INVERSE TRIGONOMETRIC FUNCTIONS

CHAPTER 02

FORMULAS

Trigonometric functions are not one-one on their natural domains, so their inverse does not exist in all values but their inverse may exists in some interval of their restricted domains. Thus, we can say that, inverse of trigonometric functions are defined within restricted domains of corresponding trigonometric functions.

Inverse of f is denoted by f^{-1}.

NOTE:

- $\sin^{-1} x \neq (\sin x)^{-1}$
- $\sin^{-1} x \neq \sin^{-1}\left(\frac{1}{x}\right)$
- $\sin^{-1} x \neq \frac{1}{\sin x}$

Domain and Principle Value Branch (Range) of Inverse Trigonometric Functions

FUNCTION	DOMAIN	PRINCIPAL VALUE BRANCH
$\sin^{-1} x$	$[-1,1]$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
$\cos^{-1} x$	$[-1,1]$	$[0,\pi]$
$\tan^{-1} x$	R	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$
$\csc^{-1} x$	$(-\infty,-1]\cup[1,\infty)$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$
$\sec^{-1} x$	$(-\infty,-1]\cup[1,\infty)$	$[0,\pi]-\left\{\frac{\pi}{2}\right\}$
$\cot^{-1} x$	R	$(0,\pi)$

ELEMENTARY PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS:

PROPERTY I

i) $\sin^{-1}(\sin\theta) = \theta, \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

ii) $\cos^{-1}(\cos\theta) = \theta, \theta \in [0, \pi]$

iii) $\tan^{-1}(\tan\theta) = \theta, \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

iv) $\cot^{-1}(\cot\theta) = \theta, \theta \in (0, \pi)$

v) $\csc^{-1}(\csc\theta) = \theta, \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

vi) $\sec^{-1}(\sec\theta) = \theta, \theta \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

vii) $\sin(\sin^{-1} x) = x, x \in [-1,1]$

viii) $\cos(\cos^{-1} x) = x, x \in [-1,1]$

ix) $\tan(\tan^{-1} x) = x, x \in R$

x) $\cot(\cot^{-1} x) = x, x \in R$

xi) $\csc(\csc^{-1} x) = x, x \in (-\infty, -1] \cup [1, \infty)$

xii) $\sec(\sec^{-1} x) = x, x \in (-\infty, -1] \cup [1, \infty)$

PROPERTY II

i) $\sin^{-1}\left(\frac{1}{x}\right) = \csc^{-1} x, x \geq 1 \text{ or } x \leq -1$

ii) $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x, x \geq 1 \text{ or } x \leq -1$

iii) $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x, x > 0 \\ -\pi + \cot^{-1} x, x < 0 \end{cases}$

PROPERTY III

i) $\sin^{-1}(-x) = -\sin^{-1} x, x \in [-1,1]$

ii) $\cos^{-1}(-x) = \pi - \cos^{-1} x, x \in [-1,1]$

iii) $\tan^{-1}(-x) = -\tan^{-1} x, x \in R$

iv) $\cot^{-1}(-x) = \pi - \cot^{-1} x, x \in R$

v) $\csc^{-1}(-x) = -\csc^{-1} x, |x| \geq 1 \text{ or } x \in (-\infty, 1] \cup [1, \infty)$

vi) $\sec^{-1}(-x) = \pi - \sec^{-1} x, |x| \geq 1 \text{or } x \in (-\infty, -1] \cup [1, \infty)$

PROPERTY IV

i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}, x \in [-1,1]$

ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}, x \in R$

iii) $\csc^{-1} x + \sec^{-1} x = \frac{\pi}{2}, x \in (-\infty, -1] \cup [1, \infty)$

PROPERTY V

i) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), xy < 1$

ii) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1-xy}\right), xy > -1$

PROPERTY VI

i) $2\tan^{-1} x = \sin^{-1}\left(\frac{2x}{1+x^2}\right), |x| \leq 1$

$or - 1 \leq x \leq 1$

ii) $2\tan^{-1} x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), x \geq 0$

iii) $2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right), -1 < x \leq 1$

Inverse Trigonometric Functions

$\sin^{-1}x \neq (\sin x)^{-1} = \dfrac{1}{\sin x}$

Trigonometric functions

(i) $\sin : \mathbb{R} \rightarrow [-1,1]$

(ii) $\cos : \mathbb{R} \rightarrow [-1,1]$

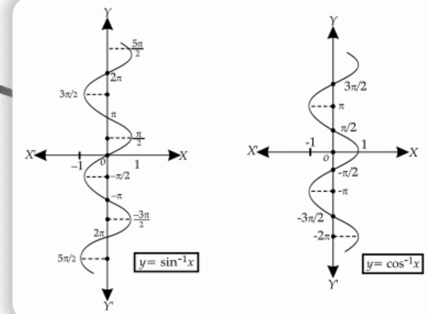
(iii) $\tan : \mathbb{R} - \left\{x : x = (2n+1)\dfrac{\pi}{2}, n \in Z\right\} \rightarrow \mathbb{R}$

(iv) $\cot : \mathbb{R} - \{x : x = n\pi, n \in Z\} \rightarrow \mathbb{R}$

(v) $\sec : \mathbb{R} - \left\{x : x = (2n+1)\dfrac{\pi}{2}, n \in Z\right\} \rightarrow \mathbb{R} - (-1,1)$

(vi) $\text{cosec} : \mathbb{R} - \{x : x = n\pi, n \in Z\} \rightarrow \mathbb{R} - (-1,1)$

Graphs of trigonometric functions



Principal value branch and principal value

The range of an inverse trigonometric function is the principal value branch and those values which lies in the principal value branch is called the principal value of that inverse trigonometric function.

If $x \geq 0$	If $x > 0$
$0 \le \sin^{-1} x \le \dfrac{\pi}{2}$	$-\dfrac{\pi}{2} \le \sin^{-1} x < 0$
$0 \le \cos^{-1} x \le \dfrac{\pi}{2}$	$\dfrac{\pi}{2} < \cos^{-1} x \le \pi$
$0 \le \tan^{-1} x \le \dfrac{\pi}{2}$	$-\dfrac{\pi}{2} \le \tan^{-1} x < 0$
$0 \le \cot^{-1} x \le \dfrac{\pi}{2}$	$\dfrac{\pi}{2} < \cot^{-1} x < \pi$
$0 \le \sec^{-1} x \le \dfrac{\pi}{2}$	$\dfrac{\pi}{2} < \sec^{-1} x \le \pi$
$0 \le \text{cosec}^{-1} x \le \dfrac{\pi}{2}$	$-\dfrac{\pi}{2} \le \text{cosec}^{-1} x < 0$

Domain and range of inverse trigonometric functions

(i) $y = \sin^{-1}x$. Domain = $[-1,1]$, Range = $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

(ii) $y = \cos^{-1}x$. Domain = $[-1,1]$ Range = $[0, \pi]$

(iii) $y = \text{cosec}^{-1}x$. Domain = $\mathbb{R} - (-1,1)$, Range = $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] - \{0\}$

(iv) $y = \sec^{-1}x$. Domain = $\mathbb{R} - (-1,1)$, Range = $[0, \pi] - \left\{\dfrac{\pi}{2}\right\}$

(v) $y = \tan^{-1}x$. Domain = $\mathbb{R}$, Range = $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

(vi) $y = \cot^{-1}x$. Domain = $\mathbb{R}$, Range = $(0, \pi)$.

Some important relations

(i) $y = \sin^{-1}x \Rightarrow x = \sin y$

(ii) $x = \sin y \Rightarrow y = \sin^{-1}x$

(iii) $\sin(\sin^{-1}x) = x, -1 \le x \le 1$

(iv) $\sin^{-1}(\sin x) = x, \dfrac{-\pi}{2} \le x \le \dfrac{\pi}{2}$

(v) $\sin^{-1}\dfrac{1}{x} = \text{cosec}^{-1}x$

(vi) $\cos^{-1}(-x) = \pi - \cos^{-1}x$

(vii) $\cos^{-1}\dfrac{1}{x} = \sec^{-1}x$

(viii) $\cot^{-1}(-x) = \pi - \cot^{-1}x$

(ix) $\tan^{-1}\dfrac{1}{x} = \cot^{-1}x, x > 0$

(x) $\sec^{-1}(-x) = \pi - \sec^{-1}x$

(xi) $\sin^{-1}(-x) = -\sin^{-1}x$

(xii) $\tan^{-1}(-x) = -\tan^{-1}x$

(xiii) $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}, -1 \le x \le 1$

(xiv) $\tan^{-1}x + \cot^{-1}x = \dfrac{\pi}{2}$

(xv) $\text{cosec}^{-1}x + \sec^{-1}x = \dfrac{\pi}{2}, |x| \ge 1$

(xvi) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\dfrac{x+y}{1-xy}, xy < 1$

(xvii) $2\tan^{-1}x = \tan^{-1}\dfrac{2x}{1-x^2}, -1<x<1$

(xviii) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\dfrac{x-y}{1+xy}, xy > 1$

(xix) $2\tan^{-1}x = \sin^{-1}\dfrac{2x}{1+x^2} = \cos^{-1}\dfrac{1-x^2}{1+x^2}$

How to understand Mind Map?

- First Level
- Second Level
- Third Level

PRACTICE QUESTIONS:

1. The number of real solutions of $\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$ is
    a) 0
    b) 1
    c) 2
    d) ∞
2. The value of $\cot^{-1}\frac{xy+1}{x-y} + \cot^{-1}\frac{yz+1}{y-z} + \cot^{-1}\frac{zx+1}{z-x}$ is
    a) 0
    b) 1
    c) $\cot^{-1}x + \cot^{-1}y + \cot^{-1}z$
    d) None of the above
3. $\tan^{-1}\frac{m}{n} - \tan^{-1}\frac{m-n}{m+n}$ is equal to
    a) $\tan^{-1}\frac{n}{m}$
    b) $\tan^{-1}\frac{m+n}{m-n}$
    c) $\frac{\pi}{4}$
    d) $\tan^{-1}\left(\frac{1}{2}\right)$
4. If $\cos^{-1}\sqrt{p} + \cos^{-1}\sqrt{1-p} + \cos^{-1}\sqrt{1-q} = \frac{3\pi}{4}$, then the value of q is
    a) 1
    b) $\frac{1}{\sqrt{2}}$
    c) $\frac{1}{3}$
    d) $\frac{1}{2}$
5. $\cos^{-1}\left(\frac{15}{17}\right) + 2\tan^{-1}\left(\frac{1}{5}\right) =$
    a) $\frac{\pi}{2}$
    b) $\cos^{-1}\left(\frac{171}{221}\right)$
    c) $\frac{\pi}{4}$
    d) None of these

6. The number of real solution of $\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$ is

a) 0
b) 1
c) 2
d) ∞

7. If $-1 \le x \le -\frac{1}{2}$, then $\cos^{-1}(4x^3 - 3x)$ equals

a) $3\cos^{-1} x$
b) $2\pi - 3\cos^{-1} x$
c) $-2\pi + 3\cos^{-1} x$
d) None of these

8. The value of $\sum_{r=0}^{\infty} \tan^{-1}\left(\frac{1}{1+r+r^2}\right)$ is equal to

a) $\frac{\pi}{2}$
b) $\frac{3\pi}{4}$
c) $\frac{\pi}{4}$
d) None of these

9. If $\tan^{-1}\frac{1-x}{1+x} = \frac{1}{2}\tan^{-1} x$, then the value of x is

a) $\frac{1}{2}$
b) $\frac{1}{\sqrt{3}}$
c) $\sqrt{3}$
d) 2

10. If $a < \frac{1}{32}$, then the number of solutions of $(\sin^{-1} x)^3 + (\cos^{-1} x)^3 = a\,\pi^3$, is

a) 0
b) 1
c) 2
d) Infinite

11. If $A = \tan^{-1} x, x \in R$, then the value of $\sin 2A$ is

a) $\frac{2x}{1-x^2}$

b) $\frac{2x}{\sqrt{1-x^2}}$

c) $\frac{2x}{1+x^2}$

d) $\frac{1-x^2}{1+x^2}$

12. If $a_1, a_2, a_3, \ldots, a_n$ are in AP with common ratio d, then $\tan\left[\tan^{-1}\frac{d}{1+a_1a_2} + \tan^{-1}\frac{d}{1+a_2a_3} + \ldots + \tan^{-1}\frac{4}{1+a_{n-1}a_n}\right]$ is equal to

a) $\frac{(n-1)d}{a_1+a_n}$

b) $\frac{(n-1)d}{1+a_1a_n}$

c) $\frac{nd}{1+a_1a_n}$

d) $\frac{a_n-a_1}{a_n+a_1}$

13. If $\theta = \sin^{-1} x + \cos^{-1} x - \tan^{-1} x, 1 \leq x < \infty$, then the smallest interval in which θ lies is

a) $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{4}$

b) $0 \leq \theta \leq \frac{\pi}{4}$

c) $-\frac{\pi}{4} \leq \theta \leq 0$

d) $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$

14. The value of $'a'$ for which $ax^2 + \sin^{-1}(x^2 - 2x + 2) + \cos^{-1}(x^2 - 2x + 2) = 0$ has a real solution, is

a) $-\frac{2}{\pi}$

b) $\frac{2}{\pi}$

c) $-\frac{\pi}{2}$

d) $\frac{\pi}{2}$

15. $\tan\left[\frac{\pi}{2}+\frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right)\right]+\tan\left[\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right)\right]$ is equal to

a) $\frac{2a}{b}$

b) $\frac{2b}{a}$

c) $\frac{a}{b}$

d) $\frac{b}{a}$

16. If $\sum_{i=1}^{20}\sin^{-1}x_i = 10\pi$,then $\sum_{i=1}^{20}x_i$ is equal to

a) 20

b) 10

c) 0

d) None of these

17. If $\tan\theta+\tan\left(\frac{\pi}{3}+\theta\right)+\tan\left(\frac{-\pi}{3}+\theta\right)=K\tan 3\,\theta$, then the value of K is

a) 1

b) 1/3

c) 3

d) None of these

18. If $[\sin^{-1}\cos^{-1}\sin^{-1}x]=1$, where [.] denotes the greatest integer function, then x belongs to the interval

a) [tan sin cos 1, tan sin cos sin 1]

b) (tan sin cos 1, tan sin cos sin 1)

c) $[-1,1]$

d) [sin cos tan 1, sin cos sin tan 1]

19. If $\sin^{-1}(2x\sqrt{1-x^2})-2\sin^{-1}x=0$, then x belongs to the interval

a) $[-1,1]$

b) $\left[-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right]$

c) $\left[-1,-\frac{1}{\sqrt{2}}\right]$

d) $\left[\frac{1}{\sqrt{2}},1\right]$

20. If $e^{[\sin^2\alpha+\sin^4\alpha+\sin^6\alpha+\ldots\infty]\log_e 2}$ is a root of equation $x^2-9x+8=0$,where $0<\alpha<\frac{\pi}{2}$,

then the principle value of $\sin^{-1}\sin\left(\frac{2\pi}{3}\right)$ is

a) α

b) 2α

c) $-\alpha$

d) -2α

21. If $\tan^{-1}\left(\frac{a}{x}\right)+\tan^{-1}\left(\frac{b}{x}\right)=\frac{\pi}{2}$,then x is equal to

a) $\sqrt{ab}$

b) $\sqrt{2ab}$

c) $2ab$

d) ab

22. The sum of series

$\tan^{-1}\frac{1}{1+1+1^2}+\tan^{-1}\frac{1}{1+2+2^2}+\tan^{-1}\frac{1}{1+3+3^2}+\ldots\infty$ is equal to

a) $\frac{\pi}{4}$

b) $\frac{\pi}{2}$

c) $\frac{\pi}{3}$

d) $\frac{\pi}{6}$

23. $\tan^{-1}\frac{c_1x-y}{c_1y+x}+\tan^{-1}\frac{c_2-c_1}{1+c_2c_1}+\tan^{-1}\frac{c_3-c_2}{1+c_3c_2}+\ldots+\tan^{-1}\frac{1}{c_n}$ is equal to

a) $\tan^{-1}\frac{y}{x}$

b) $\tan^{-1}yx$

c) $\tan^{-1}\frac{x}{y}$

d) $\tan^{-1}(x-y)$

24. If a, b, c be positive real number and the value of $\theta = \tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}} + \tan^{-1}\sqrt{\frac{b(a+b+c)}{ca}} + \tan^{-1}\sqrt{\frac{c(c+b+c)}{ab}}$. Then $\tan\theta$ is equal to

a) 0

b) 1

c) $\frac{a+b+c}{abc}$

d) None of these

25. $\tan\left[\frac{1}{2}\sin^{-1}\left(\frac{2a}{1+a^2}\right) + \frac{1}{2}\cos^{-1}\left(\frac{1-a^2}{1+a^2}\right)\right]$ is equal to

a) $\frac{2a}{1+a^2}$

b) $\frac{1-a^2}{1+a^2}$

c) $\frac{2a}{1-a^2}$

d) None of these

26. If $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1$, then x is equal to

a) 1

b) 0

c) 4/5

d) 1/5

27. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$, then the value of $x^{100} + y^{100} + z^{100} - \frac{9}{x^{101}+y^{101}+z^{101}}$, is

a) 0

b) 1

c) 2

d) 3

28. The value of $\cos[\tan^{-1}\{\sin(\cot^{-1} x)\}]$ is

a) $\sqrt{\frac{x^2+1}{x^2-1}}$

b) $\sqrt{\frac{1-x^2}{x^2+2}}$

c) $\sqrt{\frac{1-x^2}{1+x^2}}$

d) $\sqrt{\frac{x^2+1}{x^2+2}}$

29. If $\sin^{-1}\left(\frac{2x}{1+x^2}\right) + \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = 4\tan^{-1} x$, then

a) $x \in -(-\infty, -1)$

b) $x \in (1, \infty)$

c) $x \in [0, 1]$

d) $x \in [-1, 0]$

30. $\sin\left(2\sin^{-1}\sqrt{\frac{63}{65}}\right)$ is equal to

a) $\frac{2\sqrt{126}}{65}$

b) $\frac{4\sqrt{65}}{65}$

c) $\frac{8\sqrt{63}}{65}$

d) $\frac{\sqrt{63}}{65}$

31. The value of $\sin\left(4\tan^{-1}\frac{1}{3}\right) - \cos\left(2\tan^{-1}\frac{1}{7}\right)$ is

a) $\frac{3}{7}$

b) $\frac{7}{8}$

c) $\frac{8}{21}$

d) None of these

32. If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = 3\pi$, then $xy + yz + zx$ is equal to

a) 0

b) 1

c) 3

d) -3

33. For the equation $\cos^{-1}x + \cos^{-1}2x + \pi = 0$, then the number of real solutions is

a) 1

b) 2

c) 0

d) ∞

34. $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$ is equal to

a) π

b) $\frac{\pi}{2}$

c) $\frac{\pi}{3}$

d) $\frac{\pi}{4}$

35. If in a $\Delta\, ABC, \angle A = \tan^{-1}2$ and $\angle B = \tan^{-1}3$, then angle C is equal to

a) $\frac{\pi}{2}$

b) $\frac{\pi}{3}$

c) $\frac{\pi}{4}$

d) None of these

36. If $A=\tan^{-1}\left(\frac{x\sqrt{3}}{2k-x}\right)$ and $B = \tan^{-1}\left(\frac{2x-k}{k\sqrt{3}}\right)$,then the value of $A - B$ is

a) 10°

b) 45°

c) 60°

d) 30°

37. If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\pi$, then $x^4+y^4+z^4+4x^2y^2z^2=k(x^2y^2+y^2z^2+z^2x^2)$ Where k is equal to

a) 1

b) 2

c) 4

d) None of these

38. The trigonometric equation $\sin^{-1}x=2\sin^{-1}a$, has a solution for

a) $\frac{1}{2}<|a|<\frac{1}{\sqrt{2}}$

b) All real values of a

c) $|a|\leq\frac{1}{2}$

d) $|a|\geq\frac{1}{\sqrt{2}}$

39. If $-1\leq x\leq-\frac{1}{\sqrt{2}}$, then $\sin^{-1}\left(2x\sqrt{1-x^2}\right)$ equals

a) $2\sin^{-1}x$

b) $\pi-2\sin^{-1}x$

c) $-\pi-2\sin^{-1}x$

d) None of these

40. If $x^2+y^2+z^2=r^2$, then $\tan^{-1}\left(\frac{xy}{zr}\right)+\tan^{-1}\left(\frac{yz}{xr}\right)+\tan^{-1}\left(\frac{xz}{yr}\right)$ is equal to

a) π

b) $\frac{\pi}{2}$

c) 0

d) None of these

41. $\cos^{-1}\left\{\frac{1}{2}x^2+\sqrt{1-x^2}\sqrt{1-\frac{x^2}{4}}\right\}=\cos^{-1}\frac{x}{2}-\cos^{-1}x$ holds for

a) $|x|\leq1$

b) $x\in R$

c) $0\leq x\leq1$

d) $-1\leq x\leq0$

42. If θ and ϕ are the roots of the equation $8x^2 + 22x + 5 = 0$, then

a) Both $\sin^{-1}\theta$ and $\sin^{-1}\phi$ are equal

b) Both $\sec^{-1}\theta$ and $\sec^{-1}\phi$ are equal

c) Both $\tan^{-1}\theta$ and $\tan^{-1}\phi$ are equal

d) None of these

43. Cos $[\tan^{-1}\{\sin(\cot^{-1}x)\}]$ is equal to

a) $\sqrt{\frac{x^2+2}{x^2+3}}$

b) $\sqrt{\frac{x^2+2}{x^2+1}}$

c) $\sqrt{\frac{x^2+1}{x^2+2}}$

d) None of these

44. Sum of infinite terms of the series

$\cot^{-1}\left(1^2+\frac{3}{4}\right)+\cot^{-1}\left(2^2+\frac{3}{4}\right)+\cot^{-1}\left(3^2+\frac{3}{4}\right)+\ldots$ is

a) $\frac{\pi}{4}$

b) $\tan^{-1}(2)$

c) $\tan^{-1}3$

d) None of these

45. If $3\sin^{-1}\frac{2x}{1+x^2}-4\cos^{-1}\frac{1+x}{1+x^2}+2\tan^{-1}\frac{2x}{1-x^2}=\frac{\pi}{3}$, then value of x is

a) $\sqrt{3}$

b) $\frac{1}{\sqrt{3}}$

c) 1

d) None of these

46. If $[\cot^{-1}x]+[\cos^{-1}x]=0$, where x is a non-negative real number and [.] denotes the greatest integer function, then complete set of values of x is

a) $(\cos 1, 1]$

b) $(\cot 1, 1)$

c) $(\cos 1, \cot 1)$

d) None of these

47. If $\cot(\cos^{-1} x) = \sec\left(\tan^{-1} \frac{a}{\sqrt{b^2-a^2}}\right)$, then x is equal to

a) $\frac{b}{\sqrt{2b^2-a^2}}$

b) $\frac{a}{\sqrt{2b^2-a^2}}$

c) $\frac{\sqrt{2b^2-a^2}}{a}$

d) $\frac{\sqrt{2b^2-a^2}}{b}$

48. $\sum_{m=1}^{n} \tan^{-1}\left(\frac{2m}{m^4+m^2+2}\right)$ is equal to

a) $\tan^{-1}\left(\frac{n^2+n}{n^2+n+2}\right)$

b) $\tan^{-1}\left(\frac{n^2-n}{n^2-n+2}\right)$

c) $\tan^{-1}\left(\frac{n^2+n+2}{n^2+n}\right)$

d) None of these

49. If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{3\pi}{2}$

and $f(1) = 2, f(p+q) = f(p).f(q), \forall\, p, q \in R$, then$x^{f(1)} + y^{f(2)} + z^{f(3)} - \frac{(x+y+z)}{x^{f(1)}+y^{f(2)}+z^{f(3)}}$ is equal to

a) 0

b) 1

c) 2

d) 3

50. $\tan\left(\frac{\pi}{4} + \frac{1}{2}\cos^{-1} x\right) + \tan\left(\frac{\pi}{4} - \frac{1}{2}\cos^{-1} x\right), x \neq 0$ is equal to

a) x

b) $2x$

c) $\frac{2}{x}$

d) None of these

ANSWER KEY

1. C
2. A
3. C
4. D
5. D
6. C
7. C
8. A
9. B
10. A
11. C
12. B
13. B
14. C
15. B
16. A
17. C
18. A
19. B
20. A
21. A
22. A
23. C
24. A
25. C
26. D
27. A
28. D
29. C
30. A
31. D
32. C
33. C
34. D
35. C
36. D
37. B
38. C
39. C
40. B
41. A
42. C
43. C
44. B
45. B
46. B
47. A
48. A
49. C
50. C

HINTS AND SOLUTIONS

1. Given, $\tan^{-1}\sqrt{x(x+1)} = \frac{\pi}{2} - \sin^{-1}\sqrt{x^2+x+1}$

$\Rightarrow \cos^{-1}\frac{1}{\sqrt{(x^2+x)^2+1}} = \cos^{-1}\sqrt{x^2+x+1}$

$\Rightarrow \frac{1}{\sqrt{(x^2+x)^2+1}} = \sqrt{x^2+x+1}$

$\Rightarrow 1 = (x^2+x+1)[(x^2+x)^2+1]$

$\Rightarrow (x^2+x)^3 + (x^2+x)^2 + (x^2+x) + 1 = 1$

$\Rightarrow (x^2+x)\{(x^2+x)^2 + (x^2+x) + 1\} = 0$

$\Rightarrow x^2 + x = 0$

$\Rightarrow x = 0, -1$

2. $\cot^{-1}\frac{xy+1}{x-y} + \cot^{-1}\frac{yz+1}{y-z} + \cot^{-1}\frac{zx+1}{z-x}$

$= \cot^{-1} y - \cos^{-1} x + \cot^{-1} z - \cot^{-1} y + \cot^{-1} x - \cot^{-1} z$

3. $\tan^{-1} = \frac{m}{n} - \tan^{-1}\frac{m-n}{m+n}$

$= \tan^{-1}\frac{m}{n} - \tan^{-1}\frac{\frac{m}{n}-1}{1+\frac{m}{n}}$

$= \tan^{-1}\frac{m}{n} - \tan^{-1}\frac{m}{n} + \tan^{-1}(1) = \frac{\pi}{4}$

4. Let $\alpha = \cos^{-1}\sqrt{P}, \beta = \cos^{-1}\sqrt{1-P}$ and $\gamma = \cos^{-1}\sqrt{1-q}$

$\Rightarrow \cos\alpha = \sqrt{p},\ \cos\beta = \sqrt{1-p}$ and $\cos\gamma = \sqrt{1-q}$

Therefore, $\sin\alpha = \sqrt{1-p}, \sin\beta = \sqrt{p}$ and $\sin\gamma = \sqrt{q}$

The given equation may be written as

$\alpha + \beta + \gamma = \frac{3\pi}{4} \Rightarrow \alpha + \beta = \frac{3\pi}{4} - \gamma$

$\Rightarrow \cos(\alpha+\beta) = \cos\left(\frac{3\pi}{4} - \gamma\right) \Rightarrow \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= \cos\left\{\pi - \left(\frac{\pi}{4} + \gamma\right)\right\} = -\cos\left(\frac{\pi}{4} + \gamma\right) \Rightarrow \sqrt{p}\sqrt{1-p} - \sqrt{1-p}\sqrt{p} = -\left(\frac{1}{\sqrt{2}}\sqrt{1-q} - \frac{1}{\sqrt{2}}\sqrt{q}\right)$

$\Rightarrow 0 = \sqrt{1-q} - \sqrt{q} \Rightarrow 1 - q = q \Rightarrow q = \frac{1}{2}$

5. We have,

$$\cos^{-1}\left(\frac{15}{17}\right) + 2\tan^{-1}\left(\frac{1}{5}\right)$$

$$= \cos^{-1}\left(\frac{15}{17}\right) + \cos^{-1}\left(\frac{1-1/25}{1+1/25}\right)$$

$$= \cos^{-1}\left(\frac{15}{17}\right) + \cos^{-1}\left(\frac{12}{13}\right)$$

$$= \cos^{-1}\left\{\frac{15}{17} \times \frac{12}{13} - \sqrt{1-\left(\frac{15}{17}\right)^2}\sqrt{1-\left(\frac{12}{13}\right)^2}\right\} = \cos^{-1}\left(\frac{140}{221}\right)$$

6. Clearly, $x(x+1) \geq 0$ and $x^2 + x + 1 \leq 1$

$\Rightarrow x(x+1) = 0$

$\Rightarrow x = 0, -1$

When $x = 0$,

LHS=$\tan^{-1} 0 + \sin^{-1} 1 = \frac{\pi}{2}$

When $x = -1$,

LHS=$\tan^{-1} 0 + \sin^{-1} \sqrt{1-1+1}$

$= 0 + \sin^{-1}(1) = \frac{\pi}{2}$

Thus, the number of solution is 2

7. Let $\cos^{-1} x = \theta$. Then, $x = \cos\theta$

Also,

$-1 \leq x \leq -\frac{1}{2} \Rightarrow -1 \leq \cos\theta \leq -\frac{1}{2} \Rightarrow \frac{2\pi}{3} \leq \theta \leq \pi$

Now, $\cos^{-1}(4x^3 - 3x) = \cos^{-1}(\cos 3\theta) = \cos^{-1}(\cos(2\pi - 3\theta))$

$= \cos^{-1}(\cos(3\theta - 2\pi))$

$= 3\theta - 2\pi \ \left[\because \frac{2\pi}{3} \leq \theta \leq \pi \Rightarrow 0 \leq 3\theta - 2\pi \leq \pi\right] = 3\cos^{-1} x - 2\pi$

8. $\because \tan^{-1}\left(\frac{1}{1+r+r^2}\right) = \tan^{-1}\left(\frac{r+1-r}{1+r(r+1)}\right)$

$= \tan^{-1}(r+1) - \tan^{-1}(r)$

$\therefore \sum_{r=0}^{n}[\tan^{-1}(r+1) - \tan^{-1}(r)]$

$= \tan^{-1}(n+1) - \tan^{-1}(0)$

$= \tan^{-1}(n+1)$

$\Rightarrow \sum_{r=0}^{\infty} \tan^{-1}\left(\frac{1}{1+r+r^2}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$

9. Given, $\tan^{-1}\frac{1-x}{1+x} = \frac{1}{2}\tan^{-1} x$

Let $x = \tan\theta$

$\therefore \tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right) = \frac{1}{2}\tan^{-1}(\tan\theta)$

$\Rightarrow \tan^{-1}\left\{\tan\left(\frac{\pi}{4} - \theta\right)\right\} = \frac{1}{2}\tan^{-1}(\tan\theta)$

$\Rightarrow \frac{\pi}{4} - \theta = \frac{\theta}{2} \Rightarrow \theta = \frac{\pi}{6}$

$\therefore x = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

10. Given, $\tan\left\{\sec^{-1}\left(\frac{1}{x}\right)\right\} = \sin(\tan^{-1} 2)$

$\Rightarrow \tan\left(\tan^{-1}\frac{\sqrt{1-x^2}}{x}\right) = \sin\left(\sin^{-1}\frac{2}{\sqrt{1+2^2}}\right)$

$\left[\because \tan^{-1} x = \sin^{-1}\frac{x}{\sqrt{1+x^2}}\right]$

$\Rightarrow \frac{\sqrt{1-x^2}}{x} = \frac{2}{\sqrt{5}} \Rightarrow 4x^2 = 5(1 - x^2)$

$\Rightarrow x^2 = \frac{5}{9} \Rightarrow x = \frac{\sqrt{5}}{3}$

11. SOLVE ON YOUR OWN

12. Given expression

$\tan\left[\tan^{-1}\frac{a_2-a_1}{1+a_1a_2} + \tan^{-1}\frac{a_3-a_2}{1+a_2a_3} + \ldots + \tan^{-1}\frac{a_n-a_{n-1}}{1+a_na_{n-1}}\right]$

$= \tan[\tan^{-1} a_2 - \tan^{-1} a_1 + \tan^{-1} a_3 - \tan^{-1} a_2 + \ldots + \tan^{-1} a_n - \tan^{-1} a_{n-1}]$

$= \tan[\tan^{-1} a_n - \tan^{-1} a_1] = \frac{a_n - a_1}{1+a_1a_n}$

$= \frac{(n-1)d}{1+a_1a_n}$

13. $\theta = \sin^{-1} x + \cos^{-1} x - \tan^{-1} x = \frac{\pi}{2} - \tan^{-1} x$

$\left[\because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}\right]$

$\Rightarrow \theta = \cot^{-1} x$

Since, $1 \le x < \infty$, therefore $0 \le \theta \le \frac{\pi}{4}$

14. Here, $x^2 - 2x + 2 = (x-1)^2 + 1 \geq 1$

But $-1 \leq (x^2 - 2x + 2) \leq 1$

Which is possible only when

$x^2 - 2x + 2 = 1$

$\Rightarrow x = 1$

Then, $a(1)^2 + \sin^{-1}(1) + \cos^{-1}(1) = 0$

$\Rightarrow a + \frac{\pi}{2} + 0 = 0$

$\Rightarrow a = -\frac{\pi}{2}$

15. $\therefore \tan\left[\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right)\right] + \tan\left[\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right)\right]$

$= \tan\left[\frac{\pi}{4} + \phi\right] + \tan\left[\frac{\pi}{4} - \phi\right]$

$\left[\text{put} \frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right) = \phi \Rightarrow \cos 2\phi = \frac{a}{b}\right]$

$= \dfrac{1 + \tan\phi}{1 - \tan\phi} + \dfrac{1 - \tan\phi}{1 + \tan\phi}$

$= \frac{2(1+\tan^2\phi)}{1-\tan^2\phi} = \frac{2}{\cos 2\phi} = \frac{2b}{a}$

16. Since, $-\frac{\pi}{2} < \sin^{-1} x \leq \frac{\pi}{2}$

$\therefore \sin^{-1} x_i = \frac{\pi}{2}, 1 \leq i \leq 20$

$\Rightarrow x_i = 1, 1 \leq i \leq 20$

Thus, $\sum_{i=1}^{20} x_i = 20$

17. We have,

$\tan\theta + \tan\left(\frac{\pi}{3} + \theta\right) + \tan\left(\frac{-\pi}{3} + \theta\right) = K\tan 3\theta$

$\Rightarrow \tan\theta + \tan(60 + \theta) + \tan(-60 + \theta) = K\tan 3\theta$

$\Rightarrow \tan\theta + \frac{\sqrt{3}+\tan\theta}{1-\sqrt{3}\tan\theta} + \frac{\tan\theta-\sqrt{3}}{1+\sqrt{3}\tan\theta} = K\tan 3\theta$

$\Rightarrow \tan\theta + \frac{8\tan\theta}{1-3\tan^2\theta} = K\tan 3\theta$

$\Rightarrow \frac{3(3\tan\theta-\tan^3\theta)}{(1-3\tan^2\theta)} = K\tan 3\theta$

$\Rightarrow 3\tan 3\theta = K\tan 3\theta \Rightarrow K = 3$

18. We have, $1 \le \sin^{-1}\cos^{-1}\sin^{-1}\tan^{-1}x \le \frac{\pi}{2}$

$\Rightarrow \sin 1 \le \cos^{-1}\sin^{-1}\tan^{-1}x \le 1$

$\Rightarrow \cos\sin 1 \ge \sin^{-1}\tan^{-1}x \ge \cos 1$

$\Rightarrow \sin\cos\sin 1 \ge \tan^{-1}x \ge \sin\cos 1$

$\Rightarrow \tan\sin\cos\sin 1 \ge x \ge \tan\sin\cos 1$

$\therefore x \in [\tan\sin\cos 1, \tan\sin\cos\sin 1]$

19. Let $\sin^{-1}x = \theta$. Then, $x = \sin\theta$ and $\sqrt{1-x^2} = \cos\theta$

Now,

$\sin^{-1}\left(2x\sqrt{1-x^2}\right)$

$= \sin^{-1}(\sin 2\theta) = 2\theta, \text{ if } -\frac{\pi}{2} \le 2\theta \le \frac{\pi}{2}$

$= 2\sin^{-1}x, \text{ if } -\frac{\pi}{4} \le \theta \le \frac{\pi}{4} \text{ i.e. if } -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$

$\therefore \sin^{-1}\left(2x\sqrt{1-x^2}\right) - 2\sin^{-1}x = 0, \text{ if } -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$

20. Roots of equation $x^2 - 9x + 8 = 0$ are 1 and 8

Let $y = [\sin^2\alpha + \sin^4\alpha + \sin^6\alpha + \ldots\infty]\log_e 2$

$\Rightarrow y = \frac{\sin^2\alpha}{1-\sin^2\alpha}\log_e 2 = \tan^2\alpha\log_e 2$

$\Rightarrow y = \log_e 2^{\tan^2\alpha}$

$\Rightarrow e^y = 2^{\tan^2\alpha}$

According to question,

$2^{\tan^2\alpha} = 8 = 2^3 \Rightarrow \tan^2\alpha = 3$

$\Rightarrow \tan\alpha = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}$

$\therefore \sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \pi - \frac{2\pi}{3} = \frac{\pi}{3} = \alpha$

21. $\because \tan^{-1}\left(\frac{a}{x}\right) + \tan^{-1}\left(\frac{b}{x}\right) = \frac{\pi}{2}$

$\Rightarrow \tan^{-1}\left(\frac{\frac{a}{x}+\frac{b}{x}}{1-\frac{ab}{x^2}}\right) = \frac{\pi}{2}$

$\Rightarrow \frac{\frac{a}{x}+\frac{b}{x}}{1-\frac{ab}{x^2}} = \tan\frac{\pi}{2} \Rightarrow 1 - \frac{ab}{x^2} = 0$

$\Rightarrow x^2 = ab \Rightarrow x = \sqrt{ab}$

22. Given series can be rewritten as

$$\sum_{r=1}^{\infty} \tan^{-1}\left(\frac{1}{1+r+r^2}\right)$$

Now, $\tan^{-1}\left(\frac{1}{1+r+r^2}\right) = \tan^{-1}\left(\frac{r+1-r}{1+r(r+1)}\right)$

$= \tan^{-1}(r+1) - \tan^{-1}(r)$

$\therefore \sum_{r=1}^{n}[\tan^{-1}(r+1) - \tan^{-1} r]$

$= \tan^{-1}(n+1) - \tan^{-1}(1)$

$= \tan^{-1}(n+1) - \frac{\pi}{4}$

$\Rightarrow \sum_{r=1}^{\infty} \tan^{-1}\left(\frac{1}{1+r+r^2}\right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

23. $\tan^{-1}\left(\frac{c_1-y}{c_1y+x}\right) + \tan^{-1}\left(\frac{c_2-c_1}{1+c_2c_1}\right) + \tan^{-1}\left(\frac{c_3-c_2}{1+c_3c_2}\right) + \ldots + \tan^{-1}\frac{1}{c_n}$

$= \tan^{-1}\left(\frac{\frac{x}{y}-\frac{1}{c_1}}{1+\frac{x}{y}\cdot\frac{1}{c_1}}\right) + \tan^{-1}\left(\frac{\frac{1}{c_1}-\frac{1}{c_2}}{1+\frac{1}{c_1c_2}}\right) + \tan^{-1}\left(\frac{\frac{1}{c_2}-\frac{1}{c_3}}{1+\frac{1}{c_2c_3}}\right) + \ldots + \tan^{-1}\frac{1}{c_n}$

$= \tan^{-1}\frac{x}{y} - \tan^{-1}\frac{1}{c_1} + \tan^{-1}\frac{1}{c_1} - \tan^{-1}\frac{1}{c_2} + \tan^{-1}\frac{1}{c\_2}$

$- \tan^{-1}\frac{1}{c_3} + \ldots + \tan^{-1}\frac{1}{c_{n-1}} - \tan^{-1}\frac{1}{c_n} + \tan^{-1}\frac{1}{c_n}$

$= \tan^{-1}\left(\frac{x}{y}\right)$

24. $\theta = \tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}} + \tan^{-1}\sqrt{\frac{b(a+b+c)}{ca}} + \tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}$

Let $s^2 = \frac{a+b+c}{abc}$

Hence, $\theta = \tan^{-1}\sqrt{a^2s^2} + \tan^{-1}\sqrt{b^2s^2} + \tan^{-1}\sqrt{c^2s^2}$

$= \tan^{-1}(as) + \tan^{-1}(bs) + \tan^{-1}(cs)$

$= \tan^{-1}\left[\frac{as+bs+cs-abcs^3}{1-abs^2-acs^2-bcs^2}\right]$

Hence, $\tan\theta = \left[\frac{s[a+b+c)-abcs^2]}{1-(ab+bc+ca)s^2}\right]$

$= \left[\frac{s[(a+b+c)-(a+b+c)]}{1-s^2(ab+bc+ca)}\right] = 0$

25. $\tan\left[\frac{1}{2}\sin^{-1}\left(\frac{2a}{1+a^2}\right)+\frac{1}{2}\cos^{-1}\left(\frac{1-a^2}{1+a^2}\right)\right]$

$= \tan\left[\frac{1}{2}\cdot 2\tan^{-1}a+\frac{1}{2}\cdot 2\tan^{-1}a\right]$

$= \tan\left(2\tan^{-1}a\right)$

$= \tan\left[\tan^{-1}\left(\frac{2a}{1-a^2}\right)\right]$

$= \frac{2a}{1-a^2}$

26. We have,

$\sin(\sin^{-1}1/5+\cos^{-1}x)=1$

$\Rightarrow \sin^{-1}\frac{1}{5}+\cos^{-1}x=\frac{\pi}{2}$

$\Rightarrow \sin^{-1}\frac{1}{5}=\frac{\pi}{2}-\cos^{-1}x \Rightarrow \sin^{-1}\frac{1}{5}=\sin^{-1}x \Rightarrow x=\frac{1}{5}$

27. We know that $|\sin^{-1}x|\le\frac{\pi}{2}$

$\therefore \sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{3\pi}{2}$

$\Rightarrow \sin^{-1}x=\sin^{-1}y=\sin^{-1}z=\frac{\pi}{2}$

$\Rightarrow x=y=z=\sin\frac{\pi}{2}=1$

$\therefore x^{100}+y^{100}+z^{100}-\frac{9}{x^{101}+y^{101}+z^{101}}=3-\frac{9}{3}-0$

28. $\cos[\tan^{-1}\{\sin(\cot^{-1}x)\}]$

$= \cos\left[\tan^{-1}\left\{\sin\left(\sin^{-1}\frac{1}{\sqrt{1+x^2}}\right)\right\}\right]$

$= \cos\left[\tan^{-1}\frac{1}{\sqrt{1+x^2}}\right]$

$= \cos\left[\cos^{-1}\sqrt{\frac{1+x^2}{2+x^2}}\right]$

$= \sqrt{\frac{1+x^2}{2+x^2}}$

29. We know that

$\sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1}x$ for all $x \in [-1, 1]$

And,

$\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = 2\tan^{-1}x$ for all $x \in [0, \infty)$

$\therefore \sin^{-1}\left(\frac{2x}{1+x^2}\right) + \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = 4\tan^{-1}x$ for all $x \in [0, 1]$

30. $\sin\left(2\sin^{-1}\sqrt{\frac{63}{65}}\right) = \sin\left(\sin^{-1} 2\sqrt{\frac{63}{65}}\sqrt{1-\frac{63}{65}}\right)$

$= \sin\left(\sin^{-1}\frac{2\sqrt{126}}{65}\right) = \frac{2\sqrt{126}}{65}$

31. We have,

$\sin\left(4\tan^{-1}\frac{1}{3}\right)$

$= 2\sin\left(2\tan^{-1}\frac{1}{3}\right)\cos\left(2\tan^{-1}\frac{1}{3}\right)$

$= 2\sin\left(\tan^{-1}\frac{3}{4}\right)\cos\left(\tan^{-1}\frac{3}{4}\right)$

$= 2\sin\left(\sin^{-1}\frac{3}{5}\right)\cos\left(\cos^{-1}\frac{4}{5}\right) = 2 \times \frac{3}{5} \times \frac{4}{5} = \frac{24}{25}$

And,

$\cos\left(2\tan^{-1}\frac{1}{7}\right) = \cos\left(\tan^{-1}\frac{7}{24}\right) = \cos\left(\cos^{-1}\frac{24}{25}\right) = \frac{24}{25}$

Hence, the value of given expression is 0

32. Given that , $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = 3\pi$

$\because 0 \leq \cos^{-1}x \leq \pi$

Similarly, $0 \leq \cos^{-1}y \leq \pi$

And $0 \leq \cos^{-1}z \leq \pi$

Here, $\cos^{-1}x\cos^{-1}y = \cos^{-1}z = \pi$

$\Rightarrow x = y = z = \cos\pi = -1$

$\therefore xy + yz + zx = (-1)(-1) + (-1)(-1) + (-1)(-1)$ =1+1+1=3

33. Given equation is $\cos^{-1}x + \cos^{-1}2x + \pi = 0$

$\Rightarrow \cos^{-1}x + \cos^{-1}2x = -\pi$

$\Rightarrow \cos^{-1}\left(x.2x - \sqrt{1-x^2}\sqrt{1-4x^2}\right) = -\pi$

$\Rightarrow \; 2x^2 - \sqrt{1-x^2}\sqrt{1-4x^2} = -1$

$\Rightarrow \; (1+2x^2) = \sqrt{1-x^2}\sqrt{1-4x^2}$

On squaring both sides, we get

$1 + 4x^2 + 4x^2 = (1-x^2)(1-4x^2)$

$\Rightarrow \; 1 + 4x^4 + 4x^2 = 1 - 5x^2 + 4x^4$

$\Rightarrow \; 9x^2 = 0$

$\Rightarrow \; x = 0$

But $x = 0$ is not satisfied the given equation.

∴ The number of real solution is zero.

34. $4\tan^{-1}\frac{1}{5} = 2\left[2\tan^{-1}\;\frac{1}{5}\right]$

$= 2\tan^{-1}\frac{\frac{2}{5}}{1-\frac{1}{25}} = 2\tan^{-1}\frac{5}{12} = \tan^{-1}\frac{\frac{10}{12}}{1-\frac{25}{144}}$

$= \tan^{-1}\frac{120}{119}$

So, $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239} = \tan^{-1}\frac{120}{119} - \tan^{-1}\frac{1}{239}$

$= \tan^{-1}\frac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}\times\frac{1}{239}}$

$= \tan^{-1}\frac{(120\times239)-119}{(119\times239)+120}$

$= \tan^{-1}\frac{28561}{28561} = \tan^{-1}1 = \frac{\pi}{4}$

35. Given that, $\angle A = \tan^{-1}2, \angle B = \tan^{-1}3$

We know that, $\angle A + \angle B + \angle C = \pi$

$\Rightarrow \tan^{-1}2 + \tan^{-1}3 + \angle C = \pi$

$\Rightarrow \tan^{-1}\left(\frac{2+3}{1-2\times3}\right) + \angle C = \pi$

$\Rightarrow \tan^{-1}(-1) + \angle C = \pi$

$\Rightarrow \frac{3\pi}{4} + \angle C = \pi$

$\Rightarrow \angle C = \frac{\pi}{4}$

36. We know that,

$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$

$=\frac{\frac{x\sqrt{3}}{2k-x}-\frac{2x-k}{k\sqrt{3}}}{1+\frac{x\sqrt{3}}{2k-x}\cdot\frac{2x-k}{k\sqrt{3}}}=\frac{1}{\sqrt{3}}$

$\Rightarrow A-B=30^\circ$

37. We have, $\sin^{-1}x+\sin^{-1}y=\pi-\sin^{-1}z$

or $x\sqrt{1-y^2}+y\sqrt{1-x^2}=z$

or $x^2(1-y^2)=z^2+y^2(1-x^2)-2yz\sqrt{(1-x^2)}$

or $(x^2-z^2-y^2)^2=4y^2z^2(1-x^2)$

or $x^4+y^4+z^4-2x^2z^2+2y^2z^2-2x^2y^2+4x^2y^2z^2-4y^2z^2=0$

or $x^4+y^4+z^4+4x^2y^2z^2=2(x^2y^2+y^2z^2+z^2x^2)$

$\therefore\ k=2$

38. Given, $\sin^{-1}x=2\sin^{-1}a$

Since, $-\frac{\pi}{2}\le\sin^{-1}x\le\frac{\pi}{2}$

$\Rightarrow\quad -\frac{\pi}{2}\le 2\sin^{-1}a\le\frac{\pi}{2}$

$\Rightarrow\quad -\frac{\pi}{4}\le\sin^{-1}a\le\frac{\pi}{4}$

$\Rightarrow\quad \sin\left(-\frac{\pi}{4}\right)\le a\le\sin\frac{\pi}{4}$

$\Rightarrow\quad -\frac{1}{\sqrt{2}}\le a\le\frac{1}{\sqrt{2}}$

$|a|\le\frac{1}{\sqrt{2}}$

39. Let $\sin^{-1}x=\theta$. Then, $x=\sin\theta$ and $\sqrt{1-x^2}=\cos\theta$

Now,

$-1\le x\le-\frac{1}{\sqrt{2}}\Rightarrow-1\le\sin\theta\le-\frac{1}{\sqrt{2}}\Rightarrow-\frac{\pi}{2}\le\theta\le-\frac{\pi}{4}$

$\therefore\sin^{-1}\left(2x\sqrt{1-x^2}\right)=\sin^{-1}(\sin2\theta)=\sin^{-1}(-\sin(\pi+2\theta))$

$=\sin^{-1}(\sin(-\pi-2\theta))$

$=-\pi-2\theta\left[\because-\frac{\pi}{2}\le\theta\le-\frac{\pi}{4}\Rightarrow-\frac{\pi}{2}\le-\pi-2\theta\le0\right]=-\pi-2\sin^{-1}x$

40. Given that, $x^2 + y^2 + z^2 = r^2$

Now, $\tan^{-1}\left(\frac{xy}{zr}\right) + \tan^{-1}\left(\frac{yz}{xe}\right) + \tan^{-1}\left(\frac{xz}{yr}\right)$

$= \tan^{-1}\left[\frac{\frac{xy}{zr}+\frac{yz}{xr}+\frac{xz}{yr}-\frac{xyz}{r^3}}{1-\left(\frac{x^2+y^2+z^2}{r^2}\right)}\right]$

$= \tan^{-1}\left[\frac{\frac{xy}{zr}+\frac{yz}{xr}+\frac{xz}{yr}-\frac{xyz}{r^3}}{1-\frac{r^2}{r^2}}\right] = \tan^{-2}\infty = \frac{\pi}{2}$

41. Since, $0 \le \cos^{-1}\left(\frac{x^2}{2} + \sqrt{1-x^2}\sqrt{1-\frac{x^2}{4}}\right) \le \frac{\pi}{2}$

Because $\cos^{-1} x$ is in first quadrant when x is positive

And $\cos^{-1}\frac{x}{2} - \cos^{-1} x \ge 0$

So, $\cos^{-1}\frac{x}{2} \ge \cos^{-1} x$

Also, $\left|\frac{x}{2}\right| \le 1, |x| \le 1 \Rightarrow |x| \le 1$

42. $8x^2 + 22x + 5 = 0 \Rightarrow x = -\frac{1}{4}, -\frac{5}{2}$

$\therefore -1 < -\frac{1}{4} < 1$ and $-\frac{5}{2} < -1$

$\therefore \sin^{-1}\left(-\frac{1}{4}\right)$ exists but $\sin^{-1}\left(-\frac{5}{2}\right)$ does not exist.

$\sec^{-1}\left(-\frac{5}{2}\right)$ exists but $\sec^{-1}\left(-\frac{1}{4}\right)$does not exist.

$\tan^{-1}\left(-\frac{1}{4}\right)$ and $\tan^{-1}\left(-\frac{5}{2}\right)$ both exist.

43. SOLVE ON YOUR OWN

44. Here, $T_n = \cot^{-1}\left(n^2 + \frac{3}{4}\right)$

$= \tan^{-1}\left(\frac{4}{4n^2+3}\right)$

$= \tan^{-1}\left(\frac{1}{1+\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}\right)$

$= \tan^{-1}\left[\frac{\left(n+\frac{1}{2}\right)-\left(n-\frac{1}{2}\right)}{1+\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}\right]$

$= \tan^{-1}\left(n + \frac{1}{2}\right) - \tan^{-1}\left(n - \frac{1}{2}\right)$

$\therefore S_\infty = T_\infty^{-1} - \tan^{-1}\left(\frac{1}{2}\right)$

$= \frac{\pi}{2} - \tan^{-1}\left(\frac{1}{2}\right)$

$\Rightarrow S_\infty = \cot^{-1}\left(\frac{1}{2}\right)$

$\Rightarrow S_\infty = \tan^{-1}(2)$

45. $3\sin^{-1}\frac{2x}{1+x^2} - 4\cos^{-1}\frac{1-x^2}{1+x^2} + 2\tan^{-1}\frac{2x}{1-x^2} = \frac{\pi}{3}$

On putting $x = \tan\theta$, we get

$3\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) - 4\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) + 2\tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \frac{\pi}{3}$

$\Rightarrow 3\sin^{-1}(\sin 2\theta) - 4\cos^{-1}(\cos 2\theta) + 2\tan^{-1}(\tan 2\theta) = \frac{\pi}{3}$

$\Rightarrow 3(2\theta) - 4(2\theta) + 2(2\theta) = \frac{\pi}{3}$

$\Rightarrow 6\theta - 8\theta + 4\theta = \frac{\pi}{3}$

$\Rightarrow \theta = \frac{\pi}{6} \Rightarrow \tan^{-1}x = \frac{\pi}{6}$

$\Rightarrow x = \tan\frac{\pi}{6} \Rightarrow x = \frac{1}{\sqrt{3}}$

$\because 0 \le \cos^{-1}x \le \pi$

46. And $0 < \cot^{-1} x < \pi$

Given, $[\cot^{-1} x] + [\cot^{-1} x] = 0$

$\Rightarrow [\cot^{-1} x] = 0$ and $[\cos^{-1} x] = 0$

$\Rightarrow 0 < \cot^{-1} x < 1$ and $0 \leq \cos^{-1} x < 1$

$\therefore x \in (\cot 1, \infty)$ and $x \in (\cos 1, 1)$

$\Rightarrow x \in (\cot 1, 1)$

47. Given, $\cot(\cos^{-1} x) = \sec\left(\tan^{-1} \frac{a}{\sqrt{b^2-a^2}}\right)$

$\therefore \quad \cot\left(\cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\right)$

$= \sec\left(\sec^{-1} \frac{b}{\sqrt{b^2-a^2}}\right)$

$\Rightarrow \quad \frac{x}{\sqrt{1-x^2}} = \frac{b}{\sqrt{b^2-a^2}}$

$\Rightarrow \quad x^2(b^2 - a^2) = b^2 - b^2x^2$

$\Rightarrow \quad x^2(2b^2 - a^2) = b^2$

$\Rightarrow \quad x = \frac{b}{\sqrt{2b^2-a^2}}$

48. We have, $\sum_{m=1}^{n} \tan^{-1}\left(\frac{2m}{m^4+m^2+2}\right)$

$= \sum_{m=1}^{n} \tan^{-1}\left(\frac{2m}{1+(m^2+m+1)(m^2-m+1)}\right)$

$= \sum_{m=1}^{n} \tan^{-1}\left(\frac{(m^2+m+1)-(m^2-m+1)}{1+(m^2+m+1)(m^2-m+1)}\right)$

$= \sum_{m=1}^{n}[\tan^{-1}(m^2 + m + 1) - \tan^{-1}(m^2 - m + 1)]$

$= (\tan^{-1} 3 - \tan^{-1} 1) + (\tan^{-1} 7 - \tan^{-1} 3) +$

$(\tan^{-1} 13 - \tan^{-1} 7) + \ldots +$

$[\tan^{-1}(n^2 + n + 1) - \tan^{-1}(n^2 - n + 1)]$

$= \tan^{-1} \frac{n^2+n+1-1}{1+(n^2+n+1)\cdot 1} = \tan^{-1}\left(\frac{n^2+n}{2+n^2+n}\right)$

49. $\because -\frac{\pi}{2} \leq \sin^{-1} x \leq \frac{\pi}{2}, -\frac{\pi}{2} \leq \sin^{-1} y \leq \frac{\pi}{2}$

And $-\frac{\pi}{2} \leq \sin^{-1} z \leq \frac{\pi}{2}$

Given that, $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{3\pi}{2}$

Which is possible only when

$\sin^{-1} x = \sin^{-1} y = \sin^{-1} z = \frac{\pi}{2}$

Or $x = y = z = 1$

Put $p = q = 1$

Then $f(2) = f(1)f(1) = 2 \cdot 2 = 4$

And put $p = 1, q = 2$

Then, $f(3) = f(1)f(2) = 2 \cdot 2^2 = 8$

$\therefore x^{f(1)} + y^{f(2)} + z^{f(3)} - \frac{x+y+z}{x^{f(1)}+y^{f(2)}+z^{f(3)}}$

$= 1 + 1 + 1 - \frac{3}{1+1+1}$

$= 3 - 1 = 2$

50. SOLVE ON YOUR OWN.