LINEAR PROGRAMMING PROBLEMS

CHAPTER 13

LPP

LINEAR PROGRAMMING PROBLEM (LPP):

A linear programming problem is one that is concerned with finding the optimal value(maximum or minimum value) of a linear function(called objective function) of several variables(say x and y called decision variable), subject to the constraints that the variables are non-negative and satisfy a set of linear inequalities(called linear constraints).

MATHEMATICAL FORM OF LPP:

The general mathematical form of a linear programming problem may be written as Maximize or Minimize $Z = c_1x + c_2y$

Subject to constraints are $a_1x + b_1y \leq d_1$, $a_2x + b_2y \leq d_2$, etc. and non-negative restrictions are $x \geq 0, y \geq 0$.

SOME TERMS RELATED TO LPP:

i. **CONSTRAINTS:** The linear inequations or inequalities or restrictions on the variables of a linear programming problem are called constraints. The conditions $x \geq 0, y \geq 0$ are called non-negative restrictions.

ii. **OPTIMISATION PROBLEM:** A problem which seeks to maximise or minimise a linear function subject to certain constraints determined by a set of linear inequalities is called an optimisation problem. Linear programming problems are special type of optimisation problems.

iii. **OBJECTIVE FUNCTION:** A linear function of two or more variables which has to be maximised or minimised under the given restrictions in the form of linear inequations or linear constraints is called the objective function. The variables used in the objective function are called decision variables.

iv. **OPTIMAL VALUES:** The maximum or minimum value of an objective function is known as its optimal value.

v. **FEASIBLE AND INFEASIBLE REGIONS:** The common region determined by all the constraints including non-negative constraints $x, y \geq 0$ of a linear programming problem is called the feasible region or solution region. Each point in this region represents a feasible choice. The region other than feasible region is called an infeasible region.

vi. **BOUNDED AND UNBOUNDED REGIONS:** A feasible region of a system of linear inequalities is said to be bounded, if it can be enclosed within a circle. Otherwise, it is said to be unbounded region.

vii. **FEASIBLE AND INFEASIBLE SOLUTIONS:** Any solution to the given linear programming problem which also satisfies the non-negative restrictions of the problem is called a feasible solution. Any point outside the feasible region is called an infeasible solution.

viii. **OPTIMAL SOLUTION:** A feasible solution at which the objective function has optimal value is called the optimal solution of the linear programming problem.

ix. **OPTIMISATION TECHNIQUE:** The process of obtaining the optimal solution is called optimisation technique

<u>IMPORTANT THEOREMS:</u>

i. **THEOREM 1:** Let R be the feasible region (convex polygon) for a linear programming problem and Z=ax+by by the objective function. When Z has an optimal value(maximum or minimum), where the variables x and y are subject to constraints described by linear inequalities, this optimal value must occur at a corner point(vertex) of the feasible region.(A corner point of a feasible region is a point of intersection of two boundary lines in the region)

ii. **THEOREM 2:** Let R be the feasible region for a linear programming problem and Z=ax+by be the objective function. If R is bounded, then the objective function Z has both a maximum and a minimum value on R and each of these occurs at a corner point (vertex) of R.

GRAPHICAL METHOD OF SOLVING LPP:

I CORNER-POINT METHOD:

STEP 1 Formulate the given LPP in mathematical form if it is not so.

STEP 2 Convert all inequations into equations and draw their graphs. To draw the graph of a linear equation, put y=0 in it and obtain the graph representing the equation.

STEP 3 Determine the region represented by each equation. To determine the region represented by an equation replace x and y both by zero, if the inequation reduces to a valid statement, then the region containing the origin is the region represented by the given inequation. Otherwise, the region not containing the origin is the region represented by the given inequation.

STEP 4 Obtain the region in xy-plane containing all points that simultaneously satisfy all the constraints including non-negativity restrictions. The polygon region so obtained is the feasible region and is known as the convex polygon of the set of all feasible solutions of the LPP.

STEP 5 Determine the coordinates the vertices (corner points) of the convex polygon obtained in Step 2. These vertices are known as the extreme points of the set of all feasible solutions of the LPP.

STEP 6 Obtain the values of the objective function at each of the vertices of the convex polygon. The point where the objective minimum) value is the optimal solution of the given LPP.

II **<u>ISO-PROFIT OR ISO COST METHOD</u>**

STEP 1 Formulate the given LPP in mathematical form if it is not so.

STEP 2 Obtain the region in xy-plane containing all points that simultaneously satisfy all constraints including non-negativity restrictions. The polygonal region so obtained is the convex set of all feasible solutions of the given LPP and it is known as the feasible region.

STEP 3 Determine the coordinates of the vertices of the feasible region obtained in step2.

STEP 4 Give some convenient value to Z and draw the line so obtained in xy-plane.

STEP 5 If the objective function is of maximization type, then draw lines parallel to the line in step 4 and obtain a line which is farthest from the origin and has at least one common point to the feasible region.

If the objective function is of maximization type, then draw lines parallel to the line in step 4 and obtain a line which is nearest to the origin and has at least one common point to the feasible region.

STEP 6 Find the coordinates of the common point obtained in step 5. The point so obtained determine the optimal solution and the value of the objective function at these points give the optimal solution.

Linear Programming

Definition

A L.P.P. is one that is concerned with finding the optimal value (max. or min.) of a linear function of several variables (called objective function) subject to the conditions that the variables are non-negative and satisfy a set of linear inequalities (called linear constraints). Variables are sometimes called decision variables and are non-negative.

Solution of a L.P.P.

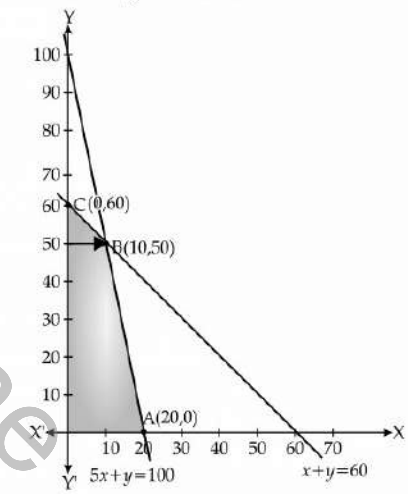
The common region determined by all the constraints including the non-negative constraint $x \geq 0, y \geq 0$ of a L.P.P. is called the feasible region (or solution region) for the problem. Points within and on the boundary of the feasible region represent feasible solutions of the constraints. Any point outside the feasible region is an infeasible solution. Any point in the feasible region that gives the optimal value (max. or min.) of the objective function is called an optimal solution.

e.g., : Max $Z = 250x + 75y$, subject to the

Constraints: $5x + y \leq 100$

$x + y \leq 60$

$x \geq 0, y \geq 0$ is an L.P.P.



Fundamental Theorems

Theorem 1 : Let R be the feasible region (convex polygon) for a L.P.P. and let $Z=ax+by$ be the objective function. When Z has an optimal value (max. or min.), where the variables x,y are subject to the constraints described by linear inequalities, this optimal value must occur at a corner point (vertex) of the feasible region,

Theorem 2: Let R be the feasible region for a L.P.P. and let $Z=ax+by$ be the objective function. If R is bounded then the objective function Z has both a max. and a min. value on R and each of these occurs at a corner point (vertex) of R. If the feasible region is unbounded, then a max. or a min. may not exist. If it exists, it must occur at a corner point of R.

Corner point method

1. Find the feasible region of the linear programming problem and determine its corner points (vertices) either by inspection or by solving the two equations of the lines intersecting at that point.
2. Evaluate the objective function $Z = ax+by$ at each corner point. Let M and m, respectively denote the largest and smallest values of these points.
3. (i) When the feasible region is **bounded,** M and m are the maximum and minimum values of Z

   (ii) In case, the feasible region is **unbounded,** we have:

   (a) M is the maximum value of Z, if the open half plane determined by $ax+by>M$ has no point in common with the feasible region. Otherwise, Z has no maximum value.

   (b) Similarly, m is the minimum value of Z, if the open half plane determined by $ax+by<m$ has no point in common with the feasible region. Otherwise, Z has no minimum value.

Trace the Mind Map

- First Level
- Second Level
- Third Level

PRACTICE QUESTIONS:

1. The solution set of the inequation $2x + y > 5$ is
   a) Half plane that consists the origin
   b) Open half plane not containing the origin
   c) Whole xy-plane except the points lying on the line 2x+y=5
   d) None of these
2. Objective function of a LPP is
   a) a constraint
   b) a function to be optimized
   c) a relation between the variables
   d) none of these
3. Which of the following sets are convex?
   a) $\{(x, y): x^2 + y^2 \geq 1\}$
   b) $\{(x, y): y^2 \geq x\}$
   c) $\{(x, y): 3x^2 + 4y^2 \geq 5\}$
   d) $\{(x, y): y \geq 2, y \leq 4\}$
4. Let $X_1 \, and \, X_2$ are optimal solutions of a LPP, then
   a) $X = \lambda X_1 + (1 - \lambda)X_2, \lambda \in R$ is also an optimal solution.
   b) $X = \lambda X_1 + (1 - \lambda)X_2, 0 \leq \lambda \leq 1$ gives an optimal solution.
   c) $X = \lambda X_1 + (1 + \lambda)X_2, 0 \leq \lambda \leq 1$ gives an optimal solution.
   d) $X = \lambda X_1 + (1 + \lambda)X_2, \lambda \in R$ gives an optimal solution.
5. The maximum value of Z=4x+2y subjected to the constraints $2x + 3y \leq 18, x + y \geq 10;\ x, y \geq 0$ is
   a) 36
   b) 40
   c) 20
   d) None of these

6. The optimal value of the objective function is attained at the points
   a) given by intersection of inequations with the axes only
   b) given by intersection of inequations with x-axis only
   c) given by corner points of the feasible region
   d) none of these
7. The maximum value of Z=4x+3y subjected to the constraints $3x + 2y \geq 160, 5x + 2y \geq 200, x + 2y \geq 80;\ x, y \geq 0$ is
   a) 320
   b) 300
   c) 230
   d) None of these
8. Consider a LPP given by

   Minimum Z=6x+10y

   Subjected to $x \geq 6; y \geq 2; 2x + y \geq 10; x, y \geq 0$

   Redundant constraints in this LPP are
   a) $x, y \geq 0$
   b) $x \geq 6, 2x + y \geq 10$
   c) $2x + y \geq 10$
   d) None of these
9. The objective function Z=4x+3y can be maximised subjected to the constraints $3x + 4y \leq 24, 8x + 6y \leq 48, x \leq 5, y \leq 6; x, y \geq 0$
   a) at only one point
   b) at two points only
   c) at an infinite number of points
   d) none of these
10. If the constraints in a linear programming problem are changed
   a) the problem is to be re-evaluated
   b) solution is not defined
   c) the objective function has to be modified
   d) the change in constraints is ignored

11. Which of the following statements is correct?

a) Every LPP admits an optimal solution

b) A LPP admits unique optimal solution

c) If a LPP admits two optimal solutions it has an infinite number of optimal solutions

d) The set of all feasible solutions of a LPP is not a converse set

12. Which of the following is not a convex set?

a) $\{(x, y): 2x + 5y < 7$

b) $\{(x, y): x^2 + y^2 \leq 4\}$

c) $\{x: |x| = 5\}$

d) $\{(x, y): 3x^2 + 2y^2 \leq 6\}$

13. By graphical method, the solution of linear programming problem

Maximize $Z = 3x_1 + 5x_2$

Subject to $3x_1 + 2x_2 \leq 18, x_1 \leq 4$, $x_2 \leq 6$, $x_1 \geq 0, x_2 \geq 0$ is

a) $x_1 = 2 , x_2 = 0 , Z = 6$

b) $x_1 = 2 , x_2 = 6 , Z = 36$

c) $x_1 = 4 , x_2 = 3 , Z = 27$

d) $x_1 = 4 , x_2 = 6 , Z = 42$

14. The region represented by the inequation system $x, y \geq 0, y \leq 6, x + y \leq 3$ is

a) unbounded in first quadrant

b) unbounded in first and second quadrant

c) bounded in first quadrant

d) none of these

15. The point at which the maximum value of x+y, subject to the constraints $x + 2 \leq 70, 2x + y \leq 95; x, y \geq 0$ is obtained, is

a) (30,25)

b) (20,35)

c) (35,20)

d) (40,15)

16. The value of the objective function is maximum under linear constraints

a) at the centre of feasible region

b) at (0,0)

c) at any vertex of feasible region

d) the vertex which is at a maximum distance from (0,0).

17. The corner points of the feasible region determined by the following system of linear inequalities: $2x + y \leq 10, x + 3y \leq 15\ ; x, y \geq 0$ are (0,0),(5,0),(3,4) and (0,5). Let Z=px+qy, where p,q>0. Condition on p and q so that the maximum of Z occurs at both (3,4) and (0,5) is

a) p=q

b) p=2q

c) p=3q

d) q=3p

18. The corner points of the feasible region determined by the system of linear constraints are : (0,10),(5,5),(15,15),(0,20). Let z=px+qy, where p,q>0. Condition on p and q so that the maximum of z occurs at both the points (15,15) and (0,20) is

a) p=q

b) p=2q

c) q=2p

d) q=3p

19. Corner points of the feasible region determined by the system of linear constraints (0,3),(1,1) and (3,0).Let z=px+qy, where p,q>0. Condition on p and q so that the minimum of z occurs at

a) p=2q

b) 2p=q

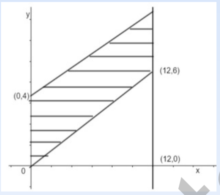
c) p=3q

d) p=q

20. Corner points of the feasible region for an LPP are: (0,2),(3,0),(6,0),(6,8) and (0,5). Let z=4x+6y the objective function. The minimum value of z occurs at
   a) (0,2) only
   b) (3,0) only
   c) the mid-point of the line segment joining the points (0,2) and (3,0) only
   d) any point on the line segment joining the points (0,2) and (3,0)

21. Corner points of the feasible region for an LPP are: (0,2),(3,0),(6,0),(6,8) and (0,5). Let z=4x+6y the objective function. Then Max.z-Min z=
   a) 60
   b) 48
   c) 42
   d) 18

22. The feasible region for an LPP is shown in Figure. Let z=3x-4y be the objective function. Maximum value of z is



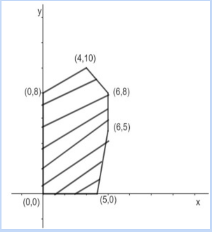
   a) 0
   b) 8
   c) 12
   d) -18

23. The corner points of the feasible region determined by the system of linear constraints are (0,0), (0,40), (20,40), (60,20),(60,0).The objective function is z=4x+3y. Compare the quantity in Column A and Column B

Column A	Column B
Maximum of z	325

   a) The quantity in Column A is greater
   b) The quantity in Column B is greater
   c) The two quantities are equal
   d) The relationship cannot be determined on the basis of information supplied.

24. The feasible region of a LPP is shown in Figure. Let $z=3x-4y$ be the objective function. Minimum of z occurs at



a) (0,0)
b) (0,8)
c) (5,0)
d) (4,10)

25. In Q. No. 24, Maximum of z occurs at

a) (5,0)
b) (6,5)
c) (6,8)
d) (4,10)

26. In Q. No. 24, (Maximum value of z + Minimum value of z) is equal to

a) 13
b) 1
c) -13
d) -17

27. The graph of the inequality $2x+3y>6$ is

a) half plane that contains the origin
b) half plane that neither contains the origin nor the points on the line $2x+3y=6$.
c) whole XOY plane excluding the points on the line $2x+3y=6$
d) entire XOY plane

28. The objective function of an LPP is

a) a constant

b) a linear function to be optimized

c) an inequality

d) a quadratic expression.

29. Which of the following is a linear objective function?

a) $Z = ax+by$

b) $Z < ax+by$

c) $Z > ax+by$

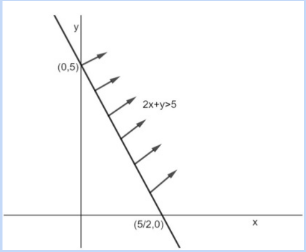
d) $Z \neq ax + by$

ANSWER KEY

1. B
2. B
3. D
4. B
5. D
6. C
7. D
8. C
9. C
10. A
11. C
12. C
13. B
14. C
15. D
16. D
17. D
18. D
19. B
20. D
21. A
22. C
23. B
24. B
25. A
26. D
27. B
28. B
29. A

HINTS AND SOLUTIONS

1. Let x,y plane



By the given equation 2x+y>5

x=0 and y=0

y>5 and x>5/2

Hence open half plane not containing origin.

2. Let, Z=CX

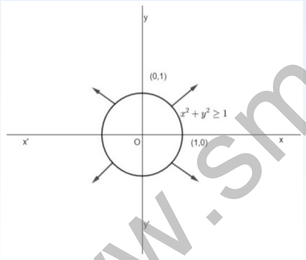
The condition may be $ax \geq b1$, $ax \leq b2$, $x \geq 0$

So the objective function of LPP is a function to be optimized.

3. A = $x^2 + y^2 \geq 1$

$x^2 + y^2 = 1$

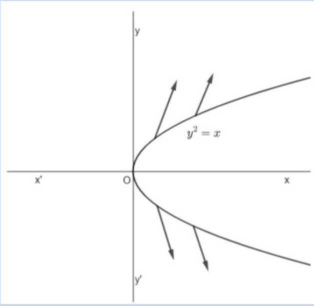
At (0,0) we have $0 \geq 1$, which is wrong. So the region of set A will not contain the origin.



Here its all points are outside the set. So option a is incorrect.

B = $y^2 \geq x$

$y^2 = x$

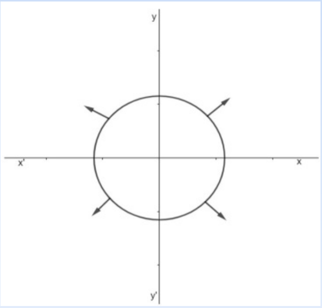


At (0,0) we have $0 \geq 0$, which is wrong. So the region of set B will not contain the origin.

Option b is also incorrect.

C = $3x^2 + 4y^2 \geq 5$

$\Rightarrow \frac{x^2}{\frac{5}{3}} + \frac{y^2}{\frac{5}{4}} \geq 1$



At (0,0) we have $0 \geq 1$, which is wrong. So the region of set C will not contain the origin.

So option c is incorrect.

Now, D = $y \geq 2, y \leq 4$

It is only possible when y=3

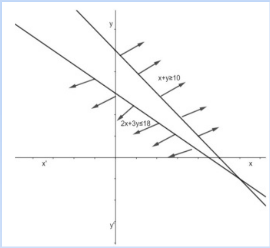
So the correct option is d.

4. A set $A_1$is convex if, for any two points $x_1, x_2 \in A$ and $\lambda \in [0,1]$ imply that $\lambda x_1 + (1-\lambda)x_2 \in A$.

   Since, here $x_1 and\ x_2$ are optimal solutions.

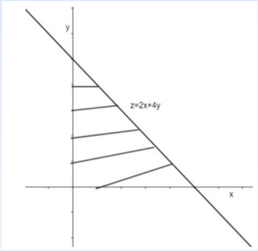
   Therefore, their convex combination will also be a optimal solution. Hence option b) is correct.

5. Z=4x+2y subject to constraints $2x + 3y \leq 18, x + y \geq 10;\ x, y \geq 0$

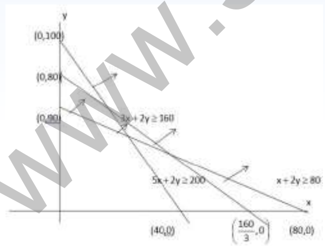
   We observe having drawn the graph the inequalities of equation, $2x + 3y \leq 18$ is inward and $x + y \geq 10$ is outward. So we are not getting feasible region and max Z cannot be determined.

6. Let we have a graph and plot some constraints:

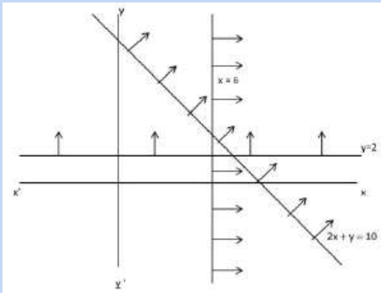
   Correct option is c

7. By the given constraints, let's plot the points in graph,

   Graph moves outwards as per given situation. Therefore we are getting unbounded region. So max Z can not be determined.

8. By the given condition:



The shaded region represent the feasible region of the given LPP. Redundant constraint is $2x + y \geq 10$

9. Let's convert the given inequalities into equations, we obtain the following equations,

3x+4y=24, 8x+6y=48, x=5, y=6, x=0, y=0

Solving for corner points we obtain (0,0), (5,0), (5,4/3), (24/7,24/7), (0,6)

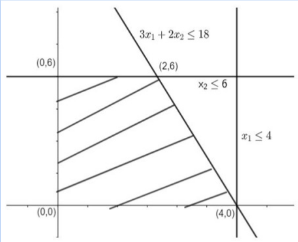
Calculating value of Z at these corner points, we see that maximum value of objective function z is 24 which is at (5,4/3) and (24/7,24/7). Thus optimal value is 24.

10. The optimization of the objective function of LPP is governed by the constraints. Therefore, if the constraints in a linear programming problem are changed, then the problem needs to be revaluated.

11. It is known that the optimal solution of an LPP either exists uniquely, does not exist or exist infinitely. So, if an LPP admits two optimal solution, it has an infinite number of optimal solution.

12. |x|=5 is not a convex set as any two points from negative and positive x-axis, if are joined will not lie in the set. Since option a,b and d is a convex set. So correct option is c).

13. By the graphical method:



Checking co-ordinates,

At (0,0), z=0

At (4,0), z=12

At (4,3), z=27

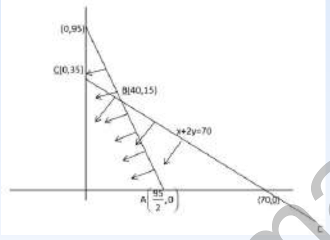
At (2,6), z=36

At (0,6), z=30

Hence zmax=36 at (2,6)

14. Draw graph and observe that every point in the first quadrant satisfies these inequalities. So the first quadrant is the region represented by inequalities.

15. Drawing graph,



The points OABC is the feasible region of LPP. Now checking the points O,A,B,C

Z(0,0)=0

Z(0,35)=35

Z(40,15)=55

Z(95/2,0)=47.5

Z is maximum at (40,15)

16. In LPP, we substitute the co-ordinates of vertices of feasible region in the objective function and then we obtain the maximum or minimum value. Therefore, the value of objective function is maximum under linear constraints at any vertex of feasible region.

17. Z=px+qy

Corner Points	Value of z
$(0,0)$	0
$(5,0)$	$5p$
$(3,4)$	$3p+4q$
$(0,5)$	$5q$

Now comparing value of z

Value on (3,4) = Value on (0,5)

3p+4q=5q

3p=q

18. Z(15,15)=Z(0,20)

15p+15q=20q

15p=5q

3p=q

19. Z(1,1)=Z(3,0)

p+q=3p

q=2p

20. Z=4x+6y

Corner points	Corresponding value of $z = 4x + 6y$
$(0, 2)$	12 (minimum)
$(3, 0)$	12 (minimum)
$(6, 0)$	24
$(6, 8)$	72 (maximum)
$(0, 5)$	30

So max Z=72

21. From previous table it is clear that minimum value is 12

Hence Max Z-MinZ=60

22.

Corner points	$z = 3x - 4y$
$(0, 4)$	-16 (minimum)
$(0, 0)$	0
$(12, 6)$	$12(3) - 4(6) = 12(\text{maximum})$

So maxZ=12

23.

Max z = 300<325

24.

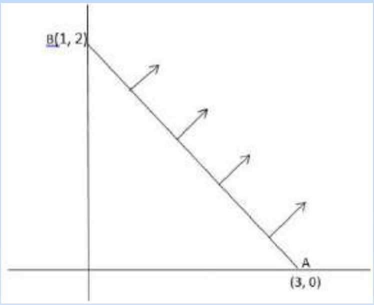
Minimum of z=-32 at (0,8)

25.

Max Z =15 at (5,0)

26. From previous table MaxZ+MinZ=15-32=-17

27.



We used dashed as y>6-2x/3

Hence half plane that neither contains origin nor the points on the line 2x+3y=6.

28. The objective function of linear programming problem is to minimize or maximize the function.

29. Objective function Z=ax+by, where a and b are constants, which has to be maximized or minimized, is called a linear objective function