^{2n}) + \omega^{2n}(\omega^n - \omega^{4n})$

$= 1(1-1) - 0 + \omega^{2n}(\omega^n - \omega^n) \quad [\because \omega^3 - 1] = 0$

35. Putting $r = 1, 2, 3, \ldots, n$ and using the formula

$\sum 1 = n$ and $\sum r = \frac{(n+1)n}{2}$

$\sum(2r-1) = 1 + 3 + 5 + \ldots = n^2$

$\therefore \sum_{r=1}^{n} \Delta_r = \begin{vmatrix} n & n & n \\ n(n+1) & n^2+n+1 & n^2+n \\ n^2 & n^2 & n^2+n+1 \end{vmatrix} = 56$

Applying $C_1 \to C_1 - C_3, C_2 \to C_2 - C_3$

$\begin{vmatrix} 0 & 0 & n \\ 0 & 1 & n^2+n \\ -n-1 & -n-1 & n^2+n+1 \end{vmatrix}$

$\Rightarrow n(n+1) = 56 \Rightarrow n^2 + n - 56 = 0 \Rightarrow (n+8)(n-7) = 0 \Rightarrow n = 7 \quad (n \neq -8)$

36. Given, $A = B^{-1} AB$

$\Rightarrow BA = AB \therefore \det(B^{-1}AB) = \det(B^{-1}BA) = \det(A)$

37. Solve it on your own

38. We have,

$\Delta=\begin{vmatrix} [x]+1 & [y] & [z] \\ [x] & [y]+1 & [z] \\ [x] & [y] & [z]+1 \end{vmatrix} \Rightarrow \Delta=\begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ [x] & [y] & [z]+1 \end{vmatrix}$

$\begin{bmatrix} \text{Applying } R_1 \to R_1 - R_3 \\ R_2 \to R_2 - R_3 \end{bmatrix}$

$\Rightarrow \Delta = [z] + 1 + [y] + [x] = [x] + [y] + [z] + 1$

Since maximum values of $[x], [y]$ and $[z]$ are 1, 0 and 2 respectively

$\therefore$ Maximum value of $\Delta = 2 + 1 + 0 + 1 = 4$

39. We have,

$$\Delta = \begin{vmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{vmatrix} \Rightarrow \Delta = \begin{vmatrix} 1 & 3 & 5 \\ 4 & 5 & 7 \\ 9 & 7 & 9 \end{vmatrix} \text{ Applying } C_2 \to C_2 - C_1 \text{ and } C_3 \to C_3 - C_2$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 3 & 2 \\ 4 & 5 & 2 \\ 9 & 7 & 2 \end{vmatrix} \quad \text{Applying } C_3 \to C_3 - C_2 \Rightarrow \Delta = 2\begin{vmatrix} 1 & 3 & 1 \\ 4 & 5 & 1 \\ 9 & 7 & 1 \end{vmatrix}$$

$$\Rightarrow \Delta = 2\begin{vmatrix} 1 & 3 & 1 \\ 3 & 2 & 0 \\ 8 & 4 & 0 \end{vmatrix} \text{ Applying } R_2 \to R_2 - R_1, R_3 \to R_3 - R_1 \Rightarrow \Delta = 2 \times -4 = -8$$

40. We have, $a = 1 + 2 + 4 + 8 + \ldots$ upto n terms

$$= 1\left(\frac{2^n - 1}{2 - 1}\right) = 2^n - 1$$

$b = 1 + 3 + 9 + \ldots$ upto n terms$=\frac{3^n-1}{2}$and $c = 1 + 5 + 25 + \ldots$ upto n terms$=\frac{5^n-1}{4}$

$$\therefore \begin{vmatrix} a & 2b & 4c \\ 2 & 2 & 2 \\ 2^n & 3^n & 5^n \end{vmatrix} = 2\begin{vmatrix} 2^n - 1 & 3^n - 1 & 5^n - 1 \\ 1 & 1 & 1 \\ 2^n & 3^n & 5^n \end{vmatrix} = 2\begin{vmatrix} 2^n & 3^n & 5^n \\ 1 & 1 & 1 \\ 2^n & 3^n & 5^n \end{vmatrix} \quad [R_1 \to R_1 +$$

$R_2] = 2 \times 0 = 0$ [$\because$ two rows are identical]

41. Applying $R_1 \to R_1 + R_3 - 2R_2$, we get

$$\Delta = \begin{vmatrix} 0 & 0 & 0 & x + z - zy \\ 4 & 5 & 6 & y \\ 5 & 6 & 7 & z \\ x & y & z & 0 \end{vmatrix} = -(x + z - 2y)\begin{vmatrix} 4 & 5 & 6 \\ 5 & 6 & 7 \\ x & y & z \end{vmatrix} \text{ [Expanding along } R_1]$$

$$= -(x + z - 2y)\begin{vmatrix} 0 & -1 & 6 \\ 0 & -1 & 7 \\ x - 2y + z & y - z & z \end{vmatrix} \begin{bmatrix} \text{Applying } C_1 \to C_1 + C_3 \\ -2C_2 \text{ and } C_2 \to C_2 - C_3 \end{bmatrix}$$

$$= -(x + z - 2y)^2 \begin{vmatrix} -1 & 6 \\ -1 & 7 \end{vmatrix} = (x - 2y + z)^2$$

42. Solve it on your own.

43. Given, $\Delta(x) = \begin{vmatrix} 1 & \cos x & 1 - \cos x \\ 1 + \sin x & \cos x & 1 + \sin x - \cos x \\ \sin x & \sin x & 1 \end{vmatrix}$

Applying $C_3 \to C_3 + C_2 - C_1$

$$= \begin{vmatrix} 1 & \cos x & 0 \\ 1 + \sin x & \cos x & 0 \\ \sin x & \sin x & 1 \end{vmatrix} = \cos x - \cos x\,(1 + \sin x) = -\cos x \, \sin x = -\frac{1}{2}\sin 2x$$

$$\therefore \int_0^{\pi/2} \Delta x \, dx = -\frac{1}{2}\int_0^{\pi/2} \sin 2x \, dx = -\frac{1}{2}\left[-\frac{\cos 2x}{2}\right]_0^{\pi/2} = -\frac{1}{2}$$

44. Applying $C_3 \to C_3 - C_1$,we get

$$\Delta = \begin{vmatrix} 1 & \alpha & \alpha^2 - 1 \\ \cos(p-d)\,a & \cos pa & 0 \\ \sin(p-d)\,a & \sin pa & 0 \end{vmatrix}$$

$= (\alpha^2 - 1)\{\sin pa \cos(p-d)\,a - \cos pa \sin(p-d)\,a\} = (\alpha^2 - 1)\sin\{-(p - d)a + pa\} \Rightarrow \Delta = (\alpha^2 - 1)\sin da$ (Which is independent of p.)

45. $$\begin{vmatrix} [e] & [\pi] & [\pi^2 - 6] \\ [\pi] & [\pi^2 - 6] & [e] \\ [\pi^2 - 6] & [e] & [\pi] \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 3 & 3 \\ 3 & 3 & 2 \\ 3 & 2 & 3 \end{vmatrix} = 2(9-4) - 3(9-6) + 3(6-9) = 10 - 9 - 9 = -8$$

46. $f(1) = \begin{vmatrix} -2 & -16 & -78 \\ -4 & -48 & -496 \\ 1 & 2 & 3 \end{vmatrix} = 2928 \Rightarrow f(3) = \begin{vmatrix} 0 & 0 & 0 \\ -2 & -32 & -392 \\ 1 & 2 & 3 \end{vmatrix} = 0$

and $f(5) = \begin{vmatrix} 2 & 32 & 294 \\ 0 & 0 & 0 \\ 1 & 2 & 3 \end{vmatrix} = 0$

$\therefore\ f(1).f(3) + f(3).f(5) + f(5).f(1) = f(1).0 + 0 + f(1).0 = 0 = f(3)$ or $f\ (5)$

47. Solve on your own.

48. $\begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^n \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ a & -b & c \end{vmatrix}^1$

$= \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^n \begin{vmatrix} a+1 & a-1 & a \\ b+1 & b-1 & -b \\ c-1 & c+1 & c \end{vmatrix}$

$== \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^{n+1} \begin{vmatrix} a+1 & a & a-1 \\ b+1 & -b & b-1 \\ c-1 & c & c+1 \end{vmatrix}$

$C_2 \leftrightarrow C_3$

$= (1 + (-1)^{n+2}) \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix}$

This is equal to zero only, if $n + 2$ is odd ie, n is an odd integer.

49. Since, $-1 \leq x < 0$

$\therefore [x] = -1$

Also, $0 \leq y < 1 \Rightarrow [y] = 0$

and $1 \leq z < 2 \Rightarrow [z] = 1$

$\therefore$ Given determinant becomes

$$\begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & 0 & 2 \end{vmatrix} = 1 = [z]$$

50. We have, $\begin{vmatrix} p & b & c \\ p+a & q+b & 2c \\ a & b & r \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} p & b & c \\ p & b & c \\ a & b & r \end{vmatrix} + \begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$$

$$\Rightarrow 0 + \begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$$

$\Rightarrow p(qr - bc) - b(ar - ac) - c(ab - aq) = 0$

$\Rightarrow -pqr + pbc + bar + acq = 0$

On simplifying, we get

$\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} = 2$