Continuity & Differentiability

Single Correct Answer Type

1. A function $f(x)$ is defined by ,

$$f(x)=\begin{cases}\frac{[x^2]-1}{x^2-1}, & for\, x^2\neq 1\\ 0 & , for\, x^2=1\end{cases} \qquad Where[.]\, denotes\, GIF$$

A) Continuous at $x=-1$ B) Discontinuous at $x=1$

C) Differentiable at $x=1$ D) None of these

Key. B

Sol.
$$f(x)=\begin{cases}\frac{[x^2]-1}{x^2-1}, & for\ x^2\neq 1\\ 0 & , for\ x^2=1\end{cases}$$

$$=\begin{cases}\frac{-1}{x^2-1} & , for\ 0<x^2<1\\ 0 & , for\ x^2=1\\ 0 & , for\ 1<x^2<2\end{cases}$$

$\therefore$ RHL at x = 1 is 0

Also LHL at x = 1 is ∞

2. If $f(x)=\operatorname{sgn}(x)$ and $g(x)=x(1-x^2)$ then $(fog)(x)$ is discontinuous at

(A) exactly one point (B)exactly two points

(C) exactly three points (D) no point.

Key. C

Sol. Given $f(x)=Sgnx=\begin{cases}-1\, if\ x<0\\ 0\, if\ x=0\\ 1\, if\ x>0\end{cases}$

And $g(x)=x(1-x^2)$

Now $fog(x)=-1$ if $x(1-x^2)<0$ solving

$= 0$ if $x(1-x^2)=0$, $x(1-x^2)<0$

$= 1$ if $x(1-x^2)>0$ we have $x\in(-1,0)\cup(1,\infty)$

$\therefore\ fog(x) = -1$ if $x \in (-1,0) \cup (1,\infty)$

$= 0$ if $x \in \{-1,0,1\}$

$= 1$ if $x \in (-\infty,-1) \cup (0,1)$

$\therefore\ fog(x)$ is discontinuous at $x = -1,0,1$

3. If $f(x)$ is a polynomial satisfying the relation $f(x) + f(2x) = 5x^2 - 18$ then $f^1(1)$ is equal to

(A) 1

(B) 3

(C) cannot be found since degree of $f(x)$ is not given

(D) 2

Key. D

Sol. Let $f(x) = ax^2 + bx + c$ (By hypothesis)

$f(x) + f(2x) = 5x^2 - 8$

$\Rightarrow f(x) = x^2 - 9 \therefore f^1(1) = 2.$

4. Let $'f'$ be a real valued function defined on the interval $(-1,1)$ such that $e^{-x}.f(x) = 2 + \int_0^x \sqrt{t^4+1}\,dt \quad \forall x \in (-1,1)$ and let $'g'$ be the inverse function of $'f'$.

Then $g^1(2)$ = _____

(A) 3 (B) $1/2$ (C) $1/3$ (D) 2

Key. C

Sol. Differentiating given equation we get

$e^{-x}.f^1(x) - e^{-x}.f(x) = \sqrt{1+x^4}$

Since $(g\,of)(x) = x.as\,'\underline{g}'$ is inverse of f.

$\Rightarrow g[f(x)] = x$

$\Rightarrow g^1[f(x)].f^1(x) = 1$

$\Rightarrow g^1[f(0)] = \dfrac{1}{f^1(0)}$

$\Rightarrow g^1(2) = \dfrac{1}{f^1(0)}$

(Here $f(0) = 2$ observe from hypothesis)

Put $x = 0$ in (1) we get $f^1(0) = 3$.

5. If $y = f(x)$ represents a straight line passing through origin and not passing through any of the points with integral Co-ordinates in the co-ordinate plane. Then the number of such continuous functions on 'R' is _____(it is known that straight line represents a function)

(A) 0 (B) finite (C) infinite (D) at most one

Key. C

Sol. $\exists$ infinitely many continuous functions of the form $f(x) = mx$. When m is Irrational, and when slope is irrational the line obviously will not pass through any of the pts in the Co-ordinate plane with integral Co-ordinates. We know a straight line is always continuous.

6. If a function $y = \phi(x)$ is defined on $[a,b]$ and $\phi(a)\phi(b) < 0$ then

(A) $\exists$ no $c \in (a,b)$ such that $\phi(c) = 0$ if and only if $'\phi'$ is continuous

(B) $\exists$ a function $\phi(x)$ differentiable on $R - \{0\}$ satisfying the given hypothesis

(C) If $\phi(c) = 0$ satisfying the given hypothesis then $\phi(x)$ must be discontinuous

(D) None of these

Key. B

Sol. Consider the function $\phi(x) = \begin{cases} \frac{1}{x}, & if \quad x \neq 0 \\ 1, & if \quad x = 0 \end{cases}$ defined on $[-1,1]$,clearly $\phi(-1) \times \phi(1) < 0$, and $\phi(x)$ is differentiable on $R1\{0\}$

But there is no point $c \in [-1,1]$ $\ni \phi(c) = 0$.

7. Let $f : R \to R$ be a differentiable function satisfying $f(y)f(x-y) = f(x)\ \forall\, x, y \in R$ and $f^1(0) = p, f^1(5) = q$ then $f(5)$ is

A. p^2/q B. p/q C. q/p D. q

Key. C

Sol. $y = 0 \Rightarrow f(0) = 1$ and $x = 0 \Rightarrow f(-y) = \frac{1}{f(y)}$.

Hence $f(x+y) = f(x)f(y)$ $f^1(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f(x) \lim_{h\to 0} \frac{f(x)-1}{h} = f(x).f^1(0) = pf(x)$ put $x = 5$ $f(5) = \frac{q}{p}$

8. If both f(x) and g(x) are differentiable functions at x = x_0, then the function defined as h(x) = maximum {f(x), g(x)} :

(A) is always differentiable at x = x_0

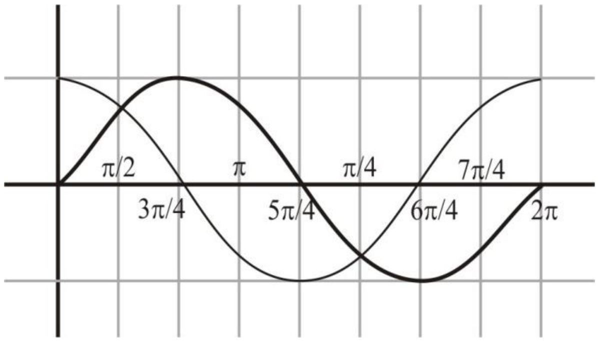
(B) is never differentiable at x = x_0

(C) is differentiable at x = x_0 provided $f(x_0) \neq g(x_0)$

(D) cannot be differentiable at x = x_0 if $f(x_0) \neq g(x_0)$

Key. C

Sol. Consider the graph of f(x) = max(sinx, cosx), which is non-differentiable at x = $\pi/4$, hence statement (A) is false. From the graph y = f(x) is differentiable at x = $\pi/2$, hence statement (B) is false.
Statement (C) is false
Statement (D) is false as consider g(x) = max (x, x^2) at x = 0, for which x = x^2 at x = 0, but f(x) is differentiable at x = 0.



9. $$\left.\begin{aligned} f(x) &= \left(\tan\left(\frac{\pi}{4}+x\right)\right)^{\frac{1}{x}} & if \quad x \neq 0 \\ &= \lambda & if \quad x = 0 \end{aligned}\right\}$$ is continuous at x = 0 then value of λ is

1) 1 2) e 3) e^2 4) 0

Key. 3

Sol. $$\lambda = \lim_{x\to 0}\left(\frac{1+\tan x}{1-\tan x}\right)^{\frac{1}{x}} = \frac{e}{e^{-1}} = e^2$$

10. $f(x) = \frac{1}{q}$ If $x = \frac{p}{q}$ where p and q are integer and $q \neq 0$, G.C.D of (p,q) = 1 and $f(x) = 0$
If x is irrational then set of continuous points of $f(x)$ is

1) all real numbers 2) all rational numbers 3) all irrational number 4) all integers

Key. 3

Sol. Let $x = \frac{p}{q}$

$f(x) = \frac{1}{q}$

When $x \to \frac{p}{q}$ $f(x) = 0$ for every irrational number $\in nbd(p/q)$

$= \frac{1}{n}\ if\ \ n = \frac{m}{n} \in nbd(p/q)$

$\frac{1}{n} \to 0\ as\ n \to \infty$ since

There ∞ - number of rational $\in nbd(p/q)$

$\therefore \lim_{x\to \frac{p}{q}} f(x) = 0$ but $f\left(\frac{p}{q}\right) = \frac{1}{q} \neq 0$

Discontinuous at every rational

If $x=\alpha$ is irrational $\Rightarrow f(\alpha)=0$

Now $\lim\limits_{x\to\alpha} f(x)$ is also O

$\therefore$ continuous for every irrational α

11. $f(x)=\max\{3-x,3+x,6\}$ is differentiable at

A) All points
B) No point
C) All points except two
D) All points expect at one point

Key. C

Sol.

$$f(x)=\begin{cases}3-x & x<-3\\ 6 & -3\le x\le 3\\ 3+x & x>3\end{cases}$$

Since these expressions are linear function in x or a constant
It is clearly differentiable at all points except at the border points at -3 and 3

At $x=-3,\ LHD=-1, RHD=0$

At $x=3,\ LHD=0, RHD=1$

$\therefore$ At $x=-3$ and x = 3 it is not differentiable

12. If ([.] denotes the greatest integer function) then $f(x)$ is

A) continuous and non-differentiable at $x=-1$ and $x=1$
B) continuous and differentiable at $x=0$
C) discontinuous at $x=1/2$
D) continuous but not differentiable at $x=2$

Key. C

Sol.

$$f(x)=\begin{cases}-1 & ,\frac{1}{2}<x<1\\ 0 & ,0<x\le\frac{1}{2}\\ 1 & ,x=0\\ 0 & ,-\frac{1}{2}\le x<0\\ -1 & ,-\frac{3}{2}<x<-\frac{1}{2}\\ 2-x & ,1\le x<2\end{cases}$$ clearly discontinuous at $x=\frac{1}{2}$

13. A function $f(x)$ is defined by ,

$$f(x)=\begin{cases}\frac{[x^2]-1}{x^2-1}, & \text{for } x^2\ne 1\\ 0, & \text{for } x^2=1\end{cases}$$, Where [.] denotes G.I.F

A) Continuous at $x=-1$
B) Discontinuous at $x=1$
C) Differentiable at $x=1$
D) None of these

Key. B

Sol.

$$f(x)=\begin{cases}\frac{[x^2]-1}{x^2-1}, & \text{for } x^2\ne 1\\ 0, & \text{for } x^2=1\end{cases}$$

$$=\begin{cases}\frac{-1}{x^2-1}, & \text{for } 0<x^2<1\\ 0, & \text{for } x^2=1\\ 0, & \text{for } 1<x^2<2\end{cases}$$

$\therefore$ RHL at x = 1 is 0

Also LHL at x = 1 is ∞

14. $f(x)=\frac{\sin 2\pi[\pi^2 x]}{5+[x^2]}$. Where [.] denotes the greatest integer function then $f(x)$ is

A) Continuous
B) Discontinuous

C) $f'(x)$ exist but $f''(x)$ does not exist D) $f'(x)$ is not differentiable

Key. A

Sol. $2\pi[\pi^2 x]$ is integral multiple of π ,there fore f(x)=0 $\forall$ x

$\Rightarrow$f(x) is constant function

$\Rightarrow$f(x) is continuous and differentiable any number of times

15. The no. of points of discontinuous of $g(x)=f(f(x))$ where $f(x)$ is defined as, $f(x)=\begin{cases}1+x, 0\le x\le 2\\ 3-x, 2<x\le 3\end{cases}$

A) 0 B) 1 C) 2 D) >2

Key. C

Sol.

$$g(x)=\begin{cases}2+x, 0\le x\le 1\\ 2-x, 1<x\le 2\\ 4-x, 2<x\le 3\end{cases}$$

16. $$Let\ f(x)=\begin{cases}x^n \sin\left(\frac{1}{x}\right), & x\ne 0\\ 0, & x=0\end{cases}$$

then f (x) is continuous but not differentiable at $x=0$, if

A) $n\in(0,1]$ B) $n\in[1,\infty)$ C) $n\in(-\infty,0)$ D) $n=0$

Key. A

Sol.

$$R.H.L = \lim_{x \to 0^+} f(x)$$
$$= \lim_{h \to 0} f(0+h)$$
$$= \lim_{h \to 0} h^n . \sin\left(\frac{1}{h}\right)$$
$$= 0^n . \sin(\infty)$$
$$= 0^n . \{-1 \text{ to } 1\}$$
$$\because V.F = f(0) = 0$$
$$\therefore n > 0 .... (1)$$
$$Rf^I(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$
$$\lim_{h \to 0} \frac{h^n \sin\left(\frac{1}{h}\right) - 0}{h}$$
$$\lim_{h \to 0} h^{n-1} \sin\left(\frac{1}{h}\right) = 0^{n-1}(-1 \text{ to } 1)$$

For not differentiable

$$n - 1 \le 0$$
$$n \le 1 .... (2)$$

From equation 1 and 2

$$0 < n \le 1$$
$$n \in (0,1]$$

17. The function f(x) is defined as

$$f(x) = \begin{cases} \frac{1}{|x|}, |x| > 2 \\ a + bx^2, |x| \le 2 \end{cases}$$ where a and b are constants. Then which one of the following is true?

A) f is differentiable at x = - 2 if and only if a =3/4, b = -1/16

B) f is differentiable at x = - 2 whatever be the values of a and b

C) f is differentiable at x = - 2 if $b = -\frac{1}{16}$, whatever be the values of a

D) f is differentiable x = - 2 if $b = \frac{1}{16}$, whatever be the values of a.

Key. A

Sol. Conceptual

18. Total number of points belonging to $(0, 2\pi)$ where $f(x) = \min\{\sin x, \cos x, 1-\sin x\}$ is not differentiable

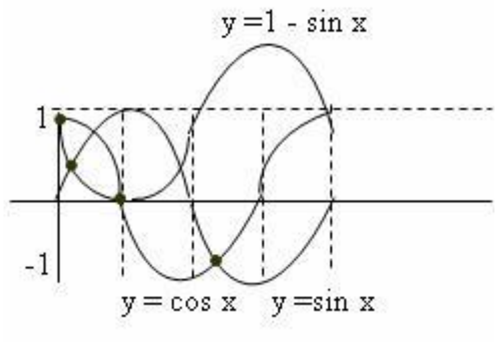
A) 2 B) 3 C) 4 D) 5

Key. B

Sol. By figure it is clear

$x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{4}$ are

The points where f(x) is not differentiable



19. If $f(x) = \begin{cases} \alpha + \frac{\sin[x]}{x} & x > 0 \\ 2 & x = 0 \\ \beta + \left[\frac{\sin x - x}{x^3}\right] & x < 0 \end{cases}$

Where [.] is G.I.F. If f(x) is continuous at x = 0 then $\beta - \alpha$ equal to

A) 1 B) -1 C) 2 D) -2

Key. A

Sol. Conceptual

$RHL(x=0) = \alpha + 0 = \alpha$

$$\frac{\sin x - x}{x^3} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - x}{x^3} = \frac{-1}{3!} + \frac{x^2}{5!} - \ldots\ldots$$

$$\lim_{x \to 0} \frac{\sin x - x}{x^3} = \frac{-1}{6}$$

$LHL = \beta - 1$

20. Given $f(x)=\begin{cases} x^2e^{2(x-1)} & 0\le x\le 1 \\ a\cos(2x-2)+bx^2 & 1<x\le 2 \end{cases}$

$f(x)$ is differentiable at $x=1$ provided

A) $a=-1, b=2$ B) $a=1, b=-2$ C) $a=-3, b=4$ D) $a=3, b=-4$

Key. A

Sol. $f(1+0)=f(1-0)\Rightarrow a+b=1$

$f^1(x)=\begin{cases} 2x^2e^{2(x-1)}+e^{2(x-1)}.2x & 0<x<1 \\ -2a\sin(2x-2)+2bx & 1<x<2 \end{cases}$

$f^1(1-0)=f^1(1+0)\Rightarrow 4=2b$

$\Rightarrow b=2,\ a=-1$

21. The function $f(x)=\dfrac{x}{1+|x|}$ is differentiable in

A) R B) $R-\{0\}$ C) $[0,\infty)$ D) $(0,\infty)$

Key. A

Sol. The function f(x) is an odd function with Range $(-1,1)\Rightarrow$ it is differentiable every where

$f'(0)=\lim_{x\to 0}\dfrac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\dfrac{1}{1+|x|}=1$

22. The domain of the derivative of the function $f(x)=\begin{cases} \tan^{-1}x & \text{if } |x|\le 1 \\ \frac{1}{2}(|x|-1) & \text{if } |x|>1 \end{cases}$ is

A) $R-\{0\}$ B) $R-\{1\}$ C) $R-\{-1\}$ D) $R-\{-1,1\}$

Key. D

Sol. The given function is $f(x)=\begin{cases}\tan^{-1}x & \text{if } |x|\le 1\\ \frac{1}{2}(|x|-1) & \text{if } |x|>1\end{cases}$

$$\Rightarrow f(x)=\begin{cases}\frac{1}{2}(-x-1) & if \quad x<-1\\ \tan^{-1}x & if \quad -1\le x\le 1\\ \frac{1}{2}(x-1) & if \quad x>1\end{cases}$$

Clearly L.H.L at $(x=-1)=\lim_{h\to 0} f(-1-h)$

R.H.L at $(x=-1)=\lim_{h\to 0} f(-1+h)=\lim_{h\to 0}\tan^{-1}(-1+h)=-\pi/4$

$\therefore$ L.H.L $\ne$ R.H.L at $x=-1$

$\therefore$ f(x) is discontinuous at $x=-1$

Also we can prove in the same way, that f(x) is discontinuous at x = 1

$\therefore$ f(x) can not be found for $x=\pm 1$ or domain of $f'(x)=R-\{-1,1\}$

23. $If \ f(x)=\frac{[x]}{|x|}, x\ne 0$ where [.] denotes the G.I.F then $f'(1)$ *is*

A) -1 B) 1 C) ∞ D) Does not exist

Key. D

Sol. $f(x)=\frac{[x]}{|x|}=\begin{cases}0 \ , \ 0<x<1\\ 1 \ , \ 1\le x<2\end{cases}$

Clearly $\lim_{x\to 1^-} f(x)=0, \ \lim_{x\to 1^+} f(x)=1$

$\therefore$ f(x) is not continuous at x = 1

f(x) is not differentiable at x = 1

$\therefore f'(1)$ does not exist

24. If f(x) = $\sin\left\{\frac{\pi}{3}[x]-x^2\right\}$ for 2 < x < 3 and ([x] denotes the G.I.F) then $f'\left(\sqrt{\frac{\pi}{3}}\right)$ is

A) $\sqrt{\frac{\pi}{3}}$ B) $-\sqrt{\frac{\pi}{3}}$ C) $-\sqrt{\pi}$ D) $\sqrt{\pi}$

Key. B

Sol. For 2 < x < 3 , we have [x] = 2

$\therefore f(x) = \sin\left(\frac{2\pi}{3} - x^2\right)$

$f^1(x) = -2x\cos\left(\frac{2\pi}{3} - x^2\right)$

$f^1\left(\sqrt{\frac{\pi}{3}}\right) = -2\sqrt{\frac{\pi}{3}}\cos\left(\frac{2\pi}{3} - \frac{\pi}{3}\right)$

$= -\sqrt{\frac{\pi}{3}}$

25. The derivation of $f(\tan x)$ with respect to $g(\sec x)$ at $x = \frac{\pi}{4}$. If $f'(1) = 2, g'(\sqrt{2}) = 4$

A) $\frac{1}{\sqrt{2}}$ B) $\sqrt{2}$ C) $\frac{1}{2}$ D) 1

Key. A

Sol. $Let\ u = f(\tan x)$

$\frac{du}{dx} = f'(\tan x).\sec^2 x$

$v = g(\sec x)$

$\frac{dv}{dx} = g'(\sec x).\sec x \tan x$

$Now\ \left(\frac{du}{dv}\right) = \frac{f'(\tan x).\sec^2 x}{g'(\sec x).\sec x \tan x} = \frac{f'(1)2}{g'(\sqrt{2}).\sqrt{2}} = \frac{2.2}{4.\sqrt{2}} = \frac{1}{\sqrt{2}}$

26. If $y = \tan^{-1}\frac{1}{x^2+x+1} + \tan^{-1}\frac{1}{x^2+3x+3} + \tan^{-1}\frac{1}{x^2+5x+7} + \ldots.\text{nterms}$ then $\frac{dy}{dx} =$

A) $\frac{1}{1+(x+n)^2} - \frac{1}{1+x^2}$ B) $\frac{1}{1+(x+n)^2} + \frac{1}{1+x^2}$ C) $\frac{1}{1-(x+n)^2} - \frac{1}{1+x^2}$ D) $\frac{1}{1-(x+n)^2} + \frac{1}{1+x^2}$

Key. A

Sol. $y = \tan^{-1}\frac{1}{x^2+x+1} + \tan^{-1}\frac{1}{x^2+3x+3} + \tan^{-1}\frac{1}{x^2+5x+7} + \ldots.\text{nterms}$

$$y=\tan^{-1}\left(\frac{(x+1)-x}{1+x(x+1)}\right)+\tan^{-1}\left(\frac{(x+2)-(x+1)}{1+(x+1)(x+2)}\right)$$
$$+\tan^{-1}\left(\frac{(x+3)-(x+2)}{1+(x+2)(x+3)}\right)+\ldots+\tan^{-1}\left(\frac{(x+n)-(x+n-1)}{1+(x+n)(x+n-1)}\right)$$

$$y=\tan^{-1}(x+1)-\tan^{-1}x+\tan^{-1}(x+2)-\tan^{-1}(x+1)+$$
$$\tan^{-1}(x+3)-\tan^{-1}(x+2)+....+\tan^{-1}(x+n)-\tan^{-1}(x+n-1)$$

$$y=\tan^{-1}(x+n)-\tan^{-1}x\Rightarrow\frac{dy}{dx}=\frac{1}{1+(x+n)^2}-\frac{1}{1+x^2}$$

27. Let f (x) = x[x] , (where [.] denotes the G.I.F) . If x is not an integer, then $f'(x)$ is

A) 2x B) x C) [x] D) 3x

Key. C

Sol. f(x)=x[x]

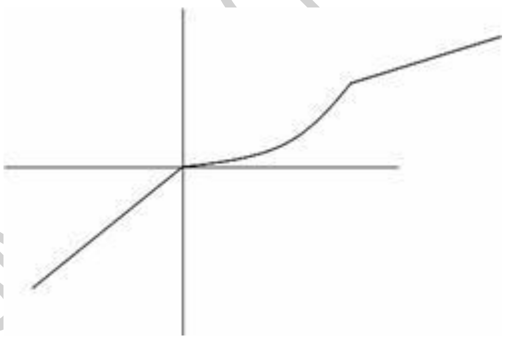
$f'(x)=[x]$

28. Number of points at which the function $f(x)=\begin{cases}\min.(x,x^2)\text{if}-\infty<x<1\\ \min.(2x-1,x^2)\text{ if }x\ge1\end{cases}$ is not derivable is

A) 0 B) 1 C) 2 D) 3

Key. C

Sol.



29. Given $f(x)=\begin{cases}x^2e^{2(x-1)} & 0\le x\le1\\ a\cos(2x-2)+bx^2 & 1<x\le2\end{cases}$

$f(x)$ is differentiable at $x=1$ provided

A) $a=-1, b=2$ B) $a=1, b=-2$ C) $a=-3, b=4$ D) $a=3, b=-4$

Key. A

Sol. $f(1+0)=f(1-0)\Rightarrow a+b=1$

$$f^1(x)=\begin{cases}2x^2e^{2(x-1)}+e^{2(x-1)}.2x & 0<x<1\\ -2a\sin(2x-2)+2bx & 1<x<2\end{cases}$$

$f^1(1-0)=f^1(1+0)\Rightarrow 4=2b$

$\Rightarrow b=2,\ a=-1$

30. The function $f(x)=\dfrac{x}{1+|x|}$ is differentiable in

A) R B) $R-\{0\}$ C) $[0,\infty)$ D) $(0,\infty)$

Key. A

Sol. The function f(x) is an odd function with Range $(-1,1)\Rightarrow$ it is differentiable every where

$$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{1}{1+|x|}=1$$

31. The value of $\lim_{x\to\infty}\left(\dfrac{a_1^{1/x}+a_2^{1/x}+......a_n^{1/x}}{n}\right)^{nx}$ is

A) $a_1+a_2+......+a_n$ B) $e^{a_1+a_2+.....+a_n}$ C) $\dfrac{a_1+a_2+.....+a_n}{n}$ D) $a_1a_2.....a_n$

Key. D

Sol. Let $x=\dfrac{1}{y}$. Then, $x\to\infty, y\to 0$

$$=\lim_{x\to\infty}\left(\frac{a_1^{1/x}+a_2^{1/x}+....+a_n^{1/x}}{n}\right)^{nx}$$

$$=\lim_{y\to 0}\left(\frac{a_1^{y}+a_2^{y}+....+a_n^{y}}{n}\right)^{n/y}=1^{\infty}$$

$$= e^{\lim_{y\to 0}\left(\frac{1+a_1^y+a_2^y+....+a_n^y-n}{n}\right)^{n/y}}$$

$$= e^{\lim_{y\to 0}\frac{n}{y}\left(\frac{a_1^y+a_2^y+......+a_n^y}{n}-1\right)}$$

$$= e^{\lim_{y\to 0}\left(\frac{a_1^y-1}{y}+\frac{a_2^y-1}{y}+......+\frac{a_n^y-1}{y}\right)}$$

$$= e^{\log a_1+\log a_2+\log a_3........+\log a_n}$$

$$= e^{\log(a_1.a_2.a_3..........a_n)} = (a_1 a_2 a_3 ......a_n)$$

32. If $f(x)=\frac{\sin\left(e^{x-2}-1\right)}{\log(x-1)}$, then $\lim_{x\to 2} f(x)$ is given by

A) -2 B) -1 C) 0 D) 1

Key. D

Sol. $\lim_{x\to 2} f(x)=\lim_{x\to 2}\frac{\sin\left(e^{x-2}-1\right)}{\log(x-1)}$

$$\lim_{x\to 2}\left[\frac{\sin\left(e^{x-2}-1\right)}{e^{x-2}-1}.\frac{e^{x-2}-1}{1}.\frac{x-2}{\log(1+(x-2))}\right]$$

$= 1.1.1 = 1$

33. The value of $\lim_{x\to\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right)$ is

A) 0 B) $\frac{1}{2}$ C) $\frac{1}{4}$ D) 1

Key. B

Sol. $\lim_{x\to\infty}\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}$

$$= \lim_{x\to\infty} \frac{\sqrt{x+\sqrt{x}}}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{1+\frac{1}{\sqrt{x}}}}{\sqrt{1+\sqrt{\frac{1}{x}+\frac{1}{\sqrt{x^3}}}}+1} = \frac{\sqrt{1+0}}{\sqrt{1+0+0}+1} = \frac{1}{2}$$

34. Let $f(x,y)$ be a periodic function satisfying the condition $f(x,y) = f(2x+2y, 2y-2x)$ for all $x, y \in R$ and let $g(x) = f(2^x, 0)$. Then the period of $g(x)$ is

A) 2 B) 6 C) 12 D) 24

Key. C

Sol. $f(x,y) = f(2x+2y, 2y-2x)$ ....(1)

$= f(2(2x+2y)+2(2y-2x),\ 2(2y-2x)-2(2x+2y))$

$= f(8y, -8x)$ ....(2)

$f(8y,-8x) = f(-64x, -64y)$ ....(3)

$f(-64x,-64y) = f(2^{12}x, 2^{12}y)$

Replace x by 2^x

$f(x,0) = f(2^{12}x, 0) = f(2^{x+12}, 0)$

$g(x) = g(x+12)$

35. The fundamental period of the function $f(x) = \left|\sin\frac{x}{2}\right| + \left|\cos|x|\right|$ is

A) 2π B) π C) 4π D) $\frac{\pi}{2}$

Key. A

Sol. The fundamental period of $\left|\sin\frac{x}{2}\right|$ is 2π and that of $\left|\cos|x|\right|$ is π. L.C.M of π and 2π is 2π

So fundamental period of $f(x)$ is 2π

36. If $\cos x = \tan y$, $\cos y = \tan z$, $\cos z = \tan x$ then the value of $\sin x$ is

A) $\sin 36^0$ B) $\cos 36^0$ C) $2\sin 18^0$ D) $2\cos 18^0$

Key. C

Sol. $\cos x = \tan y \Rightarrow \cos^2 x = \tan^2 y$

$= \sec^2 y - 1 = \cot^2 z - 1 = \operatorname{cosec}^2 z - 2 = \dfrac{1}{1-\cos^2 z} - 2 = \dfrac{1}{1-\tan^2 x} - 2$

$= \dfrac{2\tan^2 x - 1}{1-\tan^2 x}$

$\Rightarrow \cos^2 x = \dfrac{2\sin^2 x - \cos^2 x}{\cos^2 x - \sin^2 x} \Rightarrow 1-\sin^2 x = \dfrac{3\sin^2 x - 1}{1-2\sin^2 x}$

$\Rightarrow 1 - 2\sin^2 x - \sin^2 x + 2\sin^4 x = 3\sin^2 x - 1$

$\Rightarrow 2\sin^4 x - 6\sin^2 x + 2 = 0$

$\Rightarrow \sin^4 x - 3\sin^2 x + 1 = 0$

$\sin x = \dfrac{\sqrt{5}-1}{2} = 2\sin 18^0$

37. Define $f:[0,\pi] \to R$ by

$f(x) = \begin{cases} \tan^2 x\left[\sqrt{2\sin^2 x + 3\sin x + 4} - \sqrt{\sin^2 x + 6\sin x + 2}\right] & , x \neq \pi/2 \\ k & , x = \pi/2 \end{cases}$ is continuous at

$x = \dfrac{\pi}{2}$,then k =

A) $\dfrac{1}{12}$ B) $\dfrac{1}{6}$ C) $\dfrac{1}{24}$ D) $\dfrac{1}{32}$

Key. A

Sol. Let $\sin x = t$ and evaluate $\lim\limits_{t\to 1}\dfrac{t^2}{1-t^2}\left[\sqrt{2t^2+3t+4} - \sqrt{t^2+6t+2}\right]$ by rationalization

38. Let $|a_1 \sin x + a_2 \sin 2x + ... + a_8 \sin 8x| \le |\sin x|$ for $x \in R$

Define $P = a_1 + 2a_2 + 3a_3 + ... + 8a_8$ .Then P satisfies

A) $|P| \le 1$ B) $|P| < 1$ C) $|P| > 1$ D) $|P| \ge 1$

Key. A

Sol. f(x) = a_1 sinx + a_2 sin2x + ...+ a_8 sin 8x

$|a_1 + 2a_2 + ... + 8a_8| = |f'(0)| = \lim\limits_{x\to 0}\left|\dfrac{f(x)-0}{x}\right|$

$= \underset{x \to 0}{} \left|\frac{f(x)}{\sin x}\right| \left|\frac{\sin x}{x}\right|$

$= \lim_{x \to 0} \left|\frac{f(x)}{\sin x}\right| \le 1$

$|p| \le 1$

39. If $f(x) = \begin{cases} a + \frac{\sin[x]}{x}, & x > 0 \\ 2, & x = 0 \\ b + \left[\frac{\sin x - x}{x^3}\right], & x < 0 \end{cases}$ (where [.] denotes the greatest integer function). If $f(x)$ is continuous at $x = 0$, then b is equal to

A. $a - 1$ B. $a + 1$ C. $a + 2$ D. $a - 2$

Key. B

Sol. $f(0+) = \underset{x \to 0}{Lim}\, a + \frac{\sin[x]}{x} = a$

since $\underset{x \to 0}{Lim} \frac{\sin x - x}{x^3} = \frac{-1}{6}$; we get $f(0-) = b - 1$

Hence $b = a + 1$

40. If $f(x)$ is a continuous function $\forall x \in R$ and the range of $f(x) = (2, \sqrt{26})$ and $g(x) = \left[\frac{f(x)}{a}\right]$ is continuous $\forall x \in R$ (where [.] denotes the greatest integral function). Then the least positive integral value of a is

A. 2 B. 3 C. 6 D. 5

Key. C

Sol. $g(x)$ is continuous only when $\frac{f(x)}{a}$ lies between two consecutive integers Hence $(\frac{2}{a}, \frac{\sqrt{26}}{a})$ should not contain any integer. The least integral value of a is $6(\sin ce \frac{\sqrt{26}}{a} < 1)$

41. $f(x) = [x^2] - [x]^2$, then (where [.] denotes greatest integer function)

A. f is not continuous $x = 0\, and\, x = 1$

B. f is continuous at $x = 0$ but not at $x = 1$

C. f is not continuous at $x = 0$ but continuous at $x = 1$

D. f is continuous at $x = 0$ and $x = 1$

Key. C

Sol. $f(0-)=0-(-1)^2=-1 \, and \, f(0)=0$. Hence f is not continuous at $x=0$ (1) $f(1-)=0-0=0$, $f(1+)=1-1=0$ $f(1)=0$ and Thus f is continuous at $x=1$

42. Let $f(x)=\sec^{-1}([1+\sin^2 x]);$ where [.] denotes greatest integer function. Then the set of points where $f(x)$ is not continuous is

A. $\left\{\frac{n\pi}{2}, n\in I\right\}$ B. $\left\{(2n-1)\frac{\pi}{2}, n\in I\right\}$ C. $\left\{(n-1)\frac{\pi}{2}, n\in I\right\}$ D. $\{n\pi / n\in I\}$

Key. B

Sol. $f(n\pi+)=\sec^{-1}1=0$ and $f(n\pi-)=\sec^{-1}1=0$ and $f(n\pi)=0$

$\therefore f$ is continuous at $x=n\pi$

$f((2n-1)\frac{\pi}{2}+)=\sec^{-1}1=0$ but $f((2n-1)\frac{\pi}{2})=\sec^{-1}2=\frac{\pi}{3}$

$\therefore f$ is discontinuous at $x=(2n-1)\frac{\pi}{2}$ for all $n\in I$

43. The number of points at which the function $f(x)=\max.\{a-x, a+x, b\}, -\infty<x<\infty, 0<a<b$ cannot be differentiable is,

A. 2 B. 3 C. 1 D. 0

Key. A

Sol. $f(x)=\begin{cases} a-x & if \quad x<a-b \\ b & if \quad a-b\le x\le b-a \\ a+x & if \quad x>b-a \end{cases}$

Hence f is not differentiable at $x=a-b, b-a$

44. $\lim\limits_{x\to -1-}[x\sin \pi x]=$ [.] $\to$ denotes greatest integer function

1) -1 2) 1 3) 0 4) does not exist

Key. 1

Sol. $x<-1 \Rightarrow \pi x<-\pi \Rightarrow \pi x\in$ 2nd quadrant

$\Rightarrow \sin \pi x>0$

$x<0$

$\Rightarrow x\sin \pi x<0$

$[x\sin \pi x]=-1$

45. The function $f(x)=(x^2-1)|x^2-3x+2|+\cos(|x|)$ is not differentiable at

A) – 1 B) 0 C) 1 D) 2

Key. D

Sol. Here cos (|x|) = cos (± x) cos x

$f(x) = -(x^2-1)(x^2-3x+2)+\cos x, 1 \le x \le 2$

$= (x^2-1)(x^2-3x+2)+\cos x,\ x \le 1$ or $x \ge 2$

Clearly f(1) = cos 1, $\underset{x\to 1}{Lt} f(x) = \cos 1$

f(2) = cos 2, $\underset{x\to 2}{Lt} f(x) = \cos 2$

Hence f(x) is continuous at x = 1, 2

Now $f'(x) = -2x(x^2-3x+2)-(x^2-1)(2x-3)-\sin x, 1 \le x < 2$

$= 2x(x^2-3x+2)+(x^2-1)(2x-3)-\sin x, x<1$ or x > 2

$f'(1-0) = -\sin 1, f'(1+0) = -\sin 1$

$f'(2-0) = -3-\sin 2$,

$f'(2+0)+3-\sin 2$

Hence f(x) is not differentiable at x = 2.

46. If f(x) is a function such that f(0) = a, $f'(0) = ab$, $f''(0) = ab^2$, $f'''(0) = ab^3$, and so on and b > 0, where dash denotes the derivatives, then $\underset{x\to -\infty}{Lt} f(x) =$

A) ∞ B) $-\infty$ C) 0 D) none of these

Key. C

Sol. Given f(0) = a, $f'(0) = ab$, $f''(0) = ab^2$

$f'''(0) = ab^3$ and so on.

$\therefore$ $f(x) = ae^{bx}$

$\therefore$ $\underset{x\to -\infty}{Lt} f(x) = \underset{x\to -\infty}{Lt} ae^{bx} = 0\ [\because b > 0]$

47. If $f(x) = p|\sin x| + qe^{|x|} + r|x|^3$ and f(x) is differentiable at x = 0, then

A) p = q = r = 0 B) p = 0, q = 0, r = any real number

C) q = 0, r = 0, p is any real number D) r = 0, p = 0, q is any real number

Key. B

Sol. At x = 0,

L. H. derivative of p | sin x | = - p

R.H. derivative of p | sin x | = p

$\therefore$ for p | sin x| to be differentiable at x = 0, p = - p or p = 0

at x = 0, L.H. derivative of $qe^{|x|} = -q$

R.H. derivative of $qe^{|x|} = q$

For $qe^{|x|}$ to be differentiable at x = 0,

- q = q or q = 0

d.e. of r $|x|^3$ at x = 0 is 0

$\therefore$ for f (x) to be differentiable at x = 0

P = 0, q = 0 and r may be any real number.

Second Method:

$$f'(0-0) = \underset{h\to 0-0}{Lt} \frac{f(h)-f(0)}{h}$$

$$\underset{h\to 0-0}{Lt} \frac{p|\sinh| + qe^{|h|} + r|h|^3 - q}{h}$$

$$\underset{h\to 0-0}{Lt} \frac{-p\sinh + qe^{-h} - rh^3 - q}{h}$$

$$= \underset{h\to 0-0}{Lt} \left\{-p\frac{\sinh}{h} - \frac{q\left(e^{-h}-1\right)}{-h} - rh^2\right\}$$

= - p – q

Similarly, $f'(0+0) = p+q$

Since f(x) is differentiable at x = 0

$\therefore \quad f'(0-0) = f'(0+0) \Rightarrow -p-q = p+q$

$\Rightarrow \quad p+q=0$

Here r may be any real number.

$\therefore$ Correct choice is (b)

48. The number of points in (1, 3), where $f(x) = a^{\left[x^2\right]}$, $a>1$, is not differentiable where [x] denotes the integral part of x is

A) 0 B) 3 C) 5 D) 7

Key. D

Sol. Here 1 < x < 3 and in this interval x^2 is an increasing function.

$\therefore \quad 1 < x^2 < 9$

$$\left[x^2\right] = 1, 1 \le x < \sqrt{2}$$

$$= 2, \sqrt{2} \le x < \sqrt{3}$$

$$= 3, \sqrt{3} \le x < 2$$

$$= 4, 2 \le x < \sqrt{5}$$

$$= 5, \sqrt{5} \le x < \sqrt{6}$$

$$= 6, \sqrt{6} \le x < \sqrt{7}$$

$$= 7, \sqrt{7} \le x < \sqrt{8}$$

$$= 8, \sqrt{8} \le x < 3$$

Clearly $\left[x^2\right]$ and also $a^{\left[x^2\right]}$ is discontinuous and not differentiable at only 7 points $x = \sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}$

49. Let f(x) be defined in [- 2, 2] by $f(x) = \max\left(\sqrt{4-x^2}, \sqrt{1+x^2}\right), -2 \le x \le 0$

$= \min\left(\sqrt{4-x^2}, \sqrt{1+x^2}\right), 0 < x \le 2$, then f(x)

A) is continuous at all points B) has a point of discontinuity

C) is not differentiable only at one point D) is not differentiable at more than one point

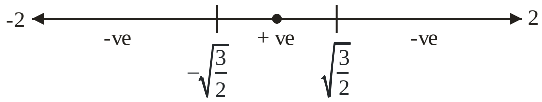
Key. B,D

Sol. $\sqrt{4-x^2} - \sqrt{1+x^2}$

$$= \frac{3-2x^2}{\sqrt{4-x^2} + \sqrt{1+x^2}}$$

$\therefore$ Sign scheme for $\left(\sqrt{4-x^2} - \sqrt{1+x^2}\right)$ is same as that of $3-2x^2$

Sign scheme for $3-2x^2$ is



$\therefore \quad f(x) = \sqrt{1+x^2}, -2 \le x \le -\sqrt{\frac{3}{2}}$

$= \sqrt{4-x^2}, -\sqrt{\frac{3}{2}} \le x \le 0$

$= \sqrt{1+x^2}, 0 < x \le \sqrt{\frac{3}{2}}$

$= \sqrt{4-x^2}, \sqrt{\frac{3}{2}} \le x \le 2$

Clearly f(x) is continuous at $x = -\sqrt{\frac{3}{2}}$ and $x = \sqrt{\frac{3}{2}}$ but it is discontinuous at x = 0

Also $f'(x) = \frac{x}{\sqrt{1+x^2}} - 2 \le x < -\sqrt{\frac{3}{2}}$

$= -\frac{x}{\sqrt{4-x^2}} - \sqrt{\frac{3}{2}} < x < 0$

$= \frac{x}{\sqrt{1+x^2}}, 0 < x < \sqrt{\frac{3}{2}}$

$= -\frac{x}{\sqrt{4-x^2}} \sqrt{\frac{3}{2}} < x \le 2$

F(x) is not differentiable at $x = \pm\sqrt{\frac{3}{2}}$ and also at x = 0 as it is discontinuous at x = 0.

50. If $f(x) = a\left|\sin^7 x\right| + be^{|x|} + c|x|^5$ and if f(x) is differentiable at x = 0, then which of the following in necessarily true

A) a = b = c = 0 B) a = 0, b = 0, $c \in R$

C) b = c = 0, $c \in R$ D) b 0 and a and $c \in R$

Key. D

Sol. $\therefore\ a\left|\sin^7 x\right|$ is differentiable at x = 0 and its d.e. is 0 for all $a \in R$ and $c|x|^5$ is differentiable at x = 0 and its d.e. is 0 for all $c \in R$.

But at x = 0, L.H. derivative of $be^{|x|} = -b$ and R.H. derivative = b

$\therefore$ for $be^{|x|}$ to be differentiable at x = 0, b = - b

$\Rightarrow \quad b = 0$

51. If [x] denotes the integral part of x and

$$f(x) = [x]\left\{\frac{\sin\dfrac{\pi}{[x+1]} + \sin\pi[x+1]}{1+[x]}\right\}$$; then

A) f(x) is continuous in R

B) f(x) is continuous but not differentiable in R

C) $f''(x)$ exists for all x in R

D) f(x) is discontinuous at all integral points in R

Key. D

Sol. $\sin\pi[x+1] = 0$.

Also [x + 1] = [x] + 1

$\therefore \quad f(x) = \dfrac{[x]}{1+[x]}\sin\dfrac{\pi}{[x]+1}$

at x = n, $n \in I,\ f(x) = \dfrac{n}{1+n}\sin\dfrac{\pi}{n+1}$

For n < x < n + 1, $n \in I$,

$$f(x) = \frac{n}{1+n}\sin\frac{\pi}{n+1}$$

For n – 1 < x < n, [x] = n – 1

$\therefore \quad f(x) = \dfrac{n-1}{n}\sin\dfrac{\pi}{n}$

Hence $\underset{x\to n=0}{Lt} f(x) = \dfrac{n-1}{n}\sin\dfrac{\pi}{4}$,

$f(n) = \dfrac{n}{1+n}\sin\dfrac{\pi}{n+1}$

$\therefore$ f(x) is discontinuous at all $n \in I$

52. In $x \in \left[0, \dfrac{\pi}{2}\right]$, let $f(x)\ \underset{n\to\infty}{Lt}\dfrac{2^x - x^n \sin x}{1+x^n}$, then

A) f(x) is a constant function

B) f(x) is continuous at x = 1

C) f(x) is discontinuous at x = 1

D) none of these

Key. C

Sol. $f(x) = \underset{n\to\infty}{Lt}\dfrac{2^x - x^n \sin x}{1+x^n}$

$$= \begin{cases} 2^x, & 0 \le x < 1 \\ \dfrac{2^x - \sin x}{2}, & x = 1 \\ -\sin x & x > 1 \end{cases}$$

Now $f(1) = \dfrac{2 - \sin 1}{2}$

$\underset{x \to 1-0}{Lt} f(x) = \underset{x \to 1-0}{Lt} 2^x = 2$

Hence f(x) is discontinuous at x = 1

53. Let f(x) = [cos x + sin x], 0 < x < 2 π, where [x] denotes the integral part of x, then the number of points of discontinuity of f (x) is

A) 3 B) 4 C) 5 D) 6

Key. C

Sol. $f(x) = \left[\sqrt{2}\cos\left(x - \dfrac{\pi}{4}\right)\right]$

But [x] is discontinuous only at integral points.

Also $-\sqrt{2} \le \sqrt{2}\cos\left(x - \dfrac{\pi}{4}\right) \le \sqrt{2}$

Integral values of $\sqrt{2}\cos\left(x - \dfrac{\pi}{4}\right)$ when

0 < x < 2π are

- 1, at $x = \pi, \dfrac{3\pi}{2}$

0, at $x = \dfrac{3\pi}{4}, \dfrac{7\pi}{4}$

1, at $x = \dfrac{\pi}{2}$

$\therefore$ In $(0, 2\pi), f(x)$ is discontinuous at $x = \dfrac{\pi}{2}, \dfrac{3\pi}{4}, \pi, \dfrac{3\pi}{2}, \dfrac{7\pi}{4}$.

54. If [x] denotes the integral part of x and in $(0, \pi)$, we define

$f(x) = \left[\dfrac{2(\sin x - \sin^n x) + |\sin x - \sin^n x|}{2(\sin x - \sin^n x) - |\sin x - \sin^n x|}\right]$. Then for n > 1.

A) f(x) is continuous but not differentiable at $x = \dfrac{\pi}{2}$

B) both continuous and differentiable at $x = \dfrac{\pi}{2}$

C) neither continuous nor differentiable at $x = \dfrac{\pi}{2}$

D) $\underset{x \to \frac{\pi}{2}}{Lt} f(x)$ exists but $\underset{x \to \frac{\pi}{2}}{Lt} f(x) \ne f\left(\dfrac{\pi}{2}\right)$

Key. B

Sol. For $0 < x < \frac{\pi}{2}$ or $\frac{\pi}{2} < x < \pi$,

0 < sin x < 1

$\therefore$ for n > 1, sin x > $\sin^4$ x

$\therefore$ $f(x) = \left[\frac{3(\sin x - \sin^n x)}{\sin x - \sin^n x}\right] = 3, x \neq \frac{\pi}{2}$

$= 3, x = \frac{\pi}{2}$

Thus in $(0, \pi), f(x) = 3$.

Hence f(x) is continuous and differentiable at $x = \frac{\pi}{2}$.

55. If [x] denotes the integral part of x and f(x) = [n + p sin x], 0 < x < π, $n \in I$ and p is a prime number, then the number of points where f(x) is not differentiable is

A) p – 1 B) p C) 2p – 1 D) 2p + 1

Key. C

Sol. [x] is not differentiable at integral points.

Also [n + p sin x] = n + [p sin x]

$\therefore$ [p sin x] is not differentiable, where

P sin x is an integer. But p is prime and 0 < sin $x \le 1 [\because 0 < x < \pi]$

$\therefore$ p sin x is an integer only when

$\sin x = \frac{r}{p}$, where 0 < r $\le$ p and $r \in N$

For r = p, sin x = 1 $\Rightarrow x = \frac{\pi}{2}$ in $(0, \pi)$

For 0 < r < p, sin $x = \frac{r}{p}$

$\therefore$ $x = \sin^{-1}\frac{r}{p}$ or $\pi - \sin^{-1}\frac{r}{p}$

Number of such values of

x = p – 1 + p – 1 = 2p – 2

$\therefore$ Total number of points where f(x) is not differentiable = 1 + 2p – 2 = 2p – 1

56. Let $f(x)$ and $g(x)$ be two differentiable functions, defined as

$f(x) = x^2 + x g'(1) + g''(2)$ and $g(x) = f(1)x^2 + x f'(x) + f''(x)$.

The value of $f(1) + g(-1)$ is

A) 0 B) 1 C) 2 D) 3

Key. C

Sol. $f(x) = x^2 + xg'(1) + g''(2)$

$f'(x) = 2x + g'(1)$

$f''(x) = 2$

$f'''(x) = 0$

and $g(x) = f(1)x^2 + xf'(x) + f''(x)$

$g(x) = f(1)x^2 + x\{2x + g'(1)\} + 2$

$= f(1)x^2 + 2x^2 + xg'(1) + 2 \; = x^2\{2 + f(1)\} + xg'(1) + 2$

$g'(x) = 2x\{2 + f(1)\} + g'(1)$

$g''(x) = 2\{2 + f(1)\}$

$\therefore f(1) + g(-1)$

$= 1 + g'(1) + g''(2) + f(1).(-1)^2 + f'(-1)(-1) + f''(-1)$

$[\because g'(2) = 4 + 2f(1)$

$f''(-1) = 2$

$f'(-1) = 1 - g'(1) + g''(2)]$

$= 1 + g'(1) + 4 + 2f(1) + f(1) - \{1 - g'(1) + g''(2)\} + 2$

$= 6 + 2g'(1) + 3f(1) - g''(2)$

$= 6 + 2g'(1) + 3f(1) - \{4 + 2f(1)\} \; = 2 + f(1) + 2g'(1)$

$f(x) = x^2 + xg'(1) + g''(2)$

$f'(x) = 2x + g'(1)$

$f''(x) = 2$

$f'''(x) = 0$

$f^{iv}(x) = 0$

$g(x) = f(1)x^2 + x.f'(x) + f''(x)$

$g'(x) = 2f(1)x + x.f''(x) + f'(x).1 + f'''(x)$

$g''(x) = 2f(1) + x.f'''(x) + f''(x).1 + f''(x) + f^{iv}(x)$

$\therefore g'(x) = 2f(1)x + 2x + 2x + g'(x) + 0$

$g'(x) = \{2f(1) + 4\}x + g'(x)$

$g''(x) = 2f(1) + 0 + 2 + 2 + 0$

$g''(x) = 4 + 2f(1)$

$\therefore f(1)+g(-1)$

$= 1+g'(1)+g''(2)+1+(-1)g'(-1)+g''(2)$

$= 2+2g''(2)+g'(1)-g'(-1)$

$= 2+2\{4+2f(1)\}+0 \quad [\because g'(1)=g'(-1)]$

$= 2+2\{0\}+(0)=2$

57. Let f(x) be a real function not identically zero, such that

$f\left(x+y^{2n+1}\right)=f(x)+\{f(y)\}^{2n+1}$; $n \in N$ and x,y are real numbers and $f'(0)\ge 0$. Find the values of f(5) and $f'(10)$.

Sol. As in the preceding example, $f'(x)=0$ or $\{f(x)\}^{2n}=x^{2n} \Rightarrow f(x)=f(0)=0$ or f(x) = x.

But f(x) is given to be not identically zero.

$\therefore$ f(x) = 0 is inadmissible. Hence f(x) = x.

$\therefore$ f(x) = 5 and $f'(10)=1$.

58. If $f(x)+f(y)=f\left(\frac{x+y}{1-xy}\right)$ for all $x, y \in R$ and $xy \ne 1$ and $\underset{x\to 0}{Lt}\frac{f(x)}{x}=2$, find $f\left(\sqrt{3}\right)$ and $f'(-2)$.

Sol. Given that $f(x)+f(y)=f\left(\frac{x+y}{1-xy}\right)$.

Putting x = 0, y = 0, we have f(0) = 0.

Differentiating both sides with respect to x, treating y as constant, we get

$$f(x)+0=f'\left(\frac{x+y}{1-xy}\right)\left\{\frac{(1-xy).1-(x+y).(-y)}{(1-xy)^2}\right\}$$

$$=f'\left(\frac{x+y}{1-xy}\right)\left\{\frac{1-xy+xy+y^2}{(1-xy)^2}\right\}=f'\left(\frac{x+y}{1-xy}\right)\left\{\frac{1+y^2}{(1-xy)^2}\right\} \quad ...(1)$$

Similarly differentiating both sides with respect to y, keeping x as constant, we get

$$f'(y)=f'\left(\frac{x+y}{1-xy}\right)\left\{\frac{1+x^2}{(1-xy)^2}\right\} \quad ...(2)$$

From (1) and (2), we get

$$\frac{f'(x)}{f'(y)}=\frac{1+y^2}{1+x^2}\Rightarrow\left(1+x^2\right)f'(x)=\left(1+y^2\right)f'(y)=k(say)\{=f'(0)\}$$

$$\Rightarrow \quad f'(x)=\frac{k}{1+x^2}\Rightarrow f(x)=k\int\frac{1}{1+x^2}dx=k\tan^{-1}x+\alpha.$$

Putting x = 0, we have $f(0)=k\times 0+\alpha\Rightarrow\alpha=0, \because f(0)=0$.

Thus $f(x)=k\tan^{-1}x$.

Again $\frac{f(x)}{x}=k\frac{\tan^{-1}x}{x}\Rightarrow\underset{x\to 0}{Lt}\frac{f(x)}{x}=k\ Lt\frac{\tan^{-1}}{x}\Rightarrow 2=k\times 1\Rightarrow k=2$.

Hence f(x) = 2 $\tan^{-1} x$.

$\therefore\ f\left(\sqrt{3}\right)=2\tan^{-1}\left(\sqrt{3}\right)=2\times\frac{\pi}{3}=\frac{2\pi}{3}$ and $f'(-2)=\frac{2}{1+(-2)^2}=\frac{2}{5}$.

59. If $2f(x)=f(xy)+f\left(\frac{x}{y}\right)$ for all $x,y\in R^+$, $f(1)=0$ and $f'(1)=1$, find f(e) and $f'(e)$.

Sol. Given $2f(x)=f(xy)+f\left(\frac{x}{y}\right)$.

Differentiating partially with respect to x (keeping y as constant), we get

$$2f'(x)=f'(xy).\ y+f'\left(\frac{x}{y}\right).\frac{1}{y} \quad ...(1)$$

Again, differentiating partially with respect to y (keeping x as constant), we get

$$0=f'(xy).x+f'\left(\frac{x}{y}\right).x\left(-\frac{1}{y^2}\right) \quad ...(2)$$

(2) $\Rightarrow$ $\frac{x}{y^2}f'\left(\frac{x}{y}\right)=xf'(xy)\Rightarrow f'\left(\frac{x}{y}\right)=y^2f'(x)$.

Hence from (1), $2f'(x)=yf'(xy)=2f'(xy)\Rightarrow f'(x)=yf'(xy)$.

Now, putting x = 1, we have $yf'(y)=f'(1)=1$.

$\Rightarrow$ $f'(y)=\frac{1}{y}\Rightarrow\int f'(y)dy=\int\frac{1}{y}dy\Rightarrow f(y)=\log y+c$.

Putting y = 1, we have f(1) = 0 + c $\Rightarrow$ 0 = c; Q f (1) = 0

$\therefore$ c = 0.

Hence f(y) = log y i.e. f(x) = log x (x > 0).

Hence f(e) = log e = 1 and $f'(e)=\frac{1}{e}$

60. A function y = f(x) is defined for all $x\in[0,1]$ and f(x) + f(y) $=f\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right)$.

And $f(0)=\frac{\pi}{2}, f\left(\frac{1}{\sqrt{2}}\right)=\frac{\pi}{4}$ Find the function y = f(x)

Sol. Given f(x) + f(y) $=f\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right)$ ...(1)

Differentiating partially with respect to x (treating y as constant), we get

$$f'(x)+0=f'\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right)\times\left\{y-\sqrt{1-y^2},\frac{-2x}{2\sqrt{1-x^2}}\right\}$$

$$\Rightarrow\quad f'(x)=f'\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right)\times\left\{\frac{y\sqrt{1-x^2}+x\sqrt{1-y^2}}{\sqrt{1-x^2}}\right\} \quad ...(2)$$

Similarly, differentiating (2) partially with respect to y (treating x as constant), we get

$$f'(y)f'\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right)\times\left\{\frac{x\sqrt{1-y^2}+y\sqrt{1+x^2}}{\sqrt{1-y^2}}\right\} \qquad ...(3)$$

Now, dividing (2) by (3), we get

$$\frac{f'(x)}{f'(y)}=\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}\Rightarrow\sqrt{1-x^2}f'(x)=\sqrt{1-y^2}f'(y)=k \text{ (say)}$$

Thus, $\sqrt{1-x^2}\,f'(x)=k\Rightarrow f'(x)=\frac{k}{1-x^2}$

$\Rightarrow$ $\int f'(x)dx=k\int\frac{1}{\sqrt{1-x^2}}dx\Rightarrow f(x)=k\sin^{-1}x+\alpha$ ...(4)

Now, x = 0 $\Rightarrow$ f(0) = k. $0+\alpha\Rightarrow\frac{\pi}{2}=\alpha$.

Again $x=\frac{1}{\sqrt{2}}\Rightarrow f\left(\frac{1}{\sqrt{2}}\right)=k\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)+\alpha$

$\Rightarrow$ $\frac{\pi}{4}=k\frac{\pi}{4}=\alpha\Rightarrow\frac{\pi}{4}=k\frac{\pi}{4}+\frac{\pi}{2},\ \because\ \alpha=\frac{\pi}{2}$

$\Rightarrow$ $k\frac{\pi}{4}=\frac{\pi}{4}-\frac{\pi}{2}=-\frac{\pi}{4}\Rightarrow k=-1$.

Hence putting k = - 1 and $\alpha=\frac{\pi}{2}$ in (4), we get $f(x)=-\sin^{-1}x+\frac{\pi}{2}=\cos^{-1}x$.

61. Let $f(x)=\underset{n\to\infty}{\text{Lt}}\sum_{r=0}^{n-1}\frac{x}{(rx+1)\{(r+1)x+1\}}$, then

A) f(x) is continuous but not differentiable at x = 0

B) f(x) is both continuous and differentiable at x = 0

C) f(x) is neither continuous not differentiable at x = 0

D) f(x) is a periodic function

Key. C

Sol. $t_{r+1}=\frac{x}{(rx+1)\{(r+1)x+1\}}$

$=\frac{(r+1)x+1-(rx+1)}{(rx+1)[(r+1)x+1]}$

$=\frac{1}{(rx+1)}-\frac{1}{(r+1)x+1}$

$\therefore\ S_n=\sum_{r=0}^{n-1}t_{r+1}\frac{1}{nx+1}$

$=1,\ x\neq 0$

= 0, x = 0

$\therefore$ $\underset{n\to\infty}{\text{Lt}}S_n=\underset{n\to\infty}{\text{Lt}}\left(1-\frac{1}{nx+1}\right)$

Thus, f(x) is neither continuous nor differentiable at x = 0.

Clearly f(x) is not a periodic function.

62. If f(x) is a polynomial function which satisfy the relation

$(f(x))^2 f'''(x) = (f''(x))^3 f'(x)$, $f'(0) = f'(1) = f'(-1) = 0$, $f(0) = 4$, $f(\pm 1) = 3$, then $f''(i)$ (where $i = \sqrt{-1}$) is equal to

(A) 10 (B) 15

(C) –16 (D) –15

Key. C

Sol. Solving the equation

We will get $f(x) = x^4 - 2x^2 + 4$

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(A) 10 (B) 15

(C) –16 (D) –15

Key. C

Sol. Solving the equation

We will get $f(x) = x^4 - 2x^2 + 4$

65. Let a function f(x) be such that $f''(x) = f'(x) + e^x$ and $f(0) = 0$, $f'(0) = 1$, then $\ln\left(\frac{(f(2))^2}{4}\right)$ equal to

(A) $\frac{1}{2}$ (B) 1

(C) 2 (D) 4

Key. D

Sol. $f''(x) - f'(x) = e^x$

put $f'(x) = v$

$$\frac{dv}{dx} + v(-1) = e^x$$

$$\Rightarrow \quad ve^{-x} = \int e^x . e^{-x} dx$$

$ve^{-x} = x + C_1$, $f'(0) = 1 \Rightarrow C_1 = 1$

$f'(x) = xe^x + e^x$

$f(x) = xe^x + C_2$

$\Rightarrow$ f(0) = 0 $\Rightarrow C_2 = 0$

$\Rightarrow f(x) = xe^x \Rightarrow f(2) = 2e^2$

$\ln\left(\frac{(f(2))^2}{4}\right) = 4.$

66. If $\int_{\sin x}^{1} t^2.f(t)dt = 1-\sin x, \forall x \in \left(0,\frac{\pi}{2}\right)$ then the value of $f\left(\frac{1}{\sqrt{3}}\right)$ is

(A) $\frac{1}{\sqrt{3}}$ (B) $\sqrt{3}$

(C) $\frac{1}{3}$ (D) 3

Key. D

Sol. $\int_{\sin x}^{1} t^2.f(t)dt = 1-\sin x$

Differentiating both sides with respect to 'x'

$0-\sin^2 x.f(\sin x).\cos x = -\cos x \Rightarrow \cos x\left[1-\sin^2 x.f(\sin x)\right] = 0$

But $\cos x \neq 0$

So, $f(\sin x) = \frac{1}{\sin^2 x}$

$f\left(\frac{1}{\sqrt{3}}\right) = 3$

67. Let $f:(0,\infty)\to R$ and $F(x) = \int_1^x f(t)dt$. If $F(x^2) = x^2(1+x)$ then f(4) equals

(A) 5/4 (B) 7 (C) 4 (D) 2

Key. C

Sol. $F'(x) = f(x)$

$F(x) = x(1+\sqrt{x}) = x + x^{3/2}$

$\therefore$ $F'(x) = f(x) = 1+\frac{3}{2}\sqrt{x}$

$\therefore$ f(4) = 4

68. If $f(x) = \int_0^x (1+t^3)^{-1/2}dt$ and g (x) is the inverse of f, then the value of $\frac{g''(x)}{g^2(x)}$ is

(A) 3/2 (B) 2/3 (C) 1/3 (D) 1/2

Key. A

Sol. $f\left(x\right)=\int\limits_{0}^{x}\left(1+t^{3}\right)^{-1/2}dt$

i.e. $f\left[g\left(x\right)\right]=\int\limits_{0}^{g(x)}\left(1+t^{3}\right)^{-1/2}dt$

i.e. $x=\int\limits_{0}^{g(x)}\left(1+t^{3}\right)^{-1/2}dt$ [$\because$ g is inverse of f $\Rightarrow$ f[g(x)] = x]

Differentiating with respect to x, we have

$1 = (1 + g^3)^{-1/2} . g'$

i.e. $(g')^2 = 1 + g^3$

Differentiating again with respect to x, we have

$2g'g''=3g^2g'$

gives $\frac{g''}{g^2}=\frac{3}{2}$

69. If $f\left(x\right)$ be positive, continuous and differentiable on the interval $\left(a,b\right)$. If $\lim\limits_{x\to a^+} f\left(x\right)=1$ and $\lim\limits_{x\to b^-} f\left(x\right)=3^{1/4}$ also $f'(x)>\left(f\left(x\right)\right)^3+\frac{1}{f\left(x\right)}$ then

a) $b-a>\frac{\pi}{24}$ b) $b-a<\frac{\pi}{24}$ c) $b-a=\frac{\pi}{12}$ d) $b-a=\frac{\pi}{24}$

Key. B

Sol. $\frac{f'(x)f\left(x\right)}{f\left(x\right)^4+1}>1$

Integrating both sides with respect to "x" from a to b

$\Rightarrow\frac{1}{2}\left[\tan^{-1}\left(\left(f\left(x\right)\right)^2\right)\right]_a^b>\left(b-a\right)$

$\Rightarrow\frac{1}{2}\left\{\pi/3-\pi/4\right\}>\left(b-a\right)$

$\Rightarrow b-a<\frac{\pi}{24}$

70. $f(x)=\left(\tan\left(\frac{\pi}{4}+x\right)\right)^{\frac{1}{x}}$ if $x\neq 0$; $=\lambda$ if $x=0$ is continuous at x = 0 then value of λ is

1) 1 2) e 3) e^2 4) 0

Key. 3

Sol. $\lambda=\lim\limits_{x\to 0}\left(\frac{1+\tan x}{1-\tan x}\right)^{\frac{1}{x}}=\frac{e}{e^{-1}}=e^2$

71. $f(x)=\frac{1}{q}$ If $x=\frac{p}{q}$ where p and q are integer and $q \neq 0$, G.C.D of (p,q) = 1 and $f(x)=0$

If x is irrational then set of continuous points of $f(x)$ is

1) all real numbers 2) all rational numbers 3) all irrational number 4) all integers

Key. 3

Sol. Let $x=\frac{p}{q}$

$f(x)=\frac{1}{q}$

When $x \to \frac{p}{q}$ $f(x)=0$ for every irrational number $\in nbd(p/q)$

$=\frac{1}{n}\ if\ n=\frac{m}{n} \in nbd(p/q)$

$\frac{1}{n} \to 0\ as\ n \to \infty$ since

There ∞ - number of rational $\in nbd(p/q)$

$\therefore \lim_{x \to \frac{p}{q}} f(x)=0$ but $f\left(\frac{p}{q}\right)=\frac{1}{q} \neq 0$

Discontinuous at every rational

If $x=\alpha$ is irrational $\Rightarrow f(\alpha)=0$

Now $\lim_{x \to \alpha} f(x)$ is also O

$\therefore$ continuous for every irrational α

72. If a function $f:[-2a,2a] \to R$ is an odd function such that $f(x)=f(2a-x)$ for $x \in [a,2a]$ and the left hand derivative at x=a is zero then left hand derivative at $x=-a$ is____

a) a b) 0 c) –a d) 1

Key. B

Sol. LHD at x = –a is $\lim_{h \to 0}\frac{f(-a)-f(-a-h)}{h}=-\lim_{h \to 0}\frac{f(a)-f(2a-a+h)}{h}$

$=-\lim_{h \to 0}\frac{f(a)-f(a-h)}{h}=0$ by hypothesis

73. Let $f(x)=\begin{cases} x^n \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0; & x=0 \end{cases}$,then f(x) is continuous but not differentiable at x = 0 if

a) $n \in (0,1]$ b) $n \in [1,\infty)$ c) $n \in (-\infty,0)$ d) n = 0

Key. A

Sol. $\lim_{x \to 0} x^n \sin\frac{1}{x}=0$ for n > 0 $\therefore$ continuous for n > 0 Similarly f(x) is non-differentiable for $n \leq 1$

$\therefore n \in (0,1]$ for f(x) to be continuous and non-differentiable at x = 0.

74. If $f(x)$ is continuous on [-2,5] and differentiable over (-2,5) and -4 £ $f'(x)$£ 3 for all x in (-2,5) then the greatest possible value of $f(5)-\ f(-\ 2)$ is

a) 7 b) 9 c) 15 d) 21

Key. D

Sol. Using LMVT in [–2, 5]

$$\frac{f(5)-f(-2)}{5-(-2)} = f^1(c); c \in (-2,5)$$

$$\therefore f(5)-f(-2) = 7f^1(c) \le 21 \text{ Since } -4 \le f^1(x) \le 3$$

$$\therefore \max\{f(5)-f(-2)\} = 21$$

75. If [.] denotes the integral part of x and $f(x) = [x]\left\{\dfrac{\sin\dfrac{\pi}{[x+1]} + \sin\pi[x+1]}{1+[x]}\right\}$, then

(A) f(x) is continuous in R
(B) f(x) is continuous but not differentiable in R
(C) f'(x) exists $\forall x \in R$
(D) $f(x)$ is discontinuous at all integral points in R

Key: D

Hint: At x = n, $f(n) = \dfrac{n}{n+1}\sin\left(\dfrac{\pi}{n+1}\right) = f(n^+)$

$$f(n) = \frac{n-1}{n}\sin\frac{\pi}{n}$$

$\Rightarrow$ f(x) is discontinuous at all $n \in 1$

76. If $f(x) = \begin{cases} x, & x \le 1 \\ x^2+bx+c, & x > 1 \end{cases}$ and $f(x)$ is differentiable for all $x \in R$, then

a) $b=-1, c \in R$ b) $c=1, b \in R$ c) $b=1, c=-1$ d) $b=-1, c=1$

Key. 4

Sol. $Lf'(1)=1,\ Rf'(1)=2+b$ $\Rightarrow b=-1$

$f(1-)=1$ AND $f(1+)=1+b+c$ $\Rightarrow c=1$

77. If $f(x) = \begin{cases} x^m \sin\dfrac{1}{x} & x \ne 0 \\ 0 & x = 0 \end{cases}$ then the interval in which m lies so that $f(x)$ is both continuous and differentiable at $x=0$ is

a) ¡ b) $(0,\infty)$ c) $(0,1]$ d) $(1,\infty)$

Key. 4

Sol. $\underset{x\to 0}{Lt} f(x) = \underset{x\to 0}{Lt} x^m \sin\frac{1}{x}$ exists if m > 0 I.E., $m\in[0,\infty)$

$f'(0) = \underset{x\to 0}{Lt}\frac{f(x)-f(0)}{x-0} = \underset{x\to 0}{Lt} x^{m-1}\sin\frac{1}{x}$ EXISTS IF M – 1 > 0 IF M > 1 OR $m\in(1,\infty)$

78. $f(x) = Max\ \{x,\ x^3\}$, then at x = 0

a) $f(x)$ is both continuous and differentiable

b) $f(x)$ is neither continuous nor differentiable

c) $f(x)$ is continuous but not differentiable

d) $f(x)$ is differentiable but not continuous

Key. 3

Sol. $f(x) = \begin{cases} x & 0\le x\le 1 \\ x^3 & -1\le x\le 0\end{cases}$ $f(0+)=0$ $f(0-)=0=f(0)$ $Lf'(0)=0$ $Rf'(0)=1$

79. $f(x) = \begin{cases} \left(\dfrac{e^{\frac{1}{x}} - e^{-\frac{1}{x}}}{e^{\frac{1}{x}} + e^{-\frac{1}{x}}}\right) & x\ne 0 \\ 0 & x=0\end{cases}$ then at x = 0

a) $f(x)$ is both continuous and differentiable

b) $f(x)$ is neither continuous nor differentiable

c) $f(x)$ is continuous but not differentiable

d) $f(x)$ is differentiable but not continuous

Key. 2

Sol. $\underset{x\to 0+}{Lt}\ e^{-\frac{1}{x}} = 0$, $\underset{x\to 0-}{Lt}\ e^{\frac{1}{x}} = 0$ $f(0-) = \underset{x\to 0-}{Lt}\left(\dfrac{e^{\frac{2}{x}}-1}{e^{\frac{2}{x}+1}}\right) = \underset{x\to 0-}{Lt}\left(\dfrac{0-1}{0+1}\right) = -1$

$f(0+) = \underset{x\to 0+}{Lt}\left(\dfrac{1-e^{-\frac{1}{x}}}{1+e^{-\frac{1}{x}}}\right) = 1$ $\underset{x\to 0}{\text{Lt}} f(x)$ DOES NOT EXIST

80. If $f\left(\dfrac{x+2y}{3}\right) = \dfrac{f(x)+2f(y)}{3}\ \forall x,y\in R$ and $f'(0)=1$; then $f(x)$ is

a). a second degree polynomial in x b). Discontinuous $\forall x \in R$

c). not differentiable $\forall x \in R$ d). a linear function in x

Key. 4

Sol. We have $f\left(\frac{x+2y}{3}\right)=\frac{f(x)+2f(y)}{3}\forall x,y\in R$ → (1) replacing x by $3x$ and putting $y=0$ in (1),

we get $f(x)=\frac{f(3x)+2f(0)}{3}$. $\Rightarrow f(3x)=3f(x)-2f(0)\to(2)$

. Now, $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ = $\lim_{h\to 0}\frac{f\left(\frac{3x+2.\frac{3h}{2}}{3}\right)-f(x)}{h}$

= $\lim_{h\to 0}\frac{\frac{f(3x)+2.f\left(\frac{3h}{2}\right)}{3}-f(x)}{h}$ (from (1))

= $\lim_{h\to 0}\frac{f(3x)+2f\left(\frac{3h}{2}\right)-3f(x)}{3h}$ = $\lim_{h\to 0}\frac{2f\left(\frac{3h}{2}\right)-2f(0)}{3h}$ (from(2))

= $\lim_{h\to 0}\frac{f\left(\frac{3h}{2}\right)-f(0)}{\frac{3h}{2}}=f'(0)=1$ (given) $\Rightarrow f'(x)=1\Rightarrow f(x)=x+c$. $\therefore f(x)$ is a linear

function in x, continuous $\forall x\in R$ and differentiable $\forall x\in R$. $\therefore$ Only 4 is correct option

81. Let f be a function defined by $f(x)=2^{|\log_2 x|}$, then at $x = 1$

(A) f is continuous as well as differentiable (B) continuous but not differentiable

(C) differentiable but not continuous (D) neither continuous nor differentiable

Key. B

Sol. $f(x)=\begin{cases}1/x, & 0<x<1\\ x, & x\ge 1\end{cases}$, f is continuous

$f'(x)=\begin{cases}-1/x^2, & 0<x<1\\ 1, & x>1\end{cases}$, f is not differentiable at x = 1.

82. If the function $f(x)=\left[\frac{(x-2)^3}{a}\right]\sin(x-2)+a\cos(x-2)$ [.] GIF, is continuous and differentiable in (4, 6), then a belongs

A) [8, 64] B) (0, 8] C) $(64,\infty)$ D) (0, 64)

Key. C

Sol. $a > (x-2)^3$

$8 \le (x-2)^3 \le 64 \Rightarrow a > 64$

83. The equation $x^7 + 3x^3 + 4x - 9 = 0$ has

A) no real root
B) all its roots real
C) a unique rational root
D) a unique irrational root

Key. D

Sol. Let $f(x) = x^7 + 3x^3 + 4x - 9$

$f^1(x) = 7x^6 + 9x^2 + 4 > 0 \ \forall x \in R$

$\therefore f$ is strictly increasing.

$\therefore$ $f(x) = 0$ has a unique real root.

$f(1)f(2) < 0$

$\therefore$ The real root belongs to the interval (1, 2). If $f(x) = 0$ has rational roots, they must be integers. But there are no integers between 1 and 2.

84. A function $f : R \to R$ is such that $f(0) = 4, f^1(x) = 1$ in $-1 < x < 1$ and $f^1(x) = 3$ in $1 < x < 3$. Also f is continuous every where. Then $f(2)$ is

A) 5
B) 7
C) 8
D) Can not be determined

Key. C

Sol. If $-1 < x < 1$ then $f(x) = x + 4$

If $1 < x < 3$ then $f(x) = 3x + c$

But f is continuous at $x = 1$

$\therefore$ $f(1) = 1 + 4 = 3 + c \Rightarrow c = 2$ and $f(1) = 5$

$\therefore$ $f(2) = 8$

85. $f(x) = a|\sin x| + be^{|x|} + c|x|^3$. If $f(x)$ is differentiable at x = 0, then

a) $a + b + c = 0$
b) $a + b = 0$ and c can be any real number
c) $b = c = 0$ and a can be any real number
d) $c = a = 0$ and b can be any real number.

Key. B

Sol. $f(x) = -a\sin x + be^{-x} - cx^3, x \le 0$

$= a\sin x + be^{x} + cx^3, x \ge 0$

Clearly continuous at 0, for differentiability $-a - b = a + b$

86. Let $f : [0,1] \to [0,1]$ be a continuous function. The equation $f(x) = x$

a) will have at least one solution.
b) will have exactly two solutions.
c) will have no solution
d) None of these

Key. A

Sol. $g(x) = f(x) - x$

$g(0)g(1) = f(0)(f(1) - 1) \le 0$

87. The value of f(0), so that the function $f(x) = \dfrac{1 - \cos(1 - \cos x)}{x^4}$ is continuous everywhere, is

a) 1/8
b) 1/2
c) 1/4
d) 1/16

Key. A

Sol. $f(0)=\lim_{h\to 0}\frac{1-\cos(1-\cos h)}{h^4}\times\frac{1+\cos(1-\cos h)}{1+\cos(1-\cos h)}$

$=\lim_{h\to 0}\frac{\sin^2(1-\cos h)}{h^4.(1+\cos(1-\cos h)}.\frac{(1-\cos h)^2}{(1-\cos h)^2}$

$=\lim_{h\to 0}\left[\frac{\sin\,(1-\cos h)}{(1-\cos h)}\right]^2\times\lim_{h\to 0}\left(\frac{1-\cos h}{h^2}\right)^2\times\lim_{h\to 0}\frac{1}{1+\cos(1-\cos h)}$

$=(1)^2\times\frac{1}{4}\times\frac{1}{2}=\frac{1}{8}$.

88. Let $f(x+y)=f(x)f(y)$ for all *x* and *y*. Suppose that *f* (3) = 3 and $f'(0)=11$ then $f'(3)$ is given by

a) 22 b) 44 c) 28 d) 33

Key. D

Sol. $\because f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$=\lim_{h\to 0}\frac{f(x)f(h)-f(x)}{h}$

$=f(x)\lim_{h\to 0}\frac{f(h)-1}{h}$

$=f(x)f'(0)$ since $1=f(0)$ [By putting $x=3, y=0$, we can show that $f(0)=1$]

$f'(3)=f(3)f'(0)$

$=3\times 11=33$.

89. Let $f(x)=[\cos x+\sin x], 0<x<2\pi,$ where $[x]$ denotes the greatest integer less than or equal to x.

The number of points of discontinuity of $f(x)$ is

a) 6 b) 5 c) 4 d)3

Key. B

Sol. $[\cos x+\sin x]=\left[\sqrt{2}\cos(x-\pi/4\right]$

We know that $[x]$ is discontinuous at integral values of *x*,

Now, $\sqrt{2}\cos(x-\pi/4)$ is an integer.

at $x=\pi/2,\ 3\pi/4,\ \pi, 3\pi/2, 7\pi/4$

90. The function f defined by $f(x)=\begin{cases}\frac{1}{2} if\ x\ is\ rational\\ \frac{1}{3} if\ x\ is\ Irrational\end{cases}$

(a) Discontinuous for all x (b) Continuous at x = 2

(c) Continuous at $x=\frac{1}{2}$ (d) Continuous at x = 3

Key. A

Sol. If x is Rational any interval there lie many rationals as well as infinitely many Irrationals

$\therefore \forall n\in N\ \exists$ an Irrational number x_n such that $x-\frac{1}{n}<x_n<x+\frac{1}{n}\Rightarrow |x_n-x|<\frac{1}{n}, \forall n$

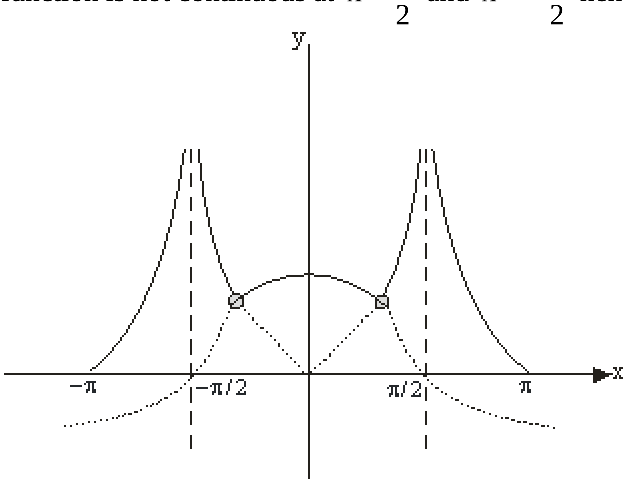
$\Rightarrow \underset{n\to\infty}{Lt} f(x_n)=\frac{1}{3}$, Similarly in case of Irrational

91. Number of points where the function f(x) = max (|tan x|, cos |x|) is non differentiable in the interval $(-\pi, \pi)$ is

A) 4 B) 6 C) 3 D) 2

Key. A

Sol. The function is not differentiable and continuous at two points between $x = -\pi/2 \,\&\, x = \pi/2$ also function is not continuous at $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$ hence at four points function is not differentiable

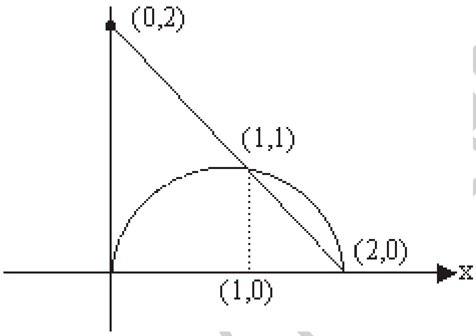


92. The function f(x) = maximum $\left\{\sqrt{x(2-x)}, 2-x\right\}$ is non-differentiable at x equal to

A) 1 B) 0.2 C) 0, 1 D) 1,2

Key. D

Sol.

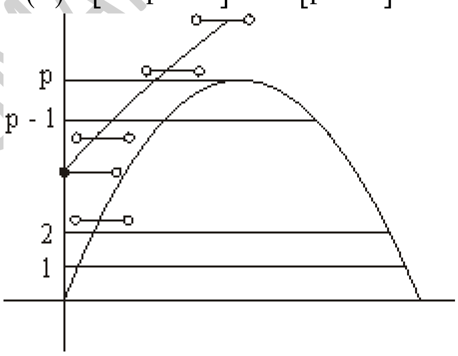


93. Let $f(x) = [n + p\sin x],\ x \in (0, \pi),\ n \in Z, p$ is a prime number and [x] is greatest integer less than or equal to x. The number of points at which f(x) is not differentiable is

A) p B) p – 1 C) 2p + 1 D) 2p – 1

Key. D

Sol. $f(x) = [n + p\sin x] = n + [p\sin x]$



$$[p\sin x]=\begin{cases}0 & 0\le \sin x<\frac{1}{p}\\ 1 & \frac{1}{p}\le \sin x<\frac{2}{p}\\ 2 & \frac{2}{p}\le \sin x<\frac{3}{p}\\ p-1 & \frac{p-1}{p}\le \sin x<1\\ p & \sin x=1\end{cases}$$

$\therefore$ Number of points of discontinuituy are 2 (p – 1) + 1 = 2p – 1 else where it is differentiable and the value = 0

94. Let $f:R\to R$ be any function and $g(x)=\frac{1}{f(x)}$. Then g is

A) onto if f is onto
B) one-one if f is one-one
C) continuous if f is continuous
D) differentiable if f is differentiable

Key. B

Sol. $f:R\to R,\quad g(x)=\frac{1}{f(x)}$

$g'(x)=-\frac{1}{f(x)^2}.f'(x)$

$\Rightarrow$ g is one – one if f is one – one

95. If f(x) = [x] (sin kx)^p is continuous for real x, then

A) $k\in\{n\pi,\ n\in I\},p>0$
B) $k\in\{2n\pi,\ n\in I\},p>0$
C) $k\in\{n\pi,n\in I\},p\in R-\{0\}$
D) $k\in\{n\pi,\ n\in I,n\ne 0\},\ p\in R-\{0\}$

Key. A

Sol. f(x) = [x] (sin kx)^p

$(\sin kx)^p$ is continuous and differentiable function $\forall\ x\in R,\ k\in R$ and p > 0.

[X] is discoutinuous at $x\in I$

For k = n π, n $\in$ I

$f(x)=[x](\sin(n\pi x))^p$

$\lim\limits_{x\to a} f(x)=0,\ a\in I$

and f(a) = 0

So. f(x) becomes coutinuous for all $x\in R$

96. $f(x)=\begin{cases}x+2 & x<0\\ -x^2-2 & 0\le x<1\\ x & x\ge 1\end{cases}$

Then the number of points of discontinuity of |f(x)| is

A) 1
B) 2
C) 3
D) none of these

Key. A

Sol. $f(x)=\begin{cases}x+2 & x<0\\ -x^2-2 & 0\le x<1\\ x & x\ge 1\end{cases}$

$$\therefore \qquad |f(x)| = \begin{cases} -x-2 & x<-2 \\ x+2 & -2\le x<0 \\ x^2+2 & 0\le x<1 \\ x & x\ge 1 \end{cases}$$

Discontinuous at x = 1

$\therefore$ number of points of discount. 1

97. $f(x) = \begin{cases} \dfrac{e^{e/x}-e^{-e/x}}{e^{1/x}+e^{-1/x}} & , \quad x\neq 0 \\ x & , \quad x=0 \end{cases}$

A) f is continuous at x, when k = 0

B) f is not continuous at x = 0 for any real k.

C) $\lim\limits_{x\to 0}$ f(x) exist infinitely

D) None of these

Key. B

Sol. $\lim\limits_{x\to 0^+}\dfrac{e^{e/x}-e^{-e/x}}{e^{1/x}+e^{-1/x}} = \lim\limits_{x\to 0^+}\dfrac{e^{\frac{e-1}{ex}}\left(1-e^{-2e/x}\right)}{\left(1+e^{-2/x}\right)} = +\infty$

$\lim\limits_{x\to 0^-}\dfrac{e^{e/x}-e^{-e/x}}{e^{1/x}+e^{-1/x}} = \lim\limits_{x\to 0^-}\dfrac{e^{-e/x}\left(e^{2e/x}-1\right)}{e^{-e/x}\left(e^{+2/x}+1\right)} = \lim\limits_{x\to 0^-} e^{-\left(\frac{e-1}{x}\right)}\left(\dfrac{e^{2e/x}-1}{e^{2/x}+1}\right) = -\infty$

Limit doesn't exist So f(x) is discoutinous

98. The correct statement for the function $f(x) = \begin{cases} x & , \quad x\in Q \\ -x & , \quad x\in R\sim Q \end{cases}$ IS

A) continuous every where

B) f(x) is a periodic function

C) discontinuous everywhere except at x = 0

D) f(x) is an even function

Key. C

Sol. $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} x = a,\ x\in Q$

$\lim\limits_{x\to a} f(x) \lim\limits_{x\to a}(-x) = -a,\ x\in R\sim Q$

The limit exists $\Leftrightarrow$ a = 0

99. If f(x) = sgn(x) and g(x) = x(1 – x^2), then the number of points of discontinuity of function f(g(x)) is

A) exact two

B) exact three

C) finite and more than 3

D) infinitely many

Key. B

Sol. $f(g(x)) = \begin{cases} 1 & , \quad x<-1 \\ 0 & , \quad x=-1 \\ -1 & , \quad -1<x<0 \\ 0 & , \quad x=0 \\ 1 & , \quad 0<x<1 \\ 0 & , \quad x=1 \\ -1 & , \quad x>1 \end{cases}$

100. The value of Argz + Arg $\bar{z}=0,\ z=x+iy,\ \forall\ x,y\in R$ is (Arg z stands for principal argument of z)

A) 0

B) Non-zero real number

C) Any real number

D) Can't say

Key. D

Sol. Let z = - 2 + 0i, then $\bar{z}=-2-0i$

$\therefore \quad Arg(z) + Arg\left(\bar{z}\right) = 2\pi \neq 0$

If z = 2 + 3i

Arg (2 – 3i) is $\tan^{-1}\left(-\frac{3}{2}\right)$

Arg(2 + 3i) + Arg (2 – 3i) = 0

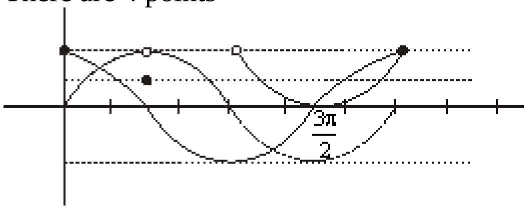
101. If f(x) = maximum $\left(\cos x, \frac{1}{2}, \{\sin x\}\right)$, $0 \le x \le 2\pi$, where { . } represents fractional part function, then number of points at which f(x) is continuous but not differentiable, is

A) 1 B) 2 C) 3 D) 4

Key. D

Sol. See figure

There are 4 points



102. Function $\begin{cases} 2x\tan x - \frac{\pi}{\cos x} & , \quad x \neq \frac{\pi}{2} \\ k & , \quad x = \frac{\pi}{2} \end{cases}$ is continuous at $x = \frac{\pi}{2}$ if k =

A) – 2 B) 2 C) $\frac{1}{2}$ D) no such values of k exists

Key. A

Sol. $\lim_{x\to\frac{\pi}{2}}\left(2x\tan x - \frac{\pi}{\cos x}\right)$

$= \lim_{x\to\frac{\pi}{2}}\left(\frac{2x\sin x - \pi}{\cos x}\right) = \lim_{h\to 0}\left(\frac{2\left(\frac{\pi}{2}+h\right)\cos h - \pi}{-\sin h}\right)$

$= \lim_{x\to 0} - \frac{2h\cos h}{\sin h} = -2 \quad \therefore \quad k = -2$

103. If $f(x) = \begin{cases} x^2\{e^{1/x}\} & x \neq 0 \\ k & x = 0 \end{cases}$ is continuous at x = 0, then

({ } denotes fractional part function)

A) It is differentiable at x = 0 B) k = 1

C) continuous but not differentiable at x = 0 D) continuous everywhere in its domain

Key. A

Sol. $\lim_{x\to 0} f(x) = 0$ { $\because$ $\lim_{x\to 0} x^2 = 0$ and $\{e^{1/x}\}$ is a bounded function}

$\lim_{x\to 0}\frac{f(0+x) - f(0)}{x} = \lim_{x\to 0} x\{e^{1/x}\} = 0$

$\therefore \quad f'(0) = 0$

not continuous at $x = \log_2 e, \log_3 e, ...$ etc.

104. Let $f(x)=a|\sin x|+be^{|x|}+c|x|^3$. If $f(x)$ is differentiable at x = 0 then

A) c = a = 0 and b can be any real number B) a + b =0 and c can be any real number
C) b = c = 0 and a can be any real number D) a = b = c = 0

Key. B

Sol. we have f(x)= $\begin{Bmatrix} -a\sin x+be^{-x}-cx^3 \; if \;\; x<0 \\ a\sin x+be^{x}+cx^3 \; if \;\; x\ge 0 \end{Bmatrix}$

f(x) is obviously continuous at zero.

L.H.D = R.H.D

$(-a\cos x-be^{-x}-2cx^2)_{x=0}=(a\cos x+be^{x}+2cx^2)_{x=0}$

$\Rightarrow -a-b=a+b$

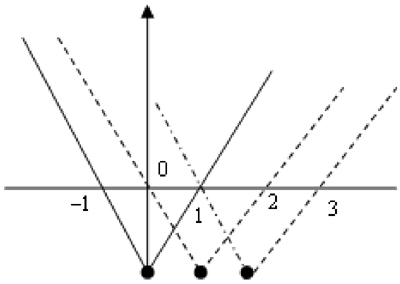
$\Rightarrow$ a+b=0 , and c can be any real number

105. The function $f(x)=\min\{|x|-1,|x-2|-1,|x-1|-1\}$ is not differentiable at

A) 2 points B) 5 points C) 4 points D) 3 points

Key. B

Sol. From the graph , it is clear that function is non-differentiable at 0, ½, 1,3/2, 2.



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