RELATIONS AND FUNCTIONS

CHAPTER 01

RELATION

Let A be a non-empty set and $R \subseteq A \times A$. Then, R is called a relation on A. If $(a, b) \in R$, then we say that a is related to b and we write aRb and if $(a, b) \notin R$, then we write a R b.

DOMAIN, RANGE AND CODOMAIN OF A RELATION:

Let R be a relation from set A to set B, such that $R = \{(a, b): a \in A \text{ and } b \in R\}$. The set of all first and second elements of the ordered pairs in R is called the domain and range respectively, i.e. Domain(R)= $\{a: (a, b) \in R\}$ and Range(R) = $\{b: (a, b) \in R\}$. The set B is called the codomain of relation R.

TYPE OF RELATIONS:

I **EMPTY(VOID) RELATION:** A relation R on a set A is called empty relation, if no element of A is related to any element of A, i.e., $R = \emptyset \subset A \times A$.

II **UNIVERSAL RELATION:** A relation R on a set A is called universal relation, if each element of A is related to every element of A, i.e. $R = A \times A$

III **IDENTITY RELATION:** A relation R on a set A is called an identity relation, if each element of A is related to itself only. It is denoted by I_A.

$$\boldsymbol{I_A = R = \{(a, a): a \in A\}}$$

IV **REFLEXIVE RELATION:** A relation R defined on set A is said to be reflexive, if $(x, x) \in R, \forall x \in A$ i.e. $xRx, \forall x \in A$.

V **SYMMETRIC RELATION:** A relation R define on set A is said to be symmetric, if $(x, y) \in R \Rightarrow (y, x) \in R, \forall x, y \in A$ i.e. $xRy \Rightarrow yRx, \forall x, y \in A$

VI **TRANSITIVE RELATION:** A relation R defined on set A is said to be transitive, if $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R, \forall x, y, z \in A$ i.e. $xRy \text{ and } yRz \Rightarrow xRz$, $\forall x, y, z \in A$

EQUIVALENCE RELATION:

A relation R on a set A is called an equivalence relation, if it is reflexive, symmetric and transitive.

EQUIVALENCE CLASSES:

Consider an arbitrary equivalence relation R on an arbitrary set X, R divides X into mutually disjoint subsets A_i, called partitions or subdivisions of X, satisfying

i) For each i all elements of A_i are related to each other i.e. $A_i \cup A_j = X, i \neq j$

ii) No element of A_i is related to any element of A_j, i.e. $A_i \cap A_j = \emptyset, i \neq j$

iii) $\cup A_i = X$ and $A_i \cap A_j = \emptyset, i \neq j$.

Here, subset A_i are also called equivalence classes.

Let A and B be two non-empty sets. Then, a rule f from A to B which associates each element $x \in A$, to a unique element of $f(x) \in B$ is called a function or mapping from A to B and we write $f: A \to B$. Here, element of A is called the domain of f i.e. dom(f)=A and element of B is called the codomain of f. Also, $\{f(x): x \in A\} \subseteq B$ is called the range of f.

TYPES OF FUNCTIONS:

I **ONE-ONE (INJECTIVE) FUNCTION:** A function $f: A \to B$ is said to be one-one if, distinct element of A have distinct images in B, i.e. $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \; or$ $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ where $x_1, x_2 \in A$

II **MANY-ONE FUNCTION:** A function $f: A \to B$ is said to be many-one, if two or more than two elements in A have the same image in B.

III **ONTO (SURJECTIVE) FUNCTION:** A function $f: A \to B$ is said to be onto or surjective, if every element in B have atleast one pre-image in A, i.e. if for each $y \in B$, there exists an element $x \in A$, such that $f(x) = y$.

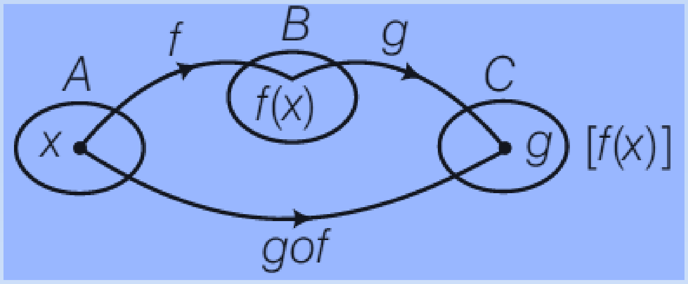
IV **INTO FUNCTION:** A function $f: A \to B$ is said to be into, if atleast one element of B do not have a pre-image in A.

V **ONE-ONE AND ONTO (BIJECTIVE) FUNCTION:** A function $f: A \to B$ is said to be bijective, if f is both one-one and onto.

COMPOSITION OF FUNCTIONS:

Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be any two functions, then the composition of f and g denoted by the function gof and defined as $gof: A \rightarrow C$, such that

$$(gof)(x) = g\{f(x)\}, \forall x \in A$$



INVERTIBLE FUNCTION:

A function $f: X \rightarrow Y$ is defined to be invertible, if there exists a function $g: Y \rightarrow X$, such that $gof = I_x$ and $fog = I_y$. Then, function g is called the inverse of f and it is denoted by f^{-1}. Also f is invertible, then f must be one-one and onto and vice-versa.

BINARY OPERATIONS:

A binary operation $*$ on a set X is a function $*: X \times X \rightarrow X$. We denote $*(a, b)$ by $a * b$, where $a, b \in X$.

COMMUTATIVE BINARY OPERATIONS:

A binary operation $*$ on a set X is said to be commutative, if $a * b = b * a, \forall a, b \in X$.

ASSOCIATIVE BINARY OPERATIONS:

A binary operation $*$ on a set X is said to be associative, if $a * (b * c) = (a * b) * c, \forall a, b, c \in X$.

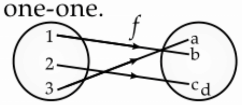
Relations and Functions

Types of Functions

One-one (injective)

$f : X \to Y$ is one-one if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ $\forall\ x_1, x_2 \in X$. Otherwise, f is many- one.

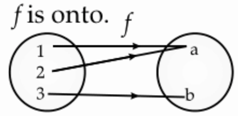
f is one-one.



(Surjective) Onto

$f : X \to Y$ is onto if every $y \in Y, \exists\ x \in X$ such that $f(x) = y$, Then f is surjective

f is onto.



Types of Relations

Reflexive relation

A relation $R : A \to A$ is reflexive if $aRa\ \forall\ a \in A$

Symmetric relation

A relation $R : A \to A$ is symmetric if $aRb \Rightarrow bRa\ \forall\ a, b \in A$

Transitive relation

A relation $R : A \to A$ is transitive if aRb and $bRc \Rightarrow aRc\ \forall\ a,b,c \in A$.

Equivalence relation

(If a relation has reflexive, symmetric and transitive relations) e.g., Let T = the set of all triangles in a plane and $R : T \to T$ defined by $R = \{(T_1, T_2)\} : T_1$ is congruent to $T_2\}$.Then, R is equivalence.

Trace the Mind Map

- First Level
- Second Level
- Third Level

PRACTICE QUESTIONS

1. If $f: R \to R$ satisfies $f(x+y) = f(x) + f(y)$, for all $x, y \in R$ and $f(1) = 7$, then $\sum_{r=1}^{n} f(r)$ is
   a) $\frac{7n}{2}$
   b) $\frac{7(n+1)}{2}$
   c) $7n(n+1)$
   d) $\frac{7n(n+1)}{2}$
2. Let $f: R \to R$ and $g: R \to R$ be given by $f(x) = 3x^2 + 2$ and $g(x) = 3x - 1$ for all $x \in R$. Then,
   a) $fog(x) = 27x^2 - 18x + 5$
   b) $fog(x) = 27x^2 + 18x - 5$
   c) $gof\ (x) = 9x^2 - 5$
   d) $gof\ (x) = 9x^2 + 15$
3. The domain of the function $f(x) = \log_{3+x}(x^2 - 1)$ is
   a) $(-3, -1) \cup (1, \infty)$
   b) $[-3, -1] \cup [1, \infty]$
   c) $(-3, -2) \cup (-2, -1) \cup (1, \infty)$
   d) $[-3, -2) \cup (-2, -1) \cup (1, \infty)$
4. If a, b are two fixed positive integers such that $f(a+x) = b + [b^3 + 1 - 3b^2 f(x) + 3b\,\{f(x)\}^2 - \{f(x)\}^3]^{1/3}$
   For all $x \in R$, then $f(x)$ is a periodic function with period
   a) a
   b) 2a
   c) b
   d) 2b

5. If $f: N \to Z$ is defined by

$$f(n) = \begin{cases} 2 \text{ if } n = 3k, k \in Z \\ 10 \text{ if } n = 3k+1, k \in Z, \\ 0 \text{ if } n = 3k+2, k \in Z \end{cases}$$

Then $\{n \in N: f(n) > 2\}$ is equal to

a) $\{3,6,4\}$

b) $\{1,4,7\}$

c) $\{4,7\}$

d) $\{7\}$

6. Let the function $f(x) = x^2 + x + \sin x - \cos x + \log(1 + |x|)$ be defined on the interval $[0, 1]$. The odd extension of $f(x)$ to the interval $[-1, 1]$ is

a) $x^2 + x + \sin x + \cos x - \log(1 + |x|)$

b) $-x^2 + x + \sin x + \cos x - \log(1 + |x|)$

c) $-x^2 + x + \sin x - \cos x + \log(1 + |x|)$

d) None of these

7. The domain of definition of $f(x) = \log_{1.7}\left(\frac{2-\phi'(x)}{x+1}\right)^{1/2}$, where $\phi(x) = \frac{x^3}{3} - \frac{3}{2}x^2 - 2x + \frac{3}{2}$, is

a) $(-\infty, -4)$

b) $(-4, \infty)$

c) $(-\infty, -1) \cup (-1, 4)$

d) $(-\infty, -1) \cup (-1, 4]$

8. If $[x]$ denotes the greatest integer $\leq x$, then

$\left[\frac{2}{3}\right] + \left[\frac{2}{3} + \frac{1}{99}\right] + \left[\frac{2}{3} + \frac{2}{99}\right] + \ldots + \left[\frac{2}{3} + \frac{98}{99}\right]$ is equal to

a) 99

b) 98

c) 66

d) 65

9. Which of the following functions have period 2π?

a) $y = \sin\left(2\pi t + \frac{\pi}{3}\right) + 2\sin\left(3\pi t + \frac{\pi}{4}\right) + 3\sin 5\pi t$

b) $y = \sin\frac{\pi}{3}t + \sin\frac{\pi}{4}t$

c) $y = \sin t + \cos 2t$

d) None of the above

10. Let R be the set of real numbers and the mapping $f: R \to R$ and $g: R \to R$ be defined by $f(x) = 5 - x^2$ and $g(x) = 3x - 4$, then the value of $(fog)(-1)$ is

a) -44

b) -54

c) -32

d) -64

11. If a function $f : [2, \infty) \to B$ defined by $f(x) = x^2 - 4x + 5$ is a bijection, then $B =$

a) R

b) $[1, \infty)$

c) $[4, \infty)$

d) $[5, \infty)$

12. The domain of definition of

$f(x) = \sqrt{\log_{0.4}\left(\frac{x-1}{x+5}\right) \times \frac{1}{x^2-36}}$, is

a) $(-\infty, 0) - \{-6\}$

b) $(0, \infty) - \{1, 6\}$

c) $(1, \infty) - \{6\}$

d) $[1, \infty) - \{6\}$

13. The set of values of x for which of the function $f(x) = \frac{1}{x} + 2^{\sin^{-1}x} + \frac{1}{\sqrt{x-2}}$ exists is

a) R

b) $R - \{0\}$

c) Φ

d) None of these

14. Suppose $f: [-2, 2] \rightarrow R$ is defined by

$f(x) = \begin{cases} -1, \text{for } -2 \leq x \leq 0 \\ x - 1 \text{ for } 0 \leq x \leq 2 \end{cases}$, then $\{x \in [-2, 2]: x \leq 0 \text{ and } f(|x|) = x\}$ is equal to

a) {-1}

b) {0}

c) $\left\{-\frac{1}{2}\right\}$

d) Φ

15. The domain of definition of the function

$f(x) = x \cdot \frac{1+2(x+4)^{-0.5}}{2-(x+4)^{0.5}} + (x + 4)^{0.5} + 4(x + 4)^{0.5}$ is

a) R

b) $(-4, 4)$

c) R^+

d) $(-4, 0) \cup (0, \infty)$

16. The domain of definition of $f(x) = \sqrt{\log_{10}(\log_{10} x) - \log_{10}(4 - \log_{10} x) - \log_{10} 3}$, is

a) $(10^3, 10^4)$

b) $[10^3, 10^4]$

c) $[10^3, 10^4)$

d) $(10^3, 10^4]$

17. Let R be the real line. Consider the following subsets of the plane $R \times R$

$S = \{(x, y): y = x + 1 \text{ and } o < x < 2\}$

$T = \{(x, y): x - y \text{ is an integer}\}$

Which of the following is true?

a) T is an equivalent relation on R but S is not

b) Neither S nor T is an equivalence relation on R

c) Both S and T are equivalence relations on R

d) S is an equivalence relations on R and T is not

18. If $f(x) = (ax^2 + b)^3$, then the function g such that $f(g(x)) = g(f(x))$ is given by

a) $g(x) = \left(\frac{b - x^{\frac{1}{3}}}{a}\right)^{\frac{1}{2}}$

b) $g(x) = \frac{1}{(a\,x^2 + b)^3}$

c) $g(x) = (a\,x^2 + b)^{\frac{1}{3}}$

d) $g(x) = \left(\frac{x^{\frac{1}{3}} - b}{a}\right)^{\frac{1}{2}}$

19. If $2f(x^2) + 3f\left(\frac{1}{x^2}\right) = x^2 - 1$ for all $x \in R - \{0\}$, then $f(x^4)$ is

a) $\frac{(1-x^4)(2x^4+3)}{5x^4}$

b) $\frac{(1+x^4)(2x^4-3)}{5x^4}$

c) $\frac{(1-x^4)(2x^4-3)}{5x^4}$

d) None of these

20. If X and Y are two non-empty sets where $f: X \to Y$ is function is defined such that $f(C) = \{f(x): x \in C\}$for $C \subseteq X$ and $f^{-1}(D) = \{x: f(x) \in D\} for\ D \subseteq Y$,

For any $A \subseteq X$ and $B \subseteq Y$, then

a) $f^{-1}(f(A)) = A$

b) $f^{-1}(f(A)) = A$ only if $f(X) = Y$

c) $f(f^{-1}(B)) = B$ only if $B \subseteq f(x)$

d) $f(f^{-1}(B)) = B$

21. If $f(x) = \log\left(\frac{1+x}{1-x}\right)$, $-1 < x < 1$, then

$f\left(\frac{3x+x^3}{1+3x^2}\right) - f\left(\frac{2x}{1+x^2}\right)$ is

a) $[f(x)]^3$

b) $[f(x)]^2$

c) $-f(x)$

d) $f(x)$

22. If $f : R \to R$ and $g : R \to R$ are given by $f(x) = |x|$ and $g(x) = [x]$ for each $x \in R$, then $\{x \in R : g(f(x)) \leq f(g(x))\} =$

a) $Z \cup (-\infty, 0)$

b) $(-\infty, 0)$

c) Z

d) R

23. Let $f : N \to Y$ be a function defined as $f(x) = 4x + 3$ where $Y = \{y \in N : y = 4x + 3 \text{ for some } x \in N\}$. Show that f is invertible and its inverse is

a) $g(y) = \frac{y-3}{4}$

b) $g(y) = \frac{3y+4}{3}$

c) $g(y) = 4 + \frac{y+3}{4}$

d) $g(y) = \frac{y+3}{4}$

24. If $f : R \to R$ is given by

$$f(x) = \begin{cases} -1, \text{when } x \text{ is rational} \\ 1, \text{when } x \text{ is irrational} \end{cases},$$

Then $(fof)(1 - \sqrt{3})$ is equal to

a) 1

b) -1

c) $\sqrt{3}$

d) 0

25. Let C denote the set of all complex numbers. The function $f : C \to C$ defined by $f(x) = \frac{ax+b}{cx+d}$ for $x \in C$, where $bd \neq 0$ reduces to a constant function if:

a) $a = c$

b) $b = d$

c) $ad = bc$

d) $ab = cd$

26. If $f(x) = \frac{2^x + 2^{-x}}{2}$, then $f(x+y)f(x-y)$ is equal to

a) $\frac{1}{2}\{f(2x) + f(2y)\}$

b) $\frac{1}{2}\{f(2x) - f(2y)\}$

c) $\frac{1}{4}\{f(2x) + f(2y)\}$

d) $\frac{1}{4}\{f(2x) - f(2y)\}$

27. If $f(x) = \sin^2 x, g(x) = \sqrt{x}$ and $h(x) = \cos^{-1} x, 0 \leq x \leq 1$, then

a) $hogof = fogoh$

b) $gofoh = fohog$

c) $fohog = hogof$

d) None of these

28. The domain of the function

$f(x) = {}^{16-x}C_{2x-1} + {}^{20-3x}P_{4x-5}$, where the symbols have their usual meanings, is the set

a) {2,3}

b) {2,3,4}

c) {1,2,3,4}

d) {1,2,3,4,5}

29. If $f(x) = \frac{4^x}{4^x+2}$, then $f\left(\frac{1}{97}\right) + f\left(\frac{2}{97}\right) + \ldots + f\left(\frac{96}{97}\right)$ is equal to

a) 1

b) 48

c) -48

d) -1

30. The period of $f(x) = \sin\left(\frac{\pi x}{n-1}\right) + \cos\left(\frac{\pi x}{n}\right), n \in Z,\ n > 2,$ is

a) $2n\,\pi(n-1)$

b) $4(n-1)\,\pi$

c) $2n\,(n-1)$

d) None of these

31. If $f(x) = \begin{cases} -1; & x < 0 \\ 0; & x = 0 \\ 1; & x > 0 \end{cases}$ and $g(x) = x(1 - x^2)$, then

a) $fog\,(x) = \begin{cases} -1; & -1 < x < 0 \text{ or } x > 1 \\ 0; & x = 0, 1, -1 \\ 1; & 0 < x < 1 \end{cases}$

b) $fog\,(x) = \begin{cases} -1; & -1 < x < 0 \\ 0; & x = 0,1,-1 \\ 1; & 0 < x < 1 \end{cases}$

c) $fog\,(x) = \begin{cases} -1; & -1 < x < 0 \text{ or } x > 1 \\ 0; & x = 0,\ 1, -1 \\ 1; & 0 < x < 1 \text{ or } x < -1 \end{cases}$

d) $fog\,(x) = \begin{cases} 1; & -1 < x < 0 \text{ or } x > 1 \\ 0; & x = 0,1,-1 \\ 1; & 0 < x < 1 \text{ or } x < -1 \end{cases}$

32. Let X and Y be subsets of R, the set of all real numbers. The function $f: X \to Y$ defined by $f(x) = x^2$ for $x \in X$ is one-one but not onto, if (Here, R^+ is the set of all positive real numbers)

a) $X = Y = R^+$

b) $X = R, Y = R^+$

c) $X = R^+, Y = R$

d) $X = Y = R$

33. Let $A = \{2, 3, 4, 5, \ldots, 16, 17, 18\}$. Let be the equivalence relation on $A \times A$, cartesian product of A and A, defined by $(a, b) \approx (c, d)$ if $ad = bc$, then the number of ordered pairs of the equivalence class of (3, 2) is

a) 4

b) 5

c) 6

d) 7

34. Let $f: [4, \infty[\to [4, \infty[$ be defined by $f(x) = 5^{x(x-4)}$ then $f^{-1}(x)$

a) $2 - \sqrt{4 + \log_5 x}$

b) $2 + \sqrt{4 + \log_5 x}$

c) $\left(\frac{1}{5}\right)^{x(x-4)}$

d) Not defined

35. If $f(x) = \frac{x}{x-1}, x \neq 1$ then

$\underbrace{(f o f o ... o f)}_{19 \text{ times}}(x)$ is equal to

a) $\frac{x}{x-1}$

b) $\left(\frac{x}{x-1}\right)^{19}$

c) $\frac{19x}{x-1}$

d) x

36. Let $f(x)$ be a real valued function defined by

$f(x + \lambda) = 1 + [2 - 5f(x) + 10\{f(x)\}^2 - 10\{f(x)\}^3 + 5\{f(x)\}^4 - \{f(x)\}^5]^{1/5}$ for all real x and some positive constant λ, then $f(x)$ is

a) A periodic function with period λ

b) A periodic function with period 2λ

c) Not a periodic function

d) A periodic function with indeterminate period

37. If $f(x) = \sqrt{|3^x - 3^{1-x}| - 2}$ and $g(x) = \tan \pi x$, then domain of $fog(x)$ is

a) $\left[n + \frac{1}{3}, n + \frac{1}{2}\right] \cup \left[n + \frac{1}{2}, n + 1\right], n \in Z$

b) $\left(nx + \frac{1}{4}, n + \frac{1}{2}\right) \cup \left(n + \frac{1}{2}, n + 1\right), n \in Z$

c) $\left(n + \frac{1}{4}, n + \frac{1}{2}\right) \cup \left[n - \frac{1}{2}, n + 1\right], n \in Z$

d) $\left[n + \frac{1}{4}, x + \frac{1}{2}\right) \cup \left(n + \frac{1}{2}, n + 2\right), n \in Z$

38. A function $f: A \to B$, where $A = \{x: -1 \le x \le 1\}$ and $B = \{y: 1 \le y \le 2\}$, is defined by the rule $y = f(x) = 1 + x^2$. Which of the following statement is true?

a) f is injective but not surjective

b) f is surjective but not injective

c) f is both injective and surjective

d) f is neither injective nor surjective

39. Let the function f, g, h are defined from the set of real numbers R to R such that $f(x) = x^2 - 1, g(x) = \sqrt{(x^2 + 1)}$ and $h(x) = \begin{cases} 0, \text{if } x < 0 \\ x, \text{if } x \ge 0 \end{cases}$, then $ho(fog)(x)$ is defined by

a) x

b) x^2

c) 0

d) None of these

40. If $f(x) = (9x + 0.5) \log_{(0.5+x)} \left(\frac{x^2+2x-3}{4x^2-4x-3}\right)$ is a real number, then x belongs to

a) $\left(-\frac{1}{2}, 1\right)$

b) $\left(-\frac{1}{2}, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \cup \left(\frac{3}{2}, \infty\right)$

c) $\left(-\frac{1}{2} - 1\right)$

d) None of these

41. The function $f(x)$ which satisfies $f(x) = f(-x) = \frac{f'(x)}{x}$, is given by

a) $f(x) = \frac{1}{2} e^{x^2}$

b) $f(x) = \frac{1}{2} e^{-x^2}$

c) $f(x) = x^2 e^{\frac{x^2}{2}}$

d) $f(x) = e^{\frac{x^2}{2}}$

42. The range of the function

$f(x) = 1 + \sin x + \sin^3 x + \sin^5 x + \cdots$ when $x \in (-\pi/2, \pi/2)$, is

a) $(0, 1)$

b) R

c) (-2,2)

d) None of these

43. A polynomial function $f(x)$ satisfies the condition $f(x)f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$

If $f(10) = 1001$, then $f(20) =$

a) 2002

b) 8008

c) 8001

d) None of these

44. If $f(x) = x^3 - x$ and $\phi(x) = \sin 2x$, then

a) $\phi(f(2)) = \sin 2$

b) $\phi(f(1)) = 1$

c) $f\left(\phi\left(\frac{\pi}{12}\right)\right) = -\frac{3}{8}$

d) $f(f(1)) = 2$

45. The interval in which the function $y = \frac{x-1}{x^2-3x+3}$ transforms the real line is

a) $(0, \infty)$

b) $(-\infty, \infty)$

c) $[0, 1]$

d) $\left[-\frac{1}{3,1}\right]$

46. Let $f: R \to, g: R \to R$ be two functions given by $f(x) = 2x - 3, g(x) = x^3 + 5$. Then, $(fog)^{-1}(x)$ is equal to

a) $\left(\frac{x+7}{2}\right)^{\frac{1}{3}}$

b) $\left(x - \frac{7}{2}\right)^{\frac{1}{3}}$

c) $\left(\frac{x-2}{7}\right)^{\frac{1}{3}}$

d) $\left(\frac{x-7}{2}\right)^{1}$

47. If $e^{f(x)} = \frac{10+x}{10-x}, x \in (-10, 10)$ and $f(x) = kf\left(\frac{200x}{100+x^2}\right)$, then k is equal to

a) 0.5

b) 0.6

c) 0.7

d) 0.8

48. If T_1 is the period of the function $f(x) = e^{3(x-[x])}$ and T_2 is the period of the function $g(x) = e^{3x-[3x]}$ ([·] denotes the greatest integer function), then

a) $T_1 = T_2$

b) $T_1 = \frac{T_2}{3}$

c) $T_1 = 3\,T_2$

d) None of these

49. If $b^2 - 4ac = 0$ and $a > 0$, then domian of the function $f(x) = \log\{(ax^2 + bx + c)(x + 1)\}$ is

a) $R - \left(-\frac{b}{2a}\right)$

b) $R - (-\infty, -1)$

c) $(-1, \infty) - \left\{-\frac{b}{2a}\right\}$

d) $R - \left(\left\{-\frac{b}{2a}\right\} \cap (-\infty, -1)\right)$

50. The function $f(x) = \log(x + \sqrt{x^2 + 1})$ is

a) An even function

b) An odd function

c) A periodic function

d) Neither an even nor an odd function

ANSWER KEY

1. D
2. A
3. C
4. B
5. B
6. B
7. C
8. C
9. C
10. A
11. B
12. C
13. C
14. C
15. D
16. C
17. A
18. D
19. A
20. C
21. D
22. A
23. A
24. B
25. C
26. A
27. D
28. A
29. B
30. C
31. C
32. C
33. C
34. D
35. A
36. B
37. B
38. B
39. B
40. B
41. D
42. B
43. C
44. C
45. D
46. D
47. A
48. C
49. C
50. B

HINTS AND SOLUTIONS

1. $\sum_{r=1}^{n} f(r) = f(1) + f(2) + f(3) + \cdots + f(n)$

$= f(1) + 2f(1) + 3f(1) + \ldots nf(n)$ [since, $f(x+y) = f(x) + f(y)$]

$= (1 + 2 + 3 + \ldots + n)f(1) = f(1)\sum n$

$= \frac{7n(n+1)}{2}$ [$\because f(1) = 7$ (given)]

2. $fog\ (x) = f\big(g(x)\big) = f(3x-1) = 3(3x-1)^2 + 2 = 27x^2 - 18x + 5$

3. The given function is defined when $x^2 - 1; 3 + x > 0$ and $3 + x \neq 1$

$\Rightarrow \quad x^2 > 1; \quad 3 + x > 0 \quad$ and $x \neq -2$

$\Rightarrow \quad -1 > x > 1; \quad x > -3, \quad x \neq -2$

$\therefore$ Domain of the function is

$D_f = (-3, -2) \cup (-2, -1) \cup (1, \infty)$

4. We have,

$f(a + x) = b + [b^3 + 1 - 3b^2 f(x) + 3b\{f(x)\}^2 - \{f(x)\}^3]^{1/3}$ for all $x \in R$

$\Rightarrow f(a + x) = b + [1 + \{b - f(x)\}^3]^{1/3}$ for all $x \in R$

$\Rightarrow f(a + x) - b = [1 - \{f(x) - b\}^3]^{1/3}$ for all $x \in R$

$\Rightarrow g(a + x) = [1 - \{g(x)\}^3]^{1/3}$ for all $x \in R$,

Where $g(x) = f(x) - 1$

$\Rightarrow g(2a + x) = [1 - \{g(a + x)\}^3]^{1/3}$ for all $x \in R$

$\Rightarrow g(2a + x) = \left[1 - \left\{1 - \left(g(x)\right)^3\right\}\right]^{1/3}$ for all $x \in R$

$\Rightarrow g(2a + x) = g(x)$ for all $x \in R$

$\Rightarrow f(2a + x) - 1 = f(x) - 1$ for all $x \in R$

$\Rightarrow f(2a + x) = f(x)$ for all $x \in R$

$\Rightarrow f(x)$ is periodic with period $2a$

5. We have

$$f(n)=\begin{cases} \ \ 2 \ \text{ if } \ n=3k, \qquad k\in Z \\ 10 \quad \text{if} \quad n=3k+1, k\in Z \\ 0 \ \text{ if } \ n=3k+2, \qquad k\in Z \end{cases}$$

For $f(n) > 2$, we take $n = 3k + 1, k \in Z$

$\Rightarrow \quad n = 1, 4, 7$

$\therefore$ Required set $\{n \in Z; f(n) > 2\} = \{1, 4, 7\}$

6. To make $f(x)$ an odd function in the interval $[-1,1]$, we re-define $f(x)$ as follows:

$$f(x)=\begin{cases} \quad f(x), \ \ 0\le x\le 1 \\ -f(-x), \qquad -1\le x<0 \end{cases}$$

$$\Rightarrow f(x)=\begin{cases} x^2+x+\sin x-\cos x+\log(1+|x|)\,, 0\le x\le 1 \\ -x^2+x+\sin x+\cos x-\log(1+|x|), -1\le x0 \end{cases}$$

Thus, the odd extension of $f(x)$ to the interval $[-1, 1]$ is

$-x^2 + x + \sin x + \cos x - \log(1 + |x|)$

7. We have,

$f(x) = \log_{1.7}\left\{\frac{2-\varphi'(x)}{x+1}\right\}$, where $\varphi(x) = \frac{x^3}{3} - \frac{3}{2}x^2 - 2x + \frac{3}{2}$

For $f(x)$ to be defined, we must have

$\frac{2-\varphi'(x)}{x+1} > 0, x \neq -1$

$\Rightarrow \frac{2-(x^2-3x-2)}{3x+1} > 0, x \neq -1$

$\Rightarrow \frac{x^2-3x-4}{x+1} < 0, x \neq -1$

$\Rightarrow \frac{(x-4)(x+1)}{x+1} < 0, x \neq -1$

$\Rightarrow x - 4 < 0, x \neq -1$

$\Rightarrow x < 4, x \neq -1$

$\Rightarrow x \in (-\infty, 4), x \neq -1 \Rightarrow x \in (-\infty, -1) \cup (-1,4)$

8. $\left[\frac{2}{3}+\frac{r}{99}\right] = \begin{cases} 0, & r < 33 \\ 1, & r \geq 33 \end{cases}$

$\therefore \quad \sum_{r=0}^{98}\left[\frac{2}{3}+\frac{r}{99}\right] = \sum_{r=0}^{32}\left[\frac{2}{3}+\frac{r}{99}\right] + \sum_{r=33}^{98}\left[\frac{2}{3}+\frac{r}{99}\right] = 0 + 66 = 66$

9. The period of the function in option (a) is 2. The period of the function in option (b) is 24.

   The period of the function in option (c) is 2π.

10. $fog(-1) = f\{g(-1)\}$

    $= f(-7) = 5 - 49 = -44$

11. We have,

    $f : [2, \infty) \to B$ such that $f(x) = x^2 - 4\,x + 5$

    Since f is a bijection. Therefore, B = Range of f

    Now,

    $f(x) = x^2 - 4x + 5 = 5 = (x-2)^2 + 1$ for all $x \in [2, \infty)$

    $\Rightarrow f(x) \geq 1$ for all $x \in [2, \infty) \Rightarrow$ Range of $f = [1, \infty)$

    Hence, $B = [1, \infty)$

12. We observe that $\frac{1}{x^2-36}$ is not defined for $x = \pm 6$

    Also, $\sqrt{\log_{0.4}\left(\frac{x-1}{x+5}\right)}$ is a real number, if

    $0 < \dfrac{x-1}{x++5} \leq 1$

    $\Rightarrow 0 < \dfrac{x-1}{x+5}$ and $\dfrac{x-1}{x+5} \leq 1$

    $\Rightarrow (x-1)(x+5) > 0$ and $1 - \dfrac{6}{x+5} \leq 1$

    $\Rightarrow (x < -5$ or $x > 1)$ and $-\dfrac{6}{x+5} \leq 0$

    $\Rightarrow (x < -5$ or $x > 1)$ and $x + 5 > 0$

    $\Rightarrow (x < -5$ or $x > 1)$ and $x > -5$

    Hence, domain of $f(x) = (1, \infty) - \{6\}$

13. Let $f(x) = g(x) + h(x) + u(x)$, where

$g(x) = \frac{1}{x}, h(x) = 2^{\sin^{-1} x}$ and $u(x) = \frac{1}{\sqrt{x-2}}$

The domain of $g(x)$ is the set of all real numbers other than zero i.e. $R - \{0\}$

The domain of $h(x)$ is the set $[-1, 1]$ and the domain of $u(x)$ is the set of all reals greater than 2, i.e., $(2, \infty)$

Therefore, domain of $f(x) = R - \{0\} \cap [-1, 1] \cap (2, \infty) = \phi$

14. Since, $x \in [-2, 2]$, $x \le 0$ and $f(|x|) = x$

For $-2 \le x \le 0$

$f(-x) = x \Rightarrow \le (-x) - 1 = x \Rightarrow x = -\frac{1}{2}$

15. We have,

$$f(x) = x \frac{1 + \frac{2}{\sqrt{x+4}}}{2 - \sqrt{x+4}} + \sqrt{x+4} + 4\sqrt{x+4}$$

Clearly, $f(x)$ is defined for $x + 4 > 0$ and $x \neq 0$. So, Domain of $f(x)$ is $(-4, 0) \cup (0, \infty)$

16. We have,

$f(x) = \sqrt{\log_{10}(\log_{10} x) - \log_{10}(4 - \log_{10} x) - \log_{10} 3}$

Clearly, $f(x)$ assumes real values, if

$\log_{10}(\log_{10} x) - \log_{10}(4 - \log_{10} x) - \log_{10} 3 \ge 0$

$\Rightarrow \log_{10}\left\{\frac{\log_{10} x}{3(4-\log_{10} x)}\right\} \ge 0$

$\Rightarrow \frac{\log_{10} x}{3(4-\log_{10} x)} \ge 1$

$\Rightarrow \frac{4\log_{10} x - 12}{3(4-\log_{10} x)} \ge 0$

$\Rightarrow \frac{\log_{10} x - 3}{\log_{10} x - 4} \le 0$

$\Rightarrow 3 \le \log_{10} x < 4 \Rightarrow 10^3 \le x < 10^4 \Rightarrow x \in [10^3, 10^4)$

Hence, domain of $f = [10^3, 10^4)$

17. Since, $(1,2) \in S$ but $(2,1) \notin S$

$\therefore S$ is not symmetric.

Hence, S is not an equivalent relation.

Given, $T = \{(x, y) : (x - y) \in I\}$

Now, $xTx \Rightarrow x - x = 0 \in I$, it is reflexive relation

Again, $xTy \Rightarrow (x - y) \in I$

$\Rightarrow y - x \in I \Rightarrow yTx$ it is symmetric relation.

Let xTy and yTz

$\therefore x - y = I_1$ and $y - z = I_2$

Now, $x - z = (x - y) + (y - z) = I_1 + I_2 \in I$

$\Rightarrow x - z \in I$

$\Rightarrow \ xTz$

$\therefore T$ is transitive.

Hence, T is an equivalent relation.

18. It can be easily checked that $g(x) = \left(\frac{x^{1/3} - b}{a}\right)^{1/2}$ satisfies the relation $fog\,(x) = gof\,(x)$

19. Given, $2f(x^2) + 3f\left(\frac{1}{x^2}\right) = x^2 - 1$ ...(i)

Replacing x by $\frac{1}{x}$, we get

$2f\left(\frac{1}{x^2}\right) + 3f(x^2) = \frac{1}{x^2} - 1$ ...(ii)

On multiplying Eq. (i) by 2, Eq. (ii) by 3 and subtracting Eq. (i) from Eq. (ii), we get

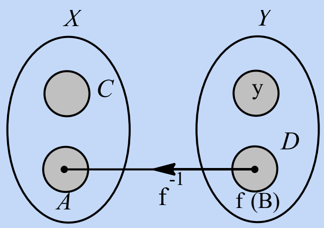
$$5f(x^2) = \frac{3}{x^2} - 1 - 2x^2$$

$$\Rightarrow \quad f(x^2) = \frac{1}{5x^2}(3 - x^2 - 2x^4)$$

$$\Rightarrow \quad f(x^4) = \frac{1}{5x^4}(3 - x^4 - 2x^8) \quad \text{[Replacing } x \text{ by } x^2]$$

$$= \frac{(1 - x^4)(2x^4 + 3)}{5x^4}$$

20. The given data is shown in the figure below



Since, $\quad f^{-1}(D) = x$

$\Rightarrow \; f(x) = D$

Now, $\quad$ if $\quad B \subset X, f(B) \subset D$

$\Rightarrow f^{-1}(f(B)) = B$

21. $\therefore \; f\left(\frac{3x+x^3}{1+3x^2}\right) - f\left(\frac{2x}{1+x^2}\right)$

$$= \log\left(\frac{1+\left(\frac{3x+x^3}{1+3x^2}\right)}{1-\left(\frac{3x+x^3}{1+3x^2}\right)}\right) - \log\left(\frac{1+\frac{2x}{1+x^2}}{1-\frac{2x}{1+x^2}}\right)$$

$$= \log\left(\frac{1+x}{1-x}\right)^3 - \log\left(\frac{1+x}{1-x}\right)^2$$

$$= \log\left(\frac{1+x}{1-x}\right) = f(x)$$

22. We have,

$f(x) = |x|$ and $g(x) = [x]$

$\therefore g(f(x)) \leq f(g(x))$

$\Rightarrow g(|x|) \leq f([x]) \Rightarrow [|x|] \leq |[x]|$

Clearly, $[|x|] = |[x]|$ for all $x \in Z$

Let $x \in (-\infty, 0)$ such that $x \notin Z$. Then, there exists positive integer k such that

$-k - 1 < x < -k$

$\Rightarrow [x] = -k - 1$ and $k < |x| < k + 1$

$\Rightarrow |[x]| = k + 1$ and $[|x|] = k$

$\Rightarrow [|x|] < |[x]|$

Hence, $[|x|] \leq \big||x|\big|$ for all $x \in Z \cup (-\infty, 0)$

i.e. $\{x \in R : g(f(x)) \leq f(g(x))\} = Z \cup (-\infty, 0)$

23. Here, $Y = \{7, 11, \ldots, \infty\}$

Let $y = 4x + 3 \Rightarrow \frac{y-3}{4}$

Inverse of $f(x)$ is $g(y) = \frac{y-3}{4}$

24. Given, $f(x) = \begin{cases} -1, \text{when } x \text{ is rational} \\ 1, \text{when } x \text{ is irrational} \end{cases}$

Now, $(fof)(1-\sqrt{3}) = f[f(1-\sqrt{3})] = f(1) = -1$

25. If $f: C \to C$ given by $f(x) = \frac{ax+b}{cx+d}$ is a constant function, then

$f(x) = \text{Constant } (= \lambda, \text{say}) \text{ for all } x \in C$

$\Rightarrow \dfrac{ax+b}{cx+d} = \lambda \text{ for all } x \in C$

$\Rightarrow (a - \lambda c)\, x + (b - \lambda d) = 0 \text{ for all } x \in C$

$\Rightarrow a - \lambda c = 0 \text{ and } b - \lambda d = 0 \Rightarrow \frac{a}{c} = \frac{b}{d} \Rightarrow ad = bc$

26. We have,

$f(x) = \frac{2^x + 2^{-x}}{2}$

$\therefore f(x+y) f(x-y) = \frac{2^{x+y} + 2^{-x-y}}{2} \times \frac{2^{x-y} + 2^{-x+y}}{2}$

$\Rightarrow f(x+y) f(x-y) = \frac{2^{2x} + 2^{-2y} + 2^{2y} + 2^{-2x}}{4}$

$\Rightarrow f(x+y) f(x-y) = \frac{1}{2}\left(\frac{2^{2x} + 2^{-2x}}{2} + \frac{2^{2y} + 2^{-2y}}{2}\right)$

$\Rightarrow f(x+y)\, f\,(x-y) = \frac{1}{2}\{f(2x) + f\,(2y)\}$

27. We have, $hogof\ (x) = \cos^{-1}(|\sin x|)$ and, $fogoh\ (x) = \sin^2\left(\sqrt{\cos^{-1} x}\right)$

Clearly, $hogof\ (x) \neq fogoh(c)$ Thus, option (a) is not correct

Now, $gofoh\ (x) = |\sin(\cos^{-1} x)| = \left|\sin\left(\sin^{-1}\sqrt{1-x^2}\right)\right| = \sqrt{1-x^2}$

and, $fohog\ (x) = \sin^2(\cos^{-1}\sqrt{x}) = 1 - \cos^2(\cos^{-1}\sqrt{x})$

$\Rightarrow fohog\ (x) = 1 - \left\{\cos\left(\cos^{-1}\sqrt{x}\right)\right\}^2 = 1 - x \therefore gofoh\ (x) \neq fohog(x)$

Thus, option (b) is correct. Also, $hogof(x) = \cos^{-1}(|\sin x|)$ and, $fohog\ (x) = 1 - x \therefore hogof(x) \neq fohog(x)$. Thus, option (c) is not correct. Hence, option (d) is correct

28. We have,

$f(x) = {}^{16-x}C_{2x-1} + {}^{20-3x}P_{4x-5}$

Clearly, $f(x)$ is defined, if

$16 - x \geq 2x - 1 > 0, 20 - 3x \geq 4x - 5 > 0$ and $x \in Z$

$\Rightarrow x \in \{1,2,3,4,5\}, x \in \{2, 3\}$ and $x \in Z$

$\Rightarrow x \in \{2,3\}$

$\therefore$ Domain $(f) = \{2,3\}$

29. $f(x) = \frac{4^x}{4^x+2}$

$\therefore \quad f(1-x) + f(x) = \frac{4^{1-x}}{4^{1-x}+2} + \frac{4^x}{4^x+2}$

$= \frac{4}{4+2.4^x} + \frac{4^x}{4^x+2} = \frac{2}{2+4^x} + \frac{4^x}{4^x+2} = 1$

By putting $x = \frac{1}{97}, \frac{2}{97}, \frac{3}{97}, \dots . \frac{48}{97}$

And adding, we get

$f\left(\frac{1}{97}\right) + f\left(\frac{2}{97}\right) + \cdots + f\left(\frac{96}{97}\right) = 48$

30. We have,

$f(x) = \sin\left(\frac{\pi x}{n-1}\right) + \cos\left(\frac{\pi x}{n}\right), n \in Z, n > 2$

Since $\sin\left(\frac{\pi x}{n-1}\right)$ and $\cos\left(\frac{\pi x}{n}\right)$ are periodic functions with period $2(n-1)$ and $2n$ respectively. Therefore, $f(x)$ is periodic with period equal to LCM of $\left(2n, 2(n-1)\right) = 2n(n-1)$

31. We have, $f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$ and $g(x) = x(1-x^2)$

$\therefore fog\ (x) = f\left(g(x)\right) \Rightarrow fog\ (x) = \begin{cases} -1, & \text{if } g(x) < 0 \\ 0, & \text{if } g(x) = 0 \\ 1, & \text{if } g(x) > 0 \end{cases}$

$\Rightarrow fog\ (x) = \begin{cases} -1, & \text{if } x \in (-1,0) \cup (1,\infty) \\ 0, & \text{if } x = 0, \pm 1 \\ 1, & \text{if } x \in (-\infty,-1) \cup (0,1) \end{cases}$

32. Clearly, $X = R^+$ and $Y = R$

33. Given, $A = \{2, 3, 4, 5, \dots, 16, 17, 18\}$

And $(a, b) = (c, d)$

$\therefore$ Equivalence class of $(3, 2)$ is

$\{(a, b) \in A \times A : (a, b) R\ (3, 2)\}$

$= \{(a, b) \in A \times A : 2a = 3b\}$

$= \left\{(a, b) \in A \times A : b = \frac{2}{3}a\right\}$

$\left\{\left(a, \frac{2}{3}a\right) : a \in A \times A\right\}$

$= \{(3, 2), (6, 4), (9, 6), (12, 8), (15, 10), (18, 12)\}$

$\therefore$ Number of ordered pairs of the equivalence class=6.

34. Given, $f(x) = 5^{x(x-4)}$ for $f: [4, \infty[\to [4, \infty[$

At $x = 4$

$f(x) = 5^{4(4-4)} = 1$

Which is not lie in the interval $[4, \infty[$

$\therefore$ Function is not bijective.

Hence, $f^{-1}(x)$ is not defined.

35. $(fof)x = f\left(\frac{x}{x-1}\right)$

$= \frac{\frac{x}{x-1}}{\left(\frac{x}{x-1}\right) - 1} = x$

$\Rightarrow (fofof)x = f(fof)x = f(x) = \frac{x}{x-1}$

$\therefore (fofof \dots 19 \text{ times})(x) = \frac{x}{x-1}$

36. We have,

$f(x+\lambda)=1+[1+\{1-f(x)\}^5]^{1/5}$

$\Rightarrow f(x+\lambda)-1=[1+\{1-f(x)\}^5]^{1/5}$

$\Rightarrow g(x+\lambda)=[1-\{g(x)\}^5]^{1/5}$, where $g(x)=f(x)-1$

$\Rightarrow g(x+2\,\lambda)=[1-\{g(x+\lambda)\}^5]^{1/5}$

$\Rightarrow g(x+2\,\lambda)=[1-[1-\{g(x)\}^5]^{1/5}$

$\Rightarrow g(x+2\,\lambda)=g(x)$

$\Rightarrow f(x+2\,\lambda)-1=f(x)-1$ for all $x\in R$

$\Rightarrow f(x+2\,\lambda)=f(x)$ for all $x\in R$

Hence, $f(x)$ is periodic with period $2\,\lambda$

37. We have,

$fog\ (x)=\sqrt{|3^{\tan\pi x}-3^{1-\tan\tan\pi}|-2}$

For $fog\ (x)$ to be defined, we must have

$|3^{\tan\pi x}-3^{1-\tan\pi x}|-2\geq 0$

$\Rightarrow \left|3^{\tan\pi x}-\dfrac{3}{3^{\tan\pi x}}\right|\geq 2$

$\Rightarrow \left|t-\dfrac{3}{t}\right|\geq 2$, where $t=3^{\tan\pi x}>0$

$\Rightarrow t-\dfrac{3}{t}\geq 2$ or $t-\dfrac{3}{t}\leq -2$

$\Rightarrow t^2-2t-3\geq 0$ or $t^2+2t-3\leq 0$

$\Rightarrow (t-3)(t+1)\geq 0$ or $(t+3)(t-1)\leq 0$

$\Rightarrow t\geq 3$ or $0<t\leq 1\quad [\because t>0]$

$\Rightarrow 3^{\tan\pi x}\geq 3$ or, $3^{\tan\pi x}\leq 1$

$\Rightarrow \tan\pi x\geq 1$ or, $\tan\pi x\leq 0$

$\Rightarrow n\,\pi+\frac{\pi}{4}\leq \pi\,x<n\pi+\frac{\pi}{2}$ or $n\,\pi-\frac{\pi}{2}<\pi\,x<\pi\,x<n\,\pi, n\in Z$

$\Rightarrow n\,\pi+\frac{\pi}{4}\leq \pi\,x<n\,\pi+\frac{\pi}{2}$ or, $n\,\pi+\frac{\pi}{2}\leq \pi\,x<(n+1)\,\pi, n\in Z$

$\Rightarrow x\in\left(n+\frac{1}{4},n+\frac{1}{2}\right)\cup\left(n+\frac{1}{2},n+1\right)$

38. Since, $A = \{x: -1 \leq x \leq 1\}$

And $B = \{y: 1 \leq y \leq 2\}$

Also, $y = f(x) = 1 + x^2$

For $x = -1, y = 1 + (-1)^2 = 2$

And for $x = 1, y = 1 + 1^2 = 2$

$\therefore f$ is not injective. (one-one)

Here, $\forall B$ their is a preimage.

Hence, f is surjective.

39. $ho(fog)(x) = hof\{g(x)\} = hof\{\sqrt{(x^2+1)}\} = h\{(\sqrt{x^2+1})^2 - 1\} = h\{x^2 + 1 - 1\} = h\{x^2\} = x^2$

40. We have,

$f(x) = (9x + 0.5)\log_{(0.5+x)}\left\{\frac{x^2+2x-3}{4x^2-4x-3}\right\}$

Clearly, $f(x)$ will assume real values, if

$0.5 + x > 0, 0.5 + x \neq 1$ and $\frac{x^2+2x-3}{4x^2-4x-3} > 0$

Clearly, $f(x)$ will assume real values, if

$0.5 + x > 0, 0.5 + x \neq 1$ and $\frac{x^2+2x-3}{4x^2-4x-3} > 0$

$\Rightarrow x > -\frac{1}{2}, x \neq \frac{1}{2}$ and $\frac{(x+3)(x-1)}{(2x-3)(2x+1)} > 0$

$\Rightarrow x > -\frac{1}{2}, x \neq \frac{1}{2}, x \neq \frac{1}{2}$

and, $x \in (-\infty, -3) \cup (-1/2, 1) \cup (3/2, \infty)$

$\Rightarrow x \in (-1/2, 1/2) \cup (1/2, 1) \cup \left(\frac{3}{2}, \infty\right)$

41. Let $f(x) = e^{x^2/2}$

$\therefore f(-x) = e^{(-x)^2} = e^{x2/2}$

And $\frac{f'(x)}{x} = \frac{1}{x}\left(e^{x^2/2}.\frac{2x}{2}\right) = e^{x^2/2} \Rightarrow \quad f(x) = f(-x) = \frac{f'(x)}{x}$

42. We have,

$f(x) = 1 + \frac{\sin x}{1-\sin^2 x} = 1 + \frac{\sin x}{\cos^2 x} = 1 + \tan x \sec x$

$\therefore f'(x) = \sec^3 x + \sec x \tan^2 x > 0$ for all $x \in (-\pi/2, \pi/2)$

$\Rightarrow f(x)$ is an increasing function on $(-\pi/2, \pi/2)$

Now, $\lim_{x\to\pi/2} f(x) = \lim_{x\to\pi/2}\left(1 + \frac{\sin x}{1-\sin^2 x}\right) = \infty$ and, $\lim_{x\to-\pi/2} f(x) = \lim_{x\to-\pi/2}\left(1 + \frac{\sin x}{1-\sin^2 x}\right) = -\infty$

Hence, range $(f) = \big(f(-\pi/2), f(\pi/2)\big) = (-\infty, \infty) = R$

43. We have,

$f(x)f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right) \Rightarrow f(x) = x^n + 1$

Now,

$f(10) = 1001 \Rightarrow 10^n + 1 = 1001 \Rightarrow n = 3$

$\therefore f(x) = x^3 + 1 \Rightarrow f(20) = 20^3 + 1 = 8001$

44. SOLVE ON YOUR OWN

45. Here, we have to find the range of the function which $\left[-\frac{1}{3}, 1\right]$

46. Since $f: R \to R$ and $g: R \to R$, given by $f(x) = 2x - 3$ and $g(x) = x^3 + 5$ respectively, are bijections. Therefore, f^{-1} and g^{-1} exist. We have,

$f(x) = 2x - 3$

$\therefore f(x) = y$

$\Rightarrow 2x - 3 = y \Rightarrow x = \frac{y+3}{2} \Rightarrow f^{-1}(y) = \frac{y+3}{2}$

Thus, f^{-1} is given by $f^{-1}(x) = \frac{x+3}{3}$ for all $x \in R$

Similarly, $g^{-1}(x) = (x-5)^{1/3}$ for all $x \in R$

Now, $(fog)^{-1}(x) = (g^{-1}of^{-1})(x) = g^{-1}(f^{-1}(x))$

$$\Rightarrow (fog)^{-1}(x) = g^{-1}\left(\frac{x+3}{2}\right) = \left(\frac{x+3}{2} - 5\right)^{1/3} = \left(\frac{x-7}{2}\right)^{1/3}$$

47. Here, $f(x) = \log \frac{10+x}{10-x}$

Given that, $f(x) = k\, f\left(\frac{200x}{100+x^2}\right)$

$\Rightarrow \log \frac{10+x}{10-x} = k.\log \left\{\frac{10+\frac{200x}{100+x^2}}{10-\frac{200x}{100+x^2}}\right\} = k \log \left(\frac{10+x}{10-x}\right)^2$

$\Rightarrow \log \frac{10+x}{10-x} = 2k \log \frac{10+x}{10-x} \Rightarrow k = 0.5$

48. We have,

$T_1 = 1$ and $T_2 = \frac{1}{3}$

Clearly, $T_1 = 3\, T_2$

49. Given,

$f(x) = \log\{(ax^2 + bx + c)(x+1)\}$

$= \log(ax^2 + bx + c) + \log(x+1)$

For $f(x)$ to be defined

$ax^2 + bx + c > 0$ and $x + 1 > 0$

$\Rightarrow \quad x > -1$

Hence, option (c) is correct.

50. Given, $f(x) = \log\left(x + \sqrt{x^2+1}\right)$

$\therefore \; f(x) + f(-x) = \log(x + \sqrt{x^2+1} + \log(-x + \sqrt{x^2+1}) = \log(1) = 0$

Hence, $f(x)$ is an odd function.