Date: 19/10/2021

Q1. Form the differential equation from the following primitives where constants are arbitrary : $y = cx + 2c^2 + c^3$

Q2. Form the differential equation from the following primitives where constants are arbitrary : $xy = a^2$

Q3. Form the differential equation from the following primitives where constants are arbitrary : $y^2 = 4ax$

Q4. Form the differential equation from the following primitives where constants are arbitrary : $e^x + ce^y = 1$

Q5. Form the differential equation from the following primitives where constants are arbitrary : $y = ax^2 + bx + c$

Q6. Form the differential equation representing the family of curves $y = A \cos (x + B)$, where A and B are parameters.

Q7. Form the differential equation of family of parabolas having vertex at the origin and axis along positive y-axis.

Q8. Form the differential equation of the family of curves represented by $y = c(x - c)^2$, where c is a parameter.

Q9. Form the differential equation representing the family of ellipses having centre at the origin and foci on X-axis.

Q10. Show that the differential equation that represents the family of all parabolas having their axis of symmetry coincident with the axis of x is $yy_2 + y_1^2 = 0$.

Q11. Form the differential equation of the family of circles in the second quadrant and touching the coordinate axes.

Q12. Form the differential equation representing the family of hyperbolas having foci on X-axis and centre at the origin.

Q13. Form the differential equation of the family of curves represented by the equation (a being the parameter):

$(x - a)^2 + 2y^2 = a^2$

Q14. Form the differential equation of the family of ellipses having foci on y-axis and centre at the origin.

Q15. Form the differential equation satisfied by the equation $y = ae^{bx}$, a and b are arbitrary constants.

Q16. Find the differential equation of all circles touching the x-axis at the origin.

Q17. Form the differential equation corresponding to $y^2 = a(b - x)(b + x)$ by eliminating parameters a and b.

Q18. Find the differential equation of all circles touching the y-axis at the origin.

Q19. Find the differential equation of all the circles in the first quadrant which touch the coordinate axes.

Q20. Obtain the differential equation of all circles of radius r.

Q21. The differential equation $x\frac{dy}{dx}\sin\left(\frac{y}{x}\right)+x-y\sin\left(\frac{y}{x}\right)=0$ is homogenous. Find the particular solution of this differential equation, given that $x = 1$ when $y = \frac{\pi}{2}$.

Q22. The differential equation $\left[x\sin^2\left(\frac{y}{x}\right)-y\right]dx + x\,dy = 0$ is homogeneous. Find the particular solution of this differential equation, given that $y = \frac{\pi}{4}$ when $x = 1$

Q23. The differential equation $2ye^{x/y}\,dx + (y - 2x\,e^{x/y})\,dy = 0$ is homogeneous. Find the particular solution of this differential equation, given that $x = 0$ when $y = 1$

Q24. Find the particular solution of the differential equation

$\frac{dx}{dy}+x\cot y=2y+y^2\cot y,(y\neq 0)$, given that $x = 0$ when $y = \frac{\pi}{2}$.

Q25. Find the particular solution of the differential equation $(\tan^{-1}y - x)dy = (1 + y^2)\,dx$, given that when $x = 0$, $y = 0$

Date: 19/10/2021

S1. We have,

$$y = cx + 2c^2 + c^3 \quad \ldots \text{(i)}$$

$$\Rightarrow \quad \frac{dy}{dx} = c$$

Substituting c in Eq. (i) is

$$y = x \cdot \frac{dy}{dx} + 2\left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^3.$$

S2. We have,

$$xy = a^2$$

$$\Rightarrow \quad x\frac{dy}{dx} + y = 0,$$ which is the required differential equation

S3. We have,

$$y^2 = 4ax \quad \ldots (1)$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = 4a$$

$$\Rightarrow \quad a = \frac{1}{2}y\frac{dy}{dx}$$

$\Rightarrow$ put the value of a in equation (1)

$$y^2 = 4\frac{1}{2}y\frac{dy}{dx} \cdot x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{2x}$$ is required differential equation

S4. We have,

$$e^x + ce^y = 1 \quad \ldots (1)$$

$$\Rightarrow \quad e^x + ce^y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad c = -e^{x-y}\frac{dx}{dy}$$

Put the value 'c' in equation 1.

$$e^x - \left(e^{x-y}\frac{dx}{dy}\right)e^y = 1$$

S5. Differentiate y three times to get $\frac{d^3y}{dx^3} = 0$ as the required differential equation.

$$y = a\,x^2 + bx + c \qquad \text{... (i)}$$

Differentiating Eq. (i) three time because Eq. (i) has three arbitraray constant.

$$\frac{dy}{dx} = 2ax + b$$

$$\frac{d^2y}{dx^2} = 2a$$

$$\frac{d^3y}{dx^3} = 0.$$

S6. We have,

$$y = A\cos(x + B), \qquad \text{... (i)}$$

Since the given equation contains two arbitrary constants, we shall differentiate it two times and we shall get a differential equation of second order.

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = -A\sin(x + B) \qquad \text{... (ii)}$$

Differentiating (ii) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -A\cos(x + B).$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -y \qquad \text{[Using (i)]}$$

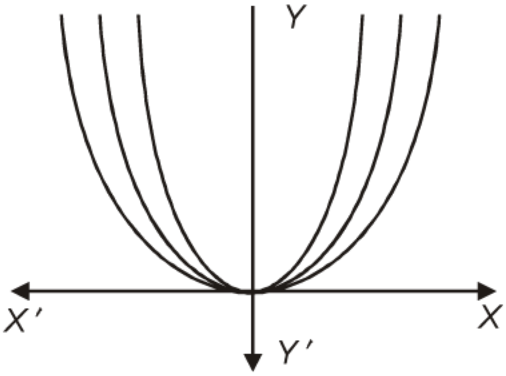
$$\Rightarrow \quad \frac{d^2y}{dx^2} + y = 0$$

which is the required differential equation of the given family of curves.

S7. The equation of the family of parabolas having vertex at the origin and axis along positive y-axis is

$$x^2 = 4ay, \text{ where } a \text{ is a parameter} \qquad \text{... (i)}$$

Differentiating w.r.t. x, we get



$$2x = 4a\frac{dy}{dx} \Rightarrow a = \frac{x}{2\left(\frac{dy}{dx}\right)}$$

Substituting the value of a in (i), we get

$$x^2 = 4 \times \frac{x}{2\left(\frac{dy}{dx}\right)} \times y \Rightarrow x\frac{dy}{dx} = 2y$$

This is the required differential equation.

S8. We have,

$$y = c(x - c)^2 \qquad \text{... (i)}$$

Since the given equation contains only one parameter, we differentiate it once, so that

$$\frac{dy}{dx} = 2c(x - c) \qquad \text{... (ii)}$$

From (i) ÷ (ii), we get

$$\frac{y}{\frac{dy}{dx}} = \frac{\not{c}(x-c)^2}{2\not{c}(x-c)}$$

$$\Rightarrow \quad \frac{y}{\frac{dy}{dx}} = \frac{x-c}{2}$$

$$\Rightarrow \quad x - c = \frac{2y}{\frac{dy}{dx}}$$

$$\Rightarrow \quad c = x - \frac{2y}{\frac{dy}{dx}}$$

Substituting this value of c in (i), we get

$$y = \left(x - \frac{2y}{\frac{dy}{dx}}\right)\left(\not{x} - \not{x} + \frac{2y}{\frac{dy}{dx}}\right)^2$$

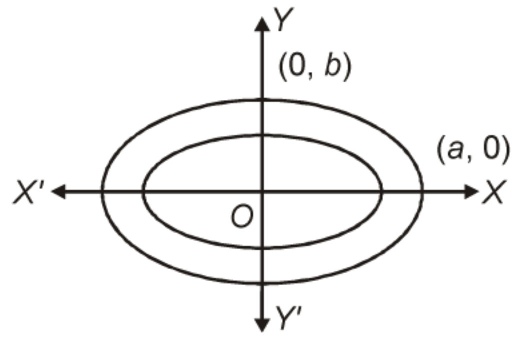
$$\Rightarrow \quad y\left(\frac{dy}{dx}\right)^3 = 4y^2\left(x\frac{dy}{dx} - 2y\right)$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^3 = 4y\left(x\frac{dy}{dx} - 2y\right)$$

Which is the required differential equation.

S9. The equation of the family of ellipses having centre at the origin and foci on X-axis is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1, \quad \text{where } (a > b) \quad \ldots \text{(i)}$$



Differentiating (i) with respect to x, we get

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx} = 0 \quad \ldots \text{(ii)}$$

$$\Rightarrow \quad \frac{x}{a^2}+\frac{y}{b^2}\frac{dy}{dx} = 0 \quad \ldots \text{(ii)}$$

Differentiating (ii) with respect to x, we get

$$\frac{1}{a^2}+\frac{1}{b^2}\left(\frac{dy}{dx}\right)^2+\frac{1}{b^2}y\frac{d^2y}{dx^2} = 0 \quad \ldots \text{(iii)}$$

Multiplying (iii) by x

$$\frac{x}{a^2}+\frac{x}{b^2}\left(\frac{dy}{dx}\right)^2+\frac{xy}{b^2}\frac{d^2y}{dx^2} = 0 \quad \ldots \text{(iv)}$$

Subtracting Eq (iv) from Eq (ii)

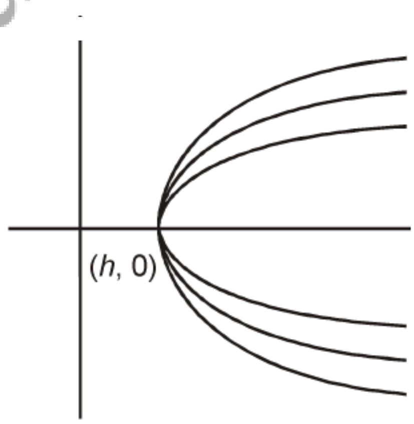
$$\frac{1}{b^2}\left(y\frac{dy}{dx}-x\left(\frac{dy}{dx}\right)^2-xy\frac{d^2y}{dx^2}\right) = 0 \Rightarrow xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx}=0$$

This is the required differential equation.

S10. The equation that represents a family of parabolas having their axis of symmetry coincident with the axis of x is

$$y^2 = 4a(x-h) \quad \ldots \text{(i)}$$

This equation contains two parameters a and h, so we will differentiate it twice to obtain a second order differential equation.



Differentiating (i) w.r.t. x, we get

$$2y\frac{dy}{dx} = 4a$$

$$y\frac{dy}{dx} = 2a \quad \ldots \text{(ii)}$$

Differentiating (ii) w.r.t. x, we get

$$y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2 = 0$$

$$\Rightarrow \quad yy_2 + y_1^2 = 0$$

which is the required differential equation.

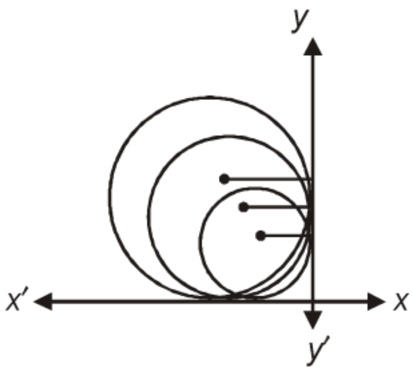
S11. The equation of the family of circles in second quadrant and touching the coordinates axes is

$$(x + a)^2 + (y - a)^2 = a^2,\ a > 0 \text{ or, } x^2 + y^2 + 2ax - 2ay + a^2 = 0 \quad \dots \text{(i)}$$

Differentiating (i) w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} + 2a - 2a\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x + y\frac{dy}{dx} + a\left(1 - \frac{dy}{dx}\right) = 0 \Rightarrow a = \frac{x + py}{p - 1}, \text{ where } p = \frac{dy}{dx}$$



Substituting the value of a in $(x + a)^2 + (y - a)^2 = a^2$, we get

$$\left(x + \frac{x + py}{p - 1}\right)^2 \left(y - \frac{x + py}{p - 1}\right)^2 = \left(\frac{x + py}{p - 1}\right)^2$$

$$\Rightarrow \frac{(px - x + x + py)^2}{(p - 1)^2} + \frac{(py - y - x - py)^2}{(p - 1)^2} = \frac{(x + py)^2}{(p - 1)^2}$$

$$\Rightarrow \quad (x + y)^2 p^2 + (y + x)^2 = (x + py)^2$$

$$\Rightarrow \quad (x + y)^2 (1 + p^2) = (x + py)^2, \text{ where } p = \frac{dy}{dx}$$

This is the required differential equation.

$$(x + y)^2\left(1 + \left(\frac{dy}{dx}\right)^2\right) = \left(x + y\frac{dy}{dx}\right)^2.$$

S12. The equation of the family of hyperbolas having foci on X-axis and centre at the origin is

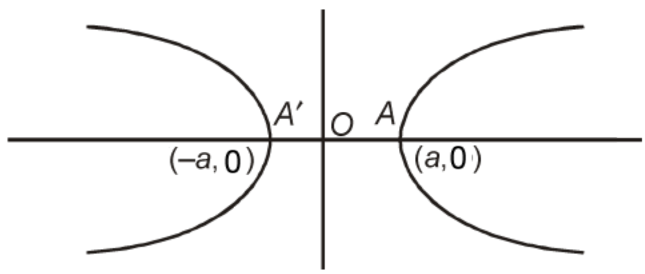
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \dots \text{(i)}$$

This equation contains two parameters a and b, so we will differentiate it twice to obtain a second order differential equation.

Differentiating w.r.t. x, we get

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{x}{a^2} - \frac{y}{b^2}\frac{dy}{dx} = 0 \quad \dots \text{(ii)}$$

$$\Rightarrow \quad \frac{1}{a^2} - \frac{1}{b^2}\left(\frac{dy}{dx}\right)^2 - \frac{y}{b^2}\frac{d^2y}{dx^2} = 0 \quad \dots \text{(iii)}$$



Multiplying (iii) by x and subtracting from (ii), we get

$$\frac{1}{b^2}\left(-y\frac{dy}{dx}+x\left(\frac{dy}{dx}\right)^2+xy\frac{d^2y}{dx^2}\right) = 0 \quad \Rightarrow \quad xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx} = 0$$

This is the required differential equation.

S13. The equation of the one parameter family of curves is

$$(x-a)^2+2y^2 = a^2 \qquad \text{... (i)}$$

Differentiating with respect to x, we get

$$2(x-a)+4y\frac{dy}{dx} = 0$$

$$\Rightarrow \qquad x-a = -2y\frac{dy}{dx}$$

$$\Rightarrow \qquad a = x+2y\frac{dy}{dx}$$

Substituting the value of a in (i), we get

$$4y^2\left(\frac{dy}{dx}\right)^2+2y^2 = \left(x+2y\frac{dy}{dx}\right)^2 \quad \Rightarrow \quad 2y^2-x^2 = 4xy\frac{dy}{dx}$$

This is the required differential equation.

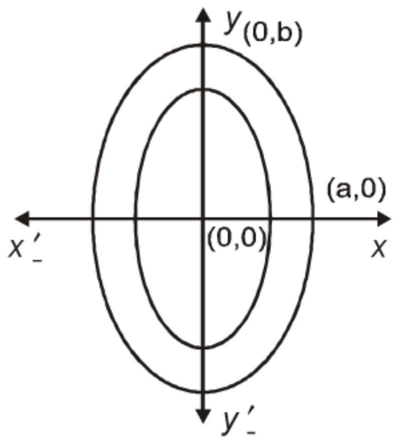
S14. The equation of the family of ellipses having centre at the origin and foci on y-axis is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1, \text{ where } b > a \qquad \text{... (i)}$$

This equation contains two parameters a and b, so we will differentiate it twice to obtain a second order differential equation.

Differentiating w.r.t. x, we get

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{x}{a^2}+\frac{y}{b^2}\frac{dy}{dx} = 0 \qquad \text{... (ii)}$$



Differentiating (ii) w.r.t. x, we get

$$\frac{1}{a^2}+\frac{1}{b^2}\left(\frac{dy}{dx}\right)^2+\frac{y}{b^2}\frac{d^2y}{dx^2} = 0 \qquad \text{... (iii)}$$

Multiplying throughout by x, we get

$$\frac{x}{a^2}+\frac{x}{b^2}\left(\frac{dy}{dx}\right)^2+\frac{xy}{b^2}\frac{d^2y}{dx^2} = 0 \qquad \text{... (iv)}$$

Subtracting Eq. (ii) from Eq. (iv), we get

$$\frac{1}{b^2}\left\{x\left(\frac{dy}{dx}\right)^2 + xy\frac{d^2y}{dx^2} - y\left(\frac{dy}{dx}\right)\right\} = 0 \Rightarrow x\left(\frac{dy}{dx}\right)^2 + xy\frac{d^2y}{dx^2} - y\frac{dy}{dx} = 0$$

This is the required differential equation.

S15. We have,

$$y = ae^{bx} \quad \text{... (i)}$$

Since there are two arbitrary constants in (i), so we shall differentiate it two times and we shall get a differential equation of second order.

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = ae^{bx} . b$$

$$\Rightarrow \quad \frac{dy}{dx} = by \quad \text{[using (i)]} \quad \text{.... (ii)}$$

Differentiating (ii) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = b\frac{dy}{dx} \quad \text{... (iii)}$$

From (ii) and (iii), we obtain

$$\frac{d^2y}{dx^2} = \left(\frac{1}{y}\frac{dy}{dx}\right)\frac{dy}{dx} \quad \left[\text{From } (ii), b = \frac{1}{y}\frac{dy}{dx}\right]$$

$$\Rightarrow \quad y\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2,$$ which is the required differential equation.

S16. The equation of the family of circles touching x-axis at the origin is

$$(x - 0)^2 + (y - a)^2 = a^2$$

$$\Rightarrow \quad x^2 + y^2 - 2ay + a^2 = a^2$$

$$x^2 + y^2 - 2ay + a^2 = a^2$$

$$\Rightarrow \quad x^2 + y^2 - 2ay = 0 \quad \text{... (i)}$$

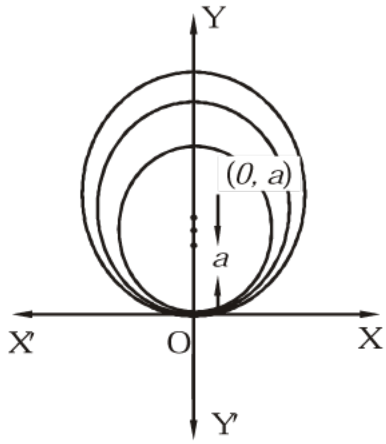
where a is a parameter.

This equation contains only one arbitrary constant, we differentiate it once w.r.t. x, so that

$$2x + 2y\frac{dy}{dx} - 2a\frac{dy}{dx} = 0$$

$$a\frac{dy}{dx} = x + y\ \frac{dy}{dx}$$

$$\Rightarrow \qquad a = \frac{x + y\left(\frac{dy}{dx}\right)}{\frac{dy}{dx}} \qquad \text{... (ii)}$$



Putting the value of *a* from Eq. (ii) in Eq. (i), we get

$$x^2 + y^2 = 2y\left(\frac{x + y\left(\frac{dy}{dx}\right)}{\frac{dy}{dx}}\right)^2 = \left(x^2 + y^2\right)\frac{dy}{dx} = 2xy + 2y^2\ \frac{dy}{dx}$$

$$\Rightarrow \qquad (x^2 - y^2)\frac{dy}{dx} = 2xy$$

This is the required differential equation.

S17. We have,

$$y^2 = a(b^2 - x^2) \qquad \text{... (i)}$$

Since there are two arbitrary constants in the given equation, so we shall differentiate it two times and we shall get a differential equation of second order.

Differentiating (i) w.r.t. x, we get

$$2y\frac{dy}{dx} = -2ax$$

$$\Rightarrow \qquad y\frac{dy}{dx} = -ax \qquad \text{... (ii)}$$

Differentiating (ii) w.r.t. x, we get

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -a \qquad \text{...(iii)}$$

From (ii) and (iii), we get

$$x\left\{y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right\} = y\frac{dy}{dx}$$ [Putting the value of a from Eq. (iii) to Eq. (ii)]

This is the required differential equation.

S18. The equation of the family of circles touching y-axis at the origin is

$$(x - a)^2 + (y - 0)^2 = a^2$$

$$x^2 + a^2 - 2ax + y^2 = a^2$$

$$\Rightarrow \qquad x^2 + y^2 - 2ax = 0 \qquad \text{... (i)}$$

where a is a parameter.

This equation contains only one arbitrary constant, we differentiate it only once w.r.t. x, so, that

$$2x + 2y\frac{dy}{dx} - 2a = 0$$

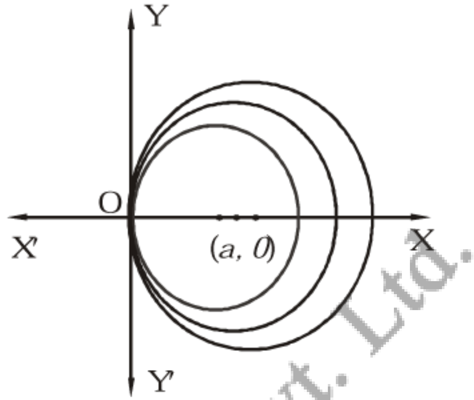
$$\Rightarrow \quad a = x + y\frac{dy}{dx} \quad \text{... (ii)}$$

Putting the value of a from Eq. (ii) in Eq. (i), we get

$$x^2 + y^2 - 2x\left(x + y\frac{dy}{dx}\right) = 0$$

$$\Rightarrow x^2 + y^2 - 2x\left(x + y\frac{dy}{dx}\right) = 0$$

$$\Rightarrow \quad y^2 - x^2 = 2xy\frac{dy}{dx}$$



This is the required differential equation.

S19. The equation of circles in the first quadrant which touch the coordinate axes is

$$(x - a)^2 + (y - a)^2 = a^2 \quad \text{...(i)}$$

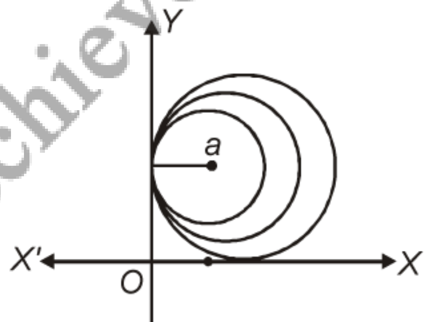
This equation contains one arbitrary constant, so we shall differentiate it once only and we shall get a differential equation of first order.

Differentiating (i) w.r.t. x, we get

$$2(x - a) + 2(y - a)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x - a + (y - a)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad -a\left(1 + \frac{dy}{dx}\right) = -\left(x + y\frac{dy}{dx}\right)$$



$$\Rightarrow \quad a = \frac{x + y\frac{dy}{dx}}{1 + \frac{dy}{dx}}$$

$$\Rightarrow \quad a = \frac{x + py}{1 + p}, \text{ where } p = \frac{dy}{dx}$$

Substituting the value of a in (i), we get

$$\left(x-\frac{x+py}{1+p}\right)^2+\left(y-\frac{x+py}{1+p}\right)^2 = \left(\frac{x+py}{1+p}\right)^2$$

$\Rightarrow$ $$\frac{(x+px-x-py)^2}{(1+p)^2}+\frac{(y+py-x-py)^2}{(1+p)^2} = \frac{(x+py)^2}{(1+p)^2}$$

$\Rightarrow$ $$(xp-py)^2+(y-x)^2 = (x+py)^2$$

$\Rightarrow$ $$(x-y)^2p^2+(x-y)^2 = (x+py)^2$$

$\Rightarrow$ $$(x-y)^2(p^2+1) = (x+py)^2$$

$\Rightarrow$ $$(x-y)^2\left(1+\left(\frac{dy}{dx}\right)^2\right) = \left(x+y\frac{dy}{dx}\right)^2$$

This is the required differential equation.

S20. The equation of the family of circles of radius r is

where a and b are parameters. ... (i)

$$(x-a)^2+(y-b)^2 = r^2$$

Since equation (i) contains two arbitrary constants, we differentiate it two times w.r.t. x and the differential equation will be of second order.

Differentiating (i) w.r.t. x, we get

$$2(x-a)+2(y-b)\frac{dy}{dx} = 0$$

$\Rightarrow$ $$(x-a)+(y-b)\frac{dy}{dx} = 0 \quad \text{... (ii)}$$

Differentiating (ii) w.r.t. x, we get

$$1+(y-b)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2 = 0$$

$$(y-b)\frac{d^2y}{dx^2} = -\left[1+\left(\frac{dy}{dx}\right)^2\right]$$

$$y-b = -\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}$$

Putting the value of $(y - b)$ in (ii), we obtain

$$x - a = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}\frac{dy}{dx}}{\frac{d^2y}{dx^2}}$$

Substituting the values of $(x - a)$ and $(y - b)$ in (i), we get

$$\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^2\left(\frac{dy}{dx}\right)^2}{\left(\frac{d^2y}{dx^2}\right)^2} + \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^2}{\left(\frac{d^2y}{dx^2}\right)^2} = r^2 = \left[1+\left(\frac{dy}{dx}\right)^2\right]^2\left[1+\left(\frac{dy}{dx}\right)^2\right] = r^2\left(\frac{d^2y}{dx^2}\right)^2$$

$$\Rightarrow \qquad \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^3 = r^2\left(\frac{d^2y}{dx^2}\right)^2$$

This is the required differential equation.

S21. Given differential equation is

$$x\frac{dy}{dx}\sin\left(\frac{y}{x}\right) = y\sin\left(\frac{y}{x}\right) - x$$

or $$\frac{dy}{dx} = \frac{y}{x} - \frac{1}{\sin\frac{y}{x}}$$

Which is a homogenous diffrential Equation

Let $$y = vx$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\Rightarrow \qquad v + x\frac{dv}{dx} = v - \frac{1}{\sin v}$$

$$\Rightarrow \qquad x\frac{dv}{dx} = -\frac{1}{\sin v}$$

$$\Rightarrow \qquad \sin v\,dv = -\frac{dx}{x}$$

Integrating on both sides, we get

$$\Rightarrow \qquad \int \sin v\,dv = \int -\frac{dx}{x}$$

$\Rightarrow \quad -\cos v = -\ln|x| + c$

$\Rightarrow \quad -\cos \dfrac{y}{x} = -\ln|x| + c \qquad \left[\because v = \dfrac{y}{x}\right]$

Given that, $\quad x = 1,\ y = \dfrac{\pi}{2}$

$\Rightarrow \quad -0 = -0 + c \quad \Rightarrow \quad c = 0$

$\therefore \quad \cos \dfrac{y}{x} = \ln|x|$

$\Rightarrow \quad x = e^{\cos(y/x)}$

S22. Given differential equation is

$$\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx + x\,dy = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y - x\sin^2\left(\frac{y}{x}\right)}{x}$$

which is a homogeneous differential equation

Put $\quad y = vx$

$\therefore \quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{vx - x\sin^2\left(\frac{vx}{x}\right)}{x}$$

$\Rightarrow \quad v + x\dfrac{dv}{dx} = v - \sin^2(v)$

$\Rightarrow \quad x\dfrac{dv}{dx} = -\sin^2(v)$

$\Rightarrow \quad \text{cosec}^2\, v\, dv = -\dfrac{dx}{x}$

Integrating on both sides, we get

$$\int \text{cosec}^2 v\, dv + \int \frac{dx}{x} = 0$$

$\Rightarrow \quad -\cot v + \log|x| = c$

$$\Rightarrow \quad -\cot\left(\frac{y}{x}\right) + \log|x| = c \qquad \left[\because v = \frac{y}{x}\right] \ldots \text{(i)}$$

Given that,

$$y = \frac{\pi}{4} \quad \text{when} \quad x = 1$$

$$-\cot\frac{\pi}{4} + \log|1| = c$$

$$\Rightarrow \quad c = -1 + 0$$

$$\Rightarrow \quad c = -1$$

$\therefore$ Required particular solution is

$$-\cot\left(\frac{y}{x}\right) + \log|x| = -1$$

$$\Rightarrow \quad 1 + \log|x| - \cot\left(\frac{y}{x}\right) = 0$$

S23. If each terms of a differential has same degree, then it is a homogeneous differential equation.

The given differential equation can be written as

$$\frac{dx}{dy} = \frac{2xe^{x/y} - y}{2ye^{\frac{x}{y}}} \qquad \ldots \text{(i)}$$

To solve it, we make the substitution

$$x = vy \qquad \ldots \text{(ii)}$$

Differentiating Eq. (ii) w.r.t. y., we get

$$\frac{dx}{dy} = v + y\frac{dv}{dy}$$

Substituting the value of x and $\frac{dx}{dy}$ in Eq. (i), we get

$$v + y\frac{dv}{dy} = \frac{2ve^{v} - 1}{2e^{v}}$$

$$\Rightarrow \quad y\frac{dv}{dy} = \frac{2ve^{v} - 1}{2e^{v}} - v$$

$$\Rightarrow \quad y\frac{dv}{dy} = -\frac{1}{2e^{v}}$$

$\Rightarrow \qquad 2e^{v}dv = \frac{-dy}{y}$

Integrating both, sides, we get

$$\int 2e^{v}dv = -\int \frac{dy}{y}$$

$\Rightarrow \qquad 2e^{v} = -\log|y| + C$

and replacing v by $\frac{x}{y}$, we get

$$2e^{\frac{x}{y}} + \log|y| = C \qquad \ldots \text{(iii)}$$

Substituting $x = 0$ and $y = 1$ in Eq. (iii), we get

$$2e^{0} + \log|1| = C \Rightarrow C = 2$$

Substituting the value of C in Eq. (iii), we get

$$2e^{\frac{x}{y}} + \log|y| = 2$$

which is particular solution of the given differential equation.

S24. Given differential equation is

$$\frac{dx}{dy} + x\cot y = 2y + y^2 \cot y, \ (y \neq 0)$$

which is a linear differential equation.

Comparing with $\frac{dx}{dy} + Px = Q$

Here, $\qquad P = \cot y$ and $Q = 2y + y^2 \cot y$

Now, $\qquad \text{I.F.} = e^{\int P dy} = e^{\int \cot y \, dy}$

$$= e^{\log \sin y} = \sin y$$

$\therefore$ Complete solution is

$$x \cdot (\text{I.F.}) = \int Q \cdot (\text{IF}) dy + c$$

$$x \cdot \sin y = \int (2y + y^2 \cot y) \cdot \sin y \, dy + c$$

$\Rightarrow \qquad x \sin y = 2\int y \sin y \, dy + \int \underset{\text{I}}{y^2} \underset{\text{II}}{\cos y} \, dy + c$

$$= 2\int y \sin y\, dy + y^2 \int \cos y\, dy - \int \left[\left(\frac{d}{dy} y^2\right)\int \cos y\, dy\right] dy + c$$

$$\left[\text{Using integration by part in second integral } i.e. \int \underset{\text{I}}{u}\underset{\text{II}}{v}\, dx = u\int v\, dx - \int \left\{\left(\frac{d}{dx}u\int vdx\right)\right\} dx\right]$$

$$= 2\int y \sin y\, dy + y^2 \sin y - 2\int y \sin y\, dy + c$$

$$= y^2 \sin y + c$$

$$\Rightarrow \quad x \sin y = y^2 \sin y + c \qquad \ldots \text{(i)}$$

Given that, $x = 0$ when $y = \frac{\pi}{2}$

$$\Rightarrow \quad 0 = \left(\frac{\pi}{2}\right)^2 \sin\frac{\pi}{2} + c \qquad \Rightarrow c = -\frac{\pi^2}{4}$$

Put this value in Eq. (i), we get

$$x \sin y = y^2 \sin y - \frac{\pi^2}{4}$$

$$\Rightarrow \quad x = y^2 - \frac{\pi^2}{4} . \operatorname{cosec} y$$

which is required particular solution of given differential equation.

S25. Given differential equation is,

$$(\tan^{-1} y - x)\, dy = (1 + y^2)\, dx$$

$$\Rightarrow \quad \frac{(\tan^{-1} y - x)}{1+y^2} = \frac{dx}{dy}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{-x}{1+y^2} + \frac{\tan^{-1} y}{1+y^2}$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{1}{1+y^2}\cdot x = \frac{\tan^{-1} y}{1+y^2}$$

which is a linear differential equation

Comparing with $\frac{dx}{dy} + Px = Q$, we get

$$P = \frac{1}{1+y^2} \quad \text{and} \quad Q = \frac{\tan^{-1} y}{1+y^2}$$

Now, $\quad \text{IF} = e^{\int P dy} = e^{\int \frac{dy}{1+y^2}} = e^{\tan^{-1} y}$

Complete solution is

$$x \cdot (\text{IF}) = \int Q \cdot (\text{IF}) dy + c$$

$\Rightarrow$ $$xe^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2} \times e^{\tan^{-1} y} + c$$

Let $$t = \tan^{-1} y, \ dt = \frac{1}{1+y^2} dy$$

$\therefore$ $$x \cdot e^{\tan^{-1} y} = \int t \cdot e^t dt + c$$

$\Rightarrow$ $$x \cdot e^{\tan^{-1} y} = t \cdot e^t - \int 1 \cdot e^t dt + c$$ [Using integration by parts]

$\Rightarrow$ $$x \cdot e^{\tan^{-1} y} = t \cdot e^t - e^t + c$$

$\Rightarrow$ $$x \cdot e^{\tan^{-1} y} = (\tan^{-1} y - 1) e^{\tan^{-1} y} + c \quad \text{... (i)}$$

Also, given that, when $x = 0$, $y = 0$

$$0 = (\tan^{-1} 0 - 1) e^{\tan^{-1} 0} + c$$

$\Rightarrow$ $$0 = (0 - 1) e^0 + c$$

$\Rightarrow$ $$0 = (0 - 1) \cdot 1 + c \Rightarrow c = 1$$

From Eq. (i), we get

$\therefore$ $$x \cdot e^{\tan^{-1} y} = (\tan^{-1} y - 1) \cdot e^{\tan^{-1} y} + 1$$

$$x = \tan^{-1} y - 1 + e^{-\tan^{-1} y}$$

which is the required particular solution cf the differential equation.

Date: 19/10/2021

Q1. Determine the order and degree of the following differential equation.

$$y + \frac{dy}{dx} = \frac{1}{4}\int y\, dx$$

Q2. Determine the order and degree of the following differential equation.

$$y = \frac{dy}{dx} + \frac{c}{\frac{dy}{dx}}$$

Q3. Determine the order and degree of the following differential equation.

$$\frac{d^2y}{dx^2} = 1 + \sqrt{\frac{dy}{dx}}$$

Q4. Determine the order and degree of the following differential equation.

$$\frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = K$$

Q5. Write the degree of the following differential equation

$$x^3\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^4 = 0.$$

Q6. Write the degree of the following differential equation

$$\left(\frac{dy}{dx}\right)^4 + 3x\frac{d^2y}{dx^2} = 0.$$

Q7. Determine the order and degree of the following differential equation.

$$\left(\frac{d^2y}{dx^2}\right)^2 + \cos\left(\frac{dy}{dx}\right) = 0.$$

Q8. Determine the order and degree of the following differential equation.

$$\frac{d^4y}{dx^4} + \sin\left(\frac{d^3y}{dx^3}\right) = 0.$$

Q9. Determine the order and degree of the following differential equation.

$$\frac{d^5y}{dx^5} + e^{\frac{dy}{dx}} + y^2 = 0.$$

Date: 19/10/2021

S1. We have,

$$y + \frac{dy}{dx} = \frac{1}{4}\int y\,dx$$

$$\Rightarrow \qquad \frac{dy}{dx} + \frac{d^2y}{dx^2} = \frac{1}{4}y$$

The highest order differential coefficient in this equation is $\frac{d^2y}{dx^2}$ and its power is 1. Therefore, the differential equation of order 2 and degree 1.

S2. The given differential equation when written as a polynomial in $\frac{dy}{dx}$ is

$$\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} + c = 0$$

The highest order differential coefficient in this equation is $\frac{dy}{dx}$ and its power is 2. Therefore, the differential equation of order 1 and degree 2.

S3. The given differential equation when written as a polynomial in derivatives becomes

$$\left(\frac{d^2y}{dx^2} - 1\right)^2 = \frac{dy}{dx} \Rightarrow \left(\frac{d^2y}{dx^2}\right)^2 - 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + 1 = 0$$

The highest order differential coefficient in this equation is $\frac{d^2y}{dx^2}$ and its power is 2. Therefore, the differential equation is of second order and second degree.

S4. The given differential equation when written as a polynomial in derivatives becomes

$$K^2\left(\frac{d^2y}{dx^2}\right)^2 = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^3$$

The highest order differential coefficient in this equation is $\frac{d^2y}{dx^2}$ and it's power is 2. Therefore, the differential equation is of second order and second degree.

S5. We know that the degree of the differential equation is the degree of the highest order derivative, when differential coefficients are made free from radicals and fractions sign

Given differential equation is

$$x^3\left(\frac{d^2y}{dx^2}\right)^2+\left(\frac{dy}{dx}\right)^4 = 0$$

Here, we see that differential coefficient is free from radical sign.

$\therefore$ Degree = 2

S6. We know that, the degree of the differential equation is the degree of the highest order derivative, when differential coefficients are made free from radicals and fractions sign

Given differential equation is

$$\left(\frac{dy}{dx}\right)^4+3x\left(\frac{d^2y}{dx^2}\right) = 0$$

Degree = 1

Here, we see that differential coefficient is free from radical sign.

S7. The order of the highest order derivative present in the given differential equation is 2. So, its order is 2. The given differential equation is not expressible as a polynomial in differential coefficients. So, its degree is not defined.

S8. The highest order derivative present in the given differential equation is 4, so the order of the given differential equation is 4. As it is not expressible as a polynomial in differential coefficients. So, its degree is not defined.

S9. The highest order differential coefficient present in the differential equation is $\frac{d^5y}{dx^5}$. So its is order 5.

As the LHS of the differential equation is not expressible as a polynomial in $\frac{dy}{dx}$. So, its degree is not defined.

Date: 19/10/2021

Q1. In a bank principal increases at the rate of 5% per year. An amount of Rs. 1000 is deposited with this bank, how much will it worth after 10 years ($e^{0.5} = 1.648$).

Q2. In a bank principal increases at the rate of r% per year. Find the value of r if Rs. 100 double itself in 10 years ($\log_e 2 = 0.6931$).

Q3. Solve : $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$

Q4. Find the particular solution of the differential equation satisfying the given condition $\frac{dy}{dx} = y \tan x$, given that $y = 1$ when $x = 0$.

Q5. Solve the following differential equation
$e^x \tan y\, dx + (1 - e^x) \sec^2 y\, dy = 0$

Q6. Solve the following differential equation
$(1 + y^2)(1 + \log x)\, dx + x dy = 0$

Q7. Find the particular solution of the differential equation
$(1 + e^{2x}) dy + (1 + y^2) e^x dx = 0$, given that $y = 1$ when $x = 0$.

Q8. Solve the following differential equation $\frac{dy}{dx} = 1 + x^2 + y^2 + x^2y^2$, given that $y = 1$ when $x = 0$

Q9. Solve :

$$e^x \sqrt{1 - y^2}\, dx + \frac{y}{x}\, dy = 0$$

Q10. Solve the differential equation $x(1 + y^2)\, dx - y(1 + x^2)\, dy = 0$, given that $y = 0$, when $x = 1$.

Q11. Solve $(x^2 - yx^2)\, dy + (y^2 + x^2y^2)\, dx = 0$

Q12. Solve $3e^x \tan y\, dx + (1 - e^x) \sec^2 y dy = 0$

Q13. Solve :

$$\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$$

Q14. Solve :

$$\frac{dy}{dx} = \frac{1 + y^2}{1 + x^2}$$

Q15. Find the equation of the curve passing through the point (0, – 2) given that at any point (x, y) on the curve the product of the slope of its tangent and y coordinate of the point is equal to the x-coordinate of the point.

Q16. At any point (x, y) of a curve, the slope of the tangent is twice the slope of the line segment joining the point of contact to the point (– 4, – 3). Find the equation of the curve given that it passes through (– 2, 1).

Q17. From the following differential equation, find the equation of curve passing through the point (1, – 1)

$$xy\frac{dy}{dx} = (x+2)(y+2).$$

Q18. Solve :

$$\frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \qquad x \neq 0.$$

Q19. Solve : $\frac{dy}{dx} + \frac{y}{x} = \cos x + \frac{\sin x}{x}, x > 0.$

Q20. Solve : $\frac{dy}{dx} + y \tan x = 2x + x^2 \tan x$

Q21. Solve the following initial value problem :

$(x + y + 1)^2 \, dy = dx, \quad y(-1) = 0$

Q22. Solve the following differential equation:

$$(x+y)^2 \frac{dy}{dx} = a^2.$$

Q23. Solve the following differential equation:

$$\frac{dy}{dx} = (4x + y + 1)^2$$

Q24. In a culture the bacteria count is 100000. The number is increased by 10% in 2 hours. In how many hours will the count reach 200000, if the rate of growth of bacteria is proportional to the number present.

Q25. Solve the following differential equation:

$$\frac{dy}{dx} = \cos(x+y)$$

Q26. Solve the following differential equation $\frac{dy}{dx}$ + y cot x = 4x cosec x, given that y = 0 when $x = \frac{\pi}{2}$.

Q27. Solve : $y\,dx + (x - y^3)dy = 0$

Q28. Solve : $y\,dx - (x + 2y^2)dy = 0$

Q29. Solve the following differential equation

$$\frac{dy}{dx} + 2y \tan x = \sin x$$

Q30. Solve the following differential equation

$(1 + x^2)\,dy + 2xy\,dx = \cot x\,dx, \ (x \neq 0)$

Q31. Solve the following differential equation $x\,dy - (y + 2x^2)\,dx = 0$.

Q32. Solve the following differential equation $(y + 3x^2)\frac{dx}{dy} = x$.

Q33. Solve the following differential equation

$$(x^2+1)\frac{dy}{dx}+2xy = \sqrt{x^2+4}$$

Q34. Solve : $(x^3 - 3xy^2)\,dx = (y^3 - 3x^2y)\,dy$.

Q35. Solve the following differential equation $x\,dy + (y - x^3)\,dx = 0$.

Q36. Solve the following differential equation

$$(x^2 - 1)\frac{dy}{dx} + 2xy = \frac{1}{x^2-1};\ |x| \neq 1$$

Q37. Solve the following differential equation

$$\cos^2 x\,\frac{dy}{dx} + y = \tan x$$

Q38. Solve the following differential equation

$$(x \log x)\frac{dy}{dx} + y = 2\log x$$

Q39. Solve the following differential equation

$$(1 + x^2)\frac{dy}{dx} + y = \tan^{-1} x$$

Q40. Solve the following differential equation

$$\frac{dy}{dx} + y = \cos x - \sin x$$

Q41. Solve the following initial value problem:

$(x^2 + 1)y' - 2xy = (x^4 + 2x^2 + 1)\cos x,\ y(0) = 0.$

Q42. Solve the following differential equation

$$\frac{dy}{dx} + y\sec x = \tan x$$

Q43. Find the general solution of the following differential equation

$$(x \log x)\cdot\frac{dy}{dx} + y = \frac{2}{x}\log x$$

Q44. Solve the following differential equation

$$(1 + x^2)\frac{dy}{dx} + 2xy = \frac{1}{1+x^2},\ \text{given } y = 0 \text{ when } x = 1.$$

Q45. Solve : $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$

Q46. Solve the following differential equation

$$x^2\frac{dy}{dx} = y^2 + 2xy$$

Q47. Solve the following differential equation

$$x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right)$$

Q48. Solve the following differential equation

$$xy\log\left(\frac{y}{x}\right)dx+\left[y^2-x^2\log\left(\frac{y}{x}\right)\right]dy=0$$

Q49. Solve the following differential equation

$$(x-y)\ \frac{dy}{dx} = x + 2y.$$

Q50. Solve the following differential equation

$$\left(x\cos\frac{y}{x}+y\sin\frac{y}{x}\right)y-\left(y\sin\frac{y}{x}-x\cos\frac{y}{x}\right)\cdot x\,\frac{dy}{dx}=0$$

Q51. Solve the following differential equation

$$ydx + x\log\left(\frac{y}{x}\right)\ dy - 2x\,dy = 0.$$

Q52. Solve the following differential equation $\frac{dy}{dx}-\frac{y}{x}+\operatorname{cosec}\left(\frac{y}{x}\right)=0.$

Q53. Solve the following differential equation

$$\left[x\sin^2\left(\frac{y}{x}\right)-y\right]dx + x\,dy = 0$$

Q54. Solve the following differential equation $xdy - ydx = \sqrt{x^2+y^2}dx.$

Q55. Solve each of the following initial value problem:

$$(xe^{y/x}+y)dx = x\,dy,\quad y(1)=1$$

Q56. Solve the following differential equation $2x^2\frac{dy}{dx}-2xy+y^2=0.$

Q57. The volume of a spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds it is 6 units. Find the radius of the balloon after t seconds.

Q58. Solve the following differential equation $x(x^2-1)\frac{dy}{dx}=1$; given that $y=0$ when $x=2$.

Q59. Solve the following differential equation

$$(x^3 + x^2 + x + 1)\ \frac{dy}{dx} = 2x^2 + x$$

Q60. Solve the following differential equation

$$\frac{dy}{dx} = \frac{x(2y-x)}{x(2y+x)}$$

if $y = 1$ when $x = 1$

Q61. Solve the following differential equation

$$\sqrt{1+x^2+y^2+x^2y^2}+xy\frac{dy}{dx}=0$$

Q62. Solve the following differential equation $(x^2-y^2)\,dx + 2xy\,dy = 0$ given that $y = 1$ when $x = 1$.

Q63. Solve the following differential equation

$$x \frac{dy}{dx} + y - x + xy \cot x = 0, \; x \neq 0$$

Q64. Solve the following differential equation

$$\sec x \frac{dy}{dx} - y = \sin x$$

Q65. Find the particular solution of the differential equation satisfying the given condition

$x^2 dy + (xy + y^2)\, dx = 0;\; y(1) = 1$

Q66. Solve the following initial value problem:

$$x \frac{dy}{dx} \sin\left(\frac{y}{x}\right) + x - y \sin\left(\frac{y}{x}\right) = 0,\; y(1) = \frac{\pi}{2}$$

Q67. Solve the following initial value problem:

$$xe^{y/x} - y \sin\left(\frac{y}{x}\right) + x \frac{dy}{dx} \sin\left(\frac{y}{x}\right) = 0,\; y(1) = 0$$

Q68. Find the particular solution of the differential equation

$\frac{dx}{dy} + x \cot y = 2y + y^2 \cot y, (y \neq 0)$, given that $x = 0$ when $y = \frac{\pi}{2}$.

Q69. Find the particular solution of the differential equation

$(\tan^{-1} y - x)\, dy = (1 + y^2)\, dx$, given that when $x = 0$, $y = 0$

Q70. Solve the following initial value problem:

$$\sqrt{1 - y^2}\, dx = (\sin^{-1} y - x)\, dy,\; y(0) = 0$$

Q71. Solve the following initial value problem:

$ye^y dx = (y^3 + 2x\, e^y) dy, \quad y(0) = 1.$

Date: 19/10/2021

S1. We have,

$$\frac{dP}{dt} = \frac{5P}{100} \Rightarrow \int\frac{dP}{P} = \int\frac{1}{20}dt \Rightarrow \log P = \frac{t}{20} + \log C \quad \text{... (i)}$$

Initially *i.e.*, at $t = 0$, $P = 1000$

$\therefore \quad \log 1000 = \log C$

Substituting the value of log C in (i), we get

$$\log P = \frac{t}{20} + \log 1000$$

Putting $t = 10$, we get

$$\log\frac{P}{1000} = 0.5 \Rightarrow \frac{P}{1000} = e^{0.5} \Rightarrow P = 1000 \times 1.648 = 1648 \text{ Rs.}$$

S2. Let P be the principal. It is given that

$$\frac{dP}{dt} = \frac{r}{100}P \Rightarrow \int\frac{dP}{P} = \int\frac{r}{100}dt \Rightarrow \log P = \frac{rt}{100} + C$$

Initially *i.e.*, at $t = 0$, let $P = P_0$. Then,

$$\log P_0 = C$$

$$\therefore \quad \log P = \frac{rt}{100} + \log P_0 \Rightarrow \log\frac{P}{P_0} = \frac{rt}{100}$$

Substituting $P_0 = 100$, $P = 2P_0 = 200$ and $t = 10$, we get

$$\log 2 = \frac{r}{10} \Rightarrow r = 10 \log 2 = 10 \times 0.6931 = 6.931$$

S3. We have,

$\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$

$$\Rightarrow \quad \sec^2 x \tan y\, dx = -\sec^2 y \tan x\, dy$$

$$\Rightarrow \quad \frac{\sec^2 x}{\tan x}dx = -\frac{\sec^2 y}{\tan y}dy$$

$$\Rightarrow \quad \int\frac{\sec^2 x}{\tan x}dx = -\int\frac{\sec^2 y}{\tan y}dy \quad \text{[Integrating both sides]}$$

$\Rightarrow \qquad \log|\tan x| = -\log|\tan y| + \log C$

$\Rightarrow \qquad \log|(\tan x)(\tan y)| = \log C$

$\Rightarrow \qquad |\tan x \tan y| = C$

It is defined for $x \in R - \left\{(2n+1)\frac{\pi}{2} : n \in Z\right\}$

Hence, $|\tan x \tan y| = C$, where $x \in R - \left\{(2n+1)\frac{\pi}{2} : n \in Z\right\}$ is the solution of the given differential equation.

S4. Given differential equation is

$$\frac{dy}{dx} = y \tan x$$

Above equation is of variable separable type and can be written as

$$\frac{dy}{y} = \tan x \, dx$$

Integrating both sides, we get

$$\int \frac{dy}{y} = \int \tan x \, dx$$

$\Rightarrow \qquad \log|y| = \log|\sec x| + c \qquad \ldots$ (i)

$$\left[\because \int \frac{1}{v} dv = \log|v| \text{ and } \int \tan x \, dx = \log|\sec x|\right]$$

Now, putting $x = 0$ and $y = 1$, we get

$$\log 1 = \log(\sec 0^\circ) + c$$

$\Rightarrow \qquad 0 = \log 1 + c \qquad [\because \sec 0^\circ = 1]$

or $\qquad c = 0 \qquad [\because \log 1 = 0]$

Putting $c = 0$ in Eq. (i), we get the required particular solution as

$$\log|y| = \log|\sec x|$$

or $\qquad y = \sec x.$

S5. Given differential equation is

$$e^x \tan y \, dx + (1 - e^x) \sec^2 y \, dy = 0$$

Above equation may be written as

$$e^x \tan y \, dx = (e^x - 1) \sec^2 y \, dy$$

$\Rightarrow \qquad \frac{e^x}{e^x - 1} dx = \frac{\sec^2 y}{\tan y} dy$

Integrating both sides, we get

$$\int \frac{e^x}{e^x - 1} dx = \int \frac{\sec^2 y}{\tan y} dy$$

Put $e^x - 1 = t$ and $\tan y = z$

$\Rightarrow \quad e^x\, dx = dt$ and $\sec^2 y\, dy = dz$

$\therefore \quad \int \frac{dt}{t} = \int \frac{dz}{z}$

$\Rightarrow \quad \log |t| = \log |z| + \log c \qquad \left[\because \int \frac{1}{x} dx = \log |x|\right]$

$\Rightarrow \quad \log |e^x - 1| = \log |c \,.\, \tan y| \qquad [\because \log m + \log n = \log mn]$

$\Rightarrow \quad e^x - 1 = c \tan y \quad \Rightarrow \quad \tan y = \frac{e^x - 1}{c}$

$\Rightarrow \quad y = \tan^{-1}\left(\frac{e^x - 1}{c}\right)$ is the required solution.

S6. Given differential equation is

$$(1 + y^2)(1 + \log x)\, dx + x\, dy = 0$$

Above equation can be written as

$$(1 + y^2)(1 + \log x)\, dx = -x\, dy$$

$\Rightarrow \quad \frac{1 + \log x}{x} dx = \frac{-dy}{1 + y^2}$

Integrating both sides, we get

$$\int \frac{1 + \log x}{x} dx = -\int \frac{dy}{1 + y^2} \quad \Rightarrow \quad \int \frac{1}{x} dx + \int \frac{\log x}{x} dx = -\int \frac{dy}{1 + y^2}$$

$\Rightarrow \quad \log|x| + \frac{(\log x)^2}{2} + c = -\tan^{-1} y$

$$\left[\because \int \frac{\log x}{x} dx \text{ put } \log x = t \Rightarrow \frac{1}{x} dx = dt \;\therefore\; \int t\, dt = \frac{t^2}{2} + c = \frac{(\log x)^2}{2} + c\right]$$

$\Rightarrow \quad \tan^{-1} y = -\left[\log |x| + \frac{(\log x)^2}{2} + c\right]$

$\Rightarrow \quad y = \tan\left[-\log |x| - \frac{(\log x)^2}{2} - c\right]$

is the required solution.

S7. Given differential equation is

$$(1 + e^{2x})\, dy + (1 + y^2)\, e^x dx = 0$$

$$(1 + e^{2x})\, dy = -(1 + y^2)\, e^x\, dx$$

$$\Rightarrow \quad \frac{dy}{1+y^2} = \frac{-e^x}{1+e^{2x}} dx$$

Integrating both sides, we get

$$\int \frac{dy}{1+y^2} = -\int \frac{e^x}{1+e^{2x}} dx$$

Put $e^x = t$

$$\Rightarrow \quad e^x dx = dt$$

Hence $\quad \tan^{-1} y = -\tan^{-1} t + c$

$$\Rightarrow \quad \tan^{-1} y = -\tan^{-1}(e^x) + c \quad \text{... (i)}$$

Now, given that $x = 0$ when $y = 1$. Putting above values in Eq. (i), we get

$$\tan^{-1} 1 = -\tan^{-1}(e^0) + c$$

$$\Rightarrow \quad \tan^{-1}\tan \frac{\pi}{4} = -\tan^{-1}(1) + c$$

$$\Rightarrow \quad \frac{\pi}{4} = -\tan^{-1}\tan \frac{\pi}{4} + c \quad \Rightarrow \frac{\pi}{4} = -\frac{\pi}{4} + c$$

$$\Rightarrow \quad c = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

Putting $c = \frac{\pi}{2}$ in Eq. (i), we get the solution of the given differential equation

$$\tan^{-1} y = -\tan^{-1} e^x + \frac{\pi}{2}$$

$$\Rightarrow \quad y = \tan\left[\frac{\pi}{2} - \tan^{-1}(e^x)\right]$$

$$= \cot[\tan^{-1}(e^x)]$$

$$= \cot\left[\cot^{-1}\left(\frac{1}{e^x}\right)\right] \quad \left[\because \tan^{-1} x = \cot^{-1}\frac{1}{x}\right]$$

$$\Rightarrow \quad y = e^{-x}.$$

S8. Given differential equation is

$$\frac{dy}{dx} = 1 + x^2 + y^2 + x^2y^2$$

$$\Rightarrow \quad \frac{dy}{dx} = (1 + x^2) + y^2(1 + x^2)$$

$\Rightarrow \quad \dfrac{dy}{dx} = (1 + x^2)(1 + y^2)$

Integrating both sides, we get

$$\int \frac{dy}{1+y^2} = \int (1+x^2)dx$$

$\Rightarrow \quad \tan^{-1} y = x + \dfrac{x^3}{3} + c \qquad \text{... (i)}$

Now, put $x = 0$ and $y = 1$, we get

$$\tan^{-1} 1 = c$$

$\Rightarrow \quad \tan^{-1} \tan \pi/4 = c$

$\Rightarrow \quad c = \pi/4$

Put $c = \pi/4$ in Eq. (i)

$\Rightarrow \quad \tan^{-1} y = x + \dfrac{x^3}{3} + \dfrac{\pi}{4}$

Hence, required solution is $y = \tan\left(x + \dfrac{x^3}{3} + \dfrac{\pi}{4}\right)$.

S9. Given that

$\Rightarrow \quad e^x\sqrt{1-y^2}\,dx + \dfrac{y}{x}dy = 0$

$\Rightarrow \quad e^x\sqrt{1-y^2}\,dx = -\dfrac{y}{x}dy$

$\Rightarrow \quad x\,e^x\,dx = -\dfrac{y}{\sqrt{1-y^2}}dy$

$\Rightarrow \quad \int \underset{\text{I}}{x}\,\underset{\text{II}}{e^x}\,dx = -\int \dfrac{y}{\sqrt{1-y^2}}dy \qquad$ [Integrating both sides]

$\Rightarrow \quad xe^x - \int e^x dx = \dfrac{1}{2}\int \dfrac{dt}{\sqrt{t}}$, where $t = 1 - y^2$, $dt = -2ydy$

Left side using by parts and Right side using substitution

$\Rightarrow \quad xe^x - e^x = \dfrac{1}{2}\left(\dfrac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right) + C$

$$\Rightarrow \quad xe^x - e^x = \frac{1}{2}\frac{t^{1/2}}{1/2} + C$$

$$\Rightarrow \quad xe^x - e^x = \sqrt{t} + C$$

$$\Rightarrow \quad xe^x - e^x = \sqrt{1-y^2} + C$$, where $x \in R$ is the required solution.

S10. The given differential equation is

$$x(1 + y^2)\,dx - y(1 + x^2)\,dy = 0$$

$$\Rightarrow \quad x(1 + y^2)dx = y(1 + x^2)dy$$

$$\Rightarrow \quad \frac{x}{1+x^2}dx = \frac{y}{1+y^2}dy$$

$$\Rightarrow \quad \frac{2x}{1+x^2}dx = \frac{2y}{1+y^2}dy$$

Integrating both sides, we get

$$\int\frac{2x}{1+x^2}dx = \int\frac{2y}{1+y^2}dy$$

$$\Rightarrow \quad \log|1 + x^2| = \log|1 + y^2| + \log C$$

$$\Rightarrow \quad \log\left|\frac{1+x^2}{1+y^2}\right| = \log C$$

$$\Rightarrow \quad \frac{1+x^2}{1+y^2} = C$$

$$\Rightarrow \quad (1 + x^2) = (1 + y^2)\,C \qquad \ldots \text{(i)}$$

It is given that when $x = 1$, where $y = 0$. So, putting $x = 1$, and $y = 0$ in (i), we get

$$(1 + 1) = (1 + 0)\,C \quad \Rightarrow \quad C = 2$$

Putting $C = 2$ in (i), we get

$$(1 + x^2) = 2(1 + y^2)$$

S11. The given differential equations is

$$x^2(1 - y)\,dy + y^2(1 + x^2)\,dx = 0$$

$$\Rightarrow \quad x^2(1 - y)\,dy = -y^2(1 + x^2)\,dx$$

$$\Rightarrow \quad \frac{1-y}{y^2}dy = -\left(\frac{1+x^2}{x^2}\right)dx, \quad \text{if} \quad x, y \neq 0$$

$$\Rightarrow \quad \left(\frac{1}{y} - \frac{1}{y^2}\right)dy = \left(\frac{1}{x^2} + 1\right)dx$$

$$\Rightarrow \quad \int\left(\frac{1}{y}-\frac{1}{y^2}\right)dy = \int\left(\frac{1}{x^2}+1\right)dx \quad \text{[Integrating both sides]}$$

$$\Rightarrow \quad \log|y| + \frac{1}{y} = -\frac{1}{x}+x+C$$

Hence, $\log|y|+\frac{1}{y}=-\frac{1}{x}+x+C, x \in R-\{0\}$ gives the general solution of the differential equation.

S12. We are given that

$$3e^x \tan y\, dx + (1-e^x)\sec^2 y\, dy = 0$$

$$\Rightarrow \quad 3e^x \tan y\, dx = -(1-e^x)\sec^2 y\, dy$$

$$\Rightarrow \quad \frac{3e^x}{1-e^x}dx = -\frac{\sec^2 y}{\tan y}dy$$

$$\Rightarrow \quad 3\int\frac{e^x}{1-e^x}dx = -\int\frac{\sec^2 y}{\tan y}dy, \text{ if } x \neq 0, y \neq 0 \quad \text{[Integrating both sides]}$$

$$\Rightarrow \quad 3\int\frac{e^x}{e^x-1}dx = \int\frac{\sec^2 y}{\tan y}dy$$

$$\Rightarrow \quad 3\log|e^x-1| = \log|\tan y| + \log C$$

$$\Rightarrow \quad \log\left(\frac{|e^x-1|^3}{|\tan y|}\right) = \log C$$

$$\Rightarrow \quad \frac{(e^x-1)^3}{\tan y} = C$$

$$\Rightarrow \quad (e^x-1)^3 = C\tan y$$

Hence, $(e^x-1)^3 = C\tan y, x \in R-\{0\}$ gives the general solution of the given differential equation.

S13. We have,

$$\Rightarrow \quad \frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$$

$$\Rightarrow \quad dy = (e^{x-y} + x^2 e^{-y})dx$$

$$\Rightarrow \quad e^y dy = (e^x + x^2)dx$$

$$\Rightarrow \quad \int e^y dy = \int(e^x+x^2)dx \quad \text{[Integrating both sides]}$$

$$\Rightarrow \quad e^y = e^x + \frac{x^3}{3} + C,$$ which is the required solution.

S14. The given differential equation is

$$\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$$

$$\Rightarrow \quad (1 + x^2)dy = (1 + y^2)dx$$

$$\Rightarrow \quad \frac{dy}{1+y^2} = \frac{dx}{1+x^2}$$

$$\Rightarrow \quad \int \frac{1}{1+y^2}\,dy = \int \frac{1}{1+x^2}\,dx$$ [Integrating both sides]

$$\Rightarrow \quad \tan^{-1}y = \tan^{-1}x + \tan^{-1} C$$

$$\Rightarrow \quad \tan^{-1}y - \tan^{-1}x = \tan^{-1} C$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{y-x}{1+xy}\right) = \tan^{-1} C$$

$$\Rightarrow \quad \frac{y-x}{1+xy} = C$$

$\Rightarrow \quad y - x = C(1 + xy)$, which is the required solution.

S15. We know that the slope of the tangent at any point (x, y) on the curve is given by $\frac{dy}{dx}$. According to the given problem, we have

$$y\frac{dy}{dx} = x \quad \text{... (i)}$$

$$\Rightarrow \quad y\,dy = x\,dx$$

On integrating both sides, we get

$$\int y\,dy = \int x\,dx$$

$$\Rightarrow \quad \frac{y^2}{2} = \frac{x^2}{2} + C \quad \text{... (ii)}$$

This is the equation of the family of curves of differential equation (i). We have to find a particular member of the family which passes through $(0, -2)$.

Substituting $x = 0$ and $y = -2$ in (ii), we get

$$\frac{4}{2} = 0 + C \quad \Rightarrow \quad C = 2$$

Putting $C = 2$ in (ii), we get

$$\frac{y^2}{2} = \frac{x^2}{2} + 2 \quad \Rightarrow \quad y^2 = x^2 + 4$$

This is the equation of the required curve.

S16. The slope of the tangent at any point $P(x, y)$ is given by $\frac{dy}{dx}$. The slope of the line segment joining $P(x, y)$ and $A(-4, -3)$ is $\frac{y+3}{x+4}$

According to the given problem, we have

$$\frac{dy}{dx} = 2\left(\frac{y+3}{x+4}\right) \quad \text{... (i)}$$

$$\frac{1}{y+3}dy = \frac{2}{x+4}dx$$

On integrating both sides, we get

$$\int\frac{1}{y+3}dy = 2\int\frac{1}{x+4}dx$$

$$\Rightarrow \quad \log(y + 3) = 2 \log(x + 4) + \log C$$

$$\Rightarrow \quad (y + 3) = C(x + 4)^2 \quad \text{... (ii)}$$

This represents the family of curves of differential equation (i). We have to find a particular member of this family which passes through $(-2, 1)$.

Substituting $x = -2$, $y = 1$ in (ii), we get

$$4 = C(-2 + 4)^2 \quad \Rightarrow \quad C = 1$$

Putting $C = 1$ in (ii), we get $y + 3 = (x + 4)^2$

This is the required equation of the curve.

S17. We have,

$$xy\frac{dy}{dx} = (x + 2)(y + 2)$$

$$\Rightarrow \quad \frac{y}{y+2}dy = \frac{x+2}{x}dx$$

$$\Rightarrow \quad \frac{y+2-2}{y+2}dy = \left(1+\frac{2}{x}\right)dx$$

$$\Rightarrow \quad \left(1-\frac{2}{y+2}\right)dy = \left(1+\frac{2}{x}\right)dx$$

$$\Rightarrow \quad \int\left(1-\frac{2}{y+2}\right)dy = \int\left(1+\frac{2}{x}\right)dx$$

$$\Rightarrow \quad y - 2\log(y + 2) = x + 2\log x + C \quad \text{... (i)}$$

Since it passes through $(1, -1)$. Putting $x = 1$ and $y = -1$ in Eq. (i), we get

$$-1 - 2\log 1 = 1 + 2\log 1 + C \quad \Rightarrow \quad C = -2$$

Putting $C = -2$ in Eq. (i), we get

$$y - x = 2[\log(y+2) + \log x] - 2$$
$$y - x + 2 = 2\log\{x(y+2)\}$$

Hence, required solution is

$$y = x + 2\log\{x(y+2)\} - 2.$$

S18. We have,

$$\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1 \quad \Rightarrow \frac{dy}{dx} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}$$

$$\Rightarrow \qquad \frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \qquad \ldots \text{(i)}$$

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$,

where $P = \frac{1}{\sqrt{x}}$ and $Q = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

Now solution of above equation is given by $y \times \text{I.F.} = \int (Q \times \text{I.F.})\, dx + C$ ... (ii)

where I.F. = integrating factor and I.F. = $e^{\int p dx}$

$$\therefore \qquad \text{I.F.} = e^{\int P\,dx} = e^{\int \frac{1}{\sqrt{x}} dx} = e^{2\sqrt{x}}$$

Putting I.F. and Q in Eq. (ii).

$$ye^{2\sqrt{x}} = \int e^{2\sqrt{x}} \cdot \frac{e^{-2\sqrt{x}}}{\sqrt{x}}\, dx + C$$

$$\Rightarrow \qquad ye^{2\sqrt{x}} = \int \frac{1}{\sqrt{x}}\, dx + C$$

$$\Rightarrow \qquad ye^{2\sqrt{x}} = 2\sqrt{x} + C$$

$\Rightarrow \qquad y = (2\sqrt{x} + C)e^{-2\sqrt{x}}$, which gives the required solution.

S19. The given differential equation is

$$\frac{dy}{dx} + \left(\frac{1}{x}\right)y = \cos x + \frac{\sin x}{x} \qquad \ldots \text{(i)}$$

This is a linear differential equation of the form

$\frac{dy}{dx} + Py = Q$, where $P = \frac{1}{x}$ and $Q = \cos x + \frac{\sin x}{x}$

$$\therefore \qquad \text{I.F.} = e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

Now solution of above equation is given by $y \times \text{I.F.} = \int (Q \times \text{I.F.})\, dx + C$ ... (ii)

where I.F =integrating factor and $\text{I.F} = e^{\int p dx}$

Putting I.F. and Q in Eq. (ii).

$$yx = \int (x \cos x + \sin x) dx + C$$

$$\Rightarrow \quad xy = \int \underset{\text{I}}{x} \underset{\text{II}}{\cos} x \, dx + \int \sin x \, dx + C$$

$$\Rightarrow \quad xy = x \sin x - \int \sin x \, dx + \int \sin x \, dx + C$$ [Integrating 1st integral by parts]

$$\Rightarrow \quad xy = x \sin x + C$$

Hence, $y = \sin x + \frac{C}{x}, x > 0$ gives the required solution.

S20. The given differential equation is

$$\frac{dy}{dx} + y \tan x = 2x + x^2 \tan x \quad \text{...(i)}$$

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$,

where $P = \tan x$ and $Q = 2x + x^2 \tan x$

Now solution of above equation is given by $y \times \text{I.F.} = \int (Q \times \text{I.F.})\, dx + C$ ... (ii)

where I.F. = integrating factor and $\text{I.F.} = e^{\int p dx}$

$$\therefore \quad \text{I.F.} = e^{\int P dx} = e^{\int \tan x dx} = e^{\log \sec x} = \sec x$$

Putting I.F. and Q in Eq. (ii).

$$y \sec x = \int (2x \sec x + x^2 \sec x \tan x) dx + C$$

$$\Rightarrow \quad y \sec x = \int 2x \sec x \, dx + \int \underset{\text{I}}{x^2} \sec \underset{\text{II}}{x} \tan x \, dx + C$$

$$\Rightarrow \quad y \sec x = \int 2x \sec x \, dx + x^2 \sec x - \int 2x \sec x \, dx + C$$

$\Rightarrow \quad y \sec x = x^2 \sec x + C$, which is the required solution.

S21. We have,

$$(x + y + 1)^2 dy = dx$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{(x+y+1)^2} \quad \ldots \text{(i)}$$

Let $x + y + 1 = v$. Then,

$$1+\frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$$

Putting $x + y + 1 = v$ and $\frac{dy}{dx} = \frac{dv}{dx} - 1$ in Eq. (i), we get

$$\therefore \quad \frac{dv}{dx} - 1 = \frac{1}{v^2}$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{1+v^2}{v^2}$$

$$\Rightarrow \quad \frac{v^2}{v^2+1}dv = dx$$

$$\Rightarrow \quad \int\frac{v^2}{v^2+1}dv = \int dx$$

$$\Rightarrow \quad \int\frac{v^2+1-1}{v^2+1}dv = \int dx$$

$$\Rightarrow \quad \int\left(1-\frac{1}{v^2+1}\right)dv = \int dx$$

$$\Rightarrow \quad \int 1dv - \int\frac{1}{v^2+1}dv = \int 1.\,dx$$

$$\Rightarrow \quad v - \tan^{-1} v = x + C$$

$$\Rightarrow \quad (x + y + 1) - \tan^{-1} (x + y + 1) = x + C$$

$$\Rightarrow \quad y + 1 - \tan^{-1}(x + y + 1) = C \quad \ldots \text{(ii)}$$

It is given that $y(-1) = 0$ *i.e.*, $y = 0$ when $x = -1$

Putting $x = -1$ and $y = 0$ in (ii), we get

$$1 - \tan^{-1} 0 = C \Rightarrow C = 1$$

Putting $C = 1$ in (ii), we get

$y = \tan^{-1}(x + y + 1) \Rightarrow x + y + 1 = \tan y$ as the required solution.

S22. Let $x + y = v$. Then,

$$1+\frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$$

Putting $x + y = v$ and $\frac{dy}{dx} = \frac{dv}{dx} - 1$ in the given differential equation, we get

$$v^2\left(\frac{dv}{dx} - 1\right) = a^2$$

$$\Rightarrow \quad v^2 \frac{dv}{dx} = a^2 + v^2$$

$$\Rightarrow \quad v^2\, dv = (a^2 + v^2)\, dx$$

$$\Rightarrow \quad \frac{v^2}{v^2 + a^2} dv = dx$$ [By separating the variables]

$$\Rightarrow \quad \frac{v^2 + a^2 - a^2}{v^2 + a^2} dv = dx$$

$$\Rightarrow \quad \left(1 - \frac{a^2}{v^2 + a^2}\right) dv = dx$$

$$\Rightarrow \quad \int 1 \cdot dv - a^2 \int \frac{1}{v^2 + a^2} dv = \int dx + C$$ [On integration]

$$\Rightarrow \quad v - \frac{a^2}{a}\tan^{-1}\left(\frac{v}{a}\right) = x + C$$

$$\Rightarrow \quad (x + y) - a\tan^{-1}\left(\frac{x + y}{a}\right) = x + C.$$

S23. Given that

$$\frac{dy}{dx} = (4x + y + 1)^2$$

Let $4x + y + 1 = v$, Then,

$$4 + \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 4$$

Putting $4x + y + 1 = v$ and $\frac{dy}{dx} = \frac{dv}{dx} - 4$ in the given differential equation, we get

$$\frac{dv}{dx} - 4 = v^2$$

$$\Rightarrow \quad \frac{dv}{dx} = v^2 + 4$$

$$\Rightarrow \quad dv = (v^2 + 4)\, dx$$

$\Rightarrow \quad \dfrac{dv}{v^2+4} = dx$

$\Rightarrow \quad \int \dfrac{1}{v^2+4}dv = \int 1 \cdot dx$ [integrating both sides]

$\Rightarrow \quad \dfrac{1}{2}\tan^{-1}\left(\dfrac{v}{2}\right) = x + C$

$\Rightarrow \quad \dfrac{1}{2}\tan^{-1}\left(\dfrac{4x+y+1}{2}\right) = x + C$, which is the required solution.

S24. Let at any time the bacteria count be N. Then,

$$\frac{dN}{dt} \propto N$$

$\Rightarrow \quad \dfrac{dN}{dt} = \lambda N \Rightarrow \int \dfrac{1}{N}dN = \int \lambda dt$

$\Rightarrow \quad \log N = \lambda t + \log C$

At $\quad t = 0, \quad N = 100000$

$\therefore \quad \log C = \log 100000$

So, $\quad \log N = \lambda t + \log 100000$

At $\quad t = 2, \quad N = 110000$

$\therefore \quad \log 110000 = 2\lambda + \log 100000 \quad \Rightarrow \quad \dfrac{1}{2}\log\dfrac{11}{10} = \lambda$

$\therefore \quad \log N = \dfrac{1}{2}\log\left(\dfrac{11}{10}\right)t + \log 100000$

When $N = 200000$, let $t = T$. Then,

$$\log 200000 = \frac{T}{2}\log\left(\frac{11}{10}\right) + \log 100000$$

$\Rightarrow \quad \log 2 = \dfrac{T}{2}\log\dfrac{11}{10} \quad \Rightarrow \quad T = 2\dfrac{\log 2}{\log\dfrac{11}{10}}$

S25. Given that

$$\frac{dy}{dx} = \cos(x+y)$$

Let $\quad x + y = v$. Then,

$$1+\frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$$

Putting $x + y = v$ and $\frac{dy}{dx} = \frac{dv}{dx} - 1$ in the given differential equation, we get

$$\frac{dv}{dx} - 1 = \cos v$$

$$\Rightarrow \quad \frac{dv}{dx} = 1 + \cos v$$

$$\Rightarrow \quad \frac{1}{2\cos^2 \frac{v}{2}} dv = dx$$

$$\Rightarrow \quad \frac{1}{2}\sec^2 \frac{v}{2} dv = dx$$

$$\Rightarrow \quad \int \frac{1}{2}\sec^2 \frac{v}{2} dv = \int 1 \cdot dx$$ [Integrating both sides]

$$\Rightarrow \quad \tan\frac{v}{2} = x + C$$

$\Rightarrow \quad \tan\left(\frac{x+y}{2}\right) = x + C$, which is the required solution.

S26. Given differential equation is

$$\frac{dy}{dx} + y \cot x = 4x \operatorname{cosec} x \quad \text{and} \quad y = 0 \quad \text{when} \quad x = \frac{\pi}{2}$$

Comparing the above equation with general form of linear differential equation of 1st order

$\frac{dy}{dx} + Py = Q$, we get

$P = \cot x$ and $Q = 4x \operatorname{cosec} x$

Now, we will find the integrating factor (I.F.).

$$\text{I.F.} = e^{\int P dx} = e^{\int \cot x\, dx} = e^{\log \sin x} = \sin x \qquad \left[\because e^{\log x} = x\right]$$

Now, solution of linear differential equation is given by

$$y \times \text{I.F.} = \int (Q \times \text{IF}) dx + c \qquad \text{...(i)}$$

$\therefore$ Putting I.F. $= \sin x$ and $Q = 4x \operatorname{cosec} x$ in Eq. (i)

we get $y.\sin x = \int 4x \operatorname{cosec} x . \sin x dx$

$\Rightarrow \qquad y \sin x = \int 4x\, dx$

$\Rightarrow \qquad y \sin x = \frac{4x^2}{2} + c$

$\Rightarrow \qquad y \sin x = 2x^2 + c$ ... (ii)

Now putting y = 0, x = $\frac{\pi}{2}$ in Eq. (ii) we get $0 = \frac{\pi^2}{2} + c \Rightarrow c = -\frac{\pi^2}{2}$

Putting $c = -\frac{\pi^2}{2}$ in Eq. (ii), we get the required solution as

$$y \sin x = 2x^2 - \frac{\pi^2}{2}$$

S27. The given differential equation is

$$y\, dx + (x - y^3)dy = 0$$

$$\frac{dx}{dy} = \frac{y^3 - x}{y}$$

$\Rightarrow \qquad \frac{dx}{dy} + \frac{x}{y} = y^2$ ... (i)

This is a linear differential equation of the form

$$\frac{dx}{dy} + Rx = S, \text{ where } R = \frac{1}{y} \text{ and } S = y^2$$

Now solution of above equation is given by x × I.F. = $\int (S \times \text{I.F.})\, dy + C$ ... (ii)

where I.F =integrating factor and I.F = $e^{\int R dy}$

$\therefore \qquad \text{I.F.} = e^{\int R dy} = e^{\int \frac{1}{y} dy} = e^{\log y} = y$

Putting I.F. and S in Eq. (ii),

$$xy = \int y^3 dy + C$$

$\Rightarrow \qquad xy = \frac{y^4}{4} + C,$ which is the required solution.

S28. The given differential equation is

$$y\, dx - (x + 2y^2)dy = 0$$

$\Rightarrow \qquad \frac{dx}{dy} = \frac{x + 2y^2}{y}$

$\Rightarrow \qquad \frac{dx}{dy} - \frac{x}{y} = 2y$

$$\Rightarrow \qquad \frac{dx}{dy} + \left(\frac{-1}{y}\right)x = 2y \qquad \text{... (i)}$$

This is a linear differential equation and is of the form

$$\frac{dx}{dy} + Rx = S, \text{ where } R = -\frac{1}{y} \text{ and } S = 2y$$

Now solution of above equation is given by

$$x \times \text{I.F.} = \int (S \times \text{I.F.})\, dy + C \qquad \text{... (ii)}$$

where I.F = integrating factor and I.F. = $e^{\int R dy}$

$$\therefore \qquad \text{I.F.} = e^{\int R dy} = e^{\int \frac{-1}{y} dy} = e^{-\log y} = e^{\log y^{-1}} = y^{-1}$$

Putting I.F. and S in Eq. (ii).

$$x \cdot \frac{1}{y} = \int 2dy + C \qquad \left[\text{Using : } x(\text{I.F.}) = \int S(I.F.)dy + C\right]$$

$$\Rightarrow \qquad \frac{x}{y} = 2y + C, \text{ which is the required solution.}$$

S29. Given differential equation is

$$\frac{dy}{dx} + 2y \tan x = \sin x \qquad \text{... (i)}$$

This is a linear differential equation of 1st order and is of the form

$$\frac{dy}{dx} + Py = Q \qquad \text{... (ii)}$$

Comparing Eqs. (i) and (ii), we get

$$P = 2 \tan x \quad \text{and} \quad Q = \sin x$$

Now, solution of above equation is given by

$$y \times \text{I.F.} = \int (Q \times \text{I.F.})\, dx + c \qquad \text{... (iii)}$$

where I.F. = Integrating factor and I.F. = $e^{\int P dx}$

Now, $\qquad \text{I.F.} = e^{\int 2\tan x\, dx} = e^{2\log|\sec x|}$

$$= e^{\log \sec^2 x} = \sec^2 x$$

Putting I.F. = $\sec^2 x$ and $Q = \sin x$ in Eq. (iii), we get

$$y \sec^2 x = \int \sin x \cdot \sec^2 x \, dx$$

$\Rightarrow$ $$y \sec^2 x = \int \frac{\sin x}{\cos^2 x} dx$$

$\Rightarrow$ $$y \sec^2 x = \int \sec x \tan x \, dx$$

$\Rightarrow$ $$y \sec^2 x = \sec x + c$$

$\therefore$ Required solution is

$$y = \frac{1}{\sec x} + \frac{c}{\sec^2 x}$$

S30. Given differential equation is

$(1 + x^2)\, dy + 2xy\, dx = \cot x\, dx$ $[x \neq 0]$

Above equation can be written as

$(1 + x^2)dy + (2xy - \cot x)\, dx = 0$

$\Rightarrow$ $(1 + x^2)\, dy = (\cot x - 2xy)\, dx$

Dividing both sides by $1 + x^2$, we get

$$dy = \frac{\cot x - 2xy}{1 + x^2} dx$$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{\cot x}{1 + x^2} - \frac{2xy}{1 + x^2}$$

$\Rightarrow$ $$\frac{dy}{dx} + \frac{2xy}{1 + x^2} = \frac{\cot x}{1 + x^2} \quad \text{... (i)}$$

This is a linear differential equation of 1st order and is of the form

$$\frac{dy}{dx} + Py = Q \quad \text{... (ii)}$$

Comparing Eqs. (i) and (ii), we get

$$P = \frac{2x}{1 + x^2} \quad \text{and} \quad Q = \frac{\cot x}{1 + x^2}$$

Now, solution of linear differential equation is given by

$$y \times \text{I.F.} = \int (Q \times \text{IF}) dx + c$$

where I.F. = Integrating factor and I.F. = $e^{\int P dx}$ ... (iii)

$\therefore$ $$\text{I.F.} = e^{\int \frac{2x}{1+x^2} dx} = e^{\log|1+x^2|} = 1 + x^2$$

$\left[\because \int \frac{2x}{1+x^2} dx, \text{ Put } 1 + x^2 = t \Rightarrow 2x\, dx = dt \Rightarrow \int \frac{dt}{t} = \log|t| = \log|1 + x^2| + c\right]$

Putting I.F. = 1 + x^2 and Q = $\frac{\cot x}{1+x^2}$ in Eq. (iii), we get

$$y\,(1 + x^2) = \int \frac{\cot x}{1+x^2} \times (1+x^2)\,dx$$

$$\Rightarrow \quad y\,(1 + x^2) = \int \cot x\,dx$$

$$\Rightarrow \quad y\,(1 + x^2) = \log|\sin x| + c \qquad \left[\because \int \cot x\,dx = \log|\sin x| + c\right]$$

Dividing both sides by 1 + x^2, we get

$$y = \frac{\log|\sin x|}{1+x^2} + \frac{c}{1+x^2}$$ is the required solution.

S31. Given differential equation is

$$xdy - (y + 2x^2)\,dx = 0$$

Above equation can be written as

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y+2x^2}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} + 2x \qquad \text{... (i)}$$

Above equation is a linear differential equation of 1st order and it is of the form

$$\frac{dy}{dx} + Py = Q \qquad \text{... (ii)}$$

Comparing Eqs. (i) and (ii), we get

$$P = \frac{1}{x} \quad \text{and} \quad Q = 2x$$

Now, solution of linear differential equation is given by

$$y \times \text{I.F.} = \int (Q \times \text{I.F.})\,dx + c \qquad \text{... (iii)}$$

where, I.F. = Integrating factor and I.F. = $e^{\int P dx}$

So, $\quad \text{IF} = e^{\int \frac{1}{x} dx} = e^{\log|x|} = x$

Putting I.F. = x and $Q = 2x$ in Eq. (iii), we get

$$y \times x = \int (2x \times x)\,dx \qquad \Rightarrow yx = \int 2x^2\,dx$$

$$\Rightarrow \quad yx = \frac{2x^3}{3} + c$$

or $\quad y = \frac{2x^2}{3} + \frac{c}{x}$ is the required solution.

S32. Given equation is $(y+3x^2)\frac{dx}{dy} = x$.

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y+3x^2}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}+3x$$

or $$\frac{dy}{dx} - \frac{y}{x} = 3x \quad \text{... (i)}$$

Above equation is linear differential equation and is of the form

$$\frac{dy}{dx}+Py = Q \quad \text{... (ii)}$$

Comparing Eqs. (i) and (ii), we get

$$P = \frac{-1}{x}, Q = 3x$$

Now, solution of linear differential equation is given by

$$y \times \text{I.F.} = \int (Q \times \text{I.F.})\, dx + c \quad \text{... (iii)}$$

where, I.F. = integrating factor and I.F. = $e^{\int P dx}$

$$\therefore \quad \text{I.F.} = e^{\int -\frac{1}{x}dx} = e^{-\log|x|} = e^{\log x^{-1}} = x^{-1}$$

$$\therefore \quad \text{I.F.} = x^{-1} = \frac{1}{x}$$

Putting I.F. = $\frac{1}{x}$ and $Q = 3x$ in Eq. (iii), we get

$$y \times \frac{1}{x} = \int 3x \cdot \frac{1}{x} dx$$

$$\Rightarrow \quad \frac{y}{x} = \int 3\, dx$$

$$\Rightarrow \quad \frac{y}{x} = 3x + c$$

$$\Rightarrow \quad y = 3x^2 + cx$$

is the required solution of given differential equation.

S33. The given differential equation is

$$(x^2+1)\frac{dy}{dx} + 2xy = \sqrt{x^2+4}$$

Dividing both sides by $(x^2 + 1)$, we get

$$\frac{dy}{dx} + \frac{2xy}{x^2+1} = \frac{\sqrt{x^2+4}}{x^2+1} \quad \text{... (i)}$$

Above equation is a linear differential equation of 1st order and is of the form

$$\frac{dy}{dx} + Py = Q \quad \text{... (ii)}$$

Comparing Eqs. (i) and (ii), we get

$$P = \frac{2x}{x^2+1} \quad \text{and} \quad Q = \frac{\sqrt{x^2+4}}{x^2+1}$$

Now, solution of this equation is given by

$$y \times \text{I.F.} = \int (Q \times \text{I.F.})\, dx + c \quad \text{... (iii)}$$

where, I.F.= Integrating factor and I.F. = $e^{\int P dx}$

$\therefore$ $$\text{I.F.} = e^{\int \frac{2x}{x^2+1} dx} = e^{\log|x^2+1|} = x^2 + 1 \quad \left[\because e^{\log x} = x\right]$$

$\therefore$ $$\text{I.F.} = x^2 + 1$$

$$\left[\because \int \frac{2x}{x^2+1} dx \ \Rightarrow \text{Put } x^2+1 = t \ \Rightarrow 2x\, dx = dt \therefore \int \frac{dt}{t} = \log|t| = \log|x^2+1| + c\right]$$

Putting I.F. = $x^2 + 1$ and $Q = \frac{\sqrt{x^2+4}}{x^2+1}$ in Eq. (iii), we get

$$y(x^2+1) = \int (x^2+1) \cdot \frac{\sqrt{x^2+4}}{x^2+1} dx$$

$\Rightarrow$ $$y(x^2+1) = \int \sqrt{x^2+4}\, dx$$

$\Rightarrow$ $$y(x^2+1) = \int \sqrt{x^2+(2)^2}\, dx$$

Now, we know that

$$\int \sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log|x+\sqrt{x^2+a^2}| + c$$

$\therefore$ We get,

$$y(x^2+1) = \frac{x}{2}\sqrt{x^2+4} + \frac{4}{2}\log|x+\sqrt{x^2+4}| + c$$

$\Rightarrow$ $$y(x^2+1) = \frac{x}{2}\sqrt{x^2+4} + 2\log|x+\sqrt{x^2+4}| + c$$ is the required solution.

S34. Given that

$$(x^3 - 3xy^2)\, dx = (y^3 - 3x^2y)\, dy$$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{x^3 - 3xy^2}{y^3 - 3x^2y}$$

The given equation is a homogeneous equation.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in it, we get

$$v + x\frac{dv}{dx} = \frac{x^3 - 3v^2x^3}{v^3x^3 - 3vx^3}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{1-3v^2}{v^3-3v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1-3v^2}{v^3-3v} - v = \frac{1-3v^2-v^4+3v^2}{v^3-3v} = \frac{1-v^4}{v^3-3v}$$

$$\Rightarrow \quad x(v^3 - 3v)dv = (1 - v^4)dx$$

$$\Rightarrow \quad \frac{v^3-3v}{1-v^4}dv = \frac{dx}{x}, x \neq 0, v \neq \pm 1$$

$$\Rightarrow \quad \int\frac{v^3-3v}{1-v^4}dv = \int\frac{dx}{x}$$ [Integrating both sides]

$$\Rightarrow \quad \int\frac{v^3}{1-v^4}dv - 3\int\frac{v}{1-v^4}dv = \int\frac{dx}{x}$$

$$\Rightarrow \quad -\frac{1}{4}\int\frac{-4v^3}{1-v^4}dv - \frac{3}{2}\int\frac{2v}{1-(v^2)^2}dv = \int\frac{dx}{x}$$

and in II[nd] Integral substitute $t = v^2$

$$dt = 2\,vdv$$

$$\Rightarrow \quad -\frac{1}{4}\int\frac{-4v^3}{1-v^4}dv - \frac{3}{2}\int\frac{dt}{1-t^2} = \int\frac{dx}{x},$$

$$\Rightarrow \quad -\frac{1}{4}\log|1-v^4| - \frac{3}{2}\times\frac{1}{2\times 1}\log\left|\frac{1+t}{1-t}\right| = \log|x| + \log C$$

$$\Rightarrow \quad -\frac{1}{4}\log|1-v^4| - \frac{3}{4}\log\left|\frac{1+v^2}{1-v^2}\right| = \log|Cx|$$

$$\Rightarrow \quad -\log|1-v^4| - 3\log\left|\frac{1+v^2}{1-v^2}\right| = 4\log|Cx|$$

$$\Rightarrow \quad \log\left|(1-v^4)^{-1}\left(\frac{1+v^2}{1-v^2}\right)^{-3}\right| = \log|(Cx)^4|$$

$$\Rightarrow \quad \frac{1}{1-v^4} \times \left(\frac{1-v^2}{1+v^2}\right)^3 = (Cx)^4$$

$$\Rightarrow \quad (1 - v^2)^2 = (1 + v^2)^4 (Cx)^4$$

$$\Rightarrow \quad 1 - v^2 = (1 + v^2)^2 (Cx)^2$$

$$\Rightarrow \quad 1 - \frac{y^2}{x^2} = \left(1 + \frac{y^2}{x^2}\right)^2 C^2 x^2 \qquad \left[\because v = \frac{y}{x}\right]$$

$$\Rightarrow \quad x^2 - y^2 = (x^2 + y^2)^2 C^2,$$

which is the required solution.

S35. Given differential equation is

$$x\,dy + (y - x^3)\,dx = 0$$

Above equation can be written as

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^3 - y}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = x^2 - \frac{y}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{y}{x} = x^2 \qquad \text{... (i)}$$

It is a linear differential equation and is of the form

$$\Rightarrow \quad \frac{dy}{dx} + Py = Q \qquad \text{... (ii)}$$

Comparing Eqs. (i) and (ii), we get

$$P = \frac{1}{x} \quad \text{and} \quad Q = x^2$$

Solution of linear differential equation is given by

$$y \times \text{I.F.} = \int (Q \times \text{I.F.})\,dx + c \qquad \text{... (iii)}$$

I.F. = Integrating factor and I.F. = $e^{\int P dx}$

$$\therefore \quad \text{I.F.} = e^{\int \frac{1}{x} dx} = e^{\log|x|} = x$$

Putting I.F. = x and $Q = x^2$ in Eq. (iii), we get

$$y \times x = \int x^2 \times x\,dx$$

$$\Rightarrow \quad yx = \int x^3 dx$$

$\Rightarrow \qquad yx = \dfrac{x^4}{4} + c$

$\Rightarrow \qquad y = \dfrac{x^3}{4} + \dfrac{c}{x}$ is the required solution.

S36. Given differential equation is

$$(x^2 - 1)\,\frac{dy}{dx} + 2xy = \frac{1}{x^2 - 1};\; |x| \neq 1$$

Dividing both sides by $x^2 - 1$, we get

$$\frac{dy}{dx} + \frac{2xy}{x^2 - 1} = \frac{1}{(x^2 - 1)^2} \qquad \dots \text{(i)}$$

This is a linear differential equation and is of the form

$$\frac{dy}{dx} + Py = Q \qquad \dots \text{(ii)}$$

Comparing Eqs. (i) and (ii), we get

$$P = \frac{2x}{x^2 - 1} \text{ and } Q = \frac{1}{(x^2 - 1)^2}$$

Now, solution of above equation is given by

$$y \times \text{I.F.} = \int (Q \times \text{I.F.})\, dx + \text{c} \qquad \dots \text{(iii)}$$

where, I.F. = Integrating factor and I.F. = $e^{\int P dx}$

$\therefore \qquad \text{I.F.} = e^{\int \frac{2x}{x^2-1} dx} = e^{\log(x^2-1)} = x^2 - 1 \qquad [\because e^{\log x} = x]$

$$\left[\because \int \frac{2x}{x^2 - 1} dx \text{ Put } x^2 - 1 = t \Rightarrow 2x\,dx = dt \;\; \therefore \int \frac{dt}{t} = \log|t| = \log|x^2 - 1| + c\right]$$

Putting I.F. = $x^2 - 1$ and $Q = \dfrac{1}{(x^2 - 1)^2}$ in Eq. (iii),

we get

$$y(x^2 - 1) = \int (x^2 - 1) \cdot \frac{1}{(x^2 - 1)^2} dx$$

$\Rightarrow \qquad y(x^2 - 1) = \displaystyle\int \frac{1}{x^2 - 1} dx$

$\Rightarrow \qquad y(x^2 - 1) = \displaystyle\int \frac{dx}{x^2 - (1)^2}$

$\Rightarrow \quad y(x^2 - 1) = \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + c \qquad \left[\because \int\frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c\right]$

Hence $y(x^2 - 1) = \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + c$ is the required solution.

S37. Given differential equation is

$$\cos^2 x \frac{dy}{dx} + y = \tan x$$

Dividing both sides by $\cos^2 x$, we get

$$\frac{dy}{dx} + \frac{y}{\cos^2 x} = \frac{\tan x}{\cos^2 x}$$

$\Rightarrow \quad \frac{dy}{dx} + y \sec^2 x = \tan x \sec^2 x \qquad \text{... (i)}$

This is a linear differential equation and it is of the form

$$\frac{dy}{dx} + Py = Q \qquad \text{... (ii)}$$

Comparing Eqs. (i) and (ii), we get

$$P = \sec^2 x \quad \text{and} \quad Q = \tan x \sec^2 x$$

Now, solution of above equation is given by

$$y \times \text{I.F.} = \int (Q \times \text{I.F.})\, dx + c \qquad \text{... (iii)}$$

where, I.F. = Integrating factor and I.F. = $e^{\int P dx}$

Now, $\quad \text{IF} = e^{\int \sec^2 x\, dx} = e^{\tan x} \qquad \left[\because \int \sec^2 x\, dx = \tan x\right]$

Putting I.F. = $e^{\tan x}$ and $Q = \tan x \cdot \sec^2 x$ in Eq. (iii), we get

$$y \cdot e^{\tan x} = \int \tan x \sec^2 x e^{\tan x} dx$$

Put $\quad \tan x = t$

$\Rightarrow \quad \sec^2 x\, dx = dt$

$\therefore \quad ye^{\tan x} = \int t \times e^t dt$

$\Rightarrow \quad ye^{\tan x} = t\int e^t dt - \int\left[\frac{d}{dt}(t)\int e^t dt\right]dt \qquad$ [By using integration by parts]

$\Rightarrow \quad ye^{\tan x} = te^t - \int 1 \times e^t dt$

$\Rightarrow \quad ye^{\tan x} = te^t - e^t + c$

$\Rightarrow \quad ye^{\tan x} = \tan \mathbf{x} \cdot \mathbf{e}^{\tan x} - \mathbf{e}^{\tan x} + \mathbf{c}$ is the required solution.

S38. The given differential equation is

$$(x \log x) \frac{dy}{dx} + y = 2 \log x$$

Dividing both sides by $x \log x$, we get

$$\frac{dy}{dx} + \frac{y}{x \log x} = \frac{2}{x} \quad \text{... (i)}$$

This is a linear differential equation which is of the form

$$\frac{dy}{dx} + Py = Q \quad \text{... (ii)}$$

Comparing Eqs. (i) and (ii), we get

$$P = \frac{1}{x \log x},\ Q = \frac{2}{x}$$

Now, solution of above equation is given by

$$y \times \text{I.F.} = \int (Q \times \text{I.F.})\, dx + c \quad \text{... (iii)}$$

where, I.F. = Integrating factor and I.F. = $e^{\int P dx}$

$$\therefore \quad \text{I.F.} = e^{\int \frac{1}{x \log x} dx} = e^{\log \log x} = \log x \quad \left[\because e^{\log x} = x\right]$$

$$\left[\because \int \frac{1}{x \log x} dx \text{ Put} \log x = t \Rightarrow \frac{1}{x} dx = dt \ \therefore \int \frac{dt}{t} = \log |t| = \log |\log x|\right]$$

Putting I.F. = $\log x$ and Q = $\frac{2}{x}$ in Eq. (iii), we get

$$y \times \log x = \int \underset{\text{II}}{\frac{2}{x}} \underset{\text{I}}{\log x}\, dx$$

$$\Rightarrow \quad y \log x = \log x \int \frac{2}{x} dx - \int \left[\frac{d}{dx}(\log x) \int \frac{2}{x} dx\right] \quad \text{[By using integration by parts]}$$

$$\Rightarrow \quad y \log x = \log x \cdot 2 \log x - \int \frac{1}{x} \cdot 2 \log x\, dx \quad \left[\because \int \frac{1}{x} dx = \log |x| + c\right]$$

$$\Rightarrow \quad y \log x = 2 (\log x)^2 - 2 \int \frac{\log x}{x} dx$$

$$\Rightarrow \quad y \log x = 2 (\log x)^2 - \frac{2(\log x)^2}{2} + c$$

$$\left[\because \int \frac{\log x}{x} dx \text{ Put } \log x = t \Rightarrow \frac{1}{x} dx = dt \ \therefore \int t\, dt = \frac{t^2}{2} = \frac{(\log x)^2}{2} + c\right]$$

Hence, the required solution is given by

$$2 (\log x) - (\log x) + \frac{c}{\log x} \quad \Rightarrow \quad y = \log x + \frac{c}{\log x}$$

S39. The given differential equation is

$$(1 + x^2)\ \frac{dy}{dx} + y = \tan^{-1} x$$

Dividing both sides by $(1 + x^2)$, we get

$$\frac{dy}{dx} + \frac{y}{1+x^2} = \frac{\tan^{-1} x}{1+x^2} \qquad \text{... (i)}$$

This is a linear differential equation of 1st order which is of the form

$$\frac{dy}{dx} + Py = Q \qquad \text{... (ii)}$$

Comparing Eqs. (i) and (ii), we get

$$P = \frac{1}{1+x^2} \quad \text{and} \quad Q = \frac{\tan^{-1} x}{1+x^2}$$

Now, solution of above equation is given by

$$y \times \text{I.F.} = \int (Q \times \text{I.F.})\, dx + c \qquad \text{... (iii)}$$

Where, I.F. = integrating factor and I.F. = $e^{\int P dx}$

$$\therefore \qquad \text{I.F.} = e^{\int \frac{1}{1+x^2} dx} = e^{\tan^{-1} x} \qquad \left[\because \int \frac{1}{1+x^2} dx = \tan^{-1} x + c\right]$$

Putting I.F. = $e^{\tan^{-1} x}$ and $Q = \frac{\tan^{-1} x}{1+x^2}$ in Eq. (iii), we get

$$y \times e^{\tan^{-1} x} = \int \frac{\tan^{-1} x}{1+x^2} \cdot e^{\tan^{-1} x} dx$$

Put $\tan^{-1} x = t$

$$\Rightarrow \qquad \frac{1}{1+x^2}\, dx = dt \qquad \therefore\ y e^{\tan^{-1} x} = \int \underset{\text{I}}{t}\, \underset{\text{II}}{e^t}\, dt$$

$$\Rightarrow \qquad y e^{\tan^{-1} x} = t \int e^t dt - \int \left[\frac{d}{dt}(t) \int e^t dt\right] dt \qquad \text{[Using integration by parts]}$$

$$\Rightarrow \qquad y e^{\tan^{-1} x} = t e^t - \int 1 \times e^t dt$$

$$\Rightarrow \qquad y e^{\tan^{-1} x} = t e^t - e^t + c$$

or

$$y e^{\tan^{-1} x} = \tan^{-1} x \cdot e^{\tan^{-1} x} - e^{\tan^{-1} x} + c$$

$$y e^{\tan^{-1} x} = e^{\tan^{-1} x} (\tan^{-1} x - 1) + c$$

S40. The given differential equation is

$$\frac{dy}{dx} + y = \cos x - \sin x \qquad \text{... (i)}$$

This is a linear differential equation of 1st order and is of the form

$$\frac{dy}{dx} + Py = Q \qquad \text{... (ii)}$$

Comparing Eqs. (i) and (ii), we get

$$P = 1 \quad \text{and} \quad Q = \cos x - \sin x$$

Now, solution of above equation is given by

$$y \times \text{I.F.} = \int (Q \times \text{I.F.})\, dx + c \qquad \text{... (iii)}$$

Where, I.F. = integrating factor

and $\qquad \text{I.F.} = e^{\int P dx}$

$\therefore \qquad \text{I.F.} = e^{\int 1 dx} = e^x$

Putting I.F. = e^x and $Q = \cos x - \sin x$ in Eq. (iii), we get

$$ye^x = \int e^x (\cos x - \sin x) dx$$

$$ye^x = \int \underset{\text{II}}{e^x} \underset{\text{I}}{\cos} x\, dx - \int e^x \sin x\, dx$$

$$ye^x = \left[\cos x \int e^x dx - \int \left\{\frac{d}{dx}(\cos x) \int e^x dx\right\} dx\right] - \int e^x \sin x\, dx$$

[Applying integration by parts in the first integral]

$\Rightarrow \qquad ye^x = [e^x \cos x - \int -\sin x \cdot e^x dx\,] - \int e^x \sin x\, dx$

$\Rightarrow \qquad ye^x = e^x \cos x + \int e^x \sin x\, dx - \int e^x \sin x\, dx + c$

$\therefore \qquad ye^x = e^x \cos x + c$

Dividing both sides by e^x, we get

$$y = \cos x + ce^{-x} \text{ is required solution.}$$

S41. $(x^2 + 1)y' - 2xy = (x^4 + 2x^2 + 1) \cos x$

$\Rightarrow \qquad \dfrac{dy}{dx} - \dfrac{2x}{x^2 + 1} y = (x^2 + 1) \cos x$

This is a linear differential equation of the form $\dfrac{dy}{dx} + Py = Q$, where $P = \frac{-2x}{x^2+1}$ and $Q = (x^2 + 1)\cos x$

Now solution of above equation is given by $y \times \text{I.F.} = \int (Q \times \text{I.F.})\, dx + C$ ... (i)

where I.F =integrating factor and I.F. = $e^{\int p dx}$

$$\therefore \quad \text{I.F.} = e^{\int \frac{-2x}{x^2+1} dx} = e^{-\log(x^2+1)} = (x^2+1)^{-1}$$

Putting I.F. and Q in Eq. (i)

$$y \times \frac{1}{x^2+1} = \int \cos x\, dx + C$$

$$\Rightarrow \quad \frac{y}{x^2+1} = \sin x + C \qquad \text{... (ii)}$$

It is given that $y(0) = 0$ *i.e.*, $y = 0$ when $x = 0$

Putting $x = 0$, $y = 0$ in (ii), we get $C = 0$

Putting $C = 0$ in (ii), we get

$$\frac{y}{x^2+1} = \sin x$$

$$\Rightarrow \quad y = (x^2 + 1)\sin x$$

Clearly, it is defined for all $x \in R$.

Hence, $y = (x^2 + 1)\sin x$, $x \in R$ gives the solution.

S42. The given differential equation is

$$\frac{dy}{dx} + y \sec x = \tan x \qquad \text{... (i)}$$

This is a linear differential equation of 1st order and is of the form

$$\frac{dy}{dx} + Py = Q \qquad \text{... (ii)}$$

Comparing Eqs. (i) and (ii), we get

$$P = \sec x, \quad Q = \tan x$$

Now, solution of above equation is given by

$$y \times \text{I.F.} = \int (Q \times \text{I.F.})\, dx + c \qquad \text{... (iii)}$$

where I.F. = Integrating factor and I.F. = $e^{\int P dx}$

$$\therefore \quad \text{I.F.} = e^{\int \sec x\, dx} = e^{\log|\sec x + \tan x|} \qquad \left[\because \int \sec x\, dx = \log|\sec x + \tan x|\right]$$

$\therefore$ I.F. = sec x + tan x

Putting I.F. = sec x + tan x and Q = tan x in Eq. (iii), we get

$\Rightarrow$ y (sec x + tan x) = $\int \tan x \cdot (\sec x + \tan x) dx$

$\Rightarrow$ y (sec x + tan x) = $\int \sec x \tan x \, dx + \int \tan^2 x \, dx$

$\Rightarrow$ y (sec x + tan x) = sec x + $\int (\sec^2 x - 1) dx$

$\Rightarrow$ y (sec x + tan x) = (sec x + tan x) – x + c, is required solution.

S43. The given differential equation is

$$(x \log x) \cdot \frac{dy}{dx} + y = \frac{2}{x} \log x$$

Dividing both sides by $x \log x$, we get

$$\frac{dy}{dx} + \frac{y}{x \log x} = \frac{2}{x^2} \frac{\log x}{\log x} = \frac{2}{x^2} \quad \ldots \text{(i)}$$

This is linear differential equation of 1st order and is of the form.

$$\frac{dy}{dx} + Py = Q \quad \ldots \text{(ii)}$$

Comparing Eqs. (i) and (ii), we get

$$P = \frac{1}{x \log x} \quad \text{and} \quad Q = \frac{2}{x^2}$$

Now, solution of above equation is given by

$$y \times \text{I.F.} = \int (Q \times \text{I.F.}) \, dx + c \quad \ldots \text{(iii)}$$

Where, I.F. = integrating factor $\text{I.F.} = e^{\int P dx}$

$\therefore$ I.F. = $e^{\int \frac{1}{x \log x} dx} = e^{\log \log x} = \log x$ $\left[\because e^{\log x} = x\right]$

$$\left[\because \int \frac{1}{x \log x} dx,\ Put\ \log x = t \Rightarrow \frac{1}{x} dx = dt \ \therefore\ \int \frac{1}{t} dt = \log|t| = \log|\log x|\right]$$

$\therefore$ I.F. = log x

Putting I.F. = log x and $Q = \frac{2}{x^2}$ in Eq. (iii), we get

$$y \log x = \int \frac{2}{x^2} \log x \, dx$$

$$\Rightarrow \quad y \log x = \log x \int \frac{2}{x^2} dx - \int \left(\frac{d}{dx}(\log x) \cdot \int \frac{2}{x^2} dx \right) dx \quad \text{[Using by parts]}$$

$\Rightarrow \quad y \log x = \log x \cdot 2\left(-\frac{1}{x}\right) - \int \frac{1}{x} \cdot 2 \times \frac{-1}{x} dx$

$\Rightarrow \quad y \log x = -\frac{2}{x}\log x - \int \frac{2}{x}\left(-\frac{1}{x}\right) dx$

$\Rightarrow \quad y \log x = -\frac{2}{x}\log x + \int \frac{2}{x^2} dx$

$\Rightarrow \quad y \log x = -\frac{2}{x}\log x - \frac{2}{x} + c.$ is the required solution

S44. Given differential equation is

$$(1+x^2)\frac{dy}{dx} + 2xy = \frac{1}{1+x^2}$$

$y = 0$ when $x = 1$

Dividing both sides by $(1 + x^2)$, we get

$$\frac{dy}{dx} + \frac{2xy}{1+x^2} = \frac{1}{(1+x^2)^2} \quad \text{... (i)}$$

Above equation is linear differential equation and is of the form

$$\frac{dy}{dx} + Py = Q \quad \text{... (ii)}$$

Comparing Eqs. (i) and (ii), we get

$$P = \frac{2x}{1+x^2} \quad \text{and} \quad Q = \frac{1}{(1+x^2)^2}$$

Now, solution of linear differential equation is given by

$$y \times \text{I.F.} = \int (Q \times \text{I.F.})\, dx + c \quad \text{... (iii)}$$

where, I.F. = Integrating factor is I.F. = $e^{\int P dx}$

So, $\quad \text{I.F.} = e^{\int \frac{2x}{1+x^2} dx} = e^{\log|1+x^2|} = 1 + x^2 \quad \left[\because e^{\log x} = x\right]$

$\left[\because \int \frac{2x}{1+x^2} dx \quad \text{Put } 1+x^2 = t \quad \Rightarrow 2x\,dx = dt \therefore \int \frac{dt}{t} = \log|t| = \log|1+x^2|\right]$

Putting I.F. = $1 + x^2$ in Eq. (iii), we get

$$y(1 + x^2) = \int \frac{1}{(1+x^2)^2} \times (1+x^2) dx$$

$\Rightarrow \quad y(1 + x^2) = \int \frac{1}{1+x^2} dx$

$\Rightarrow \quad y(1 + x^2) = \tan^{-1} x + c \quad \text{...(iv)} \quad \left[\because \int \frac{1}{1+x^2} dx = \tan^{-1} x + c\right]$

Now, putting $y = 0$ and $x = 1$, we get

$$0 = \tan^{-1}1 + c \quad \Rightarrow \quad c = \frac{-\pi}{4}$$

Putting $c = \frac{-\pi}{4}$ in Eq. (iv), we get the required solution as

$$y(1 + x^2) = \tan^{-1}x - \frac{\pi}{4}$$

or $$y = \frac{\tan^{-1}x}{1+x^2} - \frac{\pi}{4(1+x^2)}.$$

S45. The given differential equation can be written as

$$\sec^2 y\frac{dy}{dx} + x\frac{\sin 2y}{\cos^2 y} = x^3 \Rightarrow \sec^2 y\frac{dy}{dx} + 2x\tan y = x^3$$

Let $\tan y = v$. Then,

$$\sec^2 y\frac{dy}{dx} = \frac{dv}{dx} \quad \text{... (i)}$$

Putting $\tan y = v$ and $\sec^2 y\frac{dy}{dx} = \frac{dv}{dx}$ in (i), we get

$$\therefore \quad \frac{dv}{dx} + (2x)v = x^3$$

This is a linear differential equation of the form

$$\frac{dv}{dx} + Pv = Q, \text{ where } P = 2x \text{ and } Q = x^3.$$

Now solution of above equation is given by $v \times \text{I.F.} = \int (Q \times \text{I.F.})\,dx + C$ ... (ii)

where I.F = integrating factor and I.F = $e^{\int p dx}$

$$\therefore \quad \text{I.F.} = e^{\int 2x\,dx} = e^{x^2}$$

Putting I.F. and Q in Eqs (ii).

$$ve^{x^2} = \int x^3 e^{x^2} dx + C$$

$$\Rightarrow \quad ve^{x^2} = \frac{1}{2}\int te^t dt + C, \text{ where } t = x^2$$

$$\Rightarrow \quad ve^{x^2} = \frac{1}{2}(t-1)e^t + C$$

$$\Rightarrow \quad e^{x^2}\tan y = \frac{1}{2}(x^2-1)e^{x^2} + C,$$ which gives the required solution.

S46. The given differential equation is

$$x^2 \frac{dy}{dx} = y^2 + 2xy$$

This is a homogeneous differential equation as degree of each term is same in the equation.

Above equation can be written as

$$\frac{dy}{dx} = \frac{y^2 + 2xy}{x^2} \quad \text{... (i)}$$

Put $\quad y = vx \quad$ ... (ii)

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{... (iii)}$$

From Eqs. (i), (ii), and (iii), we get

$$v + x\frac{dv}{dx} = \frac{v^2x^2 + 2vx^2}{x^2} = v^2 + 2v$$

$$\Rightarrow \quad x\frac{dv}{dx} = v^2 + v \Rightarrow \frac{dv}{v^2 + v} = \frac{dx}{x}$$

Integrating both sides, we get

$$\int \frac{dv}{v^2 + v} = \int \frac{dx}{x}$$

$$\Rightarrow \quad \int \frac{dv}{v^2 + v + \frac{1}{4} - \frac{1}{4}} = \int \frac{dx}{x} \quad \Rightarrow \int \frac{dv}{\left(v + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2} = \int \frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{2 \times \frac{1}{2}} \log\left|\frac{v + \frac{1}{2} - \frac{1}{2}}{v + \frac{1}{2} + \frac{1}{2}}\right| = \log|x| + c \quad \left[\because \int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|\right]$$

$$\Rightarrow \quad \log\left|\frac{v}{v+1}\right| - \log|x| = c$$

$$\Rightarrow \quad \log\left|\frac{v}{(v+1)\cdot x}\right| = c \quad \left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

$$\Rightarrow \quad \log\left|\frac{\frac{y}{x}}{\left(\frac{y}{x} + 1\right)x}\right| = c \quad \left[\because y = vx \quad \therefore v = \frac{y}{x}\right]$$

$$\Rightarrow \quad \log\left|\frac{y}{xy + x^2}\right| = c.$$

S47. Given differential equation is

$$x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right)$$

This is a homogeneous differential equation. Above equation can be written as

$$\frac{dy}{dx} = \frac{y - x\tan\left(\frac{y}{x}\right)}{x} \qquad \text{... (i)}$$

Put $y = vx$ ... (ii)

$$\therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \text{... (iii)}$$

From Eqs. (i), (ii) and (iii), we get

$$v + x\frac{dv}{dx} = \frac{vx - x\tan v}{x} = v - \tan v$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\tan v$$

$$\Rightarrow \quad \frac{dv}{\tan v} = -\frac{dx}{x}$$

$$\Rightarrow \quad \cot v\, dv = -\frac{dx}{x}$$

Integrating both sides, we get

$$\int \cot v\, dv = -\int \frac{dx}{x}$$

$$\Rightarrow \quad \log|\sin v| = -\log|x| + c \qquad \left[\because \int \cot v\, dv = \log|\sin v| + c\right]$$

$$\Rightarrow \quad \log|\sin v| + \log|x| = c$$

or $\log|x\sin v| = c$ $[\because \log m + \log n = \log mn]$

Required solution is

$$\log\left|x\sin\frac{y}{x}\right| = c \qquad \left[\because v = \frac{y}{x}\right]$$

S48. Given differential equation is

$$xy\log\left(\frac{y}{x}\right)dx + \left[y^2 - x^2\log\left(\frac{y}{x}\right)\right]dy = 0$$

Above equation is a homogenous differential equation. This equation can be written as

$$xy\log\left(\frac{y}{x}\right)dx = \left[x^2\log\left(\frac{y}{x}\right) - y^2\right]dy$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{xy\log\left(\frac{y}{x}\right)}{x^2\log\left(\frac{y}{x}\right)-y^2} \qquad \dots \text{(i)}$$

Now, put $\qquad y = vx \qquad \dots$ (ii)

$$\Rightarrow \qquad \frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \dots \text{(iii)}$$

From Eqs. (i), (ii) and (iii), we get

$$v + x\frac{dv}{dx} = \frac{vx^2\log\left(\frac{vx}{x}\right)}{x^2\log\left(\frac{vx}{x}\right)-v^2x^2} = \frac{v\log v}{\log v - v^2} \qquad \Rightarrow x\frac{dv}{dx} = \frac{v\log v}{\log v - v^2} - v$$

$$\Rightarrow \qquad x\frac{dv}{dx} = \frac{v\log v - v\log v + v^3}{\log v - v^2}$$

$$= \frac{v^3}{\log v - v^2} \qquad \Rightarrow \quad \frac{\log v - v^2}{v^3}\text{dv} = \frac{dx}{x}$$

Integrating both sides, we get

$$\int\frac{\log v - v^2}{v^3}dv = \int\frac{dx}{x}$$

$$\Rightarrow \qquad \int\frac{\log v}{v^3}dv - \int\frac{1}{v}dv = \int\frac{dx}{x}$$

$$\Rightarrow \qquad \int \underset{\text{I}}{v^{-3}} \underset{\text{II}}{\log v}\, dv - \log|v| = \log|x| + c$$

Using integration by parts, we get

$$\log v\int v^{-3}dv - \int\left[\frac{d}{dv}(\log v)\cdot\int v^{-3}dv\right]dv = \log|v| + \log|x| + c$$

$$\Rightarrow \qquad \frac{v^{-2}}{-2}\log v - \int\frac{1v^{-2}}{(-2)v}dv = \log|v| + \log|x| + c$$

$$\Rightarrow \qquad \frac{-1}{2v^2}\log v + \frac{1}{2}\int v^{-3}dv = \log|v| + \log|x| + c$$

$$\Rightarrow \qquad \frac{-1}{2v^2}\log v + \frac{1}{2}\cdot\frac{v^{-2}}{(-2)} = \log|v| + \log|x| + c \qquad \left[\because \int x^n dx = \frac{x^{n+1}}{n+1} + c\right]$$

$\Rightarrow \quad \frac{-1}{2v^2}\log v - \frac{1}{4v^2} = \log|v| + \log|x| + c$

$\Rightarrow \quad \frac{-1}{2v^2}\log v - \frac{1}{4v^2} = \log|vx| + c \qquad [\because \log m + \log n = \log mn]$

$\Rightarrow \because \quad y = vx \Rightarrow v = \frac{y}{x}$

So, we get $\quad \frac{-1}{2}\cdot\frac{x^2}{y^2}\log\left(\frac{y}{x}\right) - \frac{1}{4}\cdot\frac{x^2}{y^2} = \log\left|\frac{y}{x}\cdot x\right| + c$

$\Rightarrow \quad \frac{-x^2}{2y^2}\log\left(\frac{y}{x}\right) - \frac{x^2}{4y^2} = \log|y| + c$

$\Rightarrow \quad \frac{-x^2}{y^2}\left[\frac{\log\left(\frac{y}{x}\right)}{2} + \frac{1}{4}\right] = \log|y| + c$

$\Rightarrow \quad \frac{x^2}{y^2}\left[2\log\left(\frac{y}{x}\right) + 1\right] + 4\log|y| = -4c$

Hence, the required solution is

$$x^2\left[2\log\left(\frac{y}{x}\right) + 1\right] + 4y^2\log|y| = 4y^2k \qquad [\text{Where, } k = -c]$$

S49. Given differential equation is

$$(x - y)\frac{dy}{dx} = x + 2y$$

$\Rightarrow \quad \frac{dy}{dx} = \frac{x + 2y}{x - y} \qquad \dots \text{(i)}$

put $\quad y = vx \qquad \dots \text{(ii)}$

$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \dots \text{(iii)}$

From Eqs. (i), (ii) and (iii), we get

$$v + x\frac{dv}{dx} = \frac{x + 2vx}{x - vx} = \frac{1 + 2v}{1 - v}$$

$\Rightarrow \quad x\frac{dv}{dx} = \frac{1 + 2v}{1 - v} - v$

$$= \frac{1 + 2v - v + v^2}{1 - v}$$

$\Rightarrow \quad x\frac{dv}{dx} = \frac{1 + v + v^2}{1 - v}$

$\Rightarrow \quad \dfrac{1-v}{v^2+v+1}dv = \dfrac{dx}{x}$

Integrating both sides, we get

$$\int \frac{1-v}{v^2+v+1}dv = \int \frac{dx}{x}$$

or $\quad I = \log|x| + c \quad \ldots$ (iv)

where, $\quad I = \displaystyle\int \frac{1-v}{v^2+v+1}dv$

Let $\quad 1 - v = A \cdot \dfrac{d}{dv}(v^2+v+1) + B$

$\Rightarrow \quad 1 - v = A(2v + 1) + B$

Compare coefficient of v and constant term on both sides.

Now, $\quad 2A = -1 \quad \Rightarrow A = -\dfrac{1}{2}$

and $\quad A + B = 1$

or $\quad B = 1 + \dfrac{1}{2} \quad \Rightarrow \quad B = \dfrac{3}{2}$

$\therefore \quad B = \dfrac{3}{2}$

So, we write $\quad 1 - v = -\dfrac{1}{2}(2v+1) + \dfrac{3}{2}$

$\therefore$ We get, $\quad I = \displaystyle\int \frac{-\frac{1}{2}(2v+1)+\frac{3}{2}}{v^2+v+1}dv$

$\Rightarrow \quad I = -\dfrac{1}{2}\displaystyle\int \frac{2v+1}{v^2+v+1}dv + \frac{3}{2}\int \frac{dv}{v^2+v+1}$

$\Rightarrow \quad I = -\dfrac{1}{2}\log|v^2+v+1| + \dfrac{3}{2}\displaystyle\int \frac{dv}{v^2+v+1+\frac{1}{4}-\frac{1}{4}}$

$\Rightarrow \quad I = -\dfrac{1}{2}\log|v^2+v+1| + \dfrac{3}{2}\displaystyle\int \frac{dv}{\left(v+\frac{1}{2}\right)^2+\frac{3}{4}}$

$\Rightarrow \quad I = -\dfrac{1}{2}\log|v^2+v+1| + \dfrac{3}{2}\displaystyle\int \frac{dv}{\left(v+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$

$$\Rightarrow \quad I = -\frac{1}{2}\log|v^2+v+1| + \frac{3}{2}\times\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{v+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+c \qquad \left[\because \int\frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}+c\right]$$

$$\Rightarrow \quad I = -\frac{1}{2}\log|v^2+v+1| + \frac{3}{\sqrt{3}}\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right)+c$$

$\therefore$ We get,
$$I = -\frac{1}{2}\log\left|\frac{y^2}{x^2}+\frac{y}{x}+1\right| + \sqrt{3}\tan^{-1}\left(\frac{\frac{2y}{x}+1}{\sqrt{3}}\right)+c \qquad \left[\because y = vx, \therefore v = \frac{y}{x}\right]$$

or
$$I = -\frac{1}{2}\log\left|\frac{y^2+xy+x^2}{x^2}\right| + \sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)+c$$

Putting value of I in Eq. (iv), we get

$-\frac{1}{2}\log\left|\frac{y^2+xy+x^2}{x^2}\right| + \sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right) = \log|x| + c$ is the required solution.

S50. Given differential equation is

$$\left(x\cos\frac{y}{x}+y\sin\frac{y}{x}\right)\cdot y - \left(y\sin\frac{y}{x}-x\cos\frac{y}{x}\right)\cdot x\frac{dy}{dx} = 0$$

This is a homogeneous differential equation. This equation can be written as

$$\left(x\cos\frac{y}{x}+y\sin\frac{y}{x}\right)\cdot y = \left(y\sin\frac{y}{x}-x\cos\frac{y}{x}\right)\cdot x\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\left[x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right]\cdot y}{\left(y\sin\frac{y}{x}-x\cos\frac{y}{x}\right)\cdot x} \qquad \text{... (i)}$$

Put $\quad y = vx \qquad$ ... (ii)

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \text{... (iii)}$$

From Eqs. (i), (ii) and (iii), we get

$$v + x\frac{dv}{dx} = \frac{(x\cos v + vx\sin v)\cdot vx}{(vx\sin v - x\cos v)\cdot x}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v}{v\sin v - \cos v}$$

$$\Rightarrow \qquad x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v}{v\sin v - \cos v} - v$$

$$\Rightarrow \qquad x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v - v^2\sin v + v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow \qquad x\frac{dv}{dx} = \frac{2v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow \qquad \frac{v\sin v - \cos v}{2v\cos v}dv = \frac{dx}{x}$$

Integrating both sides, we get

$$\int\frac{v\sin v - \cos v}{2v\cos v}dv = \int\frac{dx}{x}$$

$$\Rightarrow \qquad \int\left(\frac{v\sin v}{v\cos v} - \frac{\cos v}{v\cos v}\right)dv = 2\int\frac{dx}{x}$$

$$\Rightarrow \qquad \int\left(\tan v - \frac{1}{v}\right)dv = 2\int\frac{dx}{x}$$

$$\Rightarrow \qquad \log|\sec v| - \log|v| = 2\log|x| + c \qquad \left[\because \int\tan v\,dv = \log|\sec v|, \int\frac{1}{x}dx = \log|x|\right]$$

$$\Rightarrow \qquad \log|\sec v| - \log|v| - 2\log|x| = c$$

$$\Rightarrow \qquad \log|\sec v| - [\log|v| + \log|x|^2] = c \qquad [\because \log m^n = n\log m]$$

$$\log|\sec v| - \log|vx^2| = c \qquad [\because \log m + \log n = \log mn]$$

$$\Rightarrow \qquad \log\left|\frac{\sec v}{vx^2}\right| = c \qquad \left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

$$\Rightarrow \qquad \log\left|\frac{\sec\frac{y}{x}}{\frac{y}{x}\cdot x^2}\right| = c \qquad \left[\because y = vx \Rightarrow v = \frac{y}{x}\right]$$

$$\Rightarrow \qquad \log\left|\frac{\sec\frac{y}{x}}{xy}\right| = c \text{ is the required solution.}$$

S51. Given equation is

$$y\,dx + x\log\left(\frac{y}{x}\right)dy - 2x\,dy = 0$$

First, we find $\frac{dy}{dx}$ from above equation

$$\therefore \quad y\,dx = \left[2x - x\log\left(\frac{y}{x}\right)\right]dy$$

or
$$\frac{dy}{dx} = \frac{y}{2x - x\log\left(\frac{y}{x}\right)} \qquad \text{... (i)}$$

Given differential equation is homogeneous.

Now, put
$$y = vx \qquad \text{... (ii)}$$

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \text{... (iii)}$$

From Eqs. (i), (ii) and (iii), we get

$$v + x\frac{dv}{dx} = \frac{vx}{2x - x\log\left(\frac{vx}{x}\right)}$$

$$= \frac{v}{2 - \log v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v}{2-\log v} - v$$

$$= \frac{v - 2v + v\log v}{2 - \log v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{-v + v\log v}{2 - \log v}$$

$$\Rightarrow \quad \frac{2 - \log v}{v\log v - v}dv = \frac{dx}{x}$$

Integrating both sides, we get

$$\int \frac{2 - \log v}{v(\log v - 1)}dv = \int \frac{dx}{x}$$

Put
$$\log v = t$$

$$\Rightarrow \quad \frac{1}{v}dv = dt$$

$$\Rightarrow \quad \int \frac{2-t}{t-1}dt = \log|x| + c$$

$$\Rightarrow \quad \int \frac{1-(t-1)}{t-1}\,dt = \log |x| + c$$

$$\Rightarrow \quad \int \left(\frac{1}{t-1} - 1\right) dt = \log |x| + c$$

$$\Rightarrow \quad \log |t-1| - t = \log |x| + c$$

$$\Rightarrow \quad \log |\log v - 1| - \log v = \log |x| + c$$

$$\Rightarrow \quad \log \left|\frac{\log v - 1}{v}\right| = \log |x| + c \qquad \left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

$$\Rightarrow \quad \log \left|\frac{\log v - 1}{v}\right| - \log |x| = c$$

$$\Rightarrow \quad \log \left|\frac{\log v - 1}{vx}\right| = c$$

$\Rightarrow \log \left|\dfrac{\log \frac{y}{x} - 1}{y}\right| = c$ is the required solution.

S52. Given differential equation is

$$\frac{dy}{dx} - \frac{y}{x} + \operatorname{cosec}\left(\frac{y}{x}\right) = 0$$

Above equation can be written as

$$\frac{dy}{dx} = \frac{y}{x} - \operatorname{cosec}\left(\frac{y}{x}\right) \qquad \text{... (i)}$$

This is a homogeneous differential equation, because each term have same degree.

So, put $\quad y = vx \qquad$ ... (ii)

$$\Rightarrow \quad \frac{dy}{dx} = v + x\,\frac{dv}{dx} \qquad \text{... (iii)}$$

From Eqs. (i), (ii) and (iii), we get ... (iv)

$$v + x\frac{dv}{dx} = \frac{vx}{x} - \operatorname{cosec}\left(\frac{vx}{x}\right)$$

$$\Rightarrow \quad v + x\,\frac{dv}{dx} = v - \operatorname{cosec}(v)$$

$$\Rightarrow \quad x\,\frac{dv}{dx} = -\operatorname{cosec}(v)$$

$\Rightarrow \qquad \dfrac{dv}{\text{cosec}\,(v)} = \dfrac{-dx}{x}$

Integrating both sides, we get

$$\int \frac{dv}{\text{cosec}\,(v)} = \int -\frac{dx}{x}$$

$\Rightarrow \qquad \int \sin(v)\,dv = \int -\dfrac{dx}{x}$

$\Rightarrow \qquad -\cos(v) = -\log|x| + c \qquad \left[\because \int \sin x\,dx = -\cos x + c \text{ and } \int \dfrac{1}{x}dx = \log|x| + c\right]$

$\therefore$ Required solution is given by

$$-\cos\left(\frac{y}{x}\right) = -\log|x| + c \qquad \left[\text{put } v = \frac{y}{x}\right]$$

$\Rightarrow \qquad \cos\left(\dfrac{y}{x}\right) = (\log|x| - c)$

$\Rightarrow \qquad \dfrac{y}{x} = \cos^{-1}(\log|x| - c)$

$\Rightarrow \qquad y = x\cos^{-1}(\log|x| - c)$ is the required solution.

S53. Given differential equation is

$$\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx + x\,dy = 0$$

Above equation is a homogeneous differential equation.

This equation can be written as

$$\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx = -x\,dy$$

or $\qquad \dfrac{dy}{dx} = \dfrac{y - x\sin^2\left(\dfrac{y}{x}\right)}{x} \qquad \ldots \text{(i)}$

Now, put $y = vx \qquad \ldots \text{(ii)}$

$\Rightarrow \qquad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx} \qquad \ldots \text{(iii)}$

From Eqs. (i), (ii) and (iii), we get

$$v + x\frac{dv}{dx} = \frac{vx - x\sin^2\left(\frac{vx}{x}\right)}{x}$$

$$= v - \sin^2 v$$

$\Rightarrow$ $$x\frac{dv}{dx} = -\sin^2 v$$

$\Rightarrow$ $$\frac{dv}{\sin^2 v} = -\frac{dx}{x}$$

Integrating both sides, we get $\int \frac{dv}{\sin^2 v} = -\int \frac{dx}{x}$

$\Rightarrow$ $$\int \text{cosec}^2 v\, dv = -\int \frac{dx}{x}$$

$\Rightarrow$ $$-\cot v = -\log x + c$$

$\Rightarrow$ $$-\cot\left(\frac{y}{x}\right) = -\log x + c \qquad \left[\because y = vx \Rightarrow v = \frac{y}{x}\right]$$

$\Rightarrow$ $$\cot\left(\frac{y}{x}\right) = \log x - c$$

$\Rightarrow$ $$\frac{y}{x} = \cot^{-1}(\log x - c)$$

Hence, $y = x\,.\cot^{-1}(\log x - c)$ is the required solution.

S54. Given differential equation is

$$xdy - ydx = \sqrt{x^2 + y^2}\,dx$$

Above equation may be written as

$$\sqrt{x^2 + y^2}\,dx + ydx = xdy$$

$\Rightarrow$ $$(y + \sqrt{x^2 + y^2})dx = xdy$$

or $$\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x} \qquad \dots \text{(i)}$$

Above equation is a homogeneous differential equation, because each term have same degree.

So, put $$y = vx \qquad \dots \text{(ii)}$$

$\Rightarrow$ $$\frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \dots \text{(iii)}$$

From Eqs. (i), (ii) and (iii), we get

$$v + x\frac{dv}{dx} = \frac{vx + \sqrt{x^2 + v^2x^2}}{x}$$

$$= \frac{vx + x\sqrt{1+v^2}}{x}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = v + \sqrt{1+v^2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \sqrt{1+v^2}$$

$$\Rightarrow \quad \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$$

Integrating both sides, we get

$$\int\frac{dv}{\sqrt{1+v^2}} = \int\frac{dx}{x}$$

$$\Rightarrow \quad \log\left|v + \sqrt{1+v^2}\right| = \log|x| + c \left[\because \int\frac{dx}{\sqrt{a^2+x^2}} = \log\left|x + \sqrt{x^2+a^2}\right| \text{and} \int\frac{dx}{x} = \log|x| + c\right]$$

$$\Rightarrow \quad \log\left|\frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}}\right| = \log|x| + c \qquad \left[\because y = vx \Rightarrow v = \frac{y}{x}\right]$$

$$\Rightarrow \quad \log\frac{\left|\frac{y+\sqrt{x^2+y^2}}{x}\right|}{|x|} = c \qquad \left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$$

$$\Rightarrow \quad \frac{y+\sqrt{x^2+y^2}}{x^2} = e^c \qquad \left[\because \textit{if } \log y = x, \text{ then } y = e^x\right]$$

$$\Rightarrow \quad y + \sqrt{x^2+y^2} = x^2 \cdot e^c$$

$$\Rightarrow \quad y + \sqrt{x^2+y^2} = Ax^2 \qquad [\text{Where, } A = e^c]$$

S55. $(xe^{y/x} + y)dx = xdy$

$$\Rightarrow \quad \frac{dy}{dx} = e^{y/x} + \frac{y}{x}$$

This is a homogeneous differential equation.

Putting $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ it reduces to

$$v + x\frac{dv}{dx} = e^v + v$$

$\Rightarrow$ $$x\frac{dv}{dx} = e^v$$

$\Rightarrow$ $$e^{-v}\,dv = \frac{dx}{x}, \text{ if } x \neq 0.$$

$\Rightarrow$ $$\int e^{-v}dv = \int\frac{1}{x}dx$$

$\Rightarrow$ $$-e^{-v} = \log|x| + C$$

$\Rightarrow$ $$-e^{-y/x} = \log|x| + C \quad \ldots \text{(i)}$$

It is given that $y(1) = 1$ *i.e.*, when $x = 1$, $y = 1$

Putting $x = 1$, $y = 1$ in Eq. (i), we get

$$-e^{-1} = C$$

Putting $C = -\frac{1}{e}$ in (i), we get

$$-e^{-y/x} = \log|x| - \frac{1}{e}$$

$\Rightarrow$ $$e^{-y/x} = \frac{1}{e} - \log|x| \Rightarrow e^{-y/x} = \frac{1 - e\log|x|}{e}$$

$\Rightarrow$ $$\frac{-y}{x} = \log\left(\frac{1 - e\log|x|}{e}\right)$$

$\Rightarrow$ $$-\frac{y}{x} = \log(1 - e\log|x|) - 1$$

$\Rightarrow$ $$y = x - x\log(1 - e\log|x|)$$ as the required solution.

S56. Given differential equation is

$$2x^2\frac{dy}{dx} - 2xy + y^2 = 0$$

$\Rightarrow$ $$2x^2\frac{dy}{dx} = 2xy - y^2$$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{2xy - y^2}{2x^2} \quad \ldots \text{(i)}$$

Given differential equation is a homogeneous equation.

So, put $$y = vx \quad \ldots \text{(ii)}$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad \ldots \text{(iii)}$$

From Eqs. (i), (ii) and (iii), we have

$$v + x\frac{dv}{dx} = \frac{2vx^2 - v^2x^2}{2x^2} = \frac{2v - v^2}{2}$$

$$\therefore \quad x\frac{dv}{dx} = \frac{2v - v^2}{2} - v = \frac{2v - v^2 - 2v}{2}$$

$$x\frac{dv}{dx} = \frac{-v^2}{2} \Rightarrow \frac{2dv}{v^2} = -\frac{1}{x}dx$$

Integrating both sides, we get

$$\int\frac{2dv}{v^2} = \int\frac{-dx}{x}$$

$$\Rightarrow \quad 2\int v^{-2}dv = -\log|x| + c$$

$$\Rightarrow \quad \frac{2v^{-1}}{-1} = -\log|x| + c$$

$$\Rightarrow \quad \frac{-2}{v} = -\log|x| + c$$

$$\Rightarrow \quad \frac{-2x}{y} = -\log|x| + c \qquad \left[\because y = vx, \therefore v = \frac{y}{x}\right]$$

$$\Rightarrow \quad -2x = y(-\log|x| + c)$$

$$\therefore \quad y = \frac{-2x}{-\log|x| + c}.$$

S57. Let r be the radius and V be the volume of the balloon. Then, $V = \frac{4}{3}\pi r^3$

It is given that

$$\frac{dV}{dt} = \lambda, \quad \text{where} \quad \lambda > 0$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = \lambda$$

$$\Rightarrow \quad 4\pi r^2\frac{dr}{dt} = \lambda$$

$$\Rightarrow \quad 4\pi r^2 dr = \lambda dt$$

On integrating, we get

$$\int 4\pi r^2 dr = \int \lambda\, dt$$

$$\frac{4}{3}\pi r^3 = \lambda t + C \quad \text{... (i)}$$

At $t = 0,\ r = 3$ [Given]

$\therefore$ $$C = 36\pi$$

Putting $C = 36\pi$ in Eq. (i), we get

$$\frac{4}{3}\pi r^3 = \lambda t + 36\pi \quad \text{... (ii)}$$

At $t = 3,\ r = 6$ (Given)

$\therefore$ $$288\pi = 3\lambda + 36\pi$$

$\Rightarrow$ $$\lambda = 84\pi$$

Putting $\lambda = 84\pi$ in Eq. (ii), we get

$$\frac{4}{3}\pi r^3 = 84\pi t + 36\pi$$

$\Rightarrow$ $$r^3 = 63t + 27 \quad \Rightarrow \quad r = (63t + 27)^{1/3}$$

S58. Given differential equation is

$$x(x^2 - 1)\ \frac{dy}{dx} = 1$$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{1}{x(x^2 - 1)}$$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{1}{x(x-1)(x+1)}$$

$\Rightarrow$ $$dy = \frac{dx}{x(x-1)(x+1)}$$

Integrating both sides, we get

$$\int dy = \int \frac{dx}{x(x-1)(x+1)}$$

$\Rightarrow$ $$y = \int \frac{dx}{x(x-1)(x+1)} \quad \text{... (i)}$$

Let, $$I = \int \frac{dx}{x(x-1)(x+1)}$$

Now $$\frac{1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

$\Rightarrow$ $$1 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$$

Put $$x = 0, \quad 1 = -A \quad \Rightarrow \quad A = -1$$

Put $x = 1, \quad 1 = 2B \quad \Rightarrow \quad B = \frac{1}{2}$

Put $x = -1, \quad 1 = 2C \quad \Rightarrow \quad C = \frac{1}{2}$

$\therefore \quad A = -1, \quad B = \frac{1}{2}, \quad C = \frac{1}{2}$

We get,

$$\frac{1}{x(x-1)(x+1)} = \frac{-1}{x} + \frac{1/2}{x-1} + \frac{1/2}{x+1}$$

Integrating both sides w.r.t. x, we get

$$\int \frac{1}{x(x-1)(x+1)}dx = \int \frac{-1}{x}dx + \frac{1}{2}\int \frac{dx}{x-1} + \frac{1}{2}\int \frac{dx}{x+1}$$

$$\therefore \quad I = -\log|x| + \frac{1}{2}\log|x-1| + \frac{1}{2}\log|x+1|$$

Putting the value of I in Eq. (i), we get

$$y = -\log|x| + \frac{1}{2}\log|x-1| + \frac{1}{2}\log|x+1| + c$$

Put $y = 0$ and $x = 2$, we get

$$0 = -\log 2 + \frac{1}{2}\log 1 + \frac{1}{2}\log 3 + c$$

$$0 = -\log 2 + \log\sqrt{3} + c$$

or $\quad c = \log\frac{2}{\sqrt{3}} \qquad [\because \log 1 = 0]$

Hence, the required solution is $y = -\log|x| + \frac{1}{2}\log|x-1| + \frac{1}{2}\log|x+1| + \log\frac{2}{\sqrt{3}}$.

S59. The given differential equation is

$$(x^3 + x^2 + x + 1)\frac{dy}{dx} = (2x^2 + x)$$

This equation can be written as

$$\frac{dy}{dx} = \frac{2x^2 + x}{x^3 + x^2 + x + 1}$$

This is a variable separable type differential equation.

$\therefore$ We get, $\quad dy = \frac{2x^2 + x}{x^3 + x^2 + x + 1}dx$

Integrating both sides, we get

$$\int dy = \int \frac{2x^2+x}{x^3+x^2+x+1}dx$$

$\Rightarrow$ $$y = \int \frac{2x^2+x}{x^2(x+1)+1(x+1)}dx$$

$$y = \int \frac{2x^2+x}{(x+1)(x^2+1)}dx \quad \text{... (i)}$$

Let $$I = \int \frac{2x^2+x}{(x+1)(x^2+1)}dx$$

Using partial fractions, we get

$$\frac{2x^2+x}{(x+1)(x^2+1)} = \frac{A}{x+1}+\frac{Bx+c}{x^2+1}$$

$\Rightarrow$ $$\frac{2x^2+x}{(x+1)(x^2+1)} = \frac{A(x^2+1)+(Bx+C)(x+1)}{(x+1)(x^2+1)}$$

$\Rightarrow$ $2x^2 + x = A(x^2 + 1) + (Bx + C)(x + 1)$

$\Rightarrow$ $2x^2 + x = A(x^2 + 1) + (Bx^2 + Bx + Cx + C)$

Now, comparing coefficients of x^2, x and constant term, we get

$$A + B = 2 \quad \text{... (ii)}$$

$$B + C = 1 \quad \text{... (iii)}$$

$$A + C = 0 \quad \text{... (iv)}$$

Subtracting Eq. (iii) from Eq. (ii), we get

$$A - C = 1 \quad \text{... (v)}$$

Adding Eqs. (iv) and (v), we get

$$2A = 1 \Rightarrow A = \frac{1}{2}$$

Put $A = \frac{1}{2}$ in Eq. (ii), we get

$$\frac{1}{2} + B = 2 \Rightarrow B = 2 - \frac{1}{2} = \frac{3}{2}$$

Put $B = \frac{3}{2}$ in Eq. (iii), we get

$$\frac{3}{2} + C = 1 \Rightarrow C = 1 - \frac{3}{2} = \frac{-1}{2}$$

We get,

$$\frac{2x^2+x}{(x+1)(x^2+1)} = \frac{1/2}{x+1}+\frac{\frac{3}{2}x-\frac{1}{2}}{x^2+1}$$

Integrating both sides, we get

$$\int \frac{2x^2 + x}{(x+1)(x^2+1)} dx = \frac{1}{2}\int \frac{dx}{x+1} + \frac{3}{2}\int \frac{x}{x^2+1} dx - \frac{1}{2}\int \frac{dx}{x^2+1}$$

$$\Rightarrow \quad I = \frac{1}{2} \log|x + 1| + \frac{3}{4} \log|x^2 + 1| - \frac{1}{2} \tan^{-1} x + c$$

$$\left[\because \int \frac{x}{x^2+1} dx \text{ Put } x^2 + 1 = t \Rightarrow 2x dx = dt \therefore x dx = \frac{dt}{2} \Rightarrow \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\log t = \frac{1}{2}\log|x^2+1| + c\right]$$

Putting above value of I in Eq. (i), we get

$$y = \frac{1}{2}\log|x+1| + \frac{3}{4}\log|x^2+1| - \frac{1}{2}\tan^{-1} x + c$$

which is the required solution.

S60. Given differential equation is

$$\frac{dy}{dx} = \frac{x(2y-x)}{x(2y+x)} \quad \text{and} \quad y = 1 \text{ when } x = 1$$

Above equation can be written as

$$\frac{dy}{dx} = \frac{2xy - x^2}{2xy + x^2} \quad \text{..... (i)}$$

This is a homogeneous differential equation, because each term of numerator and denominator have same degree.

$$\text{Put} \quad y = vx \quad \text{... (ii)}$$

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{... (iii)}$$

From Eqs. (i), (ii) and (iii), we get

$$v + x\frac{dv}{dx} = \frac{2vx^2 - x^2}{2vx^2 + x^2} = \frac{2v-1}{2v+1}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{2v-1}{2v+1} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{2v - 1 - 2v^2 - v}{2v+1} = \frac{v - 1 - 2v^2}{2v+1}$$

$$\Rightarrow \quad \frac{2v+1}{2v^2 - v + 1} dv = -\frac{dx}{x}$$

Integrating both sides, we get

$$\int \frac{2v+1}{2v^2 - v + 1} dv = -\int \frac{dx}{x}$$

$$\text{or} \quad I = -\log|x| + c \quad \text{... (iv)}$$

where $$I = \int \frac{2v+1}{2v^2 - v + 1} dv$$

Now, let $$2v + 1 = A \cdot \frac{d}{dv}(2v^2 - v + 1) + B$$

$\Rightarrow$ $$2v + 1 = A(4v - 1) + B \quad \ldots \text{(v)}$$

Comparing coefficient of v and constants term, we get

$$4A = 2 \quad \Rightarrow \quad A = \frac{1}{2}$$

and $$-A + B = 1 \quad \Rightarrow \quad -\frac{1}{2} + B = 1$$

$\therefore$ $$B = \frac{3}{2}$$

$\therefore$ $$2v + 1 = \frac{1}{2}(4v - 1) + \frac{3}{2}$$

$\therefore$ We get,

$$I = \int \frac{\frac{1}{2}(4v-1) + \frac{3}{2}}{2v^2 - v + 1} dv$$

$$I = \frac{1}{2}\int \frac{4v-1}{2v^2 - v + 1} dv + \frac{3}{2}\int \frac{dv}{2v^2 - v + 1}$$

$\Rightarrow$ $$I = \frac{1}{2}\log|2v^2 - v + 1| + \frac{3}{4}\int \frac{dv}{v^2 - \frac{v}{2} + \frac{1}{2}}$$

$$\left[\because \int \frac{4v-1}{2x^2 - v + 1} dv \text{ Put } 2v^2 - v + 1 = t \Rightarrow (4v-1)dv = dt \ \therefore \int \frac{dt}{t} = \log|t| = \log|2v^2 - v + 1|\right]$$

$\Rightarrow$ $$I = \frac{1}{2}\log|2v^2 - v + 1| + \frac{3}{4}\int \frac{dv}{v^2 - \frac{1}{2}v + \frac{1}{2} + \frac{1}{16} - \frac{1}{16}}$$

$$= \frac{1}{2}\log|2v^2 - v + 1| + \frac{3}{4}\int \frac{dv}{\left(v - \frac{1}{4}\right)^2 + \frac{7}{16}}$$

$$= \frac{1}{2}\log|2v^2 - v + 1| + \frac{3}{4}\int \frac{dv}{\left(v - \frac{1}{4}\right)^2 + \left(\frac{\sqrt{7}}{4}\right)^2}$$

$$\therefore \quad I = \frac{1}{2}\log|2v^2 - v + 1| + \frac{3}{4} \times \frac{4}{\sqrt{7}}\tan^{-1}\left(\frac{v - \frac{1}{4}}{\frac{\sqrt{7}}{4}}\right) + c$$

$$\left[\because \int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$I = \frac{1}{2}\log|2v^2 - v + 1| + \frac{3}{\sqrt{7}}\tan^{-1}\left(\frac{4v - 1}{\sqrt{7}}\right)$$

Now, putting value of I in Eq. (iv), we get

$$\Rightarrow \frac{1}{2}\log|2v^2 - v + 1| + \frac{3\sqrt{7}}{7}\tan^{-1}\left(\frac{4v - 1}{\sqrt{7}}\right) = -\log|x| + c$$

$$\Rightarrow \frac{1}{2}\log\left|\frac{2y^2}{x^2} - \frac{y}{x} + 1\right| + \frac{3\sqrt{7}}{7}\tan^{-1}\left(\frac{\frac{4y}{x} - 1}{\sqrt{7}}\right) = -\log|x| + c \quad \left[\because \text{Put } v = \frac{y}{x}\right]$$

$$\Rightarrow \frac{1}{2}\log\left|\frac{2y^2}{x^2} - \frac{y}{x} + 1\right| + \frac{3\sqrt{7}}{7}\tan^{-1}\left(\frac{4y - x}{\sqrt{7} \cdot x}\right) = -\log|x| + c \quad \text{... (vi)}$$

Now, put $x = 1$ and $y = 1$, we get

$$\frac{1}{2}\log|2| + \frac{3\sqrt{7}}{7}\tan^{-1}\left(\frac{3}{\sqrt{7}}\right) = -\log 1 + c$$

$$\Rightarrow \quad \frac{1}{2}\log 2 + \frac{3\sqrt{7}}{7}\tan^{-1}\left(\frac{3}{\sqrt{7}}\right) = c \quad [\because \log 1 = 0]$$

Putting value of c in Eq. (vi), we get the particular solution as

$$\frac{1}{2}\log\left(\frac{2y^2 - xy + x^2}{x^2}\right) + \frac{3\sqrt{7}}{7}\tan^{-1}\left(\frac{4y - x}{\sqrt{7}x}\right) = -\log|x| + \frac{1}{2}\log 2 + \frac{3\sqrt{7}}{7}\tan^{-1}\left(\frac{3}{\sqrt{7}}\right)$$

$$\Rightarrow \quad \frac{1}{2}\log\left(\frac{2y^2 - xy + x^2}{x^2}\right) + \log|x| - \frac{1}{2}\log 2 = \frac{3\sqrt{7}}{7}\left[\tan^{-1}\left(\frac{3}{\sqrt{7}}\right) - \tan^{-1}\left(\frac{4y - x}{\sqrt{7}x}\right)\right]$$

$$\Rightarrow \quad \log\left(\frac{2y^2 - xy + x^2}{x^2}\right)^{1/2} + \log x - \log(2)^{1/2} = \frac{3\sqrt{7}}{7}\left[\tan^{-1}\left\{\frac{\frac{3}{\sqrt{7}} - \left(\frac{4y - x}{\sqrt{7}x}\right)}{1 + \frac{3}{\sqrt{7}} \cdot \left(\frac{4y - x}{\sqrt{7}x}\right)}\right\}\right]$$

$$\Rightarrow \quad \log \frac{x(2y^2 - xy + x^2)^{1/2}}{x} - \log\sqrt{2} = \frac{3\sqrt{7}}{7}\tan^{-1}\frac{\left[\frac{3x-4y+x}{\sqrt{7}x}\right]}{1+\left[\frac{12y-3x}{7x}\right]}$$

$$\Rightarrow \quad \log(2y^2 - xy + x^2)^{1/2} - \log\sqrt{2} = \frac{3\sqrt{7}}{7}\tan^{-1}\left[\frac{(4x-4y).\sqrt{7}}{4x+12y}\right]$$

$$\Rightarrow \quad \log\sqrt{\frac{2y^2 - xy + x^2}{2}} = \frac{3\sqrt{7}}{7}\tan^{-1}\left[\frac{(4x-4y).\sqrt{7}}{4x+12y}\right]$$

Hence, the required particular solution is

$$\log\sqrt{\frac{2y^2 - xy + x^2}{2}} = \frac{3\sqrt{7}}{7}\tan^{-1}\left[\frac{(4x-4y).\sqrt{7}}{4x+12y}\right]$$

S61. Given differential equation is

$$\sqrt{1+x^2+y^2+x^2y^2} + xy\frac{dy}{dx} = 0$$

The above equation can be written as

$$\sqrt{(1+x^2)+y^2(1+x^2)} = -\,xy\,\frac{dy}{dx}$$

$$\Rightarrow \quad \sqrt{(1+x^2)(1+y^2)} = -\,xy\,\frac{dy}{dx}$$

$$\Rightarrow \quad \sqrt{1+x^2}\cdot\sqrt{1+y^2} = -\,xy\,\frac{dy}{dx}$$

or $$\frac{y}{\sqrt{1+y^2}}dy = -\frac{\sqrt{1+x^2}}{x}dx$$

Integrating both sides, we get

$$\int\frac{y}{\sqrt{1+y^2}}dy = -\int\frac{\sqrt{1+x^2}}{x^2}\cdot x\,dx$$

$\Rightarrow$ Put $\quad 1 + y^2 = t \quad$ and $\quad 1 + x^2 = u^2$

$\Rightarrow \quad 2y\,dy = dt \quad$ and $\quad 2x\,dx = 2u\,du$

$\Rightarrow \quad y\,dy = \frac{dt}{2} \quad$ and $\quad x\,dx = u\,du$

$$\therefore \quad \frac{1}{2}\int\frac{dt}{\sqrt{t}} = -\int\frac{u}{u^2-1}\cdot u\,du$$

$$\Rightarrow \quad \frac{1}{2}\int t^{-1/2}dt = -\int\frac{u^2}{u^2-1}du$$

$\Rightarrow \quad t^{1/2} = -\int \frac{(u^2 - 1 + 1)}{u^2 - 1} du$

$\Rightarrow \quad t^{1/2} = -\int \frac{u^2 - 1}{u^2 - 1} du - \int \frac{1}{u^2 - 1} du$

$\Rightarrow \quad \sqrt{1 + y^2} = -\int 1 du - \int \frac{1}{u^2 - (1)^2} du$

$\Rightarrow \quad \sqrt{1 + y^2} = -u - \frac{1}{2}\log\left|\frac{u - 1}{u + 1}\right| + c \qquad \left[\because \int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x - a}{x + a}\right| + c\right]$

$\Rightarrow \quad \sqrt{1 + y^2} = -\sqrt{1 + x^2} - \frac{1}{2}\log\left|\frac{\sqrt{1 + x^2} - 1}{\sqrt{1 + x^2} + 1}\right| + c$ is the required solution.

S62. The given differential equation is

$(x^2 - y^2)\, dx + 2xy\, dy = 0$ and $y = 1$ when $x = 1$

This is a homogeneous differential equation as degree of each term is same.

Above equation can be written as

$(x^2 - y^2)\, dx = -2xy\, dy$

$\Rightarrow \quad \frac{dy}{dx} = \frac{y^2 - x^2}{2xy} \qquad \ldots \text{(i)}$

Now, put $\quad y = vx \qquad \ldots \text{(ii)}$

$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \ldots \text{(iii)}$

From Eqs. (i), (ii) and (iii), we get

$v + x\frac{dv}{dx} = \frac{v^2x^2 - x^2}{2vx^2} = \frac{v^2 - 1}{2v}$

$\Rightarrow \quad x\frac{dv}{dx} = \frac{v^2 - 1}{2v} - v = \frac{v^2 - 1 - 2v^2}{2v} = \frac{-1 - v^2}{2v}$

$\Rightarrow \quad \frac{2v}{v^2 + 1} dv = -\frac{dx}{x}$

Integrating both sides, we get

$\Rightarrow \quad \int \frac{2v}{v^2 + 1} dv = -\int \frac{dx}{x}$

$\Rightarrow \quad \log |v^2 + 1| = -\log |x| + c$

$\Rightarrow \quad \log |v^2 + 1| + \log |x| = c$

$\Rightarrow \quad \log\left|\frac{y^2}{x^2} + 1\right| + \log |x| = c \qquad \ldots \text{(iv)}$

Now, put $x = 1$ and $y = 1$, in Eq. (iv), we get

$$\log 2 + \log 1 = c$$

$\therefore$ $c = \log 2$ $[\because \log 1 = 0]$

Putting $c = \log 2$ in Eq. (iv), we get the required solution as

$$\log\left|\frac{y^2 + x^2}{x^2}\right| + \log x = \log 2$$

$$\Rightarrow \quad \log\left|x\left(\frac{x^2 + y^2}{x^2}\right)\right| = \log 2 \qquad [\because \log m + \log n = \log mn]$$

$$\Rightarrow \quad \log\left|\frac{x^2 + y^2}{x}\right| = \log 2$$

$$\Rightarrow \quad x^2 + y^2 = 2x.$$

S63. The given differential equation is

$$x\frac{dy}{dx} + y - x + xy\cot x = 0, \; x \neq 0$$

Above equation can be written as

$$x\frac{dy}{dx} + y(1 + x\cot x) = x$$

Dividing both sides by x, we get

$$\frac{dy}{dx} + y\left(\frac{1 + x\cot x}{x}\right) = 1$$

$$\Rightarrow \quad \frac{dy}{dx} + y\left(\frac{1}{x} + \cot x\right) = 1 \qquad \text{... (i)}$$

This equation is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q \qquad \text{... (ii)}$$

Comparing Eqs. (i) and (ii), we get

$$P = \left(\frac{1}{x} + \cot x\right) \quad \text{and} \quad Q = 1$$

Now, solution of the above equation is given by

$$y \times \text{I.F.} = \int (Q \times \text{I.F.})\, dx + c \qquad \text{... (iii)}$$

where, I.F. = Integrating factor and I.F. = $e^{\int P dx}$

$$\therefore \quad \text{I.F.} = e^{\int P dx} = e^{\int\left(\frac{1}{x} + \cot x\right)dx} = e^{\log|x| + \log \sin x} = e^{\log|x + \sin x|}$$

$$\left[\because \int \frac{1}{x} dx = \log|x| \text{ and } \int \cot x\, dx = \log|\sin x|\right]$$

$\therefore$ I.F. = $x \sin x$

Putting I.F. = $x \sin x$ and $Q = 1$ in Eq. (iii), we get

$$y x \sin x = \int 1 \cdot x \sin x\, dx$$

$$\Rightarrow \quad y x \sin x = \int \underset{\text{I}}{x} \underset{\text{II}}{\sin x}\, dx$$

$$\Rightarrow \quad yx \sin x = x \int \sin x\, dx - \int \left(\frac{d}{dx}(x) \cdot \int \sin x\, dx \right) dx$$

[By using integration by parts in $\int x \sin x\, dx$]

$$\Rightarrow \quad y x \sin x = -x \cos x + \int 1 . \cos x\, dx$$

$$\Rightarrow \quad yx \sin x = -x \cos x + \int \cos x\, dx$$

$$\Rightarrow \quad yx \sin x = -x \cos x + \sin x + c$$

Dividing both sides by $x \sin x$, we get

$$y = \frac{-x \cos x + \sin x + c}{x \sin x}$$

or $$y = -\cot x + \frac{1}{x} + \frac{c}{x \sin x}$$

is the required solution.

S64. Given differential equation is

$$\sec x \frac{dy}{dx} - y = \sin x$$

Dividing both sides by $\sec x$, we get

$$\frac{dy}{dx} - \frac{y}{\sec x} = \frac{\sin x}{\sec x}$$

or $$\frac{dy}{dx} - y \cos x = \sin x \cos x \qquad \text{... (i)}$$

This is a linear differential equation and is of the form

$$\frac{dy}{dx} + Py = Q \qquad \text{... (ii)}$$

Comparing Eqs. (i) and (ii), we get

$$P = -\cos x \quad \text{and} \quad Q = \sin x \cos x$$

Now, solution of above equation is given by

$$y \times \text{I.F.} = \int (Q \times \text{IF})dx + c \qquad \text{... (iii)}$$

where, I.F. = Integrating factor and I.F. = $e^{\int P dx}$

$\therefore$ $$\text{I.F.} = e^{\int -\cos x\, dx} = e^{-\sin x} \qquad \left[\because \int \cos x dx = \sin x + c\right]$$

Putting I.F. = $e^{-\sin x}$ and $Q = \sin x \cos x$ in Eq. (iii), we get

$$ye^{-\sin x} = \int \sin x \cos x e^{-\sin x} dx$$

Put $\sin x = t \Rightarrow \cos x\, dx = dt$

$$ye^{-\sin x} = \int \underset{\text{I}}{t}\, \underset{\text{II}}{e^{-t}}\, dt$$

$$ye^{-\sin x} = t\int e^{-t} dt - \int \left[\frac{d}{dt}(t) \int e^{-t} dt\right] dt \qquad \text{[Using integration by parts]}$$

$\Rightarrow$ $$ye^{-\sin x} = -te^{-t} + \int 1 \times e^{-t} dt$$

$$= -te^{-t} + \int e^{-t} dt$$

$\Rightarrow$ $$ye^{-\sin x} = -te^{-t} + e^{-t} + c$$

$\Rightarrow$ $$ye^{-\sin x} = -\sin x\, e^{-\sin x} + e^{-\sin x} + c \qquad [\because \sin x = t]$$

Hence, the required solution is

$$y = -\sin x + 1 + ce^{\sin x}$$

S65. Given differential equation is

$$x^2 dy + (xy + y^2)\, dx = 0$$

Since, degree of each term is same, so the above equation is a homogenous equation. This equation can be written as

$$x^2 dy = -(xy + y^2)\, dx$$

$\therefore$ $$\frac{dy}{dx} = \frac{-(xy + y^2)}{x^2} \qquad \text{... (i)}$$

Put $$y = vx \qquad \text{... (ii)}$$

$\therefore$ $$\frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \text{... (iii)}$$

From Eqs. (i), (ii) and (iii), we get

$$v + x\frac{dv}{dx} = \frac{-(vx^2 + v^2x^2)}{x^2} = -v - v^2$$

$\Rightarrow$ $$x\frac{dv}{dx} = -v^2 - 2v$$

$$\Rightarrow \quad \frac{dv}{v^2+2v} = \frac{-dx}{x}$$

Integrating both sides, we get

$$\int\frac{dv}{v^2+2v} = -\int\frac{dx}{x} \quad \Rightarrow \quad \int\frac{dv}{v^2+2v+1-1} = -\int\frac{dx}{x}$$

$$\Rightarrow \quad \int\frac{dv}{(v+1)^2-(1)^2} = -\int\frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{2}\log\left|\frac{v+1-1}{v+1+1}\right| = -\log|x| + c \qquad \left[\because \int\frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c\right]$$

$$\Rightarrow \quad \frac{1}{2}\log\left|\frac{v}{v+2}\right| = -\log|x| + c$$

$$\Rightarrow \quad \frac{1}{2}\log\left|\frac{\frac{y}{x}}{\frac{y}{x}+2}\right| = -\log|x| + c \qquad \ldots \text{(iv)}$$

Now, put $x = y = 1$ in Eqs. (iv)

$$\therefore \quad \frac{1}{2}\log\left|\frac{1}{1+2}\right| = -\log 1 + c \quad \Rightarrow \quad \frac{1}{2}\log\left|\frac{1}{3}\right| = -\log 1 + c$$

$$\Rightarrow \quad c = \frac{1}{2}\log\frac{1}{3}$$

Putting value of c in Eq. (iv), we get

$$\frac{1}{2}\log\left|\frac{y}{y+2x}\right| = -\log|x| + \frac{1}{2}\log\frac{1}{3}$$

$$\Rightarrow \quad \log\left|\frac{y}{y+2x}\right| = -2\log|x| + \log\frac{1}{3}$$

$$\Rightarrow \quad \log\frac{y}{y+2x} = \log x^{-2} + \log\frac{1}{3} \qquad [\because n\log m = \log m^n]$$

$$\Rightarrow \quad \frac{y}{y+2x} = \frac{1}{3x^2}$$

$$\Rightarrow \quad y + 2x = 3x^2y$$

$$\Rightarrow \quad y - 3x^2y = -2x$$

$$\Rightarrow \quad y(1-3x^2) = -2x$$

or $y = \dfrac{2x}{3x^2-1}$ is the required particular solution.

S66. We have,

$$x\frac{dy}{dx}\sin\left(\frac{y}{x}\right)+x-y\sin\left(\frac{y}{x}\right) = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x-y\sin\left(\frac{y}{x}\right)}{x\sin\left(\frac{y}{x}\right)}$$

This is a homogeneous differential equation.

Putting $y = vx$ and $\frac{dy}{dx}=v+x\frac{dv}{dx}$, it reduces to

$$v+x\frac{dv}{dx} = \frac{x-vx\sin v}{x\sin v}=-\frac{1-v\sin v}{\sin v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\frac{1-v\sin v}{\sin v}-v = \frac{-1+v\sin v-v\sin v}{\sin v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\frac{1}{\sin v}$$

$$\Rightarrow \quad \sin v\, dv = -\frac{1}{x}dx,\ \text{if } x\neq 0$$

$$\Rightarrow \quad \int \sin v\, dv = -\int\frac{1}{x}\,dx$$

$$\Rightarrow \quad -\cos v = -\log|x|+C$$

$$\Rightarrow \quad -\cos\left(\frac{y}{x}\right)+\log|x| = C \quad \ldots (i)$$

It is given that $y(1) = \frac{\pi}{2}$ *i.e.*, when $x = 1$, $y = \frac{\pi}{2}$

Putting $x = 1$ and $y = \frac{\pi}{2}$ in Eq. (i), we get

$$-\cos\frac{\pi}{2}+\log 1 = C \Rightarrow C = 0.$$

Putting C = 0 in (i), we get

$$-\cos\left(\frac{y}{x}\right)+\log|x| = 0$$

$$\Rightarrow \quad \log|x| = \cos\left(\frac{y}{x}\right)$$

Hence, $\log|x| = \cos\left(\frac{y}{x}\right)$, $x \neq 0$ is the required solution.

S67. $xe^{y/x} - y\sin\left(\frac{y}{x}\right) + x\frac{dy}{dx}\sin\left(\frac{y}{x}\right) = 0,\ y(1) = 0$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y\sin\left(\frac{y}{x}\right) - xe^{y/x}}{x\sin\left(\frac{y}{x}\right)}$$

This is a homogeneous differential equation.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$, it reduces to

$$v + x\frac{dv}{dx} = \frac{vx\sin v - xe^{v}}{x\sin v} = \frac{v\sin v - e^{v}}{\sin v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v\sin v - e^{v}}{\sin v} - v = \frac{v\sin v - e^{v} - v\sin v}{\sin v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\frac{e^{v}}{\sin v}$$

$$\Rightarrow \quad e^{-v}\sin v\, dv = -\frac{dx}{x}, \text{ if } x \neq 0$$

$$\Rightarrow \quad \int e^{-v}\sin v\, dv = -\int \frac{1}{x}dx$$

$$\Rightarrow \quad \frac{e^{-v}}{2}(-\sin v - \cos v) = -\log|x| + \log C \quad \left[\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2 + b^2}\right]$$

$$\Rightarrow \quad -\frac{1}{2}e^{-y/x}\left\{\sin\left(\frac{y}{x}\right) + \cos\left(\frac{y}{x}\right)\right\} = -\log|x| + \log C$$

$$\Rightarrow \quad e^{-y/x}\left\{\sin\left(\frac{y}{x}\right) + \cos\left(\frac{y}{x}\right)\right\} = 2\log|x| - 2\log C$$

$$\Rightarrow \quad e^{-y/x}\left\{\sin\left(\frac{y}{x}\right) + \cos\left(\frac{y}{x}\right)\right\} = \log|x|^2 - 2\log C \qquad \ldots \text{(ii)}$$

It is given that $y(1) = 0$ *i.e.*, $y = 0$ when $x = 1$. Putting these values in (ii), we get

$$1 = 0 - 2 \log C \Rightarrow \log C = -\frac{1}{2}$$

Putting $\log C = -\frac{1}{2}$ in (ii), we get

$$e^{-y/x}\left\{\sin\left(\frac{y}{x}\right)+\cos\left(\frac{y}{x}\right)\right\} = \log |x|^2 + 1$$

Hence, $e^{-y/x}\left\{\sin\left(\frac{y}{x}\right)+\cos\left(\frac{y}{x}\right)\right\} = 1+\log x^2, x \neq 0$ gives the required solution.

S68. Given differential equation is

$$\frac{dx}{dy} + x \cot y = 2y + y^2 \cot y, \quad (y \neq 0)$$

which is a linear differential equation.

Comparing with $\frac{dx}{dy} + Px = Q$

Here, $P = \cot y$ and $Q = 2y + y^2 \cot y$

Now, $\text{I.F.} = e^{\int P dy} = e^{\int \cot y \, dy}$

$$= e^{\log \sin y} = \sin y$$

$\therefore$ Complete solution is

$$x \cdot (\text{I.F.}) = \int Q \cdot (\text{IF}) dy + c$$

$$x \cdot \sin y = \int (2y + y^2 \cot y) \cdot \sin y \, dy + c$$

$$\Rightarrow \quad x \sin y = 2\int y \sin y \, dy + \int \underset{\text{I}}{y^2} \underset{\text{II}}{\cos y} \, dy + c$$

$$= 2\int y \sin y \, dy + y^2 \int \cos y \, dy - \int\left[\left(\frac{d}{dy} y^2\right) \int \cos y \, dy\right] dy + c$$

$$\left[\text{Using integration by part in second integral } i.e. \int \underset{\text{I}}{u} \underset{\text{II}}{v} \, dx = u\int v \, dx - \int \left\{\left(\frac{d}{dx} u \int v dx\right)\right\} dx\right]$$

$$= 2\int y \sin y \, dy + y^2 \sin y - 2\int y \sin y \, dy + c$$

$$= y^2 \sin y + c$$

$$\Rightarrow \quad x \sin y = y^2 \sin y + c \qquad \text{... (i)}$$

Given that, $x = 0$ when $y = \frac{\pi}{2}$

$$\Rightarrow \qquad 0 = \left(\frac{\pi}{2}\right)^2 \sin\frac{\pi}{2} + c \qquad \Rightarrow c = -\frac{\pi^2}{4}$$

Put this value in Eq. (i), we get

$$x \sin y = y^2 \sin y - \frac{\pi^2}{4}$$

$$\Rightarrow \qquad x = y^2 - \frac{\pi^2}{4} . \operatorname{cosec} y$$

which is required particular solution of given differential equation.

S69. Given differential equation is,

$$(\tan^{-1} y - x)\, dy = (1 + y^2)\, dx$$

$$\Rightarrow \qquad \frac{\tan^{-1} y - x}{1+y^2} = \frac{dx}{dy}$$

$$\Rightarrow \qquad \frac{dx}{dy} = \frac{-x}{1+y^2} + \frac{\tan^{-1} y}{1+y^2}$$

$$\Rightarrow \qquad \frac{dx}{dy} + \frac{1}{1+y^2} \cdot x = \frac{\tan^{-1} y}{1+y^2}$$

which is a linear differential equation

Comparing with $\frac{dx}{dy} + Px = Q$, we get

$$P = \frac{1}{1+y^2} \quad \text{and} \quad Q = \frac{\tan^{-1} y}{1+y^2}$$

Now, $\qquad \text{IF} = e^{\int P dy} = e^{\int \frac{dy}{1+y^2}} = e^{\tan^{-1} y}$

Complete solution is

$$x \cdot (\text{IF}) = \int Q \cdot (\text{IF}) dy + c$$

$$\Rightarrow \qquad x e^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2} \times e^{\tan^{-1} y}\, dy + c$$

Let $\qquad t = \tan^{-1} y,\ dt = \frac{1}{1+y^2} dy$

$$\therefore \qquad x \cdot e^{\tan^{-1} y} = \int t \cdot e^t dt + c$$

$$\Rightarrow \qquad x \cdot e^{\tan^{-1} y} = t \cdot e^t - \int 1 \cdot e^t dt + c \qquad \text{[Using integration by parts]}$$

$$\Rightarrow \qquad x \cdot e^{\tan^{-1} y} = t \cdot e^t - e^t + c$$

$\Rightarrow \quad x \cdot e^{\tan^{-1} y} = (\tan^{-1} y - 1)e^{\tan^{-1} y} + c$ ... (i)

Also, given that, when $x = 0, \quad y = 0$

$$0 = (\tan^{-1} 0 - 1)\, e^{\tan^{-1} 0} + c$$

$\Rightarrow \quad 0 = (0 - 1)\, e^0 + c$

$\Rightarrow \quad 0 = (0 - 1) \cdot 1 + c \Rightarrow c = 1$

From Eq. (i),

$\therefore \quad x \cdot e^{\tan^{-1} y} = (\tan^{-1} y - 1) \cdot e^{\tan^{-1} y} + 1$

$$x = \tan^{-1} y - 1 + e^{-\tan^{-1} y}$$

which is the required particular solution *cf* the differential equation.

S70. $\sqrt{1 - y^2}\, dx = (\sin^{-1} y - x) dy$

$\Rightarrow \quad \dfrac{dx}{dy} = \dfrac{\sin^{-1} y}{\sqrt{1-y^2}} - \dfrac{x}{\sqrt{1-y^2}}$, if $1 - y^2 \neq 0$ i.e. $y \neq \pm 1$

$\Rightarrow \quad \dfrac{dx}{dy} + \dfrac{1}{\sqrt{1-y^2}} x = \dfrac{\sin^{-1} y}{\sqrt{1-y^2}}$ ... (i)

This is a linear differential equation of the form $\dfrac{dx}{dy} + Rx = S$

where $\quad R = \dfrac{1}{\sqrt{1-y^2}}$ and $S = \dfrac{\sin^{-1} y}{\sqrt{1-y^2}}$

Now solution of above equation is given by $x \times \text{I.F.} = \int (S \times I.F)\, dy + C$ ... (ii)

where I.F. = integrating factor and $\text{I.F.} = e^{\int R dy}$

$\Rightarrow \quad \text{I.F.} = e^{\int \frac{1}{\sqrt{1-y^2}} dy} = e^{\sin^{-1} y}$

Putting I.F. and S in Eqs (ii).

$$x e^{\sin^{-1} y} = \int e^{\sin^{-1} y} \frac{\sin^{-1} y}{\sqrt{1-y^2}} dy + C$$

$\Rightarrow \quad x e^{\sin^{-1} y} = \int t e^t dt + C$, where $t = \sin^{-1} y$, $dt = \dfrac{1}{\sqrt{1-y^2}} dy$,

$$x e^{\sin^{-1} y} = t e^t - \int 1 \cdot e^t dt + C$$

$\Rightarrow \quad x e^{\sin^{-1} y} = e^t (t - 1) + C$

$\Rightarrow \quad x e^{\sin^{-1} y} = e^{\sin^{-1} y} (\sin^{-1} y - 1) + C$ ... (ii)

It is given that $y(0) = 0$ *i.e.*, $y = 0$ when $x = 0$. Putting $x = 0$, $y = 0$ in (ii), we get

$$0 = e^0(0 - 1) + C \Rightarrow C = 1$$

Putting $C = 1$ in (ii), we get

$$xe^{\sin^{-1} y} = e^{\sin^{-1} y}(\sin^{-1} y - 1) + 1$$

$$\Rightarrow \quad e^{\sin^{-1} y}(x - \sin^{-1} y + 1) = 1$$

$$\Rightarrow \quad x - \sin^{-1} y + 1 = e^{-\sin^{-1} y}, \quad y \in (-1, 1)$$

This gives the required solution.

S71. We have,

$$ye^y\, dx = (y^3 + 2x\, e^y)dy$$

$$\frac{dx}{dy} = \frac{y^3 + 2x\, e^y}{ye^y}$$

$$\Rightarrow \quad \frac{dx}{dy} = y^2 e^{-y} + \frac{2x}{y}, \text{ if } y \neq 0$$

$$\Rightarrow \quad \frac{dx}{dy} - \frac{2}{y}x = y^2\, e^{-y} \quad \text{... (i)}$$

This is a linear differential equation of the form $\frac{dx}{dy} + Rx = S$

where

$$R = \frac{-2}{y} \quad \text{and} \quad S = y^2 e^{-y}$$

Now solution of above equation is given by $x \times \text{I.F.} = \int (S \times \text{I.F.})\, dy + C$ ... (ii)

where I.F. =integrating factor and I.F = $e^{\int R dy}$

$$\Rightarrow \quad \text{I.F.} = e^{\int -\frac{2}{y} dy} = e^{-2\log y} = e^{\log y^{-2}} = y^{-2} = \frac{1}{y^2}$$

Putting I.F. and S in Eqs. (ii).

$$x\left(\frac{1}{y^2}\right) = \int e^{-y} dy + C$$

$$\Rightarrow \quad \frac{x}{y^2} = -e^{-y} + C \quad \text{... (ii)}$$

It is given that $y(0) = 1$ *i.e.*, $y = 1$ when $x = 0$.

Putting $x = 0$, $y = 1$ in (iii), we get

$$0 = -e^{-1} + C \Rightarrow C = \frac{1}{e}$$

Putting $C = \frac{1}{e}$ in (ii), we get

$$\frac{x}{y^2} = -e^{-y} + \frac{1}{e} \Rightarrow x = y^2(e^{-1} - e^{-y})$$

Hence, $x = y^2(e^{-1} - e^{-y})$, $y \neq 0$ gives the required solution.