THREE-DIMENSIONAL GEOMETRY

CHAPTER 12

3D

DIRECTION COSINES AND DIRECTION RATIOS OF A LINE:

If a line makes angles α, β and γ with X-axis, Y-axis and Z-axis respectively then $l = \cos\alpha, m = \cos\beta$ and $n = \cos\gamma$ are called the direction cosines of the line.

The relation between direction cosines of a line is $l^2 + m^2 + n^2 = 1$.

Numbers proportional to the direction cosines of a line are called the direction ratios of the line. If a,b and c are the direction ratios of a line, then $l = \frac{\pm a}{\sqrt{a^2+b^2+c^2}}, m = \frac{\pm b}{\sqrt{a^2+b^2+c^2}}, n = \frac{\pm c}{\sqrt{a^2+b^2+c^2}}$

For any line direction cosines are unique but direction ratios are not unique.

DIRECTION COSINES AND DIRECTION RATIOS OF A LINE PASSING THROUGH TWO POINTS:

The direction cosines and direction ratios of the line passing through the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are respectively given by $\frac{x_2-x_1}{PQ}, \frac{y_2-y_1}{PQ}, \frac{z_2-z_1}{PQ}$ and $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$

where, PQ = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

LINE:

A line or straight line is a curve such that all points on the line segment joining two points of it lies on it. A line in space can be determined uniquely, if

i. Its direction and the coordinates of a point on it are known.

ii. It passes through two given points.

EQUATION OF A LINE:

i. **Equation of a line passes through a given point and parallel to a given vector**

Vector Equation The equation of a line through a given point A with position vector a and parallel to a given vector b is given by $r = a + \lambda b$, where λ is a scalar.

Cartesian Equation The equation of a line passing through a point $A(x_1, y_1, z_1)$ and having direction ratios a,b and c is $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

If l,m and n are the direction cosines of the line, then equation of the line is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

ii. **Equation of a line passing through two given points**

Vector Equation The vector equation of a line passing through two points with position vectors a and b is given by $r = a + \lambda(b - a)$ where λ is a scalar.

Cartesian Equation The equation of a line passing through two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$

ANGLE BETWEEN TWO LINES:

Vector Form If θ is the angle between two lines $r = a_1 + \lambda b_1$ and $r = a_2 + \mu b_2$,

then, $\cos\theta = \left|\frac{b_1.b_2}{|b_1||b_2|}\right|$

If two lines are perpendicular, then $b_1.b_2 = 0$ and if two lines are parallel, then $b_1 = \lambda b_2$.

Cartesian Form If a_1, b_1, c_1 and a_2, b_2, c_2 are direction ratios of two lines respectively, then the angle θ between the lines is given by $\cos\theta = \left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}.\sqrt{a_2^2+b_2^2+c_2^2}}\right|$

SOME IMPORTANT RESULTS:

i. If θ is the angle between two lines with direction cosines l_1, m_1, n_1 and l_2, m_2, n_2 then $\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$.

ii. If two lines are perpendicular, then $a_1a_2 + b_1b_2 + c_1c_2 = 0$

iii. If two lines are parallel, then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

SKEW LINES:

If two lines do not meet and not parallel, then they are known as skew lines.

SHORTEST DISTANCE BETWEEN TWO SKEW-LINES:

The shortest distance between two skew-lines is the length of the line segment perpendicular to both the lines.

Vector Form Shortest distance between the skew lines $r = a_1 + \lambda b_1$ and $r = a_2 + \lambda b_2$ is given by $SD = \left|\frac{|(a_2-a_1).(b_1\times b_2)|}{|b_1\times b_2|}\right|$

The shortest distance between the parallel lines $r = a_1 + \lambda b$ and $r = a_2 + \mu b$ is given by $SD = \left|\frac{b\times(a_2-a_1)}{|b|}\right|$

Cartesian Form The shortest distance between the lines $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_1}{a_2} = \frac{y-y_1}{b_2} = \frac{z-z_1}{c_2}$ is $\left|\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}\right|$

PLANE:

A plane is a surface such that a line segment joining any two points on it lies wholly on it. A straight line which is perpendicular to every line lying on a plane is called a normal to the plane.

EQUATION OF A PLANE:

i. **Equation of a plane in normal form:**

Vector Equation Let O be the origin and $\hat{n}$ be a unit vector in the direction of the normal ON to the plane and let ON=d. Then, the equation of the plane is $r.\hat{n} = d$.

Cartesian Equation The equation of a plane is ax+by+cz+d=0, where a,b and c are the direction ratios of the normal to this plane.

ii. **Equation of a plane perpendicular to a given vector and passing through a given point**

Vector Equation The equation of a plane passing through a point a and perpendicular to the given point n is **(r-a).n=0**

Cartesian Equation The equation of a plane passing through a point (x_1, y_1, z_1) is

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

where a,b and c are direction ratios of perpendicular vector.

iii. **Equation of a plane passing through three non-collinear points**

Vector Equation The equation of plane passing through three non collinear points a,b and c is $(r - a)[(b - a) \times (c - a)] = 0$.

Cartesian Equation The equation of a plane passing through three non-collinear points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ is given by

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

If above points are collinear then $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$

iv. **Intercept form of the equation of a plane** If a plane cuts(intercepts) a,b and c with the coordinate axes X,Y and Z then the equation of the plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

v. **Equation of a plane passing through the intersection of two planes**

Vector Form The equation of plane passing through the intersection of two planes $r.n_1=d_1$ and $r.n_2=d_2$ is $r.(n_1+\lambda n_2)=d_1+\lambda d_2$, where λ is a real number.

Cartesian Form The equation of a plane passing through the intersection of the planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ is $(a_1x+b_1y+c_1z+d_1)+\lambda(a_2x+b_2y+c_2z+d_2)=0$

vi. **Equation of a plane parallel to a given plane**

Vector Form The vector equation of a plane parallel to the given plane $r.n=d_1$ is $r.n=d_2$, where d_2 is a constant determined by the given condition.

Cartesian Form The Cartesian equation of a plane parallel to the given plane $ax+by+cz+d_1=0$ is $ax+by+cz+d_2=0$

COPLANARITY OF TWO LINES :

Vector Form Two lines $r=a_1+\lambda b_1$ and $r=a_2+\lambda b_2$ are coplanar, iff $(a_2-a_1).(b_1\times b_2)=0$.

Cartesian Form: Suppose, Cartesian equations of two lines are $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$. Then, these lines are coplanar iff

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$$

ANGLE BETWEEN TWO PLANES:

Vector Form The angle between two planes is the angle between their normals.

Angle between the planes $r.n_1=d_1$ and $r.n_2=d_2$ is given by $\cos\theta=\left|\frac{n_1.n_2}{|n_1||n_2|}\right|$

Cartesian Form If θ is the angle between the planes $a_1x+b_1y+c_1z+d_1=0$

and $a_2x+b_2y+c_2z+d_2=0$ then, $\cos\theta=\left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{{a_1}^2+{b_1}^2+{c_1}^2}.\sqrt{{a_2}^2+{b_2}^2+{c_2}^2}}\right|$

NOTE : If two planes are perpendicular, then $n_1 . n_2 = 0$ or $a_1a_2 + b_1b_2 + c_1c_2 = 0$ and if they are parallel, then $n_1 \times n_2 = 0$ or $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

DISTANCE OF A POINT FROM A PLANE:

Vector Form The perpendicular distance of a point a from the plane r.n=d, where n is normal to the plane, is $\frac{|a.n-d|}{|n|}$

Cartesian Form Distance of a point P(x_1, y_1, z_1) from a plane ax+by+cz+d=0 is given by

$$p = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

ANGLE BETWEEN A LINE AND A PLANE:

The angle φ between the line $r = a + \lambda b$ and plane r.n=d is given by $\sin\varphi = \left|\frac{b.n}{|b|.|n|}\right|$

Three Dimensional Geometry

Direction ratios and direction cosines of a line

D. Cs of a line are the cosines of the angles made by the line with the positive direction of the co-ordinate axes. If l, m, n are the D. Cs of a line, then $l^2 + m^2 + n^2 = 1$. D. Cs of a line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $\frac{x_2-x_1}{PQ}, \frac{y_2-y_1}{PQ}, \frac{z_2-z_1}{PQ}$, where $PQ = \left|\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}\right|$

D.Rs of a line are the no.s which are proportional to the D.Cs of the line if l, m, n are the D.Cs and a, b, c are D.Rs of a line, then

$l = \frac{a}{\left|\sqrt{a^2+b^2+c^2}\right|}$, $m = \frac{b}{\left|\sqrt{a^2+b^2+c^2}\right|}$, $n = \frac{c}{\left|\sqrt{a^2+b^2+c^2}\right|}$

Parallel lines

(i) Two skew lines are the line segment perpendicular to both the lines

(ii) $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\mu\vec{b}_2$ is $\frac{\left|(\vec{b}_1\times\vec{b}_2).(\vec{a}_2-\vec{a}_1)\right|}{\left|(\vec{b}_1\times\vec{b}_2)\right|}$

(iii) $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ is

$$= \frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}$$

(iv) Distance between parallel lines $\vec{r}=\vec{a_1}+\lambda\vec{b}$ and $\vec{r}=\vec{a_2}+\mu\vec{b}$ is $\frac{\left|\vec{b}\times(\vec{a_2}-\vec{a_1})\right|}{\left|\vec{b}\right|}$

Skew lines

These are the lines in space which are neither parallel nor intersecting. They lie in different planes. Angle between skew lines is the angle between two intersecting lines drawn from any point (origin) parallel to each of the skew lines.

Equation of line Vector in 3D

Vector equation of a line passing through the given point whose position vector is $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r}=\vec{a}+\lambda\vec{b}$

Vector equation of a line which passes through two points whose position vectors are $\vec{a}$ and $\vec{b}$ is $\vec{r}=\vec{a}+\lambda(\vec{b}-\vec{a})$

Equation of a line through point (x_1, y_1, z_1) and having D.Cs l, m, n is $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ Also, equation of a line that passes through two points is $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$

Angle between the two lines

If l_1, m_1, n_1, l_2, m_2, n_2 are the D.Cs and $a_1, b_1, c_1, a_2, b_2, c_2$ are the D.Rs of the two lines and 'θ' is the acute angle between them, then

$\cos\theta = |l_1l_2+m_1m_2+n_1n_2| = \left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$

Trace the Mind Map
First Level · Second Level · Third Level

PRACTICE QUESTIONS

1. If the plane $2ax - 3ay + 4az + 6 = 0$ passes through the mid point of the line joining the centres of the spheres $x^2 + y^2 + z^2 + 6x - 8y - 2z = 13$ and $x^2 + y^2 + z^2 - 10x + 4y - 2z = 8$, then a equals
   a) -2
   b) 2
   c) -1
   d) 1
2. A point moves such that the sum of its distance from points $(4, 0, 0)$ and $(-4, 0, 0)$ is 10, then the locus of the point is
   a) $9x^2 - 25y^2 + 25z^2 = 225$
   b) $9x^2 + 25y^2 + 25z^2 = 225$
   c) $9x^2 + 25y^2 - 25z^2 = 225$
   d) $9x^2 + 25y^2 + 25z^2 = 225 = 0$
3. The point of intersection of the line $\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z+3}{4}$ and the plane $2x + 4y - z + 1 = 0$ is
   a) $\left(-\frac{10}{3}, \frac{3}{2}, -\frac{5}{3}\right)$
   b) $\left(-\frac{10}{3}, -\frac{3}{2}, \frac{5}{3}\right)$
   c) $\left(\frac{10}{3}, \frac{3}{2}, -\frac{5}{3}\right)$
   d) $\left(\frac{10}{3}, -\frac{3}{2}, \frac{5}{3}\right)$
4. A variable plane moves so that sum of the reciprocals of its intercepts on the coordinate axes is 1/2 Then, the plane passes through
   a) $\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$
   b) $(-1, 1, 1)$
   c) $(2, 2, 2)$
   d) $(0, 0, 0)$

5. The lines $\vec{r} = \overrightarrow{a_1} + \lambda\,\overrightarrow{b_1}$ and $\vec{r} = \overrightarrow{a_2} + \mu\,\overrightarrow{b_2}$ are coplanar if

   a) $\overrightarrow{a_1} \times \overrightarrow{a_2} = \vec{0}$

   b) $\overrightarrow{b_1} \times \overrightarrow{b_2} = 0$

   c) $(\overrightarrow{a_2} - \overrightarrow{a_1}) \times (\overrightarrow{b_1} \times \overrightarrow{b_2}) = 0$

   d) $[\overrightarrow{a_1}\ \overrightarrow{b_1}\ \overrightarrow{b_2}] = [\overrightarrow{a_2}\ \overrightarrow{b_1}\ \overrightarrow{b_2}]$

6. A straight line $\vec{r} = \vec{a} + \lambda\,\vec{b}$ meets the plane $\vec{r} \cdot \vec{n} = 0$ in P. The position vector of P is

   a) $\vec{a} + \frac{\vec{a}\cdot\vec{n}}{\vec{b}\cdot\vec{n}}\vec{b}$

   b) $\vec{a} - \frac{\vec{a}\cdot\vec{n}}{\vec{b}\cdot\vec{n}}\vec{b}$

   c) $\vec{a} - \frac{\vec{a}\cdot\vec{n}}{\vec{b}\cdot\vec{n}}\vec{b}$

   d) None of the above

7. Equation of the plane passing through line $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4}$ and perpendicular to the plane $x + 2y + z = 12$ is given by $ax + by + cz + 4 = 0$, then

   a) $a = -8, b = 2, c = -5$

   b) $a = -9, b = -2, c = -5$

   c) $a = 9, b = -2, c = -5$

   d) None of the above

8. The value of k so that the lines $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and, $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$ may be perpendicular is given by

   a) $-\frac{7}{10}$

   b) $-\frac{10}{7}$

   c) -10

   d) $\frac{10}{7}$

9. If O is the origin and A is the point (a, b, c) then the equation of the plane through A and at right angles to OA is
    a) $a(x-a)-b(y-b)-c(z-c)=0$
    b) $a(x+a)+b(y+b)+c(z+c)=0$
    c) $a(x-a)+b(y-b)+c(z-c)=0$
    d) None of the above
10. The projection of a directed line segment on the coordinate axes are 12, 4, 3. The DC's of the line are
    a) $\frac{12}{13}, -\frac{4}{13}, \frac{3}{13}$
    b) $-\frac{12}{13}, -\frac{4}{13}, \frac{3}{13}$
    c) $\frac{12}{13}, \frac{4}{13}, \frac{3}{13}$
    d) None of these
11. The line drown from $(4, -1, 2)$ the point $(-3, 2, 3)$ meets a plane at right angle at the point $(-10, 5, 4)$, then the equation of plane is
    a) $7x+3y+z+89=0$
    b) $7x-3y-z+89=0$
    c) $7x-3y+z+89=0$
    d) None of these
12. The equation of the plane passing through the point (1, 1, 1) and containing the line of intersection of the planes $x+y+z=6$ and $2x+3y+4z=12$ is
    a) $x+y+z=3$
    b) $x+2y+3z=6$
    c) $2x+3y+4z=9$
    d) $3x+4y+5z=18$
13. A line makes the same angle θ with each of the x and z axes. If the angle β, which it makes with y-axis, is such that $\sin^2\theta$, then $\cos^2\theta$ equals
    a) 2/3
    b) 1/5
    c) 3/5
    d) 2/5

14. If the straight lines $x = 1 + s, y = -3 - \lambda s, z = 1 + \lambda s$ and $x = \frac{t}{2}, y = 1 + t, z = 2 - t$ with parameters s and t respectively, are coplanar, then λ equals

a) -2

b) -1

c) $-\frac{1}{2}$

d) 0

15. Let L be the line of intersection of the planes $2x + 3y + z = 1$ and $x + 3y + 2z = 2$. If L makes an angle α with the positive x-axis, then $\cos\alpha$ equals

a) $\frac{1}{\sqrt{3}}$

b) $\frac{1}{2}$

c) 1

d) $\frac{1}{\sqrt{2}}$

16. Equation of plane containing the line $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and parallel to the line $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ is

a) $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0$

b) $\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ a_2 & b_2 & c_2 \\ x_1 & y_1 & z_1 \end{vmatrix} = 0$

c) $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$

d) None of the above

17. The line through $\hat{\imath}+3\hat{\jmath}+2\hat{k}$ and perpendicular to the lines $\vec{r}=(\hat{\imath}+2\hat{\jmath}-\hat{k})+\lambda(2\hat{\imath}+\hat{\jmath}+\hat{k})$ and, $\vec{r}=(2\hat{\imath}+6\hat{\jmath}+\hat{k})+\mu(\hat{\imath}+2\hat{\jmath}+3\hat{k})$ is

a) $\vec{r}=(\hat{\imath}+2\hat{\jmath}-\hat{k})+\lambda(-\hat{\imath}+5\hat{\jmath}-3\hat{k})$

b) $\vec{r}=\hat{\imath}+3\hat{\jmath}+2\hat{k}+\lambda(\hat{\imath}-5\hat{\jmath}+3\hat{k})$

c) $\vec{r}=\hat{\imath}+3\hat{\jmath}+2\hat{k}+\lambda(\hat{\imath}+5\hat{\jmath}+3\hat{k})$

d) $\vec{r}=\hat{\imath}+3\hat{\jmath}+2\hat{k}+\lambda(-\hat{\imath}-5\hat{\jmath}-3\hat{k})$

18. In ΔABC and mid points of the sides AB, BC and CA are respectively (1,0,0), $(0,m,0)$and $(0,0,n)$ Then, $\frac{AB^2+BC^2+CA^2}{(l^2+m^2+n^2)}$ is equal to

a) 2

b) 4

c) 8

d) 16

19. If direction cosines of two lines are proportional to $(2,3-6)$and $(3,-4,5)$ then the acute angle between then is

a) $\cos^{-1}\left(\frac{49}{36}\right)$

b) $\cos^{-1}\left(\frac{18\sqrt{2}}{35}\right)$

c) $96°$

d) $\cos^{-1}\left(\frac{18}{35}\right)$

20. If $P(x,y,z)$ is a point on the line segment joining $Q(2,24)$ and $R(3,5,6)$ such that the projections of OP on the axes are $\frac{13}{5}, \frac{19}{5}$ and $\frac{26}{5}$ respectively, then P divides QR in the ratio

a) 1:2

b) 3:2

c) 2:3

d) 1:3

21. Equation of plane containing the lines

$\vec{\mathbf{r}} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}} + \lambda(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$ and $\vec{\mathbf{r}} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}} + \mu(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$ is

a) $\vec{\mathbf{r}} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}} + \lambda(\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

b) $\vec{\mathbf{r}} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}} + \lambda(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}) + \mu(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}})$

c) $\vec{\mathbf{r}} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}} + \lambda(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) + \mu(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}})$

d) $\vec{\mathbf{r}} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}} + \lambda(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) + \mu(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

22. There is point $P(a, a, a)$ on the line passing through the origin and equally inclined with axes. The equation of plane perpendicular to OP and passing through P cuts the intercepts on axes. The sum of whose reciprocals is

a) a

b) $\frac{3}{2a}$

c) $\frac{3a}{2}$

d) $\frac{1}{a}$

23. The position vectors of points A and B are $\hat{\imath} - \hat{\jmath} + 3\,\hat{k}$ and $3\,\hat{\imath} + 3\,\hat{\jmath} + 3\,\hat{k}$ respectively. The equation of a plane is $\vec{r} \cdot (5\,\hat{\imath} + 2\,\hat{\jmath} - 7\,\hat{k}) + 9 = 0$. The points A and B

a) Lie on the plane

b) Are on the same side of the plane

c) Are on the opposite side of the plane

d) None of these

24. A sphere of constant radius $2k$ passes through the origin and meets the axes in A, B, C. The locus of the centroid of the tetrahedron ABC is

a) $x^2 + y^2 + z^2 = 4k^2$

b) $x^2 + y^2 + z^2 = k^2$

c) $2(x^2 + y^2 + z^2) = k^2$

d) None of these

25. The equation of the plane passing through three non-collinear points with position vectors $\vec{a}, \vec{b}, \vec{c}$ is

a) $\vec{r} \cdot (\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b}) = 0$

b) $\vec{r} \cdot (\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b}) = [\vec{a}\, \vec{b}\, \vec{c}]$

c) $\vec{r} \cdot (\vec{a} \times (\vec{b} + \vec{c})) = [\vec{a}\, \vec{b}\, \vec{c}]$

d) $\vec{r} \cdot (\vec{a} + \vec{b} + \vec{c}) = 0$

26. What are the DR's of vector parallel to $(2, -1, 1)$ and $(3, 4, -1)$?

a) $(1, 5 - 2)$

b) $(-2, -5, 2)$

c) $(-1, 5, 2)$

d) $(-1, -5, -2)$

27. In a three dimensional xyz-space, the equation $x^2 - 5x + 6 = 0$ represents

a) Points

b) Plane

c) Curves

d) Pair of straight lines

28. The direction cosines of a line equally inclined to three mutually perpendicular lines having direction cosines as $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$ are

a) $l_1 + l_2 + l_3, m_1 + m_2 + m_3, n_1 + n_2 + n_3$

b) $\frac{l_1+l_2+l_3}{\sqrt{3}}, \frac{m_1+m_2+m_3}{\sqrt{3}}, \frac{n_1+n_2+n_3}{\sqrt{3}}$

c) $\frac{l_1+l_2+l_3}{3}, \frac{m_1+m_2+m_3}{3}, \frac{n_1+n_2+n_3}{3}$

d) None of these

29. The equation of the plane through the point (1, 2,3) and parallel to the plane $x + 2y + 5z = 0$ is

a) $(x - 1) + 2(y - 2) + 5(z - 3) = 0$

b) $x + 2y + 5z = 14$

c) $x + 2y + 5z = 6$

d) None of the above

30. The shortest distance between the lines $\vec{r} = (5\hat{\imath} + 7\hat{\jmath} + 3\hat{k}) + \lambda(5\hat{\imath} - 16\hat{\jmath} + 7\hat{k})$ and, $\vec{r} = 9\hat{\imath} + 13\hat{\jmath} + 15\hat{k} + \mu(3\hat{\imath} + 8\hat{\jmath} - 5\hat{k})$, is

a) 10 units

b) 12 units

c) 14 units

d) None of these

31. The equation of the plane passing through the intersection of the planes $x + 2y + 3z + 4 = 0$ and $4x + 3y + 2z + 1 = 0$ and the origin, is

a) $3x + 2y + z + 1 = 0$

b) $3x + 2y + z = 0$

c) $2x + 3y + z = 0$

d) $x + y + z = 0$

32. If the plane $3x + y + 2z + 6 = 0$ is parallel to the line $\frac{3x-1}{2b} = 3 - y = \frac{z-1}{a}$, then the value of $3a + 3b$ is

a) $\frac{1}{2}$

b) $\frac{3}{2}$

c) 3

d) 4

33. The shortest distance between the skew lines $l_1: \vec{\mathbf{r}} = \vec{\mathbf{a}}_1 + \lambda\vec{\mathbf{b}}_1$ $l_2: \vec{\mathbf{r}} = \vec{\mathbf{a}}_2 + \mu\vec{\mathbf{b}}_2$ is

a) $\frac{|(\vec{\mathbf{a}}_2 - \vec{\mathbf{a}}_1)\cdot\vec{\mathbf{b}}_1 \times \vec{\mathbf{b}}_2|}{|\vec{\mathbf{b}}_1 \times \vec{\mathbf{b}}_2|}$

b) $\frac{|(\vec{\mathbf{a}}_2 - \vec{\mathbf{a}}_1)\cdot\vec{\mathbf{a}}_2 \times \vec{\mathbf{b}}_2|}{|\vec{\mathbf{b}}_1 \times \vec{\mathbf{b}}_2|}$

c) $\frac{|(\vec{\mathbf{a}}_2 - \vec{\mathbf{b}}_2)\cdot\vec{\mathbf{a}}_1 \times \vec{\mathbf{b}}_1|}{|\vec{\mathbf{b}}_1 \times \vec{\mathbf{b}}_2|}$

d) $\frac{|(\vec{\mathbf{a}}_1 - \vec{\mathbf{b}}_2)\cdot\vec{\mathbf{b}}_1 \times \vec{\mathbf{a}}_2|}{|\vec{\mathbf{b}}_1 \times \vec{\mathbf{a}}_2|}$

34. The equation of the plane through the line of intersection of planes $ax + by + cz + d = 0, a'x + b'y + c'z + d' = 0$ and parallel to the line $y = 0, z = 0$ is

a) $(ab' - a'b)x + (bc' - b'c)y + (ad' - a'd) = 0$

b) $(ab' - a'b)x + (bc' - b'c)y + (ad' - a'd)z = 0$

c) $(ab' - a'b)y + (ac' - a'c)z + (ad' - a'd) = 0$

d) None of these

35. P, Q, R, S Are four coplanar points on the sides AB, BC, CD, DA of a skew quadrilateral. The product $\frac{AP}{PB} \cdot \frac{BQ}{QC} \cdot \frac{CR}{RD} \cdot \frac{DS}{SA}$ equals

a) -2

b) -1

c) 2

d) 1

36. The coordinates of the foot of the perpendicular drawn from the point $A(1, 0, 3)$ to the join of the points $B(4, 7, 1)$ and $C(3, 5, 3)$ are

a) (5/3, 7/3, 17/3)

b) (5, 7, 17)

c) (5/7, $-7/3$, 17/3)

d) $(-5/3, 7/3, -17/3)$

37. Consider the following statements:

I Line joining (1, 2, 5); (4, 3, 2) is parallel to the line joining (5, 1, -11), (8, 2,-8)

II Three concurrent lines with DC's $(l_i, m_i, n_i) i = 1,2,3$ are coplanar, if $\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0$

III The plane $x - 2y + z = 21$ and the line $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-1}{3}$ are parallel

Which of these is/are correct?

a) (1)and (2)

b) (2) and (3)

c) (3) and (1)

d) (1), (2) and (3)

38. If the straight lines $\frac{x-1}{k}=\frac{y-2}{2}=\frac{z-3}{3}$ and $\frac{x-2}{3}=\frac{y-3}{k}=\frac{z-1}{3}$ intersect at a point, then the integer k is equal to

a) -2

b) -5

c) 5

d) 2

39. The vector equation of the plane passing through the origin and the line of intersection of the plane $\vec{\mathbf{r}}\cdot\vec{\mathbf{a}}=\lambda$ and $\vec{\mathbf{r}}\cdot\vec{\mathbf{b}}=\mu$, is

a) $\vec{\mathbf{r}}\cdot\left(\lambda\,\vec{\mathbf{a}}-\mu\,\vec{\mathbf{b}}\right)=0$

b) $\vec{\mathbf{r}}\cdot\left(\lambda\,\vec{\mathbf{b}}-\mu\,\vec{\mathbf{a}}\right)=0$

c) $\vec{\mathbf{r}}\cdot\left(\lambda\,\vec{\mathbf{b}}-\mu\,\vec{\mathbf{a}}\right)=0$

d) $\vec{\mathbf{r}}\cdot\left(\lambda\,\vec{\mathbf{b}}+\mu\,\vec{\mathbf{a}}\right)=0$

40. The line passing through the points $(5,1,a)$ and $(3,b,1)$ crosses the yz-plane at the point $\left(0,\frac{17}{2},-\frac{13}{2}\right)$. Then,

a) $a=8,b=2$

b) $a=2,b=8$

c) $a=4,b=6$

d) $a=6,b=4$

41. The equation of the plane containing the line $\vec{r}=\hat{\imath}+\hat{\jmath}+\lambda\left(2\hat{\imath}+\hat{\jmath}+4\hat{k}\right)$, is

a) $\vec{r}\cdot\left(\hat{\imath}+2\hat{\jmath}-\hat{k}\right)=3$

b) $\vec{r}\cdot\left(\hat{\imath}+2\hat{\jmath}-\hat{k}\right)=6$

c) $\vec{r}\cdot\left(-\hat{\imath}-2\hat{\jmath}+\hat{k}\right)$

d) None of these

42. The intersection of the sphere $x^2+y^2+z^2+7x-2y-z=1$ and $x^2+y^2+z^2-3x+3y+4z=-4$ is same as the intersection of one of the spheres and the plane is

a) $2x-y-z=1$

b) $-2x+y+z=1$

c) $2x-y+z=1$

d) $2x+y+z=1$

43. A line makes acute angles of α, β and γ with the coordinate axes such that $\cos\alpha\cos\beta = \cos\beta\cos\gamma = \frac{2}{9}$ and $\cos\gamma\cos\alpha = \frac{4}{9}$, Then $\cos\alpha + \cos\beta + \cos\gamma$ is equal to

a) $\frac{25}{9}$

b) $\frac{5}{9}$

c) $\frac{5}{3}$

d) $\frac{2}{3}$

44. The vector from of the sphere $2(x^2 + y^2 + z^2) - 4x + 6y + 8z - 5 = 0$ is

a) $\vec{\mathbf{r}} \cdot \left[\vec{\mathbf{r}} - \left(2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}\right)\right] = \frac{2}{5}$

b) $\vec{\mathbf{r}} \cdot \left[\vec{\mathbf{r}} - \left(2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 4\hat{\mathbf{k}}\right)\right] = \frac{1}{2}$

c) $\vec{\mathbf{r}} \cdot \left[\vec{\mathbf{r}} - \left(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}\right)\right] = \frac{5}{2}$

d) $\vec{\mathbf{r}} \cdot \left[\vec{\mathbf{r}} - \left(2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 4\hat{\mathbf{k}}\right)\right] = \frac{5}{2}$

45. Let a plane passes through the point $P(-1, -1, 1)$ and also passes through a line joining the points Q(0,1,1) and R(0,0,2). Then the distance of the plane from the point (0, 0, 0) is

a) 3

b) 0

c) $\frac{1}{\sqrt{6}}$

d) $\frac{2}{\sqrt{6}}$

46. If the direction ratio of two lines are given by $3lm - 4ln + mn = 0$ and $l + 2m + 3n = 0$, then the angle between the line is

a) $\frac{\pi}{6}$

b) $\frac{\pi}{4}$

c) $\frac{\pi}{3}$

d) $\frac{\pi}{2}$

47. A mirror and a source of light are situated at the origin O and at a point on OX respectively. A ray of light from the source strikes the mirror and is reflected. If the direction ratios of the normal to the plane are proportional to $1, -1, 1$, then direction cosines of the reflected ray are

a) $\frac{1}{2}, \frac{2}{3}, \frac{2}{3}$

b) $-\frac{1}{2}, \frac{2}{3}, \frac{2}{3}$

c) $-\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}$

d) $-\frac{1}{2}, -\frac{2}{3}, \frac{2}{3}$

48. If a sphere of radius r passes through the origin, then the extremities of the diameter parallel to x-axis lie on each of the spheres

a) $x^2 + y^2 + z^2 \pm 2rx = 0$

b) $x^2 + y^2 + z^2 \pm 2ry = 0$

c) $x^2 + y^2 + z^2 \pm 2rz = 0$

d) $x^2 + y^2 + z^2 \pm 2ry \pm 2rz =$

49. If l_1, m_1, n_1 and l_2, m_2, n_2 are direction cosines of the two lines inclined to each other at an angle θ, then the direction cosines of internal bisector of the angle between these lines are

a) $\frac{l_1+l_2}{2\sin\frac{\theta}{2}}, \frac{m_1+m_2}{2\sin\frac{\theta}{2}}, \frac{n_1+n_2}{2\sin\frac{\theta}{2}}$

b) $\frac{l_1+l_2}{2\cos\frac{\theta}{2}}, \frac{m_1+m_2}{2\cos\frac{\theta}{2}}, \frac{n_1+n_2}{2\cos\frac{\theta}{2}}$

c) $\frac{l_1-l_2}{2\sin\frac{\theta}{2}}, \frac{m_1-m_2}{2\sin\frac{\theta}{2}}, \frac{n_1-n_2}{2\sin\frac{\theta}{2}}$

d) $\frac{l_1-l_2}{2\cos\frac{\theta}{2}}, \frac{m_1-m_2}{2\cos\frac{\theta}{2}}, \frac{n_1-n_2}{2\cos\frac{\theta}{2}}$

50. The equation of the sphere touching the three coordinate planes is

a) $x^2 + y^2 + z^2 + 2a(x + y + z) + 2a^2 = 0$

b) $x^2 + y^2 + z^2 - 2a(x + y + z) + 2a^2 = 0$

c) $x^2 + y^2 + z^2 \pm 2a(x + y + z) + 2a^2 = 0$

d) $x^2 + y^2 + z^2 \pm 2ax \pm 2ay \pm 2az + 2a^2 = 0$

ANSWER KEY

1. A
2. B
3. D
4. C
5. D
6. C
7. C
8. B
9. C
10. C
11. B
12. B
13. C
14. A
15. A
16. C
17. D
18. C
19. A
20. A
21. D
22. D
23. C
24. B
25. B
26. A
27. B
28. B
29. A
30. C
31. B
32. B
33. A
34. C
35. D
36. A
37. D
38. B
39. B
40. D
41. A
42. A
43. C
44. D
45. D
46. D
47. D
48. A
49. B
50. B

HINTS AND SOLUTIONS

1. Centre of given sphere are $C_1(-3,4,1)$and $C_2(5,-2,1)$

   So, midpoint of $C_1C_2 \equiv P\left(\frac{5-3}{2},\frac{4-2}{2},\frac{1+1}{2}\right) = P(1,1,1)$

   Now, the plane $2ax-3ay+4az+6=0$ passes through the point P.

   $\therefore 2a(1)-3a(1)+4a(1)+6=0 \Longrightarrow a=-2$

2. Let the point is P whose coordinate are (x,y,z) and the given points are $A(4,0,0)$ and $(-4,0,0) \because PA+PB=10$

   $\sqrt{(x-4)^2+y^2+z^2}+\sqrt{(x+4)^2+y^2+z^2}=10$ …(i)

   Also, $[(x+4)^2+y^2+z^2]+[(x-4)^2+y^2+z^2]=16x$

   And $\left[\sqrt{(x+4)^2+y^2+z^2}+\sqrt{(x-4)^2+y^2+z^2}\right]$

   $\left[\sqrt{(x+4)^2+y^2+z^2}-\sqrt{(x-4)^2+y^2+z^2}\right]16x$

   $\left[\sqrt{(x+4)^2+y^2+z^2}-\sqrt{(x-4)^2+y^2+z^2}\right]=\frac{16x}{10}$ …(ii)

   On solving Eqs.(i) and (ii), we get

   $2\sqrt{(x+4)^2+y^2+z^2}=\frac{16x}{10}+10 \Rightarrow (x+4)^2+y^2+z^2=\left(\frac{4x}{5}+5\right)^2$

   $\Rightarrow x^2+8x+16+y^2+z^2=\frac{16x^2+625+200x}{25}$

   $\Rightarrow [25x^2+400+200x+25y^2+25z^2]-16x^2-625-200x]=0$

   $\Rightarrow 9x^2+25y^2+25z^2=225$

3. Given equation line is

   $\frac{x-1}{2}=\frac{y-2}{-3}=\frac{z+3}{4}=\lambda$ [say]

   Any point on the line is$P(2\lambda+1,-3\lambda+2,4\lambda-3)$

   Since, these point lies on the given plane.

   $\therefore 2(2\lambda+1)+4(-3\lambda+2)-(4\lambda-3)+1=0 \Longrightarrow \lambda=\frac{7}{6}$

   $\therefore$ Required point is $P\left(\frac{10}{3},-\frac{3}{2},\frac{5}{3}\right)$

4. Let $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ ….(i)

   Then, $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{2}$ …(ii)

5. Given lines pass through points $P(\overrightarrow{a_1})$ and $Q(\overrightarrow{a_2})$ and are parallel to vectors $\overrightarrow{b_1}$and $\overrightarrow{b_2}$respectively. If the lines are coplanar, then

   $\vec{P}Q \perp (\overrightarrow{b_1} \times \overrightarrow{b_2}) \Rightarrow \vec{P}Q \cdot (\overrightarrow{b_1} \times \overrightarrow{b_2}) = 0$

   $\Rightarrow (\overrightarrow{a_2} - \overrightarrow{a_1}) \cdot (\overrightarrow{b_1} \times \overrightarrow{b_2}) = 0 \Rightarrow [\overrightarrow{a_1}\ \overrightarrow{b_1}\ \overrightarrow{b_2}] = [\overrightarrow{a_2}\ \overrightarrow{b_1}\ \overrightarrow{b_2}]$

6. The straight line $\vec{r} = \vec{a} + \lambda\,\vec{b}$ meets the plane $\vec{r} \cdot \vec{n} = 0$ in P for which λ is given by

   $(\vec{a} + \lambda\,\vec{b}) \cdot \vec{n} = 0 \Rightarrow \lambda = -\dfrac{\vec{a} \cdot \vec{n}}{\vec{b} \cdot \vec{n}}$

   Thus, the position vector of P is $\vec{r} = \vec{a}\left(\frac{\vec{a}\cdot\vec{n}}{\vec{b}\cdot\vec{n}}\right)\vec{b}$

7. Equation of any plane passing through given line is

   $a(x-1) + b(y+1) + c(z-3) = 0$ …(i)

   Above plane is perpendicular to the plane

   $x + 2y + z = 12$

   $\therefore a + 2b + c = 0$

   Also, normal to the plane is perpendicular to the line

   $\therefore 2a - b + 4c = 0$

   $\Rightarrow \frac{a}{8+1} = \frac{b}{2-4} = \frac{c}{-1-4} \Rightarrow \frac{a}{9} = \frac{b}{-2} = \frac{c}{-5}$

   $\therefore 9(x-1) - 2(y+1) - 5(z-3) = 0$

   $\Rightarrow 9x - 2y - 5z + 4 = 0$

   $\therefore a = 9, b = -2, c = -5$

8. Given lines will be perpendicular, if

   $-3 \times 3k + 2k \times 1 + 2 \times -5 = 0 \Rightarrow -7k - 10 = 0 \Rightarrow k = -\frac{10}{7}$

9. Equation of any plane passing through (a, b, c) is $a'(x-a) + b'(y-b) + c'(z-c) = 0$

   DR's of $OA = (a, b, c)$

   Since, plane (i) is perpendicular to the line OA, therefore its DR's is proportional to (a, b, c) ∴ Required equation of plane is $a(x-a) + b(y-b) + c(z-c) = 0$

10. DC's of line $= \left(\frac{12}{\sqrt{12^2+4^2+3^2}}, \frac{4}{\sqrt{12^2+4^2+3^2}}, \frac{3}{\sqrt{12^2+4^2+3^2}}\right) = \left(\frac{12}{13}, \frac{4}{13}, \frac{3}{13}\right)$

11. Since, the line passing through the points $(4, -1, 2)$ and $(-3, 2, 3)$. So, the DR's of the line is $(4+3, -1-2, 2-3) ie,\ (7, -3, -1)$. Since, the line is perpendicular to the plane therefore DR's of this line is proportional to the normal of the plane.

$\therefore$ Required equation plane is

$7(x+10) - 3(y-5) - 1(z-4) = 0 \Rightarrow 7x - 3y - z + 89 = 0$

12. Equation of plane containing the line of intersection of planes is

$(x+y+z-6) + \lambda(2x+3y+4z-12) = 0$

Since, it passes through the point (1, 1, 1),

$\therefore (1+1+1-6) + \lambda\,(2+3+4-12) = 0 \Rightarrow -3 + \lambda(-3) = 0 \Rightarrow \lambda = -1$

Hence, required equation of plane is

$(x+y+z-6) - (2x+3y+4z-12) = 0$

$i.e,\ x + 2y + 3z = 6$

13. Since, $l^2 + m^2 + n^2 = 1$

$\Rightarrow \cos^2\theta + \cos^2\beta + \cos^2\theta = 1$

$[\because l = \cos\theta, m = \cos\beta, \text{given}]$

$\Rightarrow \cos^2\theta + \cos^2\beta$

But $\sin^2\beta = 3\sin^2\theta$

$\therefore\ 3\sin^2\theta = 2\cos^2\theta$

$\Rightarrow 3 = 5\cos^2\theta \Rightarrow \cos^2\theta = \frac{3}{5}$

14. Given lines can be rewritten as

$\frac{x-1}{1} = \frac{y+3}{-\lambda} = \frac{z-1}{\lambda} = s$ and $\frac{x-0}{1} = \frac{y-1}{2} = \frac{z-2}{-2} = \frac{t}{2}$

Since, two lines are coplanar.

$$\therefore \begin{vmatrix} 1-0 & -3-1 & 1-2 \\ 1 & -\lambda & \lambda \\ 1 & 2 & -2 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 1 & -4 & -1 \\ 1 & -\lambda & \lambda \\ 1 & 2 & -2 \end{vmatrix} = 0$$

$\Rightarrow 1(2\lambda - 2\lambda) + 4(-2-\lambda) - 1(2+\lambda) = 0$

$\Rightarrow -8 - 4\lambda - 2 - \lambda = 0 \Rightarrow \lambda = -2$

15. Let the direction cosines of the line L be l, m, n. Since, the line intersect the given planes, then the normal to the planes are perpendicular to the line L

$\therefore \quad 2l + 3m + n = 0 \quad \ldots(i)$

and $l + 3m + 2n = 0 \ldots.(ii)$

From Eqs. (i) and (ii), we get $\frac{l}{3} = \frac{m}{-3} = \frac{n}{3} = k$ [say]

We, know, $l^2 + m^2 + n^2 = 1$

$\therefore \quad (3k)^2 + (-3k)^2 + (3k)^2 = 1 \Longrightarrow k = \frac{1}{3\sqrt{3}}$

$\therefore l = \frac{1}{\sqrt{3}} \Longrightarrow \cos\alpha = \frac{1}{\sqrt{3}}$

16. Do it yourself.

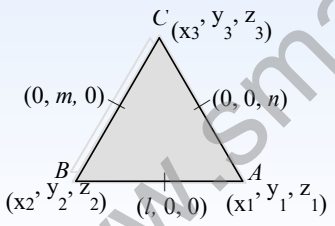
17. The required line passes through the point $\hat{\imath} + 3\hat{\jmath} + 2\hat{k}$ and is perpendicular to the lines $\vec{r} = (\hat{\imath} + 2\hat{\jmath} - \hat{k}) + \lambda(2\hat{\imath} + \hat{\jmath} + \hat{k})$ and, $\vec{r} = (2\hat{\imath} + 6\hat{\jmath} + \hat{k}) + \mu(\hat{\imath} + 2\hat{\jmath} + 3\hat{k})$

Therefore, it is parallel to the vector $\vec{b} = (2\hat{\imath} + \hat{\jmath} + \hat{k}) \times (\hat{\imath} + 2\hat{\jmath} + 3\hat{k}) = (\hat{\imath} + 5\hat{\jmath} + 3\hat{k})$

Hence, the equation of the required line is $\vec{r} = (\hat{\imath} + 3j + 2\hat{k}) + \lambda'(\hat{\imath} - 5j + 3k)$

$\Rightarrow \vec{r} = (\hat{\imath} + 3\hat{\jmath} + 2\hat{k}) + \lambda(-\hat{\imath} + 3\hat{\jmath} - 3\hat{k})$, where $\lambda = -\lambda'$

18. From the figure



$x_1 + x_2 = 2l, y_1 + y_2 = 0, z_1 + z_2 = 0,$

$x_2 + x_3 = 0, y_2 + y_3 = 2m, z_2 + z_3 = 0,$ and $x_1 + x_3 = 0, y_1 + y_3 = 0, z_1 + z_3 = 2n$

On solving, we get the coordinate are $A(l, -m, n), B(l, m, -n)$ and $C(-l, m\, n)$.

$\therefore \frac{AB^2+BC^2+CA^2}{l^2+m^2+n^2} = \frac{(4m^2+4n^2)+(4l^2+4n^2)+(4l^2+4m^2)}{l^2+m^2+n^2} = 8$

19. Since, direction cosines of two lines are proportional to $(2,3,-6)$and $(3,-4,5)$

$\therefore \cos = \frac{|2\times3+3\times(-4)-6\times5|}{\sqrt{2^2+3^2+(-6)^2}\sqrt{3^2+(-4)^2+5^2}} = \frac{|6-12-13|}{\sqrt{49}\sqrt{50}} \Longrightarrow \theta = \cos^{-1}\left(\frac{18\sqrt{2}}{35}\right)$

20. The coordinate of P are$\left(\frac{3\lambda+2}{\lambda+1}, \frac{5\lambda+2}{\lambda+1}, \frac{6\lambda+4}{\lambda+1}\right)$

$Q(2,2,4)$ —— $P(x,y,z)$ —— $R(3,5,6)$, $PQ:PR = \lambda:1$

Since, the projection of OP on x-axis is $\frac{3\lambda+2}{\lambda+1} = \frac{13}{5}$

$\Longrightarrow 15\lambda + 10 = 13\lambda + 13 \Longrightarrow \lambda = \frac{3}{2}$

21. Do it yourself

22. $\because OP = \sqrt{a^2+a^2+a^2} = \sqrt{3}\,a$

$\therefore$ DC's of OP are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

Equation of plane is

$x + y + z = 3a \Rightarrow \frac{x}{3a} + \frac{y}{3a} + \frac{z}{3a} = 1$

$\therefore$ Intersection on axes are $3a, 3a$ and $3a$ respectively

Sum of their reciprocals $= \frac{1}{3a} + \frac{1}{3a} + \frac{1}{3a} = \frac{1}{a}$

23. The position vectors of two given points are $\vec{a} = \hat{\imath} - \hat{\jmath} + 3\hat{k}$ and $\vec{b} = 3\hat{\imath} + 3\hat{\jmath} + 3\hat{k}$ and the equation of the given plane is $\vec{r}\cdot\left(5\hat{\imath} + 2\hat{\jmath} - 7\hat{k}\right) + 9 = 0$ or, $\vec{r}\cdot\vec{n} + d = 0$

We have, $\vec{a}\cdot\vec{n} + d = \left(\hat{\imath} - \hat{\jmath} + 3\hat{k}\right)\cdot\left(5\hat{\imath} + 2\hat{\jmath} - 7\hat{k}\right) + 9 = 5 - 2 - 21 + 9 < 0$

and, $\vec{b}\cdot\vec{n} + d = \left(3\hat{\imath} + 3\hat{\jmath} + 3\hat{k}\right)\cdot\left(5\hat{\imath} + 2\hat{\jmath} - 7\hat{k}\right) + 9 = 15 + 6 - 21 + 9 > 0$

So, the points $\vec{a}$ and $\vec{b}$ are on the opposite sides of the plane

24. Equation of sphere $OABC$ is $x^2 + y^2 + z^2 - ax - by - cz = 0$ where $\sqrt{\frac{a^2+b^2+c^2}{4}} = 2k$

$\Rightarrow a^2 + b^2 + c^2 = 16k^2$ …(i)

Let (α, β, γ) be the centroid of the tetrahedron $OABC$, then $\alpha = \frac{a}{4}, \beta = \frac{b}{4}, \gamma = \frac{c}{4}$

From Eq. (i), $\alpha^2 + \beta^2 + \gamma^2 = k^2$

Locus is $x^2 + y^2 + z^2 = k^2$

25. Let $P(\vec{r})$ be any point on plane

Clearly $\vec{r}-\vec{a}$ will be in linear combination of $\vec{b}-\vec{a}$ and $\vec{c}-\vec{a}$

$\Rightarrow \vec{r}-\vec{a}, \vec{b}-\vec{a}, \vec{c}-\vec{a}$ will be coplanar

$\Rightarrow (\vec{r}-\vec{a})\cdot\{(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})\}=0 \Rightarrow (\vec{r}-\vec{a})\cdot\{\vec{b}\times\vec{c}+\vec{a}\times\vec{b}+\vec{c}\times\vec{a}\}=0$

$=\vec{r}\cdot(\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b})=[\vec{a}\,\vec{b}\,\vec{c}]$

26. Required DR's are $(3-2, 4+1, -1-1)$ $ie, (1, 5, -2)$.

27. Given equation is $x^2-5x+6=0$

$\Rightarrow (x-2)(x-3)=0 \Rightarrow (x-2)=0$ or $(x-3)=0$

Which represents a plane

28. Do it yourself.

29. Equation of plane passing through the point (1, 2, 3) is

$A(x-1)+B(y-2)+C(z-3)=0$ …(i)

Since, plane (i) is parallel to plane $x+2y+5z=0$

$\Rightarrow A=1, B=2, C=5$

Putting these values in Eq. (i), we get

$(x-1)+2(y-2)+5(z-3)=0$ is the required plane

30. Do it yourself.

31. Equation of plane passing through the intersection of given planes, is

$(x+2y+3z+4)+\lambda(4x+3y+2z+1)=0$ …(i)

Plane (i) is passing through the origin $ie, (0, 0, 0)$

$\therefore 4+\lambda=0 \Rightarrow \lambda=-4$

On putting the value of λ in Eq. (i), we get

$(x+2y+3z+4)-4(4x+3y+2z+1)=0$

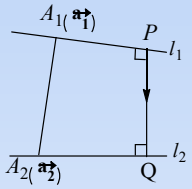
$\Rightarrow -15x-10y-5z=0 \Rightarrow 3x+2y+z=0$

32. Given line can be rewritten as $\frac{x-\frac{1}{3}}{\frac{2b}{3}} = \frac{y-3}{-1} = \frac{z-1}{a}$

Given plane $3x + y + 2z + 6 = 0$ is parallel to the above line

$\therefore \frac{2b}{3} \cdot 3 + 1 \cdot (-1) + 2 \cdot a = 0 \Rightarrow 2a + 2b = 1 \Rightarrow 3a + 3b = \frac{3}{2}$

33. Let PQ be the shortest distance vector between l_1 and l_2. Now, l_1 passes through $A_1(\vec{\mathbf{a}}_1)$ and is parallel to $\vec{\mathbf{b}}_1$ and l_2 passes through $A_2(\vec{\mathbf{a}}_2)$ and is parallel to $\vec{\mathbf{b}}_2$. Since, PQ is perpendicular to both l_1 and l_2 it is parallel to $\vec{\mathbf{b}}_1 \times \vec{\mathbf{b}}_2$



Let $\hat{\mathbf{n}}$ be the unit vector along PQ.Then, $\hat{\mathbf{n}} = \frac{\vec{\mathbf{b}}_1 \times \vec{\mathbf{b}}_2}{|\vec{\mathbf{b}}_1 \times \vec{\mathbf{b}}_2|}$

Let d be the shortest distance between the given lines l_1 and l_2

$\left|\overrightarrow{\mathbf{PQ}}\right| = d$ and $\overrightarrow{\mathbf{PQ}} = d\,\hat{\mathbf{n}}$

Next PQ being the line of shortest distance between l_1 and l_2 is the projection of the line joining the points $A_1(\vec{\mathbf{a}}_1)$ and $A_2(\vec{\mathbf{a}}_2)$ on $\hat{\mathbf{n}}$

$\left|\overrightarrow{\mathbf{PQ}}\right| = \left|\overrightarrow{\mathbf{A}_1\,\mathbf{A}_2} \cdot \hat{\mathbf{n}}\right| \Rightarrow d = \left|\frac{(\vec{\mathbf{a}}_2 - \vec{\mathbf{a}}_1) \cdot \vec{\mathbf{b}}_1 \times \vec{\mathbf{b}}_2}{|\vec{\mathbf{b}}_1 \times \vec{\mathbf{b}}_2|}\right|$

34. The equation of a plane through the line of intersection of the planes

$ax + by + cz + d = 0$ and $a'x + b'y + c'z + d' = 0$ is

$(ax + by + cz + d) + \lambda(a'x + b'y + c'z + d') = 0$

$\Rightarrow x(a + \lambda\, a') + y(b + \lambda\, b') + z(c + \lambda\, c') + d + \lambda\, d' = 0 \quad \ldots$(i)

This parallel to x-axis i.e., $\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$

$\therefore 1 + (a + \lambda\, a') + 0(b + \lambda\, b') + 0(c + \lambda\, c') = 0 \Rightarrow \lambda - \frac{a}{a'}$

Putting the value of λ in (i), the required plane is

$y(a'b - ab') + z(a'c - ac') + a'd - ad' = 0$

35. Let the vertices A, B, C, D of quadrilateral be $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$ and (x_4, y_4, z_4) the equation of plane $PQRS$ be $u \equiv ax + by + cz + d = 0$

Let $u_r = a_r x + b_r y + c_r z + d$ where $r = 1, 2, 3, 4$

Then, $\frac{AP}{PB} \cdot \frac{BQ}{QC} \cdot \frac{CR}{RD} \cdot \frac{DS}{SA} = \left(-\frac{u_1}{u_2}\right)\left(-\frac{u_2}{u_3}\right)\left(-\frac{u_3}{u_4}\right)\left(-\frac{u_4}{u_1}\right) = 1$

36. Let D be the foot of the perpendicular and let it divide BC in the ratio $\lambda : 1$. Then, the coordinates of D are $\left(\frac{3\lambda+4}{\lambda+1}, \frac{5\lambda+7}{\lambda+1}, \frac{3\lambda+1}{\lambda+1}\right)$

Now, $\vec{AD} \perp \vec{BC}$

$\Rightarrow \vec{AD} \cdot \vec{BC} = 0 \Rightarrow -(2\lambda + 3) - 2(5\lambda + 7) - 4 = 0 \Rightarrow \lambda = -7/4$

So, the coordinates of D are $(5/3, 7/3, 17/3)$

37. (1) Direction ratio of the joining the points (1, 2, 5) and (4, 3, 2) is (3, 1, -3) and direction ratios of the joining the points (5, 1, -11) and (8, 2, -8) is (3, 1, 3)

$\therefore$ These are parallel

(2) It is true

(3) Direction ratios of the plane $x - 2y + z = 21$ are (1,-2, 1) and direction ratios of the line $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-1}{3}$ are (1, 2, 3). If they are parallel, then $1(1) - 2(2) + 1(3) = 0$

38. Since two lines intersect at a point. Then shortest distance between them is zero.

$$\therefore \begin{vmatrix} k & 2 & 3 \\ 3 & k & 2 \\ 1 & 1 & -2 \end{vmatrix} = 0$$

$$\left[\therefore \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \end{vmatrix} = 0\right]$$

$\Rightarrow k(-2k - 2) - 2(-6 - 2) + 3(3 - k) = 0 \Rightarrow 2k^2 + 5k - 25 = 0$

$\Rightarrow (2k - 5)(k + 5) = 0 \Rightarrow k = \frac{5}{2}, -5$

Hence, integer value of k is -5

39. The equation of a plane through the line of intersection of the planes $\vec{\mathbf{r}} \cdot \vec{\mathbf{a}} = \lambda$ and $\vec{\mathbf{r}} \cdot \vec{\mathbf{b}} = \mu$ can be written as $(\vec{\mathbf{r}} \cdot \vec{\mathbf{a}} = \lambda) + k(\vec{\mathbf{r}} \cdot \vec{\mathbf{b}} = \mu) = 0$

$\Rightarrow \vec{\mathbf{r}} \cdot (\vec{\mathbf{a}} + k\vec{\mathbf{b}}) = \lambda + k\mu$ …(i)

This plane passes through the origin, therefore $\vec{\mathbf{O}} \cdot (\vec{\mathbf{a}} + k\vec{\mathbf{b}}) = \lambda + \mu k \Rightarrow k = -\frac{\lambda}{\mu}$

On putting the value of k in Eq. (i), the equation of the required plane is

$\vec{\mathbf{r}} \cdot (\mu\vec{\mathbf{a}} - \lambda\vec{\mathbf{b}}) = 0 \Rightarrow \vec{\mathbf{r}} \cdot (\lambda\vec{\mathbf{b}} - \mu\vec{\mathbf{a}}) = 0$

40. Equation of the line passing through $(5, 1, a)$ and $(3, b, 1)$ is $\frac{x-3}{5-3} = \frac{y-b}{1-b} = \frac{z-1}{a-1}$. (i)

Also, point $\left(0, \frac{17}{2}, -\frac{13}{2}\right)$ satisfies Eq. (i), we get $-\frac{3}{2} = \frac{\frac{17}{2}-b}{1-b} = \frac{-\frac{13}{2}-1}{a-1}$

From Ist and IIIrd terms $a - 1 = \frac{\left(-\frac{15}{2}\right)}{\left(-\frac{3}{2}\right)} \Rightarrow a = 6$

From Ist and IIIrd terms $-3(1-b) = 2\left(\frac{17}{2} - b\right) \Rightarrow b = 4$

41. The equation of a plane passing through (2, 2, 1) is $a(x-2) + b(y-2) + c(z-1) = 0$

This passes through (9, 3, 6) and is perpendicular to the plane $2x + 6y + 6z - 1 = 0$

$\therefore 7a + 1 \cdot b + 5c = 0$ and $2a + 6b + 6c = 0 \Rightarrow \frac{a}{-24} = \frac{b}{-32} = \frac{c}{40} \Rightarrow \frac{a}{3} = \frac{b}{4} = \frac{c}{-5}$

So, equation of the required plane is

$3(x-2) + 4(y-2) - 5(z-1) = 0$ or, $3x + 4y - 5z = 9$

42. Given sphere are

$S_1 \equiv x^2 + y^2 + z^2 + 7x - 2y - z = 1$

$S_2 \equiv x^2 + y^2 + z^2 - 3x + 3y + 4z = -4$

required equation of plane is

$(x^2 + y^2 + z^2 + 7x - 2y - z - 1) - (x^2 + y^2 + z^2 - 3x + 3y + 4z + 4) = 0$

$[\because S_1 - S_2 = 0] \Longrightarrow 10x - 5y - 5z = 5 \Longrightarrow 2x - y - z = 1$

43. Given, $\cos\alpha\cos\beta\cos\gamma = \frac{2}{9}$ and $\cos\gamma\cos\alpha = \frac{4}{9}$

Then, $\cos\alpha = \frac{2}{3}, \cos\beta = \frac{1}{3}$ and $\cos\gamma = \frac{2}{3}$

$\therefore \cos\alpha + \cos\beta + \cos\gamma = \frac{2}{3} + \frac{1}{3} + \frac{2}{3} = \frac{5}{3}$

44. Given, equation can be rewritten as $x^2 + y^2 + z^2 - 2x + 3y + 4z - \frac{5}{2} = 0$

Let $\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$

$\therefore$ Given equation written in vector form is $\vec{\mathbf{r}} \cdot \left[\vec{\mathbf{r}} - \left(2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 4\hat{\mathbf{k}}\right)\right] = \frac{5}{2}$

45. Let the equation of plane passing through the point $P(-1,-1,1)$ is

$a(x+1) + b(y+1) + c(z-1) = 0 \ldots.(1)$

Which passes through the points $Q(0,1,1)$and $R(0,0,2)$

$\therefore a + 2b + 0c = 0$ and $a + b + c = 0$

$\Longrightarrow \frac{a}{2-0} = -\frac{b}{1-0} = \frac{c}{1-2} \Longrightarrow \frac{a}{2} = \frac{b}{-1} = \frac{c}{-1}$

From Eq. (i)

$2(x+1) - 1(y+1) - 1(z-1) = 0 \Longrightarrow 2x - y - z + 2 = 0$

$\therefore$ Distance of plane from point (0, 0, 0)$= \frac{0+0+0+2}{\sqrt{2^2+(-1)^2+(-1)^2}} = \frac{2}{\sqrt{6}}$

46. Given, $3lm - 4ln + mn = 0$ .....(i)

and $l + 2m + 3n = 0$ ...(ii)

From Eq. (ii), $l = -(2m + 3n)$ putting in Eq. (i)

$-3(2m+3n)m + 4(2m+3n)n + mn = 0 \Longrightarrow -6m^2 + 12n^2 = 0$

$\Longrightarrow m = \pm\sqrt{2}n$

Now, $m = \sqrt{2}n$

$\Longrightarrow l = -\left(2\sqrt{2}n + 3n\right) = -\left(2\sqrt{2} + 3\right)n$

$\therefore l : m : n = -\left(3 + 2\sqrt{2}\right)n : \sqrt{2}n : n = -\left(3 + 2\sqrt{2}\right) : \sqrt{2} : 1$

Also, $m = -\sqrt{2}n \Longrightarrow l = -\left(-2\sqrt{2} + 3\right)n$

$\therefore l : m : n = -\left(3 - 2\sqrt{2}\right)n : -\sqrt{2} : n$

$= -\left(3 - 2\sqrt{2}\right) : -\sqrt{2} : 1 = \cos\theta = \frac{(3+2\sqrt{2})(3-2\sqrt{2}) + (\sqrt{2})(-\sqrt{2}) + 1\cdot 1}{\sqrt{(3+2\sqrt{2})^2 + (\sqrt{2})^2 + 1^2}\ \sqrt{(3-2\sqrt{2})^2 + (-\sqrt{2})^2 + 1^2}} = 0$

$\Longrightarrow \theta = \frac{\pi}{2}$

47. Let the source of light be situated at $A(a, 0, 0)$, where, $a \neq 0$

Let OA be the incident ray, OB be the reflected ray and ON be the normal to the mirror at $O \therefore \angle AON = \angle NOB = \frac{\theta}{2}$ (say)Direction ratios of $\vec{OA}$ are proportional to $a, 0, 0$ and so its direction cosines are 1, 0, 0Direction cosines of ON are $\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \therefore \cos\frac{\theta}{2} = \frac{1}{\sqrt{3}}$.

Let l, m, n be the direction cosines of the reflected ray OB. Then,

$\frac{l+1}{2\cos\theta/2} = \frac{1}{\sqrt{3}}, \frac{m+0}{2\cos\theta/2} = -\frac{1}{\sqrt{3}}$ and , $\frac{n+0}{2\cos\theta/2} = \frac{1}{\sqrt{3}}$

$\Rightarrow l = \frac{2}{3} - 1, m = -\frac{2}{3}, n = \frac{2}{3} \Rightarrow l = -\frac{1}{3}, m = -\frac{2}{3}, n = \frac{2}{3}$

Hence, direction cosines of the reflected ray are $-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}$

48. Let (u, v, w) be the centre of the sphere with radius r. Since, it passes through the origin $\therefore u^2 + v^2 + w^2 = r^2$. Equation of the diameter parallel to x-axis is $\frac{x-u}{1} = \frac{y-v}{0} = \frac{z-w}{0}$

As it passes through u, v, w and direction ratios of x- axis are 1, 0, 0.The extremities of diameter are the points on Eq. (i) at a distance r from the centre (u, v, w)

$\therefore$ The required extremities are $P(r + u, v, w)$ and $Q(-r + u, v, w)$

P lies on the sphere $x^2 + y^2 + z^2 - 2rx = 0$ as $(r + u)^2 + v^2 + w^2 - 2r(r + u) = 0$

Because $u^2 + v^2 + w^2 = r^2$and similarly Q lies on the sphere $x^2 + y^2 + z^2 + 2rx = 0$

49. $\because \overrightarrow{\mathbf{OC}} = \left(\frac{l_1+l_2}{2}, \frac{m_1+m_2}{2}, \frac{n_1+n_2}{2}\right)$ And $|\overrightarrow{\mathbf{OC}}| = \cos\frac{\theta}{2}$

So, direction cosines of internal angle bisector are $\frac{l_1+l_2}{2\cos\frac{\theta}{2}}, \frac{m_1+m_2}{2\cos\frac{\theta}{2}}, \frac{n_1+n_2}{2\cos\frac{\theta}{2}}$

50. Since, the given sphere touching the three coordinates planes. So, it is clear that centre is (a, a, a) and radius is $a \therefore$ The equation of sphere at the centre (a, a, a) and radius a is

$(x - a)^2 + (y - a)^2 + (z - a)^2 = a^2 \quad \Rightarrow x^2 + y^2 + z^2 - 2ax - 2ay - 2az + 3a^2 = a^2$

$\therefore x^2 + y^2 + z^2 - 2a(x + y + z) + 2a^2 = 0$ is the required equation of sphere