Date: 19/10/2021

Q1. Evaluate $\int \frac{\sin^3 x - \cos^3 x}{\sin^2 x \cos^2 x} dx$

Q2. Evaluate $\int \sin^{-1} \cos x \, dx$

Q3. Evaluate $\int \frac{2 + 3\cos x}{\sin^2 x} dx$

Q4. Evaluate $\int \frac{\cos 2x}{\sin^2 x \cos^2 x} dx$

Q5. Evaluate $\int \frac{2\cos x}{\sin^2 x} dx$

Q6. Evaluate $\int \frac{1 - \sin x}{\cos^2 x} dx$

Q7. Evaluate $\int \sqrt{\frac{1 - \cos 2x}{1 + \cos 2x}} dx$

Q8. Evaluate $\int \frac{2}{1 + \cos 2x} dx$

Q9. Evaluate $\int \sin(e^x) \, d(e^x)$

Q10. Evaluate $\int \frac{x^3 - 1}{x^2} dx$

Q11. Evaluate $\int \frac{x^3 - x^2 + x - 1}{x - 1} dx$

Q12. Evaluate $\int \frac{\sin 4x}{\sin 2x} dx$

Q13. Evaluate $\int \text{cosec}^2(3x + 2) \, dx$

Q14. Evaluate $\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} dx$

Q15. Evaluate $\int \frac{\sin^2 x}{1 + \cos x} dx$

Q16. Evaluate $\int \tan^{-1}\left(\frac{\sin 2x}{1 + \cos 2x}\right) dx$

Q17. Evaluate $\int \cot^{-1}\left(\frac{\sin 2x}{1 - \cos 2x}\right) dx$

Q18. Evaluate $\int \sin^{-1}\left(\frac{2\tan x}{1 + \tan^2 x}\right) dx$

Q19. Evaluate $\int \cos^{-1} \sin x \, dx$

Q20. Evaluate $\int \sin 3x \sin 2x \, dx$

Q21. Evaluate $\int \cos 4x \cos 2x \, dx$

Q22. Evaluate $\int \cos 4x \sin 3x \, dx$

Q23. Evaluate $\int \sin 4x \cos 3x \, dx$

Q24. Evaluate $\int \sin^2(2x+5) \, dx$

Q25. Evaluate $\int \cos^3 x \, dx$

Q26. Evaluate $\int \frac{1+\cos 4x}{\cot x - \tan x} dx$

Q27. Evaluate $\int \frac{\sin 4x}{\cos 2x} dx$

Q28. Evaluate $\int \frac{\sin 4x}{\sin x} dx$

Q29. Evaluate $\int \frac{dx}{\sqrt{x+2}-\sqrt{x+3}}$

Q30. Evaluate $\int (x^4 + x^2 + 1) \, d(x^2)$

Q31. Evaluate $\int \frac{\sec x}{\sec x + \tan x} dx$

Q32. Evaluate $\int \sin^3 x dx$

Q33. Evaluate $\int \frac{\sin x}{1+\sin x} dx$

Q34. Evaluate $\int \frac{\cos x}{1+\cos x} dx$

Q35. Evaluate $\int \frac{1}{\sin^2 x \cos^2 x} dx$.

Q36. If $f'(x) = a \sin x + b \cos x$ and $f'(0) = 4$, $f(0) = 3, f\left(\frac{\pi}{2}\right) = 5$ find $f(x)$.

Q37. If $f'(x) = 8x^3 - 2x$, $f(2) = 8$, find $f(x)$

Q38. If $f'(x) = x + b$, $f(1) = 5$, $f(2) = 13$. Find $f(x)$.

Q39. If $f'(x) = 3x^2 - \frac{2}{x^3}$ and $f(1) = 0$ find $f(x)$.

Q40. Evaluate $\int \frac{x^4 + x^2 + 1}{x^2 - x + 1} dx$

Q41. Evaluate $\int \sin^3(2x+1) \, dx$

Q42. Evaluate $\int \tan^{-1} \sqrt{\frac{1-\cos 2x}{1+\cos 2x}} \, dx$

Q43. Evaluate $\int \frac{\cos x - \cos 2x}{1-\cos x} dx$

Q44. Evaluate $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx$

Q45. Evaluate $\int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x} dx$

Q46. Evaluate $\int \tan^{-1} \sqrt{\frac{1-\sin x}{1+\sin x}} dx$

Q47. Evaluate $\int \tan^{-1}(\sec x + \tan x)\, dx$

Q48. Evaluate $\int (5x+3)\sqrt{2x-1}\, dx$

Q49. Evaluate $\int (7x-2)\sqrt{3x+2}\, dx$

Q50. Evaluate $\int \left(\frac{2+x+x^2}{x^2(2+x)} + \frac{2x-1}{(x+1)^2} \right) dx$

Q51. Evaluate $\int \frac{2x+1}{\sqrt{3x+2}} dx$

Q52. Evaluate $\int \frac{8x+13}{\sqrt{4x+7}} dx$

Q53. Evaluate $\int \frac{(x+1)}{\sqrt{2x-1}} dx$

Q54. Evaluate $\int \frac{3x+5}{\sqrt{7x+9}} dx$

Q55. Evaluate $\int \frac{2-3x}{\sqrt{1+3x}} dx$

Q56. Evaluate $\int \frac{1}{\sin(x-a)\cos(x-b)} dx$

Q57. Evaluate $\int \frac{\sin x}{\sin(x-a)} dx$

Q58. Evaluate $\int \frac{\sin(x-a)}{\sin x} dx$

Q59. Evaluate $\int \tan(x-\theta)\tan(x+\theta)\tan 2x\, dx$

Q60. Evaluate $\int \tan x \tan 2x \tan 3x\, dx$

Q61. Evaluate $\int \cos x \cos 2x \cos 3x\, dx$

Q62. Evaluate $\int \cos 2x \cos 4x \cos 6x\, dx$

Q63. Evaluate $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} dx$

Q64. Evaluate $\int \sin x \cdot \sin 2x \cdot \sin 3x\, dx$

Q65. Evaluate $\int \frac{\sin(x-a)}{\sin(x+a)} dx$

Q66. Evaluate $\int \frac{\sin(x+a)}{\sin(x+b)} dx$

Q67. Evaluate $\int \frac{1}{\sin(x-a)\sin(x-b)} dx$

Q68. Evaluate $\int \frac{1}{\cos(x-a)\cos(x-b)}\,dx$

Q69. Evaluate $\int \frac{1}{\cos(x-a)\sin(x-b)}\,dx$

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S1. Given integral is $\int \frac{\sin^3 x - \cos^3 x}{\sin^2 x \cos^2 x} dx$

$$= \int \frac{\sin^3 x}{\sin^2 x \cos^2 x} dx - \int \frac{\cos^3 x}{\sin^2 x \cos^2 x} dx$$

$$= \int \sec x \tan x \, dx - \int \text{cosec } x \cot x \, dx = \sec x + \text{cosec } x + C$$

S2. Given integral is $\int \sin^{-1} \cos x \, dx$

$$= \int \sin^{-1}\left\{\sin\left(\frac{\pi}{2} - x\right)\right\} dx = \int\left(\frac{\pi}{2} - x\right) dx = \frac{\pi}{2}x - \frac{x^2}{2} + C$$

S3. Given integral is $\int \frac{2 + 3\cos x}{\sin^2 x} dx$

$$= \int \frac{2}{\sin^2 x} dx + \int \frac{3\cos x}{\sin^2 x} dx = \int 2\text{cosec}^2 x \, dx + \int 3\cot x \, \text{cosec } x \, dx$$

$$= 2\int \text{cosec}^2 x \, dx + 3\int \text{cosec } x \cot x \, dx$$

$$= -2\cot x - 3\text{cosec } x + C$$

S4. Given integral is $\int \frac{\cos 2x}{\sin^2 x \cos^2 x} dx$

$$= \int \frac{\cos^2 x - \sin^2 x}{\sin^2 x \cos^2 x} dx$$

$$= \int\left(\frac{1}{\sin^2 x} - \frac{1}{\cos^2 x}\right) dx$$

$$= \int \text{cosec}^2 x \, dx - \int \sec^2 x \, dx = -\cot x - \tan x + C$$

S5. Given integral is $\int \frac{2\cos x}{\sin^2 x} dx$

$$= \int 2\,\text{cosec } x \cot x \, dx$$

$$= -2\,\text{cosec } x + C$$

S6. Given integral is $\int \frac{1-\sin x}{\cos^2 x}dx$

$$= \int \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}\right)dx$$

$$= \int \sec^2 x dx - \int \sec x \tan x dx$$

$$= \tan x - \sec x + C$$

S7.

$$I = \int\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\,dx = \int\sqrt{\frac{2\sin^2 x}{2\cos^2 x}}\,dx$$

$$= \int \tan x\, dx = \log(\sec x) + C$$

S8.

$$I = \int\frac{2}{1+\cos 2x}dx = \int\frac{2}{2\cos^2 x}dx$$

$$= \int \sec^2 x dx = \tan x + c$$

S9. Given integral is $\int \sin(e^x)\, d(e^x)$

Put $e^x = t$

$$\int \sin t\, dt = -\cos t + C = -\cos(e^x) + C$$

S10. Given Integral is $\int \frac{x^3-1}{x^2}dx$

$$= \int\left(\frac{x^3}{x^2} - \frac{1}{x^2}\right)dx$$

$$= \int x dx - \int\frac{1}{x^2}dx$$

$$= \frac{x^2}{2} - \frac{x^{-1}}{-1} + C$$

$$= \frac{x^2}{2} + \frac{1}{x} + C$$

S11. Given integral is $\int \frac{x^3 - x^2 + x - 1}{x-1}dx$

$$= \int \frac{x^2(x-1)+(x-1)}{x-1}dx$$

$$= \int \frac{(x^2+1)(x-1)}{x-1} dx$$

$$= \int (x^2+1)\, dx = \frac{x^3}{3} + x + C$$

S12. Given integral is $\int \frac{\sin 4x}{\sin 2x} dx$

$$= \int \frac{2\sin 2x \cos 2x}{\sin 2x} dx = 2\int \cos 2x\, dx = \sin 2x + C$$

S13. Given integral is $\int \text{cosec}^2(3x+2)\, dx$

$$= -\frac{1}{3}\cot(3x+2) + C$$

S14. Given integral is $\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} dx$

$$= \int \frac{1 - 2\sin^2 x + 2\sin^2 x}{\cos^2 x} dx = \int \frac{1}{\cos^2 x} dx$$

$$= \int \sec^2 x\, dx = \tan x + C$$

S15. Given integral is $\int \frac{1-\cos^2 x}{1+\cos x} dx$

$$= \int \frac{(1-\cos x)(1+\cos x)}{1+\cos x} dx \qquad = \int (1-\cos x) dx$$

$$= \int 1\, dx - \int \cos x\, dx = x - \sin x + C$$

S16. Given integral is $\int \tan^{-1}\left(\frac{\sin 2x}{1+\cos 2x}\right) dx$

$$= \int \tan^{-1}\left(\frac{2\sin x \cos x}{2\cos^2 x}\right) dx = \int \tan^{-1}(\tan x)\, dx$$

$$= \int x\, dx = \frac{x^2}{2} + C$$

S17. Given integral is $\int \cot^{-1}\left(\frac{\sin 2x}{1-\cos 2x}\right) dx$

$$= \int \cot^{-1}\left(\frac{2\sin x \cos x}{2\sin^2 x}\right) dx$$

$$= \int \cot^{-1}(\cot x)\,dx = \int x\,dx = \frac{x^2}{2} + C$$

S18. Given integral is $\int \sin^{-1}\left(\frac{2\tan x}{1+\tan^2 x}\right)dx$

$$= \int \sin^{-1}\sin 2x\,dx = \int 2x\,dx = x^2 + C$$

S19. Given integral is $\int \cos^{-1}\sin x\,dx$

$$= \int \cos^{-1}\cos\left(\frac{\pi}{2} - x\right)dx = \int\left(\frac{\pi}{2} - x\right)dx = \frac{\pi}{2}x - \frac{x^2}{2} + C$$

S20. Given integral is $\int \sin 3x\,\sin 2x\,dx$

$$= \frac{1}{2}\int 2\sin 3x\,\sin 2x\,dx = \frac{1}{2}\int(\cos x - \cos 5x)\,dx$$

$$= \frac{1}{2}\sin x - \frac{1}{10}\sin 5x + C$$

S21. Given integral is $\int \cos 4x\,\cos 2x\,dx$

$$= \frac{1}{2}\int 2\cos 4x\,\cos 2x\,dx = \frac{1}{2}\int(\cos 6x + \cos 2x)\,dx$$

$$= \frac{1}{12}\sin 6x + \frac{1}{4}\sin 2x + C$$

S22. Given integral is $\int \cos 4x\,\sin 3x\,dx$

$$= \frac{1}{2}\int 2\cos 4x\,\sin 3x\,dx = \frac{1}{2}\int(\sin 7x - \sin x)\,dx$$

$$= \frac{-1}{14}\cos 7x + \frac{1}{2}\cos x + C$$

S23. Given integral is $\int \sin 4x\,\cos 3x\,dx$

$$= \frac{1}{2}\int 2\sin 4x\,\cos 3x\,dx = \frac{1}{2}\int(\sin 7x + \sin x)\,dx$$

$$= \frac{-1}{14}\cos 7x - \frac{1}{2}\cos x + C$$

S24. Given integral is $\int \sin^2(2x+5)\,dx$

$$= \int \frac{1-\cos 2(2x+5)}{2}\,dx = \frac{1}{2}\int 1\,dx - \frac{1}{2}\int \cos(4x+10)\,dx$$

$$= \frac{1}{2}x - \frac{1}{8}\sin(4x+10) + C$$

S25. Given integral is $\int \cos^3 x\,dx$

$$= \int \frac{\cos 3x + 3\cos x}{4}\,dx = \frac{1}{4}\int \cos 3x\,dx + \frac{3}{4}\int \cos x\,dx$$

$$= \frac{1}{12}\sin 3x + \frac{3}{4}\sin x + C$$

S26. Given integral is $\int \frac{1+\cos 4x}{\cot x - \tan x}\,dx$

$$= \int \frac{2\cos^2 2x\,\cos x\,\sin x}{\cos^2 x - \sin^2 x}\,dx = \int \frac{\sin 2x\,\cos^2 2x}{\cos 2x}\,dx$$

$$= \frac{1}{2}\int 2\sin 2x\cos 2x\,dx = \frac{1}{2}\int \sin 4x\,dx = -\frac{1}{8}\cos 4x + C$$

S27. Given integral is $\int \frac{\sin 4x}{\cos 2x}\,dx$

$$= \int \frac{2\sin 2x\cos 2x}{\cos 2x}\,dx = 2\int \sin 2x\,dx = -\cos 2x + C$$

S28. Given integral is $\int \frac{\sin 4x}{\sin x}\,dx$

$$= \int \frac{2\sin 2x\cos 2x}{\sin x}\,dx = \int \frac{4\sin x\cos x\cos 2x}{\sin x}\,dx$$

$$= 2\int 2\cos 2x\cos x\,dx = 2\int (\cos 3x + \cos x)\,dx$$

$$= 2\left\{\frac{\sin 3x}{3} + \sin x\right\} + C$$

S29.

$$I = \int \frac{1}{\sqrt{x+2}-\sqrt{x+3}} \times \frac{\sqrt{x+2}+\sqrt{x+3}}{\sqrt{x+2}+\sqrt{x+3}}\,dx \qquad \text{[Rationalising]}$$

$$= \int \frac{\sqrt{x+2}+\sqrt{x+3}}{(x+2)-(x+3)}\,dx = -\left[\int (x+2)^{1/2}dx + \int (x+3)^{1/2}dx\right]$$

$$= -\left[\frac{(x+2)^{3/2}}{3/2} + \frac{(x+3)^{3/2}}{3/2}\right] + c = -\frac{2}{3}[(x+2)^{3/2} + (x+3)^{3/2}] + c$$

S30. Given integral is $\int (x^4 + x^2 + 1)\, d(x^2)$

Let $x^2 = t$

$$= \int (t^2 + t + 1)\, dt = \frac{t^3}{3} + \frac{t^2}{2} + t + C = \frac{x^6}{3} + \frac{x^4}{2} + x^2 + C$$

S31. Given integral is $\int \frac{\sec x}{\sec x + \tan x}\, dx$

$$= \int \frac{\sec x(\sec x - \tan x)}{(\sec x + \tan x)(\sec x - \tan x)}\, dx$$

$$= \int \frac{\sec^2 x - \sec x \tan x}{\sec^2 x - \tan^2 x}\, dx$$

$$= \int \sec^2 x\, dx - \int \sec x \tan x\, dx = \tan x - \sec x + C$$

S32.

$$I = \int \sin^3 x\, dx$$

$$= \int \frac{3 \sin x - \sin 3x}{4}\, dx$$

$$= \frac{1}{4} \int 3 \sin x\, dx - \frac{1}{4} \int \sin 3x\, dx$$

$$= \frac{1}{4}\left(-3 \cos x + \frac{\cos 3x}{3} + C \right)$$

S33.

$$I = \int \frac{1 + \sin x - 1}{1 + \sin x}\, dx = \int 1.dx - \int \frac{1}{1 + \sin x}\, dx$$

$$= x - \int \frac{1 - \sin x}{(1 + \sin x)(1 - \sin x)}\, dx$$

$$= x - \int \frac{1 - \sin x}{\cos^2 x}\, dx = x - \int \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} \right) dx$$

$$= x - \int (\sec^2 x - \sec x \tan x)\, dx$$

$$= x - \tan x + \sec x + c.$$

S34.

$$I = \int \frac{\cos x}{1 + \cos x}\, dx = \int \frac{(1 + \cos x) - 1}{1 + \cos x}\, dx$$

$$= \int 1 \cdot dx - \int \frac{1}{1 + \cos x}\, dx = x - \int \frac{1}{2 \cos^2 \frac{x}{2}}\, dx$$

$$= x - \frac{1}{2}\int \sec^2 \frac{x}{2} dx = x - \frac{1}{2}\frac{\tan x/2}{1/2} + c = x - \tan\frac{x}{2} + c$$

S35. Given integral is $\int \frac{1}{\sin^2 x \cos^2 x} dx$

$$= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx$$

$$= \int \left(\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}\right) dx$$

$$= \int \sec^2 dx + \int \text{cosec}^2 x \, dx$$

$$= \tan x - \cot x + C$$

S36.

$$f(x) = \int f'(x) dx$$

$$= \int (a \sin x + b \cos x) dx = -a \cos x + b \sin x + c$$

Now, $f(0) = -a \cos 0 + b \sin 0 + c \quad \Rightarrow \quad -a + c = 3$ ... (i)

$f\left(\frac{\pi}{2}\right) = -a \cos\frac{\pi}{2} + b \sin \frac{\pi}{2} + c \quad \Rightarrow \quad b + c = 5$ ... (ii)

$f'(0) = a \sin 0 + b \cos 0 \quad \Rightarrow \quad b = 4$ ... (iii)

From (i), (ii) and (iii) $a = -2, \ b = 4, \ c = 1$

$\therefore \quad f(x) = 2\cos x + 4 \sin x + 1$

S37.

$$f(x) = \int f'(x) dx$$

$$= \int (8x^3 - 2x) dx = \frac{8x^4}{4} - \frac{2x^2}{2} + C$$

$f(x) = 2x^4 - x^2 + C$

$f(2) = 32 - 4 + C = 8 \quad \Rightarrow \quad C = -20$

$\therefore \quad f(x) = 2x^4 - x^2 - 20$

S38.

$$f(x) = \int f'(x) dx$$

$$= \int (x + b) dx = \frac{x^2}{2} + bx + c$$

Now,

$f(1) = \frac{1}{2} + b + c = 5$ ... (i)

$f(2) = 2 + 2b + c = 13$ ... (ii)

From (i) and (ii)

$$b = \frac{13}{2}, \quad c = -2$$

$$\therefore \qquad f(x) = \frac{x^2}{2} + \frac{13x}{2} - 2.$$

S39.

$$f(x) = \int f'(x)\,dx$$

$$= \int\left(3x^2 - \frac{2}{x^3}\right)dx = 3\left(\frac{x^3}{3}\right) - 2\left(-\frac{1}{2x^2}\right) + C$$

$$= x^3 + \frac{1}{x^2} + C$$

$$f(1) = 1 + 1 + C \;\Rightarrow\; 0 = C + 2 \;\Rightarrow\; C = -2$$

$$\therefore \qquad f(x) = x^3 + \frac{1}{x^2} - 2$$

S40. Given integral is $\int \frac{x^4 + x^2 + 1}{x^2 - x + 1}\,dx$

$$= \int \frac{x^4 + 2x^2 + 1 - x^2}{x^2 - x + 1}\,dx = \int \frac{(x^2+1)^2 - x^2}{x^2 - x + 1}\,dx$$

$$= \int \frac{(x^2 + 1 - x)(x^2 + 1 + x)}{(x^2 - x + 1)}\,dx = \int (x^2 + x + 1)\,dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + x + C$$

S41. Given integral is $\int \sin^3(2x+1)\,dx$

$$= \int \frac{3\sin(2x+1) - \sin 3(2x+1)}{4}\,dx$$

$$= \frac{3}{4}\int \sin(2x+1)\,dx - \frac{1}{4}\int \sin(6x+3)\,dx$$

$$= \frac{-3}{8}\cos(2x+1) + \frac{1}{24}\cos(6x+3) + C$$

S42. Given integral is $\int \tan^{-1}\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\,dx$

$$= \int \tan^{-1}\sqrt{\frac{2\sin^2 x}{2\cos^2 x}}\,dx = \int \tan^{-1}\tan x\,dx$$

$$= \int x\,dx = \frac{x^2}{2} + C$$

S43. Given integral is $\int \frac{\cos x - \cos 2x}{1 - \cos x}\, dx$

$$= \int \frac{\cos x - (2\cos^2 x - 1)}{1 - \cos x} dx = \int \frac{-(2\cos^2 x - \cos x - 1)}{1 - \cos x} dx$$

$$= \int \frac{-(2\cos x + 1)(\cos x - 1)}{-(\cos x - 1)} dx \qquad = \int (2\cos x + 1)\, dx$$

$$= 2\int \cos x\, dx + \int 1\, dx = 2\sin x + x + C$$

S44. Given integral is $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x}\, dx$

$$= \int \frac{(\sin^2 x + \cos^2 x)^3 - 3\sin^2 x \cos^2 x(\sin^2 x + \cos^2 x)}{\sin^2 x \cos^2 x} dx$$

$[\because \quad a^3 + b^3 = (a + b)^3 - 3ab(a + b)]$

$$= \int \frac{1 - 3\sin^2 x \cos^2 x}{\sin^2 x \cos^2 x} dx = \int \left(\frac{1}{\sin^2 x \cos^2 x} - 3 \right) dx$$

$$= \int \left(\frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} - 3 \right) dx = \int (\sec^2 x + \text{cosec}^2 x - 3)\, dx$$

$$= \tan x - \cot x - 3x + C$$

S45. Given integral is $\int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x}\, dx$

$$= \int \frac{(\sin^4 x + \cos^4 x)(\sin^4 x - \cos^4 x)}{(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x} dx$$

$$= \int \frac{(\sin^4 x + \cos^4 x)(\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x)}{\sin^4 x + \cos^4 x} dx$$

$$= \int -\cos 2x\, dx = -\frac{1}{2}\sin 2x + C$$

S46. Given integral is $\int \tan^{-1} \sqrt{\frac{1 - \sin x}{1 + \sin x}}\, dx$

$$= \int \tan^{-1} \sqrt{\frac{1 - \cos\left(\frac{\pi}{2} - x\right)}{1 + \cos\left(\frac{\pi}{2} - x\right)}}\, dx = \int \tan^{-1} \sqrt{\frac{2\sin^2\left(\frac{\pi}{4} - \frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4} - \frac{x}{2}\right)}}\, dx$$

$$= \int \tan^{-1} \tan\left(\frac{\pi}{4} - \frac{x}{2}\right) dx = \int \left(\frac{\pi}{4} - \frac{x}{2}\right) dx = \frac{\pi}{4}x - \frac{x^2}{4} + C$$

S47. Given integral is $\int \tan^{-1}(\sec x + \tan x)\,dx$

$$= \int \tan^{-1}\left(\frac{1+\sin x}{\cos x}\right)dx = \int \tan^{-1}\left(\frac{1-\cos\left(\frac{\pi}{2}+x\right)}{\sin\left(\frac{\pi}{2}+x\right)}\right)dx$$

$$= \int \tan^{-1}\left(\frac{2\sin^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}{2\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}\right)dx$$

$$= \int \tan^{-1}\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)dx = \int\left(\frac{\pi}{4}+\frac{x}{2}\right)dx$$

$$= \frac{\pi}{4}x + \frac{x^2}{4} + C$$

S48. Given integral is $\int (5x+3)\sqrt{2x-1}\,dx$

Put $5x + 3 = A(2x - 1) + B$

$2A = 5$ and $-A + B = 3$

$A = \frac{5}{2}$ and $B = \frac{11}{2}$

$$= \int\left(\frac{5}{2}(2x-1)+\frac{11}{2}\right)\sqrt{2x-1}\,dx$$

$$= \frac{5}{2}\int(2x-1)^{3/2}dx + \frac{11}{2}\int(2x-1)^{1/2}dx$$

$$= \frac{5}{2}\,\frac{(2x-1)^{5/2}}{2\times\frac{5}{2}} + \frac{11}{2}\,\frac{(2x-1)^{3/2}}{2\times\frac{3}{2}} + C$$

$$= \frac{(2x-1)^{5/2}}{2} + \frac{11}{6}(2x-1)^{3/2} + C$$

S49. Given integral is $\int (7x-2)\sqrt{3x+2}\,dx$

Let $7x - 2 = A(3x + 2) + B$

$\Rightarrow \quad A = \frac{7}{3}$ and $B = \frac{-20}{3}$

$\therefore \quad \int\left(\frac{7}{3}(3x+2)-\frac{20}{3}\right)\sqrt{3x+2}\,dx$

$$= \frac{7}{3}\int(3x+2)^{3/2} - \frac{20}{3}\int(3x+2)^{1/2}dx$$

$$= \frac{7}{3}\frac{(3x+2)^{5/2}}{3\times\frac{5}{2}} - \frac{20}{3}\frac{(3x+2)^{3/2}}{\frac{3}{2}\times 3} + C$$

$$= \frac{14}{45}(3x+2)^{5/2} - \frac{40}{27}(3x+2)^{3/2} + C$$

S50. Given integral is $\int\left(\frac{2+x+x^2}{x^2(2+x)} + \frac{2x-1}{(x+1)^2}\right)dx$

$$= \int\left(\frac{(2+x)+x^2}{x^2(2+x)} + \frac{2x+2-3}{(x+1)^2}\right)dx$$

$$= \int\frac{1}{x^2}dx + \int\frac{1}{2+x}dx + 2\int\frac{1}{x+1}dx - 3\int\frac{1}{(x+1)^2}dx$$

$$= -\frac{1}{x} + \log(2+x) + 2\log(x+1) + \frac{3}{x+1} + C$$

S51. Given integral is $\int\frac{2x+1}{\sqrt{3x+2}}dx$

Let $2x + 1 = A(3x + 2) + B$

$3A = 2$ and $2A + B = 1$

$A = \frac{2}{3}$ and $B = -\frac{1}{3}$

$$= \int\frac{\frac{2}{3}(3x+2)-\frac{1}{3}}{\sqrt{3x+2}}dx = \frac{2}{3}\int(3x+2)^{1/2}dx - \frac{1}{3}\int(3x+2)^{-1/2}dx$$

$$= \frac{2}{3}\frac{(3x+2)^{3/2}}{3\times\frac{3}{2}} - \frac{1}{3}\frac{(3x+2)^{1/2}}{3\times\frac{1}{2}} + C$$

$$= \frac{4}{27}(3x+2)^{3/2} - \frac{2}{9}(3x+2)^{1/2} + C$$

S52. Given integral is $\int\frac{8x+13}{\sqrt{4x+7}}dx$

Let $8x + 13 = A(4x + 7) + B$

$4A = 8$ and $7A + B = 13$

$A = 2$ and $B = -1$

$$\int \frac{2(4x+7)-1}{\sqrt{4x+7}}\,dx = 2\int \sqrt{4x+7}\,dx - \int (4x+7)^{-1/2}\,dx$$

$$= 2\,\frac{(4x+7)^{3/2}}{4\times\frac{3}{2}} - \frac{(4x+7)^{1/2}}{4\times\frac{1}{2}} + C = \frac{(4x+7)^{3/2}}{3} - \frac{(4x+7)^{1/2}}{2} + C$$

S53. Given integral is $\int \frac{(x+1)}{\sqrt{2x-1}}\,dx$

Let $x + 1 = A(2x - 1) + B$

$2A = 1$ and $-A + B = 1$

$A = \frac{1}{2}$ and $B = \frac{3}{2}$

$$= \int \frac{\frac{1}{2}(2x-1)+\frac{3}{2}}{\sqrt{2x-1}}\,dx = \frac{1}{2}\int (2x-1)^{1/2}\,dx + \frac{3}{2}\int (2x-1)^{-1/2}\,dx$$

$$= \frac{1}{2}\,\frac{(2x-1)^{3/2}}{2\times\frac{3}{2}} + \frac{3}{2}\,\frac{(2x-1)^{1/2}}{2\times\frac{1}{2}} + C$$

$$= \frac{1}{6}(2x-1)^{3/2} + \frac{3}{2}(2x-1)^{1/2} + C$$

S54. Given integral is $\int \frac{3x+5}{\sqrt{7x+9}}\,dx$

Let $3x + 5 = A(7x + 9) + B$

$7A = 3$ and $9A + B = 5$

$A = \frac{3}{7}$ and $B = \frac{8}{7}$

$$\therefore \quad \int \frac{\frac{3}{7}(7x+9)+\frac{8}{7}}{\sqrt{7x+9}}\,dx = \frac{3}{7}\int (7x+9)^{1/2}\,dx + \frac{8}{7}\int (7x+9)^{-1/2}\,dx$$

$$= \frac{3}{7}\,\frac{(7x+9)^{3/2}}{7\times\frac{3}{2}} + \frac{8}{7}\,\frac{(7x+9)^{1/2}}{7\times\frac{1}{2}} + C$$

$$= \frac{2}{49}(7x+9)^{3/2} + \frac{16}{49}(7x+9)^{1/2} + C$$

S55. Given integral is $\int \frac{2-3x}{\sqrt{1+3x}}\,dx$

Put 2 – 3x = A (1 + 3x) + B

3A = – 3 and A + B = 2

A = – 1 and B = 3

$$\int \frac{-(1+3x)+3}{\sqrt{1+3x}}\,dx = -\int (1+3x)^{1/2}\,dx + 3\int (1+3x)^{-1/2}\,dx$$

$$\Rightarrow \quad \frac{-(1+3x)^{3/2}}{3\times\frac{3}{2}} + \frac{3(1+3x)^{1/2}}{3\times\frac{1}{2}} + C = \frac{-2}{9}(1+3x)^{3/2} + 2(1+3x)^{1/2} + C$$

S56. Given integral is $\int \frac{1}{\sin(x-a)\cos(x-b)}\,dx$

$$= \frac{1}{\cos(a-b)}\int \frac{\cos(a-b)}{\sin(x-a)\cos(x-b)}\,dx$$

$$= \frac{1}{\cos(a-b)}\int \frac{\cos\{(x-b)-(x-a)\}}{\sin(x-a)\cos(x-b)}\,dx$$

$$= \frac{1}{\cos(a-b)}\int \frac{\cos(x-b)\cos(x-a)+\sin(x-a)\sin(x-b)}{\sin(x-a)\cos(x-b)}\,dx$$

$$= \frac{1}{\cos(a-b)}\int [\cot(x-a)+\tan(x-b)]\,dx$$

$$= \frac{1}{\cos(a-b)}[\log\sin(x-a) - \log\cos(x-b)] + C$$

$$= \frac{1}{\cos(a-b)}\log\left(\frac{\sin(x-a)}{\cos(x-b)}\right) + C$$

S57. Given integral is $\int \frac{\sin x}{\sin(x-a)}\,dx$

Let $x - a = t$

$$dx = dt$$

$$= \int \frac{\sin(a+t)}{\sin t}\,dt = \int \frac{\sin a\cos t + \cos a\sin t}{\sin t}\,dt$$

$$= \int \sin a\cot t\,dt + \int \cos a\,dt$$

$$= \sin a \int \cot t \, dt + \cos a \int 1 \, dt$$

$$= \sin a \log (\sin t) + t \cos a + C$$

$$= \sin a \log \sin (x - a) + (x - a) \cos a + C$$

S58. Given integral is $\int \frac{\sin(x-a)}{\sin x} dx$

$$= \int \frac{\sin x \cos a - \cos x \sin a}{\sin x} dx$$

$$= \int \cos a \, dx - \int \sin a \cot x \, dx$$

$$= \cos a \int 1 \, dx - \sin a \int \cot x \, dx$$

$$= x \cos a - \sin a \log(\sin x) + C$$

S59. Given integral is $\int \tan (x - \theta) \tan (x + \theta) \tan 2x \, dx$

As $\tan 2x = \tan [(x - \theta) + (x + \theta)]$

$\Rightarrow$ $\tan 2x = \frac{\tan (x - \theta) + \tan (x + \theta)}{1 - \tan (x - \theta) \tan (x + \theta)}$

$\Rightarrow$ $\tan 2x - \tan (x - \theta) - \tan (x + \theta) = \tan (x + \theta) \tan (x - \theta) \tan 2x$

$\therefore$ $\int [\tan 2x - \tan(x - \theta) - \tan(x + \theta)] dx$

$\Rightarrow$ $\frac{1}{2} \log(\sec 2x) - \log \sec(x - \theta) - \log \sec(x + \theta) + C$

S60. Given integral is $\int \tan x \tan 2x \tan 3x \, dx$

As $\tan 3x = \tan(2x + x)$

$\Rightarrow$ $\tan 3x = \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$

$\Rightarrow$ $\tan 3x - \tan 3x \tan 2x \tan x = \tan 2x + \tan x$

$\Rightarrow$ $\tan 3x - \tan 2x - \tan x = \tan 3x \tan 2x \tan x$

Hence $\int \tan x \tan 2x \tan 3x \, dx = \int (\tan 3x - \tan 2x - \tan x) dx$

$\Rightarrow$ $\int \tan x \tan 2x \tan 3x \, dx = \frac{1}{3} \log(\sec 3x) - \frac{1}{2} \log(\sec 2x) - \log \sec x + C$

S61. Given integral is $\int \cos x \cos 2x \cos 3x \, dx$

$$= \frac{1}{2}\int (2\cos 2x \cos x)\cos 3x \, dx = \frac{1}{2}\int (\cos 3x + \cos x)\cos 3x \, dx$$

$$= \frac{1}{4}\int (2\cos^2 3x + 2\cos 3x \cos x)\, dx$$

$$= \frac{1}{4}\int (1 + \cos 6x + \cos 4x + \cos 2x)\, dx$$

$$= \frac{1}{4}\left[x + \frac{\sin 6x}{6} + \frac{\sin 4x}{4} + \frac{\sin 2x}{2}\right] + C$$

S62. Given integral is $\int \cos 2x \cos 4x \cos 6x \, dx$

$$= \frac{1}{2}\int (2\cos 4x \cos 2x)\cos 6x \, dx$$

$$= \frac{1}{2}\int (\cos 6x + \cos 2x)\cos 6x \, dx$$

$$= \frac{1}{4}\int (2\cos^2 6x + 2\cos 6x \cos 2x)\, dx$$

$$= \frac{1}{4}\int (1 + \cos 12x + \cos 8x + \cos 4x)\, dx$$

$$= \frac{1}{4}\left[x + \frac{\sin 12x}{12} + \frac{\sin 8x}{8} + \frac{\sin 4x}{4}\right] + C$$

S63. Let $I = \int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} dx$

$I = \int \frac{(2\cos^2 x - 1) - (2\cos^2 \alpha - 1)}{\cos x - \cos \alpha} dx$ $\quad [\because \cos 2\theta = 2\cos^2 \theta - 1]$

$I = \int \frac{2(\cos^2 x - \cos^2 \alpha)}{(\cos x - \cos \alpha)} dx$

$I = \int \frac{2(\cos x - \cos \alpha)(\cos x + \cos \alpha)}{(\cos x - \cos \alpha)} dx$

$I = \int 2(\cos x + \cos \alpha) dx$

$I = 2\int (\cos x)\, dx + \cos \alpha \int dx$

$I = 2 \sin x + x \cos \alpha + C.$

S64. $\therefore \quad \int \sin x \sin 2x \sin 3x \, dx$

$= \frac{1}{2}\int \sin x (2\sin 2x \sin 3x)\, dx$ $\quad$ [Multiply numerator & denominator by 2]

$$= \frac{1}{2}\int \sin x[\cos(2x-3x)-\cos(2x+3x)dx \qquad [\because \cos(-x)=\cos x]$$

$$= \frac{1}{2}\int \sin x\cos x dx - \frac{1}{2}\int \sin x\cos 5x\, dx$$

$$= \frac{1}{4}\int 2\sin x\cos x\, dx - \frac{1}{4}\int (2\sin x\cos 5x)dx \quad \text{[Multiply numerator \& denominator by 2]}$$

$$= \frac{1}{4}\int \sin 2x\, dx - \frac{1}{4}\int \{\sin(x+5x)+\sin(x-5x)\}dx$$

$$\left[\begin{array}{l}\because\ 2\sin x\cos x = \sin 2x \text{ and}\\ 2\sin A\cos B = \sin(A+B)+\sin(A-B)\end{array}\right]$$

$$= \frac{1}{4}\int \sin 2x\, dx - \frac{1}{4}\int [\sin 6x + \sin(-4x)]dx$$

$$= \frac{1}{4}\int \sin 2x\, dx - \frac{1}{4}\int (\sin 6x - \sin 4x)dx \qquad \left[\because\ \int \sin ax\, dx = \frac{-\cos ax}{a}+C\right]$$

$$= \frac{-\cos 2x}{8} + \frac{\cos 6x}{24} - \frac{\cos 4x}{16} + C$$

S65. $$I = \int \frac{\sin(x-a)}{\sin(x+a)}dx$$

Put $x + a = t$

$\Rightarrow$ $x = t - a$

$\Rightarrow$ $dx = dt$

$\therefore$ $$I = \int \frac{\sin(t-2a)}{\sin t}\, dt$$

$\Rightarrow$ $$I = \int \frac{\sin t\cos 2a - \cos t\sin 2a}{\sin t}dt \qquad [\because \sin(x-y) = \sin x\cdot\cos y - \cos x\cdot \sin y]$$

$$I = \int \frac{\sin t\cos 2a}{\sin t}dt - \int \frac{\cos t\sin 2a}{\sin t}dt$$

$$= \cos 2a\int dt - \sin 2a\int \cot t\, dt$$

$$= t\cos 2a - \sin 2a \ln|\sin t| + C$$

$$= (x+a)\cos 2a - \sin 2a \ln|\sin(x+a)| + C$$

S66. Given integral is $\int \frac{\sin(x+a)}{\sin(x+b)}dx$

Put $x + b = t \Rightarrow dx = dt$

$$\int \frac{\sin(t+a-b)}{\sin t}dt = \int \frac{\sin t\cos(a-b)+\cos t\sin(a-b)}{\sin t}dt$$

$\Rightarrow \quad \cos(a-b)\int 1\,dt + \sin(a-b)\int \cot t\,dt$

$\Rightarrow \quad t\cos(a-b) + \sin(a-b)\log\sin t + C$

$\Rightarrow \quad (x+b)\cos(a-b) + \sin(a-b)\log\sin(x+b) + C$

S67. Given integral is $\int \frac{1}{\sin(x-a)\sin(x-b)}\,dx$

$$= \frac{1}{\sin(a-b)}\int \frac{\sin(a-b)}{\sin(x-a)\sin(x-b)}\,dx$$

$$= \frac{1}{\sin(a-b)}\int \frac{\sin[(x-b)-(x-a)]}{\sin(x-a)\sin(x-b)}\,dx$$

$$= \frac{1}{\sin(a-b)}\int \frac{\sin(x-b)\cos(x-a)-\cos(x-b)\sin(x-a)}{\sin(x-a)\sin(x-b)}\,dx$$

$$= \frac{1}{\sin(a-b)}\int [\cot(x-a)-\cot(x-b)]\,dx$$

$$= \frac{1}{\sin(a-b)}[-\log\sin(x-b)+\log\sin(x-a)] + C$$

$$= \frac{1}{\sin(a-b)}\log\frac{\sin(x-a)}{\sin(x-b)} + C$$

S68. Given integral is $\int \frac{1}{\cos(x-a)\cos(x-b)}\,dx$

$$= \frac{1}{\sin(a-b)}\int \frac{\sin(a-b)}{\cos(x-a)\cos(x-b)}\,dx$$

$$= \frac{1}{\sin(a-b)}\int \frac{\sin[(x-b)-(x-a)]}{\cos(x-a)\cos(x-b)}\,dx$$

$$= \frac{1}{\sin(a-b)}\int \frac{\sin(x-b)\cos(x-a)-\cos(x-b)\sin(x-a)}{\cos(x-a)\cos(x-b)}\,dx$$

$$= \frac{1}{\sin(a-b)}\int [\tan(x-b)-\tan(x-a)]\,dx$$

$$= \frac{1}{\sin(a-b)}[-\log\cos(x-b)+\log\cos(x-a)] + C$$

$$= \frac{1}{\sin(a-b)}\log\left(\frac{\cos(x-a)}{\cos(x-b)}\right) + C$$

S69. Given integral is $\int \frac{1}{\cos(x-a)\sin(x-b)}\,dx$

$$= \frac{1}{\cos(a-b)}\int \frac{\cos(a-b)}{\cos(x-a)\sin(x-b)}\,dx$$

$$= \frac{1}{\cos(a-b)} \int \frac{\cos[(x-b)-(x-a)]}{\cos(x-a)\sin(x-b)}\,dx$$

$$= \frac{1}{\cos(a-b)} \int \frac{\cos(x-b)\cos(x-a)+\sin(x-b)\sin(x-a)}{\cos(x-a)\sin(x-b)}\,dx$$

$$= \frac{1}{\cos(a-b)} \int [\cot(x-b)+\tan(x-a)]\,dx$$

$$= \frac{1}{\cos(a-b)} [\log \sin (x-b) - \log \cos (x-a)] + C$$

$$= \frac{1}{\cos(a-b)} \log \left(\frac{\sin (x-b)}{\cos (x-a)}\right) + C$$

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Q1. Evaluate $\int \sin(ax+b)\cos(ax+b)dx$

Q2. Evaluate $\int \sin^2 x \cos^3 x\, dx$

Q3. Evaluate $\int \sqrt{\sin 2x}\cos 2x\, dx$

Q4. Evaluate $\int \frac{x}{e^{x^2}}dx$

Q5. Evaluate $\int \frac{e^{\tan^{-1} x}}{1+x^2}dx$

Q6. Evaluate $\int \frac{(1+\log x)^2}{x}dx$

Q7. Compute the Integral : $\int 2x\sin(x^2+1)dx$

Q8. Evalute $\int \frac{2\cos x}{3\sin^2 x}dx$

Q9. Evaluate $\int \sec^2(7-4x)dx$

Q10. Evaluate $\int \frac{\sec^2 x}{3+\tan^2 x}dx$

Q11. Evaluate $\int \frac{dx}{\sqrt{x}(1+\sqrt{x})}$.

Q12. Evaluate $\int x\sqrt{1+2x^2}dx$

Q13. Evaluate $\int \frac{2\cos 2x+\sec^2 x}{\sin 2x+\tan x-5}dx$

Q14. Evaluate $\int \frac{1+\tan x}{x+\log\sec x}dx$

Q15. Evaluate $\int \frac{2\cos x-\sin x}{\cos x+2\sin x}dx$

Q16. Evaluate $\int \frac{\cos 2x+x+1}{x^2+\sin 2x+2x}dx$

Q17. Evaluate $\int \frac{1-\sin x}{x+\cos x}dx$

Q18. Evaluate $\int \frac{e^x+1}{e^x+x}dx$

Q19. Evaluate $\int \frac{\sec x\ \text{cosec}\ x}{\log\tan x}dx$

Q20. Evaluate $\int \frac{\sec x\tan x}{3\sec x+5}dx$

Q21. Evaluate $\int \frac{1-\tan x}{1+\tan x}dx$

Q22. Evaluate $\int \frac{1+\cot x}{x+\log \sin x} dx$

Q23. Evaluate $\int \sec^4 x \tan x \, dx$.

Q24. Evaluate $\int \operatorname{cosec} x \log(\operatorname{cosec} x - \cot x) \, dx$

Q25. Evaluate $\int \frac{1+\cos x}{(x+\sin x)^3} dx$

Q26. Evaluate $\int \frac{1+\sin x}{\sqrt{x-\cos x}} dx$

Q27. Evaluate $\int \frac{(1+\sqrt{x})^2}{\sqrt{x}} dx$

Q28. Evaluate $\int \sqrt{\tan x}\,(1+\tan^2 x) \, dx$

Q29. Evaluate $\int e^{-x} \operatorname{cosec}^2 (2e^{-x}+5) \, dx$

Q30. Evaluate $\int x^3 \sin x^4 \, dx$

Q31. Evaluate $\int \frac{1-\sin 2x}{x+\cos^2 x} dx$

Q32. Evaluate $\int \frac{dx}{\sqrt{1-x^2}\,(2+3\sin^{-1} x)}$

Q33. Evaluate $\int \frac{\sin(2+3\log x)}{x} dx$

Q34. Evaluate $\int \frac{x \tan^{-1}(x^2)}{1+x^4} dx$

Q35. Evaluate $\int \frac{\sin(\tan^{-1} x)}{1+x^2} dx$

Q36. Evaluate $\int \frac{\cos\sqrt{x}}{\sqrt{x}} dx$

Q37. Evaluate $\int \frac{e^{\sqrt{x}} \cos e^{\sqrt{x}}}{\sqrt{x}} dx$

Q38. Evaluate $\int x^3 \sin(x^4+1) \, dx$

Q39. Evaluate $\int \frac{\sqrt{\tan x}}{\sin x \cos x} dx$

Q40. Evaluate $\int \{f(ax+b)\}^n f'(ax+b) \, dx$

Q41. Evaluate $\int \frac{\sec^2 \sqrt{x}}{\sqrt{x}} dx$

Q42. Evaluate $\int \frac{x \sin^{-1}(x^2)}{\sqrt{1-x^4}} dx$

Q43. Evaluate $\int e^{\cos^2 x} \sin 2x \, dx$

Q44. Evaluate $\int \log x \frac{\sin[1+(\log x)^2]}{x} dx$

Q45. Evaluate $\int \frac{\text{cosec}^2(\log x)}{x} dx$

Q46. Evaluate $\int \frac{(x+1)(x+\log x)^2}{x} dx$

Q47. Evaluate $\int \frac{(\tan^{-1} x)^2}{1+x^2} dx$

Q48. Evaluate $\int \frac{e^x(1+x)}{\cos^2(xe^x)} dx$

Q49. Evaluate $\int x^2 e^{x^3} \cos(2e^{x^3}) dx$

Q50. Evaluate $\int \frac{x^2}{(2+3x^3)^3} dx$

Q51. Evaluate $\int (x^3-1)^{1/3} x^5 dx$

Q52. Evaluate $\int \frac{x^2}{1+x^3} dx$

Q53. Evaluate $\int (4x+2)\sqrt{x^2+x+1}\, dx$

Q54. Evaluate $\int \frac{1}{x(\log x)^m} dx,\ m>0$

Q55. Evaluate $\int (ax+b)^3 dx$

Q56. Evaluate $\int \frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}} dx$

Q57. Evaluate $\int 2^x dx$

Q58. Evaluate $\int \frac{\log(\sin x)}{\tan x} dx$

Q59. Evaluate $\int \frac{\sin\sqrt{x}}{\sqrt{x}} dx$

Q60. Evaluate $\int \frac{\cos 4x - \cos 2x}{\sin 4x - \sin 2x} dx$

Q61. Evaluate $\int \frac{\sec x}{\log(\sec x + \tan x)} dx$

Q62. Evaluate $\int \frac{1}{1+e^{-x}} dx$

Q63. Evaluate $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

Q64. Evaluate $\int \frac{2\cos x - 3\sin x}{6\cos x + 4\sin x} dx$

Q65. Evaluate $\int \sec x \log(\sec x + \tan x) dx$

Q66. Evaluate $\int 5^{(x+\tan^{-1}x)} \left(\frac{x^2+2}{x^2+1}\right) dx$

Q67. Evaluate $\int \frac{(x+1)e^x}{\sin^2(xe^x)} dx$

Q68. Evaluate $\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)dx$

Q69. Evaluate $\int \frac{\log\left(1+\frac{1}{x}\right)}{x(1+x)}dx$

Q70. Evaluate $\int \frac{\tan x \sec^2 x}{(a+b\tan^2 x)^2}dx$

Q71. Evaluate $\int \frac{a}{b+ce^x}dx$

Q72. Evaluate $\int \frac{x+1}{x(x+\log x)}dx$

Q73. Evaluate $\int \frac{\text{cosec } x}{\log \tan x/2}dx$

Q74. Evaluate $\int \tan^3 x \sec^3 x\, dx$

Q75. Evaluate $\int \frac{\cos^5 x}{\sin x}dx$

Q76. Evaluate $\int \frac{\sin 2x}{a^2\sin^2 x+b^2\cos^2 x}dx$

Q77. Evaluate $\int 2x\sec^3(x^2+3)\tan(x^2+3)dx$

Q78. Evaluate $\int \frac{x^{e-1}+e^{x-1}}{x^e+e^x}dx.$

Q79. Evaluate $\int \frac{1}{\sqrt{\sin^3 x\cos x}}dx$

Q80. Evaluate $\int \frac{\sin^4 x}{\cos^8 x}dx$

Q81. Evaluate $\int \frac{\sin 2x}{\sin\left(x-\frac{\pi}{3}\right)\sin\left(x+\frac{\pi}{3}\right)}dx$

Q82. Evaluate $\int \sec^{4/3} x \operatorname{cosec}^{8/3} x\, dx$

Q83. Evaluate $\int \frac{1}{\sqrt{\sin^3 x\cos^5 x}}dx$

Q84. Evaluate $\int \tan^4 x\, dx$

Q85. Evaluate $\int \cot^4 x\, dx$

Q86. Evaluate $\int \frac{1}{\sqrt[4]{(x-1)^3(x+2)^5}}dx$

Q87. Evaluate $\int \frac{\sin 2x}{(a+b\cos x)^2}dx$

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S1. Put sin $(ax + b) = t$ so that $a \cos(ax + b)\, dx = dt$

i.e. $\cos(ax + b)\, dx = \frac{1}{a} dt.$

$$I = \int t \cdot \frac{1}{a} dt = \frac{1}{a} \cdot \frac{t^2}{2} + c = \frac{1}{2a} \sin^2(ax + b) + c$$

S2. Put $\sin x = t$ so that $\cos x\, dx = dt$

$$I = \int \sin^2 x \cdot \cos^2 x \cdot \cos x\, dx = \int t^2 (1 - t^2) dt$$

$$= \int (t^2 - t^4) dt = \frac{t^3}{3} - \frac{t^5}{5} + c = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + c$$

S3. Put $\sin 2x = t$ so that $2 \cos 2x\, dx = dt$ *i.e.* $\cos 2x\, dx = \frac{1}{2} dt$.

$$\therefore \quad \int \sqrt{\sin 2x} \cos 2x\, dx = \int \sqrt{t} \frac{1}{2} dt = \frac{1}{2} \int t^{1/2} dt$$

$$= \frac{1}{2} \frac{t^{3/2}}{3/2} + c = \frac{1}{3} t^{3/2} + c = \frac{1}{3} (\sin 2x)^{3/2} + c$$

S4. Put $x^2 = t$ so that $2x\, dx = dt$, $\quad x\, dx = \frac{1}{2} dt.$

$$\int \frac{x}{e^{x^2}} dx = \int \frac{\frac{1}{2} dt}{e^t} = \frac{1}{2} \int e^{-t} dt = \frac{1}{2} \frac{e^{-t}}{(-1)} + c = -\frac{1}{2} \frac{1}{e^t} + c = -\frac{1}{2e^{x^2}} + c$$

S5. Given Integral is $\int \frac{e^{\tan^{-1} x}}{1 + x^2} dx$

Putting $\tan^{-1} x = t$

$\Rightarrow \quad \frac{1}{1 + x^2} dx = dt$

$\therefore$ *We get* $\int \frac{e^{\tan^{-1} x}}{1 + x^2} dx = \int e^t dt$

$= e^t + C \qquad \left[\because \int e^x dx = e^x + C\right]$

$= e^{\tan^{-1} x} + C$

S6. Given integral is $\int \frac{(1+\log x)^2}{x}dx$

Putting $1 + \log x = t$

$\Rightarrow \frac{1}{x}dx = dt$

$\therefore$ We get $\int \frac{(1+\log x)^2}{x}dx = \int t^2 dt$

$$= \frac{t^3}{3} + C = \frac{(1+\log x)^3}{3} + C$$

S7. Put $x^2 + 1 = t$

so that $2x\ dx = dt.$

$\therefore \int 2x \sin(x^2+1)dx = \int \sin t\, dt = -\cos t + c = -\cos(x^2+1) + c$

S8.

$$I = \int \frac{2\cos x}{3\sin^2 x}dx$$

Let $\sin x = t$

$\cos x\ dx = dt$

$\therefore I = \int \frac{2dt}{3t^2} = \frac{2}{3}\int \frac{dt}{t^2} = \frac{2}{3}\frac{t^{-2+1}}{(-2+1)} = \frac{-2}{3t} + C$

$$= \frac{-2}{3\sin x} + C$$

S9. Given integral is $\int \sec^2(7-4x)dx$

$$= \frac{\tan(7-4x)}{-4} + C \qquad \left[\because \int \sec^2 ax\, dx = \frac{\tan ax}{a} + C\right]$$

$$= -\frac{\tan(7-4x)}{4} + C$$

S10. Given integral is $\int \frac{\sec^2 x}{3+\tan^2 x}dx$

Put $\tan x = t$

$\Rightarrow \sec^2 x\ dx = dt$

$\therefore$ We get $\int \frac{\sec^2 x}{3+\tan^2 x}dx = \int \frac{dt}{3+t^2}$

$$= \int \frac{dt}{(\sqrt{3})^2 + t^2}$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)+C \qquad \left[\because \int\frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}+C\right]$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\tan x}{\sqrt{3}}\right)+C$$

S11. Put $\sqrt{x} = t$ so that $\frac{1}{2\sqrt{x}}dx = dt$ i.e. $\frac{dx}{\sqrt{x}} = 2dt$.

$$I = \int 2\frac{dt}{1+t} = 2\int\frac{1}{1+t}dt = 2\log|1+t|+c$$

$$= 2\log|1+\sqrt{x}|+c$$

S12. Put $1+2x^2 = t$ so that $4x\,dx = dt$ i.e. $x\,dx = \frac{1}{4}dt$

$$\therefore \quad \int x\sqrt{1+2x^2}\,dx = \int\sqrt{t}\,\frac{dt}{4} = \frac{1}{4}\int t^{1/2}dt$$

$$= \frac{1}{4}\frac{t^{3/2}}{3/2}+c = \frac{1}{6}t^{3/2}+c$$

$$= \frac{1}{6}(1+2x^2)^{3/2}+c$$

S13. Given integral is $\int\frac{2\cos 2x+\sec^2 x}{\sin 2x+\tan x-5}dx$

Put $\sin 2x + \tan x - 5 = t$

$(2\cos 2x + \sec^2 x)dx = dt$

$$= \int\frac{1}{t}dt = \log t + C = \log(\sin 2x + \tan x - 5) + C$$

S14. Given integral is $\int\frac{1+\tan x}{x+\log\sec x}dx$

Put $x + \log\sec x = t$

$$= \left(1+\frac{1}{\sec x}\sec x\tan x\right)dx = dt \quad\Rightarrow\quad (1+\tan x)dx = dt$$

$$= \int\frac{1}{t}dt = \log t + C = \log(x + \log\sec x) + C$$

S15. Given integral is $\int \frac{2\cos x - \sin x}{\cos x + 2\sin x} dx$

Put $\cos x + 2\sin x = t$

$(2\cos x - \sin x)dx = dt$

$$= \int \frac{1}{t} dt = \log t + C = \log(2\sin x + \cos x) + C$$

S16. Given integral is $\int \frac{\cos 2x + x + 1}{x^2 + \sin 2x + 2x} dx$

Put $x^2 + \sin 2x + 2x = t$

$(2x + 2\cos 2x + 2)\, dx = dt$

$$= \frac{1}{2}\int \frac{1}{t} dt = \frac{1}{2}\log t + C = \frac{1}{2}\log(x^2 + \sin 2x + 2x) + C$$

S17. Given integral is $\int \frac{1 - \sin x}{x + \cos x} dx$

Put $x + \cos x = t$

$(1 - \sin x)\, dx = dt$

$$= \int \frac{1}{t} dt = \log t + C = \log(x + \cos x) + C$$

S18. Given integral is $\int \frac{e^x + 1}{e^x + x} dx$

Put $e^x + x = t$

$(e^x + 1)dx = dt$

$$= \int \frac{1}{t} dt = \log t + C = \log(e^x + x) + C$$

S19. Given integral is $\int \frac{\sec x \operatorname{cosec} x}{\log \tan x} dx$

Put $\log \tan x = t$

$\frac{1}{\tan x} \times \sec^2 x\, dx = dt \Rightarrow \sec x \operatorname{cosec} x\, dx = dt$

$$= \int \frac{1}{t} dt = \log t + C = \log(\log \tan x) + C$$

S20. Given integral is $\int \frac{\sec x \tan x}{3\sec x + 5} dx$

Put $3\sec x + 5 = t$

$3\sec x \tan x \, dx = dt$

$$= \frac{1}{3}\int \frac{1}{t} dt = \frac{1}{3}\log t + C = \frac{1}{3}\log(3\sec x + 5) + C$$

S21. Given integral is $\int \frac{1 - \tan x}{1 + \tan x} dx$

$$= \int \frac{\cos x - \sin x}{\cos x + \sin x} dx$$

Put $\cos x + \sin x = t$

$(\cos x - \sin x)dx = dt$

$$= \int \frac{1}{t} dt = \log t + C = \log(\cos x + \sin x) + C$$

S22. Put $x + \log \sin x = t$

so that $\left(1 + \frac{1}{\sin x}\cos x\right)dx = dt$ *i.e.* $(1 + \cot x)\, dx = dt$

$$I = \int \frac{dt}{t} = \log|t| + c = \log|x + \log \sin x| + c$$

S23.

$$I = \int \sec^2 x \cdot \sec^2 x \cdot \tan x \, dx = \int (1 + \tan^2 x)\tan x \sec^2 x \, dx$$

$$= \int (\tan x + \tan^3 x)\sec^2 x \, dx.$$

Put $\tan x = t$ so that $\sec^2 x \, dx = dt$.

$$= \int (t + t^3)dt = \frac{t^2}{2} + \frac{t^4}{4} + c = \frac{\tan^2 x}{2} + \frac{\tan^4 x}{4} + c$$

S24. Given integral is $\int \operatorname{cosec} x \log(\operatorname{cosec} x - \cot x)\, dx$

Put $\log(\operatorname{cosec} x - \cot x) = t$

$$\left(\frac{-\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x}{\operatorname{cosec} x - \cot x}\right)dx = dt \Rightarrow \operatorname{cosec} x \, dx = dt$$

$$\Rightarrow \int t \, dt = \frac{t^2}{2} + C = \frac{1}{2}\left[\log(\operatorname{cosec} x - \cot x)\right]^2 + C$$

S25. Given integral is $\int \frac{1+\cos x}{(x+\sin x)^3}\,dx$

Put $x + \sin x = t \Rightarrow (1 + \cos x)\,dx = dt$

$$= \int \frac{dt}{t^3} = -\frac{1}{2t^2} + C = -\frac{1}{2(x+\sin x)^2} + C$$

S26. Given integral is $\int \frac{1+\sin x}{\sqrt{x-\cos x}}\,dx$

Put $x - \cos x = t$

$(1 + \sin x)\,dx = dt$

$$= \int \frac{dt}{\sqrt{t}} = \frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = 2\sqrt{t} + \text{C} = 2\sqrt{x-\cos x} + C$$

S27. Given integral is $\int \frac{(1+\sqrt{x})^2}{\sqrt{x}}\,dx$

Put $1+\sqrt{x} = t \Rightarrow \frac{1}{2\sqrt{x}}\,dx = \text{dt}$

$$= 2\int t^2\,dt = \frac{2}{3}\,t^3 + C = \frac{2}{3}(1+\sqrt{x})^3 + C$$

S28. Given integral is $\int \sqrt{\tan x}\,(1+\tan^2 x)\,dx$

$$= \int \sqrt{\tan x}\,\sec^2 x\,dx$$

Put $\tan x = t$

$\sec^2 x\,dx = dt$

$$= \int \sqrt{t}\,dt = \frac{t^{3/2}}{3/2} + C = \frac{2}{3}(\tan \text{x})^{3/2} + C$$

S29. Given integral is $\int e^{-x}\text{cosec}^2\,(2e^{-x} + 5)\,dx$

Put $2e^{-x} + 5 = t$

$-2e^{-x}\,dx = dt$

$$= -\frac{1}{2}\int \text{cosec}^2\,t\,dt = \frac{1}{2}\cot t + C = \frac{1}{2}\cot(2e^{-x} + 5) + C$$

S30. Given integral is $\int x^3 \sin x^4 \, dx$

Put $\quad x^4 = t \quad \Rightarrow \quad 4x^3 \, dx = dt$

$$= \frac{1}{4}\int \sin t \, dt = -\frac{1}{4}\cos t + C = -\frac{1}{4}\cos x^4 + C$$

S31. Given integral is $\int \frac{1 - \sin 2x}{x + \cos^2 x} dx$

Put $\quad x + \cos^2 x = t$

$$(1 - 2\sin x \cos x)\, dx = dt \quad \Rightarrow \quad (1 - \sin 2x)\, dx = dt$$

$$= \int \frac{1}{t} dt = \log t + C = \log (x + \cos^2 x) + C$$

S32. Given integral is $\int \frac{dx}{\sqrt{1 - x^2}(2 + 3\sin^{-1} x)}$

Put $\quad 2 + 3\sin^{-1} x = t \quad \Rightarrow \quad \frac{3}{\sqrt{1 - x^2}} dx = \mathrm{dt}$

$$= \frac{1}{3}\int \frac{dt}{t} = \frac{1}{3}\log t + C = \frac{1}{3}\log (2 + 3\sin^{-1} x) + \mathrm{C}$$

S33. Given integral is $\int \frac{\sin(2 + 3\log x)}{x} dx$

Put $\quad 2 + 3 \log x = t \quad \Rightarrow \quad \frac{3}{x} dx = dt$

$$\Rightarrow \quad \frac{1}{3}\int \sin t \, dt = -\frac{1}{3}\cos t + C = -\frac{1}{3}\cos(2 + 3\log x) + C$$

S34. Given integral is $\int \frac{x \tan^{-1}(x^2)}{1 + x^4} dx$

Put $\quad \tan^{-1}(x^2) = t \quad \Rightarrow \quad \frac{2x}{1 + x^4} \mathrm{dx} = \mathrm{dt}$

$$= \frac{1}{2}\int t \, dt = \frac{1}{4}t^2 + C = \frac{[\tan^{-1}(x^2)]^2}{4} + C$$

S35. Given integral is $\int \frac{\sin(\tan^{-1} x)}{1 + x^2} dx$

Put $\tan^{-1}(x) = t \Rightarrow \dfrac{dx}{1+x^2} = dt$

$$\int \sin t\, dt = -\cos t + C = -\cos(\tan^{-1} x) + C$$

S36. Given integral is $\int \dfrac{\cos\sqrt{x}}{\sqrt{x}}\,dx$

Put $\sqrt{x} = t \Rightarrow \dfrac{1}{2\sqrt{x}}dx = dt$

$$= 2\int \cos t\, dt = 2\sin t + C = 2\sin\sqrt{x} + C$$

S37. Given integral is $\int \dfrac{e^{\sqrt{x}}\cos e^{\sqrt{x}}}{\sqrt{x}}\,dx$

Put $e^{\sqrt{x}} = t \Rightarrow \dfrac{1}{2\sqrt{x}}e^{\sqrt{x}}\,dx = dt$

$$= 2\int \cos t\, dt = 2\sin t + C = 2\sin e^{\sqrt{x}} + C$$

S38. Given integral is $\int x^3 \sin(x^4+1)\,dx$

Put $x^4 + 1 = t \Rightarrow 4x^3\,dx = dt$

$$= \frac{1}{4}\int \sin t\, dt = -\frac{1}{4}\cos(x^4+1) + C$$

S39.

$$I = \int \frac{\sqrt{\tan x}}{\frac{\sin x}{\cos x}\cdot\cos^2 x}dx = \int \frac{\sqrt{\tan x}}{\tan x}\cdot\sec^2 x\,dx = \int \frac{\sec^2 x}{\sqrt{\tan x}}dx$$

Put $\tan x = t$ so that $\sec^2 x\,dx = dt$.

$$\therefore \quad I = \int \frac{dt}{\sqrt{t}} = \int t^{\frac{-1}{2}}dt = \frac{t^{-\frac{1}{2}+1}}{\frac{-1}{2}+1} = \frac{t^{1/2}}{1/2} + c = 2\sqrt{t} + c = 2\sqrt{\tan x} + c$$

S40. Given integral is $\int \{f(ax+b)\}^n f'(ax+b)\,dx$

Put $f(ax+b) = t \Rightarrow af'(ax+b)dx = dt$

$$\Rightarrow \frac{1}{a}\int t^n\,dt = \frac{t^{n+1}}{a(n+1)} + C$$

$$\Rightarrow \frac{\{f(ax+b)\}^{n+1}}{a(n+1)} + C$$

S41. Given integral is $\int \frac{\sec^2 \sqrt{x}}{\sqrt{x}}\, dx$

Put $\sqrt{x} = t \Rightarrow \frac{1}{2\sqrt{x}} dx = dt$

$\Rightarrow \quad 2\int \sec^2 t\, dt = 2\tan t + C = 2\tan \sqrt{x} + C$

S42. Given integral is $\int \frac{x \sin^{-1}(x^2)}{\sqrt{1 - x^4}}\, dx$

Put $\sin^{-1}(x^2) = t \Rightarrow \frac{2x}{\sqrt{1-x^4}} dx = dt$

$$= \frac{1}{2}\int t\, dt = \frac{1}{4}t^2 + C = \frac{[\sin^{-1}(x^2)]^2}{4} + C$$

S43. Given integral is $\int e^{\cos^2 x} \sin 2x\, dx$

Put $\cos^2 x = t \Rightarrow -2 \sin x \cos x\, dx = dt \Rightarrow \sin 2x\, dx = -dt$

$\Rightarrow \quad -\int e^t\, dt = -e^t + C = -e^{\cos^2 x} + C$

S44. Put $(\log x)^2 = t$ so that $\frac{2(\log x)}{x} dx = dt$ i.e. $\frac{\log x}{x} dx = \frac{1}{2} dt$

$\therefore \quad I = \frac{1}{2}\int \sin(1+t)dt = -\frac{1}{2}\cos(1+t) + c$

$$= -\frac{1}{2}\cos\{1 + (\log x)^2\} + c$$

S45. Put $\log x = t$ so that $\frac{1}{x}\, dx = dt$

$$I = \int \text{cosec}^2\, t\, dt = -\cot t + c = -\cot(\log x) + c$$

S46. Put $x + \log x = t$ so that $\left(1 + \frac{1}{x}\right) dx = dt$ i.e. $\frac{(x+1)}{x} dx = dt$.

$$I = \int t^2 dt = \frac{1}{3}t^3 + c = \frac{1}{3}(x + \log x)^3 + c$$

S47. Put $\tan^{-1} x = t$ so that $\frac{1}{1+x^2} dx = dt$

$$I = \int t^2 dt = \frac{t^3}{3} + c = \frac{1}{3}(\tan^{-1} x)^3 + c$$

S48. Put $xe^x = t$ so that $(x\,e^x + e^x)\,dx = dt$ *i.e.* $e^x\,(1 + x)\,dx = dt$.

$$\therefore \quad I = \int \frac{dt}{\cos^2 t} = \int \sec^2 t\,dt = \tan t + c = \tan(x\,e^x) + c$$

S49. Put $e^{x^3} = t$ so that $3x^2 e^{x^3} dx = dt$ *i.e.* $x^2 e^{x^3} dx = \frac{1}{3} dt$.

$$\therefore \quad I = \frac{1}{3}\int \cos 2t\,dt = \frac{1}{3}\frac{\sin 2t}{2} + c = \frac{1}{6}\sin(2e^{x^3}) + c$$

S50. Let
$$I = \int \frac{x^2}{(2+3x^3)^3}\,dx$$

Put $2 + 3x^3 = t$ so that $9x^2\,dx = dt$ *i.e.* $x^2 dx = \frac{1}{9} dt$

$$I = \int \frac{\frac{1}{9}dt}{t^3} = \frac{1}{9}\int t^{-3} dt = \frac{1}{9}\frac{t^{-3+1}}{(-3+1)} + c$$

$$= \frac{1}{9}\frac{t^{-2}}{(-2)} + c = -\frac{1}{18}\frac{1}{(2+3x^3)^2} + c.$$

S51. Let
$$I = \int (x^3 - 1)^{1/3} x^3 . x^2 dx$$

Put $x^3 - 1 = t$ so that $3x^2\,dx = dt$ *i.e.* $x^2 dx = \frac{1}{3} dt$. Also $x^3 = t + 1$

$$I = \int t^{1/3}(t+1)\frac{1}{3}dt = \frac{1}{3}\int (t^{4/3} + t^{1/3})dt$$

$$= \frac{1}{3}\left[\frac{t^{7/3}}{7/3} + \frac{t^{4/3}}{4/3}\right] + c = \frac{1}{7}t^{7/3} + \frac{1}{4}t^{4/3} + c$$

$$= \frac{1}{7}(x^3 - 1)^{7/3} + \frac{1}{4}(x^3 - 1)^{4/3} + c$$

S52. Given integral is $\int \frac{x^2}{1+x^3} dx$

Putting $1 + x^3 = t$

$\Rightarrow \quad 3x^2\,dx = dt$

$\Rightarrow \quad x^2\,dx = \frac{dt}{3}$

$\therefore$ We get $\int \frac{x^2}{1+x^3} dx = \frac{1}{3}\int \frac{dt}{t}$

$$= \frac{1}{3}\log|t| + C \qquad \left[\because \int\frac{dx}{x} = \log|x| + C\right]$$

$$= \frac{1}{3}\log|1+x^3| + C$$

S53. Put $x^2 + x + 1 = t$ so that $(2x + 1)\,dx = dt$

$$I = 2\int\sqrt{x^2+x+1}(2x+1)dx = 2\int\sqrt{t}\,dt$$

$$= 2\int t^{1/2}dt = 2\frac{t^{3/2}}{3/2} + c = \frac{4}{3}t^{3/2} + c = \frac{4}{3}(x^2+x+1)^{3/2} + c$$

S54.

Let $I = \int\frac{dx}{x(\log x)^m}$

Put $\log x = t$ so that $\frac{1}{x}dx = dt$

$$I = \int\frac{dt}{t^m} = \int t^{-m}dt = \frac{t^{1-m}}{1-m} + c = \frac{(\log x)^{1-m}}{1-m} + c$$

S55.

$$I = \int(ax+b)^3\,dx$$

Let $t = ax + b$

$$\frac{dt}{dx} = a$$

$\Rightarrow$ $\frac{dt}{a} = dx$

$\therefore$ $I = \int\frac{t^3}{a}dt = \frac{1}{a}\frac{t^4}{4} + C$

$$= \frac{(ax+b)^4}{4a} + C$$

S56.

Given Integral is $\int\frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}dx$

Putting $e^{2x} + e^{-2x} = t$

$\Rightarrow$ $(2e^{2x} - 2e^{-2x})dx = dt \qquad \left[\because \frac{d}{dx}(e^{ax}) = ae^{ax}\right]$

$$(e^{2x} - e^{-2x})\,dx = \frac{dt}{2}$$

$\therefore$ We get $$\int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} dx = \frac{1}{2}\int \frac{dt}{t}$$

$$= \frac{1}{2}\log|t| + C$$

$$= \frac{1}{2}\log\left|e^{2x} + e^{-2x}\right| + C$$

S57. Let $$2^x = t$$

Taking log both sides

$$x \log 2 = \log t$$

$$\Rightarrow \quad (\log 2)\, dx = \frac{1}{t} dt$$

$$I = \int \frac{\not{t}\, dt}{\not{t} \log 2}$$

$$= \int \frac{1}{\log 2} dt$$

$$= \frac{t}{\log 2} + C = \frac{2^x}{\log 2} + C$$

S58. The differentiation of $\log(\sin x)$ is $\tan x$, which exists in denominator. So solve by substitution method.

Given integral is $\int \frac{\log(\sin x)}{\tan x} dx$

Putting $$\log \sin x = t$$

$$\Rightarrow \quad \frac{1}{\sin x} \cdot \cos x\, dx = dt$$

$$\Rightarrow \quad \cot x\, dx = dt$$

$$\Rightarrow \quad \frac{1}{\tan x} dx = dt \qquad \left[\because \int \sin x\, dx = -\cos x + C\right]$$

$\therefore$ We get $$\int \frac{\log(\sin x)}{\tan x} dx = \int t\, dt$$

$$= \frac{t^2}{2} + C$$

$$= \frac{(\log \sin x)^2}{2} + C$$

S59. Given integral is $\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx$

Put $\sqrt{x} = t$

$\Rightarrow \quad \frac{1}{2\sqrt{x}}dx = dt$

$\Rightarrow \quad \frac{1}{\sqrt{x}}dx = 2\,dt$

$\therefore$ We get $\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx = 2\int \sin t\, dt$

$= -2\cos t + C \qquad \left[\because \int \sin x\, dx = -\cos x + C\right]$

$= -2\cos\sqrt{x} + C$

S60. Given integral is $\int \frac{\cos 4x - \cos 2x}{\sin 4x - \sin 2x}dx$

$$= \int \frac{-2\sin\left(\frac{4x+2x}{2}\right)\sin\left(\frac{4x-2x}{2}\right)}{2\cos\left(\frac{4x+2x}{2}\right)\sin\left(\frac{4x-2x}{2}\right)}dx$$

$$= -\int \tan 3x\, dx = -\frac{1}{3}\log(\sec 3x) + C$$

S61. Given integral is $\int \frac{\sec x}{\log(\sec x + \tan x)}dx$

Put $\log(\sec x + \tan x) = t$

$$= \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}dx = dt \Rightarrow \sec x\, dx = dt$$

$$= \int \frac{1}{t}dt = \log t + C = \log\log(\sec x + \tan x) + C$$

S62. Given integral is $\int \frac{1}{1+e^{-x}}dx = \int \frac{e^x}{e^x+1}dx$

Put $e^x + 1 = t$

$e^x\, dx = dt$

$$= \int \frac{1}{t}dt = \log t + C = \log(e^x + 1) + C$$

S63. Given integral is $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$

Put $e^x + e^{-x} = t$

$$(e^x - e^{-x})dx = dt$$

$$= \int \frac{1}{t}dt = \log t + C = \log(e^x + e^{-x}) + C$$

S64. Put $6 \cos x + 4 \sin x = t$

so that $(-6 \sin x + 4 \cos x)\,dx = dt$ *i.e.* $(2 \cos x - 3 \sin x)\,dx = \frac{1}{2}\,dt.$

$$I = \int \frac{\frac{1}{2}dt}{t} = \frac{1}{2}\int \frac{1}{t}dt$$

$$= \frac{1}{2}\log|t| + c' = \frac{1}{2}\log|6\cos x + 4\sin x| + c'$$

$$= \frac{1}{2}\log|2\sin x + 3\cos x| + c \text{, where } c = c' + \frac{1}{2}\log 2.$$

S65. Put $\log(\sec x + \tan x) = t$

so that $\frac{1}{\sec x + \tan x}(\sec x \tan x + \sec^2 x)\,dx = dt$ *i.e.,* $\sec x\,dx = dt.$

$$I = \int t\,dt = \frac{t^2}{2} + c = \frac{1}{2}[\log(\sec x + \tan x)]^2 + c$$

S66. Given integral is $\int 5^{(x + \tan^{-1} x)}\left(\frac{x^2 + 2}{x^2 + 1}\right)dx$

Put $x + \tan^{-1} x = t \;\Rightarrow\; \left(1 + \frac{1}{x^2 + 1}\right)dx = dt \;\Rightarrow\; \left(\frac{x^2 + 2}{x^2 + 1}\right)dx = dt$

$$= \int 5^t\,dt = \frac{5^t}{\log_e 5} + C = \frac{5^{(x + \tan^{-1} x)}}{\log_e 5} + C$$

S67. Given integral is $\int \frac{(x + 1)e^x}{\sin^2(xe^x)}\,dx$

Put $xe^x = t \;\Rightarrow\; (e^x + xe^x)dx = dt$

$$= \int \frac{dt}{\sin^2 t} = \int \text{cosec}^2\, t\,dt = -\cot t + C = -\cot(xe^x) + C$$

S68. Given integral is $\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)dx$

Put $\frac{1}{x} = t \Rightarrow -\frac{1}{x^2}dx = dt$

$$= -\int \cos^2 t\, dt = -\int \frac{1+\cos 2t}{2}dt$$

$$= -\frac{1}{2}\int 1\, dt - \frac{1}{2}\int \cos 2t\, dt = -\frac{1}{2}t - \frac{1}{4}\sin 2t + C$$

$$= -\frac{1}{2x} - \frac{1}{4}\sin\left(\frac{2}{x}\right) + C$$

S69. Given integral is $\int \frac{\log\left(1+\frac{1}{x}\right)}{x(1+x)}dx$

Put $\log\left(1+\frac{1}{x}\right) = t \Rightarrow \frac{1}{1+\frac{1}{x}}\cdot\left(\frac{-1}{x^2}\right)dx = dt$

$$\frac{-dx}{x(1+x)} = dt$$

$$\Rightarrow \quad -\int t\, dt = \frac{-t^2}{2} = \frac{-1}{2}\left[\log\left(1+\frac{1}{x}\right)\right]^2 + C$$

S70. Given integral is $\int \frac{\tan x \sec^2 x}{(a+b\tan^2 x)^2}dx$

Put $a + b\tan^2 x = t$

$$2b\tan x\sec^2 x\, dx = dt \Rightarrow \tan x \sec^2 x\, dx = \frac{1}{2b}dt$$

$$\Rightarrow \quad \frac{1}{2b}\int \frac{1}{t^2}dt = \frac{-1}{2bt} + C$$

$$= \frac{-1}{2b(a+b\tan^2 x)} + C$$

S71. Given integral is $\int \frac{a}{b+ce^x}dx$

$$= \int \frac{ae^{-x}}{be^{-x}+c}dx$$

Put $be^{-x} + c = t$

$\Rightarrow$ $-be^{-x}\,dx = dt \;\Rightarrow\; e^{-x}\,dx = \frac{-1}{b}dt$

$\Rightarrow$ $\frac{-a}{b}\int\frac{1}{t}dt = \frac{-a}{b}\log t + c = \frac{-a}{b}\log(be^{-x} + c) + C$

S72. Given integral is $\int\frac{x+1}{x(x+\log x)}dx$

Put $(x + \log x) = t \;\Rightarrow\; \left(1+\frac{1}{x}\right)dx = dt$

Hence $\int\left(1+\frac{1}{x}\right)\cdot\frac{dx}{(x+\log x)} = \int\frac{1}{t}dt = \log t + C = \log(x + \log x) + C$

S73. Given integral is $\int\frac{\operatorname{cosec} x}{\log\tan x/2}dx$

Put $\log\tan\frac{x}{2} = t \;\Rightarrow\; \frac{1}{\tan x/2}\cdot\sec^2(x/2)\cdot\frac{1}{2}dx = dt$

$= \frac{1}{2\sin x/2\cos x/2}dx = dt$

$= \operatorname{cosec} x\,dx = dt$

$\Rightarrow$ $\int\frac{1}{t}dt = \log t + C = \log(\log\tan x/2) + C$

S74. Given integral is $\int\tan^3 x\sec^3 x\,dx$

$\Rightarrow$ $\int\tan^2 x\sec^2 x(\sec x\tan x)\,dx$

$\Rightarrow$ $\int(\sec^2 x - 1)\sec^2 x(\sec x\tan x)\,dx$

Put $\sec x = t$

$\Rightarrow$ $\sec x\tan x\,dx = dt$

$= \int(t^2 - 1)\,t^2\,dt = \int(t^4 - t^2)\,dt$

$= \frac{t^5}{5} - \frac{t^3}{3} + C = \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$

S75.

$I = \int\frac{\cos^5 x}{\sin x}dx = \int\frac{\cos^4 x}{\sin x}\cdot\cos x\,dx$

$= \int\frac{(\cos^2 x)^2}{\sin x}\cdot\cos x\,dx = \int\frac{(1-\sin^2 x)^2}{\sin x}\cdot\cos x\,dx$

Put sin $x = t$ so that cos $x\,dx = dt$.

$$\therefore \quad I = \int \frac{(1-t^2)^2}{t}dt = \int \frac{1-2t^2+t^4}{t}dt$$

$$= \int \frac{1}{t}dt - 2\int t\,dt + \int t^3\,dt = \log|t| - 2\cdot\frac{t^2}{2} + \frac{t^4}{4} + c$$

$$= \log|\sin x| - \sin^2 x + \frac{1}{4}\sin^4 x + c$$

S76. Put $a^2 \sin^2 x + b^2 \cos^2 x = t$

$2a^2 \sin x \cos x + 2b^2 \cos x (-\sin x)\, dx = dt$

$$\Rightarrow \quad 2\sin x \cos x\, dx = \frac{dt}{a^2-b^2} \text{ i.e. } \sin 2x\, dx = \frac{dt}{a^2-b^2}$$

$$\therefore \quad I = \int \frac{1}{a^2-b^2}\cdot\frac{1}{t}dt$$

$$= \frac{1}{a^2-b^2}\int \frac{1}{t}dt = \frac{1}{a^2-b^2}\log|t| + c$$

$$= \frac{1}{a^2-b^2}\log|a^2\sin^2 x + b^2\cos^2 x| + c$$

S77.

$$I = \int 2x\sec^3(x^2+3)\tan(x^2+3)dx$$

$$= \int \sec^2(x^2+3)\cdot 2x\sec(x^2+3)\tan(x^2+3)dx$$

Put sec $(x^2 + 3) = t$ so that $2x$ sec $(x^2 + 3)$ tan $(x^2 + 3)$ $dx = dt$.

$$\therefore \quad I = \int t^2 dt = \frac{1}{3}t^3 + c = \frac{1}{3}\sec^3(x^2+3) + c$$

S78. Put $x^e + e^x = t$.

so that $(ex^{e-1} + e^x)dx = dt$ i.e. $(x^{e-1} + e^{x-1})dx = \frac{dt}{e}$

$$\therefore \quad I = \int \frac{\frac{1}{e}dt}{t} = \frac{1}{e}\int \frac{dt}{t} = \frac{1}{e}\log|t| + c = \frac{1}{e}\log|x^e + e^x| + c.$$

S79.

$$I = \int \frac{1}{\sqrt{\frac{\sin^3 x}{\cos^3 x}\cdot\cos^4 x}}dx = \int \frac{\sec^2 x}{\sqrt{\tan^3 x}}dx$$

Put tan $x = t$ so that $\sec^2 x\, dx = dt$.

$\therefore \qquad I = \int \frac{dt}{\sqrt{t^3}} = \int t^{-3/2} dt = \frac{t^{-\frac{3}{2}+1}}{-3/2+1} + c$

$$= \frac{t^{-1/2}}{-1/2} + c = -\frac{2}{\sqrt{t}} + c = -\frac{2}{\sqrt{\tan x}} + c.$$

S80. Given integral is $\int \frac{\sin^4 x}{\cos^8 x} dx$

$\Rightarrow \qquad \int \frac{\sin^4 x / \cos^4 x}{\cos^8 x / \cos^4 x} dx \qquad$ divided by $\cos^4 x$ Nr and Dr

$\Rightarrow \qquad \int \tan^4 x \sec^4 dx = \int \tan^4 x (1 + \tan^2 x) \sec^2 x\, dx$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$\Rightarrow \qquad \int t^4 (1 + t^2)\, dt = \int (t^4 + t^6)\, dt$

$\Rightarrow \qquad \frac{t^5}{5} + \frac{t^7}{7} + C = \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C$

S81. Given integral is $\int \frac{\sin 2x}{\sin\left(x - \frac{\pi}{3}\right) \sin\left(x + \frac{\pi}{3}\right)} dx$

Put $\sin^2 x - \sin^2 \frac{\pi}{3} = t$

$\Rightarrow \qquad 2 \sin x \cos x\, dx = dt$

$\Rightarrow \qquad \sin 2x\, dx = dt$

$\Rightarrow \qquad \int \frac{dt}{t} = \log t + C \Rightarrow \log(\sin^2 x - \sin^2 \pi/3) + C$

$$\Rightarrow \log \sin\left(x + \frac{\pi}{3}\right) \sin\left(x - \frac{\pi}{3}\right) + C$$

$$\Rightarrow \log \sin\left(x + \frac{\pi}{3}\right) + \log \sin\left(x - \frac{\pi}{3}\right) + C$$

S82. Given integral is $\int \sec^{4/3} x \operatorname{cosec}^{8/3} x\, dx$

$$= \int \frac{1}{\cos^{4/3} x \sin^{8/3} x} dx$$

$= \int \frac{\sec^4 x}{\tan^{8/3} x} dx \qquad$ Divided by $\cos^4 x$ Nr and Dr.

$\Rightarrow \qquad \int \frac{(1 + \tan^2 x) \sec^2 x}{\tan^{8/3} x} dx$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$= \int \frac{1+t^2}{t^{8/3}}\, dt = \int t^{-8/3}\, dt + \int t^{-2/3}\, dt$$

$$\Rightarrow \frac{-3}{5} t^{-5/3} + 3t^{1/3} + C = \frac{-3}{5} \tan^{-5/3} x + 3\tan^{1/3} x + C$$

S83. Given integral is $\int \frac{1}{\sqrt{\sin^3 x \cos^5 x}}\, dx$

$$\Rightarrow \int \frac{1}{\sin^{3/2} x \cos^{5/2} x}\, dx$$

$$\Rightarrow \int \frac{\sec^4 x}{\tan^{3/2} x}\, dx$$ divided by $\cos^4 x$ Nr and Dr

$$\Rightarrow \int \frac{(1+\tan^2 x)\sec^2 x}{\tan^{3/2} x}\, dx$$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$\Rightarrow \int \frac{1+t^2}{t^{3/2}}\, dt = \int (t^{-3/2} + t^{1/2})\, dt$$

$$\Rightarrow \frac{-2}{\sqrt{t}} + \frac{t^{3/2}}{3/2} + C = \frac{-2}{\sqrt{\tan x}} + \frac{2}{3}(\tan x)^{3/2} + C$$

S84. Given integral is $\int \tan^4 x\, dx$

$$\Rightarrow \int \tan^2 x \tan^2 x\, dx = \int \tan^2 x \left(\sec^2 x - 1\right) dx$$

$$\Rightarrow \int \tan^2 x \sec^2 x\, dx - \int \tan^2 x\, dx$$

$$\Rightarrow \int \tan^2 x \sec^2 x\, dx - \int (\sec^2 x - 1)\, dx$$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$ in First Integration

$$\Rightarrow \int t^2\, dt - \int (\sec^2 x - 1)\, dx$$

$$\Rightarrow \frac{t^3}{3} - (\tan x - x) + C = \frac{\tan^3 x}{3} - \tan x + x + C$$

S85. Given integral is $\int \cot^4 x\, dx$

$$\Rightarrow \int \cot^2 x \cot^2 x\, dx = \int (\text{cosec}^2 x - 1) \cot^2 x\, dx$$

$\Rightarrow \quad \int \text{cosec}^2 x \cot^2 x \, dx - \int \cot^2 x \, dx$

$\Rightarrow \quad \int \text{cosec}^2 x \cot^2 x \, dx - \int (\text{cosec}^2 x - 1) \, dx$

Put $\cot x = t \Rightarrow -\text{cosec}^2 x \, dx = dt$ in First Integral

$\Rightarrow \quad -\int t^2 \, dt - \int (\text{cosec}^2 x - 1) \, dx$

$$\Rightarrow -\frac{t^3}{3} - (-\cot x - x) + C = -\frac{t^3}{3} + (\cot x + x) + C$$

$$= -\frac{1}{3}\cot^3 x + \cot x + x + C$$

S86. Given integral is $\int \frac{1}{\sqrt[4]{(x-1)^3 (x+2)^5}} dx$

$$\Rightarrow \quad \int \frac{1}{\sqrt[4]{\left(\frac{x-1}{x+2}\right)^3 (x+2)^8}} dx = \int \frac{1}{\left(\frac{x-1}{x+2}\right)^{3/4} (x+2)^2} dx$$

Put $\frac{x-1}{x+2} = t \Rightarrow 1 - \frac{3}{x+2} = t \Rightarrow \frac{3}{(x+2)^2} dx = dt$

$$\Rightarrow \quad \frac{1}{3}\int \frac{dt}{t^{3/4}} = \frac{1}{3} \frac{t^{-\frac{3}{4}+1}}{-\frac{3}{4}+1} + C = \frac{4}{3} t^{1/4} + c = \frac{4}{3}\left(\frac{x-1}{x+2}\right)^{1/4} + c$$

S87. Given integral is $\int \frac{\sin 2x}{(a + b\cos x)^2} dx$

$$\Rightarrow \quad \int \frac{2\sin x \cos x}{(a + b\cos x)^2} dx$$

Put $a + b \cos x = t$

$\Rightarrow \quad -b \sin x \, dx = dt$ and $\cos x = \frac{t-a}{b}$

$$\Rightarrow \quad \frac{-2}{b}\int \frac{1}{t^2}\left(\frac{t-a}{b}\right) dt = \frac{-2}{b^2}\int \left(\frac{1}{t} - \frac{a}{t^2}\right) dt$$

$$= \frac{-2}{b^2}\left[\log t + \frac{a}{t}\right] = \frac{-2}{b^2}\left[\log(a + b\cos x) + \frac{a}{a + b\cos x}\right] + C$$

Date: 19/10/2021

Q1. Evaluate $\int \frac{dx}{9x^2 - 4}$

Q2. Evaluate $\int \frac{dx}{4 + 9x^2}$

Q3. Evaluate $\int \frac{dx}{16 - 9x^2}$

Q4. Write the value of $\int \frac{dx}{x^2 + 16}$.

Q5. Evaluate $\int \frac{1}{9x^2 + 6x + 10} dx$

Q6. Evaluate $\int \frac{1}{4x^2 - 4x + 3} dx$

Q7. Evaluate $\int \frac{1}{2x^2 + x - 1} dx$

Q8. Evaluate $\int \frac{1}{3x^2 + 13x - 10} dx$

Q9. Evaluate $\int \frac{1}{x^2 + 6x + 13} dx$

Q10. Evaluate $\int \frac{\cos x}{\sin^2 x + 4\sin x + 5} dx$

Q11. Evaluate $\int \frac{1}{x[6(\log x)^2 + 7\log x + 2]} dx$

Q12. Evaluate $\int \frac{e^x}{e^{2x} + 5e^x + 6} dx$

Q13. Evaluate $\int \frac{x\,dx}{x^4 - x^2 + 1}$

Q14. Evaluate $\int \frac{x}{x^4 + 2x^2 + 3} dx$

Q15. Evaluate $\int \frac{x}{3x^4 - 18x^2 + 11} dx$

Q16. Evaluate $\int \frac{e^x}{(1 + e^x)(2 + e^x)} dx$

Q17. Evaluate $\int \frac{x + 2}{2x^2 + 6x + 5} dx$

Q18. Evaluate $\int \frac{x^2(x^4 + 4)}{(x^2 + 4)} dx$

Q19. Evaluate $\int \frac{5x - 2}{1 + 2x + 3x^2} dx$

Q20. Evaluate $\int \frac{x + 3}{x^2 - 2x - 5} dx$

Q21. Evaluate $\int \frac{x-1}{3x^2-4x+3}\,dx$

Q22. Evaluate $\int \frac{2x-3}{x^2+3x-18}\,dx$

Q23. Evaluate $\int \frac{4x+1}{x^2+3x+2}\,dx$

Q24. Evaluate $\int \frac{x^2}{x^2+6x-12}\,dx$

Q25. Evaluate $\int \frac{\cos x}{\cos 3x}\,dx$

Q26. Evaluate $\int \frac{(x-1)^2}{x^2+2x+2}\,dx$

Q27. Evaluate $\int \frac{x^2+1}{x^2-5x+6}\,dx$

Q28. Evaluate $\int \frac{x^2+5x+3}{x^2+3x+2}\,dx$

Q29. Evaluate $\int \frac{1-3x}{3x^2+4x+2}\,dx$

Q30. Evaluate $\int \frac{x^3+x^2+2x+1}{x^2-x+1}\,dx$

Date: 19/10/2021

S1. Given integral is $\int \frac{dx}{9x^2 - 4}$

$$= \frac{1}{9}\int \frac{dx}{x^2 - \frac{4}{9}} = \frac{1}{9}\cdot\frac{1}{2\times\frac{2}{3}}\log\left(\frac{x-\frac{2}{3}}{x+\frac{2}{3}}\right) + C$$

$$= \frac{1}{12}\log\left(\frac{3x-2}{3x+2}\right) + C$$

S2. Given integral is $\int \frac{dx}{4 + 9x^2}$

$$= \frac{1}{9}\int \frac{dx}{x^2 + \frac{4}{9}} = \frac{1}{9}\int \frac{dx}{x^2 + \left(\frac{2}{3}\right)^2}$$

$$= \frac{1}{9}\cdot\frac{1}{\frac{2}{3}}\tan^{-1}\left(\frac{x}{2/3}\right) = \frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right) + C$$

S3. Given integral is $\int \frac{dx}{16 - 9x^2}$

$$= \frac{1}{9}\int \frac{dx}{\frac{16}{9} - x^2} = \frac{1}{9}\int \frac{dx}{\left(\frac{4}{3}\right)^2 - x^2} = \frac{1}{9}\cdot\frac{1}{2\left(\frac{4}{3}\right)}\log\left(\frac{\frac{4}{3}+x}{\frac{4}{3}-x}\right) + C$$

$$= \frac{1}{24}\log\left(\frac{4+3x}{4-3x}\right) + C$$

S4. Given integral is

$$\int \frac{dx}{x^2 + 16} = \int \frac{dx}{x^2 + (4)^2}$$

$$= \frac{1}{4}\tan^{-1}\frac{x}{4} + C \qquad \left[\because \int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C\right]$$

S5. Given integral is $\int \frac{1}{9x^2+6x+10}\,dx$

$$= \frac{1}{9}\int \frac{1}{x^2+\frac{2}{3}x+\frac{10}{9}}\,dx = \frac{1}{9}\int \frac{1}{x^2+\frac{2}{3}x+\frac{1}{9}-\frac{1}{9}+\frac{10}{9}}\,dx$$

$$= \frac{1}{9}\int \frac{1}{\left(x+\frac{1}{3}\right)^2+1^2}\,dx = \frac{1}{9}\tan^{-1}\left(x+\frac{1}{3}\right) = \frac{1}{9}\tan^{-1}\left(\frac{3x+1}{3}\right)+C$$

S6. Given integral is $\int \frac{1}{4x^2-4x+3}\,dx$

$$= \frac{1}{4}\int \frac{1}{x^2-x+\frac{3}{4}}\,dx = \frac{1}{4}\int \frac{1}{x^2-x+\frac{1}{4}-\frac{1}{4}+\frac{3}{4}}\,dx$$

$$= \frac{1}{4}\int \frac{1}{\left(x-\frac{1}{2}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}\,dx = \frac{1}{4}\cdot\frac{1}{1/\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{1}{2}}{1/\sqrt{2}}\right)+C$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right)+C$$

S7. Given integral is $\int \frac{1}{2x^2+x-1}\,dx$

$$= \frac{1}{2}\int \frac{dx}{x^2+\frac{x}{2}-\frac{1}{2}} = \frac{1}{2}\int \frac{dx}{x^2+\frac{x}{2}+\left(\frac{1}{4}\right)^2-\left(\frac{1}{4}\right)^2-\frac{1}{2}}$$

$$= \frac{1}{2}\int \frac{dx}{\left(x+\frac{1}{4}\right)^2-\left(\frac{3}{4}\right)^2} = \frac{1}{2}\cdot\frac{1}{2\cdot\left(\frac{3}{4}\right)}\log\left|\frac{x+\frac{1}{4}-\frac{3}{4}}{x+\frac{1}{4}+\frac{3}{4}}\right|+C$$

$$= \frac{1}{3}\log\left|\frac{x-\frac{1}{2}}{x+1}\right|+C = \frac{1}{3}\log\left|\frac{2x-1}{2(x+1)}\right|+C$$

S8. Given integral is $\int \frac{1}{3x^2+13x-10}\,dx$

$$= \frac{1}{3}\int \frac{1}{x^2+\frac{13x}{3}-\frac{10}{3}}\,dx = \frac{1}{3}\int \frac{1}{x^2+\frac{13x}{3}+\left(\frac{13}{6}\right)^2-\left(\frac{13}{6}\right)^2-\frac{10}{3}}\,dx$$

$$= \frac{1}{3}\int \frac{1}{\left(x+\frac{13}{6}\right)^2-\left(\frac{17}{6}\right)^2}\,dx = \frac{1}{3}\cdot\frac{1}{2\cdot\left(\frac{17}{6}\right)}\log\left(\frac{x+\frac{13}{6}-\frac{17}{6}}{x+\frac{13}{6}+\frac{17}{6}}\right)+C$$

$$= \frac{1}{17}\log\left(\frac{x-\frac{2}{3}}{x+5}\right)+C = \frac{1}{17}\log\left(\frac{3x-2}{3(x+5)}\right)+C$$

S9. Given integral is $\int \frac{1}{x^2+6x+13}\,dx$

$$= \int \frac{1}{x^2+6x+9-9+13}\,dx = \int \frac{1}{(x+3)^2+2^2}\,dx$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x+3}{2}\right)+C$$

S10. Given integral is $\int \frac{\cos x}{\sin^2 x+4\sin x+5}\,dx$

Put $\sin x = t \Rightarrow \cos x\,dx = dt$

$$\Rightarrow \int \frac{dt}{t^2+4t+5} = \int \frac{dt}{t^2+4t+4+1}$$

$$= \int \frac{dt}{(t+2)^2+1} = \tan^{-1}(t+2)+C$$

$$= \tan^{-1}(\sin x+2)+C$$

S11. Given integral is $\int \frac{1}{x[6(\log x)^2+7\log x+2]}\,dx$

Put $\log x = t \Rightarrow \frac{1}{x}dx = dt$

$$\Rightarrow \int \frac{1}{6t^2+7t+2}\,dt = \frac{1}{6}\int \frac{1}{t^2+\frac{7}{6}t+\frac{1}{3}}\,dt$$

$$\Rightarrow \frac{1}{6}\int \frac{1}{t^2+\frac{7}{6}t+\left(\frac{7}{12}\right)^2-\left(\frac{7}{12}\right)^2+\frac{1}{3}}\,dt$$

$$\Rightarrow \frac{1}{6}\int \frac{1}{\left(t+\frac{7}{12}\right)^2-\left(\frac{1}{12}\right)^2}\,dt = \frac{1}{6}\times\frac{1}{2\cdot\left(\frac{1}{12}\right)}\log\left(\frac{t+\frac{7}{12}-\frac{1}{12}}{t+\frac{7}{12}+\frac{1}{12}}\right)+C$$

$$\Rightarrow \log\left(\frac{2t+1}{3t+2}\right)+C = \log\left(\frac{2\log x+1}{3\log x+2}\right)+C$$

S12. Given integral is $\int \frac{e^x}{e^{2x}+5e^x+6}\,dx$

Put $e^x = t \Rightarrow e^x\,dx = dt$

$$\Rightarrow \quad \int \frac{dt}{t^2+5t+6} = \int \frac{dt}{t^2+5t+\left(\frac{5}{2}\right)^2-\left(\frac{5}{2}\right)^2+6}$$

$$\Rightarrow \quad \int \frac{dt}{\left(t+\frac{5}{2}\right)^2-\left(\frac{1}{2}\right)^2} = \frac{1}{2\cdot\frac{1}{2}}\log\left(\frac{t+\frac{5}{2}-\frac{1}{2}}{t+\frac{5}{2}+\frac{1}{2}}\right)+C$$

$$= \log\left(\frac{t+2}{t+3}\right)+C = \log\left(\frac{e^x+2}{e^x+3}\right)+C$$

S13. Put $x^2 = t$ so that $2x\,dx = dt$ *i.e.* $x\,dx = \frac{1}{2}\,dt.$

$$\therefore \quad I = \int \frac{\frac{1}{2}dt}{t^2-t+1} = \frac{1}{2}\int \frac{dt}{\left(t^2-t+\frac{1}{4}\right)+\frac{3}{4}}$$

$$= \frac{1}{2}\int \frac{dt}{\left(t-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{2}\cdot\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\frac{t-1/2}{\frac{\sqrt{3}}{2}}+c$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\frac{2t-1}{\sqrt{3}}+c = \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x^2-1}{\sqrt{3}}+c.$$

S14. Given integral is $\int \frac{x}{x^4+2x^2+3}\,dx$

Put $x^2 = t \Rightarrow 2x\,dx = dt$

$$\Rightarrow \quad \frac{1}{2}\int \frac{dt}{t^2+2t+3} = \frac{1}{2}\int \frac{dt}{t^2+2t+1+2}$$

$$\Rightarrow \quad \frac{1}{2}\int \frac{dt}{(t+1)^2+(\sqrt{2})^2} = \frac{1}{2}\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t+1}{\sqrt{2}}\right) = \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2+1}{\sqrt{2}}\right)+C$$

S15. Given integral is $\int \frac{x}{3x^4-18x^2+11}\,dx$

Put $x^2 = t \Rightarrow 2x\,dx = dt$

$$\frac{1}{2}\int\frac{dt}{3t^2-18t+11} = \frac{1}{6}\int\frac{dt}{t^2-6t+\frac{11}{3}}$$

$$= \frac{1}{6}\int\frac{dt}{t^2-6t+9-9+\frac{11}{3}} = \frac{1}{6}\int\frac{dt}{(t-3)^2-\frac{16}{3}}$$

$$= \frac{1}{6}\int\frac{dt}{(t-3)^2-\left(\frac{4}{\sqrt{3}}\right)^2} = \frac{1}{6}\cdot\frac{1}{2\cdot\frac{4}{\sqrt{3}}}\log\left(\frac{t-3-\frac{4}{\sqrt{3}}}{t-3+\frac{4}{\sqrt{3}}}\right)+C$$

$$= \frac{\sqrt{3}}{48}\log\left(\frac{x^2-3-\frac{4}{\sqrt{3}}}{x^2-3+\frac{4}{\sqrt{3}}}\right)+C$$

S16. Given integral is $\int\frac{e^x}{(1+e^x)(2+e^x)}\,dx$

Put $e^x = t \Rightarrow e^x\,dx = dt$

$$\int\frac{dt}{(1+t)(2+t)} = \int\frac{dt}{t^2+3t+2}$$

$$= \int\frac{dt}{t^2+3t+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2+2} = \int\frac{dt}{\left(t+\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}$$

$$= \frac{1}{2\cdot\frac{1}{2}}\log\left(\frac{t+\frac{3}{2}-\frac{1}{2}}{t+\frac{3}{2}+\frac{1}{2}}\right)+C = \log\left(\frac{t+1}{t+2}\right)+C$$

$$= \log\left(\frac{e^x+1}{e^x+2}\right)+C$$

S17. Given integral is $\int\frac{x+2}{2x^2+6x+5}\,dx$

Put $x + 2 = A\frac{d}{dx}(2x^2 + 6x + 5) + B$

$x + 2 = A(4x + 6) + B$

$$4A = 1 \text{ and } 6A + B = 2$$

$$A = \frac{1}{4} \text{ and } B = \frac{1}{2}$$

$$= \int \frac{\frac{1}{4}(4x+6) + \frac{1}{2}}{2x^2+6x+5} \, dx$$

$$= \frac{1}{4}\int \frac{4x+6}{2x^2+6x+5}\, dx + \frac{1}{2}\int \frac{dx}{2x^2+6x+5}$$

$$= \frac{1}{4}\log(2x^2+6x+5) + \frac{1}{4}\int \frac{dx}{x^2+3x+\frac{9}{4}-\frac{9}{4}+\frac{5}{2}}$$

$$= \frac{1}{4}\log(2x^2+6x+5) + \frac{1}{4}\int \frac{dx}{\left(x+\frac{3}{2}\right)^2+\left(\frac{1}{2}\right)^2}$$

$$= \frac{1}{4}\log(2x^2+6x+5) + \frac{1}{4}\cdot\frac{1}{1/2}\tan\left(\frac{x+\frac{3}{2}}{\frac{1}{2}}\right) + C$$

$$= \frac{1}{4}\log(2x^2+6x+5) + \frac{1}{2}\tan^{-1}(2x+3) + C$$

S18. Given integral is $\int \frac{x^2(x^4+4)}{(x^2+4)}\, dx$

$$= \int \frac{x^6+4x^2}{x^2+4}\, dx = \int\left(x^4 - 4x^2 + 20 - \frac{80}{x^2+4}\right)dx$$

$$= \int (x^4-4x^2+20)\, dx - 80\int \frac{dx}{x^2+4}$$

$$= \frac{x^5}{5} - \frac{4}{3}x^3 + 20x - 80\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

$$= \frac{x^5}{5} - \frac{4}{3}x^3 + 20x - 40\tan^{-1}\left(\frac{x}{2}\right) + C$$

S19. $I = \int \frac{5x-2}{1+2x+3x^2}\, dx$

Let $5x - 2 = A\frac{d}{dx}(1+2x+3x^2) + B$

$5x - 2 = A(2 + 6x) + B$

Comparing the coefficient, we get

$$5 = 6A$$

$\Rightarrow$ $A = \dfrac{5}{6}$, $2A + B = -2$

$\Rightarrow$ $B = -2A - 2$

$$B = -\frac{5}{3} - 2 = -\frac{11}{3} \qquad \left[\because \; A = \frac{5}{6}\right]$$

Hence $I = \displaystyle\int \frac{\frac{5}{6}(2+6x)}{1+2x+3x^2}\,dx - \int \frac{\frac{11}{3}}{1+2x+3x^2}\,dx$

Let $I_1 = \dfrac{5}{6}\displaystyle\int \frac{2+6x}{1+2x+3x^2}\,dx$

Put $1 + 2x + 3x^2 = t$

$\Rightarrow$ $(2 + 6x)\mathrm{d}x = dt$

$\therefore$ $I_1 = \dfrac{5}{6}\displaystyle\int \frac{dt}{t} = \frac{5}{6}\ln t + C_1 = \frac{5}{6}\ln|1+2x+3x^2| + C_1$

and let $I_2 = \dfrac{11}{3}\displaystyle\int \frac{dx}{3x^2+2x+1} = \frac{11}{9}\int \frac{dx}{x^2 + \frac{2}{3}x + \frac{1}{3} + \frac{1}{9} - \frac{1}{9}}$

[Making a perfect square in denominator]

$$= \frac{11}{9}\int \frac{dx}{\left(x+\frac{1}{3}\right)^2 + \frac{2}{9}}$$

$$= \frac{11}{9}\cdot\frac{1}{\frac{\sqrt{2}}{3}}\tan^{-1}\left|\frac{x+\frac{1}{3}}{\frac{\sqrt{2}}{3}}\right| + C_2 \qquad \left[\because \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$= \frac{11}{3\sqrt{2}}\tan^{-1}\left|\left(\frac{3x+1}{\sqrt{2}}\right)\right| + C_2$$

$\Rightarrow$ As $I = I_1 - I_2$

$$= \frac{5}{6}\ln\left|1+2x+3x^2\right| - \frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C \qquad [\because \text{Put} \;\; C = C_1 - C_2]$$

S20. Given $I = \displaystyle\int \frac{x+3}{x^2-2x-5}\,dx$

Let $x + 3 = A + B\,\dfrac{d}{dx}(x^2 - 2x - 5)$

$$x + 3 = A + B\,(2x - 2)$$

Equating the coefficients of like terms, we get

$$2B = 1 \quad \Rightarrow \quad B = \frac{1}{2}$$

and $\quad A - 2B = 3$

$\therefore \quad A - 2 \times \frac{1}{2} = 3$

$\Rightarrow \quad A = 3 + 1 = 4$

$$\therefore \quad I = \frac{\int 4 + \frac{1}{2}(2x-2)}{x^2 - 2x - 5}\, dx$$

$$I = 4\int \frac{1}{x^2 - 2x - 5}\, dx + \frac{1}{2}\int \frac{2x-2}{x^2 - 2x - 5}\, dx$$

Put $x^2 - 2x - 5 = t$ in second integral

$\therefore \quad (2x - 2)\, dx = dt$

$$\therefore \quad I = 4\int \frac{dx}{(x-1)^2 - 1 - 5} + \frac{1}{2}\int \frac{dt}{t}$$

$$= 4\int \frac{dx}{(x-1)^2 - (\sqrt{6})^2} + \frac{1}{2}\log|t| + C \quad \left[\because \int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C\right]$$

$$= 4 \cdot \frac{1}{2\sqrt{6}} \log\left|\frac{x - 1 - \sqrt{6}}{x - 1 + \sqrt{6}}\right| + \frac{1}{2}\log\left|x^2 - 2x - 5\right| + C$$

$$= \frac{2}{\sqrt{6}} \log\left|\frac{x - 1 - \sqrt{6}}{x - 1 + \sqrt{6}}\right| + \frac{1}{2}\log\left|x^2 - 2x - 5\right| + C$$

S21. Given integral is $\int \frac{x-1}{3x^2 - 4x + 3}\, dx$

$$x - 1 = A\frac{d}{dx}(3x^2 - 4x + 3) + B$$

$$x - 1 = A(6x - 4) + B$$

$$6A = 1 \text{ and } -4A + B = -1$$

$$A = \frac{1}{6} \text{ and } B = -\frac{1}{3}$$

$$= \int \frac{\frac{1}{6}(6x-4) - \frac{1}{3}}{3x^2 - 4x + 3}\, dx$$

$$= \frac{1}{6}\int \frac{6x-4}{3x^2 - 4x + 3}\, dx - \frac{1}{3}\int \frac{dx}{3x^2 - 4x + 3}$$

$$= \frac{1}{6}\log(3x^2 - 4x + 3) - \frac{1}{9}\int \frac{dx}{x^2 - \frac{4}{3}x + \frac{4}{9} - \frac{4}{9} + 1}$$

$$= \frac{1}{6}\log(3x^2 - 4x + 3) - \frac{1}{9}\int \frac{dx}{\left(x - \frac{2}{3}\right)^2 + \left(\frac{\sqrt{5}}{3}\right)^2}$$

$$= \frac{1}{6}\log(3x^2 - 4x + 3) - \frac{1}{9}\cdot\frac{1}{\frac{\sqrt{5}}{3}}\tan^{-1}\left(\frac{x - \frac{2}{3}}{\frac{\sqrt{5}}{3}}\right) + C$$

$$= \frac{1}{6}\log(3x^2 - 4x + 3) - \frac{1}{3\sqrt{5}}\tan^{-1}\left(\frac{3x - 2}{\sqrt{5}}\right) + C$$

S22. Given integral is $\int \frac{2x - 3}{x^2 + 3x - 18}\,dx$

Put $2x - 3 = A\frac{d}{dx}(x^2 + 3x - 18) + B$

$2x - 3 = A(2x + 3) + B$

$2A = 2$ and $3A + B = -3$

$A = 1$ and $B = -6$

$\therefore$ $\int \frac{(2x + 3) - 6}{x^2 + 3x - 18}\,dx$

$$= \int \frac{(2x + 3)}{x^2 + 3x - 18}\,dx - 6\int \frac{dx}{x^2 + 3x - 18}$$

$$= \log(x^2 + 3x - 18) - 6\int \frac{dx}{x^2 + 3x + \frac{9}{4} - \frac{9}{4} - 18}$$

$$= \log(x^2 + 3x - 18) - 6\int \frac{dx}{\left(x + \frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2}$$

$$= \log(x^2 + 3x - 18) - 6\cdot\frac{1}{2\cdot\frac{9}{2}}\log\left(\frac{x + \frac{3}{2} - \frac{9}{2}}{x + \frac{3}{2} + \frac{9}{2}}\right) + C$$

$$= \log(x^2 + 3x - 18) - \frac{2}{3}\log\left(\frac{x - 3}{x + 6}\right) + C$$

S23. Given integral is $\int \frac{4x+1}{x^2+3x+2}dx$

Put $4x + 1 = A\frac{d}{dx}(x^2 + 3x + 2) + B$

$4x + 1 = A[2x + 3] + B$

$2A = 4$ and $3A + B = 1$

$A = 2$ and $B = -5$

$$= \int \frac{2(2x+3)-5}{x^2+3x+2}dx$$

$$= 2\int \frac{2x+3}{x^2+3x+2}dx - 5\int \frac{dx}{x^2+3x+2}$$

$$= 2\log(x^2+3x+2) - 5\int \frac{dx}{x^2+3x+\frac{9}{4}-\frac{9}{4}+2}$$

$$= 2\log(x^2+3x+2) - 5\int \frac{dx}{\left(x+\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}$$

$$= 2\log(x^2+3x+2) - 5\cdot\frac{1}{2\cdot\frac{1}{2}}\log\left\{\frac{\left(x+\frac{3}{2}\right)-\frac{1}{2}}{\left(x+\frac{3}{2}\right)+\frac{1}{2}}\right\}+C$$

$$= 2\log(x^2+3x+2) - 5\log\left(\frac{x+1}{x+2}\right)+C$$

S24. $\because \quad \frac{x^2}{x^2+6x-12} = 1 + \frac{12-6x}{x^2+6x-12}$

Let $I = \int \frac{x^2}{x^2+6x-12}dx$

$$= \left(1+\frac{12-6x}{x^2+6x-12}\right)dx$$

$$= \int 1\cdot dx - 6\int \frac{x-2}{x^2+6x-12}dx$$

Put $x - 2 = A\frac{d}{dx}(x^2 + 6x - 12) + B$

Comparing the coff. of x and constant term both side, we get

$$x - 2 = A(2x + 6) + B$$

$$x - 2 = 2Ax + 6A + B$$

$$2A = 1, \quad 6A + B = -2$$

$$A = \frac{1}{2}, \quad 6A + B = -2$$

$$A = \frac{1}{2}, \quad 6 \times \frac{1}{2} + B = -2$$

$$A = \frac{1}{2}, \quad B = -5$$

$$\therefore \quad x - 2 = \frac{1}{2}(2x + 6) - 5$$

$$= x - 6\int \frac{\frac{1}{2}(2x+6) - 5}{x^2 + 6x - 12}dx$$

$$= x - 3\int \frac{2x+6}{x^2 + 6x - 12}dx + 30\int \frac{dx}{(x^2 + 6x + 9) - 21}$$

$$= x - 3\log|x^2 + 6x - 12| + 30\int \frac{dx}{(x+3)^2 - (\sqrt{21})^2}$$

$$= x - 3\log|x^2 + 6x - 12| + \frac{30}{2\sqrt{21}}\log\left|\frac{x + 3 - \sqrt{21}}{x + 3 + \sqrt{21}}\right| + c \quad \left[\because \int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|\right]$$

$$= x - 3\log|x^2 + 6x - 12| + \frac{15}{\sqrt{21}}\log\left|\frac{x + 3 - \sqrt{21}}{x + 3 + \sqrt{21}}\right| + c$$

S25.

$$I = \int \frac{\cos x}{\cos 3x}dx = \int \frac{\cos x}{4\cos^3 x - 3\cos x}dx$$

$$= \int \frac{1}{4\cos^2 x - 3}dx$$

$$= \int \frac{1}{4\cos^2 x - 3(\sin^2 x + \cos^2 x)}dx$$

$$= \int \frac{dx}{\cos^2 x - 3\sin^2 x}$$

$$= \int \frac{\sec^2 x\, dx}{1 - 3\tan^2 x}$$ [Dividing num. and Denom. by $\cos^2 x$]

Put $\tan x = t$ so that $\sec^2 x\, dx = dt$.

$$\therefore \quad I = \int \frac{dt}{1 - 3t^2} = \frac{1}{3}\int \frac{dt}{\frac{1}{3} - t^2}$$

$$= \frac{1}{3}\int \frac{dt}{\left(\frac{1}{\sqrt{3}}\right)^2 - t^2}$$

$$= \frac{1}{3} \cdot \frac{1}{2\left(\frac{1}{\sqrt{3}}\right)} \log\left|\frac{\frac{1}{\sqrt{3}} + t}{\frac{1}{\sqrt{3}} - t}\right| + c$$

$$= \frac{1}{2\sqrt{3}} \log\left|\frac{1 + \sqrt{3}t}{1 - \sqrt{3}t}\right| + c$$

$$= \frac{1}{2\sqrt{3}} \log\left|\frac{1 + \sqrt{3}\tan x}{1 - \sqrt{3}\tan x}\right| + c$$

S26. Given integral is $\int \frac{(x-1)^2}{x^2 + 2x + 2}\, dx$

$$= \int \frac{x^2 - 2x + 1}{x^2 + 2x + 2}\, dx = \int\left(1 - \frac{4x + 1}{x^2 + 2x + 2}\right) dx$$

$$= \int 1\, dx - \int \frac{4x + 1}{x^2 + 2x + 2}\, dx = I_1 - I_2$$

For I_2 Let $4x + 1 = A\frac{d}{dx}(x^2 + 2x + 2) + B$

$$4x + 1 = A(2x + 2) + B$$

$$2A = 4 \text{ and } 2A + B = 1$$

$$A = 2 \text{ and } B = -3$$

$$I_2 = \int \frac{2(2x + 2) - 3}{x^2 + 2x + 2}\, dx$$

$$= 2\int \frac{2x + 2}{x^2 + 2x + 2}\, dx - 3\int \frac{dx}{x^2 + 2x + 2}$$

$$= 2\log(x^2 + 2x + 2) - 3\int \frac{dx}{(x+1)^2 + (1)^2}$$

$$= 2\log(x^2 + 2x + 2) - 3\tan^{-1}(x + 1) + C_1$$

$$\therefore \quad \int \frac{(x-1)^2}{x^2+2x+2}\,dx = I_1 - I_2$$

Put $(C = -C_1)$

$$= x - 2\log(x^2+2x+2) + 3\tan^{-1}(x+1) + C$$

S27. Given integral is $\int \frac{x^2+1}{x^2-5x+6}\,dx$

$$= \int\left(1+\frac{5x-5}{x^2-5x+6}\right)dx = \underset{I_1}{\int 1\,dx} + 5\underset{I_2}{\int \frac{x-1}{x^2-5x+6}\,dx}$$

For I_2 Let $x-1 = A\frac{d}{dx}(x^2-5x+6) + B$

$$x-1 = A(2x-5) + B$$

$$2A = 1 \text{ and } -5A + B = -1$$

$$A = \frac{1}{2} \text{ and } B = \frac{3}{2}$$

$$\therefore \quad I_2 = \int \frac{\frac{1}{2}(2x-5)+\frac{3}{2}}{x^2-5x+6}\,dx$$

$$= \frac{1}{2}\int \frac{2x-5}{x^2-5x+6}\,dx + \frac{3}{2}\int \frac{dx}{x^2-5x+6}$$

$$= \frac{1}{2}\log(x^2-5x+6) + \frac{3}{2}\int \frac{dx}{x^2-5x+\frac{25}{4}-\frac{25}{4}+6}$$

$$= \frac{1}{2}\log(x^2-5x+6) + \frac{3}{2}\int \frac{dx}{\left(x-\frac{5}{2}\right)^2-\left(\frac{1}{2}\right)^2}$$

$$= \frac{1}{2}\log(x^2-5x+6) + \frac{3}{2}\cdot\frac{1}{2\cdot\frac{1}{2}}\log\left(\frac{x-\frac{5}{2}-\frac{1}{2}}{x-\frac{5}{2}+\frac{1}{2}}\right) + C$$

$$= \frac{1}{2}\log(x^2-5x+6) + \frac{3}{2}\log\left(\frac{x-3}{x-2}\right) + C$$

$$\therefore \quad \int \frac{x^2+1}{x^2-5x+6}\,dx = I_1 + 5I_2$$

$$= x + \frac{5}{2}\log(x^2-5x+6) + \frac{15}{2}\log\left(\frac{x-3}{x-2}\right) + C$$

S28. Given integral is $\int \frac{x^2 + 5x + 3}{x^2 + 3x + 2} dx$

$$= \int \left(1 + \frac{2x+1}{x^2 + 3x + 2}\right) dx = \underbrace{\int 1\, dx}_{I_1} + \underbrace{\int \frac{2x+1}{x^2+3x+2} dx}_{I_2}$$

For I_2

Let $2x + 1 = A \frac{d}{dx}(x^2 + 3x + 2) + B$

$2x + 1 = A(2x + 3) + B$

$2A = 2$ and $3A + B = 1$

$A = 1$ and $B = -2$

$\therefore \quad I_2 = \int \frac{(2x+3) - 2}{x^2 + 3x + 2} dx$

$$= \int \frac{2x+3}{x^2+3x+2} dx - 2\int \frac{dx}{x^2 + 3x + 2}$$

$$= \log(x^2 + 3x + 2) - 2\int \frac{dx}{x^2 + 3x + \frac{9}{4} - \frac{9}{4} + 2}$$

$$= \log(x^2 + 3x + 2) - 2\int \frac{dx}{\left(x + \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}$$

$$= \log(x^2 + 3x + 2) - 2 \cdot \frac{1}{2 \cdot \frac{1}{2}} \log\left(\frac{x + \frac{3}{2} - \frac{1}{2}}{x + \frac{3}{2} + \frac{1}{2}}\right) + C$$

$$= \log(x^2 + 3x + 2) - 2 \cdot \log\left(\frac{x+1}{x+2}\right) + C$$

$\therefore \quad \int \frac{x^2 + 5x + 3}{x^2 + 3x + 2} dx = I_1 + I_2$

$$= x + \log(x^2 + 3x + 2) - 2\log\left(\frac{x+1}{x+2}\right) + C$$

S29. Given integral is $\int \frac{1 - 3x}{3x^2 + 4x + 2} dx$

Put $1 - 3x = A \frac{d}{dx}(3x^2 + 4x + 2) + B$

$1 - 3x = A(6x + 4) + B$

$$6A = -3 \text{ and } 4A + B = 1$$

$$A = -\frac{1}{2} \text{ and } B = 3$$

$$= \int \frac{\frac{-1}{2}(6x+4)+3}{3x^2+4x+2}\,dx$$

$$= -\frac{1}{2}\int \frac{6x+4}{3x^2+4x+2}\,dx + 3\int \frac{dx}{3x^2+4x+2}$$

$$= -\frac{1}{2}\log(3x^2+4x+2) + \frac{3}{3}\int \frac{dx}{x^2+\frac{4x}{3}+\frac{2}{3}}$$

$$= -\frac{1}{2}\log(3x^2+4x+2) + \int \frac{dx}{x^2+\frac{4}{3}x+\frac{4}{9}-\frac{4}{9}+\frac{2}{3}}$$

$$= -\frac{1}{2}\log(3x^2+4x+2) + \int \frac{dx}{\left(x+\frac{2}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2}$$

$$= -\frac{1}{2}\log(3x^2+4x+2) + \frac{1}{\frac{\sqrt{2}}{3}}\tan^{-1}\left(\frac{x+\frac{2}{3}}{\frac{\sqrt{2}}{3}}\right) + C$$

$$= -\frac{1}{2}\log(3x^2+4x+2) + \frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{3x+2}{\sqrt{2}}\right) + C$$

S30. Given integral is $\int \frac{x^3+x^2+2x+1}{x^2-x+1}\,dx$

$$= \int\left(x+2+\frac{3x-1}{x^2-x+1}\right)dx = \underbrace{\int (x+2)\,dx}_{I_1} + \underbrace{\int \frac{3x-1}{x^2-x+1}\,dx}_{I_2}$$

For I_2 Let $3x-1 = A\frac{d}{dx}(x^2-x+1)+B$

$$3x-1 = A(2x-1)+B$$

$$2A = 3 \text{ and } -A+B = -1$$

$$A = \frac{3}{2} \text{ and } B = \frac{1}{2}$$

$$\therefore \quad I_2 = \int \frac{\frac{3}{2}(2x-1) + \frac{1}{2}}{x^2 - x + 1} dx$$

$$= \frac{3}{2} \int \frac{2x-1}{x^2-x+1} dx + \frac{1}{2} \int \frac{dx}{x^2-x+1}$$

$$= \frac{3}{2} \log(x^2 - x + 1) + \frac{1}{2} \int \frac{dx}{x^2 - x + \frac{1}{4} - \frac{1}{4} + 1}$$

$$= \frac{3}{2} \log(x^2 - x + 1) + \frac{1}{2} \int \frac{dx}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{3}{2} \log(x^2 - x + 1) + \frac{1}{2} \cdot \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1}\left(\frac{\left(x - \frac{1}{2}\right)}{\frac{\sqrt{3}}{2}}\right) + C$$

$$= \frac{3}{2} \log(x^2 - x + 1) + \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

$$\therefore \quad \int \frac{x^3 + x^2 + 2x + 1}{x^2 - x + 1} dx = I_1 + I_2$$

$$= \frac{x^2}{2} + 2x + \frac{3}{2} \log(x^2 - x + 1) + \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

Date: 19/10/2021

Q1. Evaluate $\int \frac{dx}{\sqrt{16x^2+25}}$

Q2. Evaluate $\int \frac{dx}{\sqrt{1-x^2}}$

Q3. Evaluate $\int \frac{1}{\sqrt{9-25x^2}}dx$

Q4. Evaluate $\int \frac{1}{\sqrt{1+4x^2}}dx$

Q5. Evaluate $\int \frac{dx}{\sqrt{4x^2-9}}$

Q6. Evaluate $\int \frac{1}{\sqrt{9+8x-x^2}}dx$

Q7. Evaluate $\int \frac{1}{\sqrt{16-6x-x^2}}dx$

Q8. Evaluate $\int \frac{1}{\sqrt{7-6x-x^2}}dx$

Q9. Evaluate $\int \frac{1}{\sqrt{(x-1)(x-2)}}dx$

Q10. Evaluate $\int \frac{dx}{\sqrt{8+3x-x^2}}$

Q11. Evaluate $\int \frac{1}{\sqrt{3x^2+5x+7}}dx$

Q12. Evaluate $\int \frac{2x}{\sqrt{1-x^2-x^4}}dx$

Q13. Evaluate $\int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}}dx$

Q14. Evaluate $\int \frac{\cos x}{\sqrt{\sin^2 x-2\sin x-3}}dx$

Q15. Evaluate $\int \sqrt{\frac{x}{a^3-x^3}}dx$

Q16. Evaluate $\int \sqrt{\sec x-1}\,dx$

Q17. Evaluate $\int \frac{\sin 2x}{\sqrt{\sin^4 x+4\sin^2 x-2}}dx$

Q18. Evaluate $\int \frac{\sin x-\cos x}{\sqrt{\sin 2x}}dx$

Q19. Evaluate $\int \frac{(\sin x+\cos x)}{\sqrt{\sin 2x}}dx$

Q20. Evaluate $I = \int \frac{\sin 2x}{\sqrt{\cos^4 x-\sin^2 x-4}}dx$

Q21. Evaluate $\int \frac{dx}{x^{2/3}\sqrt{x^{2/3}-4}}$

Q22. Evaluate $\int \sqrt{\frac{a-x}{a+x}}\,dx$

Q23. Evaluate $\int \sqrt{e^x-1}\,dx$

Q24. Evaluate $\int \left(\sqrt{\tan x}+\sqrt{\cot x}\right)dx$

Q25. Evaluate $\int \frac{x+2}{\sqrt{x^2+5x+6}}dx$

Q26. Evaluate $\int \frac{2x+3}{\sqrt{x^2+4x+2}}dx$

Q27. Evaluate $\int \frac{x+1}{\sqrt{4+5x-x^2}}dx$

Q28. Evaluate $\int \frac{3x+1}{\sqrt{5-2x-x^2}}dx$

Q29. Evaluate $\int \frac{2x+1}{\sqrt{x^2+4x+3}}dx$

Q30. Evaluate $\int \frac{2x+3}{\sqrt{x^2+4x+5}}dx$

Q31. Evaluate $\int \frac{dx}{\sqrt{5-4x-2x^2}}$

Q32. Evaluate $\int \frac{x+2}{\sqrt{x^2+2x+3}}dx$

Q33. Evaluate $\int \frac{3x+5}{\sqrt{x^2-8x+7}}dx$

Q34. Evaluate $\int \frac{6x+7}{\sqrt{(x-5)(x-4)}}dx$

Q35. Evaluate $\int \frac{2x+5}{\sqrt{7-6x-x^2}}dx$

Q36. Evaluate the following $\int \frac{x+2}{\sqrt{(x-2)(x-3)}}dx$

Q37. Evaluate $\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx$

Date: 19/10/2021

S1. Given integral is $\int \frac{dx}{\sqrt{16x^2+25}}$

$$= \frac{1}{4}\int \frac{dx}{\sqrt{x^2+\frac{25}{16}}} = \frac{1}{4}\int \frac{dx}{\sqrt{x^2+\left(\frac{5}{4}\right)^2}} = \frac{1}{4}\log\left(x+\sqrt{x^2+\left(\frac{5}{4}\right)^2}\right)+C$$

S2. Given integral is $\int \frac{dx}{\sqrt{1-x^2}}$

$$= \int \frac{dx}{\sqrt{(1)^2-x^2}} = \sin^{-1} x + c \qquad \left[\because \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c\right]$$

S3.

$$I = \frac{1}{5}\int \frac{1}{\sqrt{\frac{9}{25}-x^2}}dx = \frac{1}{5}\int \frac{1}{\sqrt{\left(\frac{3}{5}\right)^2-x^2}}dx$$

$$= \frac{1}{5}\cdot\sin^{-1}\frac{x}{\frac{3}{5}}+c = \frac{1}{5}\sin^{-1}\frac{5x}{3}+c$$

S4.

$$I = \frac{1}{2}\int \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2+x^2}} = \frac{1}{2}\log\left|x+\sqrt{\frac{1}{4}+x^2}\right|+c'$$

$$= \frac{1}{2}\log|2x+\sqrt{4x^2+1}|+c, \text{ where } c = c' - \frac{1}{2}\log 2.$$

S5. Given integral is $\int \frac{dx}{\sqrt{4x^2-9}}$

$$= \frac{1}{2}\int \frac{dx}{\sqrt{x^2-\frac{9}{4}}} = \frac{1}{2}\int \frac{dx}{\sqrt{x^2-\left(\frac{3}{2}\right)^2}} = \frac{1}{2}\log\left(x+\sqrt{x^2-\frac{9}{4}}\right)+C$$

S6. Given integral is $\int \frac{1}{\sqrt{9+8x-x^2}}\,dx$

$$= \int \frac{1}{\sqrt{-(x^2-8x-9)}}\,dx = \int \frac{dx}{\sqrt{-(x^2-8x+16-16-9)}}$$

$$= \int \frac{dx}{\sqrt{-\{(x-4)^2-5^2\}}} = \int \frac{dx}{\sqrt{5^2-(x-4)^2}}$$

$$= \sin^{-1}\left(\frac{x-4}{5}\right)+C$$

S7. Given integral is $\int \frac{1}{\sqrt{16-6x-x^2}}\,dx$

$$= \int \frac{1}{\sqrt{-(x^2+6x-16)}}\,dx = \int \frac{dx}{\sqrt{-(x^2+6x+9-9-16)}}$$

$$= \int \frac{dx}{\sqrt{-\left\{(x+3)^2-5^2\right\}}} = \int \frac{dx}{\sqrt{5^2-(x+3)^2}}$$

$$= \sin^{-1}\left(\frac{x+3}{5}\right)+C$$

S8. Given integral is $\int \frac{1}{\sqrt{7-6x-x^2}}\,dx$

$$= \int \frac{1}{\sqrt{-(x^2+6x-7)}}\,dx$$

$$= \int \frac{1}{\sqrt{-(x^2+6x+9-9-7)}}\,dx = \int \frac{1}{\sqrt{-\{(x+3)^2-16\}}}\,dx$$

$$= \int \frac{dx}{\sqrt{16-(x+3)^2}} = \sin^{-1}\left(\frac{x+3}{4}\right)+C$$

S9. Given integral is $\int \frac{1}{\sqrt{(x-1)(x-2)}}\,dx$

$$\Rightarrow \quad \int \frac{1}{\sqrt{x^2-3x+2}}\,dx = \int \frac{1}{\sqrt{x^2-3x+\frac{9}{4}-\frac{9}{4}+2}}\,dx$$

$$= \int \frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}\,dx$$

$$= \log\left[\left(x - \frac{3}{2}\right) + \sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\right] + C$$

$$= \log\left[\left(x - \frac{3}{2}\right) + \sqrt{x^2 - 3x + 2}\right] + C$$

S10. Given integral is $\int \frac{dx}{\sqrt{8 + 3x - x^2}}$

$$= \int \frac{dx}{\sqrt{-(x^2 - 3x - 8)}} = \int \frac{dx}{\sqrt{-\left(x^2 - 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 - 8\right)}}$$

$$= \int \frac{dx}{\sqrt{-\left\{\left(x - \frac{3}{2}\right)^2 - \left(\frac{\sqrt{41}}{2}\right)^2\right\}}} = \int \frac{dx}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2}}$$

$$= \sin^{-1}\left(\frac{x - \frac{3}{2}}{\frac{\sqrt{41}}{2}}\right) + C = \sin^{-1}\left(\frac{2x - 3}{\sqrt{41}}\right) + C$$

S11. Given integral is $\int \frac{1}{\sqrt{3x^2 + 5x + 7}}\, dx$

$$= \frac{1}{\sqrt{3}} \int \frac{dx}{\sqrt{x^2 + \frac{5}{3}x + \frac{7}{3}}} = \frac{1}{\sqrt{3}} \int \frac{dx}{\sqrt{x^2 + \frac{5}{3}x + \left(\frac{5}{6}\right)^2 - \left(\frac{5}{6}\right)^2 + \frac{7}{3}}}$$

$$= \frac{1}{\sqrt{3}} \int \frac{dx}{\sqrt{\left(x + \frac{5}{6}\right)^2 + \left(\frac{\sqrt{59}}{6}\right)^2}}$$

$$= \frac{1}{\sqrt{3}} \log\left\{\left(x + \frac{5}{6}\right) + \sqrt{\left(x + \frac{5}{6}\right)^2 + \left(\frac{\sqrt{59}}{6}\right)^2}\right\} + C$$

$$= \frac{1}{\sqrt{3}} \log\left\{x + \frac{5}{6} + \sqrt{x^2 + \frac{5}{3}x + \frac{7}{3}}\right\} + C$$

S12. Given integral is $\int \frac{2x}{\sqrt{1 - x^2 - x^4}}\, dx$

Put $x^2 = t \;\Rightarrow\; 2x\, dx = dt$

$$\Rightarrow \quad \int \frac{dt}{\sqrt{1-t-t^2}} = \int \frac{dt}{\sqrt{-(t^2+t-1)}}$$

$$\Rightarrow \quad \int \frac{dt}{\sqrt{-\left(t^2+t+\frac{1}{4}-\frac{1}{4}-1\right)}} = \frac{dt}{\sqrt{-\left[\left(t+\frac{1}{2}\right)^2-\frac{5}{4}\right]}}$$

$$\Rightarrow \quad \int \frac{dt}{\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2-\left(t+\frac{1}{2}\right)^2}} = \sin^{-1}\left(\frac{t+\frac{1}{2}}{\frac{\sqrt{5}}{2}}\right)+C$$

$$= \sin^{-1}\left(\frac{2t+1}{\sqrt{5}}\right)+C = \sin^{-1}\left(\frac{2x^2+1}{\sqrt{5}}\right)+C$$

S13. Given integral is $\int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}}\,dx$

Put $e^x = t \quad \Rightarrow \quad e^x\,dx = dt$

$$\Rightarrow \quad \int \frac{dt}{\sqrt{5-4t-t^2}} = \int \frac{dt}{\sqrt{-(t^2+4t-5)}}$$

$$\Rightarrow \quad \int \frac{dt}{\sqrt{-(t^2+4t+4-4-5)}} = \frac{dt}{\sqrt{-\left[(t+2)^2-(3)^2\right]}}$$

$$\Rightarrow \quad \int \frac{dt}{\sqrt{3^2-(t+2)^2}} = \sin^{-1}\left(\frac{t+2}{3}\right)+C$$

$$= \sin^{-1}\left(\frac{e^x+2}{3}\right)+C$$

S14. Given integral is $\int \frac{\cos x}{\sqrt{\sin^2 x-2\sin x-3}}\,dx$

Put $\sin x = t \quad \Rightarrow \quad \cos x\,dx = dt$

$$I = \int \frac{dt}{\sqrt{t^2-2t-3}} = \int \frac{dt}{\sqrt{t^2-2t+1-1-3}}$$

$$= \int \frac{dt}{\sqrt{(t-1)^2-(2)^2}} = \log\left\{(t-1)+\sqrt{(t-1)^2-(2)^2}\right\}+C$$

$$= \log\left\{(t-1)+\sqrt{t^2-2t-3}\right\}+C$$

$$= \log\left\{(\sin x-1)+\sqrt{\sin^2 x-2\sin x-3}\right\}+C$$

S15. Given integral is $\int \sqrt{\frac{x}{a^3 - x^3}}\, dx$

$$= \int \frac{\sqrt{x}}{\sqrt{(a^{3/2})^2 - (x^{3/2})^2}}\, dx$$

Put $x^{3/2} = t \;\Rightarrow\; \frac{3}{2}\, x^{1/2}\, dx = dt$

$$= \frac{2}{3} \int \frac{dt}{\sqrt{(a^{3/2})^2 - t^2}} = \frac{2}{3} \sin^{-1}\left(\frac{t}{a^{3/2}}\right) + C$$

$$= \frac{2}{3} \sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right) + C$$

S16. Given integral is $\int \sqrt{\sec x - 1}\, dx$

$$= \int \sqrt{\frac{1-\cos x}{\cos x}}\, dx = \int \sqrt{\frac{(1-\cos x)(1+\cos x)}{\cos(1+\cos x)}}\, dx$$

$$= \int \sqrt{\frac{1-\cos^2 x}{\cos x + \cos^2 x}}\, dx = \int \frac{\sin x}{\sqrt{\cos x + \cos^2 x}}\, dx$$

Put $\cos x = t \;\Rightarrow\; -\sin x\, dx = dt$

$$= \int \frac{-dt}{\sqrt{t^2 + t}} = -\int \frac{dt}{\sqrt{t^2 + t + \frac{1}{4} - \frac{1}{4}}}$$

$$= -\int \frac{dt}{\sqrt{\left(t + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} = -\log\left\{\left(t + \frac{1}{2}\right) + \sqrt{\left(t + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\right\} + C$$

$$= -\log\left\{\left(\cos x + \frac{1}{2}\right) + \sqrt{\cos^2 x + \cos x}\right\} + C$$

S17. Given integral is $\int \frac{\sin 2x}{\sqrt{\sin^4 x + 4\sin^2 x - 2}}\, dx$

Put $\sin^2 x = t \;\Rightarrow\; 2 \sin x \cos x\, dx = dt$

$$\Rightarrow \quad \int \frac{dt}{\sqrt{t^2 + 4t - 2}} = \int \frac{dt}{\sqrt{t^2 + 4t + 4 - 4 - 2}}$$

$$\Rightarrow \quad \int \frac{dt}{\sqrt{(t+2)^2 - (\sqrt{6})^2}} = \log\left\{(t+2) + \sqrt{(t+2)^2 - (\sqrt{6})^2}\right\} + C$$

$$\Rightarrow \qquad = \log\left\{(t+2)+\sqrt{t^2+4t-2}\right\} + C$$

$$= \log\left\{(\sin^2 x+2)+\sqrt{\sin^4 x+4\sin^2 x-2}\right\}+C$$

S18. Let

$$I = \int\frac{\sin x-\cos x}{\sqrt{\sin 2x}}dx$$

$$= \int\frac{\sin x-\cos x}{\sqrt{1+\sin 2x-1}}dx$$

$$= \int\frac{\sin x-\cos x}{\sqrt{\sin^2 x+\cos^2 x+2\sin x\cos x-1}}dx$$

$$= \int\frac{\sin x-\cos x}{\sqrt{(\sin x+\cos x)^2-1}}dx$$

Put $\sin x + \cos x = t$

$\Rightarrow \quad (\cos x - \sin x)dx = dt$

$$\therefore \quad I = \int\frac{-dt}{\sqrt{t^2-1}}$$

$$= -\log(t+\sqrt{t^2-1})+C \quad \left[\because \int\frac{dx}{\sqrt{x^2-a^2}} = \log(x+\sqrt{x^2-a^2})\right]$$

[Where, $t = \sin x + \cos x$]

$$\therefore \quad I = -\log|(\sin x+\cos x)+\sqrt{(\sin x+\cos x)^2-1}|+C$$

$$= -\log|\sin x+\cos x)+\sqrt{\sin 2x}|+C$$

S19.

$$I = \int\frac{\sin x+\cos x}{\sqrt{\sin 2x}}dx$$

Put $\sin x - \cos x = t$

$\Rightarrow \quad (\cos x + \sin x)\, dx = dt$

Also, $(\sin x - \cos x)^2 = t^2$

$\Rightarrow \quad \sin^2 x + \cos^2 x - 2\sin x\cos x = t^2$

$\Rightarrow \quad 1 - \sin 2x = t^2$

$\Rightarrow \quad \sin 2x = 1 - t^2 \qquad [\because \ 2\sin x\cos x = \sin 2x]$

$$\therefore \quad I = \int\frac{dt}{\sqrt{1-t^2}}$$

$$= \sin^{-1} t + C = \sin^{-1}(\sin x-\cos x)+C$$

$$\left[\because \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C\right]$$

S20. Let $$I = \int \frac{\sin 2x}{\sqrt{\cos^4 x - \sin^2 x - 4}}\, dx$$

Put $\cos^2 x = t$ so that $-2\cos x \sin x\, dx = dt$ *i.e.* $\sin 2x\, dx = -dt.$

$$I = \int \frac{-dt}{\sqrt{t^2-(1-t)-4}} = -\int \frac{dt}{\sqrt{t^2+t-5}}$$

$$= -\int \frac{dt}{\sqrt{\left(t+\frac{1}{2}\right)^2 - 5 - \frac{1}{4}}} = -\int \frac{dt}{\sqrt{\left(t+\frac{1}{2}\right)^2 - \left(\frac{\sqrt{21}}{2}\right)^2}}$$

$$= -\log\left|\left(t+\frac{1}{2}\right) + \sqrt{\left(t+\frac{1}{2}\right)^2 - \left(\frac{\sqrt{21}}{2}\right)^2}\right| + c$$

$$\left[\therefore \int \frac{dx}{\sqrt{x^2-a^2}} = \log\left(x+\sqrt{x^2+a^2}\right) + c\right]$$

$$= -\log\left|\left(\cos^2 x + \frac{1}{2}\right) + \sqrt{\left(\cos^2 x + \frac{1}{2}\right)^2 - \frac{21}{4}}\right| + c$$

$$= -\log\left|\left(\cos^2 x + \frac{1}{2}\right) + \sqrt{\cos^4 x + \cos^2 x - 5}\right| + c$$

S21. Given integral is $\int \frac{dx}{x^{2/3}\sqrt{x^{2/3}-4}}$

Put $x^{1/3} = t \Rightarrow \frac{1}{3}x^{1-\frac{1}{3}}\, dx = dt$

$$\Rightarrow \quad \frac{1}{3x^{2/3}}\, dx = dt$$

$$\Rightarrow \quad 3\int \frac{dt}{\sqrt{t^2-4}} = 3\log\left\{t+\sqrt{t^2-4}\right\} + C$$

$$= 3\log\left\{x^{1/3} + \sqrt{x^{2/3}-4}\right\} + C$$

S22. Given integral is $\int \sqrt{\frac{a-x}{a+x}}\, dx$

$$= \int \sqrt{\frac{(a-x)(a-x)}{(a+x)(a-x)}}\, dx = \int \frac{a-x}{\sqrt{a^2-x^2}}\, dx$$

$$= a\int\frac{dx}{\sqrt{a^2-x^2}} - \int\frac{x}{\sqrt{a^2-x^2}}\,dx$$

$$= a\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}\underbrace{\int\frac{-2x}{\sqrt{a^2-x^2}}\,dx}_{I_1}$$

For I_1,

Put $a^2 - x^2 = t \;\Rightarrow\; -2x\,dx = dt$

$$= a\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}\int\frac{dt}{\sqrt{t}} + C = a\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}\times 2\sqrt{t} + C$$

$$= a\sin^{-1}\frac{x}{a} + \sqrt{a^2-x^2} + C$$

S23. Put $e^x - 1 = t$ so that $e^x\,dx = dt$

i.e., $\quad dx = \dfrac{dt}{1+t}$

$\therefore \quad I = \displaystyle\int\sqrt{t}\,\frac{dt}{1+t} = \int\frac{\sqrt{t}}{1+t}\,dt$

Put $\sqrt{t} = z$, *i.e.*, $t = z^2$ so that $dt = 2z\,dz$.

$\therefore \quad I = \displaystyle\int\frac{z}{1+z^2}(2z\,dz) = 2\int\frac{z^2}{1+z^2}\,dz$

$$= 2\int\frac{(1+z^2)-1}{1+z^2}\,dz = 2\left[\int 1.dz - \int\frac{1}{1+z^2}\,dz\right] \quad \left[\because \int\frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]$$

$$= 2[z - \tan^{-1}z] + c = 2\left[\sqrt{t} - \tan^{-1}\sqrt{t}\right] + c$$

$$= 2\sqrt{e^x-1} - 2\tan^{-1}(\sqrt{e^x-1}) + c$$

S24. $\quad I = \displaystyle\int\left(\sqrt{\tan x} + \sqrt{\cot x}\right)dx$

$$= \int\left(\frac{\sqrt{\sin x}}{\sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x}}\right)dx$$

$$= \int\frac{\sin x + \cos x}{\sqrt{\sin x\cos x}}\,dx$$

Put $\quad \sin x - \cos x = t$

$\therefore \quad (\cos x + \sin x)\,dx = dt$

Also, $\quad (\sin x - \cos x)^2 = \sin^2 x + \cos^2 x - 2\sin x\cos x = t^2$

$\Rightarrow \quad 1 - 2\sin x\cos x = t^2$

$$1 - t^2 = 2\sin x\cos x$$

$\therefore \quad \sin x \cos x = \frac{1-t^2}{2}$

$$\therefore \quad I = \int \frac{dt}{\sqrt{\frac{1-t^2}{2}}} = \sqrt{2}\int \frac{dt}{\sqrt{1-t^2}}$$

$$= \sqrt{2}\sin^{-1} t + C \qquad \left[\because \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C\right]$$

$$= \sqrt{2}\sin^{-1}(\sin x - \cos x) + C$$

S25. Given integral is $\int \frac{x+2}{\sqrt{x^2+5x+6}}\,dx$

$$\text{Let} \quad x + 2 = A\frac{d}{dx}(x^2 + 5x + 6) + B$$

$$x + 2 = A(2x + 5) + B$$

$$2A = 1 \text{ and } 5A + B = 2$$

$$A = \frac{1}{2} \text{ and } B = -\frac{1}{2} \qquad \int \frac{\frac{1}{2}(2x+5) - \frac{1}{2}}{\sqrt{x^2+5x+6}}\,dx$$

$$\Rightarrow \quad = \frac{1}{2}\int \frac{2x+5}{\sqrt{x^2+5x+6}}\,dx - \frac{1}{2}\int \frac{dx}{\sqrt{x^2+5x+6}} = \frac{1}{2}I_1 - \frac{1}{2}I_2$$

For I_1, Put $x^2 + 5x + 6 = t$

$$(2x + 5)dx = dt$$

$$I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{x^2+5x+6}$$

$$I_2 = \int \frac{dx}{\sqrt{x^2 + 5x + \frac{25}{4} - \frac{25}{4} + 6}}$$

$$= \int \frac{dx}{\sqrt{\left(x+\frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} = \log\left\{\left(x+\frac{5}{2}\right) + \sqrt{\left(x+\frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\right\} + C$$

$$= \log\left\{\left(x+\frac{5}{2}\right) + \sqrt{x^2+5x+6}\right\} + C$$

$$\int \frac{x+2}{\sqrt{x^2+5x+6}}\,dx = \frac{1}{2}I_1 - \frac{1}{2}I_2$$

$$= \frac{1}{2}\times 2\sqrt{x^2+5x+6} - \frac{1}{2}\log\left\{\left(x+\frac{5}{2}\right)+\sqrt{x^2+5x+6}\right\}+C$$

$$= \sqrt{x^2+5x+6} - \frac{1}{2}\log\left\{\left(x+\frac{5}{2}\right)+\sqrt{x^2+5x+6}\right\}+C$$

S26. Given integral is $\int \frac{2x+3}{\sqrt{x^2+4x+2}}\,dx$

Let $2x + 3 = A\frac{d}{dx}(x^2 + 4x + 2) + B$

$$2x + 3 = A(2x + 4) + B$$

$$2A = 2 \text{ and } 4A + B = 3$$

$$A = 1 \text{ and } B = -1$$

$$\int \frac{(2x+4)-1}{\sqrt{x^2+4x+2}}\,dx = \underbrace{\int \frac{2x+4}{\sqrt{x^2+4x+2}}\,dx}_{I_1} - \underbrace{\int \frac{dx}{\sqrt{x^2+4x+2}}}_{I_2}$$

For I_1

Put $x^2 + 4x + 2 = t$

$$(2x + 4)\,dx = dt$$

$$I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{x^2+4x+2}$$

$$I_2 = \int \frac{dx}{\sqrt{x^2+4x+4-4+2}}$$

$$= \int \frac{dx}{\sqrt{(x+2)^2-(\sqrt{2})}}$$

$$= \log\left\{(x+2)+\sqrt{(x+2)^2-(\sqrt{2})^2}\right\}+C$$

$$= \log\left\{(x+2)+\sqrt{x^2+4x+2}\right\}+C$$

$\therefore$ $$\int \frac{2x+3}{\sqrt{x^2+4x+2}}\,dx = I_1 - I_2$$

$$= 2\sqrt{x^2+4x+2} - \log\left\{(x+2)+\sqrt{x^2+4x+2}\right\}+C$$

S27. Given integral is $\int \frac{x+1}{\sqrt{4+5x-x^2}}\,dx$

Put $\quad x + 1 = A\frac{d}{dx}(4 + 5x - x^2) + B$

$$x + 1 = A(5 - 2x) + B$$

$$-2A = 1 \text{ and } 5A + B = 1$$

$$A = \frac{-1}{2} \text{ and } B = \frac{7}{2}$$

$\Rightarrow$ $$\int \frac{\frac{-1}{2}(5-2x)+\frac{7}{2}}{\sqrt{4+5x-x^2}}\,dx$$

$\Rightarrow$ $$\frac{-1}{2}\int \frac{(5-2x)}{\sqrt{4+5x-x^2}}\,dx + \frac{7}{2}\int \frac{dx}{\sqrt{4+5x-x^2}}$$

$$= -\frac{1}{2}I_1 + \frac{7}{2}I_2$$

For $\quad I_1 \qquad 4 + 5x - x^2 = t$

$$(5 - 2x)\,dx = dt$$

$$I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{4+5x-x^2}$$

$$I_2 = \int \frac{dx}{\sqrt{-(x^2-5x-4)}}$$

$$= \int \frac{dx}{\sqrt{-\left(x^2-5x+\frac{25}{4}-\frac{25}{4}-4\right)}}$$

$$= \int \frac{dx}{\sqrt{-\left\{\left(x-\frac{5}{2}\right)^2-\left(\frac{\sqrt{41}}{2}\right)^2\right\}}}$$

$$= \int \frac{dx}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2-\left(x-\frac{5}{2}\right)^2}}$$

$$= \sin^{-1}\left(\frac{\left(x-\frac{5}{2}\right)}{\frac{\sqrt{41}}{2}}\right)+C = \sin^{-1}\left(\frac{2x-5}{\sqrt{41}}\right)+C$$

$$\therefore \quad \int \frac{x+1}{\sqrt{4+5x-x^2}}\,dx = -\frac{1}{2}I_1 + \frac{7}{2}I_2$$

$$= \frac{-1}{2} \times 2\sqrt{4+5x-x^2} + \frac{7}{2}\sin^{-1}\left(\frac{2x-5}{\sqrt{41}}\right) + C$$

$$= -\sqrt{4+5x-x^2} + \frac{7}{2}\sin^{-1}\left(\frac{2x-5}{\sqrt{41}}\right) + C$$

S28. Given integral is $\int \frac{3x+1}{\sqrt{5-2x-x^2}}\,dx$

Put $3x + 1 = A\frac{d}{dx}(5 - 2x - x^2) + B$

$$3x + 1 = A(-2 - 2x) + B$$

$$-2A = 3 \text{ and } -2A + B = 1$$

$$A = \frac{-3}{2} \text{ and } B = -2$$

$$= \int \frac{-\frac{3}{2}(-2-2x)-2}{\sqrt{5-2x-x^2}}\,dx$$

$$= -\frac{3}{2}\int \frac{(-2-2x)}{\sqrt{5-2x-x^2}}\,dx - 2\int \frac{dx}{\sqrt{5-2x-x^2}}$$

$$= -\frac{3}{2}I_1 - 2I_2$$

For I_1

Put $5 - 2x - x^2 = t \Rightarrow (-2 - 2x)\,dx = dt$

$$I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{5-2x-x^2}$$

and $$I_2 = \int \frac{dx}{\sqrt{-(x^2+2x-5)}} = \int \frac{dx}{\sqrt{-(x^2+2x+1-6)}}$$

$$= \int \frac{dx}{\sqrt{-\left\{(x+1)^2 - (\sqrt{6})^2\right\}}} = \int \frac{dx}{\sqrt{(\sqrt{6})^2 - (x+1)^2}}$$

$$= \sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right) + C$$

$$\therefore \quad \int \frac{3x+1}{\sqrt{5-2x-x^2}}\,dx = \frac{-3}{2} I_1 - 2I_2$$

$$= \frac{-3}{2} \times 2\sqrt{5-2x-x^2} - 2\sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right) + C$$

$$= -3\sqrt{5-2x-x^2} - 2\sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right) + C$$

S29. Given integral is $\int \frac{2x+1}{\sqrt{x^2+4x+3}}\,dx$

Put $2x + 1 = A\frac{d}{dx}(x^2 + 4x + 3) + B$

$2x + 1 = A(2x + 4) + B$

$2A = 2$ and $4A + B = 1$

$A = 1$ and $B = -3$

$$\Rightarrow \quad \int \frac{(2x+4)-3}{\sqrt{x^2+4x+3}}\,dx$$

$$= \underbrace{\int \frac{2x+4}{\sqrt{x^2+4x+3}}\,dx}_{I_1} \underbrace{- 3\int \frac{dx}{\sqrt{x^2+4x+3}}}_{-3I_2}$$

For I_1

Put $x^2 + 4x + 3 = t \Rightarrow (2x + 4)dx = dt$

$$I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{x^2+4x+3}$$

$$I_2 = \int \frac{dx}{\sqrt{x^2+4x+3}} = \int \frac{dx}{\sqrt{x^2+4x+4-1}}$$

$$= \int \frac{dx}{\sqrt{(x+2)^2-(1)^2}}$$

$$= \log\left\{(x+2) + \sqrt{(x+2)^2-(1)^2}\right\} + C$$

$$= \log\left\{(x+2) + \sqrt{x^2+4x+3}\right\} + C$$

$$\therefore \quad \int \frac{2x+1}{\sqrt{x^2+4x+3}}\,dx = I_1 - 3I_2$$

$$= 2\sqrt{x^2+4x+3} - 3\log\left\{(x+2) + \sqrt{x^2+4x+3}\right\} + C$$

S30. Given integral is $\int \frac{2x+3}{\sqrt{x^2+4x+5}}\,dx$

Put $2x + 3 = A\frac{d}{dx}(x^2 + 4x + 5) + B$

$$2x + 3 = A(2x + 4) + B$$

$$2A = 2 \text{ and } 4A + B = 3$$

$$A = 1 \text{ and } B = -1$$

$$\int \frac{(2x+4)-1}{\sqrt{x^2+4x+5}}\,dx = \underbrace{\int \frac{2x+4}{\sqrt{x^2+4x+5}}\,dx}_{I_1} - \underbrace{\int \frac{dx}{\sqrt{x^2+4x+5}}}_{I_2}$$

For I_1

Put $x^2 + 4x + 5 = t \Rightarrow (2x + 4)\,dx = dt$

$\therefore$ $$I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{x^2+4x+5} + C_1$$

$$I_2 = \int \frac{dx}{\sqrt{x^2+4x+5}} = \int \frac{dx}{\sqrt{x^2+4x+4+1}}$$

$$= \int \frac{dx}{\sqrt{(x+2)^2+(1)^2}}$$

$$= \log\left\{(x+2+\sqrt{(x+2)+(1)^2}\right\} + C_2$$

$$= \log\left\{(x+2+\sqrt{x^2+4x+5}\right\} + C_2$$

$\therefore$ $$\int \frac{2x+3}{\sqrt{x^2+4x+5}}\,dx = I_1 - I_2$$

$$= 2\sqrt{x^2+4x+5} - \log\left\{\sqrt{x^2+4x+5} + (x+2)\right\} + C \quad (\text{Put } C = C_1 + C_2)$$

S31. Let $$I = \int \frac{dx}{\sqrt{5-4x-2x^2}}$$

$$= \int \frac{dx}{\sqrt{-2\left(x^2+2x-\frac{5}{2}\right)}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{-\left[x^2 + 2x - \frac{5}{2} + 1 - 1\right]}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{-\left[(x^2 + 2x + 1) - \frac{5}{2} - 1\right]}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{-\left[(x+1)^2 - \left(\frac{5}{2} + 1\right)\right]}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{-\left[(x+1)^2 - \frac{7}{2}\right]}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\left(\sqrt{\frac{7}{2}}\right)^2 - (x+1)^2}}$$

$$= \frac{1}{\sqrt{2}} \cdot \sin^{-1}\left(\frac{x+1}{\sqrt{\frac{7}{2}}}\right) + C \qquad \left[\because \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a}\right]$$

$$= \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{\sqrt{2}(x+1)}{\sqrt{7}}\right) + C$$

S32. $$I = \int \frac{x+2}{\sqrt{x^2 + 2x + 3}} dx$$

$$I = \int \frac{x+1}{\sqrt{x^2 + 2x + 3}} dx + \int \frac{dx}{\sqrt{x^2 + 2x + 3}}$$

Let $I = I_1 + I_2$ ... (i)

$$I_1 = \int \frac{(x+1)dx}{\sqrt{x^2 + 2x + 3}} = \int \frac{t\,dt}{t} = [t] + C_1 \qquad \left[\begin{array}{ll} Let & t^2 = x^2 + 2x + 3 \\ & 2t\,dt = (2x+2)dx \\ \Rightarrow & t\,dt = (x+1)dx \end{array}\right]$$

$$= \sqrt{x^2 + 2x + 3} + C_1$$

and $$I_2 = \int \frac{dx}{\sqrt{x^2 + 2x + 3}} = \int \frac{dx}{\sqrt{(x+1)^2 + (\sqrt{2})^2}}$$

$$= \log\{(x+1) + \sqrt{x^2 + 2x + 3}\} + C_2 \left[\because \int \frac{dx}{\sqrt{x^2 + a^2}} = \log(x + \sqrt{x^2 + a^2})\right]$$

Putting the value of I_1 and I_2 in Eq. (i), we get

$$I = \sqrt{x^2+2x+3} + C_1 + \log\left\{(x+1)+\sqrt{x^2+2x+3}\right\} + C_2$$

$$I = \sqrt{x^2+2x+3} + \log\left\{(x+1)+\sqrt{x^2+2x+3}\right\} + C$$

Where, $C = C_1 + C_2$

S33. Let $I = \int \frac{3x+5}{\sqrt{x^2-8x+7}} dx$

Let $3x + 5 = A \cdot \frac{d}{dx}(x^2 - 8x + 7) + B$

or $3x + 5 = A(2x - 8) + B$

$\Rightarrow$ $3x + 5 = 2Ax + (B - 8A)$ ... (i)

Comparing coefficient of x and constant on both sides, we get

$2A = 3$ ... (ii)

and $-8A + B = 5$ ... (iii)

From Eq. (ii), we get $A = \frac{3}{2}$

Putting $A = \frac{3}{2}$ in Eq. (iii), we get

$$-8A + B = 5$$

$$-8\left(\frac{3}{2}\right) + B = 5$$

$\Rightarrow$ $-12 + B = 5$

or $B = 17$

Putting the values of A and B in Eq. (i), we get

$3x + 5 = \frac{3}{2}(2x - 8) + 17$ ... (iv)

Hence, the given integral can be written as

$I = \int \frac{\frac{3}{2}(2x-8)+17}{\sqrt{x^2-8x+7}} dx$ [Using Eq. (iv)

$\Rightarrow$ $I = \frac{3}{2}\int \frac{2x-8}{\sqrt{x^2-8x+7}} dx + 17\int \frac{dx}{\sqrt{x^2-8x+7}}$

or $I = \frac{3}{2} I_1 + 17 I_2$ ... (v) where, $I_1 = \int \frac{2x-8}{\sqrt{x^2-8x+7}}\,dx$

Put $x^2 - 8x + 7 = t$

$\Rightarrow$ $(2x - 8)\,dx = dt$

$\therefore$ $I_1 = \int \frac{dt}{\sqrt{t}} = \int t^{-1/2} dt$

$$I_1 = \frac{t^{1/2}}{1/2}$$

$\therefore$ $I_1 = 2t^{1/2}$

$$I_1 = 2\sqrt{t}$$

$$I_1 = 2\sqrt{x^2-8x+7}$$

Also $I_2 = \int \frac{dx}{\sqrt{x^2-8x+7}}$

$$I_2 = \int \frac{dx}{\sqrt{x^2-8x+7+16-16}}$$

$\Rightarrow$ $I_2 = \int \frac{dx}{\sqrt{(x-4)^2-9}}$

$$= \int \frac{dx}{\sqrt{(x-4)^2-(3)^2}}$$

$\therefore$ $I_2 = \log|(x-4)+\sqrt{(x-4)^2-(3)^2}|$ $\left[\because \int \frac{1}{\sqrt{x^2-a^2}}dx = \log\left|x+\sqrt{x^2-a^2}\right|\right]$

Hence, putting the values of I_1 and I_2 in Eq. (v), we get,

$$I = \frac{3}{2}(2\sqrt{x^2-8x+7}) + 17\log\left|(x-4)+\sqrt{(x-4)^2-9}\right| + C$$

or $I = 3\sqrt{x^2-8x+7} + 17\log\left|(x-4)+\sqrt{(x-4)^2-9}\right| + C$

S34. Let $I = \int \frac{6x+7}{\sqrt{(x-5)(x-4)}}dx$

Given integral can be written as

$$I = \int \frac{6x+7}{\sqrt{(x^2-9x+20}}dx$$

Let $6x + 7 = A\cdot\frac{d}{dx}(x^2-9x+20)+B$

$\Rightarrow \quad 6x + 7 = A(2x - 9) + B \quad$ ...(i)

Comparing coefficient of x and constant, we get

$$2A = 6 \quad \Rightarrow A = 3$$

and $\quad -9A + B = 7$

$\Rightarrow \quad -9(3) + B = 7 \quad [\because A = 3]$

$\Rightarrow \quad -27 + B = 7$

or $\quad B = 34$

Putting value of A and B in Eq. (i), we get

$$6x + 7 = 3(2x - 9) + 34$$

$\therefore$ Given integral can be written as

$$I = \int \frac{3(2x-9)+34}{\sqrt{x^2-9x+20}}dx$$

$\Rightarrow \quad I = 3\int \frac{2x-9}{\sqrt{x^2-9x+20}}dx + 34\int \frac{dx}{\sqrt{x^2-9x+20}}$

or $\quad I = 3I_1 + 34I_2 \quad$ ... (ii)

Where, $\quad I_1 = \int \frac{2x-9}{\sqrt{x^2-9x+20}}dx$

Put $\quad x^2 - 9x + 20 = t$

$\Rightarrow \quad (2x - 9)\, dx = dt$

$\Rightarrow \quad I_1 = \int \frac{dt}{\sqrt{t}} = \int t^{-1/2}dt = 2t^{1/2} = 2\sqrt{t}$

$$= 2\sqrt{x^2-9x+20} \quad \text{... (iii)}$$

and $\quad I_2 = \int \frac{dx}{\sqrt{x^2-9x+20}}$

$$I_2 = \int \frac{dx}{\sqrt{x^2-9x+20+\frac{81}{4}-\frac{81}{4}}}$$

$$= \int \frac{dx}{\sqrt{\left(x-\frac{9}{2}\right)^2+\left(20-\frac{81}{4}\right)}}$$

$\Rightarrow \quad I_2 = \int \frac{dx}{\sqrt{\left(x-\frac{9}{2}\right)^2-\frac{1}{4}}}$

$\therefore \qquad I_2 = \log\left|\left(x - \frac{9}{2}\right) + \sqrt{\left(x - \frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\right| \qquad \text{... (iv)}$

$$\left[\because \int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + C\right]$$

Putting values of I_1 and I_2 from Eqs. (iii) and (iv) in Eq. (ii), we get

$$I = 3(2\sqrt{x^2 - 9x + 20}) + 34\log\left[\left(x - \frac{9}{2}\right) + \sqrt{\left(x - \frac{9}{2}\right)^2 - \frac{1}{4}}\right] + C$$

or $\qquad I = 6\sqrt{x^2 - 9x + 20} + 34\log\left|x - \frac{9}{2} + \sqrt{\left(x - \frac{9}{2}\right)^2 - \frac{1}{4}}\right| + C$

S35. Let $\qquad I = \int \frac{2x + 5}{\sqrt{7 - 6x - x^2}}\,dx$

Given integral can be written as

$$I = \int \frac{-(-2x - 6) - 1}{\sqrt{7 - 6x - x^2}}\,dx$$

$$= \int \frac{-2x - 6}{\sqrt{7 - 6x - x^2}}\,dx - \int \frac{dx}{\sqrt{7 - 6x - x^2}}$$

$\therefore \qquad I = I_1 - I_2 \qquad \text{... (i)}$

Now, consider $\qquad I_1 = \int \frac{-2x - 6}{\sqrt{7 - 6x - x^2}}\,dx$

Put $7 - 6x - x^2 = t$

$\Rightarrow \qquad (-6 - 2x)\,dx = dt$

$\therefore \qquad I_1 = \int \frac{dt}{\sqrt{t}} = \int t^{-1/2}dt$

$$= 2\sqrt{t} = 2\sqrt{7 - 6x - x^2}$$

Again, $\qquad I_2 = \int \frac{dx}{\sqrt{7 - 6x - x^2}} = \int \frac{dx}{\sqrt{-(-7 + 6x + x^2 + 9 - 9)}}$

$$= \int \frac{dx}{\sqrt{-[(x + 3)^2 - 16]}}$$

$$\Rightarrow \quad I_2 = \int \frac{dx}{\sqrt{(4)^2 - (x+3)^2}} = \sin^{-1}\left(\frac{x+3}{4}\right) + C \qquad \left[\because \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C\right]$$

Putting values of I_1 and I_2 in Eq. (i), we get

Hence, $$I = 2\sqrt{7 - 6x - x^2} - \sin^{-1}\left(\frac{x+3}{4}\right) + C$$

S36. Let $$I = \frac{x+2}{\sqrt{(x-2)(x-3)}}\,dx$$

$$= \int \frac{x+2}{\sqrt{x^2 - 5x + 6}}\,dx$$

Let $$x + 2 = A\,\frac{d}{dx}(x^2 - 5x + 6) + B$$

$\therefore$ $$x + 2 = A\,(2x - 5) + B$$

Equating the coefficient of like terms from both sides, we get

$$2A = 1 \text{ and } -5A + B = 2$$

$\therefore$ $$A = \frac{1}{2} \text{ and } -5\left(\frac{1}{2}\right) + B = 2$$

$\Rightarrow$ $$B = 2 + \frac{5}{2} = \frac{9}{2}$$

$\therefore$ $$I = \int \frac{\frac{1}{2}(2x-5)}{\sqrt{x^2 - 5x + 6}}\,dx + \int \frac{\frac{9}{2}}{\sqrt{x^2 - 5x + 6}}\,dx$$

$$= \frac{1}{2}\int \frac{2x-5}{\sqrt{x^2 - 5x + 6}}\,dx + \frac{9}{2}\int \frac{dx}{\sqrt{x^2 - 5x + 6}}$$

Put $x^2 - 5x + 6 = t$ in first integral

$\therefore$ $$(2x - 5)\,dx = dt$$

$\therefore$ $$I = \frac{1}{2}\int \frac{dt}{\sqrt{t}} + \frac{9}{2}\int \frac{dx}{\sqrt{\left(x - \frac{5}{2}\right)^2 - \frac{25}{4} + 6}}$$

$$= \frac{1}{2}\int t^{-1/2}dt + \frac{9}{2}\int \frac{dx}{\sqrt{\left(x - \frac{5}{2}\right)^2 - \frac{1}{4}}}$$

$$= \frac{1}{2}\,\frac{t^{1/2}}{\frac{1}{2}} + \frac{9}{2}.\log\left|\left(x - \frac{5}{2}\right) + \sqrt{\left(x - \frac{5}{2}\right)^2 - \frac{1}{4}}\right| + C$$

$$\left[\because \int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left\{x + \sqrt{x^2 - a^2}\right\}\right]$$

$$= \sqrt{x^2 - 5x + 6} + \frac{9}{2}\log\left|\left(x - \frac{5}{2}\right) + \sqrt{x^2 - 5x + 6}\right| + C$$

$$= \sqrt{x^2 - 5x + 6} + \frac{9}{2}\log\left|\left(x - \frac{5}{2}\right) + \sqrt{(x-2)(x-3)}\right| + C$$

S37. Let $I = \int \frac{5x+3}{\sqrt{x^2 + 4x + 10}}\,dx$

Let $5x + 3 = A \cdot \frac{d}{dx}(x^2 + 4x + 10) + B = A(2x + 4) + B$ ... (i)

Comparing coefficient of x and constant, we get

$$2A = 5$$

$$A = \frac{5}{2}$$

and $4A + B = 3$

$\Rightarrow$ $4\left(\frac{5}{2}\right) + B = 3$

$\Rightarrow$ $10 + B = 3$ or $B = -7$

Putting the values of A and B in Eq. (i) we get

$$5x + 3 = \frac{5}{2}(2x + 4) - 7$$

$\therefore$ Given integral can be written as

$$I = \int \frac{\frac{5}{2}(2x+4) - 7}{\sqrt{x^2 + 4x + 10}}\,dx$$

$$= \frac{5}{2}\int \frac{2x+4}{\sqrt{x^2 + 4x + 10}}\,dx - 7\int \frac{dx}{\sqrt{x^2 + 4x + 10}}$$

let $I_1 = \frac{5}{2}\int \frac{2x+4}{\sqrt{x^2 + 4x + 10}}\,dx$

$\Rightarrow$ $I_1 = \frac{5}{2}\int \frac{dt}{\sqrt{t}} = \frac{5}{2}\int t^{-1/2} = \frac{5}{2} 2\sqrt{t}$ [Put $x^2 + 4x + 10 = t \Rightarrow (2x + 4)\,dx = dt$]

$I_1 = \frac{5}{2} \times 2\sqrt{t} = 5\sqrt{x^2 + 4x + 10}$ ... (ii)

Also, let $I_2 = -7\int \frac{1}{\sqrt{x^2 + 4x + 10}}\,dx$

$$= -7\int \frac{1}{\sqrt{x^2+4x+10+4-4}}\,dx$$

$$= -7\int \frac{1}{\sqrt{(x+2)^2+(\sqrt{6})^2}}\,dx$$

$$I_2 = -7\log|(x+2)+\sqrt{x^2+4x+10}|$$

$$\left[\because \int \frac{dx}{\sqrt{x^2+a^2}} = \log|x+\sqrt{x^2+a^2}|+C\right]$$

$\therefore$ We have, $I = I_1 + I_2$

$$\therefore \int \frac{5x+3}{\sqrt{x^2+4x+10}}\,dx = 5\sqrt{x^2+4x+10} - 7\log|(x+2)+\sqrt{x^2+4x+10}|+C$$

Date: 19/10/2021

Q1. Evaluate $\int \sin\sqrt{x}\, dx$

Q2. Evaluate $\int \sec^3 x\, dx$

Q3. Evaluate $\int x\cos 2x \sin 4x\, dx$

Q4. Evaluate $\int x.\log(x+1)\, dx$

Q5. Evaluate $\int (\tan^{-1} x^2)x\, dx$.

Q6. Evaluate $\int (\log x)^2\, dx$

Q7. Evaluate $\int \sin^{-1} x\, dx$

Q8. Evaluate $\int \tan^{-1} x\, dx$

Q9. Evaluate $\int \sec^{-1} x\, dx$

Q10. Evaluate $\int \frac{x-\sin x}{1-\cos x}\, dx$

Q11. Evaluate $\int \frac{\sin^{-1} x}{(1-x^2)^{3/2}}\, dx$

Q12. Evaluate $\int x\cot^{-1} x\, dx$

Q13. Evaluate $\int \sin x \log\cos x\, dx$

Q14. Evaluate $\int \tan^{-1}\sqrt{\frac{1-x}{1+x}}\, dx$

Q15. Evaluate $\int \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx$

Q16. Evaluate $\int \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) dx$

Q17. Evaluate $\int \cos^{-1}(4x^3-3x)\, dx$

Q18. Evaluate $\int \frac{x^2\tan^{-1} x}{1+x^2}\, dx$

Q19. Evaluate $\int \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$

Q20. Evaluate $\int \sin^{-1}(3x-4x^3)\, dx$

Q21. Evaluate $\int x\left(\frac{\sec 2x-1}{\sec 2x+1}\right) dx$

Q22. Evaluate $\int x \tan^2 x \, dx$

Q23. Evaluate $\int \frac{x + \sin x}{1 + \cos x} dx$

Q24. Evaluate $\int x \tan^{-1} x dx$

Q25. Evaluate $\int \cos 2x \log \sin x \, dx$.

Q26. Evaluate $\int \log(1 + x^2) dx$

Q27. Evaluate $\int \frac{\log(x+2)}{(x+2)^2} dx$.

Q28. Evaluate $\int \sin^{-1} \sqrt{x} \, dx$

Q29. Evaluate $\int x^3 \tan^{-1} x \, dx$

Q30. Evaluate $\int x^2 \tan^{-1} x \, dx$

Q31. Evaluate $\int \frac{x^2 \sin^{-1} x}{(1 - x^2)^{3/2}} dx$

Q32. Evaluate $\int \frac{x^3 \sin^{-1} x^2}{\sqrt{1 - x^4}} dx$

Q33. Evaluate $\int \tan^{-1} \sqrt{x} \, dx$

Q34. Evaluate $\int (\sin^{-1} x)^2 dx$.

Q35. Evaluate $\frac{x \tan^{-1} x}{(1 + x^2)^{1/2}} dx$.

Q36. Evaluate $\int \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right) dx$.

Q37. Evaluate $\int \left[\log(1 + \cos x) - x \tan\frac{x}{2}\right] dx$.

Q38. Evaluate $\int \frac{x^2}{(x \sin x + \cos x)^2} dx$

Q39. Evaluate $\int \frac{x \sin^{-1} x}{\sqrt{1 - x^2}} dx$

Q40. Evaluate $\int x \sin^{-1} x \, dx$

Q41. Evaluate $\int e^{2x} \sin x \, dx$

Q42. Evaluate $\int \frac{\tan^{-1} x}{x^2} dx$.

Q43. Evaluate $\int \frac{\sin^{-1} \sqrt{x} - \cos^{-1} \sqrt{x}}{\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x}} dx$

Q44. Evaluate $\int \cos 2\theta \ln\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right) d\theta$.

Q45. Evaluate $\int \frac{\sqrt{1-\sin x}}{1+\cos x} e^{-x/2}\, dx$

Q46. Evaluate $\int \frac{\sin^{-1} x}{x^2} dx$

Q47. Evaluate $\int \sec^{-1} \sqrt{x}\, dx$

Q48. Evaluate $\int \sin^{-1} \sqrt{\frac{x}{x+a}}\, dx$

Date: 19/10/2021

S1. Put $\sqrt{x} = t$ *i.e.* $x = t^2$ so that $dx = 2t\,dt$.

$$\therefore \quad \int \sin\sqrt{x}\,dx = \int \sin t \cdot 2t\,dt = 2\int \underset{I}{t}\,\underset{II}{\sin t}\,dt$$

$$\left[\because \int \underset{I}{f(x)}\,\underset{II}{g(x)}\,dx = f(x)\int g(x)\,dx - \int\left(\frac{d}{dx}f(x)\int g(x)\,dx\right)dx\right] \qquad \text{[Integrating by parts]}$$

$$= 2\left[t(-\cos t) - \int (1)(-\cos t)dt\right] \qquad \text{[Integrating by parts]}$$

$$= -2t\cos t + 2\int \cos t\,dt$$

$$= -2t\cos t + 2\sin t + c$$

$$= -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + c$$

$$= 2(\sin\sqrt{x} - \sqrt{x}\cos\sqrt{x}) + c\,.$$

S2.

$$\int \sec^3 x\,dx = \int \underset{I}{\sec x}\cdot\underset{II}{\sec^2 x}\,dx$$

$$\left[\because \int \underset{I}{f(x)}\,\underset{II}{g(x)}\,dx = f(x)\int g(x)\,dx - \int\left(\frac{d}{dx}f(x)\int g(x)\,dx\right)dx\right] \qquad \text{[Integrating by parts]}$$

$$= \sec x\cdot\tan x - \int \sec x\tan x\cdot\tan x\,dx$$

$$= \sec x\cdot\tan x - \int \sec x\tan^2 x\,dx$$

$$= \sec x\cdot\tan x - \int \sec x(\sec^2 x - 1)\,dx$$

$$= \sec x\cdot\tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

$$\therefore \quad 2\int \sec^3 x\,dx = \sec x\tan x + \log|\sec x + \tan x| + c'.$$

Hence $\int \sec^3 x\,dx = \frac{1}{2}\sec x\tan x + \frac{1}{2}\log|\sec x + \tan x| + c$, where, $\frac{c'}{2} = c$.

S3.

$$\int x\cos 2x\sin 4x\,dx = \frac{1}{2}\int x(2\sin 4x\cos 2x)dx$$

$$= \frac{1}{2}\int \underset{I}{x}\,\underset{II}{(\sin 6x + \sin 2x)}dx \qquad [\because 2\sin A\cos B = \sin(A+B) + \sin(A-B)]$$

$$\left[\because \int \underset{I}{f(x)}\, \underset{II}{g(x)}\, dx = f(x)\int g(x)\,dx - \int\left(\frac{d}{dx}f(x)\int g(x)dx\right)dx\right] \quad \text{[Integrating by parts]}$$

$$= \frac{1}{2}\left[x\left(-\frac{\cos 6x}{6} - \frac{\cos 2x}{2}\right) - \int 1\cdot\left(-\frac{\cos 6x}{6} - \frac{\cos 2x}{2}\right)dx\right]$$

$$= \frac{1}{2}\left[-x\left(\frac{\cos 6x}{6} + \frac{\cos 2x}{2}\right) + \int 1\cdot\left(\frac{\cos 6x}{6} + \frac{\cos 2x}{2}\right)dx\right]$$

$$= \frac{1}{2}\left[-x\left(\frac{\cos 6x}{6} + \frac{\cos 2x}{2}\right) + \frac{\sin 6x}{36} + \frac{\sin 2x}{4}\right] + c$$

S4. Let $I = \int \underset{II}{x}.\log(\underset{I}{x+1})\,dx$

Using integration by parts, we get

$$I = \log(x+1)\int x\,dx - \int\left[\frac{d}{dx}\log(x+1)\int x\,dx\right]dx$$

$$= \frac{x^2}{2}\log(x+1) - \frac{1}{2}\int\frac{1}{x+1}\cdot\frac{x^2}{2}dx$$

$$= \frac{x^2}{2}\log(x+1) - \frac{1}{2}\int\left(x-1+\frac{1}{x+1}\right)dx$$

$$I = \frac{x^2}{2}\log(x+1) - \frac{1}{2}\left[\frac{x^2}{2} - x + \log(x+1)\right] + C \quad \left[\because \int\frac{dx}{x} = \log x + C\right]$$

$$I = \frac{x^2}{2}\log(x+1) - \frac{x^2}{4} + \frac{x}{2} - \frac{1}{2}\log(x+1) + C$$

$$I = \frac{1}{2}(x^2-1)\log(x+1) - \frac{x^2}{4} + \frac{x}{2} + C$$

S5. Let $I = \int(\tan^{-1}x^2)x\,dx$

Put $x^2 = t$ so that $2x\,dx = dt$ *i.e.* $x\,dx = \frac{1}{2}dt.$

$\therefore$ $I = \int\tan^{-1}t\cdot\frac{1}{2}dt = \frac{1}{2}\int\tan^{-1}t\cdot 1\,dt$

$$\left[\because \int \underset{I}{f(x)}\, \underset{II}{g(x)}\, dx = f(x)\int g(x)\,dx - \int\left(\frac{d}{dx}f(x)\int g(x)dx\right)dx\right] \quad \text{[Integrating by parts]}$$

$$= \frac{1}{2}\left[\tan^{-1}t\cdot t - \int\frac{1}{1+t^2}\cdot t\,dt\right]$$

$$= \frac{1}{2}\left[t\tan^{-1}t - \frac{1}{2}\int\frac{2t}{1+t^2}dt\right]$$

$$= \frac{1}{2}\left[t\tan^{-1}t - \frac{1}{2}\log|1+t^2|\right] + c$$

$$= \frac{1}{2}x^2\tan^{-1}x^2 - \frac{1}{4}\log|1+x^4| + c$$

$$= \frac{1}{2}x^2\tan^{-1}x^2 - \frac{1}{4}\log(1+x^4) + c$$

S6. $\int \underset{I}{(\log x)^2} \cdot \underset{II}{1}\, dx$

$$= (\log x)^2 \cdot \int 1\, dx - \int\left[\frac{d}{dx}(\log x)^2 \cdot \int 1\, dx\right] dx$$

$$= x(\log x)^2 - \int 2\log x \cdot \frac{1}{x}\cdot x\, dx$$

$$= x(\log x)^2 - 2\int \underset{\text{I}}{\log x} \cdot \underset{\text{II}}{1}\, dx$$

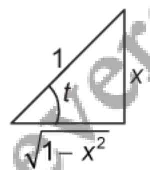
$$= x(\log x)^2 - 2\left[\log x \cdot \int 1\, dx - \int\left\{\frac{d}{dx}\log x \cdot \int 1\, dx\right\} dx\right]$$

$$= x(\log x)^2 - 2\left[x\log x - \int \frac{1}{x}\cdot x\, dx\right]$$

$$= x(\log x)^2 - 2x\log x + 2x + C$$

S7. Given integral is $\int \sin^{-1} x\, dx$

Put $\sin^{-1} x = t \Rightarrow x = \sin t \Rightarrow dx = \cos t\, dt$



$\Rightarrow$ $\int \underset{\text{I}}{t}\, \underset{\text{II}}{\cos t}\, dt$

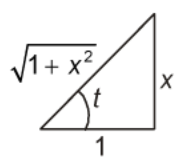
$$= t\int \cos t\, dt - \int\left[\frac{d}{dt}t \int \cos t\, dt\right] dt$$

$$= t\sin t - \int \sin t\, dt$$

$$= t\sin t + \cos t + C$$

$$= x\sin^{-1}x + \sqrt{1-x^2} + C$$

S8. Put $\tan^{-1} x = t \Rightarrow x = \tan t \Rightarrow dx = \sec^2 t\, dt$



$\Rightarrow$ $= \int \underset{\text{I}}{t}\, \underset{\text{II}}{\sec^2 t}\, dt$

$$= t\cdot\int \sec^2 t\, dt - \int\left[\frac{d}{dt}t \cdot \int \sec^2 t\, dt\right] dt$$

$$= t\tan t - \int \tan t\, dt$$

$$= t\tan t + \log\cos t + C$$

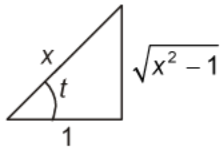
$$= x\tan^{-1}x + \log\left(\frac{1}{\sqrt{1+x^2}}\right) + C$$

$$= x\tan^{-1}x - \frac{1}{2}\log(1+x^2) + C$$

S9. $\int \sec^{-1} x\, dx$

Put $\sec^{-1} x = t \Rightarrow x = \sec t \Rightarrow dx = \sec t \tan t\, dt$

$$= \int \underset{\text{I}}{t}\,\underset{\text{II}}{(\sec t\tan t)}\,dt$$



$$= t\cdot\int \sec t\tan t\,dt - \int\left[\frac{d}{dt}t\cdot\int\sec t\tan t\,dt\right]dt$$

$$= t\sec t - \int \sec t\,dt$$

$$= t\sec t - \log(\sec t + \tan t) + C$$

$$= x\sec^{-1}x - \log(x + \sqrt{x^2-1}) + C$$

S10. Given integral is $\int \frac{x-\sin x}{1-\cos x}\,dx$

$$= \int\frac{x}{1-\cos x}\,dx - \int\frac{\sin x}{1-\cos x}\,dx$$

$$= \int\frac{x}{2\sin^2\frac{x}{2}}\,dx - \int\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\,dx$$

$$= \int\frac{x}{2}\operatorname{cosec}^2\frac{x}{2}\,dx - \int\cot\frac{x}{2}\,dx$$

$$= \frac{1}{2}\int \underset{\text{I}}{x}\,\underset{\text{II}}{\operatorname{cosec}^2\frac{x}{2}}\,dx - \int\cot\frac{x}{2}\,dx$$

$$= \frac{1}{2}\left[x\cdot\int\operatorname{cosec}^2\frac{x}{2}\,dx - \int\left[\frac{d}{dx}x\cdot\int\operatorname{cosec}^2\frac{x}{2}\right]dx - \int\cot\frac{x}{2}\,dx\right.$$

$$= \frac{1}{2}\left[x\left(-2\cot\frac{x}{2}\right) - \int 1\cdot\left(-2\cot\frac{x}{2}\right)dx\right] - \int \cot\frac{x}{2}\,dx$$

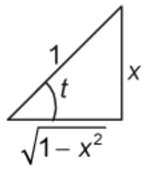
$$= -x\cot\frac{x}{2} + \int \cot\frac{x}{2}\,dx - \int \cot\frac{x}{2}\,dx + C$$

$$= -x\cot\frac{x}{2} + C$$

S11. Given integral is $\int \frac{\sin^{-1} x}{(1-x^2)^{3/2}}\,dx$

Put $\sin^{-1} x = t \Rightarrow x = \sin t \Rightarrow dx = \cos t\,dt$

$$= \int \frac{t\cos t}{(1-\sin^2 t)^{3/2}}\,dt = \int \frac{t\cos t}{\cos^3 t}\,dt$$



$$= \int \underset{I}{t}\,\underset{II}{\sec^2 t}\,dt$$

$$= t\cdot\int \sec^2 t\,dt - \int\left[\frac{d}{dt}t\cdot\int \sec^2 t\,dt\right]dt$$

$$= t\tan t - \int \tan t\,dt$$

$$= t\tan t + \log\cos t + C$$

$$= \sin^{-1} x\cdot\frac{x}{\sqrt{1-x^2}} + \log\sqrt{1-x^2} + C$$

$$= \frac{x\sin^{-1} x}{\sqrt{1-x^2}} + \frac{1}{2}\log(1-x^2) + C$$

S12. $\int \underset{II}{x}\,\underset{I}{\cot^{-1} x}\,dx$

$$= \cot^{-1} x\cdot\int x\,dx - \int\left[\frac{d}{dx}\cot^{-1} x\cdot\int x\,dx\right]dx$$

$$= \frac{x^2}{2}\cot^{-1} x - \int\left(-\frac{1}{1+x^2}\right)\cdot\frac{x^2}{2}\,dx = \frac{x^2}{2}\cot^{-1} x + \frac{1}{2}\int\frac{x^2}{x^2+1}\,dx$$

$$= \frac{x^2}{2}\cot^{-1} x + \frac{1}{2}\int\frac{x^2+1-1}{x^2+1}\,dx$$

$$= \frac{x^2}{2}\cot^{-1} x + \frac{1}{2}\int\left(1-\frac{1}{x^2+1}\right)dx$$

$$= \frac{x^2}{2}\cot^{-1} x + \frac{1}{2}x - \frac{1}{2}\tan^{-1} x + C$$

S13. Given integral is $\int \sin x \log \cos x \, dx$

Put $\cos x = t \Rightarrow -\sin x \, dx = dt$

$$= -\int \underset{I}{\log t} \; \underset{II}{1} \, dt$$

$$= -\left[\log t \int 1\, dt - \int \left[\frac{d}{dt}\log t \int 1\, dt\right] dt\right]$$

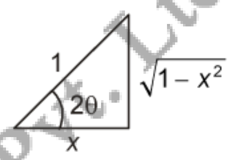
$$= -t \log t + \int \frac{1}{t} \cdot t \, dt$$

$$= -t \log t + t + C$$

$$= -\cos x \log \cos x + \cos x + C$$

$$= \cos x\,(1 - \log \cos x) + C$$

S14. Given integral is $\int \tan^{-1}\sqrt{\frac{1-x}{1+x}}\, dx$



Put $x = \cos 2\theta \Rightarrow dx = -2 \sin 2\theta \, d\theta$

$$= \int \tan^{-1}\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}} \times (-2\sin 2\theta)\, d\theta$$

$$= \int \tan^{-1}\sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}} \times (-2\sin 2\theta)\, d\theta$$

$$= -2\int \tan^{-1}(\tan\theta)\sin 2\theta \, d\theta$$

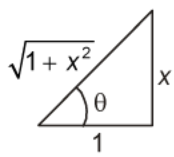
$$= -2\int \underset{I}{\theta} \; \underset{II}{\sin 2\theta}\, d\theta$$

$$= -2\left[\theta \cdot \int \sin 2\theta\, d\theta - \int\left\{\frac{d}{d\theta}\theta \cdot \int \sin 2\theta\, d\theta\right\} d\theta\right]$$

$$= -2\left[-\theta\frac{\cos 2\theta}{2} - \int -\frac{\cos 2\theta}{2}\, d\theta\right]$$

$$= \theta\cos 2\theta - \frac{\sin 2\theta}{2} + C = \frac{1}{2}x\cos^{-1}x - \frac{\sqrt{1-x^2}}{2} + C$$

S15. Given integral is $\int \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx$



Put $x = \tan\theta \Rightarrow dx = \sec^2\theta \, d\theta$

$$= \int \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)\sec^2\theta \, d\theta$$

$$= \int \tan^{-1}(\tan 2\theta)\sec^2\theta\, d\theta$$

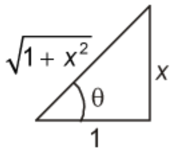
$$= 2\int \underset{\text{I}}{\theta}\ \underset{\text{II}}{\sec^2\theta}\, d\theta$$

$$= 2\left[\theta\cdot\int\sec^2\theta\, d\theta - \int\left\{\frac{d}{d\theta}\theta\cdot\int\sec^2\theta\, d\theta\right\}d\theta\right]$$

$$= 2\left[\theta\tan\theta - \int\tan\theta\, d\theta\right] = 2\,[\theta\tan\theta - \log\sec\theta] + C$$

$$= 2\left[x\tan^{-1}x - \log\sqrt{1+x^2}\right] + C$$

S16. Given integral is $\int\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)dx$



Put $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

$$= \int\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)\sec^2\theta\, d\theta$$

$$= \int\cos^{-1}(\cos 2\theta)\sec^2\theta\, d\theta$$

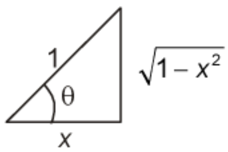
$$= 2\int \underset{\text{I}}{\theta}\ \underset{\text{II}}{\sec^2\theta}\, d\theta$$

$$= 2\left[\theta\cdot\int\sec^2\theta\, d\theta - \int\left\{\frac{d}{d\theta}\theta\cdot\int\sec^2\theta\, d\theta\right\}d\theta\right]$$

$$= 2\left[\theta\tan\theta - \int\tan\theta\, d\theta\right] = 2\,[\theta\tan\theta - \log\sec\theta] + C$$

$$= 2\left[x\tan^{-1}x - \log\sqrt{1+x^2}\right] + C$$

S17. Given integral is $\int\cos^{-1}(4x^3 - 3x)\, dx$



Put $x = \cos\theta \Rightarrow dx = -\sin\theta\, d\theta$

$$= \int\cos^{-1}(4\cos^3\theta - 3\cos\theta)\cdot(-\sin\theta)\, d\theta$$

$$= -\int\cos^{-1}(\cos 3\theta)\cdot\sin\theta\, d\theta$$

$$= -3\int \underset{\text{I}}{\theta}\ \underset{\text{II}}{\sin\theta}\, d\theta$$

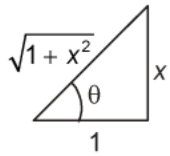
$$= -3\left[\theta\cdot\int\sin\theta\, d\theta - \int\left\{\frac{d}{d\theta}\theta\int\sin\theta\, d\theta\right\}d\theta\right]$$

$$= -3\left[-\theta\cos\theta - \int(-\cos\theta)\, d\theta\right]$$

$$= 3\left[\theta\cos\theta - \int\cos\theta\,d\theta\right] = 3\,[\theta\cos\theta - \sin\theta] + C$$

$$= 3\left[x\cos^{-1}x - \sqrt{1-x^2}\right] + C$$

S18. Given integral is $\int \frac{x^2\tan^{-1}x}{1+x^2}\,dx$



Put $\tan^{-1}x = \theta \Rightarrow x = \tan\theta \Rightarrow dx = \sec^2\theta\,d\theta$

$$= \int\frac{\theta\tan^2\theta\cdot\sec^2\theta\,d\theta}{1+\tan^2\theta} = \int\frac{\theta\tan^2\theta\cdot\sec^2\theta}{\sec^2\theta}d\theta$$

$$= \int\theta\cdot\tan^2\theta\,d\theta$$

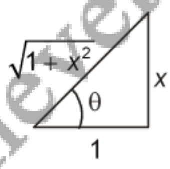
$$= \int\underset{\text{I}}{\theta}\,\underset{\text{II}}{\sec^2\theta}\,d\theta - \int\theta\,d\theta$$

$$= \theta\cdot\int\sec^2\theta\,d\theta - \int\left[\frac{d}{d\theta}\theta\cdot\int\sec^2\theta\,d\theta\right]d\theta - \int\theta\,d\theta$$

$$= \theta\cdot\tan\theta - \log\sec\theta - \frac{\theta^2}{2} + C$$

$$= x\tan^{-1}x - \log\sqrt{1+x^2} - \left(\frac{\tan^{-1}x}{2}\right)^2 + C$$

S19. Given integral is $\int\sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$



Put $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$

$$= \int\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\sec^2\theta\,d\theta$$

$$= \int\sin^{-1}(\sin 2\theta)\sec^2\theta\,d\theta = \int 2\,\theta\sec^2\theta\,d\theta$$

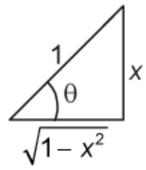
$$= 2\int\underset{\text{I}}{\theta}\,\underset{\text{II}}{\sec^2\theta}\,d\theta$$

$$= 2\left[\theta\cdot\int\sec^2\theta\,d\theta - \int\left\{\frac{d}{d\theta}\theta\cdot\int\sec^2\theta\,d\theta\right\}d\theta\right]$$

$$= 2\left[\theta\tan\theta - \int\tan\theta\,d\theta\right]$$

$$= 2\,[\theta\tan\theta - \log\sec\theta] + C = 2[x\tan^{-1}x - \log\sqrt{1+x^2}\,] + C$$

S20. Given integral is $\int \sin^{-1}(3x - 4x^3)\, dx$



Put $x = \sin\theta \;\Rightarrow\; dx = \cos\theta\, d\theta$

$$= \int \sin^{-1}(3\sin\theta - 4\sin^3\theta)\cos\theta\, d\theta$$

$$= \int \sin^{-1}(\sin 3\theta)\cos\theta\, d\theta$$

$$= \int 3\theta\cos\theta\, d\theta = 3\int \underset{\text{I}}{\theta}\, \underset{\text{II}}{\cos\theta}\, d\theta$$

$$= 3\left[\theta\int\cos\theta\, d\theta - \int\left\{\frac{d}{d\theta}\theta\int\cos\theta\, d\theta\right\}d\theta\right]$$

$$= 3\left[\theta\sin\theta - \int\sin\theta\, d\theta\right]$$

$$= 3\,[\theta\sin\theta + \cos\theta] + C = 3\,[x\sin^{-1}x + \sqrt{1-x^2}\,] + C$$

S21. Given integral is $\int x\left(\dfrac{\sec 2x - 1}{\sec 2x + 1}\right)dx$

$$= \int x\left(\frac{1-\cos 2x}{1+\cos 2x}\right)dx = \int x\,\frac{2\sin^2 x}{2\cos^2 x}\,dx = \int x\tan^2 x\, dx$$

$$= \int x\,(\sec^2 x - 1)\, dx$$

$$= \int \underset{\text{I}}{x}\, \underset{\text{II}}{\sec^2 x}\, dx - \int x\, dx$$

$$= x\cdot\int\sec^2 x\, dx - \int\left[\frac{d}{dx}x\int\sec^2 x\, dx\right]dx - \frac{x^2}{2}$$

$$= x\cdot\tan x - \int\tan x\, dx - \frac{x^2}{2} + C$$

$$= x\tan x - \log\sec x - \frac{x^2}{2} + C$$

S22. Given integral is $\int x\tan^2 x\, dx$

$$\int x\,(\sec^2 x - 1)\, dx$$

$$= \int \underset{\text{I}}{x}\, \underset{\text{II}}{\sec^2 x}\, dx - \int x\, dx$$

$$= x\cdot\int\sec^2 x\, dx - \int\left[\frac{d}{dx}x\int\sec^2 x\, dx\right]dx - \frac{x^2}{2}$$

$$= x\cdot\tan x - \int \tan x\,dx - \frac{x^2}{2} + C$$

$$= x\tan x - \log\sec x - \frac{x^2}{2} + C$$

S23. Given integral is $\int \frac{x+\sin x}{1+\cos x}\,dx$

$$= \int \frac{x}{1+\cos x}\,dx + \int \frac{\sin x}{1+\cos x}\,dx$$

$$= \int \frac{x}{2\cos^2\frac{x}{2}}\,dx + \int \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\,dx$$

$$= \frac{1}{2}\int x\sec^2\frac{x}{2}\,dx + \int \tan\frac{x}{2}\,dx$$

$$= \frac{1}{2}\left[x\cdot\int \sec^2\frac{x}{2}\,dx - \int \frac{d}{dx}x\int \sec^2\frac{x}{2}\,dx\right] + \int \tan\frac{x}{2}\,dx$$

$$= \frac{1}{2}\left[x\cdot 2\tan\frac{x}{2} - \int 2\tan\frac{x}{2}\,dx\right] + \int \tan\frac{x}{2}\,dx$$

$$= x\tan\frac{x}{2} - \int \tan\frac{x}{2}\,dx + \int \tan\frac{x}{2}\,dx + C$$

$$= x\tan\frac{x}{2} + C$$

S24. $$\int x\tan^{-1}x\,dx = \int \underset{I}{\tan^{-1}x}\cdot \underset{II}{x}\,dx$$

$$\left[\because \int \underset{I}{f(x)}\,\underset{II}{g(x)}\,dx = f(x)\int g(x)\,dx - \int\left(\frac{d}{dx}f(x)\int g(x)\,dx\right)dx\right]$$ [Integrating by parts]

$$= \tan^{-1}x\cdot\frac{x^2}{2} - \int \frac{1}{1+x^2}\cdot\frac{x^2}{2}\,dx$$

$$= \frac{x^2\tan^{-1}x}{2} - \frac{1}{2}\int \frac{(1+x^2)-1}{1+x^2}\,dx$$

$$= \frac{x^2\tan^{-1}x}{2} - \frac{1}{2}\int 1\cdot dx + \frac{1}{2}\int \frac{1}{1+x^2}\,dx$$

$$= \frac{x^2\tan^{-1}x}{2} - \frac{x}{2} + \frac{1}{2}\tan^{-1}x + c = (x^2+1)\frac{\tan^{-1}x}{2} - \frac{x}{2} + c$$

S25. $$I = \int \cos 2x \log \sin x \, dx = \int \underset{I}{\log \sin x} \cdot \underset{II}{\cos 2x} \, dx$$

$$\left[\because \int \underset{I}{f(x)} \underset{II}{g(x)} \, dx = f(x) \int g(x) \, dx - \int \left(\frac{d}{dx} f(x) \int g(x) dx \right) dx \right]$$ [Integrating by parts]

$$= \log \sin x \cdot \frac{\sin 2x}{2} - \int \frac{1}{\sin x} \cdot \cos x \cdot \frac{\sin 2x}{2} dx$$

$$= \frac{1}{2} \sin 2x \log \sin x - \int \frac{\cos x}{\sin x} \cdot \sin x \cos x \, dx$$

$$= \frac{1}{2} \sin 2x \log \sin x - \int \cos^2 x \, dx$$

$$= \frac{1}{2} \sin 2x \log \sin x - \frac{1}{2} \int (1 + \cos 2x) \, dx$$

$$= \frac{1}{2} \sin 2x \log \sin x - \frac{1}{2} \left(x + \frac{\sin 2x}{2} \right) + c$$

$$= \frac{1}{2} \sin 2x \log \sin x - \frac{1}{2} (x + \sin x \cos x) + c$$

S26. $$I = \int \log (1 + x^2) dx = \int \underset{I}{\log(1 + x^2)}.1 \, dx$$

$$= \log(1 + x^2) \cdot x - \int \frac{2x}{1 + x^2} \cdot x \, dx$$

$$\left[\because \int \underset{I}{f(x)} \underset{II}{g(x)} \, dx = f(x) \int g(x) \, dx - \int \left(\frac{d}{dx} f(x) \int g(x) dx \right) dx \right]$$ [Integrating by parts]

$$= x \log (1 + x^2) - 2 \int \frac{x^2}{1 + x^2} dx = x \log(1 + x^2) - 2 \left(1 - \frac{1}{1 + x^2} \right) dx$$

$$= x \log (1 + x^2) - 2x + 2 \tan^{-1} x + c.$$

S27. Put $x + 2 = t$ so that $dx = dt$.

$$I = \int \frac{\log t}{t^2} dt = \int \underset{I}{\log t} \cdot \underset{II}{t^{-2}} dt$$

$$\left[\because \int \underset{I}{f(x)} \underset{II}{g(x)} \, dx = f(x) \int g(x) \, dx - \int \left(\frac{d}{dx} f(x) \int g(x) dx \right) dx \right]$$ [Integrating by parts]

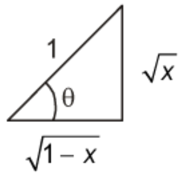
$$= -\log t \cdot \frac{t^{-1}}{1} - \int \frac{1}{t} \cdot \frac{t^{-1}}{-1} dt$$

$$= -\frac{\log t}{t} + \int t^{-2} dt = -\frac{\log t}{t} + \frac{t^{-1}}{-1} + c$$

$$= \frac{-\log t - 1}{t} + c = -\left[\frac{\log(x+2)+1}{x+2}\right] + c.$$

S28. Given integral is $\int \sin^{-1}\sqrt{x}\, dx$

Put $\sin^{-1}\sqrt{x} = \theta \Rightarrow x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\, d\theta$



$$= \int \theta \cdot 2\sin\theta\cos\theta\, d\theta = \int \underset{\text{I}}{\theta} \underset{\text{II}}{\sin 2\theta}\, d\theta$$

$$= \theta \int \sin 2\theta\, d\theta - \int \left[\frac{d}{d\theta}\theta \int \sin 2\theta\, d\theta\right] d\theta$$

$$= \frac{-\theta\cos 2\theta}{2} + \frac{1}{2}\int \cos 2\theta\, d\theta = \frac{-\theta\cos 2\theta}{2} + \frac{1}{4}\sin 2\theta + C$$

$$= \frac{-\theta(1-2\sin^2\theta)}{2} + \frac{\sin\theta\cos\theta}{2} + C$$

$$= \frac{\theta(2\sin^2\theta - 1)}{2} + \frac{\sin\theta\cos\theta}{2} + C$$

$$= \frac{\sin^{-1}\sqrt{x}\,(2x-1)}{2} + \frac{\sqrt{x}\sqrt{1-x}}{2} + C$$

$$= \frac{(2x-1)\sin^{-1}\sqrt{x}}{2} + \frac{\sqrt{x-x^2}}{2} + C$$

S29. Given integral is $\int \underset{\text{II}}{x^3} \underset{\text{I}}{\tan^{-1} x}\, dx$

$$= \tan^{-1}x \int x^3\, dx - \int \left[\frac{d}{dx}\tan^{-1}x \cdot \int x^3\, dx\right] dx$$

$$= \frac{x^4}{4}\tan^{-1}x - \frac{1}{4}\int \frac{x^4}{x^2+1}\, dx$$

$$= \frac{x^4}{4}\tan^{-1}x - \frac{1}{4}\int \left(x^2 - 1 + \frac{1}{x^2+1}\right) dx$$

$$= \frac{x^4}{4}\tan^{-1}x - \frac{1}{4}\left[\frac{x^3}{3} - x + \tan^{-1}x\right]$$

$$= \frac{x^4}{4}\tan^{-1}x - \frac{1}{12}x^3 + \frac{1}{4}x - \frac{1}{4}\tan^{-1}x + C$$

S30. Given integral is $\int \underset{\text{II}}{x^2} \underset{\text{I}}{\tan^{-1} x}\, dx$

$$= \tan^{-1} x \cdot \int x^2\, dx - \int \left[\frac{d}{dx} \tan^{-1} x \cdot \int x^2\, dx \right] dx$$

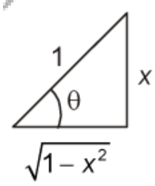
$$= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int \frac{x^3}{x^2+1}\, dx$$

$$= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int \left(x - \frac{x}{x^2+1} \right) dx$$

$$= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int x\, dx + \frac{1}{6} \int \frac{2x}{x^2+1}\, dx$$

$$= \frac{x^3}{3} \tan^{-1} x - \frac{1}{6} x^2 + \frac{1}{6} \log (x^2+1) + C$$

S31. Given integral is $\int \frac{x^2 \sin^{-1} x}{(1-x^2)^{3/2}}\, dx$



Put $\sin^{-1} x = \theta \;\Rightarrow\; x = \sin\theta \;\Rightarrow\; dx = \cos\theta\, d\theta$

$$= \int \frac{\theta \sin^2\theta \cos\theta}{(1-\sin^2\theta)^{3/2}}\, d\theta = \int \frac{\theta \sin^2\theta \cos\theta}{(\cos^2\theta)^{3/2}}\, d\theta$$

$$= \int \frac{\theta \sin^2\theta}{\cos^2\theta}\, d\theta = \int \theta \tan^2\theta\, d\theta$$

$$= \int \theta (\sec^2\theta - 1)\, d\theta = \int \underset{\text{I}}{\theta} \underset{\text{II}}{\sec^2\theta}\, d\theta - \int \theta\, d\theta$$

$$= \theta \cdot \int \sec^2\theta\, d\theta - \int \left[\frac{d}{d\theta} \theta \int \sec^2\theta\, d\theta \right] d\theta - \frac{\theta^2}{2}$$

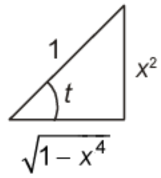
$$= \theta \tan\theta - \int \tan\theta\, d\theta - \frac{\theta^2}{2}$$

$$= \theta \tan\theta - \log \sec\theta - \frac{\theta^2}{2} + C$$

$$= \frac{x}{\sqrt{1-x^2}} \sin^{-1} x - \log \frac{1}{\sqrt{1-x^2}} - \frac{(\sin^{-1} x)^2}{2} + C$$

$$= \frac{x \sin^{-1} x}{\sqrt{1-x^2}} + \frac{1}{2} \log (1-x^2) - \frac{1}{2} (\sin^{-1} x)^2 + C$$

S32. Given integral is $\int \frac{x^3 \sin^{-1} x^2}{\sqrt{1-x^4}}\, dx$



Put $\sin^{-1} x^2 = t \Rightarrow x^2 = \sin t \Rightarrow 2x\, dx = \cos t\, dt$

$$= \frac{1}{2}\int \frac{t \sin t \cos t}{\sqrt{1-\sin^2 t}}\, dt = \frac{1}{2}\int \frac{t \sin t \cos t}{\cos t}\, dt$$

$$= \frac{1}{2}\int \underset{\text{I}}{t} \underset{\text{II}}{\sin t}\, dt$$

$$= \frac{1}{2}\left[t \cdot \int \sin t\, dt - \int\left[\frac{d}{dt} t \int \sin t\, dt\right] dt\right]$$

$$= \frac{1}{2}\left[-t \cos t - \int -\cos t\, dt\right] = \frac{1}{2}[-t \cos t + \sin t] + C$$

$$= \frac{1}{2}[-\sqrt{1-x^4}\ \sin^{-1} x^2 + x^2] + C$$

S33.
$$I = \int \tan^{-1}\sqrt{x}\, dx = \int \underset{I}{\tan^{-1}\sqrt{x}} \cdot \underset{II}{1}\, dx$$

$$\left[\because \int \underset{I}{f(x)}\, \underset{II}{g(x)}\, dx = f(x)\int g(x)\, dx - \int\left(\frac{d}{dx} f(x) \int g(x)\, dx\right) dx\right]$$ [Integrating by parts]

$$= \tan^{-1}\sqrt{x} \cdot x - \int \frac{1}{1+x}\left(\frac{1}{2\sqrt{x}}\right) \cdot x\, dx$$

$$= x \cdot \tan^{-1}\sqrt{x} - \frac{1}{2}\int \frac{\sqrt{x}}{1+x}\, dx \quad \text{... (i)}$$

Let $I_1 = \int \frac{\sqrt{x}}{1+x}\, dx$.

Put $\sqrt{x} = t$ *i.e.* $x = t^2$ so that $dx = 2t\, dt$.

$$I_1 = \int \frac{t}{1+t^2}(2t)\, dt = 2\int \frac{t^2}{1+t^2}\, dt$$

$$= 2\int \frac{1+t^2-1}{1+t^2}\, dt = 2\int 1 \cdot dt - 2\int \frac{1}{1+t^2}\, dt$$

$$= 2t - 2\tan^{-1} t = 2\sqrt{x} - 2\tan^{-1}\sqrt{x}.$$

$\therefore$ From (i), $I = x \tan^{-1}\sqrt{x} - \frac{1}{2}(2\sqrt{x} - 2\tan^{-1}\sqrt{x}) + c$

$$= (x+1)\tan^{-1}\sqrt{x} - \sqrt{x} + c.$$

S34.

$$I = \int (\sin^{-1} x)^2 dx = \int (\underset{I}{\sin^{-1} x^2}) \cdot \underset{II}{1 dx}$$

$$\left[\because \int \underset{I}{f(x)}\, \underset{II}{g(x)}\, dx = f(x)\int g(x)\, dx - \int\left(\frac{d}{dx} f(x) \int g(x) dx\right) dx\right] \quad \text{[Integrating by parts]}$$

$$= (\sin^{-1} x)^2 \cdot x - \int 2(\sin^{-1} x)\frac{1}{\sqrt{1-x^2}} \cdot x\, dx$$

$$= x(\sin^{-1} x)^2 + \int (\underset{I}{\sin^{-1} x})[\underset{II}{(1-x^2)^{-\frac{1}{2}}}(-2x)]\, dx$$

$$= x(\sin^{-1} x)^2 + \sin^{-1} x \cdot \frac{\sqrt{1-x^2}}{1/2} - \int \frac{1}{\sqrt{1-x^2}} \cdot \frac{\sqrt{1-x^2}}{1/2}\, dx$$

$$\left[\because \int [f(x)]^n f'(x) dx = \frac{f(x)}{n+1} + c\right]$$

$$= x(\sin^{-1} x)^2 + 2\sqrt{1-x^2}\, \sin^{-1} x - \int \frac{1}{\sqrt{1-x^2}} \cdot \frac{\sqrt{1-x^2}}{1/2} dx$$

$$= x(\sin^{-1} x)^2 + 2\sqrt{1-x^2}\, \sin^{-1} x - 2\int 1 \cdot dx$$

$$= x(\sin^{-1} x)^2 + 2\sqrt{1-x^2}\, \sin^{-1} x - 2x + c$$

S35. Let

$$I = \int \frac{x \tan^{-1} x}{(1+x^2)^{1/2}}\, dx$$

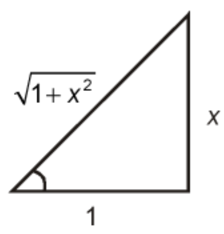
Put $\tan^{-1}x = t$, *i.e.* $x = \tan t$ so that $\frac{1}{1+x^2}\, dx = dt$.

$$\therefore \quad I = \int \frac{t.\tan t}{(1+\tan^2 t)^{1/2}}\, dt = \int \frac{t \tan t}{\sec t} dt = \int t \sin t\, dt$$

$$\left[\because \int \underset{I}{f(x)}\, \underset{II}{g(x)}\, dx = f(x)\int g(x)\, dx - \int\left(\frac{d}{dx} f(x) \int g(x) dx\right) dx\right] \quad \text{[Integrating by parts]}$$

$$= t(-\cos t) - \int (1)(-\cos t) dt$$

$$= -t \cos t + \sin t + c$$



from the triangle $\quad \cos t = \frac{1}{\sqrt{1+x^2}}, \sin t = \frac{x}{\sqrt{1+x^2}}$

$$= -\frac{\tan^{-1} x}{\sqrt{1+x^2}} + \frac{x}{\sqrt{1+x^2}} + c$$

S36. Let

$$I = \int \tan^{-1}\left(\frac{3x - x^3}{1-3x^2}\right) dx$$

Put $x = \tan\theta$ so that $dx = \sec^2\theta\, d\theta$.

$\therefore$ $$I = \int \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right)\cdot \sec^2\theta\, d\theta = \int \tan^{-1}(\tan 3\theta)\cdot\sec^2\theta\, d\theta$$

$$= \int 3\theta \sec^2\theta\, d\theta = 3\int \underset{I}{\theta}\, \underset{II}{\sec^2\theta}\, d\theta$$

$$\left[\because \int \underset{I}{f(x)}\, \underset{II}{g(x)}\, dx = f(x)\int g(x)\, dx - \int\left(\frac{d}{dx}f(x)\int g(x)\, dx\right)dx\right]$$

$$= 3\left[\theta\cdot\tan\theta - \int(1)\tan\theta\, d\theta\right]$$

$$= 3[\theta\tan\theta + \log|\cos\theta|] + c = 3\left[x\tan^{-1}x - \frac{1}{2}\log|\sec^2\theta|\right] + c$$

$$= 3\left[x\tan^{-1}x - \frac{1}{2}\log|1+\tan^2\theta|\right] + c = 3x\tan^{-1}x - \frac{3}{2}\log|1+x^2| + c$$

$$= 3x\tan^{-1}x - \frac{3}{2}\log(1+x^2) + c.$$

$[\because x^2 \ge 0 \Rightarrow 1+x^2 > 0 \quad \therefore |1+x^2| = 1+x^2]$

S37. $$I = \int\left[\log(1+\cos x) - x\tan\frac{x}{2}\right]dx$$

$$= \int \underset{I}{\log(1+\cos x)}\cdot \underset{II}{1}\, dx - \int x\tan\frac{x}{2}\, dx$$

$$= x\log(1+\cos x) - \int\left[\frac{1}{1+\cos x}(-\sin x)\right]\cdot x\, dx - \int x\tan\frac{x}{2}\, dx$$

[Integrating I^{st} Integrand by parts]

$$= x\log(1+\cos x) + \int\frac{2\sin x/2\cos x/2}{2\cos^2 x/2}x\, dx - \int x\tan\frac{x}{2}dx$$

$$= x\log(1+\cos x) + \int x\tan\frac{x}{2}dx - \int x\tan\frac{x}{2}dx$$

$$= x\log(1+\cos x) + c$$

S38. Let $$I = \int\frac{x^2}{(x\sin x+\cos x)^2}dx$$

$$= \int \underset{II}{\frac{x\cos x}{(x\sin x+\cos x)^2}}\left(\underset{I}{x\sec x}\right)dx \qquad \dots \text{(i)}$$

Let $$I_1 = \int\frac{x\cos x}{(x\sin x+\cos x)^2}dx$$

$x\sin x + \cos x = t$

$(x\cos x + \sin x - \sin x)dx = t$

$\Rightarrow \qquad x \cos x \, dx = dt$

$\therefore \qquad I_1 = \int \frac{dt}{t^2} = \frac{-1}{t} = \frac{-1}{x \sin x + \cos x}$

Now, integrating (i) by parts

$$I = x \sec x \int \frac{x \cos x}{(x \sin x + \cos x)^2} dx - \int \left\{ \frac{d}{dx}(x \sec x) \int \left(\frac{x \cos x}{(x \sin x + \cos x)^2} dx \right) \right\} dx$$

$$= x \sec x \frac{(-1)}{x \sin x + \cos x} - \int (1 \cdot \sec x + x \sec x \tan x) \frac{-dx}{x \sin x + \cos x}$$

$$= \frac{-x \sec x}{x \sin x + \cos x} + \int \sec^2 x dx$$

$$= \frac{-x \sec x}{x \sin x + \cos x} + \tan x + C$$

S39. Let $\qquad I = \int \frac{x \sin^{-1} x}{\sqrt{1 - x^2}} dx$

Put $\qquad \sin^{-1} x = t$

$\Rightarrow \qquad x = \sin t$

$\Rightarrow \qquad \frac{1}{\sqrt{1 - x^2}} dx = dt$

We get $\qquad I = \int \underset{\text{I}}{t} \underset{\text{II}}{\sin t} \, dt$

Using integration by parts taking 't' as the first function and sin t as the second function, we get

$$I = t \int \sin t \, dt - \int \left(\frac{d}{dt}(t) \cdot \int \sin t \, dt \right) dt$$

$$I = -t \cos t - \int 1 \times (-\cos t) dt$$

$$= -t \cos t + \int \cos t \, dt$$

$\therefore \qquad I = -t \cos t + \sin t + C$

$\Rightarrow \qquad I = -t\sqrt{1 - \sin^2 t} + \sin t + C \qquad \left[\because \cos^2 t = 1 - \sin^2 t \; \therefore \cos t = \sqrt{1 - \sin^2 t} \right]$

Hence, $\qquad I = -\sin^{-1} x \sqrt{1 - x^2} + x + C \qquad \left[\because t = \sin^{-1} x \text{ and } x = \sin t \right]$

S40. Using integration by parts

$$\int \underset{\text{I}}{u} \cdot \underset{\text{II}}{v}\, dx = \left[u \int v\, dx - \int \left\{ \frac{d}{dx} u \int v\, dx \right\} dx \right]$$

Let $$I = \int \sin_I^{-1} x \; x_{II}\, dx$$

Taking $\sin^{-1} x$ as 1st function and x as 2nd function and using the rule of integration by parts, we get

$$I = \sin^{-1} x \cdot \frac{x^2}{2} - \int \frac{1}{\sqrt{1-x^2}} \cdot \frac{x^2}{2} dx + C$$

$$I = \frac{x^2}{2} \sin^{-1} x - \frac{1}{2} \int \frac{x^2}{\sqrt{1-x^2}} dx + C$$

$$I = \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \frac{1 - x^2 - 1}{\sqrt{1-x^2}} dx$$

$$I = \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \frac{1-x^2}{\sqrt{1-x^2}} dx - \frac{1}{2} \int \frac{dx}{\sqrt{1-x^2}}$$

$$I = \frac{1}{2} \left[x^2 \sin^{-1} x + \int \sqrt{1-x^2}\, dx - \int \frac{dx}{\sqrt{1-x^2}} \right]$$

$$I = \frac{1}{2} \left[x^2 \sin^{-1} x + \frac{x}{2} \cdot \sqrt{1-x^2} + \frac{1}{2} \sin^{-1} x - \sin^{-1} x \right] + C$$

$$\left[\because \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C \right]$$

$$I = \frac{1}{2} \left[x^2 \sin^{-1} x + \frac{x}{2} \cdot \sqrt{1-x^2} - \frac{1}{2} \sin^{-1} x \right] + C$$

$$\left[\because \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C \right]$$

$$= \frac{x^2}{2} \sin^{-1} x + \frac{x}{4} \sqrt{1-x^2} - \frac{1}{4} \sin^{-1} x + C$$

S41. Let $$I = \int \underset{\text{II}}{e^{2x}} \underset{\text{I}}{\sin} x\, dx$$

Taking $\sin x$ as I function and e^{2x} *as* II function and integrating by parts, we get

$$I = \sin x \int e^{2x} dx - \int \left\{ \frac{d}{dx} (\sin x) \int e^{2x} dx \right\} dx$$

$$\Rightarrow \quad I = \frac{\sin x \cdot e^{2x}}{2} - \int \frac{\cos x \cdot e^{2x}}{2} dx$$

$$\Rightarrow \quad I = \frac{e^{2x}\sin x}{2} - \frac{1}{2}\int e^{2x}\cos x\,dx$$

$$\Rightarrow \quad I = \frac{e^{2x}\sin x}{2} - \frac{1}{2}I_1 \qquad \text{... (i)}$$

where, $\quad I_1 = \int \underset{II}{e^{2x}} \underset{I}{\cos} x\,dx$

Integrating by parts again by taking $\cos x$ as I function and e^{2x} as II function, we get

$$I_1 = \cos x\int e^{2x}dx - \int\left(\frac{d}{dx}(\cos x)\int e^{2x}dx\right)dx$$

$$\Rightarrow \quad I_1 = \frac{\cos x\cdot e^{2x}}{2} - \int\frac{-\sin x\cdot e^{2x}}{2}dx$$

$$\Rightarrow \quad I_1 = \frac{e^{2x}\cos x}{2} + \frac{1}{2}\int e^{2x}\sin x\,dx$$

$$\Rightarrow \quad I_1 = \frac{e^{2x}\cos x}{2} + \frac{1}{2}I \qquad \left[\because \int e^{2x}\sin x\,dx = I\right] \quad \text{... (ii)}$$

Putting the value of I_1 from Eq. (ii) in Eq. (i) we get

$$I = \frac{e^{2x}\sin x}{2} - \frac{1}{2}\left[\frac{e^{2x}\cos x}{2} + \frac{1}{2}I\right]$$

$$\Rightarrow \quad I = \frac{e^{2x}\sin x}{2} - \frac{e^{2x}\cos x}{4} - \frac{1}{4}I$$

$$\Rightarrow \quad I + \frac{1}{4}I = e^{2x}\left(\frac{2\sin x - \cos x}{4}\right)$$

$$\Rightarrow \quad \frac{5I}{4} = e^{2x}\left(\frac{2\sin x - \cos x}{4}\right)$$

$$\Rightarrow \quad I = \frac{4}{5}e^{2x}\left(\frac{2\sin x - \cos x}{4}\right)$$

Hence, $\quad I = \frac{1}{5}e^{2x}(2\sin x - \cos x)$

S42. $\int\frac{\tan^{-1}x}{x^2}dx = \int \underset{I}{\tan^{-1}x}\cdot \underset{II}{x^{-2}}dx$

$$\left[\because f(x)\,g(x)dx = f(x)\int g(x)dx - \left[\int\frac{d}{dx}f(x)\int g(x)dx\right]dx\right] \qquad \text{[Integrating by parts]}$$

$$= \tan^{-1}x\cdot\frac{x^{-2+1}}{-2+1} - \int\frac{1}{1+x^2}\cdot\frac{x^{-2+1}}{-2+1}dx$$

$$= \frac{-\tan^{-1}x}{x} + \int\frac{1}{x(1+x^2)}dx \qquad \text{... (i)}$$

Now $$\frac{1}{x(1+x^2)} \equiv \frac{A}{x}+\frac{Bx+C}{1+x^2}$$

$\Rightarrow$ $$1 \equiv A(1+x^2)+(Bx+C)x$$

Putting $x = 0 \Rightarrow A = 1$.

Comparing coeffs. of x^2, $0 = A + B \Rightarrow B = -A$

Comparing coeffs. of x, $0 = C \Rightarrow C = 0$.

$\therefore$ $$\frac{1}{x(1+x^2)} = \frac{1}{x}-\frac{x}{1+x^2}$$

$\therefore$ From Eq. (i), we get

$$\int \frac{\tan^{-1}x}{x^2}dx = -\frac{\tan^{-1}x}{x}+\int\left(\frac{1}{x}-\frac{x}{1+x^2}\right)dx$$

$$= -\frac{\tan^{-1}x}{x}+\int\frac{1}{x}dx-\frac{1}{2}\int\frac{2x}{1+x^2}dx$$

$$= \frac{-\tan^{-1}x}{x}+\log|x|-\frac{1}{2}\log(1+x^2)+c$$

$$= \frac{-\tan^{-1}x}{x}+\log\frac{|x|}{\sqrt{1+x^2}}+c.$$

S43.

$$I = \int\frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}dx = \int\frac{\sin^{-1}\sqrt{x}-\left(\frac{\pi}{2}-\sin^{-1}\sqrt{x}\right)}{\frac{\pi}{2}}dx$$

$$\left[\because \sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}=\frac{\pi}{2}\right]$$

$$= \frac{2}{\pi}\int\left(2\sin^{-1}\sqrt{x}-\frac{\pi}{2}\right)dx = \frac{4}{\pi}\int\sin^{-1}\sqrt{x}\,dx-\int 1\cdot dx \qquad \text{... (i)}$$

Let $$I_1 = \int \sin^{-1}\sqrt{x}\,dx$$

Let $$\sin^{-1}\sqrt{x} = \theta$$

$\Rightarrow$ $$\sqrt{x} = \sin\theta$$

$$\frac{1}{2\sqrt{x}}dx = \cos\theta\, d\theta$$

$$dx = 2\sqrt{x}\cos\theta\, d\theta$$

$$dx = 2\sin\theta\cos\theta\, d\theta$$

$$I_1 = \int\theta.2\sin\theta\cos\theta\,d\theta = \int \underset{I}{\theta}\,\underset{II}{\sin 2\theta}\,d\theta$$

$$= \theta\left(\frac{-\cos 2\theta}{2}\right) - \int (1)\frac{(-\cos 2\theta)}{2}\,d\theta \qquad \text{[Integrating by parts]}$$

$$= -\frac{\theta \cos 2\theta}{2} + \frac{1}{2}\int \cos 2\theta\, d\theta = -\frac{\theta}{2}\cos 2\theta + \frac{\sin 2\theta}{4}$$

$$= -\frac{1}{2}\theta(1 - 2\sin^2\theta) + \frac{1}{2}\sin\theta\cos\theta$$

$$= -\frac{1}{2}\sin^{-1}\sqrt{x}\,(1-2x) + \frac{1}{2}\sqrt{x}\,\sqrt{1-x}$$

Putting in Eq. (i),
$$I = \frac{4}{\pi}\left[-\frac{1}{2}(1-2x)\sin^{-1}\sqrt{x} + \frac{1}{2}\sqrt{x - x^2}\right] - x + c$$

$$= \frac{2}{\pi}\left[\sqrt{x-x^2} - (1-2x)\sin^{-1}\sqrt{x}\right] - x + c$$

S44. Let
$$I = \int \cos 2\theta \ln\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right) d\theta$$

$$= \int \underset{I}{\ln\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right)} \underset{II}{\cos 2\theta}\, d\theta$$

$$\left[\because \int \underset{I}{f(x)}\,\underset{II}{g(x)}\,dx = f(x)\int g(x)\,dx - \int\left(\frac{d}{dx}f(x)\int g(x)\,dx\right)dx\right] \qquad \text{[Integrating by parts]}$$

$$= \ln\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right)\cdot\frac{\sin 2\theta}{2} - \int \frac{2}{\cos 2\theta}\cdot\frac{\sin 2\theta}{2}\,d\theta$$

$$\left[\because \frac{d}{d\theta}\left(\ln\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right)\right) = \frac{d}{d\theta}[\log(\cos\theta + \sin\theta) - \log(\cos\theta - \sin\theta)]\right]$$

$$\left[\because \frac{d}{d\theta}\left(\ln\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right)\right) = \frac{d}{d\theta}[\log(\cos\theta + \sin\theta) - \log(\cos\theta - \sin\theta)]\right]$$

$$= \frac{1}{\cos\theta + \sin\theta}(-\sin\theta + \cos\theta) - \frac{1}{\cos\theta - \sin\theta}(-\sin\theta - \cos\theta)$$

$$= \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} + \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} = \frac{(\cos\theta - \sin\theta)^2 + (\cos\theta + \sin\theta)^2}{\cos^2\theta - \sin^2\theta}$$

$$= \frac{2(\cos^2\theta + \sin^2\theta)}{\cos 2\theta} = \frac{2(1)}{\cos 2\theta} = \frac{2}{\cos 2\theta}$$

$\therefore$
$$I = \frac{1}{2}\sin 2\theta \ln\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right) + \frac{1}{2}\int \frac{-2\sin 2\theta}{\cos 2\theta}\,d\theta$$

$$= \frac{1}{2}\sin 2\theta \ln\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right) + \frac{1}{2}\ln|\cos 2\theta| + c$$

S45. Given integral is $\int \frac{\sqrt{1-\sin x}}{1+\cos x}\, e^{-x/2}\, dx$

Put $\frac{-x}{2} = t \Rightarrow dx = -2\, dt$

$$= -2\int \frac{\sqrt{1-\sin(-2t)}}{1+\cos(-2t)} \cdot e^t\, dt$$

$$= -2\int \frac{\sqrt{1+\sin 2t}}{1+\cos 2t} \cdot e^t\, dt$$

$$= -2\int \frac{\sqrt{\sin^2 t + \cos^2 t + 2\sin t\cos t}}{2\cos^2 t}\, e^t\, dt$$

$$= -\int \frac{\sin t + \cos t}{\cos^2 t}\, e^t\, dt = -\int e^t (\tan t \sec t + \sec t)\, dt$$

$$= -\left[\int e^t \tan t \sec t\, dt + \int \underset{II}{e^t}\ \underset{I}{\sec t}\, dt\right]$$

$$= -\left[\int e^t \sec t \tan t\, dt + \sec t \int e^t\, dt - \int\left[\frac{d}{dx}\sec t \cdot \int e^t\, dt\right] dt\right]$$

$$= -\left[\int e^t \sec t \tan t\, dt + e^t \sec t - \int e^t \sec t \tan t\, dt\right]$$

$$= -e^t \sec t + C = -e^{-x/2} \sec x/2 + C$$

S46.

$$I = \int \underset{I}{\sin^{-1} x} \cdot \underset{II}{x^{-2}}\, dx \qquad \left[\because f(x)g(x)dx = f(x)\int g(x)dx - \left[\int \frac{d}{dx} f(x) \int g(x)dx\right]dx\right]$$

[Integrating by parts]

$$= \sin^{-1} x \cdot \frac{x^{-2+1}}{-2+1} - \int \frac{1}{\sqrt{1-x^2}} \cdot \frac{x^{-2+1}}{-2+1}\, dx$$

$$= -\frac{\sin^{-1} x}{x} + \int \frac{1}{x\sqrt{1-x^2}}\, dx \qquad \dots (i)$$

Let $I_1 = \int \frac{1}{x\sqrt{1-x^2}}\, dx.$

Put $x = \frac{1}{t}$ so that $dx = -\frac{1}{t^2}\, dt$

$$I_1 = \int \frac{-\frac{1}{t^2}\, dt}{\frac{1}{t}\sqrt{1-\frac{1}{t^2}}} = -\int \frac{dt}{\sqrt{t^2-1}}$$

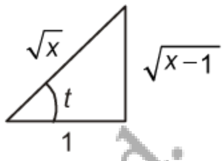
$$= -\log|t+\sqrt{t^2-1}| = -\log\left|\frac{1}{x}+\sqrt{\frac{1}{x^2}-1}\right|$$

$$\left[\because \int \frac{dx}{\sqrt{x^2-a^2}} = \log|x+\sqrt{x^2-a^2}|\right]$$

$$= -\log\left|\frac{1+\sqrt{1-x^2}}{x}\right|$$

From Eq. (i), $I = -\frac{\sin^{-1}x}{x} - \log\left|\frac{1+\sqrt{1-x^2}}{x}\right| + c$.

S47. Given integral is $\int \sec^{-1}\sqrt{x}\, dx$



Put $\sec^{-1}\sqrt{x} = t \Rightarrow x = \sec^2 t \Rightarrow dx = 2\sec t \cdot \sec t \tan t\, dt$

$\Rightarrow$ $$\int t \cdot 2\sec^2 t \tan t\, dt$$

$\Rightarrow$ $$2\int \underset{\text{I}}{t} \underset{\text{II}}{\tan t \sec^2 t}\, dt \qquad \dots \text{(i)}$$

Integrating IInd part,

Now, $$\int \tan t \sec^2 t\, dt$$

Let $\tan t = z \Rightarrow \sec^2 t\, dt = dz$

$$\int z\, dz = \frac{z^2}{2} = \frac{\tan^2 t}{2} + C$$

From (i)

$\Rightarrow$ $$2\left[t \int \tan t \sec^2 t\, dt - \int\left[\frac{d}{dt} t \int \tan t \sec^2 t\, dt\right] dt\right]$$

$$= 2\left[\frac{t\tan^2 t}{2} - \frac{\tan^2 t}{2} dt\right] = 2\left[\frac{t\tan^2 t}{2} - \frac{1}{2}\int(\sec^2 t - 1)\, dt\right]$$

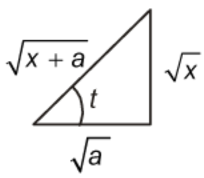
$$= 2\left[\frac{t\tan^2 t}{2} - \frac{1}{2}\tan t + \frac{1}{2}t\right] + C$$

$$= t\tan^2 t - \tan t + t + C$$

$$= (x-1)\sec^{-1}\sqrt{x} - \sqrt{x-1} + \sec^{-1}\sqrt{x} + C$$

$$= x\sec^{-1}\sqrt{x} - \sqrt{x-1} + C$$

S48. Given integral is $\int \sin^{-1}\sqrt{\frac{x}{x+a}}\, dx$



Put $x = a\tan^2 t \Rightarrow dx = a\, 2\tan t \sec^2 t\, dt$

$$= \int \sin^{-1} \sqrt{\frac{a \tan^2 t}{a \tan^2 t + a}} \times a\, 2 \tan t \sec^2 t\, dt$$

$$= \int \sin^{-1} \sqrt{\frac{\tan^2 t}{\sec^2 t}} \times 2a \tan t \sec^2 t\, dt$$

$$= 2a \int \sin^{-1}(\sin t) \tan t \sec^2 t\, dt$$

$$= 2a \int \underset{\text{I}}{t} \underset{\text{II}}{\tan t \sec^2 t}\, dt \qquad \dots \text{(i)}$$

Now integrating II[nd] part

$$= \int \tan t \sec^2 t\, dt$$

Put $\tan t = z \Rightarrow \sec^2 t\, dt = dz$

$$\int z\, dz = \frac{z^2}{2} + C = \frac{\tan^2 t}{2} + C$$

From (i)

$$= 2a\left[t \int \tan t \sec^2 t\, dt - \int\left[\frac{d}{dt} t \cdot \int \tan t \sec^2 t\, dt\right] dt\right]$$

$$= 2a\left[\frac{t \tan^2 t}{2} - \int \frac{\tan^2 t}{2}\, dt\right]$$

$$= 2a\left[\frac{t \tan^2 t}{2} - \frac{1}{2} \int (\sec^2 t - 1)\, dt\right]$$

$$= 2a\left[\frac{t \tan^2 t}{2} - \frac{1}{2} \tan t + \frac{1}{2} t\right] + C$$

$$= \frac{2a}{2}\left[\frac{x}{a} \tan^{-1} \sqrt{\frac{x}{a}} - \sqrt{\frac{x}{a}} + \tan^{-1} \sqrt{\frac{x}{a}}\right] + C$$

$$= a\left[\frac{x}{a} \tan^{-1} \sqrt{\frac{x}{a}} - \sqrt{\frac{x}{a}} + \tan^{-1} \sqrt{\frac{x}{a}}\right] + C$$

Date: 19/10/2021

Q1. Evaluate $\int \frac{dx}{\cos x\,(\sin x + 2\cos x)}$

Q2. Evaluate $\int \frac{2\sin x\cos x}{\sin^4 x + \cos^4 x}\,dx$

Q3. Evaluate $\int \frac{dx}{2 - 3\cos 2x}$

Q4. Evaluate $\int \frac{dx}{(\sin^2 x - 2\cos^2 x)(2\sin^2 x + \cos^2 x)}$

Q5. Evaluate $\int \frac{dx}{3 + \sin 2x}$

Q6. Evaluate $\int \frac{dx}{1 + 3\sin^2 x + 8\cos^2 x}$

Q7. Evaluate $\int \frac{dx}{4\sin^2 x + 5\cos^2 x}$

Q8. Evaluate $\int \frac{dx}{a^2\sin^2 x + b^2\cos^2 x}$

Q9. Evaluate $\int \frac{\sin x}{\sin 3x}\,dx$

Q10. Evaluate $\int \frac{dx}{\cos 2x + 3\sin^2 x}$

Date: 19/10/2021

S1. Given integral is $\int \frac{dx}{\cos x\,(\sin x + 2\cos x)}$

[Dividing Num. & Denom. by $\cos^2 x$]

$$= \int \frac{\sec^2 x\, dx}{\tan x + 2}$$

Put $\tan x + 2 = t \Rightarrow \sec^2 x\, dx = dt$

$$\Rightarrow \qquad \int \frac{dt}{t} = \log t + C = \log(\tan x + 2) + C$$

S2. Given integral is $\int \frac{\sin 2x}{\sin^4 x + \cos^4 x}\, dx$

$$= \int \frac{2\sin x \cos x}{\sin^4 x + \cos^4 x}\, dx$$

[Dividing Num. & Denom. by $\cos^4 x$]

$$= \int \frac{2\tan x \sec^2 x\, dx}{\tan^4 x + 1}$$

Put $\tan^2 x = t \Rightarrow 2\tan x \sec^2 x\, dx = dt$

$$= \int \frac{dt}{t^2 + 1} = \tan^{-1} t + C = \tan^{-1}(\tan^2 x) + C$$

S3. Given integral is $\int \frac{dx}{2 - 3\cos 2x}$

$$= \int \frac{dx}{2 - 3(\cos^2 x - \sin^2 x)} = \int \frac{dx}{2 - 3\cos^2 x + 3\sin^2 x}$$

[Dividing Num. & Denom. by $\cos^2 x$]

$$= \int \frac{\sec^2 x\, dx}{2\sec^2 x - 3 + 3\tan^2 x} = \int \frac{\sec^2 x\, dx}{2(1 + \tan^2 x) - 3 + 3\tan^2 x}$$

$$= \int \frac{\sec^2 x\, dx}{5\tan^2 x - 1}$$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$= \int \frac{dt}{5t^2 - 1} = \frac{1}{5}\int \frac{dt}{t^2 - \left(\frac{1}{\sqrt{5}}\right)^2}$$

$$= \frac{1}{5}\cdot\frac{1}{2\cdot\frac{1}{\sqrt{5}}}\log\left(\frac{t-\frac{1}{\sqrt{5}}}{t+\frac{1}{\sqrt{5}}}\right)+C = \frac{1}{2\sqrt{5}}\log\left(\frac{\sqrt{5}t-1}{\sqrt{5}t+1}\right)+C$$

$$= \frac{1}{2\sqrt{5}}\log\left(\frac{\sqrt{5}\tan x-1}{\sqrt{5}\tan x+1}\right)+C$$

S4. Given integral is $\int\frac{dx}{(\sin^2x-2\cos^2x)(2\sin^2x+\cos^2x)}$

$$= \int\frac{\sec^2 x\,dx}{(\tan^2x-2)(2\tan^2x+1)} \qquad \text{[Dividing Num. \& Denom. by } \cos^2 x]$$

Put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$$= \int\frac{dt}{2t^2-3t-2}$$

$$= \frac{1}{2}\int\frac{dt}{t^2-\frac{3}{2}t-1} = \frac{1}{2}\int\frac{dt}{t^2-\frac{3}{2}t+\frac{9}{16}-\frac{9}{16}-1}$$

$$= \frac{1}{2}\int\frac{dt}{\left(t-\frac{3}{4}\right)^2-\left(\frac{5}{4}\right)^2}$$

$$= \frac{1}{2}\cdot\frac{1}{2\cdot\frac{5}{4}}\log\left(\frac{t-\frac{3}{4}-\frac{5}{4}}{t-\frac{3}{4}+\frac{5}{4}}\right)+C = \frac{1}{5}\log\left(\frac{t-2}{t+1}\right)+C$$

$$= \frac{1}{5}\log\left(\frac{\tan x-2}{\tan x+1}\right)+C$$

S5. Given integral is $\int\frac{dx}{3+\sin 2x}$

$$= \int\frac{dx}{3+2\sin x\cos x}$$

[Dividing Num. & Denom. by $\cos^2 x$]

$$= \int\frac{\sec^2 x\,dx}{3\sec^2 x+2\tan x} = \int\frac{\sec^2 x\,dx}{3(1+\tan^2 x)+2\tan x}$$

$$= \int\frac{\sec^2 x}{3\tan^2 x+2\tan x+3}dx$$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$\Rightarrow \int \frac{dt}{3t^2 + 2t + 3} = \frac{1}{3}\int \frac{dt}{t^2 + \frac{2}{3}t + 1}$$

$$= \frac{1}{3}\int \frac{dt}{t^2 + \frac{2}{3}t + \frac{1}{9} - \frac{1}{9} + 1} = \frac{1}{3}\int \frac{dt}{\left(t + \frac{1}{3}\right)^2 + \left(\frac{2\sqrt{2}}{3}\right)^2}$$

$$= \frac{1}{3} \cdot \frac{1}{\frac{2\sqrt{2}}{3}} \tan^{-1}\left(\frac{t + \frac{1}{3}}{\frac{2\sqrt{2}}{3}}\right) + C = \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{3\tan x + 1}{2\sqrt{2}}\right) + C$$

S6. Given integral is $\int \frac{dx}{1 + 3\sin^2 x + 8\cos^2 x}$

[Dividing Num. & Denom. by $\cos^2 x$]

$$= \int \frac{\sec^2 x\, dx}{\sec^2 x + 3\tan^2 x + 8} = \int \frac{\sec^2 x\, dx}{1 + \tan^2 x + 3\tan^2 x + 8}$$

$$= \int \frac{\sec^2 x\, dx}{4\tan^2 x + 9}$$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$= \int \frac{dt}{4t^2 + 9} = \frac{1}{4}\int \frac{dt}{t^2 + \frac{9}{4}}$$

$$= \frac{1}{4}\int \frac{dt}{t^2 + \left(\frac{3}{2}\right)^2} = \frac{1}{4} \cdot \frac{1}{\frac{3}{2}} \tan^{-1}\left(\frac{t}{\frac{3}{2}}\right) + C$$

$$= \frac{1}{6} \tan^{-1}\left(\frac{2\tan x}{3}\right) + C$$

S7.

$$I = \int \frac{dx}{4\sin^2 x + 5\cos^2 x} = \int \frac{\sec^2 x\, dx}{4\tan^2 x + 5}$$

[Dividing Num. & Denom. by $\cos^2 x$]

Put $\tan x = t$ so that $\sec^2 x\, dx = dt$

$$\therefore \quad I = \int \frac{dt}{4t^2 + 5} = \frac{1}{4}\int \frac{dt}{\frac{5}{2} + t^2} = \frac{1}{4}\int \frac{dt}{\left(\frac{\sqrt{5}}{2}\right)^2 + t^2}$$

$$= \frac{1}{4}\cdot\frac{1}{\frac{\sqrt{5}}{2}}\tan^{-1}\frac{t}{\frac{\sqrt{5}}{2}} + c = \frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{2}{\sqrt{5}}\tan x\right) + c$$

S8. Given integral is $\int \frac{dx}{a^2\sin^2 x + b^2\cos^2 x}$

[Dividing Num. & Denom. by $\cos^2 x$]

$$= \int \frac{\sec^2 x}{a^2\tan^2 x + b^2}\,dx$$

Put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$$\int \frac{dt}{a^2t^2 + b^2} = \frac{1}{a^2}\int \frac{dt}{t^2 + \left(\frac{b}{a}\right)^2}$$

$$= \frac{1}{a^2}\cdot\frac{1}{\frac{b}{a}}\tan^{-1}\left(\frac{t}{b/a}\right) + C = \frac{a}{b}\tan^{-1}\left(\frac{a\tan x}{b}\right) + C$$

S9. Given integral is $\int \frac{\sin x}{\sin 3x}\,dx$

$$= \int \frac{\sin x}{3\sin x - 4\sin^3 x}\,dx = \int \frac{1}{3 - 4\sin^2 x}\,dx$$

$$= \int \frac{\sec^2 x}{3\sec^2 x - 4\tan^2 x}\,dx = \int \frac{\sec^2 x}{3(1 + \tan^2 x) - 4\tan^2 x}\,dx$$

$$= \int \frac{\sec^2 x}{3 - \tan^2 x}\,dx$$

Put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$$= \int \frac{dt}{3 - t^2} = \int \frac{dt}{(\sqrt{3})^2 - t^2} = \frac{1}{2\sqrt{3}}\log\left(\frac{\sqrt{3} + t}{\sqrt{3} - t}\right) + C$$

$$= \frac{1}{2\sqrt{3}}\log\left(\frac{\sqrt{3} + \tan x}{\sqrt{3} - \tan x}\right) + C$$

S10. Given integral is $\int \frac{dx}{\cos 2x + 3\sin^2 x}$

$$= \int \frac{dx}{\cos^2 x - \sin^2 x + 3\sin^2 x} = \int \frac{dx}{\cos^2 x + 2\sin^2 x}$$

[Dividing Num. & Denom. by $\cos^2 x$]

$$\Rightarrow \qquad \int \frac{\sec^2 x\, dx}{1 + 2\tan^2 x}$$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$\int \frac{dt}{1+2t^2} = \frac{1}{2}\int \frac{dt}{t^2 + \left(\frac{1}{\sqrt{2}}\right)^2}$$

$$= \frac{1}{2}\cdot\frac{1}{\frac{1}{\sqrt{2}}}\tan^{-1}\left(\frac{t}{\frac{1}{\sqrt{2}}}\right) + c = \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\tan x) + C$$

Date: 19/10/2021

Q1. Evaluate $\int \frac{d\theta}{3+2\sin\theta+\cos\theta}$.

Q2. Evaluate $\int \frac{d\theta}{1-2\sin\theta}$

Q3. Evaluate $\int \frac{1}{1+2\cos\theta}\,d\theta$

Q4. Evaluate $\int \frac{1}{1+\sin x+\cos x}\,dx$

Q5. Evaluate $\int \frac{dx}{13+3\cos x+4\sin x}$

Q6. Evaluate $\int \frac{1}{5+4\cos x}\,dx$

Q7. Evaluate $\int \frac{dx}{2+\cos x}$

Q8. Evaluate $\int \frac{1}{2+\sin x+\cos x}\,dx$

Q9. Evaluate $\int \frac{dx}{\sin x+\sqrt{3}\cos x}$

Q10. Evaluate $\int \frac{dx}{5+7\cos x+\sin x}$

Q11. Evaluate $\int \frac{3\sin x+2\cos x}{3\cos x+2\sin x}\,dx$

Q12. Evaluate $\int \frac{1+\sin x}{\sin x(1+\cos x)}\,dx$.

Q13. Evaluate $\int \frac{1}{a+b\tan x}\,dx$

Q14. Evaluate $\int \frac{4\sin x+5\cos x}{5\sin x+4\cos x}\,dx$.

Q15. Evaluate $\int \frac{1}{1+\tan x}\,dx$

Q16. Evaluate $\int \frac{dx}{1+\cot x}$

Q17. Evaluate $\int \frac{2\tan x+3}{3\tan x+4}\,dx$.

Q18. Evaluate $\int \frac{5\cos x+6}{2\cos x+\sin x+3}\,dx$.

Q19. Evaluate $\int \frac{3\cos x+2}{\sin x+2\cos x+3}\,dx$.

Q20. Evaluate $\int \frac{3+2\cos x+4\sin x}{2\sin x+\cos x+3}\,dx$

Date: 19/10/2021

S1.

Put $\cos\theta = \dfrac{1-\tan^2\dfrac{\theta}{2}}{1+\tan^2\dfrac{\theta}{2}} = \dfrac{1-t^2}{1+t^2}$

and $\sin\theta = \dfrac{2\tan\dfrac{\theta}{2}}{1+\tan^2\dfrac{\theta}{2}} = \dfrac{2t}{1+t^2}$

where $t = \tan\dfrac{\theta}{2}$ so that $dt = \dfrac{1}{2}\sec^2\dfrac{\theta}{2}d\theta$

i.e., $dt = \dfrac{1}{2}\left(1+\tan^2\dfrac{\theta}{2}\right)\cdot d\theta = \dfrac{1}{2}(1+t^2)\,d\theta$

or $2dt = (1+t^2)d\theta$

$\Rightarrow$ $d\theta = \dfrac{2dt}{1+t^2}$

$$\int\frac{d\theta}{3+2\sin\theta+\cos\theta} = \int\frac{1}{3+2\left(\dfrac{2t}{1+t^2}\right)+\left(\dfrac{1-t^2}{1+t^2}\right)}\cdot\frac{2dt}{1+t^2}$$

$$= 2\int\frac{dt}{3+3t^2+4t+1-t^2} = 2\int\frac{dt}{4+4t+2t^2}$$

$$= \int\frac{dt}{t^2+2t+2} = \int\frac{dt}{(t+1)^2+(1)^2}$$

$$= \tan^{-1}(t+1)+c = \tan^{-1}\left(\tan\frac{\theta}{2}+1\right)+c \quad \left[\because \int\frac{dx}{x^2-a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}+c\right]$$

S2.

Put $\sin\theta = \dfrac{2\tan\dfrac{\theta}{2}}{1+\tan^2\dfrac{\theta}{2}} = \dfrac{2t}{1+t^2}$

where $t = \tan\frac{\theta}{2}$ so that $dt = \frac{1}{2}\sec^2\frac{\theta}{2}d\theta$

i.e., $dt = \frac{1}{2}\left(1+\tan^2\frac{\theta}{2}\right)\cdot d\theta = \frac{1}{2}(1+t^2)\,d\theta$

or $2dt = (1 + t^2)d\theta$

$\Rightarrow$ $d\theta = \frac{2dt}{1+t^2}$

$$\int\frac{d\theta}{1-2\sin\theta} = \int\frac{1}{1-2\cdot\frac{2t}{1+t^2}}\cdot\frac{2dt}{1+t^2}$$

$$= 2\int\frac{1}{1+t^2-4t}\,dt = 2\int\frac{dt}{(t^2-4t+4)-3} = 2\int\frac{dt}{(t-2)^2-(\sqrt{3})^2}$$

Put $t = 2 + z$ so that $dt = dz$.

$\therefore$ $$I = 2\int\frac{dz}{z^2-(\sqrt{3})^2} = \frac{2}{2\sqrt{3}}\log\left|\frac{z-\sqrt{3}}{z+\sqrt{3}}\right| + c$$

$$= \frac{1}{\sqrt{3}}\log\left|\frac{t-2-\sqrt{3}}{t-2+\sqrt{3}}\right| + c = \frac{1}{\sqrt{3}}\log\left|\frac{\tan\frac{\theta}{2}-2-\sqrt{3}}{\tan\frac{\theta}{2}-2+\sqrt{3}}\right| + c$$

S3. Let $I = \int\frac{1}{1+2\cos\theta}\,d\theta$

Put $\cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{1-t^2}{1+t^2}$

where $t = \tan\frac{\theta}{2}$ so that $dt = \frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta$

i.e., $dt = \frac{1}{2}\left(1+\tan^2\frac{\theta}{2}\right)\cdot d\theta = \frac{1}{2}(1+t^2)\,d\theta$

or $2dt = (1 + t^2)d\theta$

$\Rightarrow$ $d\theta = \frac{2dt}{1+t^2}$

$$\int\frac{d\theta}{1+2\cos\theta} = \int\frac{1}{1+2\left(\frac{1-t^2}{1+t^2}\right)}\cdot\frac{2dt}{1+t^2}$$

$$= 2\int\frac{dt}{1+t^2+2-2t^2} = 2\int\frac{dt}{(\sqrt{3})^2-t^2}$$

$$= 2\cdot\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+t}{\sqrt{3}-t}\right| + c = \frac{1}{\sqrt{3}}\log\left|\frac{\sqrt{3}+\tan\frac{\theta}{2}}{\sqrt{3}-\tan\frac{\theta}{2}}\right| + c$$

$$\left[\because \int\frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c\right]$$

S4. Given integral is $\int\frac{1}{1+\sin x+\cos x}\,dx$

Put $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$= \int\frac{dx}{1+\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}+\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$$

$$= \int\frac{\left(1+\tan^2\frac{x}{2}\right)}{1+2\tan\frac{x}{2}+\tan^2\frac{x}{2}+1-\tan^2\frac{x}{2}}\,dx = \frac{1}{2}\int\frac{\sec^2\frac{x}{2}}{1+\tan\frac{x}{2}}\,dx$$

Put $\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$

$$= \frac{1}{2}\int\frac{2\,dt}{1+t} = \log(1+t) + C = \log(\tan x/2 + 1) + C$$

S5. Given integral is $\int\frac{dx}{13+3\cos x+4\sin x}$

Put $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and $\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \cos x$

$$= \int\frac{dx}{13+3\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)+4\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}$$

$$= \int \frac{\left(1+\tan^2\frac{x}{2}\right)dx}{13+13\tan^2\frac{x}{2}+3-3\tan^2\frac{x}{2}+8\tan\frac{x}{2}}$$

$$= \int \frac{\sec^2\frac{x}{2}}{10\tan^2\frac{x}{2}+8\tan\frac{x}{2}+16}dx$$

Put $\tan\frac{x}{2} = t \quad \Rightarrow \quad \frac{1}{2}\sec^2\frac{x}{2}dx = dt$

$$= 2\int\frac{dt}{10t^2+8t+16} = \frac{2}{10}\int\frac{dt}{t^2+\frac{8}{10}t+\frac{16}{10}}$$

$$= \frac{1}{5}\int\frac{dt}{t^2+\frac{4}{5}t+\frac{4}{25}-\frac{4}{25}+\frac{8}{5}} = \frac{1}{5}\int\frac{dt}{\left(t+\frac{2}{5}\right)^2+\left(\frac{6}{5}\right)^2}$$

$$= \frac{1}{5}\cdot\frac{1}{\frac{6}{5}}\tan^{-1}\left(\frac{t+\frac{2}{5}}{\frac{6}{5}}\right)+C = \frac{1}{6}\tan^{-1}\left(\frac{5t+2}{6}\right)+C$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{5\tan\frac{x}{2}+2}{6}\right)+C$$

S6. Given integral is $\int\frac{1}{5+4\cos x}dx$

Put $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$= \int\frac{1}{5+4\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}dx = \int\frac{\left(1+\tan^2\frac{x}{2}\right)}{5+5\tan^2\frac{x}{2}+4-4\tan^2\frac{x}{2}}dx$$

$$= \int\frac{\sec^2\frac{x}{2}}{\tan^2\frac{x}{2}+9}dx$$

Put $\tan\frac{x}{2} = t \quad \Rightarrow \quad \frac{1}{2}\sec^2\frac{x}{2}dx = dt$

$$= 2\int\frac{dt}{t^2+9} = \frac{2}{3}\tan^{-1}\left(\frac{t}{3}\right)+C$$

$$= \frac{2}{3}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{3}\right)+C$$

S7. Given integral is $\int\frac{dx}{2+\cos x}$

Put $\cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$= \int\frac{dx}{2+\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}} = \int\frac{1+\tan^2\frac{x}{2}}{2+2\tan^2\frac{x}{2}+1-\tan^2\frac{x}{2}}dx$$

$$= \int\frac{\sec^2\frac{x}{2}}{\tan^2\frac{x}{2}+3}dx$$

Put $\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}dx = dt$

$$= 2\int\frac{dt}{t^2+(\sqrt{3})^2} = 2\cdot\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)+C$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{\sqrt{3}}\right)+C$$

S8. Given integral is $\int\frac{1}{2+\sin x+\cos x}dx$

Put $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and $\cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$= \int \frac{1}{2 + \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} + \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}\,dx$$

$$= \int \frac{\left(1+\tan^2\frac{x}{2}\right)dx}{2+2\tan^2\frac{x}{2}+2\tan\frac{x}{2}+1-\tan^2\frac{x}{2}}$$

$$= \int \frac{\sec^2\frac{x}{2}}{\tan^2\frac{x}{2}+2\tan\frac{x}{2}+3}\,dx$$

Put $\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}dx = dt$

$$= \int \frac{2\,dt}{t^2+2t+3} = 2\int \frac{dt}{t^2+2t+1+2}$$

$$= 2\int \frac{dt}{(t+1)^2+(\sqrt{2})^2} = 2\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t+1}{\sqrt{2}}\right)+C$$

$$= \sqrt{2}\tan^{-1}\left(\frac{\tan\frac{x}{2}+1}{\sqrt{2}}\right)+C$$

S9. Given integral is $\int \frac{dx}{\sin x + \sqrt{3}\cos x}$

Put $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$= \int \frac{dx}{\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} + \sqrt{3}\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)} = \int \frac{\left(1+\tan^2\frac{x}{2}\right)dx}{2\tan\frac{x}{2}+\sqrt{3}-\sqrt{3}\tan^2\frac{x}{2}}$$

$$= \int \frac{\sec^2\frac{x}{2}}{2\tan\frac{x}{2}+\sqrt{3}-\sqrt{3}\tan^2\frac{x}{2}}\,dx$$

Put $\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$

$$= 2\int\frac{dt}{2t+\sqrt{3}-\sqrt{3}\,t^2} = \frac{-2}{\sqrt{3}}\int\frac{dt}{t^2-\frac{2}{\sqrt{3}}t-1}$$

$$= \frac{-2}{\sqrt{3}}\int\frac{dt}{t^2-\frac{2}{\sqrt{3}}t+\frac{1}{3}-\frac{1}{3}-1} = \frac{-2}{\sqrt{3}}\int\frac{dt}{\left(t-\frac{1}{\sqrt{3}}\right)^2-\left(\frac{2}{\sqrt{3}}\right)^2}$$

$$= \frac{-2}{\sqrt{3}}\cdot\frac{1}{2\cdot\frac{2}{\sqrt{3}}}\log\left(\frac{t-\frac{1}{\sqrt{3}}-\frac{2}{\sqrt{3}}}{t-\frac{1}{\sqrt{3}}+\frac{2}{\sqrt{3}}}\right)+C = -\frac{1}{2}\log\left(\frac{\sqrt{3}t-3}{\sqrt{3}\,t+1}\right)$$

$$= \frac{1}{2}\log\left(\frac{1+\sqrt{3}\tan\frac{x}{2}}{\sqrt{3}\tan\frac{x}{2}-3}\right)+C$$

S10. Given integral is $\int\frac{dx}{5+7\cos x+\sin x}$

Put $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$= \int\frac{dx}{5+7\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)+\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$$

$$= \int\frac{\left(1+\tan^2\frac{x}{2}\right)dx}{5+5\tan^2\frac{x}{2}+7-7\tan^2\frac{x}{2}+2\tan\frac{x}{2}}$$

$$= \int\frac{\sec^2\frac{x}{2}}{12+2\tan\frac{x}{2}-2\tan^2\frac{x}{2}}dx$$

Put $\tan\frac{x}{2}=t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}dx = dt$

$$= 2\int\frac{dt}{12+2t-2t^2} = -\frac{2}{2}\int\frac{dt}{t^2-t-6}$$

$$= -\int\frac{dt}{t^2-t+\frac{1}{4}-\frac{1}{4}-6} = -\int\frac{dt}{\left(t-\frac{1}{2}\right)^2-\left(\frac{5}{2}\right)^2}$$

$$= -\frac{1}{2\cdot\frac{5}{2}}\log\left(\frac{t-\frac{1}{2}-\frac{5}{2}}{t-\frac{1}{2}+\frac{5}{2}}\right)+C$$

$$= -\frac{1}{5}\log\left(\frac{t-3}{t+2}\right)+C = \frac{1}{5}\log\left(\frac{\tan\frac{x}{2}+2}{\tan\frac{x}{2}-3}\right)+C$$

S11. Given integral is $\int\frac{3\sin x+2\cos x}{3\cos x+2\sin x}\,dx$

Let $\quad 3\sin x + 2\cos x = A\frac{d}{dx}(3\cos x + 2\sin x) + B(3\cos x + 2\sin x)$

$$3\sin x + 2\cos x = A(2\cos x - 3\sin x) + B(3\cos x + 2\sin x)$$

$$2A + 3B = 2 \quad \text{and} \quad -3A + 2B = 3$$

$\Rightarrow$ $$B = \frac{12}{13} \quad \text{and} \quad A = \frac{-5}{13}$$

$$= \int\frac{-\frac{5}{13}(2\cos x - 3\sin x)+\frac{12}{13}(3\cos x+2\sin x)}{3\cos x+2\sin x}\,dx$$

$$= \frac{-5}{13}\int\frac{2\cos x-3\sin x}{3\cos x+2\sin x}\,dx+\frac{12}{13}\int 1\,dx$$

$$= \frac{-5}{13}\log(3\cos x+2\sin x)+\frac{12}{13}\int 1\,dx$$

$$= \frac{-5}{13}\log(3\cos x+2\sin x)+\frac{12}{13}x+C$$

S12. Given integral is $\int\frac{1+\sin x}{\sin x(1+\cos x)}\,dx$

Put $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and $\cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$= \int\frac{\left(1+\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}{\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)\left(1+\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}\,dx$$

$$= \int \frac{\left(1+\tan^2\frac{x}{2}+2\tan\frac{x}{2}\right)\left(1+\tan^2\frac{x}{2}\right)}{2\tan\frac{x}{2}\left(1+\tan^2\frac{x}{2}+1-\tan^2\frac{x}{2}\right)}dx$$

$$= \int \frac{\left(1+\tan\frac{x}{2}\right)^2 \sec^2\frac{x}{2}}{4\tan\frac{x}{2}}dx$$

Put $\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}dx = dt$

$$= 2\int\frac{(1+t)^2}{4t}dt = \frac{1}{2}\int\frac{1+t^2+2t}{t}dt$$

$$= \frac{1}{2}\int\left(\frac{1}{t}+t+2\right)dt = \frac{1}{2}\left\{\log t+\frac{t^2}{2}+2t\right\}+C$$

$$= \frac{1}{2}\left\{\log\tan\frac{x}{2}+\frac{\tan^2\frac{x}{2}}{2}+2\tan\frac{x}{2}\right\}+C$$

S13. Let $I = \int\frac{1}{a+b\tan x}dx = \int\frac{1}{a+b\frac{\sin x}{\cos x}}dx = \int\frac{\cos x}{a\cos x+b\sin x}dx$ ... (i)

Put $\cos x = A(a\cos x + b\sin x) + B\frac{d}{dx}(a\cos x+b\sin x)$

$\cos x = A(a\cos x + b\sin x) + B(-a\sin x + b\cos x)$

Comparing, coff. of sin *x* and cos *x* both side

$1 = Aa + Bb$ ... (ii)

and $0 = Ab - Ba$ ... (iii)

From Eq. (ii) $A = \frac{a}{b}B$

$\frac{B}{b}a^2 + Bb = 1$ Putting in Eq. (i)

$\Rightarrow B = \frac{b}{a^2+b^2}$

$\therefore A = \frac{a}{b}\cdot\frac{b}{a^2+b^2} = \frac{a}{a^2+b^2}$

$\therefore$ From Eq. (i), $I = \int\frac{A(a\cos x+b\sin x)+B(-a\sin x+b\cos x)}{a\cos x+b\sin x}dx$

$$= A\int 1\cdot dx + B\int \frac{-a\sin x + b\cos x}{a\cos x + b\sin x}\,dx$$

$$= Ax + B\log|a\cos x + b\sin x| + c$$

$$= \frac{a}{a^2+b^2}x + \frac{b}{a^2+b^2}\log|a\cos x + b\sin x| + c$$

$$= \frac{1}{a^2+b^2}[ax + b\log|a\cos x + b\sin x|] + c$$

S14. Let $I = \int \frac{4\sin x + 5\cos x}{5\sin x + 4\cos x}\,dx$

Put $4\sin x + 5\cos x = A(5\sin x + 4\cos x) + B\frac{d}{dx}(5\sin x + 4\cos x)$

$4\sin x + 5\cos x = A(5\sin x + 4\cos x) + B(5\cos x - 4\sin x)$

Comparing, coff. of $\sin x$ and $\cos x$ both side

$4 = 5A - 4B$ ... (i)

and $5 = 4A + 5B$ ... (ii)

Solving Eqs. (i) and (ii), $A = \frac{40}{41}$, $B = \frac{9}{41}$.

$\therefore$ $I = \int \frac{A(5\sin x + 4\cos x) + B(5\cos x - 4\sin x)}{5\sin x + 4\cos x}\,dx$

$$= A\int 1\cdot dx + B\int\frac{5\cos x - 4\sin x}{5\sin x + 4\cos x}\,dx = Ax + B\log|5\sin x + 4\cos x| + c$$

$$= \frac{40}{41}x + \frac{9}{41}\log|5\sin x + 4\cos x| + c$$

S15. Given integral is $\int \frac{1}{1+\tan x}\,dx$

$$= \int \frac{1}{1+\frac{\sin x}{\cos x}}\,dx = \int \frac{\cos x}{\cos x + \sin x}\,dx$$

Put $\cos x = A\frac{d}{dx}(\cos x + \sin x) + B(\cos x + \sin x)$

$\cos x = A(\cos x - \sin x) + B(\cos x + \sin x)$

$A + B = 1$ and $-A + B = 0$

$$A = \frac{1}{2} \text{ and } B = \frac{1}{2}$$

$$\therefore \quad \int \frac{\frac{1}{2}(\cos x - \sin x) + \frac{1}{2}(\cos x + \sin x)}{\cos x + \sin x}\, dx$$

$$= \frac{1}{2}\int \frac{\cos x - \sin x}{\cos x + \sin x}\, dx + \frac{1}{2}\int \frac{\cos x + \sin x}{\cos x + \sin x}\, dx$$

$$= \frac{1}{2}\log(\cos x + \sin x) + \frac{1}{2}\int 1\, dx$$

$$= \frac{1}{2}\log(\cos x + \sin x) + \frac{1}{2}x + C$$

S16. Given integral is $\int \frac{dx}{1 + \cot x}$

$$= \int \frac{dx}{1 + \frac{\cos x}{\sin x}} = \int \frac{\sin x}{\sin x + \cos x}\, dx$$

Put $\sin x = A \frac{d}{dx}(\cos x + \sin x) + B(\cos x + \sin x)$

$\sin x = A(\cos x - \sin x) + B(\cos x + \sin x)$

$A + B = 0$ and $B - A = 1$

$$A = \frac{-1}{2} \text{ and } B = \frac{1}{2}$$

$$\Rightarrow \quad \int \frac{-\frac{1}{2}(\cos x - \sin x) + \frac{1}{2}(\cos x + \sin x)}{\cos x + \sin x}\, dx$$

$$= -\frac{1}{2}\int \frac{\cos x - \sin x}{\cos x + \sin x}\, dx + \frac{1}{2}\int \frac{\cos x + \sin x}{\cos x + \sin x}\, dx$$

$$= \frac{-1}{2}\log(\cos x + \sin x) + \frac{1}{2}\int 1\, dx$$

$$= \frac{-1}{2}\log(\cos x + \sin x) + \frac{1}{2}x + C$$

S17. Given integral is $\int \frac{2\tan x + 3}{3\tan x + 4}\, dx$

$$\Rightarrow \quad \int \frac{2\frac{\sin x}{\cos x} + 3}{3\frac{\sin x}{\cos x} + 4}\, dx = \int \frac{2\sin x + 3\cos x}{3\sin x + 4\cos x}\, dx$$

Put $2\sin x + 3\cos x = A\dfrac{d}{dx}(3\sin x + 4\cos x) + B(3\sin x + 4\cos x)$

$2\sin x + 3\cos x = A(3\cos x - 4\sin x) + B(3\sin x + 4\cos x)$

$3A + 4B = 3 \quad \text{and} \quad -4A + 3B = 2$

$A = \dfrac{1}{25} \quad \text{and} \quad B = \dfrac{18}{25}$

$$= \int \frac{\frac{1}{25}(3\cos x - 4\sin x) + \frac{18}{25}(3\sin x + 4\cos x)}{3\sin x + 4\cos x}\,dx$$

$$= \frac{1}{25}\int \frac{3\cos x - 4\sin x}{3\sin x + 4\cos x}\,dx + \frac{18}{25}\int \frac{3\sin x + 4\cos x}{3\sin x + 4\cos x}\,dx$$

$$= \frac{1}{25}\log(3\sin x + 4\cos x) + \frac{18}{25}\int 1\,dx$$

$$= \frac{1}{25}\log(3\sin x + 4\cos x) + \frac{18}{25}x + C$$

S18. Given integral is $\displaystyle\int \frac{5\cos x + 6}{2\cos x + \sin x + 3}\,dx$

Put $5\cos x + 6 = A\dfrac{d}{dx}(2\cos x + \sin x + 3) + B(2\cos x + \sin x + 3) + D$

$5\cos x + 6 = A(\cos x - 2\sin x) + B(2\cos x + \sin x + 3) + D$

$A + 2B = 5, \quad -2A + B = 0 \quad \text{and} \quad 3B + D = 6$

$A = 1, \quad B = 2, \quad D = 0$

$\Rightarrow \quad \displaystyle\int \frac{(\cos x - 2\sin x) + 2(2\cos x + \sin x + 3) + 0}{2\cos x + \sin x + 3}\,dx$

$\Rightarrow \quad \displaystyle\int \frac{\cos x - 2\sin x}{2\cos x + \sin x + 3}\,dx + 2\int \frac{2\cos x + \sin x + 3}{2\cos x + \sin x + 3}\,dx$

$\Rightarrow \quad \log(2\cos x + \sin x + 3) + 2\int 1\,dx$

$\Rightarrow \quad \log(2\cos x + \sin x + 3) + 2x + C$

S19. Given integral is $\displaystyle\int \frac{3\cos x + 2}{\sin x + 2\cos x + 3}\,dx$

Put $3\cos x + 2 = A\dfrac{d}{dx}(\sin x + 2\cos x + 3) + B(\sin x + 2\cos x + 3) + D$

$3\cos x + 2 = A(\cos x - 2\sin x) + B(\sin x + 2\cos x + 3) + D$

$$A + 2B = 3, \; -2A + B = 0 \quad \text{and} \quad 3B + D = 2$$

After solving above eq., we get

$$A = \frac{3}{5}, \quad B = \frac{6}{5} \quad \text{and} \quad D = \frac{-8}{5}$$

$$= \int \frac{\frac{3}{5}(\cos x - 2\sin x) + \frac{6}{5}(\sin x + 2\cos x + 3) - \frac{8}{5}}{\sin x + 2\cos x + 3}\, dx$$

$$= \frac{3}{5}\int \frac{\cos x - 2\sin x}{\sin x + 2\cos x + 3}\, dx + \frac{6}{5}\int \frac{\sin x + 2\cos x + 3}{\sin x + 2\cos x + 3}\, dx$$

$$- \frac{8}{5}\int \frac{dx}{\sin x + 2\cos x + 3}$$

$$= \frac{3}{5}\log(\sin x + 2\cos x + 3) + \frac{6}{5}\int 1\, dx - \frac{8}{5}\int \frac{dx}{\sin x + 2\cos x + 3}$$

$$= \frac{3}{5}\log(\sin x + 2\cos x + 3) + \frac{6}{5}x - \frac{8}{5}I_1$$

$$I_1 = \int \frac{dx}{\sin x + 2\cos x + 3}$$

Put $\sin x = \dfrac{2\tan\frac{x}{2}}{1 + \tan^2\frac{x}{2}}$ and $\cos x = \dfrac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}}$

$$I_1 = \int \frac{dx}{\dfrac{2\tan\frac{x}{2}}{1 + \tan^2\frac{x}{2}} + 2\left(\dfrac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}}\right) + 3}$$

$$= \int \frac{\left(1 + \tan^2\frac{x}{2}\right)}{2\tan\frac{x}{2} + 2 - 2\tan^2\frac{x}{2} + 3 + 3\tan^2\frac{x}{2}}\, dx$$

$$= \int \frac{\sec^2\frac{x}{2}}{\tan^2\frac{x}{2} + 2\tan\frac{x}{2} + 5}\, dx$$

Put $\tan\frac{x}{2} = t \;\Rightarrow\; \frac{1}{2}\sec^2\frac{x}{2}\, dx = dt$

$$= 2\int \frac{dt}{t^2 + 2t + 5} = 2\int \frac{dt}{t^2 + 2t + 1 + 4}$$

$$= 2\int\frac{dt}{(t+1)^2+(2)^2} = 2\cdot\frac{1}{2}\tan^{-1}\left(\frac{t+1}{2}\right)+C$$

$$= \tan^{-1}\left(\frac{\tan\frac{x}{2}+1}{2}\right)+C$$

$$\therefore \int\frac{3\cos x+2}{\sin x+2\cos x+3}\,dx = \frac{3}{5}\log(\sin x+2\cos x+3)+\frac{6}{5}x-\frac{8}{5}\tan^{-1}\left(\frac{\tan\frac{x}{2}+1}{2}\right)+C$$

S20. Given integral is $\int\frac{3+2\cos x+4\sin x}{2\sin x+\cos x+3}\,dx$

Put $3 + 2\cos x + 4\sin x = A\frac{d}{dx}(2\sin x+\cos x+3) + B(2\sin x+\cos x+3) + D$

$$3 + 2\cos x + 4\sin x = A(2\cos x - \sin x) + B(2\sin x + \cos x + 3) + D$$

$$2A + B = 2,\quad -A + 2B = 4 \quad\text{and}\quad 3B + D = 3$$

$$A = 0,\quad B = 2,\quad D = -3$$

$$= 2\int\frac{2\sin x+\cos x+3}{2\sin x+\cos x+3}\,dx - 3\int\frac{dx}{2\sin x+\cos x+3}$$

$$= 2\int 1\,dx - 3I_1$$

$$I_1 = \int\frac{dx}{2\sin x+\cos x+3}$$

Put $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$I_1 = \int\frac{dx}{2\cdot\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}+\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}+3}$$

$$= \int\frac{\left(1+\tan^2\frac{x}{2}\right)}{4\tan\frac{x}{2}+1-\tan^2\frac{x}{2}+3+3\tan^2\frac{x}{2}}\,dx$$

$$= \int\frac{\sec^2\frac{x}{2}}{2\tan^2\frac{x}{2}+4\tan\frac{x}{2}+4}\,dx$$

Put $\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$

$$I_1 = 2\int\frac{dt}{2t^2+4t+4} = \int\frac{dt}{t^2+2t+2}$$

$$= \int\frac{dt}{t^2+2t+1+1} = \int\frac{dt}{(t+1)^2+(1)^2}$$

$$= \tan^{-1}(t+1) + C = \tan^{-1}(\tan x/2 + 1) + C$$

$$\therefore \quad \int\frac{3+2\cos x+4\sin x}{2\sin x+\cos x+3}\,dx = 2x - 3\tan^{-1}(\tan x/2 + 1) + C$$

Date: 19/10/2021

Q1. Evaluate $\int (\tan \log x + \sec^2 \log x)\, dx$

Q2. Evaluate $\int e^x (\tan x + \log \sec x) dx$ or $\int e^x (\tan x - \log \cos x) dx$

Q3. Evaluate $\int \left[\frac{1}{\log x} - \frac{1}{(\log x)^2} \right] dx$

Q4. Evaluate $\int e^x \frac{x-1}{(x+1)^3}\, dx$

Q5. Evaluate $\int e^x \left(\tan^{-1} x + \frac{1}{1+x^2} \right) dx$

Q6. Evaluate $\int e^x \left(\frac{\sin x \cos x - 1}{\sin^2 x} \right) dx$

Q7. Evaluate $\int e^x \left(\frac{1}{x^2} - \frac{2}{x^3} \right) dx$

Q8. Evaluate $\int e^x\, (\cot x + \log \sin x)\, dx$

Q9. Evaluate $\int \frac{e^x}{x} [x (\log x)^2 + 2 \log x]\, dx$

Q10. Evaluate $\int e^x \left(\frac{2 + \sin 2x}{1 + \cos 2x} \right) dx$

Q11. Evaluate $\int e^x \frac{x}{(x+1)^2}\, dx$

Q12. Evaluate $\int e^x [\sec x + \log(\sec x + \tan x)] dx$

Q13. Evaluate $\int e^x \frac{2-x}{(1-x)^2}\, dx$

Q14. Evaluate $\int e^x \frac{x-3}{(x-1)^3}\, dx$

Q15. Evaluate $\int e^{2x}\, (-\sin x + 2 \cos x)\, dx$

Q16. Evaluate $\int e^x \frac{\sqrt{1-x^2} \sin^{-1} x + 1}{\sqrt{1-x^2}}\, dx$

Q17. Evaluate $\int e^x \frac{1+x}{(2+x)^2}\, dx$

Q18. Evaluate $\int e^x \frac{(1-x)^2}{(1+x^2)^2}\, dx$

Q19. Evaluate $\int \frac{\log x}{(1+\log x)^2}\, dx$

Q20. Evaluate $\int e^x \dfrac{x^2+1}{(x+1)^2}\,dx$

Q21. Evaluate $\int (\sin\log x + \cos\log x)\,dx$

Q22. Evaluate $\int e^x\left(\dfrac{1-\sin x}{1-\cos x}\right)dx$

Q23. Evaluate $\int e^x\left(\dfrac{\sin 4x-4}{1-\cos 4x}\right)dx$

Q24. Evaluate $\int \left(\dfrac{1+\sin x}{1+\cos x}\right)e^x\,dx$

Q25. Evaluate $\int e^{2x}\left(\dfrac{1+\sin 2x}{1+\cos 2x}\right)dx$

Q26. Evaluate $\int \dfrac{xe^{2x}}{(1+2x)^2}\,dx$

Q27. Evaluate $\int e^x\left(\log x+\dfrac{1}{x^2}\right)dx$

Q28. Evaluate $\int\left[\log(\log x)+\dfrac{1}{(\log x)^2}\right]dx$

Q29. Evaluate $\int \dfrac{2+\sin x}{1+\cos x}e^{x/2}\,dx$

Date: 19/10/2021

S1. Given integral is $\int (\tan \log x + \sec^2 \log x)\, dx$

Put $\log x = t \Rightarrow x = e^t \Rightarrow dx = e^t\, dt$

$$= \int e^t (\tan t + \sec^2 t)\, dt$$

$$= \int e^t \tan t\, dt + \int \underset{I}{e^t} \underset{II}{\sec^2 t}\, dt$$

$$= \int e^t \tan t\, dt + e^t \int \sec^2 t\, dt - \int \left[\frac{d}{dt} e^t \cdot \int \sec^2 t\, dt \right] dt$$

$$= \int e^t \tan t\, dt + e^t \tan t - \int e^t \tan t\, dt$$

$$= e^t \tan t + C = x \tan \log x + C$$

S2. $\int e^x (\tan x - \log \cos x) dx = \int e^x (\tan x + \log \sec x) dx$

$$= \int \underset{I}{\log \sec x} \cdot \underset{II}{e^x}\, dx + \int \tan x . e^x dx$$

$$= \log \sec x . e^x - \int \left(\frac{1}{\sec x} \cdot \sec x \tan x \right) e^x dx + \int \tan x \cdot e^x dx$$

[Integrating first integral by parts]

$$= e^x \log \sec x - \int \tan x \cdot e^x dx + \int \tan x \cdot e^x dx$$

$$= e^x \log \sec x + c$$

S3. Put $\log x = t$ *i.e* $x = e^t$ so that $dx = e^t\, dt.$

$$\therefore \qquad I = \int \left(\frac{1}{t} - \frac{1}{t^2} \right) e^t dt = \int \frac{1}{t} e^t dt - \int \frac{1}{t^2} e^t dt$$

$$= \frac{1}{t} \cdot e^t - \int \left(-\frac{1}{t^2} \right) e^t dt - \int \frac{1}{t^2} e^t dt$$ [Integrating first integral by parts]

$$= \frac{1}{t} \cdot e^t + \int \frac{1}{t^2} \cdot e^t dt - \int \frac{1}{t^2} e^t dt$$

$$= \frac{1}{t} \cdot e^t + c = \frac{x}{\log x} + c.$$

S4. $$\int e^x \frac{x-1}{(x+1)^3}\,dx = \int e^x \frac{(x+1)-2}{(x+1)^3}dx = \int e^x\left[\frac{1}{(x+1)^2}-\frac{2}{(x+1)^3}\right]dx$$

$$= \int \underset{I}{\frac{1}{(x+1)^2}}\cdot \underset{II}{e^x}\,dx - \int \frac{2}{(x+1)^3}e^x dx$$

$$= \frac{1}{(x+1)^2}\cdot e^x - \int \frac{-2}{(x+1)^3}\,e^x\,dx - \int \frac{2}{(x+1)^3}e^x dx$$

[Integrating first integral by parts]

$$= \frac{e^x}{(x+1)^2} + c$$

S5. Given integral is $\int e^x\left(\tan^{-1} x + \frac{1}{1+x^2}\right)dx$

$$= \int \underset{II}{e^x}\,\underset{I}{\tan^{-1} x}\,dx + \int e^x\,\frac{1}{1+x^2}\,dx$$

$$= \tan^{-1} x \int e^x\,dx - \int\left[\frac{d}{dx}\tan^{-1} x \cdot \int e^x\,dx\right]dx + \int e^x\cdot\frac{1}{1+x^2}\,dx$$

$$= e^x \tan^{-1} x - \int e^x\cdot\frac{1}{1+x^2}\,dx + \int e^x\frac{1}{1+x^2}\,dx$$

$$= e^x \tan^{-1} x + C$$

S6. Given integral is $\int e^x\left(\frac{\sin x\cos x - 1}{\sin^2 x}\right)dx$

$$= \int e^x(\cot x - \operatorname{cosec}^2 x)\,dx$$

$$= \int e^x \cot x\,dx + \int \underset{\text{I}}{e^x}\,\underset{\text{II}}{(-\operatorname{cosec}^2 x)}\,dx$$

$$= \int e^x \cot x\,dx + e^x\cdot\int -\operatorname{cosec}^2 x\,dx - \int\left[\frac{d}{dx}e^x\cdot\int(-\operatorname{cosec}^2 x)\,dx\right]dx$$

$$= \int e^x \cot x\,dx + e^x\cot x - \int e^x\cot x\,dx$$

$$= e^x \cot x + C$$

S7. $$\int \underset{II}{e^x}\,\underset{I}{\frac{1}{x^2}}\,dx - \int e^x\,\frac{2}{x^3}\,dx$$

$$= \frac{1}{x^2}\int e^x\,dx - \int\left[\frac{d}{dx}\frac{1}{x^2}\int e^x\,dx\right]dx - \int e^x\,\frac{2}{x^3}\,dx$$

$$= e^x \frac{1}{x^2} + \int \frac{2}{x^3} e^x \, dx - \int \frac{2}{x^3} e^x \, dx$$

$$= e^x \cdot \frac{1}{x^2} + C$$

S8. Given integral is $\int e^x (\cot x + \log \sin x) \, dx$

$$= \int e^x \cot x \, dx + \int \underset{II}{e^x} \underset{I}{\log \sin x} \, dx$$

$$= \int e^x \cot x \, dx + \log \sin x \int e^x \, dx - \int \left[\frac{d}{dx} \log \sin x \int e^x \, dx \right] dx$$

$$= \int e^x \cot x \, dx + e^x \log \sin x - \int e^x \cot x \, dx$$

$$= e^x \log \sin x + C$$

S9. Given integral is $\int \frac{e^x}{x} [x (\log x)^2 + 2 \log x] \, dx$

$$\Rightarrow \quad \int \underset{II}{e^x} \underset{I}{(\log x)^2} \, dx + \int \frac{e^x \, 2 \log x}{x} \, dx$$

$$= (\log x)^2 \cdot \int e^x \, dx - \int \left[\frac{d}{dx} (\log x)^2 \cdot \int e^x \, dx \right] dx + \int \frac{2 \, e^x \, \log x}{x} \, dx$$

$$= e^x (\log x)^2 - \int \frac{2 \log x}{x} e^x \, dx + \int \frac{2 \log x}{x} e^x \, dx$$

$$= e^x (\log x)^2 + C$$

S10. Given integral is $\int e^x \left(\frac{2 + \sin 2x}{1 + \cos 2x} \right) dx$

$$= \int e^x \left(\frac{2 + 2 \sin x \cos x}{2 \cos^2 x} \right) dx$$

$$= \int e^x (\sec^2 x + \tan x) \, dx$$

$$= \int \underset{I}{e^x} \underset{II}{\sec^2 x} \, dx + \int e^x \tan x \, dx$$

$$= e^x \int \sec^2 x \, dx - \int \left[\frac{d}{dx} e^x \int \sec^2 x \, dx \right] dx + \int e^x \tan x \, dx$$

$$= e^x \tan x - \int e^x \tan x \, dx + \int e^x \tan x \, dx + C$$

$$= e^x \tan x + C$$

S11. Given integral is $\int e^x \frac{x}{(x+1)^2}dx$

$$= \int e^x \frac{x+1-1}{(x+1)^2}dx = \int e^x \left\{\frac{1}{x+1} - \frac{1}{(x+1)^2}\right\}dx$$

$$= \int \underset{II}{e^x} \underset{I}{\frac{1}{x+1}}dx + \int e^x \frac{(-1)}{(x+1)^2}dx$$

$$= \frac{1}{x+1}\int e^x\,dx - \int\left[\frac{d}{dx}\left(\frac{1}{x+1}\right)\cdot \int e^x\,dx\right]dx + \int \frac{e^x(-1)}{(x+1)^2}dx$$

$$= \frac{e^x}{x+1} - \int \frac{e^x\,(-1)}{(x+1)^2}dx + \int \frac{e^x\,(-1)}{(x+1)^2}dx + C = \frac{e^x}{x+1} + C$$

S12. $\int e^x[\sec x + \log(\sec x + \tan x)]dx$

$$= \int \log(\sec \underset{I}{x} + \tan x)\cdot \underset{II}{e^x}\,dx + \int \sec x \cdot e^x dx$$

$$\left[\because \int \underset{I}{f(x)}\,\underset{II}{g(x)}\,dx = f(x)\int g(x)\,dx - \int\left(\frac{d}{dx}f(x)\int g(x)\,dx\right)dx\right]$$

$$= \log(\sec x + \tan x)\cdot e^x$$

$$- \int \frac{1}{\sec x + \tan x}\left(\sec x \tan x + \sec^2 x\right)e^x dx + \int \sec x \cdot e^x dx$$

[Integrating first integral by parts]

$$= e^x \log(\sec x + \tan x) - \int \sec x \cdot e^x dx + \int \sec x \cdot e^x dx$$

$$= e^x \log(\sec x + \tan x) + c$$

S13. Given integral is $\int e^x \frac{2-x}{(1-x)^2}dx$

$$= \int e^x \frac{1-x+1}{(1-x)^2}dx = \int e^x\left[\frac{1-x}{(1-x)^2} + \frac{1}{(1-x)^2}\right]dx$$

$$= \int e^x \frac{1}{1-x}dx + \int e^x \frac{1}{(1-x)^2}dx$$

$$= \frac{1}{1-x}\int e^x\,dx - \int\left[\frac{d}{dx}\left(\frac{1}{1-x}\right)\cdot \int e^x\,dx\right]dx + \int e^x \frac{1}{(1-x)^2}dx$$

$$= \frac{e^x}{1-x} - \int e^x \frac{1}{(1-x)^2}\,dx + \int e^x \frac{1}{(1-x)^2}\,dx$$

$$= \frac{e^x}{1-x} + C$$

S14. Given integral is $\int e^x \frac{x-3}{(x-1)^3}\,dx$

$$= \int e^x \frac{(x-1)-2}{(x-1)^3}\,dx = \int e^x \left[\frac{1}{(x-1)^2} - \frac{2}{(x-1)^3}\right] dx$$

$$= \int e^x \frac{1}{(x-1)^2}\,dx - \int e^x \frac{2}{(x-1)^3}\,dx$$

$$\Rightarrow \quad \frac{1}{(x-1)^2}\int e^x\,dx - \int\left[\frac{d}{dx}\frac{1}{(x-1)^2}\cdot \int e^x\,dx\right]dx - \int e^x \frac{2}{(x-1)^3}\,dx$$

$$\Rightarrow \quad e^x \frac{1}{(x-1)^2} + \int e^x \frac{2}{(x-1)^3}\,dx - \int e^x \frac{2}{(x-1)^3}\,dx$$

$$\Rightarrow \quad e^x \cdot \frac{1}{(x-1)^2} + C$$

S15. Given integral is $\int e^{2x}\,(-\sin x + 2\cos x)\,dx$

Put $2x = t \;\Rightarrow\; dx = \frac{1}{2}\,dt$

$$= \frac{1}{2}\int e^t \left(-\sin\frac{t}{2} + 2\cos\frac{t}{2}\right) dt$$

$$= \frac{1}{2}\left[-\int e^t \sin\frac{t}{2}\,dt + \int \underset{\text{II}}{e^t} \cdot \underset{\text{I}}{2\cos\frac{t}{2}}\,dt\right]$$

$$= \frac{1}{2}\left[-\int e^t \sin\frac{t}{2}\,dt + 2\cos\frac{t}{2}\int e^t\,dt - \int\left(\frac{d}{dt}2\cos\frac{t}{2}\int e^t\,dt\right)dt\right]$$

$$= \frac{1}{2}\left[-\int e^t \sin\frac{t}{2}\,dt + 2\,e^t\cos\frac{t}{2} + \int e^t \sin\frac{t}{2}\,dt\right]$$

$$= e^t \cos\frac{t}{2} + C = e^{2x}\cos x + C$$

S16. Given integral is $\int e^x \frac{\sqrt{1-x^2}\,\sin^{-1}x + 1}{\sqrt{1-x^2}}\,dx$

$$= \int e^x \left(\sin^{-1}x + \frac{1}{\sqrt{1-x^2}}\right) dx$$

$$= \int e^x \sin^{-1} x \, dx + \int e^x \frac{1}{\sqrt{1-x^2}} \, dx$$

$$= \underset{\text{II}}{\int e^x} \underset{\text{I}}{\sin^{-1} x} \, dx + \int e^x \frac{1}{\sqrt{1-x^2}} \, dx$$

$$\Rightarrow \quad \sin^{-1} x \int e^x \, dx - \int \left[\frac{d}{dx} \sin^{-1} x \cdot \int e^x \, dx \right] dx + \int e^x \frac{1}{\sqrt{1-x^2}} \, dx$$

$$\Rightarrow \quad e^x \sin^{-1} x - \int e^x \frac{1}{\sqrt{1-x^2}} \, dx + \int e^x \frac{1}{\sqrt{1-x^2}} \, dx$$

$$= e^x \sin^{-1} x + C$$

S17. Given integral is $\int e^x \frac{1+x}{(2+x)^2} \, dx$

$$= \int e^x \frac{2+x-1}{(2+x)^2} \, dx = \int e^x \left[\frac{2+x}{(2+x)^2} - \frac{1}{(2+x)^2} \right] dx$$

$$= \underset{II}{\int e^x} \cdot \underset{I}{\frac{1}{2+x}} \, dx - \int e^x \frac{1}{(2+x)^2} \, dx$$

$$= \frac{1}{2+x} \int e^x \, dx - \int \left[\frac{d}{dx} \left(\frac{1}{2+x} \right) \int e^x \, dx \right] dx - \int e^x \frac{1}{(2+x)^2} \, dx$$

$$= e^x \cdot \frac{1}{2+x} + \int \frac{1}{(2+x)^2} e^x \, dx - \int e^x \frac{1}{(2+x)^2} \, dx$$

$$= \frac{e^x}{2+x} + C$$

S18. Given integral is $\int e^x \frac{(1-x)^2}{(1+x^2)^2} \, dx$

$$= \int e^x \left(\frac{1+x^2-2x}{(1+x^2)^2} \right) dx = \int e^x \left[\frac{1+x^2}{(1+x^2)^2} - \frac{2x}{(1+x^2)^2} \right] dx$$

$$= \int e^x \left[\frac{1}{1+x^2} - \frac{2x}{(1+x^2)^2} \right] dx$$

$$= \underset{II}{\int e^x} \underset{I}{\frac{1}{1+x^2}} \, dx - \int e^x \frac{2x}{(1+x^2)^2} \, dx$$

$$= \frac{1}{1+x^2}\int e^x\,dx - \int\left[\frac{d}{dx}\frac{1}{1+x^2}\int e^x\,dx\right]dx - \int e^x\,\frac{2x}{(1+x^2)^2}\,dx$$

$$= e^x\cdot\frac{1}{1+x^2} + \int e^x\cdot\frac{2x}{(1+x^2)^2}\,dx - \int e^x\,\frac{2x}{(1+x^2)^2}\,dx$$

$$= \frac{e^x}{1+x^2} + C$$

S19. Given integral is $\int \frac{\log x}{(1+\log x)^2}\,dx$

Put $\log x = t \Rightarrow x = e^t \Rightarrow dx = e^t\,dt$

$$= \int \frac{t\,e^t}{(t+1)^2}\,dt = \int e^t\,\frac{t+1-1}{(t+1)^2}\,dt = \int \underset{II}{e^t}\,\underset{I}{\frac{1}{t+1}}\,dt - \int e^t\,\frac{1}{(t+1)^2}\,dt$$

$$= \frac{1}{t+1}\int e^t\,dt - \int\left[\frac{d}{dx}\left(\frac{1}{t+1}\right)\int e^t\,dt\right]dt - \int e^t\,\frac{1}{(t+1)^2}\,dt$$

$$= \frac{e^t}{t+1} + \int e^t\cdot\frac{1}{(t+1)^2}\,dt - \int e^t\cdot\frac{1}{(t+1)^2}\,dt$$

$$= \frac{e^t}{t+1} + C = \frac{x}{\log x + 1} + C$$

S20. Given integral is $\int e^x\,\frac{x^2+1}{(x+1)^2}\,dx$

$$= \int e^x\left(1 - \frac{2x}{(x+1)^2}\right)dx = \int e^x\,dx - 2\int e^x\,\frac{x}{(x+1)^2}\,dx$$

$$= e^x - 2\int e^x\,\frac{x+1-1}{(x+1)^2}\,dx$$

$$= e^x - 2\left[\int \underset{II}{e^x}\cdot\underset{I}{\frac{1}{x+1}}\,dx - \int e^x\,\frac{1}{(x+1)^2}\,dx\right]$$

$$= e^x - 2\left[\frac{1}{x+1}\int e^x\,dx - \int\left\{\frac{d}{dx}\left(\frac{1}{x+1}\right)\int e^x\,dx\right\}dx - \int\frac{e^x}{(x+1)^2}\,dx\right]$$

$$= e^x - 2\left[\frac{e^x}{x+1} + \int\frac{e^x}{(x+1)^2}\,dx - \int\frac{e^x}{(x+1)^2}\,dx\right]$$

$$= e^x - \frac{2e^x}{x+1} + C$$

S21. Given integral is $\int(\sin\log x + \cos\log x)\,dx$

Put $\log x = t \Rightarrow x = e^t \Rightarrow dx = e^t\,dt$

$$= \int e^t(\sin t + \cos t)\,dt$$

$$= \underset{II}{\int e^t \sin t\,dt} + \underset{I}{\int e^t \cos t\,dt}$$

$$= \sin t \int e^t\,dt - \int\left[\frac{d}{dx}\sin t \int e^t\,dt\right]dt + \int e^t \cos t\,dt$$

$$= e^t \sin t - \int e^t \cos dt + \int e^t \cos t\,dt$$

$$= e^t \sin t + C$$

$$= x \sin(\log x) + C$$

S22. $$\int e^x \left(\frac{1 - 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right)dx$$

$$= \int e^x\left(\frac{1}{2}\cos ec^2\frac{x}{2} - \cot\frac{x}{2}\right)dx$$

$$= -\underset{\text{II}}{\int e^x \cot\frac{x}{2}\,dx} + \frac{1}{2}\underset{\text{I}}{\int e^x \operatorname{cosec}^2\frac{x}{2}\,dx}$$

$$= -\left\{\cot\frac{x}{2}\int e^x\,dx - \int\left[\frac{d}{dx}\cot\frac{x}{2}\int e^x\,dx\right]dx\right\} + \frac{1}{2}\int e^x \operatorname{cosec}^2\frac{x}{2}\,dx$$

$$= -e^x\cot\frac{x}{2} - \frac{1}{2}\int e^x \operatorname{cosec}^2\frac{x}{2}\,dx + \frac{1}{2}\int e^x \operatorname{cosec}^2\frac{x}{2}\,dx$$

$$= -e^x\cot\frac{x}{2} + C$$

S23. Let $$I = \int e^x\left(\frac{\sin 4x - 4}{1 - \cos 4x}\right)dx$$

$$= \int e^x\left(\frac{2\sin 2x\cos 2x - 4}{2\sin^2 2x}\right)dx$$

$$= \int e^x\left(\frac{2\sin 2x\cos 2x}{2\sin^2 2x} - \frac{4}{2\sin^2 2x}\right)dx$$

$$= \int e^x(\cot 2x - 2\operatorname{cosec}^2 2x)\,dx$$

Comparing by

$$= \int e^x [f(x) + f'(x)] dx = e^x f(x) + C$$

Here, $f(x) = \cot 2x$

$\therefore \quad f'(x) = -2 \operatorname{cosec}^2 2x$

$\therefore \quad I = e^x \cot 2x + C$

S24.

$$I = \int \left(\frac{1 + \sin x}{1 + \cos x} \right) e^x dx$$

$$= \int \frac{1 + 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}} . e^x dx \qquad \left[\because \sin m = 2\sin\frac{m}{2}\cos\frac{m}{2} \text{ and } \cos m = 2\cos^2\frac{m}{2} - 1 \right]$$

$$= \int \left(\frac{1}{2}\sec^2\frac{x}{2} + \tan\frac{x}{2} \right) e^x dx$$

$$= \int e^x \cdot \left(\tan\frac{x}{2} + \frac{1}{2}\sec^2\frac{x}{2} \right) dx$$

Compare it with $\int e^x [f(x) + f'(x)] dx = e^x \cdot f(x) + C$ $\quad \left[\because f(x) = \tan\frac{x}{2} \Rightarrow f'(x) = \frac{1}{2}\sec^2\frac{x}{2} \right]$

$\Rightarrow \quad I = e^x \cdot \tan\frac{x}{2} + C$

S25. Let

$$I = \int e^{2x} \left(\frac{1 + \sin 2x}{1 + \cos 2x} \right) dx$$

$$= \int e^{2x} \left[\frac{1 + 2\sin x \cos x}{2\cos^2 x} \right] dx \qquad \left[\because \sin 2x = 2\sin x \cos x \text{ and } 1 + \cos 2x = 2\cos^2 x \right]$$

$$= \int e^{2x} \left[\frac{1}{2\cos^2 x} + \frac{2\sin x \cos x}{2\cos^2 x} \right] dx$$

$$= \int \frac{1}{2} \underset{\text{I}}{e^{2x}} \underset{\text{II}}{\sec^2 x}\, dx + \int e^{2x} \tan x\, dx$$

Using integration by parts in first integral, we get

$$I = \frac{1}{2} e^{2x} \int \sec^2 x\, dx - \int \frac{1}{2} e^{2x} \cdot 2\tan x\, dx + \int e^{2x} \tan x\, dx + C$$

$$= \frac{1}{2} e^{2x} \tan x - \int e^{2x} \tan x\, dx + \int e^{2x} \tan x\, dx + C$$

$$= \frac{1}{2} e^{2x} \tan x + C$$

S26.

$$I = \int \frac{xe^{2x}}{(1+2x)^2}\,dx$$

put $\quad 2x = t$

$$dx = \frac{1}{2}dt$$

$$= \int \frac{t}{2}\frac{e^t}{(1+t)^2}\cdot\frac{dt}{2} = \frac{1}{4}\int e^t\,\frac{(1+t)-1}{(1+t)^2}\,dt$$

$$= \frac{1}{4}\int e^t\left(\frac{1}{1+t}-\frac{1}{(1+t)^2}\right)dt$$

$$= \frac{1}{4}\int\left[\frac{1}{1+t}\,e^t\,dt - \int\frac{1}{(1+t)^2}\,e^t dt\right]$$

$$= \frac{1}{4}\left[\frac{1}{1+t}\cdot e^t - \int -\frac{1}{(1+t)^2}e^t dt \;-\int\frac{1}{(1+t)^2}e^t\,dt\right]$$

(Integrating first integral by parts)

$$= \frac{1}{4}\frac{e^t}{(1+t)} + c = \frac{e^{2x}}{4(1+2x)} + c.$$

S27. Given integral is $\int e^x\left(\log x + \frac{1}{x^2}\right)dx$

$$= \int e^x\left(\log x + \frac{1}{x} - \frac{1}{x} + \frac{1}{x^2}\right)dx$$

$$= \underbrace{\int e^x\left(\log x + \frac{1}{x}\right)dx}_{I_1} - \underbrace{\int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx}_{I_2} \qquad \text{... (i)}$$

$$I_1 = \int \underset{II}{e^x}\,\underset{I}{\log x}\,dx + \int e^x\cdot\frac{1}{x}\,dx$$

$$= \log x\int e^x\,dx - \int\left[\frac{d}{dx}\log x\cdot\int e^x\,dx\right]dx + \int e^x\cdot\frac{1}{x}\,dx$$

$$= e^x\log x - \int e^x\frac{1}{x}\,dx + \int e^x\,\frac{1}{x}\,dx$$

$$= e^x\log x + C_1$$

$$I_2 = \int \underset{II}{e^x}\,\underset{I}{\frac{1}{x}}\,dx - \int e^x\,\frac{1}{x^2}\,dx$$

$$= \frac{1}{x}\int e^x\,dx - \int\left[\frac{d}{dx}\left(\frac{1}{x}\right)\cdot\int e^x\,dx\right]dx - \int e^x\,\frac{1}{x^2}\,dx$$

$$= e^x \cdot \frac{1}{x} + \int e^x \cdot \frac{1}{x^2}\,dx - \int e^x \frac{1}{x^2}\,dx$$

$$= e^x \cdot \frac{1}{x} + C_2$$

From (i) $\quad I_1 - I_2 = e^x \log x - e^x \frac{1}{x} + C \quad$ (Where $C = C_1 - C_2$)

$$= e^x\left(\log x - \frac{1}{x}\right) + C$$

S28. Let $\quad I = \int\left[\log(\log x) + \frac{1}{(\log x)^2}\right]dx$

$$= \int \underset{\text{I}}{\log(\log x)} \cdot \underset{\text{II}}{1}\,dx + \int \frac{1}{(\log x)^2}\,dx$$

Using integration by parts in first integral, we get

$$I = \log(\log x)\int 1\,dx - \int\left[\frac{d}{dx}\log(\log x)\int 1\,dx\right]dx + \int\frac{1}{(\log x)^2}\,dx + C$$

$$= \log(\log x)\cdot x - \int \frac{1}{\log x}\cdot\frac{1}{x}\cdot x\,dx + \int\frac{1}{(\log x)^2}\,dx + C$$

$$= x\log(\log x) - \int \underset{\text{I}}{(\log x)^{-1}}\,\underset{\text{II}}{1}\,dx + \int\frac{1}{(\log x)^2}\,dx + C$$

Again, applying integration by parts in the middle integral

$$= x\log(\log x) - \left[(\log x)^{-1}\int 1\,dx - \int\left[\frac{d}{dx}(\log x)^{-1}\int 1\,dx\right]dx\right] + \int\frac{1}{(\log x)^2}\,dx + C$$

$$= x\log(\log x) - \left[\frac{x}{\log x} - \int -(\log x)^{-2}\cdot\frac{1}{x}\cdot x\,dx\right] + \int\frac{1}{(\log x)^2}\,dx + C$$

$$= x\log(\log x) - \frac{x}{\log x} - \int\frac{1}{(\log x)^2}\,dx + \int\frac{1}{(\log x)^2}\,dx + C$$

$$= x\log(\log x) - \frac{x}{\log x} + C$$

S29. $\quad I = \int \frac{2+\sin x}{1+\cos x}\,e^{x/2}\,dx$

$$= \int \frac{2\left(\cos^2\frac{x}{2} + \sin^2\frac{x}{2}\right) + 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\cdot e^{x/2}\,dx$$

$$= \int \frac{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + \sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}} \cdot e^{x/2}dx$$

$$= \int\left(1 + \tan^2\frac{x}{2} + \tan\frac{x}{2}\right)e^{x/2}dx = \int\left(\sec^2\frac{x}{2} + \tan\frac{x}{2}\right)e^{x/2}dx.$$

Put $\frac{x}{2} = t$ *i.e.* $x = 2t$ so that $dx = 2dt$.

$$I = \int(\sec^2 t + \tan t)e^t(2dt)$$

$$= 2\left[\int \underset{I}{\tan t}\cdot \underset{II}{e^t}\,dt + \int \sec^2 t \cdot e^t dt\right]$$

$$= 2\left[\tan t \cdot e^t - \int \sec^2 t \cdot e^t dt + \int \sec^2 t \cdot e^t dt\right]$$

[Integrating first integral by parts]

$$= 2\left[\tan t \cdot e^t\right] + c$$

$$= 2\tan\frac{x}{2} \cdot e^{x/2} + c.$$

Date: 19/10/2021

Q1. Evaluate $\int \sqrt{4+x^2}\,dx$

Q2. Evaluate $\int \sqrt{1-x^2}\,dx$

Q3. Evaluate $\int \sqrt{x^2+x+1}\,dx$

Q4. Evaluate $\int \sqrt{(x-3)(5-x)}\,dx$

Q5. Evaluate $\int \sqrt{7x-10-x^2}\,dx$

Q6. Evaluate $\int \sqrt{1+x-2x^2}\,dx$

Q7. Evaluate $\int \sqrt{3+2x-x^2}\,dx$

Q8. Evaluate $\int \sqrt{2x^2+3x+4}\,dx$

Q9. Evaluate $\int \sqrt{3-2x-2x^2}\,dx$

Q10. Evaluate $\int x\sqrt{x^4+1}\,dx$

Q11. Evaluate $\int x^2\sqrt{a^6-x^6}\,dx$

Q12. Evaluate $\int \frac{\sqrt{16+(\log x)^2}}{x}\,dx$

Q13. Evaluate $\int x\sqrt{1+x-x^2}\,dx.$

Q14. Evaluate $\int (2x+3)\sqrt{x^2+4x+3}\,dx$

Q15. Evaluate $\int (2x-5)\sqrt{x^2-4x+3}\,dx$

Q16. Evaluate $\int (x-2)\sqrt{2x^2-6x+5}\,dx$

Q17. Evaluate $\int (x+2)\sqrt{x^2+x+1}\,dx$

Q18. Evaluate $\int (2x-5)\sqrt{2+3x-x^2}\,dx$

Q19. Evaluate $\int (x+1)\sqrt{x^2-x+1}\,dx$

Q20. Evaluate $\int (4x+1)\sqrt{x^2-x-2}\,dx$

Q21. Evaluate $\int (x+1)\sqrt{1-x-x^2}\,dx$

Date: 19/10/2021

S1.

$$I = \int\sqrt{2^2 + x^2}\cdot dx = \frac{x\sqrt{4+x^2}}{2} + \frac{4}{2}\log\left|x+\sqrt{4+x^2}\right| + c$$

$$= \frac{1}{2}x\sqrt{4+x^2} + 2\log|x+\sqrt{4+x^2}| + c$$

S2.

$$I = \int\sqrt{1^2 - x^2}dx = \frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}\frac{x}{1} + c$$

$$= \frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}x + c$$

S3. Given integral is $\int\sqrt{x^2+x+1}\,dx$

$$= \int\sqrt{x^2+x+\frac{1}{4}-\frac{1}{4}+1}\,dx = \int\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx$$

$$= \frac{1}{2}\left(x+\frac{1}{2}\right)\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} + \frac{3}{8}\log\left\{\left(x+\frac{1}{2}\right)+\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\} + C$$

$$= \frac{(2x+1)}{4}\sqrt{x^2+x+1} + \frac{3}{8}\log\left\{\frac{2x+1}{2}+\sqrt{x^2+x+1}\right\} + C$$

S4. Given integral is $\int\sqrt{(x-3)(5-x)}\,dx$

$$= \int\sqrt{-x^2+8x-15}\,dx = \int\sqrt{-(x^2-8x+16-16+15)}\,dx$$

$$= \int\sqrt{-\{(x-4)^2-1^2\}}\,dx = \int\sqrt{1^2-(x-4)^2}\,dx$$

$$= \frac{1}{2}(x-4)\sqrt{1^2-(x-4)^2} + \frac{1}{2}(1)^2\sin^{-1}\left(\frac{x-4}{1}\right) + C$$

$$= \frac{1}{2}(x-4)\sqrt{(x-3)(5-x)} + \frac{1}{2}\sin^{-1}(x-4) + C$$

S5. Given integral is $\int\sqrt{7x-10-x^2}\,dx$

$$= \int \sqrt{-(x^2 - 7x + 10)}\, dx = \int \sqrt{-\left(x^2 - 7x + \frac{49}{4} - \frac{49}{4} + 10\right)}\, dx$$

$$= \int \sqrt{-\left\{\left(x - \frac{7}{2}\right)^2 - \left(\frac{3}{2}\right)^2\right\}}\, dx = \int \sqrt{\left(\frac{3}{2}\right)^2 - \left(x - \frac{7}{2}\right)^2}\, dx$$

$$= \frac{1}{2}\left(x - \frac{7}{2}\right)\sqrt{\left(\frac{3}{2}\right)^2 - \left(x - \frac{7}{2}\right)^2} + \frac{1}{2}\left(\frac{3}{2}\right)^2 \sin^{-1}\left(\frac{x - \frac{7}{2}}{\frac{3}{2}}\right) + C$$

$$= \frac{1}{4}(2x - 7)\sqrt{7x - 10 - x^2} + \frac{9}{8}\sin^{-1}\left(\frac{2x - 7}{3}\right) + C$$

S6. Given integral is $\int \sqrt{1 + x - 2x^2}\, dx$

$$= \int \sqrt{-2\left(x^2 - \frac{x}{2} - \frac{1}{2}\right)}\, dx = \int \sqrt{-2\left(x^2 - \frac{x}{2} + \frac{1}{16} - \frac{1}{16} - \frac{1}{2}\right)}\, dx$$

$$= \int \sqrt{2}\sqrt{-\left\{\left(x - \frac{1}{4}\right)^2 - \left(\frac{3}{4}\right)^2\right\}}\, dx = \sqrt{2}\int \sqrt{\left(\frac{3}{4}\right)^2 - \left(x - \frac{1}{4}\right)^2}\, dx$$

$$= \sqrt{2}\left\{\frac{1}{2}\left(x - \frac{1}{4}\right)\sqrt{\left(\frac{3}{4}\right)^2 - \left(x - \frac{1}{4}\right)^2} + \frac{1}{2}\left(\frac{3}{4}\right)^2 \sin^{-1}\left(\frac{\left(x - \frac{1}{4}\right)}{\frac{3}{4}}\right)\right\} + C$$

$$= \sqrt{2}\left[\frac{(4x - 1)}{8}\sqrt{\frac{1}{2} + \frac{x}{2} - x^2} + \frac{9}{32}\sin^{-1}\left(\frac{4x - 1}{3}\right)\right] + C$$

S7. Given integral is $\int \sqrt{3 + 2x - x^2}\, dx$

$$= \int \sqrt{-(x^2 - 2x - 3)}\, dx = \int \sqrt{-(x^2 - 2x + 1 - 1 - 3)}\, dx$$

$$= \int \sqrt{-\{(x - 1)^2 - (2)^2\}}\, dx = \int \sqrt{(2)^2 - (x - 1)^2}\, dx$$

$$= \frac{1}{2}(x - 1)\sqrt{(2)^2 - (x - 1)^2} + \frac{1}{2}(2)^2 \sin^{-1}\left(\frac{x - 1}{2}\right) + C$$

$$= \frac{1}{2}(x - 1)\sqrt{3 + 2x - x^2} + 2\sin^{-1}\left(\frac{x - 1}{2}\right) + C$$

S8. Given integral is $\int\sqrt{2x^2+3x+4}\,dx$

$$\Rightarrow \qquad \sqrt{2}\int\sqrt{x^2+\frac{3}{2}x+2}\,dx$$

$$= \sqrt{2}\int\sqrt{x^2+\frac{3}{2}x+\frac{9}{16}-\frac{9}{16}+2}\,dx$$

$$= \sqrt{2}\int\sqrt{\left(x+\frac{3}{4}\right)^2+\left(\frac{\sqrt{23}}{4}\right)^2}\,dx$$

$$= \sqrt{2}\left\{\frac{1}{2}\left(x+\frac{3}{4}\right)\sqrt{\left(x+\frac{3}{4}\right)^2+\left(\frac{\sqrt{23}}{4}\right)^2}+\frac{1}{2}\left(\frac{\sqrt{23}}{4}\right)^2\log\left[\left(x+\frac{3}{4}\right)\sqrt{\left(x+\frac{3}{4}\right)^2+\frac{\sqrt{23}}{4}}\right]\right\}$$

$$= \sqrt{2}\left\{\left(\frac{4x+3}{8}\right)\sqrt{x^2+\frac{3}{2}x+2}+\frac{23}{32}\log\left[\left(x+\frac{3}{4}\right)+\sqrt{x^2+\frac{3}{2}x+2}\right]\right\}+C$$

S9. Given integral is $\int\sqrt{3-2x-2x^2}\,dx$

$$= \int\sqrt{-2\left(x^2+x-\frac{3}{2}\right)}\,dx = \sqrt{2}\int\sqrt{-\left(x^2+x+\frac{1}{4}-\frac{1}{4}-\frac{3}{2}\right)}\,dx$$

$$= \sqrt{2}\int\sqrt{-\left\{\left(x+\frac{1}{2}\right)^2-\left(\frac{\sqrt{7}}{2}\right)^2\right\}} = \sqrt{2}\int\sqrt{\left(\frac{\sqrt{7}}{2}\right)^2-\left(x+\frac{1}{2}\right)^2}\,dx$$

$$= \sqrt{2}\left[\frac{1}{2}\left(x+\frac{1}{2}\right)\sqrt{\left(\frac{\sqrt{7}}{2}\right)^2-\left(x+\frac{1}{2}\right)^2}+\frac{1}{2}\left(\frac{\sqrt{7}}{2}\right)^2\sin^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{7}}{2}}\right)\right]+C$$

$$= \sqrt{2}\left[\left(\frac{2x+1}{4}\right)\sqrt{\frac{3}{2}-x-x^2}+\frac{7}{8}\sin^{-1}\left(\frac{2x+1}{\sqrt{7}}\right)\right]+C$$

S10. Given integral is $\int x\sqrt{x^4+1}\,dx$

Put $x^2 = t \Rightarrow 2x\,dx = dt$

$$= \frac{1}{2}\int\sqrt{t^2+1}\,dt = \frac{1}{2}\left[\frac{1}{2}t\sqrt{t^2+1}+\frac{1}{2}(1)^2\log\left(t+\sqrt{t^2+1}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}x^2\sqrt{x^4+1}+\frac{1}{2}\log\left(x^2+\sqrt{x^4+1}\right)\right]+C$$

S11. Given integral is $\int x^2 \sqrt{a^6 - x^6}\, dx$

Put $x^3 = t \Rightarrow 3x^2\, dx = dt$

$$= \frac{1}{3}\int\sqrt{a^6 - t^2}\, dt = \frac{1}{3}\int\sqrt{(a^3)^2 - t^2}\, dt$$

$$= \frac{1}{3}\left[\frac{1}{2}t\sqrt{(a^3)^2 - t^2} + \frac{1}{2}(a^3)^2 \sin^{-1}\left(\frac{t}{a^3}\right)\right] + C$$

$$= \frac{1}{3}\left[\frac{1}{2}t\sqrt{a^6 - t^2} + \frac{a^6}{2}\sin^{-1}\left(\frac{t}{a^3}\right)\right] + C$$

$$= \frac{1}{3}\left[\frac{x^3}{2}\sqrt{a^6 - x^6} + \frac{a^6}{2}\sin^{-1}\left(\frac{x^3}{a^3}\right)\right] + C$$

S12. Given integral is $\int \frac{\sqrt{16 + (\log x)^2}}{x}\, dx$

Put $\log x = t \Rightarrow \frac{1}{x}\, dx = dt$

$$= \int\sqrt{16 + t^2}\, dt = \int\sqrt{(4)^2 + t^2}\, dt$$

$$= \frac{1}{2}t\sqrt{(4)^2 + t^2} + \frac{1}{2}(4)^2 \log\left(t + \sqrt{(4)^2 + t^2}\right) + C$$

$$= \frac{1}{2}t\sqrt{16 + t^2} + 8\log\left(t + \sqrt{t^2 + 16}\right) + C$$

$$= \frac{1}{2}\log x\sqrt{(\log x)^2 + 16} + 8\log\left(\log x + \sqrt{(\log x)^2 + 16}\right) + C$$

S13.

$$I = \int x\sqrt{1 + x - x^2}\, dx = \int \sqrt{1 + x - x^2}\left[-\frac{1}{2}(1 - 2x) + \frac{1}{2}\right]dx$$

$$= -\frac{1}{2}\int(1 + x - x^2)^{1/2}(1 - 2x)dx + \frac{1}{2}\int\sqrt{1 + x - x^2}dx$$

$$= -\frac{1}{2}\frac{(1 + x - x^2)^{3/2}}{\frac{3}{2}} + \frac{1}{2}\int\sqrt{\frac{5}{4} - \left(x^2 - x + \frac{1}{4}\right)}dx$$

$$= -\frac{1}{3}(1 + x - x^2)^{3/2} + \frac{1}{2}\int\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2}\, dx$$

$$= -\frac{1}{3}(1+x-x^2)^{3/2} + \frac{1}{2}\left[\frac{\left(x-\frac{1}{2}\right)\sqrt{1+x-x^2}}{2} + \frac{5}{8}\sin^{-1}\frac{x-\frac{1}{2}}{\frac{\sqrt{5}}{2}}\right] + c$$

$$= -\frac{1}{3}(1+x-x^2)^{3/2} + \frac{1}{8}(2x-1)\sqrt{1+x-x^2} + \frac{5}{16}\sin^{-1}\frac{2x-1}{\sqrt{5}} + c.$$

S14. Given integral is $\int (2x+3)\sqrt{x^2+4x+3}\,dx$

Put $2x + 3 = A\frac{d}{dx}(x^2 + 4x + 3) + B$

$2x + 3 = A(2x + 4) + B$

$2A = 2$ and $4A + B = 3$

$A = 1$ and $B = -1$

$$= \int [(2x+4)-1]\sqrt{x^2+4x+3}\,dx$$

$$= \underbrace{\int (2x+4)\sqrt{x^2+4x+3}\,dx}_{I_1} - \int \sqrt{x^2+4x+3}\,dx$$

For I_1,

Put $x^2 + 4x + 3 = t \Rightarrow (2x + 4)\,dx = dt$

$$= \int \sqrt{t}\,dt - \int \sqrt{x^2+4x+4-1}\,dx$$

$$= \frac{t^{3/2}}{3/2} - \int \sqrt{(x+2)^2-(1)^2}\,dx$$

$$= \frac{2}{3}t^{3/2} - \left[\frac{1}{2}(x+2)\sqrt{(x+2)^2-(1)^2} - \frac{1}{2}(1)^2\log\left\{(x+2)+\sqrt{(x+2)^2-(1)^2}\right\}\right]$$

$$= \frac{2}{3}(x^2+4x+3)^{3/2} - \frac{1}{2}(x+2)\sqrt{x^2+4x+3} + \frac{1}{2}\log\left\{(x+2)+\sqrt{x^2+4x+3}\right\} + C$$

S15. Given $\int (2x-5)\sqrt{x^2-4x+3}\,dx$

Put $2x - 5 = A\frac{d}{dx}(x^2 - 4x + 3) + B$

$2x - 5 = A(2x - 4) + B$

$2A = 2$ and $-4A + B = -5$

$$A = 1 \quad \text{and} \quad B = -1$$

$$= \int [(2x-4) - 1]\sqrt{x^2 - 4x + 3}\, dx$$

$$= \underbrace{\int (2x-4)\sqrt{x^2 - 4x + 3}\, dx}_{I_1} - \int \sqrt{x^2 - 4x + 3}\, dx$$

For I_1,

Put $x^2 - 4x + 3 = t$

$$(2x-4)\, dx = dt$$

$$= \int \sqrt{t}\, dt - \int \sqrt{x^2 - 4x + 4 - 1}\, dx$$

$$= \frac{t^{3/2}}{3/2} - \int \sqrt{(x-2)^2 - (1)^2}\, dx$$

$$= \frac{2}{3} t^{3/2} - \left[\frac{1}{2}(x-2)\sqrt{(x-2)^2 - (1)^2} - \frac{1}{2}(1)^2 \log\left\{(x-2) + \sqrt{(x-2)^2 - (1)^2}\right\}\right] + C$$

$$= \frac{2}{3}(x^2 - 4x + 3)^{3/2} - \frac{1}{2}(x-2)\sqrt{x^2 - 4x + 3} + \frac{1}{2}\log\left\{(x-2) + \sqrt{x^2 - 4x + 3}\right\} + C$$

S16. Given integral is $\int (x-2)\sqrt{2x^2 - 6x + 5}\, dx$

Put $x - 2 = A\frac{d}{dx}(2x^2 - 6x + 5) + B$

$$x - 2 = A(4x - 6) + B$$

$$4A = 1 \quad \text{and} \quad -6A + B = -2$$

$$A = \frac{1}{4} \quad \text{and} \quad B = -\frac{1}{2}$$

$$= \int \left[\frac{1}{4}(4x-6) - \frac{1}{2}\right]\sqrt{2x^2 - 6x + 5}\, dx$$

$$= \frac{1}{4}\underbrace{\int (4x-6)\sqrt{2x^2 - 6x + 5}\, dx}_{I_1} - \frac{1}{2}\int \sqrt{2x^2 - 6x + 5}\, dx$$

For I_1,

Put $2x^2 - 6x + 5 = t \Rightarrow (4x-6)\, dx = dt$

$$= \frac{1}{4}\int \sqrt{t}\, dt - \frac{1}{2}\int \sqrt{2}\sqrt{x^2 - 3x + \frac{5}{2}}\, dx$$

$$= \frac{1}{4}\frac{t^{3/2}}{3/2} - \frac{\sqrt{2}}{2}\int\sqrt{x^2 - 3x + \frac{9}{4} - \frac{9}{4} + \frac{5}{2}}\,dx$$

$$= \frac{1}{6}t^{3/2} - \frac{1}{\sqrt{2}}\int\sqrt{\left(x - \frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2}\,dx$$

$$= \frac{1}{6}t^{3/2} - \frac{1}{\sqrt{2}}\left[\frac{1}{2}\left(x - \frac{3}{2}\right)\sqrt{\left(x - \frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2} + \frac{1}{2}\left(\frac{1}{2}\right)^2 \log\left\{\left(x - \frac{3}{2}\right) + \sqrt{\left(x - \frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2}\right\}\right] + C$$

$$= \frac{1}{6}(2x^2 - 6x + 5)^{3/2} - \frac{1}{\sqrt{2}}\left[\frac{(2x-3)}{4}\sqrt{x^2 - 3x + \frac{5}{2}} + \frac{1}{8}\log\left\{\frac{2x-3}{2} + \sqrt{x^2 - 3x + \frac{5}{2}}\right\}\right] + C$$

S17. Given integral is $\int (x+2)\sqrt{x^2 + x + 1}\,dx$

Put $x + 2 = A\frac{d}{dx}(x^2 + x + 1) + B$

$x + 2 = A(2x + 1) + B$

$2A = 1$ and $A + B = 2$

$A = \frac{1}{2}$ and $B = \frac{3}{2}$

$$= \int\left[\frac{1}{2}(2x+1) + \frac{3}{2}\right]\sqrt{x^2 + x + 1}\,dx$$

$$= \frac{1}{2}\underbrace{\int(2x+1)\sqrt{x^2 + x + 1}\,dx}_{I_1} + \frac{3}{2}\int\sqrt{x^2 + x + 1}\,dx$$

For I_1,

Put $x^2 + x + 1 = t \Rightarrow (2x + 1)\,dx = dt$

$$= \frac{1}{2}\int\sqrt{t}\,dt + \frac{3}{2}\int\sqrt{\left(x^2 + x + \frac{1}{4} - \frac{1}{4} + 1\right)}\,dx$$

$$= \frac{1}{2}\frac{t^{3/2}}{3/2} + \frac{3}{2}\int\sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,dx$$

$$= \frac{1}{3}t^{3/2} + \frac{3}{2}\left[\frac{1}{2}\left(x + \frac{1}{2}\right)\sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^2 \log\left\{\left(x + \frac{1}{2}\right) + \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right\}\right]$$

$$= \frac{1}{3}(x^2+x+1)^{3/2} + \frac{3}{8}(2x+1)\sqrt{x^2+x+1} + \frac{9}{16}\log\left\{\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right] + C$$

S18. Given integral is $\int (2x-5)\sqrt{2+3x-x^2}\,dx$

Put $2x - 5 = A\frac{d}{dx}(2 + 3x - x^2) + B$

$2x - 5 = A(3 - 2x) + B$

$-2A = 2$ and $3A + B = -5$

$A = -1$ and $B = -2$

$$= \int \{-(3-2x)-2\}\sqrt{2+3x-x^2}\,dx$$

$$= -\underbrace{\int (3-2x)\sqrt{2+3x-x^2}\,dx}_{I_1} - 2\int \sqrt{2+3x-x^2}\,dx$$

For I_1,

Put $2 + 3x - x^2 = t \Rightarrow (3 - 2x)\,dx = dt$

$$= -\int \sqrt{t}\,dt - 2\int \sqrt{-(x^2-3x-2)}\,dx$$

$$= -\frac{t^{3/2}}{3/2} - 2\int \sqrt{-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}-2\right)}\,dx$$

$$= -\frac{2}{3}t^{3/2} - 2\int \sqrt{-\left\{\left(x-\frac{3}{2}\right)^2-\left(\frac{\sqrt{17}}{2}\right)^2\right\}}\,dx$$

$$= -\frac{2}{3}(2+3x-x^2)^{3/2} - 2\int \sqrt{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}\,dx$$

$$= -\frac{2}{3}(2+3x-x^2)^{3/2} - 2\left[\frac{1}{2}\left(x-\frac{3}{2}\right)\sqrt{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{3}{2}\right)} + \frac{1}{2}\left(\frac{\sqrt{17}}{2}\right)^2\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{17}}{2}}\right)\right] + C$$

$$= -\frac{2}{3}(2+3x-x^2)^{3/2} - \frac{(2x-3)}{2}\sqrt{2+3x-x^2} - \frac{17}{4}\sin^{-1}\left(\frac{2x-3}{\sqrt{17}}\right) + C$$

S19. Given integral is $\int (x+1)\sqrt{x^2-x+1}\,dx$

Put $x + 1 = A(x^2 - x + 1) + B$

$$x + 1 = A(2x - 1) + B$$

$$2A = 1 \quad \text{and} \quad -A + B = 1$$

$$A = \frac{1}{2} \quad \text{and} \quad B = \frac{3}{2}$$

$$= \int\left\{\frac{1}{2}(2x-1) + \frac{3}{2}\right\}\sqrt{x^2 - x + 1}\, dx$$

For I_1,

$$= \frac{1}{2}\int \underbrace{(2x-1)\sqrt{x^2 - x + 1}\, dx}_{I_1} + \frac{3}{2}\int\sqrt{x^2 - x + 1}\, dx$$

Put $x^2 - x + 1 = t \Rightarrow (2x - 1)\, dx = dt$

$$= \frac{1}{2}\int\sqrt{t}\, dt + \frac{3}{2}\int\sqrt{x^2 - x + \frac{1}{4} - \frac{1}{4} + 1}\, dx$$

$$= \frac{1}{2}\frac{t^{3/2}}{3/2} + \frac{3}{2}\int\sqrt{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\, dx$$

$$= \frac{1}{3}t^{3/2} + \frac{3}{2}\left[\frac{1}{2}\left(x - \frac{1}{2}\right)\sqrt{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^2 \log\left\{\left(x - \frac{1}{2}\right)\sqrt{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right\}\right]$$

$$= \frac{1}{3}(x^2 - x + 1)^{3/2} + \frac{3}{8}(2x - 1)\sqrt{x^2 - x + 1} + \frac{9}{16}\log\left\{\left(x - \frac{1}{2}\right) + \sqrt{x^2 - x + 1}\right\} + C$$

S20. Given integral is $\int(4x + 1)\sqrt{x^2 - x - 2}\, dx$

Put $4x + 1 = A\frac{d}{dx}(x^2 - x - 2) + B$

$$4x + 1 = A(2x - 1) + \text{B}$$

$$2A = 4 \quad \text{and} \quad -A + B = 1$$

$$A = 2 \quad \text{and} \quad B = 3$$

$$= \int[2(2x - 1) + 3]\sqrt{x^2 - x - 2}\, dx$$

$$= 2\int \underbrace{(2x - 1)\sqrt{x^2 - x - 2}\, dx}_{I_1} + 3\int\sqrt{x^2 - x - 2}\, dx$$

For I_1,

Put $x^2 - x - 2 = t \Rightarrow (2x - 1)\, dx = dt$

$$= 2\int\sqrt{t}\,dt + 3\int\sqrt{x^2 - x + \frac{1}{4} - \frac{1}{4} - 2}\,dx$$

$$= 2\,\frac{t^{3/2}}{3/2} + 3\int\sqrt{\left(x - \frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2}\,dx$$

$$= \frac{4}{3}t^{3/2} + 3\left[\frac{1}{2}\left(x - \frac{1}{2}\right)\sqrt{\left(x - \frac{1}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)^2} - \frac{1}{2}\left(\frac{3}{2}\right)^2 \log\left\{\left(x - \frac{1}{2}\right) + \sqrt{x^2 - x - 2}\right\} + C\right.$$

$$= \frac{4}{3}(x^2 - x - 2)^{3/2} + \frac{3}{4}(2x - 1)\sqrt{x^2 - x - 2} - \frac{27}{8}\log\left\{\left(x - \frac{1}{2}\right) + \sqrt{x^2 - x - 2}\right\} + C$$

S21. Given integral is $\int(x + 1)\sqrt{1 - x - x^2}\,dx$

Put $x + 1 = A\frac{d}{dx}(1 - x - x^2) + B$

$x + 1 = A(-1 - 2x) + B$

$-2A = 1$ and $-A + B = 1$

$A = -\frac{1}{2}$ and $B = \frac{1}{2}$

$$= \int\left[-\frac{1}{2}(-1 - 2x) + \frac{1}{2}\right]\sqrt{1 - x - x^2}\,dx$$

$$= -\frac{1}{2}\underbrace{\int(-1 - 2x)\sqrt{1 - x - x^2}\,dx}_{I_1} - \frac{1}{2}\int\sqrt{1 - x - x^2}\,dx$$

For I_1,

Put $1 - x - x^2 = t \Rightarrow (-1 - 2x)\,dx = dt$

$$= -\frac{1}{2}\int\sqrt{t}\,dt + \frac{1}{2}\int\sqrt{-\left(x^2 + x + \frac{1}{4} - \frac{1}{4} + 1\right)}\,dx$$

$$= -\frac{1}{2}\frac{t^{3/2}}{3/2} + \frac{1}{2}\int\sqrt{-\left\{\left(x + \frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2\right\}}\,dx$$

$$= \frac{-t^{3/2}}{3} + \frac{1}{2}\int\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 - \left(x + \frac{1}{2}\right)^2}\,dx$$

$$= -\frac{1}{3}(1-x-x^2)^{3/2} + \frac{1}{4}\left(x+\frac{1}{2}\right)\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 - \left(x+\frac{1}{2}\right)^2} + \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)^2 \sin^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$$

$$= -\frac{1}{3}(1-x-x^2)^{3/2} + \left(\frac{2x+1}{8}\right)\sqrt{1-x-x^2} + \frac{9}{16}\sin^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

Date: 19/10/2021

Q1. Evaluate $\int \frac{x^2+1}{x^4+1} dx$

Q2. Evaluate $\int \frac{1}{x^4-1} dx$

Q3. Evaluate $\int \frac{x^2+4}{x^4+16} dx$

Q4. Evaluate $\int \frac{x^2+1}{x^4+x^2+1} dx$

Q5. Evaluate $\int \frac{x^2-1}{x^4+x^2+1} dx$

Q6. Evaluate $\int \frac{x^2+1}{x^4+7x^2+1} dx$

Q7. Evaluate $\int \frac{x^2-3x+1}{x^4+x^2+1} dx$

Q8. Evaluate $\int \frac{x^2}{x^4+1} dx$

Q9. Evaluate $\int \frac{(x-1)^2}{x^4+x^2+1} dx$

Q10. Evaluate $\int \frac{dx}{\sin^4 x+\cos^4 x}$

Date: 19/10/2021

S1. Let $$I = \int \frac{x^2+1}{x^4+1}\,dx$$

Dividing numerator and denominator by x^2 we get

$$I = \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx$$

$$= \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}+2-2}\,dx$$

$$= \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+(\sqrt{2})^2}\,dx$$

Put $$x-\frac{1}{x} = t$$

$$\Rightarrow \quad \left(1+\frac{1}{x^2}\right)dx = dt$$

$$\therefore \quad I = \int \frac{dt}{t^2+(\sqrt{2})^2} = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)+C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right)+C \qquad \left[\because \int\frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}+C\right]$$

$$\therefore \quad I = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{x\sqrt{2}}\right)+C$$

S2. Given $\int \frac{dx}{x^4-1}$

Let $$\frac{1}{x^4-1} = \frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{x^2+1} \qquad \dots \text{(i)}$$

$$1 = A(x+1)(x^2+1)+B(x-1)(x^2+1)+C(x^2-1) \qquad \dots \text{(ii)}$$

Put $x = 1$ in Eq. (ii)	Put $x = -1$ in Eq. (ii)	Equating constant term
$4A = 1$	$-4B = 1$	$A - B - C = 1$
$A = \frac{1}{4}$	$B = -\frac{1}{4}$	$\frac{1}{4} + \frac{1}{4} - C = 1$
		$C = -\frac{1}{2}$

Put the value of A, B, C in Eq. (i)

$$\frac{1}{(x^2-1)(x^2+1)} = \frac{1}{4(x-1)} - \frac{1}{4(x+1)} - \frac{1}{2(x^2+1)}$$

$$\int \frac{dx}{x^4-1} = \frac{1}{4}\int \frac{dx}{x-1} - \frac{1}{4}\int \frac{dx}{x+1} - \frac{1}{2}\int \frac{dx}{x^2+1}$$

$$= \frac{1}{4}\log(x-1) - \frac{1}{4}\log(x+1) - \frac{1}{2}\tan^{-1}x + C$$

$$= \frac{1}{4}\log\left(\frac{x-1}{x+1}\right) - \frac{1}{2}\tan^{-1}x + C.$$

S3. Let $I = \int \frac{x^2+4}{x^4+16}dx$

Dividing numerator and Denominator by x^2, we get

$$I = \int \frac{1+\frac{4}{x^2}}{x^2+\frac{16}{x^2}}dx = \int \frac{1+\frac{4}{x^2}}{\left(x-\frac{4}{x}\right)^2+8}dx$$

Put $x - \frac{4}{x} = t$

$\Rightarrow$ $\left(1+\frac{4}{x^2}\right)dx = dt$

$\Rightarrow$ $I = \int \frac{dt}{t^2+8} = \int \frac{dt}{t^2+(2\sqrt{2})^2}$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{t}{2\sqrt{2}}\right) + C \qquad \left[\because \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{4}{x}}{2\sqrt{2}}\right) + C$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-4}{2\sqrt{2}x}\right) + C$$

S4. Given integral is $\int \frac{x^2+1}{x^4+x^2+1}dx$

$$= \int \frac{1+\frac{1}{x^2}}{x^2+1+\frac{1}{x^2}}dx \qquad \text{[Divided by } x^2 \text{ Nr. and Dr.]}$$

$$= \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2+1}dx$$

Put $x-\frac{1}{x}=t \Rightarrow \left(1+\frac{1}{x^2}\right)dx = dt$

$$= \int \frac{dt}{t^2+3} = \int \frac{dt}{t^2+(\sqrt{3})^2}$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)+C = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{3}}\right)+C$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{3}\,x}\right)+C$$

S5. Given $\int \frac{x^2-1}{x^4+x^2+1}dx$

$$\Rightarrow \int \frac{1-\frac{1}{x^2}}{x^2+1+\frac{1}{x^2}}dx \qquad \{\text{Divided by } x^2\ Nr \text{ and } Dr\}$$

$$\int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-2+1}dx = \int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-(1)^2}dx$$

Put $x+\frac{1}{x}=t \Rightarrow \left(1-\frac{1}{x^2}\right)dx = dt$

$$\int \frac{dt}{t^2-(1)^2} = \frac{1}{2}\log\left(\frac{t-1}{t+1}\right)+C$$

$$= \frac{1}{2}\log\frac{\left(x+\frac{1}{x}-1\right)}{\left(x+\frac{1}{x}+1\right)}+C$$

$$= \frac{1}{2}\log\left(\frac{x^2-x+1}{x^2+x+1}\right)+C.$$

S6. Given integral is $\int \frac{x^2+1}{x^4+7x^2+1}dx$

$$= \int \frac{1+\frac{1}{x^2}}{x^2+7+\frac{1}{x^2}}dx \qquad \text{[Divided by } x^2 \text{ Nr. and Dr.]}$$

$$= \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2+7}dx$$

Put $x-\frac{1}{x}=t \Rightarrow \left(1+\frac{1}{x^2}\right)dx = dt$

$$= \int \frac{dt}{t^2+9} = \frac{1}{3}\tan^{-1}\frac{t}{3}+C = \frac{1}{3}\tan^{-1}\left(\frac{x-\frac{1}{x}}{3}\right)+C$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{x^2-1}{3x}\right)+C$$

S7. Given integral is $\int \frac{x^2-3x+1}{x^4+x^2+1}dx$

$$= \underbrace{\int \frac{x^2+1}{x^4+x^2+1}dx}_{I_1} - 3\underbrace{\int \frac{x}{x^4+x^2+1}dx}_{I_2}$$

$$I_1 = \int \frac{x^2+1}{x^4+x^2+1}dx = \int \frac{1+\frac{1}{x^2}}{x^2+1+\frac{1}{x^2}}dx$$

$$= \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2+1}dx$$

Put $x-\frac{1}{x}=t \Rightarrow \left(1+\frac{1}{x^2}\right)dx = dt$

$$= \int \frac{dt}{t^2+3} = \frac{dt}{t^2+(\sqrt{3})^2}$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right) + C = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x - \frac{1}{x}}{\sqrt{3}}\right) + C_1$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x^2 - 1}{\sqrt{3}\,x}\right) + C_1$$

$$I_2 = \int \frac{x}{x^4 + x^2 + 1}\,dx$$

Put $x^2 = z \Rightarrow 2x\,dx = dz$

$$= \frac{1}{2}\int \frac{dz}{z^2 + z + 1} = \frac{1}{2}\int \frac{dz}{z^2 + z + \frac{1}{4} - \frac{1}{4} + 1}$$

$$= \frac{1}{2}\int \frac{dz}{\left(z + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{2}\cdot\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{z + \frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C_2$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2z + 1}{\sqrt{3}}\right) + C_2 = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2 + 1}{\sqrt{3}}\right) + C_2$$

$\therefore$
$$I_1 - 3I_2 = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x^2 - 1}{\sqrt{3}\,x}\right) - \frac{3}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2 + 1}{\sqrt{3}}\right) + C \qquad (C = C_1 - C_2)$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x^2 - 1}{\sqrt{3}\,x}\right) - \sqrt{3}\tan^{-1}\left(\frac{2x^2 + 1}{\sqrt{3}}\right) + C$$

S8. Given $\int \frac{x^2}{x^4 + 1}\,dx$

$\Rightarrow \int \frac{1}{x^2 + \frac{1}{x^2}}\,dx$ {Divided by x^2 *Nr* and *Dr*}

$$\frac{1}{2}\int \frac{2}{x^2 + \frac{1}{x^2}}\,dx = \frac{1}{2}\int \frac{1 + \frac{1}{x^2} + 1 - \frac{1}{x^2}}{x^2 + \frac{1}{x^2}}\,dx$$

$$\Rightarrow \frac{1}{2}\int \frac{1 + \frac{1}{x^2}}{x^2 + \frac{1}{x^2}}\,dx + \frac{1}{2}\int \frac{1 - \frac{1}{x^2}}{x^2 + \frac{1}{x^2}}\,dx$$

$$\Rightarrow \quad \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2}\,dx \;+\; \frac{1}{2}\int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-2}\,dx$$

I_1 $\qquad\qquad$ I_2

For I_1, $\qquad\qquad$ For I_2,

put $x - \frac{1}{x} = t$ $\qquad\qquad$ $x + \frac{1}{x} = z$

$\left(1+\frac{1}{x^2}\right)dx = dt$ $\qquad\qquad$ $\left(1-\frac{1}{x^2}\right)dx = dz$

$$\Rightarrow \quad \frac{1}{2}\int \frac{dt}{t^2+\left(\sqrt{2}\right)^2} \;+\; \frac{1}{2}\int \frac{dz}{z^2-\left(\sqrt{2}\right)^2}$$

$$\Rightarrow \quad \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right) \;+\; \frac{1}{2}\times\frac{1}{2\sqrt{2}}\log\left(\frac{z-\sqrt{2}}{z+\sqrt{2}}\right)+C$$

$$\Rightarrow \quad \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right) \;+\; \frac{1}{4\sqrt{2}}\log\left(\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right)+C$$

$$\Rightarrow \quad \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) \;+\; \frac{1}{4\sqrt{2}}\log\left(\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right)+C.$$

S9. Given integral is $\int \frac{(x-1)^2}{x^4+x^2+1}\,dx$

$$= \int \frac{x^2-2x+1}{x^4+x^2+1}\,dx = \underbrace{\int \frac{x^2+1}{x^4+x^2+1}\,dx}_{I_1} - 2\underbrace{\int \frac{x}{x^4+x^2+1}\,dx}_{I_2}$$

$$I_1 = \int \frac{x^2+1}{x^4+x^2+1}\,dx = \int \frac{1+\frac{1}{x^2}}{x^2+1+\frac{1}{x^2}}\,dx = \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2+1}\,dx$$

Put $x - \frac{1}{x} = t \Rightarrow \left(1+\frac{1}{x^2}\right)dx = dt$, $\int \frac{dt}{t^2+3} = \frac{dt}{t^2+(\sqrt{3})^2}$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)+C_1 = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{3}}\right)+C_1$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{3}\,x}\right)+C_1$$

$$I_2 = \int \frac{x}{x^2 + x^2 + 1} dx$$

Put $x^2 = z \Rightarrow 2x\, dx = dz$

$$= \frac{1}{2}\int \frac{dz}{z^2 + z + 1} = \frac{1}{2}\int \frac{dz}{z^2 + z + \frac{1}{4} - \frac{1}{4} + 1}$$

$$= \frac{1}{2}\int \frac{dz}{\left(z + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{2}\cdot\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{z + \frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C_2$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2z+1}{\sqrt{3}}\right) + C_2 = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right) + C_2$$

$$\therefore \quad I_1 - 2I_2 = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{3}\,x}\right) - \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right) + C \qquad (C = C_1 - C_2)$$

S10. Given integral is $\int \frac{dx}{\sin^4 x + \cos^4 x}$

$$= \int \frac{\frac{1}{\cos^4 x}}{\frac{\sin^4 x + \cos^4 x}{\cos^4 x}} dx = \int \frac{\sec^4 x}{\tan^4 x + 1} dx = \int \frac{\sec^2 x \cdot \sec^2 x}{\tan^4 x + 1} dx$$

$$= \int \left(\frac{1 + \tan^2 x}{1 + \tan^4 x}\right)\sec^2 x\, dx$$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$= \int \frac{1+t^2}{1+t^4} dt = \int \frac{1 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2}} dt \qquad \text{(divided by } t^2 \text{ Nr. and Dr.)}$$

$$= \int \frac{1 + \frac{1}{t^2}}{\left(t - \frac{1}{t}\right)^2 + 2} dt$$

Put $t - \frac{1}{t} = z \Rightarrow \left(1 + \frac{1}{t^2}\right) dt = dz$

$$\int \frac{dz}{z^2 + (\sqrt{2})^2} = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{z}{\sqrt{2}}\right) + C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t - \frac{1}{t}}{\sqrt{2}}\right) + C = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t^2 - 1}{\sqrt{2}\,t}\right) + C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan^2 x - 1}{\sqrt{2}\tan x}\right) + C$$

Date: 19/10/2021

Q1. Evaluate $\int \frac{x^2+1}{x^4+x^2+1}dx$

Q2. Evaluate $\int \frac{x^2+1}{x^4+1}dx$

Q3. Evaluate $\int \frac{x^2+4}{x^4+16}dx$

Q4. Evaluate $\int \frac{x^2-1}{x^4+x^2+1}dx$

Q5. Evaluate $\int \frac{x^2+1}{x^4+7x^2+1}dx$

Q6. Evaluate $\int \frac{x^2-1}{x^4+1}dx$

Q7. Evaluate $\int \frac{1}{x^4-1}dx$

Q8. Evaluate $\int \frac{x^2-3x+1}{x^4+x^2+1}dx$

Q9. Evaluate $\int \frac{x^2}{x^4+1}dx$

Q10. Evaluate $\int \frac{(x-1)^2}{x^4+x^2+1}dx$

Q11. Evaluate $\int \frac{dx}{\sin^4 x+\cos^4 x}$

Q12. Evaluate $\int \frac{x^2}{x^4+x^2+1}dx$

Q13. Evaluate $\int \frac{1}{x^4+1}dx$

Q14. Evaluate $\int \frac{1}{x^4+3x^2+1}dx$

Q15. Evaluate $\int \frac{1}{x^4+x^2+1}dx$

Q16. Evaluate $\int \sqrt{\cot\theta}\,d\theta$

Q17. Evaluate $\int \sqrt{\tan\theta}\,d\theta$

Date: 19/10/2021

S1. Given integral is $\int \frac{x^2+1}{x^4+x^2+1}dx$

$$= \int \frac{1+\frac{1}{x^2}}{x^2+1+\frac{1}{x^2}}dx \qquad \text{[Divided by } x^2 \text{ Nr. and Dr.]}$$

$$= \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2+1}dx$$

Put $x-\frac{1}{x} = t \Rightarrow \left(1+\frac{1}{x^2}\right)dx = dt$

$$= \int \frac{dt}{t^2+3} = \int \frac{dt}{t^2+(\sqrt{3})^2}$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)+C = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{3}}\right)+C$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{3}\,x}\right)+C$$

S2. Let $I = \int \frac{x^2+1}{x^4+1}dx$

Dividing numerator and denominator by x^2 we get

$$I = \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx$$

$$= \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}+2-2}dx$$

$$= \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+(\sqrt{2})^2}dx$$

Put $\qquad x - \frac{1}{x} = t$

$\Rightarrow \qquad \left(1 + \frac{1}{x^2}\right)dx = dt$

$\therefore \qquad I = \int \frac{dt}{t^2 + (\sqrt{2})^2} = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + C$

$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x - \frac{1}{x}}{\sqrt{2}}\right) + C \qquad \left[\because \int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C\right]$

$\therefore \qquad I = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x^2 - 1}{x\sqrt{2}}\right) + C$

S3. Let $\qquad I = \int \frac{x^2 + 4}{x^4 + 16} dx$

Dividing numerator and Denominator by x^2, we get

$$I = \int \frac{1 + \frac{4}{x^2}}{x^2 + \frac{16}{x^2}} dx = \int \frac{1 + \frac{4}{x^2}}{\left(x - \frac{4}{x}\right)^2 + 8} dx$$

Put $\qquad x - \frac{4}{x} = t$

$\Rightarrow \qquad \left(1 + \frac{4}{x^2}\right)dx = dt$

$\Rightarrow \qquad I = \int \frac{dt}{t^2 + 8} = \int \frac{dt}{t^2 + (2\sqrt{2})^2}$

$= \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{t}{2\sqrt{2}}\right) + C \qquad \left[\because \int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$

$= \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{x - \frac{4}{x}}{2\sqrt{2}}\right) + C$

$= \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{x^2 - 4}{2\sqrt{2}x}\right) + C$

S4. Given $\quad \int \frac{x^2 - 1}{x^4 + x^2 + 1} dx$

$\Rightarrow \quad \int \frac{1 - \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx \qquad$ {Divided by x^2 *Nr* and *Dr*}

$$\int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-2+1}\,dx = \int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-(1)^2}\,dx$$

Put $\quad x+\frac{1}{x} = t \quad \Rightarrow \quad \left(1-\frac{1}{x^2}\right)dx = dt$

$$\int \frac{dt}{t^2-(1)^2} = \frac{1}{2}\log\left(\frac{t-1}{t+1}\right)+C$$

$$= \frac{1}{2}\log\frac{\left(x+\frac{1}{x}-1\right)}{\left(x+\frac{1}{x}+1\right)}+C$$

$$= \frac{1}{2}\log\left(\frac{x^2-x+1}{x^2+x+1}\right)+C.$$

S5. Given integral is $\int \frac{x^2+1}{x^4+7x^2+1}\,dx$

$$= \int \frac{1+\frac{1}{x^2}}{x^2+7+\frac{1}{x^2}}\,dx$$ [Divided by x^2 Nr. and Dr.]

$$= \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2+7}\,dx$$

Put $\quad x-\frac{1}{x} = t \quad \Rightarrow \quad \left(1+\frac{1}{x^2}\right)dx = dt$

$$= \int \frac{dt}{t^2+9} = \frac{1}{3}\tan^{-1}\frac{t}{3}+C = \frac{1}{3}\tan^{-1}\left(\frac{x-\frac{1}{x}}{3}\right)+C$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{x^2-1}{3x}\right)+C$$

S6. Given integral is $\int \frac{x^2-1}{x^4+1}\,dx$

$$= \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx$$ [Divided by x^2 Nr. and Dr.]

$$= \int \frac{1 - \frac{1}{x^2}}{\left(x + \frac{1}{x}\right)^2 - 2} dx$$

Put $x + \frac{1}{x} = t \Rightarrow \left(1 - \frac{1}{x^2}\right) dx = dt$

$$= \int \frac{dt}{t^2 - 2} = \int \frac{dt}{t^2 - (\sqrt{2})^2}$$

$$= \frac{1}{2\sqrt{2}} \log \left(\frac{t - \sqrt{2}}{t + \sqrt{2}}\right) + C$$

$$= \frac{1}{2\sqrt{2}} \log \left(\frac{x + \frac{1}{x} - \sqrt{2}}{x + \frac{1}{x} + \sqrt{2}}\right) + C$$

$$= \frac{1}{2\sqrt{2}} \log \left(\frac{x^2 - \sqrt{2}\,x + 1}{x^2 + \sqrt{2}\,x + 1}\right) + C$$

S7. Given $\int \frac{dx}{x^4 - 1}$

Let $$\frac{1}{x^4 - 1} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x^2 + 1} \quad \text{... (i)}$$

$$1 = A(x + 1)(x^2 + 1) + B(x - 1)(x^2 + 1) + C(x^2 - 1) \quad \text{... (ii)}$$

Put $x = 1$ in Eq. (ii)

$4A = 1$

$A = \frac{1}{4}$

Put $x = -1$ in Eq. (ii)

$-4B = 1$

$B = -\frac{1}{4}$

Equating constant term

$A - B - C = 1$

$\frac{1}{4} + \frac{1}{4} - C = 1$

$C = -\frac{1}{2}$

Put the value of A, B, C in Eq. (i)

$$\frac{1}{(x^2 - 1)(x^2 + 1)} = \frac{1}{4(x - 1)} - \frac{1}{4(x + 1)} - \frac{1}{2(x^2 + 1)}$$

$$\int \frac{dx}{x^4 - 1} = \frac{1}{4} \int \frac{dx}{x - 1} - \frac{1}{4} \int \frac{dx}{x + 1} - \frac{1}{2} \int \frac{dx}{x^2 + 1}$$

$$= \frac{1}{4} \log(x - 1) - \frac{1}{4} \log(x + 1) - \frac{1}{2} \tan^{-1} x + C$$

$$= \frac{1}{4} \log\left(\frac{x - 1}{x + 1}\right) - \frac{1}{2} \tan^{-1} x + C .$$

S8. Given integral is $\int \frac{x^2 - 3x + 1}{x^4 + x^2 + 1} dx$

$$= \underbrace{\int \frac{x^2 + 1}{x^4 + x^2 + 1} dx}_{I_1} - 3\underbrace{\int \frac{x}{x^4 + x^2 + 1} dx}_{I_2}$$

$$I_1 = \int \frac{x^2 + 1}{x^4 + x^2 + 1} dx = \int \frac{1 + \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx$$

$$= \int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 2 + 1} dx$$

Put $x - \frac{1}{x} = t \Rightarrow \left(1 + \frac{1}{x^2}\right) dx = dt$

$$= \int \frac{dt}{t^2 + 3} = \frac{dt}{t^2 + (\sqrt{3})^2}$$

$$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{t}{\sqrt{3}}\right) + C = \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x - \frac{1}{x}}{\sqrt{3}}\right) + C_1$$

$$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x^2 - 1}{\sqrt{3}\,x}\right) + C_1$$

$$I_2 = \int \frac{x}{x^4 + x^2 + 1} dx$$

Put $x^2 = z \Rightarrow 2x\,dx = dz$

$$= \frac{1}{2}\int \frac{dz}{z^2 + z + 1} = \frac{1}{2}\int \frac{dz}{z^2 + z + \frac{1}{4} - \frac{1}{4} + 1}$$

$$= \frac{1}{2}\int \frac{dz}{\left(z + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{2} \cdot \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1}\left(\frac{z + \frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C_2$$

$$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2z + 1}{\sqrt{3}}\right) + C_2 = \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x^2 + 1}{\sqrt{3}}\right) + C_2$$

$\therefore$ $$I_1 - 3I_2 = \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x^2 - 1}{\sqrt{3}\,x}\right) - \frac{3}{\sqrt{3}} \tan^{-1}\left(\frac{2x^2 + 1}{\sqrt{3}}\right) + C \qquad (C = C_1 - C_2)$$

$$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x^2 - 1}{\sqrt{3}\,x}\right) - \sqrt{3} \tan^{-1}\left(\frac{2x^2 + 1}{\sqrt{3}}\right) + C$$

S9. Given $\int \frac{x^2}{x^4+1}\,dx$

$$\Rightarrow \int \frac{1}{x^2+\frac{1}{x^2}}\,dx$$ {Divided by x^2 *Nr* and *Dr*}

$$\frac{1}{2}\int \frac{2}{x^2+\frac{1}{x^2}}\,dx = \frac{1}{2}\int \frac{1+\frac{1}{x^2}+1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx$$

$$\Rightarrow \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx + \frac{1}{2}\int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx$$

$$\Rightarrow \underbrace{\frac{1}{2}\int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2}\,dx}_{I_1} + \underbrace{\frac{1}{2}\int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-2}\,dx}_{I_2}$$

For I_1, For I_2,

put $x-\frac{1}{x}=t$ $x+\frac{1}{x}=z$

$\left(1+\frac{1}{x^2}\right)dx = dt$ $\left(1-\frac{1}{x^2}\right)dx = dz$

$$\Rightarrow \frac{1}{2}\int \frac{dt}{t^2+\left(\sqrt{2}\right)^2} + \frac{1}{2}\int \frac{dz}{z^2-\left(\sqrt{2}\right)^2}$$

$$\Rightarrow \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + \frac{1}{2}\times\frac{1}{2\sqrt{2}}\log\left(\frac{z-\sqrt{2}}{z+\sqrt{2}}\right)+C$$

$$\Rightarrow \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right) + \frac{1}{4\sqrt{2}}\log\left(\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right)+C$$

$$\Rightarrow \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) + \frac{1}{4\sqrt{2}}\log\left(\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right)+C.$$

S10. Given integral is $\int \frac{(x-1)^2}{x^4+x^2+1}\,dx$

$$= \int \frac{x^2-2x+1}{x^4+x^2+1}\,dx = \underbrace{\int \frac{x^2+1}{x^4+x^2+1}\,dx}_{I_1} - 2\underbrace{\int \frac{x}{x^4+x^2+1}\,dx}_{I_2}$$

$$I_1 = \int \frac{x^2+1}{x^4+x^2+1}\,dx = \int \frac{1+\frac{1}{x^2}}{x^2+1+\frac{1}{x^2}}\,dx = \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2+1}\,dx$$

Put $x-\frac{1}{x} = t \Rightarrow \left(1+\frac{1}{x^2}\right)dx = dt$, $\int \frac{dt}{t^2+3} = \frac{dt}{t^2+(\sqrt{3})^2}$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)+C_1 = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{3}}\right)+C_1$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{3}\,x}\right)+C_1$$

$$I_2 = \int \frac{x}{x^2+x^2+1}\,dx$$

Put $x^2 = z \Rightarrow 2x\,dx = dz$

$$= \frac{1}{2}\int \frac{dz}{z^2+z+1} = \frac{1}{2}\int \frac{dz}{z^2+z+\frac{1}{4}-\frac{1}{4}+1}$$

$$= \frac{1}{2}\int \frac{dz}{\left(z+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{2}\cdot\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{z+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+C_2$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2z+1}{\sqrt{3}}\right)+C_2 = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right)+C_2$$

$$\therefore \quad I_1-2I_2 = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{3}\,x}\right)-\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right)+C \qquad (C = C_1-C_2)$$

S11. Given integral is $\int \frac{dx}{\sin^4 x+\cos^4 x}$

$$= \int \frac{\frac{1}{\cos^4 x}}{\frac{\sin^4 x+\cos^4 x}{\cos^4 x}}\,dx = \int \frac{\sec^4 x}{\tan^4 x+1}\,dx = \int \frac{\sec^2 x\cdot\sec^2 x}{\tan^4 x+1}\,dx$$

$$= \int \left(\frac{1+\tan^2 x}{1+\tan^4 x}\right)\sec^2 x\,dx$$

Put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$$= \int \frac{1+t^2}{1+t^4}\,dt = \int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}\,dt \qquad \text{(divided by } t^2 \text{ Nr. and Dr.)}$$

$$= \int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+2}\,dt$$

Put $t-\frac{1}{t} = z \Rightarrow \left(1+\frac{1}{t^2}\right)dt = dz$

$$\int \frac{dz}{z^2+(\sqrt{2})^2} = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{z}{\sqrt{2}}\right)+C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t-\frac{1}{t}}{\sqrt{2}}\right)+C = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t^2-1}{\sqrt{2}\,t}\right)+C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan^2 x-1}{\sqrt{2}\tan x}\right)+C$$

S12. Let $I = \int \frac{x^2}{x^4+x^2+1}\,dx = \frac{1}{2}\int \frac{2x^2}{x^4+x^2+1}\,dx$

$$I = \frac{1}{2}\int \frac{x^2+1+x^2-1}{x^4+x^2+1}\,dx$$

$$I = \frac{1}{2}\int \frac{x^2+1}{x^4+x^2+1}\,dx + \frac{1}{2}\int \frac{x^2-1}{x^4+x^2+1}\,dx$$

$$I = \frac{1}{2}I_1 + \frac{1}{2}I_2 \qquad \dots (i)$$

Consider $I_1 = \int \frac{x^2+1}{x^4-x^2+1}\,dx$

Dividing numerator and denominator by x^2, we get

$$I_1 = \int \frac{\frac{x^2}{x^2}+\frac{1}{x^2}}{\frac{x^4}{x^2}+\frac{x^2}{x^2}+\frac{1}{x^2}}\,dx$$

$$I_1 = \int \frac{1+\frac{1}{x^2}}{x^2+1+\frac{1}{x^2}}\,dx$$

$$I_1 = \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2+1}\,dx$$

$$I_1 = \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+(\sqrt{3})^2}dx$$

Put $x - \frac{1}{x} = t \Rightarrow \left(1+\frac{1}{x^2}\right)dx = dt$

$\therefore$ $$I_1 = \int \frac{dt}{t^2+(\sqrt{3})^2}$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}} \qquad \left[\because \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\frac{\left(x-\frac{1}{x}\right)}{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{x\sqrt{3}}\right)$$

Now, $$I_2 = \int \frac{x^2-1}{x^4+x^2+1}dx$$

$$= \int \frac{\frac{x^2}{x^2}-\frac{1}{x^2}}{\frac{x^4}{x^2}+\frac{x^2}{x^2}+\frac{1}{x^2}}dx$$

$$= \int \frac{1-\frac{1}{x^2}}{x^2+1+\frac{1}{x^2}}dx$$

$$= \int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-2+1}dx$$

$$= \int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-1}dx$$

Put $x+\frac{1}{x} = t$

$\Rightarrow$ $\left(1-\frac{1}{x^2}\right)dx = dt$

$\therefore$ $$I_2 = \int \frac{dt}{t^2-1} = \int \frac{dt}{t^2-(1)^2} \qquad \left[\because \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+C\right]$$

$$= \frac{1}{2\times 1}\log\left|\frac{t-1}{t+1}\right|$$

$$= \frac{1}{2}\log\left|\frac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1}\right| = \frac{1}{2}\log\left|\frac{x^2+1-x}{x^2+1+x}\right|$$

Now, put the values of I_1 and I_2 in Eq. (i), we get

$$\therefore \quad I = \frac{1}{2}\cdot\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{x\sqrt{3}}\right)+\frac{1}{2}\cdot\frac{1}{2}\log\left|\frac{x^2-x+1}{x^2+x+1}\right|+C$$

$$\therefore \quad I = \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{x\sqrt{3}}\right)+\frac{1}{4}\log\left|\frac{x^2-x+1}{x^2+x+1}\right|+C$$

S13. Given $\int \frac{1}{x^4+1}\,dx$

$$\Rightarrow \int \frac{\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx$$ {Divided by x^2 *Nr* and *Dr*}

$$\Rightarrow \frac{1}{2}\int \frac{\frac{2}{x^2}}{x^2+\frac{1}{x^2}}\,dx = \frac{1}{2}\int \frac{-1+\frac{1}{x^2}+1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx$$

$$\Rightarrow \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx - \frac{1}{2}\int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx$$

$$\Rightarrow \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2}\,dx - \frac{1}{2}\int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-2}\,dx$$

I_1 $\qquad$ I_2

For I_1,

put $x-\frac{1}{x} = t$

$\left(1+\frac{1}{x^2}\right)dx = dt$

For I_2,

$x+\frac{1}{x} = z$

$\left(1-\frac{1}{x^2}\right)dx = dz$

$$\Rightarrow \quad \frac{1}{2}\int \frac{dt}{t^2+\left(\sqrt{2}\right)^2} - \frac{1}{2}\int \frac{dz}{z^2-\left(\sqrt{2}\right)^2}$$

$$\Rightarrow \quad \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right) - \frac{1}{2}\times\frac{1}{2\sqrt{2}}\log\left(\frac{z-\sqrt{2}}{z+\sqrt{2}}\right)+C$$

$$\Rightarrow \quad \frac{1}{2\sqrt{2}}\tan^{-1}\frac{\left(x-\frac{1}{x}\right)}{\sqrt{2}} - \frac{1}{4\sqrt{2}}\log\left(\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right)+C$$

$$\Rightarrow \quad \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) - \frac{1}{4\sqrt{2}}\log\left(\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right)+C.$$

S14. Given integral is $\int \frac{1}{x^4+3x^2+1}\,dx$

$$= \int \frac{\frac{1}{x^2}}{x^2+3+\frac{1}{x^2}}\,dx \qquad \text{(divided by } x^2 \text{ Nr. and Dr.)}$$

$$= \frac{1}{2}\int \frac{\frac{2}{x^2}}{x^2+\frac{1}{x}+3}\,dx = \frac{1}{2}\int \frac{1+\frac{1}{x^2}-1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}+3}\,dx$$

$$= \frac{1}{2}\underbrace{\int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2+3}\,dx}_{I_1} - \frac{1}{2}\underbrace{\int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-2+3}\,dx}_{I_2}$$

For I_1, put $x-\frac{1}{x}=t \Rightarrow \left(1+\frac{1}{x^2}\right)dx = dt$; For I_2, put $x+\frac{1}{x}=z \Rightarrow \left(1-\frac{1}{x^2}\right)dx = dz$

$$= \frac{1}{2}\int \frac{dt}{t^2+(\sqrt{5})^2} - \frac{1}{2}\int \frac{dz}{z^2+1}$$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{t}{\sqrt{5}}\right) - \frac{1}{2}\tan^{-1} z + C$$

$$= \frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{5}}\right) - \frac{1}{2}\tan^{-1}\left(x+\frac{1}{x}\right)+C$$

$$= \frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{5}\,x}\right) - \frac{1}{2}\tan^{-1}\left(\frac{x^2+1}{x}\right)+C$$

S15. Given integral is $\int \frac{1}{x^4 + x^2 + 1} dx$

$$= \int \frac{\frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx \qquad \{\text{Divided by } x^2 \text{ Nr. and Dr.}\}$$

$$= \frac{1}{2}\int \frac{\frac{2}{x^2}}{x^2 + \frac{1}{x^2} + 1} dx = \frac{1}{2}\int \frac{1 + \frac{1}{x^2} - 1 + \frac{1}{x^2}}{x^2 + \frac{1}{x^2} + 1} dx$$

$$= \frac{1}{2}\int \frac{1 + \frac{1}{x^2}}{x^2 + \frac{1}{x^2} + 1} dx - \frac{1}{2}\int \frac{1 - \frac{1}{x^2}}{x^2 + \frac{1}{x^2} + 1} dx$$

$$= \frac{1}{2}\underbrace{\int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 2 + 1} dx}_{I_2} - \frac{1}{2}\underbrace{\int \frac{1 - \frac{1}{x^2}}{\left(x + \frac{1}{x}\right) - 2 + 1} dx}_{I_2}$$

For I_1, put $x - \frac{1}{x} = t \Rightarrow \left(1 + \frac{1}{x^2}\right) dx = dt$; For I_2, put $x + \frac{1}{x} = z \Rightarrow \left(1 - \frac{1}{x^2}\right) dx = dz$

$$= \frac{1}{2}\int \frac{dt}{t^2 + 3} - \frac{1}{2}\int \frac{dz}{z^2 - 1} = \frac{1}{2}\int \frac{dt}{t^2 + (\sqrt{3})^2} - \frac{1}{2}\int \frac{dz}{z^2 - 1}$$

$$= \frac{1}{2\sqrt{3}} \tan^{-1}\left(\frac{t}{\sqrt{3}}\right) - \frac{1}{2}\log\left(\frac{z - 1}{z + 1}\right) + C$$

$$= \frac{1}{2\sqrt{3}} \tan^{-1}\left(\frac{x - \frac{1}{x}}{\sqrt{3}}\right) - \frac{1}{2}\log\left(\frac{x + \frac{1}{x} - 1}{x + \frac{1}{x} + 1}\right) + C$$

$$= \frac{1}{2\sqrt{3}} \tan^{-1}\left(\frac{x^2 - 1}{\sqrt{3}\,x}\right) - \frac{1}{2}\log\left(\frac{x^2 + 1 - x}{x^2 + 1 + x}\right) + C$$

S16. Given $\int \sqrt{\cot\theta}\, d\theta$

Let $\cot\theta = x^2 \Rightarrow -\operatorname{cosec}^2\theta\, d\theta = 2x\,dx$

$$\Rightarrow \qquad d\theta = -\frac{2x}{\operatorname{cosec}^2\theta} dx$$

$$\Rightarrow \qquad d\theta = -\frac{2x}{1 + \cot^2\theta} dx$$

$$\Rightarrow \qquad d\theta = -\frac{2x}{1 + x^4} dx$$

$$-\int \frac{\sqrt{x^2}\,2x}{x^4+1}\,dx = -\int \frac{2x^2}{x^4+1}\,dx$$

$$= -\int \frac{2}{x^2+\frac{1}{x^2}}\,dx \qquad \{\text{Divided by } x^2 \text{ in both } Nr \text{ and } Dr\}$$

$$= -\int \frac{1+\frac{1}{x^2}+1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx$$

$$= -\int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx - \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx$$

$$= -\int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2}\,dx - \int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-2}\,dx$$

I_1 $\qquad$ I_2

For I_1, $\qquad$ For I_2,

put $x-\frac{1}{x}=t$ $\qquad$ put $x+\frac{1}{x}=z$

$\left(1+\frac{1}{x^2}\right)dx = dt$ $\qquad$ $\left(1-\frac{1}{x^2}\right)dx = dz$

$$= -\int \frac{dt}{t^2+(\sqrt{2})^2} - \int \frac{dz}{t^2-(\sqrt{2})^2}$$

$$= -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right) - \frac{1}{2\sqrt{2}}\log\left(\frac{z-\sqrt{2}}{z+\sqrt{2}}\right)+C$$

$$= -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right) - \frac{1}{2\sqrt{2}}\log\left(\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right)+C$$

$$= -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) - \frac{1}{2\sqrt{2}}\log\left(\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right)+C$$

$$= -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\cot^2\theta-1}{\sqrt{2}\cot\theta}\right) - \frac{1}{2\sqrt{2}}\log\left(\frac{\cot^2\theta-\sqrt{2}\cot\theta+1}{\cot^2\theta+\sqrt{2}\cot\theta+1}\right)+C.$$

S17. Given $\int \sqrt{\tan\theta}\, d\theta$

Let, $\tan\theta = x^2 \Rightarrow \sec^2\theta\, d\theta = 2x\,dx$

$$d\theta = \frac{2x}{\sec^2\theta}dx$$

$$d\theta = \frac{2x}{1+\tan^2\theta}dx$$

$$d\theta = \frac{2x}{1+x^4}dx$$

$$\int \frac{\sqrt{x^2}\cdot 2x\,dx}{x^4+1} = \int \frac{2x^2\,dx}{x^4+1}$$

$$= \int \frac{2}{x^2+\frac{1}{x^2}}dx \qquad \{\text{Divided by } x^2 \text{ in both } Nr \text{ and } Dr\}$$

$$= \int \frac{1+\frac{1}{x^2}+1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx$$

$$= \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx + \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx$$

$$= \underbrace{\int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2}dx}_{I_1} + \underbrace{\int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-2}dx}_{I_2}$$

For I_1, For I_2,

put $x-\frac{1}{x} = t$ put $x+\frac{1}{x} = z$

$$\left(1+\frac{1}{x^2}\right)dx = dt \qquad \left(1-\frac{1}{x^2}\right)dx = dz$$

$$= \int \frac{dt}{t^2+(\sqrt{2})^2} + \int \frac{dz}{z^2-(\sqrt{2})^2}$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + \frac{1}{2\sqrt{2}} \log\left(\frac{z-\sqrt{2}}{z+\sqrt{2}}\right) + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right) + \frac{1}{2\sqrt{2}} \log\left(\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right) + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) + \frac{1}{2\sqrt{2}} \log\left(\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right) + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\tan^2\theta-1}{\sqrt{2}\tan\theta}\right) + \frac{1}{2\sqrt{2}} \log\left(\frac{\tan^2\theta-\sqrt{2}\tan\theta+1}{\tan^2\theta+\sqrt{2}\tan\theta+1}\right) + C.$$

Date: 19/10/2021

Q1. Evaluate $\int \frac{x^2+x+1}{(x-1)^3}\,dx$

Q2. Evaluate $\int \frac{2x}{(x^2+1)(x^2+2)}\,dx$

Q3. Evaluate $\int \frac{x^3}{(x-1)(x-2)}\,dx$

Q4. Evaluate $\int \frac{3x+1}{(x-2)^2(x+2)}\,dx$

Q5. Evaluate $\int \frac{x^2}{(x^2+1)(3x^2+4)}\,dx$

Q6. Evaluate $\int \frac{dx}{(x-1)^2(x+2)}$

Q7. Evaluate $\int \frac{(2x-1)}{(x-1)(x+2)(x-3)}\,dx$

Q8. Evaluate $\int \frac{2x+1}{(x-2)(x-3)}\,dx$

Q9. Evaluate $\int \frac{18}{(x+2)(x^2+4)}\,dx$

Q10. Evaluate $\int \frac{x^2+1}{(x^2+2)(2x^2+1)}\,dx$

Q11. Evaluate $\int \frac{x^2}{(x^2+1)(x^2+4)}\,dx$

Q12. Evaluate $\int \frac{x}{(x-1)(x^2+4)}\,dx$

Q13. Evaluate $\int \frac{8}{(x+2)(x^2+4)}\,dx$

Q14. Evaluate $\int \frac{3x+5}{x^3-x^2-x+1}\,dx$

Q15. Evaluate $\int \frac{2x^2+1}{x^2(x^2+4)}\,dx$

Q16. Evaluate $\int \frac{dx}{x(x^5+3)}$

Q17. Evaluate $\int \frac{1-\cos x}{\cos x\,(1+\cos x)}\,dx$

Q18. Evaluate $\int \frac{1}{x\log x\,(2+\log x)}\,dx$

Q19. Evaluate $\int \frac{\sin 2x}{(1+\sin x)(2+\sin x)}\,dx$

Q20. Evaluate $\int \frac{5x}{(x+1)(x^2-4)}dx$

Q21. Evaluate $\int \frac{\cos x}{(1-\sin x)(2-\sin x)}dx$

Q22. Evaluate $\int \frac{2}{(1-x)(1+x^2)}dx$

Q23. Evaluate $\int \frac{x^2+1}{(x^2+4)(x^2+25)}dx$

Q24. Evaluate $\int \frac{x^2}{(x^2+4)(x^2+9)}dx$

Q25. Evaluate $\int \frac{dx}{x(x^3+8)}$

Q26. Evaluate $\int \frac{dx}{x(x^3+1)}$

Q27. Evaluate $\int \frac{1}{\cos x(5-4\sin x)}dx$

Q28. Evaluate $\int \frac{x^2}{(x-1)^3(x+1)}dx$

Q29. Evaluate $\int \frac{1-x^2}{x(1-2x)}dx$

Q30. Evaluate $\int \frac{dx}{(x^2+1)(x^2+2)}$

Q31. Evaluate $\int \frac{2x}{(x^2+1)(x^2+3)}dx$

Q32. Evaluate $\int \frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}dx$

Q33. Evaluate $\int \frac{x^2+1}{(x-1)^2(x+3)}dx$

Q34. Evaluate $\int \frac{(3\sin\phi-2)\cos\phi}{5-\cos^2\phi-4\sin\phi}d\phi$

Q35. Evaluate $\int \frac{1}{\sin x+\sin 2x}dx$

Q36. Evaluate $\int \frac{\cos x}{(1-\sin x)^3(2+\sin x)}dx$

Q37. Evaluate $\int \frac{x+1}{x(1+xe^x)}dx$

Q38. Evaluate $\int \frac{dx}{\sin x(3+2\cos x)}$

Q39. Evaluate $\int \frac{\cos\theta d\theta}{(2+\sin\theta)(3+4\sin\theta)}$

Q40. Evaluate $\int \frac{4x^4+3}{(x^2+2)(x^2+3)(x^2+4)}dx$

Q41. Evaluate $\int \frac{\tan x + \tan^3 x}{1 + \tan^2 x} dx$

Q42. Evaluate $\int \frac{x^2 + x + 1}{(x+2)(x^2+1)} dx$

Q43. Evaluate $\int \frac{2\sin 2\theta - \cos\theta}{6 - \cos^2\theta - 4\sin\theta} d\theta$.

Date: 19/10/2021

S1. Given $\int \frac{x^2+x+1}{(x-1)^3}\,dx$

Put $x-1=t \Rightarrow dx = dt$

Let $\int \frac{(t+1)^2+(t+1)+1}{t^3}\,dt = \int \frac{t^2+2t+1+t+1+1}{t^3}\,dt$

$$\int \frac{t^2+3t+3}{t^3}\,dt = \int \frac{1}{t}\,dt + 3\int \frac{1}{t^2}\,dt + 3\int \frac{1}{t^3}\,dt$$

$$= \log t - \frac{3}{t} - \frac{3}{2t^2} + C$$

$$= \log(x-1) - \frac{3}{(x-1)} - \frac{3}{2}\cdot\frac{1}{(x-1)^2} + C.$$

S2. Given $\int \frac{2x}{(x^2+1)(x^2+2)}\,dx$

Put $x^2 = t \Rightarrow 2x\,dx = dt$

$$= \int \frac{dt}{(t+1)(t+2)}$$

Let $\frac{1}{(t+1)(t+2)} = \frac{A}{(t+1)} + \frac{B}{(t+2)}$ … (i)

$$\frac{1}{(t+1)(t+2)} = \frac{A(t+2)+B(t+1)}{(t+1)(t+2)}$$

$1 = A(t+2) + B(t+1)$ … (ii)

Put $t=-2$ in Eq. (ii): $B=-1$ | Put $t=-1$ in Eq. (ii): $A=1$

Putting the value of A and B in Eq. (i)

$$\frac{1}{(t+1)(t+2)} = \frac{1}{(t+1)} - \frac{1}{(t+2)}$$

$\therefore$ $\int \frac{dt}{(t+1)(t+2)} = \int \frac{1}{t+1}\,dt - \int \frac{1}{t+2}\,dt$

$$= \log(t+1) - \log(t+2) + C$$

$$= \log(x^2+1) - \log(x^2+2) + C.$$

S3. Given $\int \frac{x^3}{(x-1)(x-2)}\,dx$

[$\because$ Degree of *Nr* > Degree of *Dr*, therefore divide]

$$\because \qquad \frac{x^3}{(x-1)(x-2)} = x+3+\frac{7x-6}{(x-1)(x-2)}$$

Let $$\frac{7x-6}{(x-1)(x-2)} = \frac{A}{x-1}+\frac{B}{x-2} \qquad \text{... (i)}$$

$$\frac{7x-6}{(x-1)(x-2)} = \frac{A(x-2)+B(x-1)}{(x-1)(x-2)}$$

$$7x-6 = A(x-2)+B(x-1) \qquad \text{... (ii)}$$

Put $x = 2$ in Eq. (ii) $B = 8$	Put $x = 1$ in Eq. (ii) $A = -1$

Putting the value of *A*, *B* in Eq. (i),

$$\frac{7x-6}{(x-1)(x-2)} = -\frac{1}{x-1}+\frac{8}{x-2}$$

$$\therefore \qquad \int \frac{x^3}{(x-1)(x-2)}\,dx = \int\left(x+3-\frac{1}{x-1}+\frac{8}{x-2}\right)dx$$

$$= \frac{x^2}{2}+3x-\log(x-1)+8\log(x-2)+C.$$

S4. Given $\int \frac{3x+1}{(x-2)^2(x+2)}\,dx$

Let $$\frac{3x+1}{(x-2)^2(x+2)} = \frac{A}{(x-2)}+\frac{B}{(x-2)^2}+\frac{C}{(x+2)} \qquad \text{... (i)}$$

$$\frac{3x+1}{(x-2)^2(x+2)} = \frac{A(x-2)(x+2)+B(x+2)+C(x-2)^2}{(x-2)^2(x+2)}$$

$$3x+1 = A(x-2)(x+2)+B(x+2)+C(x-2)^2 \qquad \text{... (ii)}$$

Put $x = 2$ in Eq. (ii) $4B = 7$ $B = \frac{7}{4}$	Put $x = -2$ in Eq. (ii) $16C = -5$ $C = -\frac{5}{16}$	Comparing the coff. of x^2 both side $A + C = 0$ $A = \frac{5}{16}$

Put the value of *A*, *B*, *C* in Eq. (i)

$$\frac{3x+1}{(x-2)^2(x+2)} = \frac{5}{16}\frac{1}{(x-2)}+\frac{7}{4}\frac{1}{(x-2)^2}-\frac{5}{16}\frac{1}{(x+2)}$$

$$\int \frac{3x+1}{(x-2)^2(x+2)}\,dx = \frac{5}{16}\int \frac{1}{(x-2)}\,dx + \frac{7}{4}\int \frac{1}{(x-2)^2}\,dx - \frac{5}{16}\int \frac{1}{(x+2)}\,dx$$

$$= \frac{5}{16}\log(x-2) - \frac{7}{4}\frac{1}{(x-2)} - \frac{5}{16}\log(x+2) + C.$$

S5. Given $\int \frac{x^2}{(x^2+1)(3x^2+4)}\,dx$

Put $x^2 = y$ So that $\frac{x^2}{(x^2+1)(3x^2+4)} = \frac{y}{(y+1)(3y+4)}$

Let $$\frac{y}{(y+1)(3y+4)} = \frac{A}{y+1} + \frac{B}{(3y+4)} \quad \text{... (i)}$$

$$y = A(3y+4) + B(y+1) \quad \text{... (ii)}$$

Put $y = -1$ in Eq. (ii)	Put $y = -\frac{4}{3}$ in Eq. (ii)
$A = -1$	$-\frac{B}{3} = -\frac{4}{3}$
	$B = 4$

Put A, B in Eq. (i)

$$\therefore \quad \frac{y}{(y+1)(3y+4)} = \frac{-1}{(y+1)} + \frac{4}{(3y+4)}$$

$$\therefore \int \frac{x^2}{(x^2+1)(3x^2+4)}\,dx = -\int \frac{1}{x^2+1}\,dx + 4\int \frac{dx}{3x^2+4}$$

$$= -\tan^{-1}x + \frac{4}{3}\int \frac{dx}{x^2+\left(\frac{2}{\sqrt{3}}\right)^2}$$

$$= -\tan^{-1}x + \frac{4}{3}\cdot\frac{1}{\frac{2}{\sqrt{3}}}\tan^{-1}\left(\frac{x}{2/\sqrt{3}}\right) + C$$

$$= -\tan^{-1}x + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}x}{2}\right) + C.$$

S6. Given $\int \frac{dx}{(x-1)^2(x+2)}$

Let $$\frac{1}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2} \quad \text{... (i)}$$

$$1 = A(x-1)(x+2) + B(x+2) + C(x-1)^2 \quad \text{... (ii)}$$

Put $x = 1$ in Eq. (ii)

$3B = 1$

$B = \frac{1}{3}$

Put $x = -2$ in Eq. (ii)

$9C = 1$

$C = \frac{1}{9}$

Equating the coff. of x^2 both side

$A + C = 0 \Rightarrow C = -A$

$A = -\frac{1}{9}$

Putting the value of A, B and C in Eq. (i),

$$\frac{1}{(x-1)^2(x+2)} = -\frac{1}{9(x-1)} + \frac{1}{3}\frac{1}{(x-1)^2} + \frac{1}{9(x+2)}$$

$$\therefore \quad \int \frac{dx}{(x-1)^2(x+2)} = -\frac{1}{9}\int \frac{dx}{x-1} + \frac{1}{3}\int \frac{dx}{(x-1)^2} + \frac{1}{9}\int \frac{dx}{(x+2)}$$

$$= -\frac{1}{9}\log(x-1) - \frac{1}{3(x-1)} + \frac{1}{9}\log(x+2) + C.$$

S7. Given $\int \frac{(2x-1)}{(x-1)(x+2)(x-3)}\,dx$

Let $$\frac{(2x-1)}{(x-1)(x+2)(x-3)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x-3} \quad \text{... (i)}$$

$$\frac{(2x-1)}{(x-1)(x+2)(x-3)} = \frac{A(x+2)(x-3) + B(x-1)(x-3) + C(x-1)(x+2)}{(x-1)(x+2)(x-3)}$$

$$2x - 1 = A(x+2)(x-3) + B(x-1)(x-3) + C(x-1)(x+2) \quad \text{... (ii)}$$

Put $x = -2$ in Eq. (ii)

$-5 = (-3)(-5)B$

$B = -\frac{1}{3}$

Put $x = 3$ in Eq. (ii)

$5 = C(2)(5)$

$C = \frac{1}{2}$

Put $x = 1$ in Eq. (ii)

$1 = A(3)(-2)$

$A = -\frac{1}{6}$

Put the value of A, B and C in Eq. (i)

$$\frac{(2x-1)}{(x-1)(x+2)(x-3)} = -\frac{1}{6}\frac{1}{(x-1)} - \frac{1}{3}\frac{1}{(x+2)} + \frac{1}{2}\cdot\frac{1}{(x-3)}$$

$$\therefore \quad \int \frac{(2x-1)}{(x-1)(x+2)(x-3)}\,dx = -\frac{1}{6}\int \frac{1}{x-1}\,dx - \frac{1}{3}\int \frac{1}{x+2}\,dx + \frac{1}{2}\int \frac{1}{x-3}\,dx$$

$$= -\frac{1}{6}\log(x-1) - \frac{1}{3}\log(x+2) + \frac{1}{2}\log(x-3) + C.$$

S8. Given $\int \frac{2x+1}{(x-2)(x-3)}\,dx$

Let $$\frac{2x+1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3} \quad \text{... (i)}$$

$$2x + 1 = A(x-3) + B(x-2) \quad \text{... (ii)}$$

Put $x = 2$ in Eq. (ii)

$$-A = 5$$
$$A = -5$$

Put $x = 3$ in Eq. (ii)

$$B = 7$$

Put value of A, B in Eq. (i)

$$\frac{2x+1}{(x-2)(x-3)} = \frac{-5}{x-2} + \frac{7}{x-3}$$

$$\int \frac{2x+1}{(x-2)(x-3)}\,dx = -5\int \frac{dx}{x-2} + 7\int \frac{dx}{x-3}$$
$$= -5\log(x-2) + 7\log(x-3) + C.$$

S9. Given $\int \frac{18}{(x+2)(x^2+4)}\,dx$

Let $$\frac{18}{(x+2)(x^2+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+4} \quad \text{... (i)}$$

$$18 = A(x^2+4) + (Bx+C)(x+2) \quad \text{... (ii)}$$
$$= x^2(A+B) + x(2B+C) + 4A + 2C$$

Equating the coff. of x^2, x and constant term

$A + B = 0$, $\quad 2B + C = 0$, $\quad 4A + 2C = 18$

$A = -B$, $\quad C = -2B$, $\quad 2A + C = 9$

$\therefore \quad -4B = 9$, $\quad A = \frac{9}{4}$, $\quad C = \frac{9}{2}$

$B = -\frac{9}{4}$

Putting the value of A, B, C in Eq. (i)

$$\int \frac{18}{(x+2)(x^2+4)}\,dx = \int \frac{9\,dx}{4(x+2)} + \int \frac{-\frac{9}{4}x + \frac{9}{2}}{x^2+4}\,dx$$

$$= \int \frac{9}{4(x+2)}\,dx - \frac{9}{4}\int \frac{x}{x^2+4}\,dx + \frac{9}{2}\int \frac{dx}{x^2+4}$$

$$= \frac{9}{4}\log(x+2) - \frac{9}{8}\log(x^2+4) + \frac{9}{2}\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

$$= \frac{9}{4}\log(x+2) - \frac{9}{8}\log(x^2+4) + \frac{9}{4}\tan^{-1}\left(\frac{x}{2}\right) + C.$$

S10. Given $\int \frac{x^2+1}{(x^2+2)(2x^2+1)}\,dx$

Put $x^2 = y$ then $\frac{y+1}{(y+2)(2y+1)} = \frac{A}{y+2} + \frac{B}{2y+1}$ ... (i)

$y + 1 = A(2y + 1) + B(y + 2)$ ... (ii)

Put $y = -2$ in Eq. (ii)	Put $y = -\frac{1}{2}$ in Eq. (ii)
$-3A = -1$	$\frac{1}{2} = \frac{3B}{2}$
$A = \frac{1}{3}$	$B = \frac{1}{3}$

Put the value of A and B in Eq. (i)

$$\frac{y+1}{(y+2)(2y+1)} = \frac{1}{3(y+2)} + \frac{1}{3(2y+1)}$$

$$\therefore \int \frac{x^2+1}{(x^2+2)(2x^2+1)}\,dx = \frac{1}{3}\int \frac{dx}{x^2+2} + \frac{1}{3}\int \frac{dx}{2x^2+1}$$

$$= \frac{1}{3}\cdot\frac{1}{\sqrt{2}}\ \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + \frac{1}{6}\int \frac{dx}{x^2+\left(\frac{1}{\sqrt{2}}\right)^2}$$

$$= \frac{1}{3\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + \frac{1}{6}\cdot\frac{1}{\frac{1}{\sqrt{2}}}\tan^{-1}\left(\frac{x}{1/\sqrt{2}}\right) + C$$

$$= \frac{1}{3\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + \frac{1}{3\sqrt{2}}\tan^{-1}(\sqrt{2}x) + C.$$

S11. Given $\int \frac{x^2}{(x^2+1)(x^2+4)}\,dx$

Let $x^2 = y$ then $\frac{y}{(y+1)(y+4)} = \frac{A}{y+1} + \frac{B}{y+4}$ ... (i)

$y = A(y + 4) + B(y + 1)$ ... (ii)

Put $y = -1$ in Eq. (ii)	Put $y = -4$ in Eq. (ii)
$3A = -1$	$-3B = -4$
$A = -\frac{1}{3}$	$B = \frac{4}{3}$

Put the value of A and B in Eq. (i)

$$\frac{y}{(y+1)(y+4)} = -\frac{1}{3(y+1)} + \frac{4}{3(y+4)}$$

$$\therefore \quad \int \frac{x^2}{(x^2+1)(x^2+4)}\,dx = -\frac{1}{3}\int \frac{dx}{x^2+1} + \frac{4}{3}\int \frac{dx}{x^2+4}$$

$$= -\frac{1}{3}\tan^{-1}x + \frac{4}{3}\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

$$= -\frac{1}{3}\tan^{-1}x + \frac{2}{3}\tan^{-1}\left(\frac{x}{2}\right) + C.$$

S12. Given $\int \frac{x}{(x-1)(x^2+4)}\,dx$

Let $$\frac{x}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4} \quad \text{...(i)}$$

$$x = A(x^2+4) + (Bx+C)(x-1)$$

Comparing the coff. of x^2, x and constant term

$A + B = 0,$ $\quad -B + C = 1,$ $\quad 4A - C = 0$

$B = -A,$ $\quad A + 4A = 1,$ $\quad C = 4A$

$B = -\frac{1}{5},$ $\quad A = \frac{1}{5},$ $\quad C = \frac{4}{5}$

Put the value of A, B and C in Eq. (i) we get

$$\frac{x}{(x-1)(x^2+4)} = \frac{1}{5(x-1)} - \frac{1}{5}\frac{(x-4)}{(x^2+4)}$$

$$\int \frac{x}{(x-1)(x^2+4)}\,dx = \frac{1}{5}\int \frac{dx}{(x-1)} - \frac{1}{5}\int \frac{x}{(x^2+4)}\,dx + \frac{4}{5}\int \frac{dx}{(x^2+4)}$$

$$= \frac{1}{5}\log(x-1) - \frac{1}{10}\log(x^2+4) + \frac{4}{5}\cdot\frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

$$= \frac{1}{5}\log(x-1) - \frac{1}{10}\log(x^2+4) + \frac{2}{5}\tan^{-1}\left(\frac{x}{2}\right) + C$$

S13. Given $\int \frac{8}{(x+2)(x^2+4)}\,dx$

Let $$\frac{8}{(x+2)(x^2+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+4} \quad \text{... (i)}$$

$$8 = A(x^2 + 4) + (Bx + C) \cdot (x + 2) \quad \text{... (ii)}$$

Comparing the coff. of x^2, x and constant term

$$A + B = 0, \qquad 2B + C = 0, \qquad 4A + 2C = 8$$
$$A = -B, \qquad C = -2B, \qquad 2A + C = 4$$
$$-4B = 4, \qquad A = 1, \qquad C = 2$$
$$B = -1$$

Put the value of A, B and C in Eq. (i)

$$\frac{8}{(x+2)(x^2+4)} = \frac{1}{x+2} + \frac{-x+2}{x^2+4}$$

$$\int \frac{8}{(x+2)(x^2+4)}\,dx = \int \frac{1}{(x+2)}\,dx + \int \frac{-x+2}{x^2+4}\,dx$$

$$= \int \frac{1}{(x+2)}\,dx - \int \frac{x}{x^2+4}\,dx + 2\int \frac{dx}{x^2+4}$$

$$= \log(x+2) - \frac{1}{2}\log(x^2+4) + 2 \cdot \frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

$$= \log(x+2) - \frac{1}{2}\log(x^2+4) + \tan^{-1}\frac{x}{2} + C.$$

S14. Given $\int \frac{3x+5}{x^3 - x^2 - x + 1}\,dx$

Let $$\frac{3x+5}{x^3 - x^2 - x + 1} = \frac{3x+5}{(x^2-1)(x-1)} = \frac{3x+5}{(x-1)^2(x+1)}$$

Again $$\frac{3x+5}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} \quad \text{... (i)}$$

$$3x + 5 = A(x^2 - 1) + B(x + 1) + C(x - 1)^2 \quad \text{... (ii)}$$

Equating the coff. of x^2, x and constant term

$$A + C = 0, \qquad B - 2C = 3, \qquad -A + B + C = 5$$
$$C = -A$$

$\Rightarrow$ $$A = -\frac{1}{2}, \qquad B = 4, \qquad C = \frac{1}{2}$$

Put A, B and C in Eq. (i)

$$\int \frac{3x+5}{(x-1)^2(x+1)}\,dx = -\frac{1}{2}\int \frac{dx}{x-1} + 4\int \frac{dx}{(x-1)^2} + \frac{1}{2}\int \frac{dx}{x+1}$$

$$= -\frac{1}{2}\log(x-1) - \frac{4}{(x-1)} + \frac{1}{2}\log(x+1) + C.$$

S15. Let $$I = \int \frac{2x^2+1}{x^2(x^2+4)}dx$$

Put $x^2 = t$ and then using partial fraction method

$$\frac{2t+1}{t(t+4)} = \frac{A}{t} + \frac{B}{t+4}$$

$\Rightarrow$ $$2t + 1 = A(t+4) + Bt$$

On comparing coefficient, we get

$$2 = A + B \quad \text{and} \quad 1 = 4A$$

$\Rightarrow$ $$A = \frac{1}{4}$$

$\therefore$ $$B = 2 - A = 2 - \frac{1}{4} = \frac{7}{4}$$

$\therefore$ $$\frac{2x^2+1}{x^2(x^2+4)} = \frac{1}{4x^2} + \frac{7}{4(x^2+4)}$$

$$I = \frac{1}{4}\int\frac{dx}{x^2} + \frac{7}{4}\int\frac{dx}{x^2+4}$$

$$= -\frac{1}{4x} + \frac{7}{4}\times\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C \qquad \left[\because \int\frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C\right]$$

$$= -\frac{1}{4x} + \frac{7}{8}\tan^{-1}\left(\frac{x}{2}\right) + C$$

S16. Let $$I = \int\frac{dx}{x(x^5+3)} = \int\frac{x^4}{x^5(x^5+3)}dx$$

[$\because$ Multiplying numerator and denominator by x^4]

put $$x^5 = t$$

$\Rightarrow$ $$5x^4\,dx = dt$$

$\Rightarrow$ $$x^4 dx = \frac{1}{5}dt$$

$$I = \int\frac{dt}{5t(t+3)}$$

$$I = \frac{1}{5}\int\frac{1}{3}\left[\frac{1}{t} - \frac{1}{t+3}\right]dt$$

$$I = \frac{1}{15}[\log|t| - \log|t+3| + C$$

$$I = \frac{1}{15}\log\left[\frac{t}{t+3}\right] + C$$

$$I = \frac{1}{15}\log\left[\frac{x^5}{x^5+3}\right] + C \qquad [\because \text{Put } t = x^5]$$

S17. Given $\int \frac{1-\cos x}{\cos x\,(1+\cos x)}\,dx$

Let $\cos x = y$

$$\frac{1-\cos x}{\cos x\,(1+\cos x)} = \frac{1-y}{y\,(1+y)}$$

Let $$\frac{1-y}{y\,(1+y)} = \frac{A}{y} + \frac{B}{1+y} \qquad \text{... (i)}$$

$$\frac{1-y}{y\,(1+y)} = \frac{A(1+y)+By}{y\,(1+y)}$$

$$1 - y = A\,(1 + y) + By \qquad \text{... (ii)}$$

Put $y = 0$ in Eq. (ii)
$A = 1$

Put $y = -1$ in Eq. (ii)
$B = -2$

Put A and B in Eq. (i)

$$\frac{1-y}{y\,(1+y)} = \frac{1}{y} - \frac{2}{1+y}$$

$$\therefore \int \frac{1-\cos x}{\cos x\,(1+\cos x)}\,dx = \int \frac{dx}{\cos x} - 2\int \frac{dx}{1+\cos x}$$

$$= \int \sec x\,dx - 2\int \frac{dx}{2\cos^2 x/2}$$

$$= \log(\sec x + \tan x) - \int \sec^2 x/2\,dx$$

$$= \log(\sec x + \tan x) - 2\tan x/2 + C.$$

S18. Given $\int \frac{1}{x\log x\,(2+\log x)}\,dx$

Put $\log x = t \Rightarrow \frac{1}{x}dx = dt$

$$\Rightarrow \quad \int \frac{1}{t(2+t)}\,dt$$

Let $$\frac{1}{t(2+t)} = \frac{A}{t} + \frac{B}{t+2} \quad \text{... (i)}$$

$$1 = A(t+2) + Bt \quad \text{... (ii)}$$

Put $t = -2$ in Eq. (ii)

$$1 = -2B$$

$$B = -\frac{1}{2}$$

Put $t = 0$ in Eq. (ii)

$$2A = 1$$

$$A = \frac{1}{2}$$

Put the value of A and B in Eq. (i)

$$\frac{1}{t(2+t)} = \frac{1}{2t} - \frac{1}{2(t+2)}$$

$$\therefore \quad \int \frac{dt}{t(t+2)} = \frac{1}{2}\int \frac{dt}{t} - \frac{1}{2}\int \frac{dt}{t+2}$$

$$= \frac{1}{2}\log t - \frac{1}{2}\log(t+2) + C$$

$$= \frac{1}{2}\log\left(\frac{t}{t+2}\right) + C$$

$$= \frac{1}{2}\log\left(\frac{\log x}{\log x + 2}\right) + C.$$

S19. Given $\int \frac{\sin 2x}{(1+\sin x)(2+\sin x)}\,dx$

$$\int \frac{2\sin x \cos x}{(1+\sin x)(2+\sin x)}\,dx$$

Put $\sin x = t \quad \Rightarrow \quad \cos x\,dx = dt$

$$\Rightarrow \quad \int \frac{2t\,dt}{(1+t)(2+t)}$$

Let $$\frac{2t}{(t+1)(t+2)} = \frac{A}{t+1} + \frac{B}{t+2} \quad \text{... (i)}$$

$$2t = A(t+2) + B(t+1) \quad \text{... (ii)}$$

Put $t = -1$ in Eq. (ii) | Put $t = -2$ in Eq. (ii)

$A = -2$ | $B = 4$

Put the value of A and B in Eq. (i)

$$\frac{2t}{(t+1)(t+2)} = -\frac{2}{(t+1)} + \frac{4}{(t+2)}$$

$$\therefore \quad \int \frac{2t\,dt}{(t+1)(t+2)} = -2\int \frac{dt}{(t+1)} + 4\int \frac{dt}{(t+2)}$$

$$= -2\log(t+1) + 4\log(t+2) + C$$

$$= -2\log(\sin x + 1) + 4\log(\sin x + 2) + C.$$

S20. Given $\int \frac{5x}{(x+1)(x^2-4)}\,dx$

Let $$\frac{5x}{(x+1)(x^2-4)} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{x+2} \quad \text{... (i)}$$

$$5x = A(x^2-4) + B(x+1)(x+2) + C(x+1)(x-2) \quad \text{... (ii)}$$

Put $x = -1$ in Eq. (ii) | Put $x = 2$ in Eq. (ii) | Put $x = -2$ in Eq. (ii)

$-3A = -5$ | $10 = 12B$ | $-10 = 4C$

$A = \frac{5}{3}$ | $B = \frac{5}{6}$ | $C = -\frac{5}{2}$

Put the value of A, B and C in Eq. (i)

$$\frac{5x}{(x+1)(x^2-4)} = \frac{5}{3(x+1)} + \frac{5}{6(x-2)} - \frac{5}{2(x+2)}$$

$$\int \frac{5x}{(x+1)(x^2-4)}\,dx = \frac{5}{3}\int \frac{dx}{(x+1)} + \frac{5}{6}\int \frac{dx}{(x-2)} - \frac{5}{2}\int \frac{dx}{(x+2)}$$

$$= \frac{5}{3}\log(x+1) + \frac{5}{6}\log(x-2) - \frac{5}{2}\log(x+2) + C.$$

S21. Given $\int \frac{\cos x}{(1-\sin x)(2-\sin x)}\,dx$

Put $\sin x = t \Rightarrow \cos x\,dx = dt$

$$\int \frac{dt}{(1-t)(2-t)} = \int \frac{dt}{(t-1)(t-2)}$$

Let $$\frac{1}{(t-1)(t-2)} = \frac{A}{(t-1)} + \frac{B}{(t-2)} \quad \text{... (i)}$$

$$1 = A(t-2) + B(t-1) \quad \text{... (ii)}$$

Put $t = 1$ in Eq. (ii)

$A = -1$

Put $t = 2$ in Eq. (ii)

$B = 1$

Put A, B in Eq. (i)

$$\frac{1}{(t-1)(t-2)} = \frac{-1}{(t-1)} + \frac{1}{(t-2)}$$

$$\int \frac{dt}{(t-1)(t-2)} = -\int \frac{1}{(t-1)}\,dt + \int \frac{dt}{(t-2)}$$

$$= -\log(t-1) + \log(t-2) + C$$

$$= -\log(\sin x - 1) + \log(\sin x - 2) + C$$

$$= \log\frac{(\sin x - 2)}{(\sin x - 1)} + C.$$

S22. Let $$I = \int \frac{2}{(1-x)(1+x^2)}\,dx$$

Let $$\frac{2}{(1-x)(1+x^2)} = \frac{A}{1-x} + \frac{Bx+C}{1+x^2} \quad \text{... (i)}$$

$$\Rightarrow \quad \frac{2}{(1-x)(1+x^2)} = \frac{A(1+x^2) + (Bx+C)(1-x)}{(1-x)(1+x^2)}$$

$$\Rightarrow \quad 2 = A(1+x^2) + (Bx+C)(1-x)$$

$$\Rightarrow \quad 2 = A + Ax^2 + Bx + C - Bx^2 - Cx$$

$$\Rightarrow \quad 2 = (A-B)x^2 + (B-C)x + (A+C)$$

Comparing coefficient of x^2, x and constant terms on both sides, we get

$$A - B = 0 \quad \text{... (ii)}$$

$$B - C = 0 \quad \text{... (iii)}$$

$$A + C = 2 \quad \text{... (iv)}$$

Adding Eqs. (ii) and (iii), we get

$$A - C = 0 \quad \text{... (v)}$$

Adding Eqs. (iv) and (v), we get

$$2A = 2$$

$$\Rightarrow \quad A = 1$$

Putting $A = 1$ in Eq. (ii), we get

$$B = 1$$

Putting $B = 1$ in Eq. (iii), we get

$$C = 1$$

Hence, $A = 1, \quad B = 1 \quad \text{and} \quad C = 1$

$\because$ Eq. (i) becomes

$$\frac{2}{(1-x)(1+x^2)} = \frac{1}{1-x} + \frac{x+1}{1+x^2}$$

Integrating both sides w.r.t. x, we get

$$\int \frac{2}{(1-x)(1+x^2)}\,dx = \int \frac{1}{1-x}\,dx + \int \frac{x+1}{1+x^2}\,dx$$

$$= -\log|1-x| + \int \frac{x}{1+x^2}\,dx + \int \frac{1}{1+x^2}\,dx$$

$$= -\log|1-x| + \frac{1}{2}\log|1+x^2| + \tan^{-1}x + C$$

$$\left[\begin{array}{ll} \because & \int \frac{x}{1+x^2}\,dx \\ \text{Put} & 1+x^2 = t \\ \therefore & 2x\,dx = dt \quad or \quad x\,dx = dt/2 \\ & = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\log|t| = \frac{1}{2}\log|1+x^2| \end{array}\right]$$

Hence, $I = -\log|1-x| + \frac{1}{2}\log|1+x^2| + \tan^{-1}x + C$

S23. $$I = \int \frac{x^2+1}{(x^2+4)(x^2+25)}\,dx$$

Put $\frac{x^2+1}{(x^2+4)(x^2+25)} = \frac{A}{x^2+4} + \frac{B}{x^2+25}$ [By partial fraction]

At, $x = 0$

$\Rightarrow \quad \frac{A}{4} + \frac{B}{25} = \frac{1}{100}$

$\Rightarrow \quad 25A + 4B = 1$ ... (i)

At, $x = 1$

$\Rightarrow \quad \frac{2}{5 \times 26} = \frac{A}{5} + \frac{B}{26}$

$\Rightarrow \quad \frac{A}{5} + \frac{B}{26} = \frac{1}{65}$

$\Rightarrow \quad 13A + \frac{5}{2}B = 1$

$\Rightarrow \quad 26A + 5B = 2 \quad \text{... (ii)}$

Solving Eqs. (i) and (ii), we get

$$A = -\frac{1}{7},\ B = \frac{8}{7}$$

$$\therefore \quad I = -\frac{1}{7}\int\frac{dx}{x^2+4} + \frac{8}{7}\int\frac{dx}{x^2+25}$$

$$= -\frac{1}{7}\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + \frac{8}{7}\cdot\frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right) + C \quad \left[\because \int\frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C\right]$$

$$= -\frac{1}{14}\tan^{-1}\left(\frac{x}{2}\right) + \frac{8}{35}\tan^{-1}\left(\frac{x}{5}\right) + C$$

S24.

$$I = \frac{1}{2}\int\frac{2x^2+4+9-4-9}{(x^2+4)(x^2+9)}dx$$

$$= \frac{1}{2}\int\frac{x^2+4}{(x^2+4)(x^2+9)}dx + \frac{1}{2}\int\frac{x^2+9}{(x^2+4)(x^2+9)}dx - \frac{1}{2}\int\frac{13}{(x^2+4)(x^2+9)}dx$$

$$= \frac{1}{2}\int\frac{dx}{x^2+9} + \frac{1}{2}\int\frac{dx}{x^2+4} - \frac{1}{2}\int\frac{13\,dx}{(x^2+4)(x^2+9)}$$

$$\left[\because \frac{1}{(x^2+4)(x^2+9)} = \frac{1}{5}\left(\frac{1}{(x^2+4)} - \frac{1}{(x^2+9)}\right)\right]$$

$$= \frac{1}{2}\cdot\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + \frac{1}{2}\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}\cdot\frac{13}{5}\int\left(\frac{1}{(x^2+4)} - \frac{1}{(x^2+9)}\right)dx$$

$$\left[\because \int\frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{x}{3}\right) + \frac{1}{4}\tan^{-1}\left(\frac{x}{2}\right) - \frac{13}{10}\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + \frac{13}{10}\cdot\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{x}{3}\right) + \frac{1}{4}\tan^{-1}\left(\frac{x}{2}\right) - \frac{13}{20}\tan^{-1}\left(\frac{x}{2}\right) + \frac{13}{30}\tan^{-1}\frac{x}{3} + C$$

$$= \tan^{-1}\left(\frac{x}{3}\right)\left(\frac{1}{6} + \frac{13}{30}\right) + \tan^{-1}\left(\frac{x}{2}\right)\left(\frac{1}{4} - \frac{13}{20}\right) + C$$

$$= \tan^{-1}\left(\frac{x}{3}\right)\left(\frac{5+13}{30}\right) + \tan^{-1}\left(\frac{x}{2}\right)\left(\frac{5-13}{20}\right) + C$$

$$= \frac{18}{30}\tan^{-1}\left(\frac{x}{3}\right) - \frac{8}{20}\tan^{-1}\left(\frac{x}{2}\right) + C$$

$$= \frac{3}{5}\tan^{-1}\left(\frac{x}{3}\right) - \frac{2}{5}\tan^{-1}\left(\frac{x}{2}\right) + C$$

S25. Let $I = \int \frac{dx}{x(x^3+8)} = \int \frac{dx}{x(x^3+2^3)}$

$$I = \int \frac{dx}{x(x+2)(x^2-2x+4)}$$

Now, by partial fraction method

Let $\frac{1}{x(x+2)(x^2-2x+4)} = \frac{A}{x} + \frac{B}{x+2} + \frac{Cx+D}{x^2-2x+4}$ ... (i)

$\Rightarrow \quad 1 = A(x+2)(x^2-2x+4) + Bx(x^2-2x+4) + (Cx+D)(x^2+2x)$

$\Rightarrow \quad 1 = A(x^3 - 2x^2 + 4x + 2x^2 - 4x + 8) + B(x^3 - 2x^2 + 4x) + (Cx^3 + 2Cx^2 + Dx^2 + 2Dx)$

$\Rightarrow \quad 1 = A(x^3+8) + B(x^3 - 2x^2 + 4x) + (Cx^3 + 2Cx^2 + Dx^2 + 2Dx)$

$\Rightarrow \quad 1 = (A+B+C)x^3 + (-2B+2C+D)x^2 + (4B+2D)x + 8A$

On Comparing the like power of x, we get

$A + B + C = 0$ ... (ii)

$\Rightarrow \quad -2B + 2C + D = 0$ ... (iii)

$\Rightarrow \quad 4B + 2D = 0$ and $8A = 1$ ... (iv)

$\Rightarrow \quad A = \frac{1}{8}$

From Eqs. (iii) and (iv), we get

$-2B + 2C - 2B = 0$

$\Rightarrow \quad -4B + 2C = 0$

$\Rightarrow \quad C = 2B$ ... (v)

Put the values of C and A in Eq. (ii), we get

$\frac{1}{8} + B + 2B = 0$

$\Rightarrow \quad B = -\frac{1}{24}$ and $C = -\frac{1}{12}$

and $D = \dfrac{1}{12}$

Now, substituting the values of *A, B, C* and *D* in the eq. (i) , we get

$$\frac{1}{x(x+2)(x^2-2x+4)} = \frac{1}{8x} - \frac{1}{24(x+2)} + \frac{\frac{-x}{12}+\frac{1}{12}}{x^2-2x+4}$$

$$\therefore \quad I = \int\left[\frac{1}{8x} - \frac{1}{24(x+2)} + \frac{\frac{-x}{12}+\frac{1}{12}}{x^2-2x+4}\right]dx$$

$$= \frac{1}{8}\int\frac{dx}{x} - \frac{1}{24}\int\frac{dx}{x+2} - \frac{1}{12}\int\frac{x-1}{x^2-2x+4}\,dx$$

$$= \frac{1}{8}\log|x| - \frac{1}{24}\log|x+2| - \frac{1}{24}\int\frac{2x-2}{x^2-2x+4}\,dx$$

$$= \frac{1}{8}\log|x| - \frac{1}{24}\log|x+2| - \frac{1}{24}\log|x^2-2x+4| + C$$

$$= \frac{1}{8}\log|x| - \frac{1}{24}\log|(x+2)(x^2-2x+4)| + C$$

$$[\because \log m + \log n = \log mn]$$

$$= \frac{1}{8}\log|x| - \frac{1}{24}\log|(x^3+2^3)| + C$$

$$[\because \ (a+b)(a^2-ab+b^2) = a^3+b^3]$$

$$= \frac{1}{8}\left\{\log|x| - \frac{1}{3}\log|x^3+8|\right\} + C$$

$$= \frac{1}{8}\{\log|x| - \log|x^3+8|^{1/3}\} + C$$

$$= \frac{1}{8}\log\left|\frac{x}{(x^3+8)^{1/3}}\right| + C$$

S26. Let $I = \displaystyle\int\frac{dx}{x(x^3+1)}$

$$= \int\frac{dx}{x(x+1)(x^2-x+1)}$$

By partial fraction method

$$\frac{1}{x(x+1)(x^2-x+1)} = \frac{A}{x}+\frac{B}{x+1}+\frac{Cx+D}{x^2-x+1}$$

$$\Rightarrow \quad 1 = (x+1)(x^2-x+1)A + Bx(x^2-x+1)+x(x+1)(Cx+D)$$

$$\Rightarrow \quad 1 = (x^3-x^2+x+x^2-x+1)A + B(x^3-x^2+x)+(x^2+x)(Cx+D)$$

$$\Rightarrow \quad 1 = A(x^3+1)+B(x^3-x^2+x)+(Cx^3+Dx^2+Cx^2+Dx)$$

$$\Rightarrow \quad 1 = (A+B+C)x^3+(-B+D+C)x^2+(B+D)x+A$$

Comparing on both sides, we get

$$A + B + C = 0 \quad \text{... (i)}$$

$$-B + D + C = 0 \quad \text{... (ii)}$$

$$B + D = 0 \quad \text{... (iii)}$$

and $A = 1$ ... (iv)

From Eqs. (ii) and (iii),

$$C - 2B = 0 \quad \text{... (v)}$$

From Eqs. (i) and (iv),

$$B + C = -1 \quad \text{... (vi)}$$

From Eqs. (v) and (vi)

$$B = -\frac{1}{3} \text{ and } D = \frac{1}{3},\ C = -\frac{2}{3}$$

$$\therefore \quad \int\frac{dx}{x(x^3+1)} = \int\left[\frac{1}{x}-\frac{1}{3(x+1)}+\frac{-\frac{2x}{3}+\frac{1}{3}}{x^2-x+1}\right]dx$$

$$= \int\frac{dx}{x}-\frac{1}{3}\int\frac{dx}{x+1}+\frac{1}{3}\int\frac{1-2x}{x^2-x+1}dx$$

$$= \log|x|-\frac{1}{3}\log|x+1|-\frac{1}{3}\int\frac{2x-1}{x^2-x+1}dx$$

$$= \log|x|-\frac{1}{3}\log|x+1|-\frac{1}{3}\int\frac{dt}{t} \qquad \begin{bmatrix} Let & t = x^2-x+1 \\ \Rightarrow & dt=(2x-1)dx\end{bmatrix}$$

$$= \log|x|-\frac{1}{3}\log|x+1|-\frac{1}{3}\log|t|+C$$

$$= \log|x|-\frac{1}{3}\log|x+1|-\frac{1}{3}\log|x^2-x+1|+C \qquad [\because \text{ Put } t=x^2-x+1]$$

$$= \log|x|-\frac{1}{3}\log|(x+1)(x^2-x+1)|+C \qquad [\because \log m+\log n=\log mn]$$

$$= \log|x| - \frac{1}{3}\log|x^3+1| + C$$

$$= \log|x| - \log|x^3+1|^{1/3} + C$$

$$= \log\frac{|x|}{|x^3+1|^{1/3}} + C$$

S27. Given $\int \frac{1}{\cos x(5-4\sin x)}\,dx$

$$\int \frac{\cos x}{\cos^2 x(5-4\sin x)}\,dx = \int \frac{\cos x\,dx}{(1-\sin^2 x)(5-4\sin x)}$$

Put $\sin x = t \Rightarrow \cos x\,dx = dt$

$$\int \frac{dt}{(1-t^2)(5-4t)}$$

Let $\frac{1}{(1-t^2)(5-4t)} = \frac{A}{1-t} + \frac{B}{1+t} + \frac{C}{5-4t}$ ... (i)

$$1 = A(1+t)(5-4t) + B(5-4t)(1-t) + C(1-t^2) \quad \text{... (ii)}$$

Put $t = 1$ in Eq. (ii)	Put $t = -1$ in Eq. (ii)	Put $t = \frac{5}{4}$ in Eq. (ii)
$2A = 1$	$18B = 1$	$-\frac{9}{16}C = 1$
$A = \frac{1}{2}$	$B = \frac{1}{18}$	$C = -\frac{16}{9}$

Put the value of A, B and C in Eq. (i)

$$\therefore \quad \frac{1}{(1-t^2)(5-4t)} = \frac{1}{2(1-t)} + \frac{1}{18}\frac{1}{(1+t)} - \frac{16}{9}\cdot\frac{1}{(5-4t)}$$

$$\therefore \quad \int \frac{dt}{(1-t^2)(5-4t)} = \frac{1}{2}\int\frac{dt}{1-t} + \frac{1}{18}\int\frac{dt}{1+t} - \frac{16}{9}\int\frac{dt}{5-4t}$$

$$= -\frac{1}{2}\log(1-t) + \frac{1}{18}\log(1+t) + \frac{16}{9\times 4}\log(5-4t) + C$$

$$= -\frac{1}{2}\log(1-\sin x) + \frac{1}{18}\log(1+\sin x) + \frac{4}{9}\log(5-4\sin x) + C.$$

S28. Given $\int \frac{x^2}{(x-1)^3(x+1)}\,dx$

Let $\frac{x^2}{(x-1)^3(x+1)} = \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}+\frac{D}{x+1}$ ... (i)

$$x^2 = A(x-1)^2(x+1) + B(x-1)(x+1) + C(x+1) + D(x-1)^3 \qquad \text{... (ii)}$$

Put $x = 1$ in Eq. (ii)

$2C = 1$

$C = \frac{1}{2}$

Put $x = -1$ in Eq. (ii)

$-8D = 1$

$D = -\frac{1}{8}$

Put $x = 0$ in Eq. (ii)

$A - B + C - D = 0$

$A - B = -\frac{5}{8}$

Put $x = 2$ in Eq. (ii)

$$3A + 3B + 3C + D = 4 \quad \Rightarrow \quad A + B = \frac{7}{8}$$

Now $A - B = -\frac{5}{8}$ and $A + B = \frac{7}{8}$ $\Rightarrow$ $A = \frac{1}{8}$, $B = \frac{3}{4}$

Put the value of A, B, C and D in Eq. (i)

$$\frac{x^2}{(x-1)^3(x+1)} = \frac{1}{8(x-1)}+\frac{3}{4(x-1)^2}+\frac{1}{2(x-1)^3}-\frac{1}{8(x+1)}$$

$$\int \frac{x^2}{(x-1)^3(x+1)}\,dx = \frac{1}{8}\int\frac{1}{(x-1)}\,dx+\frac{3}{4}\int\frac{1}{(x-1)^2}\,dx+\frac{1}{2}\int\frac{1}{(x-1)^3}\,dx-\frac{1}{8}\int\frac{1}{(x+1)}\,dx$$

$$= \frac{1}{8}\log(x-1)-\frac{3}{4(x-1)}-\frac{1}{4(x-1)^2}-\frac{1}{8}\log(x+1)+C$$

$$= \frac{1}{8}\log\left(\frac{x-1}{x+1}\right)-\frac{3}{4(x-1)}-\frac{1}{4(x-1)^2}+C.$$

S29. Let

$$I = \int\frac{1-x^2}{x(1-2x)}\,dx = \int\frac{1-x^2}{x-2x^2}\,dx$$

$$= \int\left[\frac{1}{2}+\frac{1-\frac{1}{2}x}{x(1-2x)}\right]dx$$

$$\Rightarrow \quad I = \frac{1}{2}\int dx + \int\frac{1-\frac{1}{2}x}{x(1-2x)}\,dx \qquad \text{... (i)}$$

Let $\frac{1-\frac{1}{2}x}{x(1-2x)} = \frac{A}{x}+\frac{B}{1-2x}$

$$1-\frac{1}{2}x = A(1-2x)+Bx \quad \text{... (ii)}$$

Putting $x = 0$ and $x = \frac{1}{2}$ in Eq. (ii), we get

$$1 - 0 = A(1 - 0) + 0$$

$\Rightarrow \quad A = 1$

and $\quad 1-\frac{1}{2}\left(\frac{1}{2}\right) = A\left[1-2\left(\frac{1}{2}\right)\right]+B\left(\frac{1}{2}\right)$

$$1-\frac{1}{4} = A(1-1)+\frac{1}{2}B$$

$\Rightarrow \quad \frac{3}{4} = \frac{1}{2}B \Rightarrow B=\frac{3}{2}$

$\therefore$ From Eq. (ii), we get

$$I = \frac{1}{2}\int dx + \int\frac{1}{x}dx + \int\frac{3/2}{1-2x}dx$$

$$= \frac{1}{2}x+\log|x|+\frac{3}{2}\frac{\log|1-2x|}{-2}+C$$

$$= \frac{1}{2}x+\log|x|-\frac{3}{4}\log|1-2x|+C$$

S30. Let $\quad I = \int\frac{dx}{(x^2+1)(x^2+2)}$

For $\frac{1}{(x^2+1)(x^2+2)}$, put $x^2 = t$

$\therefore \quad \frac{1}{(x^2+1)(x^2+2)} = \frac{1}{(t+1)(t+2)}$

Let $\quad \frac{1}{(t+1)(t+2)} = \frac{A}{t+1}+\frac{B}{t+2}$

$\therefore \quad 1 = A(t + 2) + B(t + 1)$

Putting $t = -1$, we get

$$1 = A(-1 + 2) \Rightarrow A = 1$$

and put $t = -2$, we get

$$1 = B(-2+1)$$

$$\Rightarrow \quad B = -1$$

$$\therefore \quad \frac{1}{(t+1)(t+2)} = \frac{1}{(t+1)} - \frac{1}{(t+2)}$$

$$\therefore \quad I = \int \frac{1}{x^2+1}dx - \int \frac{1}{x^2+2}dx \qquad [\because t = x^2]$$

$$= \int \frac{1}{x^2+(1)^2}dx - \int \frac{1}{x^2+(\sqrt{2})^2}dx$$

$$= \frac{1}{1}\tan^{-1}\left(\frac{x}{1}\right) - \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C \qquad \left[\because \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C\right]$$

$$= \tan^{-1}x - \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$$

Alternate Method:

Let $I = \int \frac{dx}{(x^2+1)(x^2+2)}$ $\left[\because \frac{1}{(x^2+1)(x^2+2)} = \frac{x^2+2-x^2-1}{(x^2+1)(x^2+2)} = \frac{(x^2+2)}{(x^2+1)(x^2+2)} - \frac{x^2+1}{(x^2+1)(x^2+2)}\right]$

$$= \int \left[\frac{1}{x^2+1} - \frac{1}{x^2+2}\right]dx$$

$$= \int \frac{dx}{x^2+1} - \int \frac{dx}{x^2+(\sqrt{2})^2} \qquad \left[\because \int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$= \tan^{-1}x - \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C$$

S31. Let $I = \int \frac{2x}{(x^2+1)(x^2+3)}dx$

Put $x^2 = t \Rightarrow 2x\,dx = dt$

$$\therefore \quad I = \int \frac{1}{(t+1)(3+t)}dt$$

Let $\frac{1}{(t+1)(3+t)} = \frac{A}{t+1} + \frac{B}{3+t}$ ... (i)

$$1 = A(3+t) + B(1+t)$$

Put $t = -3$, we get

$$1 = -2B \Rightarrow B = -\frac{1}{2}$$

Now, put $t = -1$, we get

$$1 = 2A$$

$\Rightarrow$ $$A = \frac{1}{2}$$

Putting $A = \frac{1}{2}$ and $B = -\frac{1}{2}$ in Eq. (i), we get

$$\frac{1}{(1+t)(3+t)} = \frac{1/2}{1+t} + \frac{-1/2}{3+t}$$

$$\int \frac{1}{(1+t)(3+t)}\,dt = \frac{1}{2}\int \frac{1}{1+t}\,dt - \frac{1}{2}\int \frac{1}{3+t}\,dt$$

$$= \frac{1}{2}\log|1+t| - \frac{1}{2}\log|3+t| \qquad \left[\because \int \frac{dx}{x} = \log|x| + C\right]$$

$$= \frac{1}{2}\log\left|1+x^2\right| - \frac{1}{2}\log\left|3+x^2\right| + C$$

$\therefore$ $$I = \frac{1}{2}\log\left|\frac{1+x^2}{3+x^2}\right| + C \qquad \left[\because \log m - \log n = \log\frac{m}{n}\right]$$

S32. Given $\int \frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}\,dx$

Let $\frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)} = 1 + \frac{A}{x-4} + \frac{B}{x-5} + \frac{C}{x-6}$ ... (i)

$$\frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)} = \frac{(x-4)(x-5)(x-6) + A(x-5)(x-6) + B(x-4)(x-6) + C(x-4)(x-5)}{(x-4)(x-5)(x-6)}$$

$$(x-1)(x-2)(x-3) = (x-4)(x-5)(x-6) + A(x-5)(x-6) + B(x-4)(x-6) + C(x-4)(x-5) \quad \text{... (ii)}$$

Put $x = 4$ in Eq. (ii)	Put $x = 5$ in Eq. (ii)	Put $x = 6$ in Eq. (ii)
$A = 3$	$B = -24$	$C = 30$

Put A, B and C in Eq. (i)

$$\frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)} = 1 + \frac{3}{x-4} - \frac{24}{x-5} + \frac{30}{x-6}$$

$$\int \frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}\,dx = \int 1\,dx + 3\int \frac{1}{x-4}\,dx - 24\int \frac{1}{x-5}\,dx + 30\int \frac{1}{x-6}\,dx$$

$$= x + 3\log(x-4) - 24\log(x-5) + 30\log(x-6) + C.$$

S33. Let $$I = \int \frac{x^2+1}{(x-1)^2(x+3)}\,dx$$

Let $$= \frac{x^2+1}{(x-1)^2(x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3} \quad \text{... (i)}$$

$$\Rightarrow \quad \frac{x^2+1}{(x-1)^2(x+3)} = \frac{A(x-1)(x+3)+B(x+3)+C(x-1)^2}{(x-1)^2(x+3)}$$

$$\Rightarrow \quad x^2+1 = A(x-1)(x+3)+B(x+3)+C(x-1)^2$$

$$\Rightarrow \quad x^2+1 = A(x^2+2x-3)+B(x+3)+C(x^2+1-2x)$$

$$\Rightarrow \quad x^2+1 = (A+C)x^2+(2A+B-2C)x+(-3A+3B+C)$$

Comparing coefficients of x^2, x and constant on both sides, we get

$$A + C = 1 \quad \text{... (ii)}$$

$$2A + B - 2C = 0 \quad \text{... (iii)}$$

$$-3A + 3B + C = 1 \quad \text{... (iv)}$$

Multiply Eq. (iii) by 3 and subtracting it from Eq. (iv) we get

$$-3A + 3B + C = 1$$

$$\underset{-}{}6A \underset{+}{} \underset{-}{}3B \underset{+}{-} 6C = 0$$

$$-9A + 7C = 1 \quad \text{... (v)}$$

Multiply Eq. (ii) by 7 and subtracting it from Eq. (v) we get

$$-9A + 7C = 1$$

$$\underset{-}{}7A \underset{+}{} \underset{-}{}7C = \underset{-}{}7$$

$$-16A = -6 \quad \text{... (vi)}$$

$$\therefore \quad A = \frac{6}{16} = \frac{3}{8}$$

Put $A = \frac{3}{8}$ in Eq. (ii), we get

$$\frac{3}{8} + C = 1$$

$$\Rightarrow \quad C = 1 - \frac{3}{8} = \frac{5}{8}$$

Put $A = \frac{3}{8}$ and $C = \frac{5}{8}$ in Eq. (iii), we get

$$\frac{3}{4} + B - \frac{5}{4} = 0$$

$$\Rightarrow \quad B - \frac{2}{4} = 0$$

$$\Rightarrow \quad B = \frac{2}{4} = \frac{1}{2}$$

$\therefore \qquad A = \frac{3}{8}, \quad B = \frac{1}{2} \quad \text{and} \quad C = \frac{5}{8}$

$\therefore$ Eq. (i) becomes

$$\frac{x^2+1}{(x-1)^2(x+3)} = \frac{3/8}{x-1} + \frac{1/2}{(x-1)^2} + \frac{5/8}{x+3}$$

Integrating both sides, we get

$$\int \frac{x^2+1}{(x-1)^2(x+3)}\,dx = \frac{3}{8}\int \frac{dx}{x-1} + \frac{1}{2}\int \frac{dx}{(x-1)^2} + \frac{5}{8}\int \frac{dx}{x+3}$$

$$= \frac{3}{8}\log|x-1| + \frac{1}{2}\left(\frac{-1}{x-1}\right) + \frac{5}{8}\log|x+3| + C$$

Hence, $\qquad I = \frac{3}{8}\log|x-1| - \frac{1}{2(x-1)} + \frac{5}{8}\log|x+3| + C$

S34. $\qquad I = \int \frac{(3\sin\phi - 2)\cos\phi}{5 - \cos^2\phi - 4\sin\phi}\,d\phi$

$$= \int \frac{(3\sin\phi - 2)\cos\phi}{5 - (1 - \sin^2\phi) - 4\sin\phi}\,d\phi = \int \frac{(3\sin\phi - 2)\cos\phi}{4 - 4\sin\phi + \sin^2\phi}\,d\phi$$

Put $\sin\phi = t$ so that $\cos\phi\, d\phi = dt$.

$$I = \int \frac{3t-2}{4-4t+t^2}\,dt = \int \frac{3t-2}{(t-2)^2}\,dt = \int \frac{3(t-2)+4}{(t-2)^2}\,dt$$

$$= 3\int \frac{1}{t-2}\,dt + 4\int (t-2)^{-2}\,dt$$

$$= 3\log|t-2| + 4 \cdot \frac{(t-2)^{-2+1}}{-2+1} + c$$

$$= 3\log|t-2| - \frac{4}{t-2} + c$$

$$= 3\log|\sin\phi - 2| - \frac{4}{\sin\phi - 2} + c$$

$$= 3\log(2 - \sin\phi) + \frac{4}{2 - \sin\phi} + c$$

$$[\because \sin\phi - 2 < 0 \Rightarrow |\sin\phi - 2| = -(\sin\phi - 2) = 2 - \sin\phi]$$

S35.

$$I = \int \frac{1}{\sin x + \sin 2x} dx$$

$$= \int \frac{1}{\sin x + 2\sin x \cos x} dx = \int \frac{1}{\sin x (1 + 2\cos x)} dx$$

Put $\cos x = t$ so that $-\sin x\, dx = dt$ *i.e.* $dx = -\frac{dt}{\sin x}$.

$\therefore$
$$I = \int \frac{1}{\sin x(1+2t)}\left(-\frac{dt}{\sin x}\right)$$

$$= -\int \frac{dt}{\sin^2 x(1+2t)} = -\int \frac{dt}{(1-\cos^2 x)(1+2t)}$$

$$= -\int \frac{dt}{(1-t^2)(1+2t)} = -\int \frac{dt}{(1-t)(1+t)(1+2t)}$$

Let
$$\frac{1}{(1-t)(1+t)(1+2t)} = \frac{A}{1-t} + \frac{B}{1+t} + \frac{C}{1+2t} \quad \text{...(i)}$$

$\Rightarrow$
$$1 = A(1+t)(1+2t) + B(1-t)(1+2t) + C(1-t)(1+t)$$

Putting $t = 1$, $\quad 1 = A(2)(3) \quad \Rightarrow \quad A = \frac{1}{6}$.

Putting $t = -1$, $\quad 1 = B(2)(-1) \quad \Rightarrow \quad B = -\frac{1}{2}$.

Putting $t = -\frac{1}{2}$, $\quad 1 = C\left(\frac{3}{2}\right)\left(\frac{1}{2}\right) \quad \Rightarrow \quad C = \frac{4}{3}$.

$\therefore$
$$\frac{1}{(1-t)(1+t)(1+2t)} = \frac{1/6}{1-t} + \frac{-1/2}{1+t} + \frac{4/3}{1+2t}$$

$\therefore$
$$-\int \frac{1}{(1-t)(1+t)(1+2t)} dt = -\frac{1}{6}\int \frac{1}{1-t} dt + \frac{1}{2}\int \frac{1}{1+t} dt - \frac{2}{3}\int \frac{2}{1+2t} dt$$

$$= \frac{1}{6}\log|1-t| + \frac{1}{2}\log|1+t| - \frac{2}{3}\log|1+2t| + c$$

$\therefore$ Hence
$$I = \frac{1}{6}\log|1-\cos x| + \frac{1}{2}\log|1+\cos x| - \frac{2}{3}\log|1+2\cos x| + c$$

S36. Given $\int \frac{\cos x}{(1-\sin x)^3(2+\sin x)}\,dx$

Put $\sin x = t \;\Rightarrow\; \cos x\,dx = dt$

$$\int \frac{dt}{(1-t)^3(2+t)}$$

Let $\frac{1}{(1-t)^3(2+t)} = \frac{A}{(1-t)} + \frac{B}{(1-t)^2} + \frac{C}{(1-t)^3} + \frac{D}{(2+t)}$ ... (i)

$$1 = A(1-t)^2(2+t) + B(1-t)(2+t) + C(2+t) + D(1-t)^3 \quad \text{... (ii)}$$

Put $t = 1$ in Eq. (ii)

$3C = 1$

$C = \frac{1}{3}$

Put $t = -2$ in Eq. (ii)

$-27D = 1$

$D = -\frac{1}{27}$

Comparing the coff. of t^2 and t^3

$A - D = 0$ and

$3D - B = 0$

$\therefore \quad A = \frac{1}{27}, \quad B = -\frac{1}{9}$

Put the value of A, B, C and D in Eq. (i)

$$\frac{1}{(1-t)^3(2+t)} = \frac{1}{27(1-t)} - \frac{1}{9(1-t)^2} + \frac{1}{3(1-t)^3} - \frac{1}{27(2+t)}$$

$$\int \frac{dt}{(1-t)^3(2+t)} = \frac{1}{27}\int \frac{dt}{1-t} - \frac{1}{9}\int \frac{dt}{(1-t)^2} + \frac{1}{3}\int \frac{dt}{(1-t)^3} - \frac{1}{27}\int \frac{dt}{2+t}$$

$$= -\frac{1}{27}\log(1-t) - \frac{1}{9}\times\frac{-1}{-(1-t)} + \frac{1}{3}\times\frac{-1}{-2(1-t)^2} - \frac{1}{27}\log(2+t)$$

$$= -\frac{1}{27}\log(1-t) - \frac{1}{9(1-t)} + \frac{1}{6(1-t)^2} - \frac{1}{27}\log(2+t) + C$$

$$= -\frac{1}{27}\log(1-\sin x) - \frac{1}{9(1-\sin x)} + \frac{1}{6(1-\sin x)^2} - \frac{1}{27}\log(2+\sin x) + C.$$

S37. Given $\int \frac{(x+1)\,dx}{x(1+xe^x)}$

Put $xe^x = t \;\Rightarrow\; (xe^x + e^x)\,dx = dt$

$(x+1)\,e^x\,dx = dt$

$$= \int \frac{dt}{t(t+1)}$$

Let $\frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1}$ ... (i)

$$1 = A(t+1) + Bt \quad \text{... (ii)}$$

Put $t = 0$ in Eq. (ii) | Put $t = -1$ in Eq. (ii)

$A = 1$ | $B = -1$

Put A, B in Eq. (i)

$$\frac{1}{t(t+1)} = \frac{1}{t} - \frac{1}{t+1}$$

$$\int \frac{dt}{t(t+1)} = \int \frac{dt}{t} - \int \frac{dt}{t+1}$$

$$= \log t - \log(t+1) + C$$

$$= \log xe^x - \log(1 + xe^x) + C$$

$$= \log\left(\frac{xe^x}{1+xe^x}\right) + C.$$

S38. Given $\int \frac{dx}{\sin x(3+2\cos x)}$

$$\Rightarrow \int \frac{\sin x\, dx}{\sin^2 x(3+2\cos x)} = \int \frac{\sin x\, dx}{(1-\cos^2 x)(3+2\cos x)}$$

Put $\cos x = t \Rightarrow -\sin x\, dx = dt$

$$\Rightarrow -\int \frac{dt}{(1-t^2)(3+2t)} = \int \frac{dt}{(t^2-1)(2t+3)}$$

$$\Rightarrow \frac{1}{(t^2-1)(2t+3)} = \frac{A}{t-1} + \frac{B}{t+1} + \frac{C}{(2t+3)} \quad \text{... (i)}$$

$$1 = A(t+1)(2t+3) + B(t-1)(2t+3) + C(t^2-1) \quad \text{... (ii)}$$

Put $t = 1$ in Eq. (ii) | Put $t = -1$ in Eq. (ii) | Put $t = -3/2$

$10A = 1$ | $-2B = 1$ | $\frac{5}{4}C = 1$

$A = \frac{1}{10}$ | $B = -\frac{1}{2}$ | $C = \frac{4}{5}$

Put A, B and C in Eq. (i),

$$\frac{1}{(t^2-1)(2t+3)} = \frac{1}{10(t-1)} - \frac{1}{2(t+1)} + \frac{4}{5(2t+3)}$$

$$\therefore \quad \int \frac{dt}{(t^2-1)(2t+3)} = \frac{1}{10}\int \frac{dt}{t-1} - \frac{1}{2}\int \frac{dt}{t+1} + \frac{4}{5}\int \frac{dt}{2t+3}$$

$$= \frac{1}{10}\log(t-1) - \frac{1}{2}\log(t+1) + \frac{4}{5\times 2}\log(2t+3) + C$$

$$= \frac{1}{10}\log(\cos x - 1) - \frac{1}{2}\log(\cos x + 1) + \frac{2}{5}\log(2\cos x + 3) + C.$$

S39. Given $\int \frac{\cos\theta d\theta}{(2+\sin\theta)(3+4\sin\theta)}$

Put $\sin\theta = t \Rightarrow \cos\theta d\theta = dt$

$$\int \frac{\cos\theta d\theta}{(2+\sin\theta)(3+4\sin\theta)} = \int \frac{dt}{(2+t)(3+4t)}$$

Let

$$\frac{1}{(2+t)(3+4t)} = \frac{A}{2+t} + \frac{B}{3+4t} \quad \text{... (i)}$$

$$\frac{1}{(2+t)(3+4t)} = \frac{A(3+4t)+B(2+t)}{(2+t)(3+4t)}$$

$$1 = A(3+4t) + B(2+t) \quad \text{... (ii)}$$

Put $t = -3/4$ in Eq. (ii)

$B = \frac{4}{5}$

Put $t = -2$ in Eq. (ii)

$A = -\frac{1}{5}$

Put the value of A and B in Eq. (i)

$$\therefore \quad \frac{1}{(2+t)(3+4t)} = -\frac{1}{5}\cdot\frac{1}{(2+t)} + \frac{4}{5}\cdot\frac{1}{(3+4t)}$$

$$\therefore \quad \int \frac{1}{(2+t)(3+4t)}dt = -\frac{1}{5}\int \frac{dt}{2+t} + \frac{4}{5}\int \frac{dt}{3+4t}$$

$$= -\frac{1}{5}\log(2+t) + \frac{4}{5}\cdot\frac{1}{4}\log(3+4t) + C$$

$$= -\frac{1}{5}\log(2+\sin\theta) + \frac{1}{5}\log(3+4\sin\theta) + C.$$

S40. Given $\int \frac{4x^4+3}{(x^2+2)(x^2+3)(x^2+4)}\,dx$

Put $x^2 = y$

Then $\frac{4x^4+3}{(x^2+2)(x^2+3)(x^2+4)} = \frac{4y^2+3}{(y+2)(y+3)(y+4)}$

Let $\frac{4y^2+3}{(y+2)(y+3)(y+4)} = \frac{A}{y+2} + \frac{B}{y+3} + \frac{C}{y+4}$ ... (i)

$$4y^2 + 3 = A\,(y+3)(y+4) + B\,(y+2)(y+4) + C\,(y+2)(y+3) \quad \text{... (ii)}$$

Put $y = -3$ in Eq. (ii)	Put $y = -4$ in Eq. (ii)	Put $y = -2$ in Eq. (ii)
$-B = 39$	$2C = 67$	$2A = 19$
$B = -39$	$C = \frac{67}{2}$	$A = \frac{19}{2}$

Put A, B, C in Eq. (i)

$$\therefore \quad \frac{4y^4+3}{(y+2)(y+3)(y+4)} = \frac{19}{2(y+2)} - \frac{39}{(y+3)} + \frac{67}{2(y+4)}$$

$$\therefore \int \frac{4x^4+3}{(x^2+2)(x^2+3)(x^2+4)}\,dx = \frac{19}{2}\int\frac{dx}{x^2+2} - 39\int\frac{dx}{x^2+3} + \frac{67}{2}\int\frac{dx}{x^2+4}$$

$$= \frac{19}{2}\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) - 39\cdot\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + \frac{67}{2\times 2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

$$= \frac{19}{2\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) - \frac{39}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + \frac{67}{4}\tan^{-1}\left(\frac{x}{2}\right) + C.$$

S41. Let $I = \int\frac{\tan x + \tan^3 x}{1+\tan^3 x}\,dx$

$$= \int\frac{\tan x(1+\tan^2 x)}{1+\tan^3 x}\,dx$$

$$= \int\frac{\tan x\,.\sec^2 x}{1+\tan^3 x}\,dx \qquad [\because 1+\tan^2 x = \sec^2 x]$$

Put $\tan x = t$

$\Rightarrow \quad \sec^2 x\, dx = dt$

$\therefore \quad I = \int \frac{t\,dt}{1+t^3} \qquad \left[\because (a^3+b^3) = (a+b)(a^2-ab+b^2)\right]$

$= \int \frac{t\,dt}{(1+t)(1+t^2-t)}$

Let $\frac{t}{(1+t)(1+t^2-t)} = \frac{A}{1+t} + \frac{Bt+C}{t^2-t+1}$

$t = A(t^2 - t + 1) + (Bt + C)(1 + t)$ ...(i)

$\Rightarrow \quad t = (A + B)t^2 + (-A + B + C)t + (A + C)$

Putting $t = -1$ in eq. (i) we get

$-1 = A(1 + 1 + 1) + 0$

$\Rightarrow \quad A = \frac{-1}{3}$

Equating the coefficient of t^2 and constant terms

$\therefore \quad A + B = 0$

$\Rightarrow \quad \frac{-1}{3} + B = 0$

$\therefore \quad B = \frac{1}{3}$

$A + C = 0$

$\Rightarrow \quad -\frac{1}{3} + C = 0$

$\therefore \quad C = \frac{1}{3}$

$\therefore \quad I = \int \frac{-1/3}{1+t}dt + \int \frac{\frac{1}{3}t + \frac{1}{3}}{t^2-t+1}dt$

$= -\frac{1}{3}\int \frac{dt}{1+t} + \frac{1}{3}\int \frac{t+1}{t^2-t+1}dt$

$= -\frac{1}{3}\int \frac{dt}{1+t} + \frac{1}{6}\int \frac{2t-1}{t^2-t+1}dt + \frac{3}{6}\int \frac{dt}{t^2-t+1}$

Let $z = t^2 - t + 1$

$dz = (2t - 1)dt$

$\therefore \quad \int \frac{2t-1}{t^2-t+1}dt = \int \frac{dz}{z} = \log z$

$$= -\frac{1}{3}\log|1+t| + \frac{1}{6}\log|t^2 - t + 1| + \frac{1}{2}\int \frac{dt}{\left(t-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= -\frac{1}{3}\log|1+\tan x| + \frac{1}{6}\log|\tan^2 x - \tan x + 1| + \frac{1}{2}\cdot\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{t-\frac{1}{2}}{\sqrt{3}/2}\right) + C$$

$[\because t = \tan x]$

$$= -\frac{1}{3}\log|1+\tan x| + \frac{1}{6}\log|\tan^2 x - \tan x + 1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan x - 1}{\sqrt{3}}\right) + C$$

S42. $\int \frac{x^2 + x + 1}{(x+2)(x^2+1)} dx$

Decompose the rational fraction into partial fractions,

Let, $$\frac{x^2+x+1}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1} \quad ...(i)$$

$$= \frac{A(x^2+1) + (Bx+C)(x+2)}{(x+2)(x^2+1)}$$

$\therefore$ $x^2 + x + 1 = A(x^2 + 1) + (Bx + C)(x + 2)$ ... (ii)

$\Rightarrow$ $x^2 + x + 1 = (A + B)x^2 + (C + 2B)x + (A + 2C)$

Putting $x = -2$ in Eq. (ii), we get

$4 - 2 + 1 = A(4 + 1) + 0$

$\Rightarrow$ $3 = 5A$

$\Rightarrow$ $A = \frac{3}{5}$

Equating the coefficient of x^2 and constant terms, we get

$A + B = 1$ ...(iii)

$A + 2C = 1$ ...(iv)

Putting $A = \frac{3}{5}$ in Eq. (iii), we get

$\frac{3}{5} + B = 1$

$\Rightarrow \quad B = 1-\frac{3}{5}=\frac{2}{5}$

Putting $A = \frac{3}{5}$ in Eq. (iv), we get

$$\frac{3}{5}+2C = 1$$

$\Rightarrow \quad 2C = 1 - \frac{3}{5}$

$\Rightarrow \quad 2C = \frac{2}{5}$

$\Rightarrow \quad C = \frac{1}{5}$

Putting A,B,C in eq. (i)

$$\therefore \quad I = \int\frac{3/5}{x+2}dx + \int\frac{\frac{2}{5}x+\frac{1}{5}}{x^2+1}dx$$

$$I = \frac{3}{5}\int\frac{dx}{x+2}+\frac{2}{5}\int\frac{xdx}{x^2+1}+\frac{1}{5}\int\frac{dx}{x^2+1}$$

Put $\quad x^2 + 1 = t \quad \Rightarrow 2x\,dx = dt$

$$\therefore \quad I = \frac{3}{5}\int\frac{dx}{x+2}+\frac{1}{5}\int\frac{dt}{t}+\frac{1}{5}\int\frac{dx}{(x^2)+(1)^2}$$

$$\left[\because \int\frac{dx}{x}=\log x \text{ and} \int\frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$= \frac{3}{5}\log|x+2|+\frac{1}{5}\log|t|+\frac{1}{5}\cdot\frac{1}{1}\tan^{-1}\frac{x}{1}+C$$

$$= \frac{3}{5}\log|x+2|+\frac{1}{5}\log|x^2+1|+\frac{1}{5}\tan^{-1}x+C$$

S43. $$I = \int\frac{2\sin 2\theta-\cos\theta}{6-\cos^2\theta-4\sin\theta}d\theta = \int\frac{4\sin\theta\cos\theta-\cos\theta}{6-(1-\sin^2\theta)-4\sin\theta}d\theta$$

$$= \int\frac{(4\sin\theta-1).\cos\theta}{5-4\sin\theta+\sin^2\theta}d\theta$$

Put $\sin\theta = t$ so that $\cos\theta\, d\theta = dt$.

$$\therefore \quad I = \int\frac{(4t-1)}{5-4t+t^2}dt = \int\frac{2(2t-4)+7}{t^2-4t+5}dt$$

$$= 2\int\frac{2t-4}{t^2-4t+5}dt+7\int\frac{1}{(t-2)^2+1}dt$$

$= 2\log|t^2 - 4t + 5| + 7\tan^{-1}(t-2) + c$

$= 2\log|\sin^2\theta - 4\sin\theta + 5| + 7\tan^{-1}(\sin\theta - 2) + c$

Date: 19/10/2021

Q1. Evaluate $\int \frac{\cos x}{\sin x + \sqrt{\sin x}} dx$

Q2. Evaluate $\int \frac{dx}{\sqrt{(2x - x^2)^3}}$

Q3. Evaluate $\int \frac{1}{(x^2 - 1)\sqrt{x - 1}} dx$

Q4. Evaluate $\int \frac{1}{(x^2 + 4)\sqrt{x^2 - 4}} dx$

Q5. Evaluate

$$\int \left(\frac{1 - \sqrt{x}}{1 + \sqrt{x}}\right)^{1/2} \frac{dx}{x}.$$

Q6. Evaluate $\int \frac{\sqrt{x^2 + 1}}{x} dx$

Date: 19/10/2021

S1. Let $$I = \int \frac{\cos x}{\sin x + \sqrt{\sin x}}\, dx$$

Put $\sin x = t$ so that $\cos x\, dx = dt$.

$$\therefore \quad I = \int \frac{dt}{t + \sqrt{t}}$$

Put $\sqrt{t} = u$ *i.e.* $t = u^2$ so that $dt = 2u\, du$.

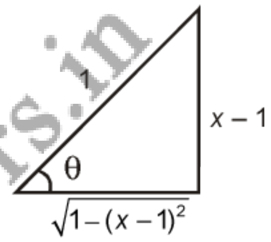
$$\therefore \quad I = \int \frac{2u\, du}{u^2 + u} = 2\int \frac{du}{u+1} = 2\log|u+1| + c$$

$$= 2\log|\sqrt{t}+1| + c = 2\log|\sqrt{\sin x}+1| + c$$

S2. Let $$I = \int \frac{dx}{\sqrt{(2x-x^2)^3}} = \int \frac{dx}{[1-(x^2-2x+1)]^{3/2}} = \int \frac{dx}{[1-(x-1)^2]^{3/2}}$$

Put $x - 1 = \sin\theta$ so that $dx = \cos\theta\, d\theta$.

$$I = \int \frac{\cos\theta d\theta}{(1-\sin^2\theta)^{3/2}} = \int \frac{\cos\theta}{(\cos^2\theta)^{3/2}} d\theta = \int \frac{\cos\theta}{\cos^3\theta} d\theta$$



$$= \int \frac{1}{\cos^2\theta} d\theta = \int \sec^2\theta\, d\theta = \tan\theta + c$$

$$= \frac{x-1}{\sqrt{1-(x-1)^2}} + c = \frac{x-1}{\sqrt{2x-x^2}} + c.$$

S3. Let $$I = \int \frac{1}{(x^2-1)\sqrt{x-1}}\, dx$$

Put $x = 1 + t^2$

so that $dx = 2t\, dt$.

$$I = \int \frac{2t\, dt}{[(1+t^2)^2 - 1]t} = \int \frac{2\, dt}{[t^4 + 1 + 2t^2 - 1]}$$

$$= \int \frac{2dt}{t^4 + 2t^2} = \int \frac{t^2 + 2 - t^2}{t^2(t^2+2)}\, dt$$

$$= \int \frac{t^2+2}{t^2(t^2+2)}\,dt - \int \left(\frac{t^2}{t^2(t^2+2)}\right) dt$$

$$= \int \left(\frac{1}{t^2} - \frac{1}{t^2+2}\right) dt = \int t^{-2} dt - \int \frac{1}{(\sqrt{2})^2 + t^2}\,dt$$

$$= \frac{t^{-2+1}}{-2+1} - \frac{1}{\sqrt{2}} \tan^{-1} \frac{t}{\sqrt{2}} + c$$

$$= -\frac{1}{t} - \frac{1}{\sqrt{2}} \tan^{-1} \frac{t}{\sqrt{2}} + c$$

$$= -\frac{1}{\sqrt{x-1}} - \frac{1}{\sqrt{2}} \tan^{-1} \sqrt{\frac{x-1}{2}} + c$$

S4. Let $$I = \int \frac{1}{(x^2+4)\sqrt{x^2-4}}\,dx$$

Put $x = \frac{1}{t}$ so that $dx = -\frac{1}{t^2}dt.$

$$I = \int \frac{-\frac{1}{t^2}dt}{\left(\frac{1}{t^2}+4\right)\sqrt{\frac{1}{t^2}-4}}$$

$$= -\int \frac{t\,dt}{(1+4t^2)\sqrt{1-4t^2}}$$

Put $\sqrt{1-4t^2} = u$ *i.e.*, $1-4t^2 = u^2$

so that $-8t\,dt = 2u\,du$ $\Rightarrow t\,dt = -\frac{1}{4}\,u\,du$

Also $1 + 4t^2 = 1 + 1 - u^2 = 2 - u^2$

$\therefore$ $$I = -\int \frac{-\frac{1}{4}u\,du}{(2-u^2)\cdot u} = \frac{1}{4}\int \frac{du}{(\sqrt{2})^2 - u^2}$$

$$= \frac{1}{4}\cdot\frac{1}{2\sqrt{2}} \log\left|\frac{\sqrt{2}+u}{\sqrt{2}-u}\right| + c \qquad \left[\because \int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c\right]$$

$$= \frac{1}{8\sqrt{2}} \log\left|\frac{\sqrt{2}+\sqrt{1-4t^2}}{\sqrt{2}-\sqrt{1-4t^2}}\right| + c$$

$$= \frac{1}{8\sqrt{2}}\log\left|\frac{\sqrt{2}+\sqrt{1-\frac{4}{x^2}}}{\sqrt{2}-\sqrt{1-\frac{4}{x^2}}}\right|+c$$

$$= \frac{1}{8\sqrt{2}}\log\left|\frac{\sqrt{2}x+\sqrt{x^2-4}}{\sqrt{2}x-\sqrt{x^2-4}}\right|+c$$

S5. Let $$I = \int\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)^{1/2}\frac{dx}{x}.$$

Put $x = t^2$ so that $dx = 2t\,dt$.

$\therefore$ $$I = \int\sqrt{\frac{1-t}{1+t}}\cdot\frac{2t\,dt}{t^2} = 2\int\frac{1-t}{t\sqrt{1-t^2}}dt$$

$$= 2\int\frac{1}{t\sqrt{1-t^2}}dt - 2\int\frac{1}{\sqrt{1-t^2}}dt \qquad \text{...(i)}$$

Let $$I_1 = \int\frac{1}{t\sqrt{1-t^2}}dt$$

Put $t = \frac{1}{z}$ so that $dt = -\frac{1}{z^2}dz.$

$\therefore$ $$I_1 = \int\frac{-\frac{1}{z^2}dz}{\frac{1}{z}\sqrt{1-\frac{1^2}{z^2}}} = -\int\frac{dz}{\sqrt{z^2-1}} \qquad \left[\therefore \frac{dx}{\sqrt{x^2-a^2}} = \log|x+\sqrt{x^2+a^2}|+c\right]$$

$$= -\log\left|z+\sqrt{z^2-1}\right| = -\log\left|\frac{1}{t}+\sqrt{\frac{1}{t^2}-1}\right|$$

$$= -\log\left|\frac{1+\sqrt{1-t^2}}{t}\right|$$

and $$I_2 = \int\frac{1}{\sqrt{1-t^2}}dt = \sin^{-1}t$$

$\therefore$ From (i), $$I = -2\log\left|\frac{1+\sqrt{1-t^2}}{t}\right| - 2\sin^{-1}t + c$$

$$= -2\log\left|\frac{1+\sqrt{1-x}}{\sqrt{x}}\right| - 2\sin^{-1}\sqrt{x} + c.$$

S6. Let $I = \int \frac{\sqrt{x^2+1}}{x}\,dx = \int \frac{x^2+1}{x\sqrt{x^2+1}}\,dx = \int \frac{x}{\sqrt{x^2+1}}\,dx + \int \frac{1}{x\sqrt{x^2+1}}\,dx$ ...(i)

Now let $I_1 = \int \frac{x}{\sqrt{x^2+1}}\,dx = \int \frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x)\,dx = \frac{1}{2}\cdot\frac{(x^2+1)^{\frac{1}{2}}}{\frac{1}{2}}$

$= (x^2+1)^{\frac{1}{2}} = \sqrt{x^2+1}$

And for $I_2 = \int \frac{1}{x\sqrt{x^2+1}}\,dx$

put $x = \frac{1}{t}$ so that $dx = \frac{-1}{t^2}dt$

$\therefore$ $I_2 = \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{\frac{1}{t^2}+1}} = -\int \frac{dt}{\sqrt{1+t^2}} = -\log\left|t+\sqrt{1+t^2}\right|$

$\left[\therefore \int \frac{dx}{\sqrt{a^2+x^2}} = \log\left(x+\sqrt{a^2+x^2}\right)+c\right]$

$= -\log\left|\frac{1}{x}+\sqrt{1+\frac{1}{x^2}}\right|$

$= -\log\left|\frac{1+\sqrt{1+x^2}}{x}\right|.$

$\therefore$ From (i), $I = \sqrt{x^2+1} - \log\left|\frac{1+\sqrt{1+x^2}}{x}\right| + c.$

Date: 19/10/2021

Q1. Evaluate

$$\int_0^{\frac{\pi}{4}} \tan x \, dx$$

Q2. Prove that

$$\int_0^{\pi} \sin^2 x \cos^2 x \, dx = \frac{\pi}{8}.$$

Q3. Evaluate

$$\int_0^{\frac{\pi}{4}} \left(2\sec^2 x + x^3 + 2\right) dx.$$

Q4. Evaluate

$$\int_0^1 \frac{1}{1+x^2} dx$$

Q5. Evaluate

$$\int_2^3 \frac{1}{x} dx$$

Q6. Evaluate

$$\int_0^2 \sqrt{4-x^2} \, dx$$

Q7. Evaluate

$$\int_1^{\sqrt{3}} \frac{dx}{1+x^2}$$

Q8. Evaluate

$$\int_0^{1/\sqrt{2}} \frac{1}{\sqrt{1-x^2}} dx$$

Q9. Evaluate

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin x} \cos x \, dx.$$

Q10. Evaluate

$$\int_0^1 \frac{e^x}{1+e^{2x}} dx.$$

Q11. Evaluate

$$\int_{2}^{3} \frac{x}{x^2+1} dx.$$

Q12. Evaluate

$$\int_{-1}^{1} x^3 (x^4+1)^3 \, dx.$$

Q13. Evaluate

$$\int_{-1}^{1} \frac{dx}{x^2+2x+5}.$$

Q14. Prove that

$$\int_{0}^{2\pi} \cos^5 x \, dx = 0.$$

Q15. Evaluate

$$\int_{0}^{1} |x-5| \, dx.$$

Q16. Evaluate

$$\int_{0}^{1} \frac{\tan^{-1} x}{1+x^2} \, dx.$$

Q17. Evaluate

$$\int_{0}^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} \, dx.$$

Q18. Evaluate

$$\int_{2}^{5} |x-1| \, dx.$$

Q19. Evaluate

$$\int_{0}^{\frac{\pi}{6}} (1-\cos 3\theta) \sin 3\theta \, d\theta.$$

Q20. Evaluate

$$\int_{0}^{\frac{\pi}{3}} \frac{\sec x \tan x}{1+\sec^2 x} dx.$$

Q21. Evaluate

$$\int_{-\pi/4}^{\pi/4} \sin^3 x \, dx$$

Q22. Evaluate

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \, dx.$$

Q23. Evaluate

$$\int_{2}^{3} \frac{dx}{x^2 - 1}$$

Q24. If $\int_{0}^{k} \frac{dx}{2+8x^2} = \frac{\pi}{16}$, find the value of k.

Q25. Evaluate

$$\int_{0}^{1} \left(xe^x + \cos \frac{\pi x}{4} \right) dx.$$

Q26. Evaluate

$$\int_0^1 \frac{2x}{1+x^2} dx$$

Q27. Evaluate

$$\int_{3}^{4} \frac{dx}{\sqrt{x^2+4}}.$$

Q28. Evaluate

$$\int_{0}^{\frac{\pi}{4}} \sin^3 2t \cos 2t \, dt.$$

Q29. Evaluate

$$\int_{4}^{9} \frac{\sqrt{x}}{\left(30 - x^{\frac{3}{2}}\right)^2} dx.$$

Q30. Prove that

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x \, dx = \frac{\pi}{4} - \frac{1}{2}.$$

Q31. Evaluate

$$\int_{1}^{2} e^{2x} \left(\frac{1}{x} - \frac{1}{2x^2} \right) dx$$

Q32. Evaluate

$$\int_{0}^{1} x(1-x)^5 \, dx$$

Q33. Evaluate

$$\int_0^{\pi} |\cos x|\, dx$$

Q34. Prove that

$$\int_0^{\frac{\pi}{2}} \sin^2 x\, dx = \frac{\pi}{4}.$$

Q35. Evaluate

$$\int_0^{\frac{\pi}{2}} \log\left(\frac{3+5\cos x}{3+5\sin x}\right) dx.$$

Q36. Evaluate

$$\int_{-1}^{1} e^{|x|}\, dx$$

Q37. Evaluate

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(5\sin^3 x + 8\sin x + 4\cos^2 x\right) dx.$$

Q38. Evaluate

$$\int_0^{\pi/2} \frac{x+\sin x}{1+\cos x}\, dx$$

Q39. If $f(x)$ is of the form $f(x) = a + bx + cx^2$, show that:

$$\int_0^1 f(x)\, dx = \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right].$$

Q40. Prove that

$$\int_0^1 \sin^{-1} x\, dx = \frac{\pi}{2} - 1.$$

Q41. Prove that

$$\int_0^{\frac{\pi}{4}} 2\tan^3 x\, dx = 1 - \log 2.$$

Q42. Evaluate

$$\int_{\pi/2}^{\pi} e^x \left(\frac{1-\sin x}{1-\cos x}\right) dx$$

Q43. Evaluate

$$\int_0^2 \frac{1}{4+x-x^2}\, dx$$

Q44. Evaluate

$$\int_0^1 \frac{x^4+1}{x^2+1}dx$$

Q45. Evaluate

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{(1+\cos x)^2}\,dx.$$

Q46. Evaluate

$$\int_{\pi/4}^{\pi/2} \cos 2x \log \sin x\, dx$$

Q47. Evaluate

$$\int_0^{2\pi} e^x \sin\left(\frac{\pi}{4}+\frac{x}{2}\right)dx$$

Q48. If f and g are continuous on $[0, a]$ and satisfy $f(x) = f(a-x)$ and $g(x) + g(a-x) = 2$, show that

$$\int_0^a f(x)\,g(x)\,dx = \int_0^a f(x)\,dx$$

Q49. Using properties of definite integrals, evaluate the following

$$\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx$$

Q50. Evaluate

$$\int_1^3 \left[|x-1|+|x-2|+|x-3|\right]dx$$

Q51. Evaluate

$$\int_{-1}^2 |x^3-x|\,dx$$

Q52. Evaluate

$$\int_0^2 |x^2+2x-3|\,dx$$

Q53. Evaluate

$$\int_0^1 |5x-3|\,dx$$

Q54. Evaluate

$$\int_{-5}^5 |x+2|\,dx$$

Q55. Evaluate

$$\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{\sin^4 x+\cos^4 x}\,dx.$$

Q56. Find the value of

$$\int_{\frac{1}{3}}^{1} \frac{(x - x^3)^{\frac{1}{3}}}{x^4}\,dx$$

Q57. Evaluate

$$\int_0^1 \tan^{-1}\left(\frac{2x}{1-x^2}\right)dx.$$

Q58. Evaluate

$$\int_0^1 \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)dx.$$

Q59. Evaluate

$$\int_0^{\pi/4} \tan^3 x\,dx$$

Q60. Evaluate

$$\int_0^{\pi/2} \frac{1}{4\sin^2 x + 5\cos^2 x}\,dx$$

Q61. Evaluate

$$\int_0^1 \frac{x\tan^{-1} x}{(1+x^2)^{3/2}}\,dx$$

Q62. Evaluate

$$\int_0^{1/\sqrt{2}} \frac{\sin^{-1} x}{(1-x^2)^{3/2}}\,dx$$

Q63. Evaluate

$$\int_0^{\pi/2} \frac{\cos\theta}{(1+\sin\theta)(2+\sin\theta)}\,d\theta$$

Q64. Evaluate

$$\int_0^{\pi/2} \sqrt{\cos\theta}\,\sin^3\theta\,d\theta$$

Q65. Evaluate

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$$

Q66. Evaluate

$$\int_0^{\pi/4} \log(1+\tan x)dx$$

Q67. Evaluate

$$\int_0^{\pi} \frac{x\tan x}{\sec x\,\mathrm{cosec}\,x}dx$$

Q68. Using properties of integrals, evaluate:

$$\int_{-\pi/2}^{\pi/2} f(x)\,dx\,, \quad \text{where} \quad f(x) = \sin|x| + \cos|x|$$

Q69. Evaluate

$$\int_0^{2\pi} \frac{1}{1+e^{\sin x}}\,dx$$

Q70. Evaluate

$$\int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x}+e^{-\cos x}}\,dx$$

Q71. Prove that

$$\int_0^{\pi} x\,f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx.$$

Q72. Evaluate

$$\int_{-\pi/4}^{\pi/4} \frac{x+\frac{\pi}{4}}{2-\cos 2x}\,dx$$

Q73. Prove that

$$\int_0^{2\pi} \frac{x\cos x}{1+\cos x} = 2\pi^2.$$

Q74. Evaluate

$$\int_2^{8} \frac{\sqrt[3]{x+1}}{\sqrt[3]{x+1}+\sqrt[3]{11-x}}\,dx.$$

Q75. Evaluate

$$\int_0^{\pi} x\sin^3 x\,dx.$$

Q76. Evaluate

$$\int_0^{1} \frac{\log(1+x)}{1+x^2}dx$$

Q77. Evaluate

$$\int_0^{\frac{\pi}{2}} \log\tan x\,dx.$$

Q78. Evaluate

$$\int_0^{\pi/2} \frac{\cos^5 x}{\sin^5 x+\cos^5 x}\,dx$$

Q79. Evaluate

$$\int_0^{\frac{\pi}{2}} \frac{dx}{5+4\sin x}.$$

Q80. Evaluate

$$\int_0^{\infty} \frac{1}{(x^2 + a^2)(x^2 + b^2)} dx$$

Q81. Evaluate

$$\int_0^{\pi/2} \frac{1}{2 \cos x + 4 \sin x} dx$$

Q82. Evaluate

$$\int_0^{\pi/2} (\sqrt{\tan x} + \sqrt{\cot x})\, dx$$

Q83. Evaluate

$$\int_0^{\pi/2} \frac{\cos x}{(3 \cos x + \sin x)} dx$$

Q84. Evaluate :

$$\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4 \sin^2 x} dx.$$

Q85. Evaluate

$$\int_0^1 \log\left(\frac{1}{x} - 1\right) dx$$

Q86. Find the value of

$$\int_0^{\frac{\pi}{2}} \log\left(\frac{4 + 3 \sin x}{4 + 3 \cos x}\right) dx$$

Q87. Evaluate

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \quad \text{or} \quad \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{1}{2}} x}{\sin^{\frac{1}{2}} x + \cos^{\frac{1}{2}} x} dx.$$

Q88. Evaluate

$$\int_0^{\pi/2} \frac{\cos x}{1 + \cos x + \sin x} dx$$

Q89. If $I_n = \int_0^{\pi/4} \tan^n x\, dx$, prove that $I_n + I_{n+2} = \frac{1}{n+1}$.

Q90. Prove that

$$\int_0^1 \sin^{-1}\left(\frac{2x}{1 + x^2}\right) dx = \frac{\pi}{2} - \log 2.$$

Q91. Evaluate

$$\int_0^{\pi/2} \frac{\sin^2 x}{1 + \sin x \cos x} dx$$

Q92. Evaluate

$$\int_0^{\pi/2} \log \sin x \, dx$$

Q93. Evaluate

$$\int_0^1 \cot^{-1}[1 - x + x^2] dx$$

Q94. Evaluate

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x} dx.$$

Q95. Evaluate

$$\int_1^4 (|x-1| + |x-2| + |x-4|) dx$$

Q96. Evaluate

$$\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{\cos^2 x + \sin^4 x} dx.$$

Q97. Prove that

$$\int_0^{\pi} \frac{x \tan x}{\sec x + \cos x} dx = \frac{\pi^2}{4}.$$

Q98. Evaluate

$$\int_0^{\pi} \frac{x}{1 + \sin x} dx$$

Q99. Evaluate

$$\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx$$

Q100Evaluate

$$\int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx$$

Q101Evaluate

$$\int_{-a}^{a} \sqrt{\frac{a-x}{a+x}} dx$$

Q102Evaluate

$$\int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\tan x}}$$

Q103Evaluate

$$\int_{-1}^{\frac{3}{2}} | x \sin \pi x | dx.$$

Q104Evaluate

$$\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}\,dx$$

Q105Evaluate

$$\int_0^{\pi/2} 2\sin x \cos x \tan^{-1}(\sin x)dx$$

Q106Evaluate

$$\int_0^{\pi} \frac{x\,dx}{a^2\cos^2 x + b^2\sin^2 x}$$

Q107Prove that :

$$\int_0^{2\pi} \frac{x\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\,dx = \pi^2$$

Q108Show that:

$$\int_0^{\pi/2} f(\sin 2x)\sin x\,dx = \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x\,dx$$

Q109Evaluate

$$\int_0^{\pi/2} \frac{x\sin x\cos x}{\sin^4 x + \cos^4 x}\,dx$$

Q110Evaluate

$$\int_0^1 x\left(\tan^{-1} x\right)^2 dx = \frac{\pi}{4}\left(\frac{\pi}{4}-1\right)+\frac{1}{2}\log 2.$$

Q111Evaluate

$$\int_0^{\pi/2} \frac{x}{\sin x + \cos x}\,dx$$

Q112Evaluate

$$\int_0^{a} \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx$$

Q113Evaluate

$$\int_0^{\pi/2} (2\log\sin x - \log\sin 2x)dx$$

Q114Prove that

$$\int_0^{\frac{\pi}{2}} \log(\tan x + \cot x)\,dx = \pi\log 2.$$

Q115Evaluate

$$\int_0^{\pi} \log(1+\cos x)\,dx.$$

Q116 Prove that :

$$\int_0^{\frac{\pi}{4}} \left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx = \sqrt{2}\,\frac{\pi}{2}.$$

Q117 Evaluate :

$$\int_0^{\pi} \frac{x}{1+\sin\alpha \sin x}\, dx.$$

Q118 Evaluate

$$\int_0^{\pi} \frac{x}{1+\sin^2 x}\, dx.$$

Q119 Evaluate

$$\int_0^{\pi} \frac{x}{4-\cos^2 x}\, dx.$$

Q120 Evaluate

$$\int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin 2x}}\, dx$$

Date: 19/10/2021

S1.

$$\int_0^{\frac{\pi}{4}} \tan x\, dx = \left[-\log|\cos x|\right]_0^{\frac{\pi}{4}}$$

$$= -\log\left|\cos\frac{\pi}{4}\right| + \log|\cos 0| = -\log\left|\frac{1}{\sqrt{2}}\right| + \log|1|$$

$$= -\log\frac{1}{\sqrt{2}} + \log 1 = -\left(\log 1 - \log\sqrt{2}\right) + \log 1$$

$$= -\left(0 - \log\sqrt{2}\right) + 0 = \log\sqrt{2} = \frac{1}{2}\log 2.$$

S2.

$$I = \frac{1}{4}\int_0^{\pi} 4\sin^2 x\cos^2 x\, dx = \frac{1}{4}\int_0^{\pi}\sin^2 2x\, dx$$

$$= \frac{1}{4}\int_0^{\pi}\frac{1-\cos 4x}{2}\, dx = \frac{1}{8}\left[x - \frac{\sin 4x}{4}\right]_0^{\pi}$$

$$= \frac{1}{8}\left[(\pi - 0) - (0 - 0)\right] = \frac{\pi}{8}$$

S3.

$$I = \int_0^{\frac{\pi}{4}} (2\sec^2 x + x^3 + 2)dx$$

$$I = \left[2\tan x + \frac{x^4}{4} + 2x\right]_0^{\frac{\pi}{4}}$$

$$= \left[2\tan\frac{\pi}{4} + \frac{\pi^4}{1024} + 2\left(\frac{\pi}{4}\right)\right] - [0 + 0 + 0]$$

$$= 2(1) + \frac{\pi^4}{1024} + \frac{\pi}{2} = \frac{\pi^4}{1024} + \frac{\pi}{2} + 2$$

S4. Given integral

$$\int_0^1 \frac{1}{1+x^2}dx = [\tan^{-1} x]_0^1 \qquad \left[\because \int \frac{1}{1+x^2}dx = \tan^{-1} x + C\right]$$

$$= \tan^{-1} 1 - \tan^{-1} 0$$

$$= \tan^{-1}\tan\frac{\pi}{4} - \tan^{-1}\tan 0^\circ \qquad \left[\because 1 = \tan\frac{\pi}{4} \text{ and } 0 = \tan 0^\circ\right]$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

S5.

$$\int_2^3 \frac{1}{x}dx = \log|x|_2^3 = \log 3 - \log 2$$

$$= \log\frac{3}{2} \qquad \left[\because \log m - \log n = \log\frac{m}{n}\right]$$

S6.

$$\therefore \int_0^2 \sqrt{4-x^2}\,dx = \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2 \qquad \left[\because \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c\right]$$

$$= 0 + 2\sin^{-1} 1$$

$$= 2 \times \frac{\pi}{2} = \pi.$$

S7. Given integral is $\int_1^{\sqrt{3}} \frac{dx}{1+x^2}$

$$\therefore \int_1^{\sqrt{3}} \frac{dx}{1+x^2} = [\tan^{-1} x]_1^{\sqrt{3}} \qquad \left[\because \int\frac{dx}{1+x^2} = \tan^{-1} x + C\right]$$

$$= \tan^{-1}\sqrt{3} - \tan^{-1} 1$$

$$= \tan^{-1}\tan\frac{\pi}{3} - \tan^{-1}\frac{\pi}{4} \qquad \left[\because \sqrt{3} = \tan\frac{\pi}{3} \text{ and } 1 = \tan\frac{\pi}{4}\right]$$

$$= \frac{\pi}{3} - \frac{\pi}{4} = \frac{4\pi - 3\pi}{12} = \frac{\pi}{12}$$

S8. Given integral is $\int_0^{1/\sqrt{2}} \frac{1}{\sqrt{1-x^2}}dx$

$$= [\sin^{-1} x]_0^{1/\sqrt{2}} \qquad \left[\because \int\frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + C\right]$$

$$= \sin^{-1}\frac{1}{\sqrt{2}} - \sin^{-1} 0$$

$$= \sin^{-1}\sin\frac{\pi}{4} - \sin^{-1}\sin 0 \qquad \left[\because \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4} \text{ and } 0 = \sin 0^\circ\right]$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

S9. Put $\sin x = t$ so that $\cos x\, dx = dt$

When $x = 0$, $t = 0$. When $x = \frac{\pi}{2}$, $t = \sin\frac{\pi}{2} = 1$

$$I = \int_0^1 t^{\frac{1}{2}} dt = \left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 = \frac{2}{3}[1-0] = \frac{2}{3}$$

S10. Let $$I = \int_0^1 \frac{e^x}{1+e^{2x}} dx$$

Put $e^x = t$ so that $e^x\, dx = dt$

When $x = 0$, $t = e^0 = 1$. When $x = 1$, $t = e^1 = e$

$$I = \int_1^e \frac{dt}{1+t^2} = \left[\tan^{-1} t\right]_1^e = \tan^{-1}e - \tan^{-1}1$$

$$= \tan^{-1} e - \frac{\pi}{4}$$

S11. Put $x^2 + 1 = t$ so that $2x\, dx = dt$ *i.e.* $x\, dx = \frac{1}{2}dt$

When $x = 2$, $t = 4 + 1 = 5$; when $x = 3$, $t = 9 + 1 = 10$

$$I = \int_5^{10} \frac{\frac{1}{2}dt}{t} = \frac{1}{2}\left[\log|t|\right]_5^{10}$$

$$= \frac{1}{2}[\log 10 - \log 5] = \frac{1}{2}\log\frac{10}{5} = \frac{1}{2}\log 2$$

S12. Put $x^4 + 1 = t$ so that $4x^3\, dx = dt$ *i.e.*, $x^3\, dx = \frac{1}{4}dt$

When $x = -1$, $t = 1 + 1 = 2$

When $x = 1$, $t = 1 + 1 = 2$

$$I = \int_2^2 t^3 \frac{dt}{4} = \frac{1}{4}\int_2^2 t^3 dt = 0$$ [Both limits are same]

S13.

$$I = \int_{-1}^{1} \frac{dx}{(x+1)^2 + 2^2} = \left[\frac{1}{2}\tan^{-1}\frac{x+1}{2}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\tan^{-1}(1) - \tan^{-1}(0)\right] = \frac{1}{2}\left[\frac{\pi}{4} - 0\right] = \frac{\pi}{8}$$

S14. Since $\cos^5(2\pi - x) = (-\cos x)^5 = -\cos^5 x$ $\left[\therefore \int_0^{2a} f(x)dx = 0, \text{ if } f(2a - x) = -f(x)\right]$

$\therefore$ $$\int_0^{2\pi} \cos^5 x\, dx = 0.$$

S15.
$$I = \int_0^1 |x - 5|\, dx \qquad [\because x < 1 \Rightarrow x < 5 \Rightarrow x - 5 < 0 \Rightarrow |x - 5| = -(x - 5)]$$
$$= \int_0^1 -(x - 5)\, dx = -\left[\frac{x^2}{2} - 5x\right]_0^1 = -\left[\frac{1}{2} - 5(1)\right] = \frac{9}{2}$$

S16. Put $\tan^{-1} x = t \Rightarrow x = \tan t$

So that $\frac{dx}{1+x^2} = dt$.

When $x = 0$, $t = \tan^{-1}(0) = 0$.

When $x = 1$, $t = \tan^{-1}(1) = \frac{\pi}{4}$

$$I = \int_0^{\frac{\pi}{4}} t\, dt = \left[\frac{t^2}{2}\right]_0^{\frac{\pi}{4}} = \frac{1}{2}\left[\frac{\pi^2}{16} - 0\right] = \frac{\pi^2}{32}$$

S17. Put $\cos x = t$ so that $-\sin x\, dx = dt$ *i.e.*, $\sin x\, dx = -dt$

When $x = 0$, $t = \cos 0 = 1$. When $x = \frac{\pi}{2}$, $t = \cos\frac{\pi}{2} = 0$

$$I = \int_1^0 \frac{-dt}{1+t^2} = -\int_1^0 \frac{dt}{1+t^2}$$
$$= -\left[\tan^{-1} t\right]_1^0 = -[\tan^{-1}(0) - \tan^{-1}(1)]$$
$$= -\left[0 - \frac{\pi}{4}\right] = \frac{\pi}{4}$$

S18. Let $$I = \int_2^5 |x - 1|\, dx$$
$$= \int_2^5 (x - 1)\, dx \qquad [\because x > 1 \Rightarrow x - 1 > 0 \therefore |x - 1| = x - 1]$$

$$= \left[\frac{x^2}{2} - x\right]_2^5 = \left(\frac{25}{2} - 5\right) - \left(\frac{4}{2} - 2\right)$$

$$= \frac{15}{2} - 0 = \frac{15}{2}.$$

S19. Put $1 - \cos 3\theta = t$

So that $3 \sin 3\theta \, d\theta = dt$ *i.e.*, $\sin 3\theta \, d\theta = \frac{1}{3} dt$

When $\theta = 0$, $t = 1 - 1 = 0$.

When $\theta = \frac{\pi}{6}$, $t = 1 - \cos\frac{\pi}{2} = 1 - 0 = 1$

$$I = \int_0^1 t \frac{dt}{3} = \frac{1}{3}\left[\frac{t^2}{2}\right]_0^1 = \frac{1}{3}\left[\frac{1}{2} - 0\right] = \frac{1}{6}$$

S20. Let $I = \int_0^{\frac{\pi}{3}} \frac{\sec x \tan x}{1 + \sec^2 x} dx$

Put $\sec x = t$ so that $\sec x \tan x \, dx = dt$

When $x = 0$, $t = \sec 0 = 1$. When $x = \frac{\pi}{3}$, $t = \sec\frac{\pi}{3} = 2$

$$I = \int_1^2 \frac{dt}{1 + t^2} = [\tan^{-1} t]_1^2 = \tan^{-1} 2 - \tan^{-1} 1 = \tan^{-1} 2 - \frac{\pi}{4}$$

S21. As $\int_{-a}^{a} f(x)\,dx = 0$, if $f(-x) = -f(x)$

Let $f(x) = \sin^3 x$

and $f(-x) = \sin^3(-x)$

$= -\sin^3 x$

$= -f(x)$

$\therefore \int_{-\pi/4}^{\pi/4} \sin^3 x \, dx = 0$

S22. Here, $f(x) = \cos x$

Since $f(-x) = \cos(-x) = \cos x = f(x)$,

$\therefore \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \, dx = 2\int_0^{\frac{\pi}{2}} \cos x \, dx = 2\,[\sin x]_0^{\frac{\pi}{2}}$

$$= 2\left[\sin\frac{\pi}{2} - \sin 0\right] = 2\ (1 - 0) = 2.$$

S23.

$$\int_2^3 \frac{dx}{x^2-1} = \left[\frac{1}{2(1)}\log\left|\frac{x-1}{x+1}\right|\right]_2^3 = \frac{1}{2}\left[\log\left|\frac{3-1}{3+1}\right| - \log\left|\frac{2-1}{2+1}\right|\right]$$

$$\left[\because \int\frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)\right]$$

$$= \frac{1}{2}\left[\log\left|\frac{2}{4}\right| - \log\left|\frac{1}{3}\right|\right] = \frac{1}{2}\left[\log\frac{1}{2} - \log\frac{1}{3}\right]$$

$$= \frac{1}{2}[-\log 2 + \log 3] = \frac{1}{2}\log\frac{3}{2}.$$

S24.

$$\int_0^k \frac{dx}{2+8x^2} = \frac{1}{8}\int_0^k \frac{dx}{\left(\frac{1}{2}\right)^2 + x^2} = \frac{1}{8}\cdot\frac{1}{\frac{1}{2}}\left[\tan^{-1}\frac{x}{\frac{1}{2}}\right]_0^k$$

$$= \frac{1}{4}[\tan^{-1}2x]_0^k = \frac{1}{4}\left[\tan^{-1}2k - \tan^{-1}0\right]$$

$$= \frac{1}{4}\left[\tan^{-1}2k - 0\right] = \frac{1}{4}\tan^{-1}2k$$

By the question, $\frac{1}{4}\tan^{-1}2k = \frac{\pi}{16} \Rightarrow \tan^{-1}2k = \frac{\pi}{4}$

$\Rightarrow$ $2k = \tan\frac{\pi}{4} \Rightarrow 2k = 1$

Hence, $k = \frac{1}{2}$

S25.

$$\int_0^1\left(xe^x + \cos\frac{\pi x}{4}\right)dx = \int_0^1 xe^x\ dx + \int_0^1 \cos\frac{\pi x}{4}dx$$

$$= [xe^x]_0^1 - \int_0^1 (1)\,e^x dx + \left[\frac{\sin\frac{\pi x}{4}}{\frac{\pi}{4}}\right]_0^1 \qquad \text{[Integrating first by parts]}$$

$$= [xe^x - e^x]_0^1 + \frac{4}{\pi}\left[\sin\frac{\pi x}{4}\right]_0^1$$

$$= [(e-e)-(0-1)] + \frac{4}{\pi}\left[\sin\frac{\pi}{4} - \sin 0\right]$$

$$= 0 + 1 + \frac{4}{\pi}\left[\frac{1}{\sqrt{2}} - 0\right] = 1 + \frac{2\sqrt{2}}{\pi}.$$

S26. The differentiation of denominator is in numerator. So, solve substitution method.

Given Integral is $\int_0^1 \frac{2x}{1+x^2}dx$

Put $1 + x^2 = t$ L.L. = when $x = 0$, $t = 1$

$\Rightarrow$ $2x\,dx = dt$ *U.L.* = When $x = 1$, $t = 2$

$\therefore$ *We get* $\int_1^2 \frac{dt}{t} = [\log t]_1^2$

$= \log 2 - \log 1$

$= \log 2$ $[\because \log 1 = 0]$

S27.

$$\int_3^4 \frac{dx}{\sqrt{x^2+4}} = \left[\log\left|x + \sqrt{x^2+4}\right|\right]_3^4 \qquad \left[\because \int \frac{dx}{\sqrt{x^2+a^2}} = \log\left(x + \sqrt{x^2+a^2}\right)\right]$$

$$= \log\left|4 + \sqrt{16+4}\right| - \log\left|3 + \sqrt{9+4}\right|$$

$$= \log\left|\frac{4+\sqrt{20}}{3+\sqrt{13}}\right| = \log\frac{4+\sqrt{20}}{3+\sqrt{13}}$$

S28. Put $\sin 2t = z$ so that $2\cos 2t\,dt = dz$

$$\cos 2t\,dt = \frac{1}{2}dz$$

When $t = 0$, $z = 0$. When $t = \frac{\pi}{4}$, $z = \sin\frac{\pi}{2} = 1$.

$$I = \int_0^1 z^3 \frac{1}{2}dz = \frac{1}{2}\int_0^1 z^3\,dz$$

$$= \frac{1}{2}\left[\frac{z^4}{4}\right]_0^1 = \frac{1}{8}[1-0] = \frac{1}{8}$$

S29.
Put $30 - x^{\frac{3}{2}} = t$ so that $0 - \frac{3}{2}x^{\frac{1}{2}}dx = dt$

i.e., $\sqrt{x}dx = -\frac{2}{3}dt.$

When $x = 4$, $t = 30 - 4^{\frac{3}{2}} = 30 - 8 = 22$

When $x = 9$, $t = 30 - 9^{\frac{3}{2}} = 30 - 27 = 3$

$$I = \int_{22}^{3} \frac{-\frac{2}{3}dt}{t^2} = -\frac{2}{3}\int_{22}^{3} t^{-2}dt$$

$$= -\frac{2}{3}\left[\frac{t^{-1}}{-1}\right]_{22}^{3} = \frac{2}{3}\left[\frac{1}{t}\right]_{22}^{3} = \frac{2}{3}\left[\frac{1}{3} - \frac{1}{22}\right]$$

$$= \frac{2}{3}\left[\frac{19}{66}\right] = \frac{19}{99}$$

S30.

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x\, dx = 2\int_{0}^{\frac{\pi}{4}} \sin^2 x\, dx \qquad [\because f(x) = \sin^2 x \text{ is an even funtion}]$$

$$= 2\int_{0}^{\frac{\pi}{4}} \frac{1-\cos 2x}{2}\, dx = \int_{0}^{\frac{\pi}{4}} (1-\cos 2x)\, dx$$

$$= \left[x - \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}} = \left(\frac{\pi}{4} - \frac{\sin\frac{\pi}{2}}{2}\right) - (0-0) = \frac{\pi}{4} - \frac{1}{2}$$

S31. Given integral is $\int_1^2 e^{2x}\left(\frac{1}{x} - \frac{1}{2x^2}\right)dx$

$$= \int_1^2 \underset{\text{II}}{e^{2x}}\, \underset{\text{I}}{\frac{1}{x}}\, dx - \int_1^2 e^{2x}\, \frac{1}{2x^2}\, dx$$

$$= \left[\frac{1}{2x}e^{2x}\right]_1^2 + \int_1^2 \frac{1}{2x^2}e^{2x}\, dx - \int_1^2 e^{2x}\, \frac{1}{2x^2}\, dx$$

$$= \left(\frac{1}{4}e^4 - \frac{1}{2}e^2\right) = \frac{e^4 - 2e^2}{4}$$

S32.

$$\int_0^1 (1-x)\,(1-(1-x))^5 dx = \int_0^1 (1-x)(1-(1-x))^5 dx = \int_0^1 (x)^5(1-x)dx$$

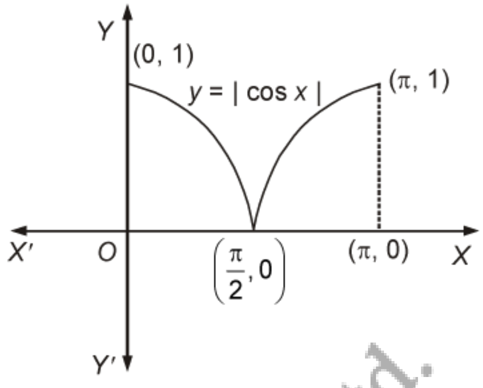
$$= \int_0^1 x^5 dx - \int_0^1 x^6 dx$$

$$\left[\frac{x^6}{6}\right]_0^1 - \left[\frac{x^7}{7}\right]_0^1 = \left[\frac{1}{6}-0\right]-\left[\frac{1}{7}-0\right]$$

$$= \frac{1}{6}-\frac{1}{7}=\frac{7-6}{42}=\frac{1}{42}.$$

S33.

Clearly, $$|\cos x| = \begin{cases} \cos x \text{ when } 0 \le x \le \frac{\pi}{2} \\ -\cos x \text{ when } \frac{\pi}{2} \le x \le \pi \end{cases}$$



$\therefore$ $$\int_0^{\pi} |\cos x|\, dx = \int_0^{\pi/2} |\cos x|\, dx + \int_{\pi/2}^{\pi} |\cos x|\, dx$$

$\Rightarrow$ $$\int_0^{\pi} |\cos x|\, dx = \int_0^{\pi/2} \cos x\, dx + \int_{\pi/2}^{\pi} (-\cos x)\, dx$$

$\Rightarrow$ $$\int_0^{\pi} |\cos x|\, dx = [\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{\pi} = 1 + 1 = 2$$

S34.

$$I = \int_0^{\frac{\pi}{2}} \sin^2 x\, dx \qquad \text{... (i)}$$

$\therefore$ $$I = \int_0^{\frac{\pi}{2}} \sin^2\left(\frac{\pi}{2} - x\right) dx = \int_0^{\frac{\pi}{2}} \cos^2 x\, dx \qquad \text{... (ii)}$$

Adding (i) and (ii), $$2I = \int_0^{\frac{\pi}{2}} \left(\sin^2 x + \cos^2 x\right) dx = \int_0^{\frac{\pi}{2}} 1 \cdot dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

Hence, $I = \frac{\pi}{4}$.

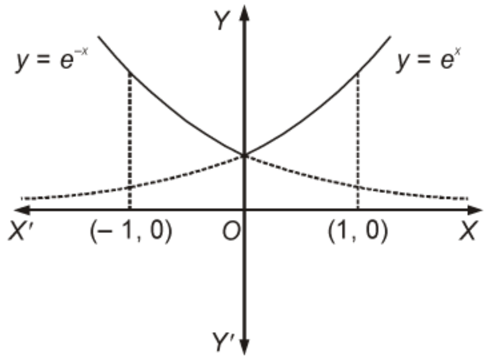
S35.

$$I = \int_0^{\frac{\pi}{2}} \log\left(\frac{3 + 5\cos\left(\frac{\pi}{2} - x\right)}{3 + 5\sin\left(\frac{\pi}{2} - x\right)}\right) dx \qquad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$= \int_0^{\frac{\pi}{2}} \log\left(\frac{3 + 5\sin x}{3 + 5\cos x}\right) dx = -I$$

$$2I = 0 \Rightarrow I = 0$$

S36. Clearly, $|x| = \begin{cases} x & \text{when } x \geq 0 \\ -x & \text{when } x < 0 \end{cases}$



$$\therefore \quad \int_{-1}^{1} e^{|x|}\,dx = \int_{-1}^{0} e^{|x|}\,dx + \int_{0}^{1} e^{|x|}\,dx$$

$$\Rightarrow \quad \int_{-1}^{1} e^{|x|}\,dx = \int_{-1}^{0} e^{-x}\,dx + \int_{0}^{1} e^{x}\,dx$$

$$\Rightarrow \quad \int_{-1}^{1} e^{|x|}\,dx = \left[-e^{-x}\right]_{-1}^{0} + \left[e^{x}\right]_{0}^{1}$$

$$\Rightarrow \quad \int_{-1}^{1} e^{|x|}\,dx = (-1 + e^{1}) + (e^{1} - 1) = 2e - 2$$

S37. $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(5\sin^3 x + 8\sin x + 4\cos^2 x\right) dx$$

$$= 5\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^3 x\,dx + 8\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x\,dx + 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 x\,dx$$

$$= 5(0) + 8(0) + 8\int_{0}^{\frac{\pi}{2}} \cos^2 x\,dx$$

[$\therefore$ $\sin^3 x$, $\sin x$ are odd functions and $\cos^2 x$ is an even function]

$$= 8\int_{0}^{\frac{\pi}{2}} \cos^2 x\,dx = 4\int_{0}^{\frac{\pi}{2}} (1 + \cos 2x)\,dx$$

$$= 4\left[x + \frac{\sin 2x}{2}\right]_{0}^{\frac{\pi}{2}} = 4\left[\left(\frac{\pi}{2} + \frac{\sin \pi}{2}\right) - \left(0 + \frac{\sin 0}{2}\right)\right]$$

$$= 4\left[\frac{\pi}{2}\right] = 2\pi.$$

S38. Consider $$I = \int_{0}^{\pi/2} \frac{x + \sin x}{1 + \cos x}\,dx$$

$$I = \int_{0}^{\pi/2} \frac{x + 2\sin\frac{x}{2}.\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\,dx \quad \left[\because \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} \text{ and } 1 + \cos x = 2\cos^2\frac{x}{2}\right]$$

$$I = \frac{1}{2}\int_0^{\pi/2} x\sec^2\frac{x}{2}dx + \int_0^{\pi/2}\tan\frac{x}{2}dx$$

$$I = \frac{1}{2}\left[\left[\frac{x\tan\frac{x}{2}}{\frac{1}{2}}\right]_0^{\pi/2} - \int_0^{\pi/2}\frac{\tan\frac{x}{2}}{\frac{1}{2}}dx\right] + \int_0^{\pi/2}\tan\frac{x}{2}dx$$

[Using integration by parts in the integral $\int \underset{I}{x}\sec\underset{II}{c^2}\frac{x}{2}dx$]

$$I = \left[x.\tan\frac{x}{2}\right]_0^{\pi/2} - \int_0^{\pi/2}\tan\frac{x}{2}dx + \int_0^{\pi/2}\tan\frac{x}{2}dx$$

$$I = \frac{\pi}{2}\cdot\tan\frac{\pi}{4} - 0$$

$\therefore$ $$I = \frac{\pi}{2}$$

S39. We have : $f(x) = a + bx + cx^2$

$\therefore$ $f(0) = a, f\left(\frac{1}{2}\right) = a + \frac{b}{2} + \frac{c}{4},\ f(1) = a + b + c$

$\therefore$ $$\text{R.H.S.} = \frac{1}{6}\left[a + 4\left(a + \frac{b}{2} + \frac{c}{4}\right) + (a + b + c)\right]$$

$$= \frac{1}{6}[6a + 3b + 2c] = a + \frac{1}{2}b + \frac{1}{3}c \qquad \text{... (i)}$$

and $$\text{LHS} = \int_0^1 f(x)\,dx = \int_0^1 (a + bx + cx^2)\,dx$$

$$= \left[ax + \frac{bx^2}{2} + \frac{cx^3}{3}\right]_0^1 = \left(a + \frac{b}{2} + \frac{c}{3}\right) - (0 + 0 + 0)$$

$$= a + \frac{1}{2}b + \frac{1}{3}c \qquad \text{... (ii)}$$

From (i) and (ii), L.H.S. = R.H.S.

S40. $$\int_0^1 \sin^{-1} x\,dx = \int_0^1 \underset{I}{\sin^{-1}} x.\underset{II}{1}\,dx$$

$$\left[\therefore \int \underset{I}{f(x)}\,\underset{II}{g(x)}\,dx = f(x)\int g(x)\,dx - \int\left(\frac{d}{dx}f(x)\int g(x)dx\right)dx\right]$$ [Integrating by parts]

$$= \left[\sin^{-1} x . x\right]_0^1 - \int_0^1 \frac{1}{\sqrt{1-x^2}}(x).dx$$

$$= \left[x \sin^{-1} x\right]_0^1 + \frac{1}{2}\int_0^1 \left(1-x^2\right)^{-\frac{1}{2}}(-2x)\,dx$$

$$\left[\therefore \int \left[f(x)\right]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + c\right]$$

$$= \left[(1)\sin^{-1}(1) - 0\right] + \frac{1}{2}\left[\frac{\left(1-x^2\right)^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^1$$

$$= \sin^{-1}(1) + \left[0 - (1)^{\frac{1}{2}}\right] = \frac{\pi}{2} - 1.$$

S41.

$$\int_0^{\frac{\pi}{4}} 2\tan^3 x\,dx = 2\int_0^{\frac{\pi}{4}} \tan x \,.\, \tan^2 x\,dx$$

$$= 2\int_0^{\frac{\pi}{4}} \tan x\left(\sec^2 x - 1\right)dx = 2\int_0^{\frac{\pi}{4}} \tan x . \sec^2 x\,dx - 2\int_0^{\frac{\pi}{4}} \tan x\,dx$$

$$= 2\left[\frac{\tan^2 x}{2}\right]_0^{\frac{\pi}{4}} + 2\left[\log|\cos x|\right]_0^{\frac{\pi}{4}}$$

$$= \left(\tan^2\frac{\pi}{4} - \tan^2 0\right) + 2\left[\log\left|\cos\frac{\pi}{4}\right| - \log|\cos 0|\right]$$

$$= 1 - 0 + 2\left[\log\frac{1}{\sqrt{2}} - \log 1\right] = 1 + 2\left[\log 1 - \log\sqrt{2} - \log 1\right]$$

$$= 1 + 2\left[0 - \frac{1}{2}\log 2 - 0\right] = 1 - \log 2.$$

S42. Given integral is $\int_{\pi/2}^{\pi} e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$

$$= \int_{\pi/2}^{\pi} e^x\left(\frac{1 - 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right)dx$$

$$= \int_{\pi/2}^{\pi} e^x \left(\underset{f}{-\cot\frac{x}{2}} + \underset{f'}{\frac{1}{2}\operatorname{cosec}^2\frac{x}{2}} \right) dx$$

$$= -\int_{\pi/2}^{\pi} \underset{\text{II}}{e^x} \underset{\text{I}}{\cot\frac{x}{2}}\, dx + \int_{\pi/2}^{\pi} \frac{1}{2} e^x \operatorname{cosec}^2\frac{x}{2}\, dx$$

$$= -\left[\cot\frac{x}{2} e^x\right]_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi} \frac{1}{2} e^x \operatorname{cosec}^2\frac{x}{2}\, dx + \int_{\pi/2}^{\pi} \frac{1}{2} e^x \operatorname{cosec}^2\frac{x}{2}\, dx$$

$$= -[0 - e^{\pi/2}] = e^{\pi/2}$$

S43. Given integral is $\int_0^2 \frac{1}{4 + x - x^2}\, dx$

$$= -\int_0^2 \frac{1}{x^2 - x - 4}\, dx = -\int_0^2 \frac{1}{\left(x - \frac{1}{2}\right)^2 - \left(\frac{\sqrt{17}}{2}\right)^2}\, dx$$

$$= \int_0^2 \frac{1}{\left(\frac{\sqrt{17}}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2}\, dx = \frac{1}{\sqrt{17}}\left[\log\left(\frac{\sqrt{17} + 2x - 1}{\sqrt{17} - 2x + 1}\right)\right]_0^2$$

$$= \frac{1}{\sqrt{17}}\left\{\log\frac{\sqrt{17} + 3}{\sqrt{17} - 3} - \log\frac{\sqrt{17} - 1}{\sqrt{17} + 1}\right\}$$

$$= \frac{1}{\sqrt{17}}\left\{\log\frac{26 + 6\sqrt{17}}{8} - \log\frac{18 - 2\sqrt{17}}{16}\right\}$$

$$= \frac{1}{\sqrt{17}}\log\left(\frac{52 + 12\sqrt{17}}{18 - 2\sqrt{17}}\right) = \frac{1}{\sqrt{17}}\log\left(\frac{52 + 12\sqrt{17}}{18 - 2\sqrt{17}} \times \frac{18 + 2\sqrt{17}}{18 + 2\sqrt{17}}\right)$$

$$= \frac{1}{\sqrt{17}}\log\left(\frac{1344 + 320\sqrt{17}}{256}\right) = \frac{1}{\sqrt{17}}\log\left(\frac{21 + 5\sqrt{17}}{4}\right)$$

S44. Let $I = \int_0^1 \frac{x^4 + 1}{x^2 + 1} dx$

$\Rightarrow$ $I = \int_0^1 \frac{(x^4 - 1) + 2}{x^2 + 1} dx$

$\Rightarrow$ $= \int_0^1 \frac{(x^2 - 1)(x^2 + 1) + 2}{x^2 + 1} dx$ $\quad \left[\because (a^2 - b^2) = (a - b)(a + b)\right]$

$$= \int_0^1 \left[\frac{(x^2 - 1)(x^2 + 1)}{x^2 + 1} + \frac{2}{x^2 + 1}\right] dx$$

$\Rightarrow \quad I = \int_0^1 \left[x^2 - 1 + \frac{2}{x^2+1}\right] dx$

$= \left[\frac{x^3}{3} - x + 2\tan^{-1} x\right]_0^1$

$\therefore \quad I = \frac{1}{3} - 1 + 2\tan^{-1} 1 - 0$

$= \frac{-2}{3} + 2 \times \frac{\pi}{4} = \frac{3\pi - 4}{6}$

S45. $I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{(1+\cos x)^2} dx = \int_0^{\frac{\pi}{2}} \frac{1-\cos^2 x}{(1+\cos x)^2} dx = \int_0^{\frac{\pi}{2}} \frac{(1-\cos x)(1+\cos x)}{(1+\cos x)^2} dx$

$= \int_0^{\frac{\pi}{2}} \frac{1-\cos x}{1+\cos x} dx = \int_0^{\frac{\pi}{2}} \frac{2\sin^2 \frac{x}{2}}{2\cos^2 \frac{x}{2}} dx = \int_0^{\frac{\pi}{2}} \tan^2 \frac{x}{2} dx$

$= \int_0^{\frac{\pi}{2}} \left(\sec^2 \frac{x}{2} - 1\right) dx = \left[\frac{\tan \frac{x}{2}}{\frac{1}{2}} - x\right]_0^{\frac{\pi}{2}}$

$= \left[2\tan \frac{x}{2} - x\right]_0^{\frac{\pi}{2}} = \left[2\tan \frac{\pi}{4} - \frac{\pi}{2}\right] - (0 - 0)$

$= 2(1) - \frac{\pi}{2} = 2 - \frac{\pi}{2}.$

S46. We have,

$I = \int_{\pi/4}^{\pi/2} \underset{\text{II}}{\cos 2x} \underset{\text{I}}{\log \sin x}\, dx$

$\Rightarrow \quad I = \left[\frac{1}{2}(\log \sin x) \sin 2x\right]_{\pi/4}^{\pi/2} - \int_{\pi/4}^{\pi/2} \frac{1}{2} \cot x \sin 2x\, dx$

$\Rightarrow \quad I = \left[0 - \frac{1}{2}\log\left(\frac{1}{\sqrt{2}}\right)\right] - \int_{\pi/4}^{\pi/2} \cos^2 x\, dx$

$\Rightarrow \quad I = \frac{1}{4}\log 2 - \frac{1}{2}\int_{\pi/4}^{\pi/2} (1 + \cos 2x)\, dx$

$\Rightarrow \quad I = \frac{1}{4}\log 2 - \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_{\pi/4}^{\pi/2}$

$$\Rightarrow \quad I = \frac{1}{4}\log 2 - \frac{1}{2}\left[\left(\frac{\pi}{2}+0\right)-\left(\frac{\pi}{4}+\frac{1}{2}\right)\right]$$

$$\Rightarrow \quad I = \frac{1}{4}\log 2 - \frac{\pi}{8}+\frac{1}{4}$$

S47. Let $I = \int_0^{2\pi} \underset{\text{II}}{e^x} \underset{\text{I}}{\sin}\left(\frac{\pi}{4}+\frac{x}{2}\right)dx$

On integrating by parts, we get

$$I = \left[\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cdot e^x\right]_0^{2\pi} - \frac{1}{2}\int_0^{2\pi}\underset{\text{II}}{e^x}\ \underset{\text{I}}{\cos}\left(\frac{\pi}{4}+\frac{x}{2}\right)dx$$

$$\Rightarrow \quad I = \left[\sin\frac{5\pi}{4}e^{2\pi} - \sin\frac{\pi}{4}\right] - \left[\left\{e^x\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}_0^{2\pi} + \frac{1}{2}\int_0^{2\pi}e^x\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)dx\right]$$

$$\Rightarrow \quad I = \left(-\frac{e^{2\pi}}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right)-\frac{1}{2}\left[\left(-\frac{e^{2\pi}}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right)+\frac{1}{2}\mathrm{I}\right]$$

$$\Rightarrow \quad I = -\left(\frac{e^{2\pi}+1}{\sqrt{2}}\right)+\left(\frac{e^{2\pi}+1}{2\sqrt{2}}\right)-\frac{1}{4}\mathrm{I}$$

$$\Rightarrow \quad I + \frac{1}{4}I = \frac{e^{2\pi}+1}{2\sqrt{2}}(1-2) \quad \Rightarrow \quad I = -\frac{\sqrt{2}}{5}(e^{2\pi}+1)$$

S48. We have,

$$\int_0^a f(x)\,g(x)\,dx = \int_0^a f(a-x)\cdot g(a-x)\,dx$$

$$\Rightarrow \quad \int_0^a f(x)\,g(x)\,dx = \int_0^a f(x)\{2-g(x)\}\,dx \qquad [\because \quad g(a-x) = 2-g(x)]$$

$$\Rightarrow \quad \int_0^a f(x)\,g(x)\,dx = 2\int_0^a f(x)\,dx - \int_0^a f(x)\,g(x)\,dx$$

$$\Rightarrow \quad 2\int_0^a f(x)\,g(x)\,dx = 2\int_0^a f(x)\,dx$$

$$\Rightarrow \quad \int_0^a f(x)\,g(x)\,dx = \int_0^a f(x)\,dx$$

S49. Using the property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ and solve it.

Let $$I = \int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx \quad \text{... (i)}$$

$\therefore$ $$I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{a-(a-x)}}dx \quad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}dx \quad \text{... (ii)}$$

Adding Eqs. (i) and (ii), we get

$$2I = \int_0^a \frac{\sqrt{x}+\sqrt{a-x}}{\sqrt{x}+\sqrt{a-x}}dx$$

$$2I = \int_0^a dx$$

$$2I = [x]_0^a$$

$$2I = a - 0 = a$$

$\therefore$ $$I = \frac{a}{2}$$

S50. $$I = \int_1^3 \left[|x-1| + |x-2| + |x-3|\right]dx$$

Here, redefined the integrand between the limit of integration

$$1 < x < 2 \Rightarrow \begin{cases} |x-1| = x-1 \\ |x-2| = -(x-2) \\ |x-3| = -(x-3) \end{cases}$$

$$2 < x < 3 \Rightarrow \begin{cases} |x-1| = x-1 \\ |x-2| = (x-2) \\ |x-3| = -(x-3) \end{cases}$$

$\therefore$ $$I = \int_1^2 (x-1+2-x+3-x)\,dx + \int_2^3 (x-1+x-2+3-x)dx$$

$$= \int_1^2 (4-x)dx + \int_2^3 xdx$$

$$= \left[4x - \frac{x^2}{2}\right]_1^2 + \left[\frac{x^2}{2}\right]_2^3$$

$$= \left(8-\frac{4}{2}\right) - \left(4-\frac{1}{2}\right) + \left(\frac{9}{2}-2\right)$$

$$= 6 - \frac{7}{2} + \frac{5}{2} = \frac{5}{2} + \frac{5}{2} = 5$$

S51. Let $$I = \int_{-1}^{2} |x^3 - x|\, dx$$

We observe that

$$|x^3 - x| = \begin{cases} (x^3 - x), & \text{when } -1 < x < 0 \\ -(x^3 - x), & \text{when } 0 < x < 1 \\ (x^3 - x), & \text{when } 1 < x < 2 \end{cases}$$

Hence, $$I = \int_{-1}^{0} (x^3 - x)dx + \int_{0}^{1} -(x^3 - x)dx + \int_{1}^{2} (x^3 - x)dx$$

$$= \left(\frac{x^4}{4} - \frac{x^2}{2}\right)_{-1}^{0} - \left(\frac{x^4}{4} - \frac{x^2}{2}\right)_{0}^{1} + \left(\frac{x^4}{4} - \frac{x^2}{2}\right)_{1}^{2}$$

$$= \left[0 - \left(\frac{1}{4} - \frac{1}{2}\right)\right] - \left[\left(\frac{1}{4} - \frac{1}{2}\right) - 0\right] + \left[\left(\frac{16}{4} - \frac{4}{2}\right)\right] - \left[\left(\frac{1}{4} - \frac{1}{2}\right)\right]$$

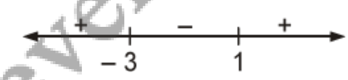
$$= -\frac{1}{4} + \frac{1}{2} - \frac{1}{4} + \frac{1}{2} + 4 - 2 - \frac{1}{4} + \frac{1}{2}$$

$$= -\frac{3}{4} + \frac{3}{2} + 2 = \frac{-3 + 6 + 8}{4} = \frac{11}{4}$$

S52. Let $I = \int_{0}^{2} |x^2 + 2x - 3|\, dx$

We have, $x^2 + 2x - 3 = (x + 3)(x - 1)$.

The signs of $x^2 + 2x - 3$ for different values of x are shown in given figure.



$\therefore$ $$|x^2 + 2x - 3| = \begin{cases} -(x^2 + 2x - 3) & \text{if } 0 < x < 1 \\ (x^2 + 2x - 3) & \text{if } 1 \le x \le 2 \end{cases}$$

$\Rightarrow$ $$I = \int_{0}^{2} |x^2 + 2x - 3|\, dx$$

$\Rightarrow$ $$I = \int_{0}^{1} |x^2 + 2x - 3|\, dx + \int_{1}^{2} |x^2 + 2x - 3|\, dx$$

$\Rightarrow$ $$I = \int_{0}^{1} -(x^2 + 2x - 3)\, dx + \int_{1}^{2} (x^2 + 2x - 3)\, dx$$

$\Rightarrow$ $$I = -\left[\frac{x^3}{3} + x^2 - 3x\right]_0^1 + \left[\frac{x^3}{3} + x^2 - 3x\right]_1^2$$

$\Rightarrow$ $$I = -\left[\left(\frac{1}{3} + 1 - 3\right) - 0\right] + \left[\left(\frac{8}{3} + 4 - 6\right) - \left(\frac{1}{3} + 1 - 3\right)\right] = \frac{5}{3} + \frac{2}{3} + \frac{5}{3} = 4$$

S53.

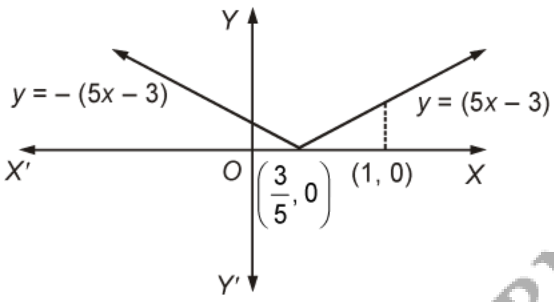
Clearly, $$|5x-3| = \begin{cases} -(5x-3) & \text{when } 5x-3<0, \text{ } i.e., x < \dfrac{3}{5} \\ 5x-3, & \text{when } 5x-3 \geq 0, \text{ } i.e., x \geq \dfrac{3}{5} \end{cases}$$

$\therefore$ $$I = \int_0^1 |5x-3|\, dx$$

$\Rightarrow$ $$I = \int_0^{3/5} -|5x-3|\, dx + \int_{3/5}^1 |5x-3|\, dx$$

$\Rightarrow$ $$I = \int_0^{3/5} -(5x-3)\, dx + \int_{3/5}^1 (5x-3)\, dx$$

$\Rightarrow$ $$I = \left[3x - \frac{5x^2}{2}\right]_0^{3/5} + \left[\frac{5x^2}{2} - 3x\right]_{3/5}^1 = \left(\frac{9}{5} - \frac{9}{10}\right) + \left(-\frac{1}{2} + \frac{9}{10}\right) = \frac{13}{10}$$



S54.

$$\because |x+2| = \begin{cases} x+2, & \text{if } x \geq -2 \\ -(x+2) & \text{if } x < -2 \end{cases}$$

Let $$I = \int_{-5}^{5} |x+2|\, dx$$

$$I = \int_{-5}^{-2} |x+2|\, dx + \int_{-2}^{5} |x+2|\, dx \qquad \left[\because \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx\right]$$

(a < c < b)

$$= \int_{-5}^{-2} -(x+2)dx + \int_{-2}^{5} (x+2)dx$$

$$= -\left(\frac{x^2}{2} + 2x\right)_{-5}^{-2} + \left(\frac{x^2}{2} + 2x\right)_{-2}^{5}$$

$$= -\left[\left(\frac{4}{2} - 4\right) - \left(\frac{25}{2} - 10\right)\right] + \left[\left(\frac{25}{2} + 10\right) - \left(\frac{4}{2} - 4\right)\right]$$

$$= -\left[-2 - \frac{5}{2}\right] + \left[\frac{45}{2} + 2\right]$$

$$= 2+\frac{5}{2}+\frac{45}{2}+2$$

$$= 4+25=29$$

S55. Put $\sin^2 x = t$ so that $2\sin x\cos x\,dx = dt$ *i.e.*, $\sin 2x\,dx = dt$

When $x = 0$, $t = \sin^2 0 = 0$. When $x = \frac{\pi}{2}$, $t = \sin^2\frac{\pi}{2} = 1$

$$I = \int_0^1 \frac{dt}{(\sin^2 x)^2+(1-\sin^2 x)^2} = \int_0^1 \frac{dt}{t^2+(1-t)^2}$$

$$= \int_0^1 \frac{dt}{t^2+1-2t+t^2} = \int_0^1 \frac{dt}{2t^2-2t+1}$$

$$= \frac{1}{2}\int_0^1 \frac{dt}{t^2-t+\frac{1}{2}} = \frac{1}{2}\int_0^1 \frac{dt}{\left(t-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}$$

$$\left[\because \int\frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c\right]$$

$$= \frac{1}{2}\cdot\frac{1}{\frac{1}{2}}\left[\tan^{-1}\frac{\left(t-\frac{1}{2}\right)}{\frac{1}{2}}\right]_0^1 = \left[\tan^{-1}(2t-1)\right]_0^1$$

$$= \tan^{-1}(1) - \tan^{-1}(-1)$$

$$= \tan^{-1}(1) + \tan^{-1}(1)$$

$$= \frac{\pi}{4}+\frac{\pi}{4} = \frac{\pi}{2}$$

S56.

Let $I = \int_{\frac{1}{3}}^{1}\frac{(x-x^3)^{\frac{1}{3}}}{x^4}\,dx = \int_{\frac{1}{3}}^{1}\frac{x\left(\frac{1}{x^2}-1\right)^{\frac{1}{3}}}{x^4}\,dx = \int_{\frac{1}{3}}^{1}\frac{(x^{-2}-1)^{\frac{1}{3}}}{x^3}\,dx$

Put $x^{-2}-1 = t$

So that $-\frac{2}{x^3}\,dx = dt \Rightarrow \frac{1}{x^3}\,dx = \frac{dt}{-2}$

Limit: when $x = \frac{1}{3}$, $t = x^{-2}-1 = \left(\frac{1}{3}\right)^{-2}-1 = 9-1 = 8$

when $x = 1$, $t = x^{-2}-1 = (1)^{-2}-1 = 0$

$$I = \int_8^0 t^{\frac{1}{3}} \cdot \frac{dt}{-2}$$

$$= -\frac{1}{2}\left[\frac{t^{\frac{1}{3}+1}}{\frac{1}{3}+1}\right]_8^0 = -\frac{1}{2}\left[\frac{3}{4}t^{\frac{4}{3}}\right]_8^0 = -\frac{1}{2}\left[0 - \frac{3}{4}(8)^{\frac{4}{3}}\right]$$

$$= \frac{1}{2} \times \frac{3}{4} \times 16 = 6.$$

S57. Let $I = \int_0^1 \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx$

Put $x = \tan\theta \quad \Rightarrow \quad dx = \sec^2\theta d\theta$

Also lower limit: when $x = 0 \quad \Rightarrow \quad \theta = \tan^{-1} 0 = 0$

and upper limit: when $x = 1 \quad \Rightarrow \quad \theta = \tan^{-1} 1 = \frac{\pi}{4}$

$\therefore$ $$I = \int_0^{\pi/4} \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) \cdot \sec^2\theta d\theta$$

$$= 2\int_0^{\pi/4} \theta \sec^2\theta d\theta \qquad \left[\because \quad \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}\right]$$

$$= 2\left\{[\theta\tan\theta]_0^{\pi/4} - \int_0^{\pi/4} 1.\tan\theta d\theta\right\}$$

$$= 2\left\{\frac{\pi}{4} - [\log\sec\theta]_0^{\pi/4}\right\}$$

$$= \frac{\pi}{2} - 2\left[\log\sqrt{2} - \log 1\right] = \frac{\pi}{2} - \log 2.$$

S58. Put $x = \tan\theta$ so that $dx = \sec^2\theta\ d\theta$.

When $x = 0$, $\tan\theta = 0 \Rightarrow \theta = 0$. When $x = 1$, $\tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

$\therefore$ $$I = \int_0^{\frac{\pi}{4}} \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) \cdot \sec^2\theta\, d\theta = \int_0^{\frac{\pi}{4}} \cos^{-1}(\cos 2\theta)\sec^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} 2\theta\sec^2\theta\, d\theta = 2\left[\theta\tan\theta - \int(1)\tan\theta\, d\theta\right]_0^{\frac{\pi}{4}}$$

$$\left[\therefore \int \underset{I}{f(x)}\, \underset{II}{g(x)}\, dx = f(x)\int g(x)\,dx - \int\left(\frac{d}{dx}f(x)\int g(x)\,dx\right)dx\right]$$ [Integrating by parts]

$$= 2\left[\theta \tan\theta + \log|\cos\theta|\right]_0^{\frac{\pi}{4}}$$

$$= 2\left[\left(\frac{\pi}{4}\right)\tan\frac{\pi}{4} + \log\left|\cos\frac{\pi}{4}\right| - (0 + \log|\cos 0^\circ|)\right]$$

$$= 2\left[\frac{\pi}{4}(1) + \log\left(\frac{1}{\sqrt{2}}\right) - \log 1\right] = 2\left[\frac{\pi}{4} + \log 1 - \frac{1}{2}\log 2 - 0\right]$$

$$= 2\left[\frac{\pi}{4} + 0 - \frac{1}{2}\log 2\right] = \frac{\pi}{2} - \log 2$$

S59. We have, $I = \int_0^{\pi/4} \tan^3 x\, dx$

$\Rightarrow$ $I = \int_0^{\pi/4} \tan^2 x \tan x\, dx = \int_0^{\pi/4} (\sec^2 x - 1)\tan x\, dx$

$\Rightarrow$ $I = \int_0^{\pi/4} \sec^2 x \tan x\, dx - \int_0^{\pi/4} \tan x\, dx$

Let $\tan x = t$. Then, $d(\tan x) = dt \Rightarrow \sec^2 x\, dx = dt$

Now $x = 0, \Rightarrow t = 0$, and $x = \frac{\pi}{4} \Rightarrow t = 1$

$\Rightarrow$ $I = \int_0^{\pi/4} \tan^3 x\, dx \Rightarrow I = \int_0^1 t\, dt - \int_0^{\pi/4} \tan x\, dx = \left[\frac{t^2}{2}\right]_0^1 - \left[\log \sec x\right]_0^{\pi/4}$

$\Rightarrow$ $I = \left(\frac{1}{2} - 0\right) - \log\sec\frac{\pi}{4} + \log\sec 0$

$\Rightarrow$ $I = \frac{1}{2} - \log\sqrt{2} + \log 1 = \frac{1}{2} - \frac{1}{2}\log 2 = \frac{1}{2}(1 - \log 2)$

S60. We have, $\int_0^{\pi/2} \frac{1}{4\sin^2 x + 5\cos^2 x}\, dx$

$\Rightarrow$ $I = \int_0^{\pi/2} \frac{\sec^2 x}{4\tan^2 x + 5}\, dx$ [Dividing Nr and Dr by $\cos^2 x$]

Let $\tan x = t$. Then, $d(\tan x) = dt \Rightarrow \sec^2 x\, dx = dt$

Also, at $x = 0 \Rightarrow t = \tan 0 = 0$ and $x = \frac{\pi}{2} \Rightarrow t = \tan\frac{\pi}{2} = \infty$

$\therefore$ $I = \int_0^{\infty} \frac{dt}{4t^2 + 5}$

$$\Rightarrow \quad I = \frac{1}{4}\int_0^\infty \frac{1}{t^2 + \left(\frac{\sqrt{5}}{2}\right)^2} dt = \frac{1}{4} \cdot \frac{1}{\left(\frac{\sqrt{5}}{2}\right)} \left[\tan^{-1}\left(\frac{t}{\frac{\sqrt{5}}{2}}\right)\right]_0^\infty$$

$$\Rightarrow \quad I = \frac{1}{2\sqrt{5}}\left[\tan^{-1}\left(\frac{2t}{\sqrt{5}}\right)_0^\infty\right] = \frac{1}{2\sqrt{5}}[\tan^{-1}\infty - \tan^{-1} 0] = \frac{1}{2\sqrt{5}} \times \frac{\pi}{2} = \frac{\pi}{4\sqrt{5}}$$

S61. Let $\tan^{-1} x = \theta$ or, $x = \tan\theta$. Then, $dx = \sec^2\theta \, d\theta$

Now, $x = 0 \Rightarrow \tan\theta = 0 \Rightarrow \theta = 0$, and $x = 1 \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

$$\therefore \quad I = \int_0^1 \frac{x \tan^{-1} x}{(1+x^2)^{3/2}} dx$$

$$\Rightarrow \quad I = \int_0^{\pi/4} \frac{\theta \tan\theta}{\sec^3\theta} \sec^2\theta \, d\theta \quad \Rightarrow \quad I = \int_0^{\pi/4} \underset{\text{I}}{\theta} \underset{\text{II}}{\sin\theta} \, d\theta$$

$$\Rightarrow \quad I = \left[-\theta\cos\theta\right]_0^{\pi/4} - \int_0^{\pi/4} (-\cos\theta) d\theta = \left[-\theta\cos\theta\right]_0^{\pi/4} + \left[\sin\theta\right]_0^{\pi/4}$$

$$\Rightarrow \quad I = \left(-\frac{\pi}{4\sqrt{2}} - 0\right) + \left(\frac{1}{\sqrt{2}} - 0\right) = \frac{4-\pi}{4\sqrt{2}}$$

S62. Let $\sin^{-1} x = \theta$ or, $x = \sin\theta$. Then, $dx = d(\sin\theta) = \cos\theta \, d\theta$

Now, $x = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$, and $x = \frac{1}{\sqrt{2}} \Rightarrow \sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$

$$\therefore \quad I = \int_0^{1/\sqrt{2}} \frac{\sin^{-1} x}{(1-x^2)^{3/2}} dx$$

$$\Rightarrow \quad I = \int_0^{\pi/4} \frac{\theta}{\cos^3\theta} \cos\theta \, d\theta = \int_0^{\pi/4} \underset{\text{I}}{\theta} \underset{\text{II}}{\sec^2\theta} \, d\theta$$

$$\Rightarrow \quad I = \left[\theta\tan\theta\right]_0^{\pi/4} - \int_0^{\pi/4} 1 \cdot \tan\theta \, d\theta$$

$$\Rightarrow \quad I = \left[\theta\tan\theta\right]_0^{\pi/4} + \left[\log\cos\theta\right]_0^{\pi/4}$$

$$\Rightarrow \quad I = \left(\frac{\pi}{4} - 0\right) + \left(\log\left(\frac{1}{\sqrt{2}}\right) - \log 1\right) = \frac{\pi}{4} - \frac{1}{2}\log 2$$

S63. Let $\sin\theta = t$. Then, $d(\sin\theta) = dt \Rightarrow \cos\theta \, d\theta = dt$

When $\theta = 0$, $t = \sin 0 = 0$. When $\theta = \frac{\pi}{2}$, $t = \sin\frac{\pi}{2} = 1$

$\therefore \qquad I = \int_0^{\pi/2} \frac{\cos\theta}{(1+\sin\theta)(2+\sin\theta)}\, d\theta$

$\Rightarrow \qquad I = \int_0^1 \frac{1}{(1+t)(2+t)}\, dt$

$\Rightarrow \qquad I = \int_0^1 \left\{\frac{1}{1+t} - \frac{1}{2+t}\right\} dt$

$\Rightarrow \qquad I = \left[\log(1+t)\right]_0^1 - \left[\log(2+t)\right]_0^1$

$\Rightarrow \qquad I = (\log 2 - \log 1) - (\log 3 - \log 2)$

$\Rightarrow \qquad I = \log 2 - \log 3 + \log 2 = 2\log 2 - \log 3 = \log\left(\frac{4}{3}\right)$

S64. Let $\cos\theta = t$. Then, $d(\cos\theta) = dt \;\Rightarrow\; -\sin\theta\, d\theta = dt$

When $\theta = 0$, $t = \cos 0 = 1$

When $\theta = \frac{\pi}{2}$, $t = \cos\frac{\pi}{2} = 0$

$$\therefore \int_0^{\pi/2} \sqrt{\cos\theta}\, \sin^3\theta\, d\theta = \int_1^0 \sqrt{t}\, \sin^3\theta \left(\frac{-dt}{\sin\theta}\right) = -\int_1^0 \sqrt{t}\, \sin^2\theta\, dt$$

$$= -\int_1^0 \sqrt{t}\,(1-t^2)\, dt = -\int_1^0 (\sqrt{t} - t^{5/2})\, dt$$

$$= -\left[\frac{2}{3}t^{3/2} - \frac{2}{7}t^{7/2}\right]_1^0 = -\left[0 - \left(\frac{2}{3} - \frac{2}{7}\right)\right] = \frac{8}{21}$$

S65. Let $\qquad I = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x} dx \qquad \ldots \text{(i)}$

$I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)} dx \qquad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

$I = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x} dx \qquad \ldots \text{(ii)}$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi} \frac{(x + \pi - x)\sin x}{1+\cos^2 x} dx$$

$I = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx \qquad [\because \text{Let } t = \cos x,\ dt = -\sin x dx]$

$$\begin{bmatrix} \text{upper limit} \to \cos\pi = -1 \\ \text{Lower limit} \to \cos 0 = 1 \end{bmatrix}$$

$$I = \frac{\pi}{2}\int_{1}^{-1}\frac{-dt}{1+t^2} = \frac{\pi}{2}\int_{-1}^{1}\frac{dt}{1+t^2} \qquad \left[\because \int_a^b f(x)dx = -\int_b^a f(x)dx\right]$$

$$= \frac{\pi}{2}[\tan^{-1} t]_{-1}^{1} = \frac{\pi}{2}[\tan^{-1}(1) - \tan^{-1}(-1)]$$

$$= \frac{\pi}{2}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi}{2}\left(\frac{\pi}{4}+\frac{\pi}{4}\right) = \frac{\pi}{2}.\frac{\pi}{2}$$

$$= \frac{\pi^2}{4}$$

S66. Let
$$I = \int_0^{\pi/4}\log(1+\tan x)dx \qquad \ldots \text{(i)}$$

$$I = \int_0^{\pi/4}\log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right]dx \qquad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$I = \int_0^{\pi/4}\log\left(1+\frac{1-\tan x}{1+\tan x}\right)dx$$

$$I = \int_0^{\pi/4}\log\left(\frac{2}{1+\tan x}\right)dx$$

$$I = \int_0^{\pi/4}\log 2dx - \int_0^{\pi/4}\log(1+\tan x)dx$$

$$I = \log 2\cdot[x]_0^{\pi/4} - I \qquad \text{[using (i)]}$$

$\Rightarrow$
$$2I = \frac{\pi}{4}\log 2$$

$\Rightarrow$
$$I = \frac{\pi}{8}\log 2$$

S67. Let
$$I = \int_0^{\pi}\frac{x\tan x}{\sec x\,\mathrm{cosec}\,x}dx$$

$$= \int_0^{\pi}\frac{x(\sin x/\cos x)}{(1/\cos x)(1/\sin x)}dx$$

$$I = \int_0^{\pi} x\sin^2 x dx \qquad \ldots\text{(i)}$$

$\Rightarrow$
$$I = \int_0^{\pi}(\pi - x)\sin^2(\pi - x)dx \qquad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \int_0^{\pi}(\pi - x)\sin^2 x\,dx \qquad \ldots\text{(ii)}$$

Adding Eqs. (i) and (ii), we get

$$2I = \pi\int_0^{\pi}\sin^2 x\,dx$$

$$= \frac{\pi}{2}\int_0^{\pi}(1-\cos 2x)dx \qquad \left[\because \sin^2 x = \frac{1-\cos 2x}{2}\right]$$

$\Rightarrow$ $$2I = \frac{\pi}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{\pi}{2}[\pi] = \frac{\pi^2}{2}$$

$\Rightarrow$ $$I = \frac{\pi^2}{4}$$

S68. We have, $$I = \int_{-\pi/2}^{\pi/2} f(x)\,dx$$

where, $$f(x) = \sin|x| + \cos|x|$$

$$f(-x) = \sin|-x| + \cos|-x| = \sin|x| + \cos|x| = f(x)$$

$\therefore$ $$\int_{-\pi/2}^{\pi/2} f(x)\,dx = 2\int_0^{\pi/2} f(x)\,dx = 2\int_0^{\pi/2}[\sin|x| + \cos|x|]\,dx$$

$$= 2\int_0^{\pi/2}[\sin x + \cos x]\,dx \qquad \left[\because \ |x| = x \text{ in } 0 < x < \frac{\pi}{2}\right]$$

$$= 2[-\cos x + \sin x]_0^{\pi/2} = 2\left[\left(-\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right) - \left(-\cos 0 + \sin 0\right)\right]$$

$$= 2[(-0+1) - (-1+0)] = 4.$$

S69. $$I = \int_0^{2\pi} \frac{1}{1+e^{\sin x}}\,dx \qquad \text{... (i)}$$

$$I = \int_0^{2\pi} \frac{dx}{1+e^{\sin x(2\pi - x)}} \qquad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$I = \int_0^{2\pi} \frac{dx}{1+e^{-\sin x}} = \int_0^{2\pi} \frac{e^{\sin x}}{e^{\sin x}+1}\,dx \qquad \text{... (ii)}$$

Adding Eqs. (i) and (ii), we get

$$I + I = \int_0^{2\pi} \frac{dx}{1+e^{\sin x}} + \int_0^{2\pi} \frac{e^{\sin x}}{1+e^{\sin x}}\,dx$$

$\Rightarrow$ $$2I = \int_0^{2\pi} \frac{(1+e^{\sin x})}{(1+e^{\sin x})}\,dx$$

$\Rightarrow$ $$2I = \int_0^{2\pi} 1.dx = [x]_0^{2\pi} = 2\pi - 0$$

$\Rightarrow$ $$I = \pi$$

S70. $$I = \int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}}\,dx \qquad \text{... (i)}$$

$$I = \int_0^{\pi} \frac{e^{\cos(\pi - x)}}{e^{\cos(\pi - x)} + e^{-\cos(\pi - x)}}\,dx \qquad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$I = \int_0^{\pi} \frac{e^{-\cos x}}{e^{-\cos x} + e^{\cos x}} dx \qquad [\because \cos(\pi - x) = -\cos x] \quad \ldots \text{(ii)}$$

Adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi} \frac{e^{\cos x} + e^{-\cos x}}{e^{\cos x} + e^{-\cos x}} dx$$

$\Rightarrow$ $$2I = \int_0^{\pi} 1 dx$$

$\Rightarrow$ $$2I = [x]_0^{\pi}$$

Hence, $$I = \frac{\pi}{2}$$

S71. Let $$I = \int_0^{\pi} x f(\sin x)\, dx \qquad \ldots \text{(i)} \qquad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right],$$

$$I = \int_0^{\pi} (\pi - x) [\sin(\pi - x)]\, dx$$

$$= \int_0^{\pi} (\pi - x) f(\sin x)\, dx \qquad \ldots \text{(ii)}$$

Adding Eq. (i) and Eq. (ii), we get

$$2I = \int_0^{\pi} (x + \pi - x) f(\sin x)\, dx = \pi \int_0^{\pi} f(\sin x)\, dx$$

Hence $$I = \frac{\pi}{2} \int_0^{\pi} f(\sin x)\, dx.$$

S72. Let $$I = \int_{-\pi/4}^{\pi/4} \frac{x + \frac{\pi}{4}}{2 - \cos 2x} dx. \quad \text{Then,}$$

$$I = \int_{-\pi/4}^{\pi/4} \frac{x}{2 - \cos 2x} dx + \frac{\pi}{4} \int_{-\pi/4}^{\pi/4} \frac{1}{2 - \cos 2x} dx$$

$\Rightarrow$ $$I = 0 + \frac{\pi}{2} \int_0^{\pi/4} \frac{1}{2 - \cos 2x} dx \qquad \left[\begin{array}{l} \text{As } \frac{x}{2 - \cos 2x} \text{ is an odd function} \\ \text{and } \frac{1}{2 - \cos 2x} \text{ is an even function} \end{array}\right]$$

$$I = \frac{\pi}{2} \int_0^{\pi/4} \frac{1}{2 - \left(\frac{1 - \tan^2 x}{1 + \tan^2 x}\right)} dx$$

$\Rightarrow \quad I = \frac{\pi}{2}\int_0^{\pi/4}\frac{1+\tan^2 x}{1+3\tan^2 x}\,dx$

$\Rightarrow \quad I = \frac{\pi}{2}\int_0^{\pi/4}\frac{\sec^2 x}{1^2+(\sqrt{3}\tan x)^2}\,dx$

$\Rightarrow \quad I = \frac{\pi}{2\sqrt{3}}\int_0^{\pi/4}\frac{1}{1^2+(\sqrt{3}\tan x)^2}\,d(\sqrt{3}\tan x)$

$\Rightarrow \quad I = \frac{\pi}{2\sqrt{3}}\left[\tan^{-1}(\sqrt{3}\tan x)\right]_0^{\pi/4}$

$\Rightarrow \quad I = \frac{\pi}{2\sqrt{3}}(\tan^{-1}\sqrt{3}-\tan^{-1}0) = \frac{\pi}{2\sqrt{3}}\times\frac{\pi}{3} = \frac{\pi^2}{6\sqrt{3}}$

S73. Let $\quad I = \int_0^{2\pi}\frac{x\cos x}{1+\cos x}\,dx = \int_0^{2\pi}\frac{(2\pi-x)\cos(2\pi-x)}{1+\cos(2\pi-x)}\,dx \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$= \int_0^{2\pi}\frac{(2\pi-x)\cos x}{1+\cos x}\,dx = 2\pi\int_0^{2\pi}\frac{\cos x}{1+\cos x}\,dx - I$

$\Rightarrow \quad 2I = 2\pi\int_0^{2\pi}\frac{1+\cos x-1}{1+\cos x}\,dx = 2\pi\left[\int_0^{2\pi}1.\,dx - \int_0^{2\pi}\frac{1}{1+\cos x}\,dx\right]$

$= 2\pi\left[[x]_0^{2\pi} - \int_0^{2\pi}\frac{1}{2\cos^2\frac{x}{2}}\,dx\right]$

$= 2\pi\left[(2\pi-0) - \int_0^{2\pi}\sec^2\frac{x}{2}\cdot\frac{1}{2}\,dx\right] = 4\pi^2 - \pi\left[\frac{\tan\frac{x}{2}}{\frac{1}{2}}\right]_0^{2\pi}$

$= 4\pi^2 - 2\pi\,[\tan\pi - \tan 0]$

$= 4\pi^2 - 2\pi\,(0-0) = 4\pi^2$

Hence $\quad I = 2\pi^2$.

S74. Let $\quad I = \int_2^8\frac{\sqrt[3]{x+1}}{\sqrt[3]{x+1}+\sqrt[3]{11-x}}\,dx \quad \ldots \text{(i)}$

$\therefore \quad I = \int_2^8\frac{\sqrt[3]{(10-x)+1}}{\sqrt[3]{(10-x)+1}+\sqrt[3]{11-(10-x)}}\,dx \quad \left[\because \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx\right]$

$\Rightarrow \quad I = \int_2^8\frac{\sqrt[3]{11-x}}{\sqrt[3]{11-x}+\sqrt[3]{x+1}}\,dx \quad \ldots \text{(ii)}$

Adding Eq. (i) and Eq. (ii), we get

$$2I = \int_2^8 \frac{\sqrt[3]{x+1} + \sqrt[3]{11-x}}{\sqrt[3]{11-x} + \sqrt[3]{x+1}}\, dx = \int_2^8 1.\, dx$$

$$= [x]_2^8 = 8 - 2 = 6$$

Hence $I = 3$

S75. $$I = \int_0^\pi x \sin^3 x\, dx = \int_0^\pi (\pi - x)\sin^3(\pi - x)\, dx \qquad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$= \int_0^\pi (\pi - x)\sin^3 x dx = \pi \int_0^\pi \sin^3 x\, dx - I$$

$\Rightarrow$ $$2I = \pi \int_0^\pi \sin^3 x\, dx \quad \Rightarrow \quad I = \frac{\pi}{2}\int_0^\pi \sin^3 x\, dx$$

$\Rightarrow$ $$I = \frac{\pi}{2}\int_0^\pi \left(\frac{3\sin x - \sin 3x}{4}\right) dx$$

$$= \frac{\pi}{2}\left[\frac{3}{4}\int_0^\pi \sin x\, dx - \frac{1}{4}\int_0^\pi \sin 3x\, dx\right]$$

$$= \frac{\pi}{2}\left[\frac{3}{4}\{-\cos x\}_0^\pi - \frac{1}{4}\left\{-\frac{\cos 3x}{3}\right\}_0^\pi\right]$$

$$= \frac{3\pi}{8}(-\cos\pi + \cos 0) + \frac{\pi}{24}(\cos 3\pi - \cos 0)$$

$$= \frac{3\pi}{8}(1+1) + \frac{\pi}{24}(-1-1) = \frac{3\pi}{4} - \frac{\pi}{12}$$

$$= \frac{9\pi - \pi}{12} = \frac{8\pi}{12} = \frac{2\pi}{3}$$

S76. $$I = \int_0^1 \frac{\log(1+x)}{1+x^2} dx$$

Put $x = \tan\theta$ $\Rightarrow dx = \sec^2\theta\, d\theta$

When x = 0 $\Rightarrow \theta = 0$

at x = 1 $\Rightarrow \theta = \frac{\pi}{4}$

$\therefore$ $$I = \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{1+\tan^2\theta}\sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} \frac{\log(1+\tan\theta)\times\sec^2\theta}{\sec^2\theta} d\theta$$

$$= \int_0^{\pi/4} \log(1+\tan\theta)\,d\theta \qquad ...(i) \qquad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \int_0^{\pi/4} \log\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right)d\theta$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right)d\theta$$

$$= \int_0^{\pi/4} \log 2d\theta - \int_0^{\pi/4} \log(1+\tan\theta)d\theta$$

$\Rightarrow \qquad I = \log 2\cdot[\theta]_0^{\pi/4} - I \qquad$ from (i)

$\Rightarrow \qquad 2I = \frac{\pi}{4}\log 2$

$$I = \frac{\pi}{8}\log 2$$

S77.

$$I = \int_0^{\frac{\pi}{2}} \log\tan x\,dx$$

$$= \int_0^{\frac{\pi}{2}} \log\tan\left(\frac{\pi}{2}-x\right)dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\frac{\pi}{2}} \log\cot x\,dx = -\int_0^{\frac{\pi}{2}} \log\tan x\,dx = -I$$

$\Rightarrow \qquad 2I = 0$

$\Rightarrow \qquad I = 0$

S78. Let $\qquad I = \int_0^{\pi/2} \frac{\cos^5 x}{\sin^5 x + \cos^5 x}\,dx \qquad$ ... (i)

$\Rightarrow \qquad I = \int_0^{\pi/2} \frac{\cos^5\left(\frac{\pi}{2}-x\right)}{\sin^5\left(\frac{\pi}{2}-x\right)+\cos^5\left(\frac{\pi}{2}-x\right)}dx \qquad \left[\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$$= \int_0^{\pi/2} \frac{\sin^5 x}{\cos^5 x + \sin^5 x}dx \Rightarrow I = \int_0^{\pi/2} \frac{\sin^5 x}{\sin^5 x + \cos^5 x}\,dx \qquad ... (ii)$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sin^5 x + \cos^5 x}{\sin^5 x + \cos^5 x}\,dx = \int_0^{\pi/2} 1dx = [x]_0^{\pi/2} = \frac{\pi}{2}$$

$\Rightarrow \qquad I = \left(\frac{\pi}{2\cdot 2}\right) = \frac{\pi}{4}.$

S79. Put $t = \tan\frac{x}{2}$

$\Rightarrow \quad x = 2\tan^{-1} t$

so that $\quad dx = \frac{2dt}{1+t^2}$

Also $\quad \sin x = \frac{2t}{1+t^2}$

When $x = 0$, $t = 0$; when $x = \frac{\pi}{2}$, $t = \tan\frac{\pi}{4} = 1$

$$\therefore \quad \int_0^{\frac{\pi}{2}} \frac{dx}{5+4\sin x} = \int_0^1 \frac{\frac{2dt}{1+t^2}}{5+4\frac{2t}{1+t^2}} = \int_0^1 \frac{2dt}{5+5t^2+8t}$$

$$= \frac{2}{5}\int_0^1 \frac{dt}{t^2+\frac{8}{5}t+1} = \frac{2}{5}\int_0^1 \frac{dt}{\left(t^2+\frac{8}{5}t+\frac{16}{25}\right)+\left(1-\frac{16}{25}\right)}$$

$$= \frac{2}{5}\int_0^1 \frac{dt}{\left(t+\frac{4}{5}\right)^2+\left(\frac{3}{5}\right)^2} = \frac{2}{5}\cdot\frac{1}{\frac{3}{5}}\left[\tan^{-1}\frac{t+\frac{4}{5}}{\frac{3}{5}}\right]_0^1$$

$$\left[\because \int\frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c\right]$$

$$= \frac{2}{3}\left[\tan^{-1}\frac{5t+4}{3}\right]_0^1 = \frac{2}{3}\left[\tan^{-1}(3) - \tan^{-1}\left(\frac{4}{3}\right)\right]$$

$$= \frac{2}{3}\tan^{-1}\frac{3-\frac{4}{3}}{1+3\cdot\frac{4}{3}} = \frac{2}{3}\tan^{-1}\frac{\frac{5}{3}}{5} = \frac{2}{3}\tan^{-1}\frac{1}{3}$$

S80. Let $x^2 = y$. Then,

$$\frac{1}{(x^2+a^2)(x^2+b^2)} = \frac{1}{(y+a^2)(y+b^2)}$$

Let $\frac{1}{(x^2+a^2)(x^2+b^2)} = \frac{A}{y+a^2} + \frac{B}{y+b^2}$ ... (i)

$\Rightarrow \qquad 1 = A(y + b^2) + B(y + a^2) \qquad \text{... (ii)}$

Putting $y = -a^2$ and $y = -b^2$ successively in (ii), we get

$$A = \frac{1}{b^2 - a^2} \text{ and } B = \frac{1}{a^2 - b^2}$$

Substituting the values of A and B in (i), we obtain

$$\frac{1}{(y+a^2)(y+b^2)} = \frac{1}{a^2-b^2}\left[\frac{1}{y+b^2} - \frac{1}{y+a^2}\right]$$

$$\Rightarrow \frac{1}{(x^2+a^2)(x^2+b^2)} = \frac{1}{a^2-b^2}\left[\frac{1}{x^2+b^2} - \frac{1}{x^2+a^2}\right] \qquad [\because \; y = x^2]$$

$$\therefore \int_0^\infty \frac{1}{(x^2+a^2)(x^2+b^2)}\,dx$$

$$= \frac{1}{a^2-b^2}\int_0^\infty \left(\frac{1}{x^2+b^2} - \frac{1}{x^2+a^2}\right)dx$$

$$= \frac{1}{a^2-b^2}\left[\left(\frac{1}{b}\tan^{-1}\frac{x}{b} - \frac{1}{a}\tan^{-1}\frac{x}{a}\right)_0^\infty\right]$$

$$= \frac{1}{a^2-b^2}\left[\left(\frac{1}{b}\tan^{-1}\infty - \frac{1}{a}\tan^{-1}\infty\right) - \left(\frac{1}{b}\tan^{-1}0 - \frac{1}{a}\tan^{-1}0\right)\right]$$

$$= \frac{1}{a^2-b^2}\left[\left(\frac{\pi}{2b} - \frac{\pi}{2a}\right) - (0-0)\right] = \frac{\pi}{2ab(a+b)}$$

S81. We have, $I = \int_0^{\pi/2} \frac{1}{2\cos x + 4\sin x}dx \Rightarrow I = \int_0^{\pi/2} \frac{1}{\frac{2\left(1-\tan^2\frac{x}{2}\right)}{1+\tan^2\frac{x}{2}} + \frac{4\left(2\tan\frac{x}{2}\right)}{1+\tan^2\frac{x}{2}}}dx$

$$\Rightarrow \qquad I = \int_0^{\pi/2} \frac{1+\tan^2\frac{x}{2}}{2-2\tan^2\frac{x}{2}+8\tan\frac{x}{2}}dx = \int_0^{\pi/2}\frac{\sec^2\frac{x}{2}}{2-2\tan^2\frac{x}{2}+8\tan\frac{x}{2}}dx$$

Let $\tan\frac{x}{2} = t$. Then, $d\left(\tan\frac{x}{2}\right) = dt \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}dx = dt$ or, $dx = 2\frac{dt}{\sec^2\frac{x}{2}}$

Also, $x = 0 \Rightarrow t = \tan 0 = 0$ and $x = \frac{\pi}{2} \Rightarrow t = \tan\frac{\pi}{4} = 1$

$$\therefore \quad I = \int_0^1 \frac{\sec^2\frac{x}{2}}{2-2t^2+8t}\cdot\frac{2\,dt}{\sec^2\frac{x}{2}} \Rightarrow I = \int_0^1 \frac{1}{1-t^2+4t}\,dt$$

$$\Rightarrow \quad I = \int_0^1 \frac{1}{-[t^2-4t-1]}\,dt \Rightarrow I = \int_0^1 \frac{1}{-[t^2-4t+4-4-1]}\,dt$$

$$\Rightarrow \quad I = \int_0^1 \frac{dt}{-[(t-2)^2-5]} \Rightarrow I = \int_0^1 \frac{1}{(\sqrt{5})^2-(t-2)^2}\,dt$$

$$\Rightarrow \quad I = \frac{1}{2\sqrt{5}}\left[\log\left|\frac{\sqrt{5}+t-2}{\sqrt{5}-t+2}\right|\right]_0^1 = \frac{1}{2\sqrt{5}}\left[\log\left(\frac{\sqrt{5}-1}{\sqrt{5}+1}\right) - \log\left(\frac{\sqrt{5}-2}{\sqrt{5}+2}\right)\right]$$

$$\Rightarrow \quad I = \frac{1}{2\sqrt{5}}\left[\log\left\{\frac{(\sqrt{5}-1)(\sqrt{5}+2)}{(\sqrt{5}+1)(\sqrt{5}-2)}\right\}\right] = \frac{1}{2\sqrt{5}}\log\left(\frac{3+\sqrt{5}}{3-\sqrt{5}}\right)$$

$$\Rightarrow \quad I = \frac{1}{2\sqrt{5}}\log\left(\frac{3+\sqrt{5}}{3-\sqrt{5}}\times\frac{3+\sqrt{5}}{3+\sqrt{5}}\right) = \frac{1}{2\sqrt{5}}\log\left(\frac{3+\sqrt{5}}{2}\right)^2 = \frac{2}{2\sqrt{5}}\log\left(\frac{3+\sqrt{5}}{2}\right)$$

$$\Rightarrow \quad I = \frac{1}{\sqrt{5}}\log\left(\frac{3+\sqrt{5}}{2}\right)$$

S82.

$$I = \int_0^{\pi/2} (\sqrt{\tan x} + \sqrt{\cot x})\,dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \left\{\frac{\sqrt{\sin x}}{\sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x}}\right\} dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}}\,dx$$

$$\Rightarrow \quad I = \sqrt{2}\int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{2\sin x \cos x}}\,dx$$

$$\Rightarrow \quad I = \sqrt{2}\int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{1-(\sin x - \cos x)^2}}\,dx$$

Let $\sin x - \cos x = t$. Then, $d(\sin x - \cos x) = dt \Rightarrow (\cos x + \sin x)dx = dt$

Now, $x = 0, \Rightarrow t = -1$, and $x = \frac{\pi}{2} \Rightarrow t = 1$

$$\therefore \quad I = \int_0^{\pi/2} \left\{\sqrt{\tan x} + \sqrt{\cot x}\right\} dx$$

$\Rightarrow \quad I = \sqrt{2}\int_{-1}^{1}\frac{dt}{\sqrt{1-t^2}} = \sqrt{2}\left[\sin^{-1} t\right]_{-1}^{1}$

$\Rightarrow \quad I = \sqrt{2}\,[\sin^{-1} 1 - \sin^{-1}(-1)] = \sqrt{2}[2\sin^{-1}(1)] = 2\sqrt{2}\left(\frac{\pi}{2}\right) = \sqrt{2}\,\pi$

S83. Let $\cos x = K(3\cos x + \sin x) + L\frac{d}{dx}(3\cos x + \sin x)$. Then,

$$\cos x = K(3\cos x + \sin x) + L(-3\sin x + \cos x)$$

Comparing coefficient of $\cos x$ and $\sin x$, we get

$$3K + L = 1 \text{ and } K - 3L = 0$$

Solving these two equations, we have

$$K = \frac{3}{10} \text{ and } L = \frac{1}{10}$$

$\therefore \quad \cos x = \frac{3}{10}(3\cos x + \sin x) + \frac{1}{10}(-3\sin x + \cos x)$

$\therefore \quad I = \int_0^{\pi/2}\frac{\cos x}{(3\cos x + \sin x)}dx$

$\Rightarrow \quad I = \int_0^{\pi/2}\frac{\frac{3}{10}(3\cos x + \sin x) + \frac{1}{10}(-3\sin x + \cos x)}{3\cos x + \sin x}dx$

$\Rightarrow \quad I = \frac{3}{10}\int_0^{\pi/2}\frac{3\cos x + \sin x}{3\cos x + \sin x}dx + \frac{1}{10}\int_0^{\pi/2}\frac{-3\sin x + \cos x}{3\cos x + \sin x}dx$

$\Rightarrow \quad I = \frac{3}{10}\int_0^{\pi/2}1\cdot dx + \frac{1}{10}\int_0^{\pi/2}\frac{-3\sin x + \cos x}{3\cos x + \sin x}dx$

$\Rightarrow \quad I = \frac{3}{10}\left[x\right]_0^{\pi/2} + \frac{1}{10}\left[\log|3\cos x + \sin x|\right]_0^{\pi/2}$

$\Rightarrow \quad I = \frac{3}{10}\left(\frac{\pi}{2} - 0\right) + \frac{1}{10}(\log 1 - \log 3) = \frac{3\pi}{20} - \frac{1}{10}\log 3$

S84.

Let $\quad I = \int_0^{\frac{\pi}{2}}\frac{\cos^2 x}{\cos^2 x + 4\sin^2 x}dx \quad \ldots \text{(i)}$

$= \int_0^{\frac{\pi}{2}}\frac{\sec^2 x\,dx}{\sec^2 x + 4\tan^2 x\sec^2 x}$ [Dividing numerator and denominator by $\cos^4 x$]

Put $\tan x = t$ so that $\sec^2 x\,dx = dt$

When $x = 0$, $t = 0$. When $x = \frac{\pi}{2}$, $t \to \infty$

$\therefore$ $$I = \int_0^{\infty} \frac{dt}{(1+t^2)+4t^2(1+t^2)} = \int_0^{\infty} \frac{dt}{(1+t^2)(1+4t^2)}$$

$$= \int_0^{\infty} \left(\frac{-\frac{1}{3}}{1+t^2} + \frac{\frac{4}{3}}{1+4t^2} \right) dt$$ [Partial Fraction]

$$= -\frac{1}{3}\left[\tan^{-1} t\right]_0^{\infty} + \frac{4}{3} \cdot \frac{1}{4} \int_0^{\infty} \frac{dt}{\left(\frac{1}{2}\right)^2 + t^2}$$ $\left[\therefore \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]$

$$= -\frac{1}{3}(\tan^{-1}\infty - \tan^{-1} 0) + \frac{1}{3} \cdot \frac{1}{\frac{1}{2}} \left[\tan^{-1} \frac{t}{\frac{1}{2}}\right]_0^{\infty}$$

$$= -\frac{1}{3}\left[\frac{\pi}{2} - 0\right] + \frac{2}{3}[\tan^{-1} 2t]_0^{\infty}$$

$$= -\frac{\pi}{6} + \frac{2}{3}\left(\tan^{-1}\infty - \tan^{-1} 0^{\circ}\right) = -\frac{\pi}{6} + \frac{2}{3}\left(\frac{\pi}{2} - 0\right)$$

$$= -\frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{6}$$

S85. Let $$I = \int_0^1 \log\left(\frac{1}{x} - 1\right) dx$$

$\Rightarrow$ $$I = \int_0^1 \log\left(\frac{1-x}{x}\right) dx \quad \text{... (i)}$$

$\Rightarrow$ $$I = \int_0^1 \log\left[\frac{1-(1-x)}{1-x}\right] dx$$ $\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

$\Rightarrow$ $$I = \int_0^1 \log\left(\frac{x}{1-x}\right) dx \quad \text{... (ii)}$$

Adding Eqs. (i) and (ii), we get

$$2I = \int_0^1 \log\left(\frac{1-x}{x}\right) dx + \int_0^1 \log\left(\frac{x}{1-x}\right) dx$$

$\Rightarrow$ $$2I = \int_0^1 \left[\log\left(\frac{1-x}{x}\right) + \log\left(\frac{x}{1-x}\right)\right] dx$$

$$= \int_0^1 \log\left[\left(\frac{1-x}{x} \times \frac{x}{1-x}\right)\right] dx$$ $[\because \log m + \log n = \log(m \times n)]$

or $$2I = \int_0^1 \log 1 \, dx$$

$\Rightarrow \qquad 2I = \int_0^1 0\,dx = 0 \qquad [\because \log 1 = 0]$

$\therefore \qquad I = 0$

S86. Let

$$I = \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx \qquad \ldots \text{(i)}$$

$$I = \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\sin\left(\frac{\pi}{2}-x\right)}{4+3\cos\left(\frac{\pi}{2}-x\right)}\right)dx$$

$$= \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\cos x}{4+3\sin x}\right)dx \qquad \ldots \text{(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx + \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\cos x}{4+3\sin x}\right)dx$$

$$= \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\sin x}{4+3\cos x}\right)\left(\frac{4+3\cos x}{4+3\sin x}\right)dx$$

$$= \int_0^{\frac{\pi}{2}} \log(1)\,dx = \int_0^{\frac{\pi}{2}} (0) = 0$$

$\Rightarrow \qquad I = 0.$

Thus, $\int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx = 0.$

S87. Let

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx \quad \text{or} \quad \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{1}{2}} x}{\sin^{\frac{1}{2}} x + \cos^{\frac{1}{2}} x}dx \qquad \ldots \text{(i)}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}dx = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx \qquad \ldots \text{(ii)}$$

Adding Eq. (i) and Eq. (ii),

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx = \int_0^{\frac{\pi}{2}} 1.dx = [x]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

Hence $\quad I = \frac{\pi}{4}$

S88. Let $\quad I = \int_0^{\pi/2} \frac{\cos x}{1 + \cos x + \sin x}\, dx \quad \ldots \text{(i)}$

Then, $\quad I = \int_0^{\pi/2} \frac{\cos\left(\frac{\pi}{2} - x\right)}{1 + \cos\left(\frac{\pi}{2} - x\right) + \sin\left(\frac{\pi}{2} - x\right)}\, dx$

$$I = \int_0^{\pi/2} \frac{\sin x}{1 + \sin x + \cos x}\, dx \quad \ldots \text{(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\cos x + \sin x}{1 + \sin x + \cos x}\, dx \quad \Rightarrow \quad 2I = \int_0^{\pi/2} \frac{1 + \sin x + \cos x - 1}{1 + \sin x + \cos x}\, dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \left\{1 - \frac{1}{1 + \sin x + \cos x}\right\} dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} 1 \cdot dx - \int_0^{\pi/2} \frac{1}{1 + \sin x + \cos x}\, dx$$

$$\Rightarrow \quad 2I = [x]_0^{\pi/2} - \int_0^{\pi/2} \frac{1 + \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2} + 2\tan \frac{x}{2} + 1 - \tan^2 \frac{x}{2}}\, dx$$

$$\Rightarrow \quad 2I = \frac{\pi}{2} - \int_0^{\pi/2} \frac{\sec^2 \frac{x}{2}}{2 + 2\tan \frac{x}{2}}\, dx = \frac{\pi}{2} - \int_0^1 \frac{2\, dt}{2 + 2t}$$

where $t = \tan \frac{x}{2}$

$dt = \frac{1}{2} \sec^2 \frac{x}{2}\, dx$

$$\Rightarrow \quad 2I = \frac{\pi}{2} - [\log(1 + t)]_0^1 = \frac{\pi}{2} - \log 2$$

$$\Rightarrow \quad I = \frac{\pi}{4} - \frac{1}{2} \log 2$$

S89. We have, $\quad I_n = \int_0^{\pi/4} \tan^n x\, dx$

$$\therefore \quad I_{n+2} = \int_0^{\pi/4} \tan^{n+2} x\, dx$$

Now, $I_n + I_{n+2} = \int_0^{\pi/4} \tan^n x\, dx + \int_0^{\pi/4} \tan^{n+2} x\, dx$

$\Rightarrow$ $I_n + I_{n+2} = \int_0^{\pi/4} \left(\tan^n x + \tan^{n+2} x\right) dx$

$\Rightarrow$ $I_n + I_{n+2} = \int_0^{\pi/4} \tan^n x\,(1 + \tan^2 x)\, dx$

$\Rightarrow$ $I_n + I_{n+2} = \int_0^{\pi/4} \tan^n x \sec^2 x\, dx$

Let $t = \tan x$. Then, $dt = \sec^2 x\, dx$

Also, at $x = 0 \Rightarrow t = \tan 0 = 0$ and, $x = \frac{\pi}{4} \Rightarrow t = \tan \frac{\pi}{4} = 1$

$\Rightarrow$ $I_n + I_{n+2} = \int_0^1 t^n\, dt \Rightarrow I_n + I_{n+2} = \left[\frac{t^{n+1}}{n+1}\right]_0^1$

$\Rightarrow$ $I_n + I_{n+2} = \frac{1}{n+1}$ **Proved**

S90. Let $I = \int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$

Putting $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

Lower limit, $x = 0 \Rightarrow \theta = \tan^{-1} 0 = 0$

Upper limit, $x = 1 \Rightarrow \theta = \tan^{-1} 1 = \frac{\pi}{4}$

$\therefore$ $I = \int_0^{\pi/4} \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\cdot \sec^2\theta\, d\theta$

$= \int_0^{\pi/4} \sin^{-1}(\sin 2\theta)\sec^2\theta\, d\theta$ $\left[\because \sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}\right]$

$= 2\int_0^{\pi/4} \theta \sec^2\theta\, d\theta = 2\left([\theta\tan\theta]_0^{\pi/4} - \int_0^{\pi/4} 1.\tan\theta\, d\theta\right)$

$= 2\left[\left(\frac{\pi}{4} - \theta\right) + \int_0^{\pi/4} \frac{-\sin\theta}{\cos\theta}\, d\theta\right] = 2\left(\frac{\pi}{4} + [\log\cos\theta]_0^{\pi/4}\right)$

$= 2\left[\frac{\pi}{4} + \log\cos\frac{\pi}{4} - \log\cos 0\right]$

$= \frac{\pi}{2} + 2\log\frac{1}{\sqrt{2}} - 2\times 0 = \frac{\pi}{2} - \log 2$. **Proved.**

S91. Let $$I = \int_0^{\pi/2} \frac{\sin^2 x}{1+\sin x \cos x}\,dx \text{ . Then,} \qquad \dots \text{(i)}$$

$$I = \int_0^{\pi/2} \frac{\sin^2\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}\,dx \qquad \left[\text{Using}: \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$\Rightarrow$ $$I = \int_0^{\pi/2} \frac{\cos^2 x}{1+\cos x \sin x}\,dx \qquad \dots \text{(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sin^2 x + \cos^2 x}{1+\sin x \cos x}\,dx = \int_0^{\pi/2} \frac{1}{1+\sin x \cos x}\,dx$$

$\Rightarrow$ $$2I = \int_0^{\pi/2} \frac{\sec^2 x}{\sec^2 x + \tan x}\,dx = \int_0^{\pi/2} \frac{\sec^2 x}{1+\tan^2 x + \tan x}\,dx$$

[Dividing Nr and Dr by $\cos^2 x$]

Let $\tan x = t$. Then, $d(\tan x) = dt \Rightarrow \sec^2 x\,dx = dt$

Now, $x = 0 \Rightarrow t = \tan 0 = 0$ and $x = \frac{\pi}{2} \Rightarrow t = \tan\frac{\pi}{2} = \infty$

$\Rightarrow$ $$2I = \int_0^{\infty} \frac{dt}{1+t^2+t} \Rightarrow 2I = \int_0^{\infty} \frac{dt}{\left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$\Rightarrow$ $$2I = \frac{1}{\frac{\sqrt{3}}{2}}\left[\tan^{-1}\left(\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\right]_0^{\infty}$$

$\Rightarrow$ $$2I = \frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)\right]_0^{\infty} = \frac{2}{\sqrt{3}}\left[\tan^{-1}\infty - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right]$$

$\Rightarrow$ $$2I = \frac{2}{\sqrt{3}}\left[\frac{\pi}{2}-\frac{\pi}{6}\right] = \frac{2\pi}{3\sqrt{3}}$$

$\Rightarrow$ $$I = \frac{\pi}{3\sqrt{3}}$$

S92. Let $$I = \int_0^{\pi/2} \log \sin x\,dx \qquad \dots \text{(i)}$$

$$= \int_0^{\pi/2} \log \sin\left(\frac{\pi}{2}-x\right)dx$$

$\Rightarrow \quad I = \int_0^{\pi/2} \log\cos x\, dx \quad$ ... (ii)

Adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi/2} (\log\sin x + \log\cos x)\, dx$$

$$= \int_0^{\pi/2} \log\sin x\cos x \times \frac{2}{2} dx$$

$$= \int_0^{\pi/2} \log\sin 2x dx - \left[\int_0^{\pi/2} \log 2 dx\right]$$

Put $2x = t$ in first integral

$\therefore \quad dx = \frac{dt}{2}$

When $\quad x = 0 \Rightarrow t = 0$

and $\quad x = \frac{\pi}{2} \Rightarrow t = \pi$

$\Rightarrow \quad 2I = \frac{1}{2}\int_0^{\pi} \log\sin t\, dt - \frac{\pi}{2}\log 2$

$\Rightarrow \quad 2I = \frac{2}{2}\int_0^{\pi/2} \log\sin t\, dt - \frac{\pi}{2}\log 2$

$$\left\{\because \int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx, \text{ if } f(2a-x) = f(x) \atop = 0 \quad ,\text{if } f(2a-x) = -f(x)\right\}$$

$\Rightarrow \quad 2I = \int_0^{\pi/2} \log\sin x\, dx - \frac{\pi}{2}\log 2 \quad \left[\because \int_0^{a} f(x)dx = \int_0^{a} f(t)dt\right]$

$\Rightarrow \quad 2I = I - \left(\frac{\pi}{2}\right)\log 2$

$\Rightarrow \quad I = -\frac{\pi}{2}\log 2$

S93. Let $\quad I = \int_0^{1} \cot^{-1}[1 - x + x^2]dx$

$= \int_0^{1} \tan^{-1}\frac{1}{1 - x + x^2} dx \quad \left[\because \cot^{-1} x = \tan^{-1}\frac{1}{x}\right]$

$= \int_0^{1} \tan^{-1}\left[\frac{x + (1-x)}{1 - x(1-x)}\right] dx$

$= \int_0^{1} [\tan^{-1} x + \tan^{-1}(1-x)]dx \quad \left[\because \tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1} x + \tan^{-1} y\right]$

$$= \int_0^1 \tan^{-1} x\,dx + \int_0^1 \tan^{-1}(1-(1-x))dx$$

$$\left[\text{Applying } \int_0^a f(x)dx = \int_0^a f(a-x)dx \text{ in 2nd integral}\right]$$

$$= \int_0^1 \tan^{-1} x dx + \int_0^1 \tan^{-1} x dx$$

$$= 2\int_0^1 \tan^{-1} x dx = 2\int_0^1 \underset{\text{I}}{\tan^{-1} x} \cdot \underset{\text{II}}{1} dx$$

Applying integration by parts, we get

$$= \left[2\left[x\tan^{-1} x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}dx\right] \qquad \left[\because \int \underset{\text{I}}{u}\,\underset{\text{II}}{v}\,dx = u\int v dx - \int\left\{\frac{d}{dx}u\int v dx\right\}dx\right]$$

$$= 2\left[[1\cdot\tan^{-1}(1) - 0] - \frac{1}{2}[\log(1+x^2)]_0^1\right] \quad \left[\because \int\frac{x}{(1+x^2)}dx = \frac{1}{2}\int\frac{2x}{1+x^2}dx = \frac{1}{2}\log(1+x^2)\right]$$

$$= 2\left[\frac{\pi}{4} - \frac{1}{2}(\log 2 - \log 1)\right]$$

$$= 2\left[\frac{\pi}{4} - \frac{1}{2}\log 2\right]$$

$$I = \frac{\pi}{2} - \log 2$$

S94.

Let $$I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x}dx \qquad \dots \text{(i)}$$

$\therefore$ $$I = \int_0^{\frac{\pi}{2}} \frac{\sin^2\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$\Rightarrow$ $$I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos x + \sin x}dx \qquad \dots \text{(ii)}$$

Adding Eq. (i) and Eq. (ii)

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x + \cos^2 x}{\cos x + \sin x}dx = \int_0^{\frac{\pi}{2}} \frac{1}{\sin x + \cos x}dx$$

$$= \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}} \frac{dx}{\cos x\cdot\frac{1}{\sqrt{2}} + \sin x\cdot\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}} \frac{dx}{\cos\left(x-\frac{\pi}{4}\right)}$$

$$= \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}} \sec\left(x - \frac{\pi}{4}\right)dx$$

$$= \frac{1}{\sqrt{2}}\left[\log\left|\sec\left(x-\frac{\pi}{4}\right)+\tan\left(x-\frac{\pi}{4}\right)\right|\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{\sqrt{2}}\left[\log\left|\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right| - \log\left|\sec\left(-\frac{\pi}{4}\right)+\tan\left(-\frac{\pi}{4}\right)\right|\right]$$

$$= \frac{1}{\sqrt{2}}\left[\log\left|\sqrt{2}+1\right| - \log\left|\sqrt{2}-1\right|\right] = \frac{1}{\sqrt{2}}\left[\log\frac{\sqrt{2}+1}{\sqrt{2}-1}\times\frac{\sqrt{2}+1}{\sqrt{2}+1}\right]$$

$$= \frac{1}{\sqrt{2}}\log\frac{\left(\sqrt{2}+1\right)^2}{2-1} = \frac{1}{\sqrt{2}}\log\frac{\left(\sqrt{2}+1\right)^2}{1} = \frac{2}{\sqrt{2}}\log\left(\sqrt{2}+1\right)$$

Hence $I = \sqrt{2}\,\log\left(\sqrt{2}+1\right)$

S95. Given integral is $\int_1^4 (|x-1|+|x-2|+|x-4|)dx$

Here, $f(x) = |x - 1| + |x - 2| + |x - 4|$

First, we split the given function into three subinterval as follows

when $1 < x < 2$

$$f(x) = (x-1) - (x-2) - (x-4) = 5 - x \qquad \text{... (i)}$$

when $2 < x < 3$

$$f(x) = (x-1) + (x-2) - (x-4) = x + 1 \qquad \text{... (ii)}$$

when $3 < x < 4$

$$f(x) = (x-1) + (x-2) - (x-4) = x + 1 \qquad \text{... (iii)}$$

Hence, given integral I can be written as

$$I = \int_1^2 (5-x)dx + \int_2^4 (x+1)dx$$

$$\left[\begin{array}{l}\because \text{From Eqs. (i), (ii) and (iii)} \\ \text{We have } f(x) = \begin{cases} 5-x; & 1<x<2 \\ x+1; & 2<x<4 \end{cases}\end{array}\right]$$

$$\therefore \quad I = \left[5x - \frac{x^2}{2}\right]_1^2 + \left[\frac{x^2}{2}+x\right]_2^4$$

$$= \left[\left(10-\frac{4}{2}\right)-\left(5-\frac{1}{2}\right)\right] + \left[\left(\frac{16}{2}+4\right)-\left(\frac{4}{2}+2\right)\right]$$

$$= \left(10-2-5+\frac{1}{2}\right)+(8+4-2-2)$$

$$= 3+\frac{1}{2}+8$$

$$= \frac{6+1+16}{2}=\frac{23}{2}$$

Hence, $I = \frac{23}{2}$

S96.

Let $I = \int_0^{\frac{\pi}{4}} \frac{\sin x+\cos x}{\cos^2 x+\sin^4 x}\,dx = \int_0^{\frac{\pi}{4}} \frac{4(\sin x+\cos x)}{4(\cos^2 x+\sin^4 x)}\,dx$

$$= \int_0^{\frac{\pi}{4}} \frac{4(\sin x+\cos x)}{2(1+\cos 2x)+(1-\cos 2x)^2}\,dx \qquad \left[\begin{aligned}&\therefore 2\cos^2 x = 1+\cos 2x\\ &\text{and } 2\sin^2 x = 1-\cos 2x\end{aligned}\right]$$

$$= \int_0^{\pi/4} \frac{4(\sin x+\cos x)}{2+2\cos 2x+1+\cos^2 2x-2\cos 2x}\,dx$$

$$= 4\int_0^{\frac{\pi}{4}} \frac{\sin x+\cos x}{3+\cos^2 2x}\,dx = 4\int_0^{\frac{\pi}{4}} \frac{\sin x+\cos x}{4-\sin^2 2x}\,dx$$

Put $\sin x - \cos x = t$ so that $(\cos x + \sin x)\,dx = dt$

When $x = 0$, $t = \sin 0 - \cos 0 = 0 - 1 = -1$. When $x = \frac{\pi}{4}$, $t = \sin\frac{\pi}{4} - \cos\frac{\pi}{4} = 0$

Also, $(\sin x - \cos x)^2 = t^2 \Rightarrow 1 - 2\sin x\cos x = t^2 \Rightarrow \sin 2x = 1 - t^2$.

$$\therefore \quad I = 4\int_{-1}^{0} \frac{dt}{4-(1-t^2)^2} = 4\int_{-1}^{0} \frac{dt}{(2-(1-t^2))(2+1-t^2)} = 4\int_{-1}^{0} \frac{dt}{(3-t^2)(1+t^2)}$$

$$= \int_{-1}^{0} \frac{(1+t^2)+(3-t^2)}{(1+t^2)(3-t^2)}\,dt = \int_{-1}^{0}\left[\frac{1}{(3-t^2)}+\frac{1}{(1+t^2)}\right]dt$$

$$= \frac{1}{2\sqrt{3}}\left[\log\left|\frac{\sqrt{3}+t}{\sqrt{3}-t}\right|\right]_{-1}^{0} + \left[\tan^{-1} t\right]_{-1}^{0} \qquad \left[\begin{aligned}&\therefore \int\frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)\\ &\text{and } \int\frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}+c\end{aligned}\right]$$

$$= \frac{1}{2\sqrt{3}}\left[\log(1) - \log\frac{\sqrt{3}-1}{\sqrt{3}+1}\right] + \left[0 - \left(\frac{-\pi}{4}\right)\right]$$

$$= \frac{1}{2\sqrt{3}}\left[0 - \log\frac{\sqrt{3}-1}{\sqrt{3}+1}\right] + \frac{\pi}{4}$$

$$= \frac{1}{2\sqrt{3}}\log\frac{\sqrt{3}+1}{\sqrt{3}-1} + \frac{\pi}{4}$$

$$= \frac{\pi}{4} + \frac{1}{2\sqrt{3}}\log\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$$

S97. Let $I = \int_0^{\pi} \frac{x \tan x}{\sec x + \cos x}\, dx$

$$I = \int_0^{\pi} \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \cos(\pi - x)}\, dx \qquad \left[\therefore \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \int_0^{\pi} \frac{(\pi - x)(-\tan x)}{-\sec x - \cos x}\, dx$$

$$= \int_0^{\pi} \frac{(\pi - x)\tan x}{\sec x + \cos x}\, dx = \pi\int_0^{\pi} \frac{\tan x}{\sec x + \cos x}\, dx - I$$

$\Rightarrow$ $$2I = \pi\int_0^{\pi} \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x} + \cos x}\, dx = \pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\, dx \qquad \text{...(i)}$$

$\Rightarrow$ $$I = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\, dx$$

Now $$I_1 = \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\, dx = 2\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\, dx$$

Put $\cos x = t$ so that $-\sin x\, dx = dt$ i.e. $\sin x\, dx = -dt$.

When $x = 0$, $t = 1$. when $x = \frac{\pi}{2}, t = 0$

$$I_1 = 2\int_1^0 \frac{-dt}{1+t^2} = 2\int_0^1 \frac{dt}{1+t^2}$$

$$= 2\left[\tan^{-1} t\right]_0^1 = 2\,[\tan^{-1}(1) - \tan^{-1}(0)]$$

$$= 2\left[\frac{\pi}{4} - 0\right] = \frac{\pi}{2}.$$

Form (i), $\quad I = \frac{\pi}{2}\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4}.$

S98. Let
$$I = \int_0^{\pi} \frac{x}{1+\sin x}\,dx \quad \text{... (i)}$$
$$I = \int_0^{\pi} \frac{\pi - x}{1+\sin(\pi - x)}\,dx \quad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$
$$= \int_0^{\pi} \frac{\pi - x}{1+\sin x}\,dx \quad \text{... (ii)} \quad [\because \sin(\pi - x) = \sin x]$$

Adding Eqs. (i) and (ii), we get
$$2I = \int_0^{\pi} \frac{x + \pi - x}{1+\sin x}\,dx$$
$$= \pi\int_0^{\pi} \frac{1}{1+\sin x}\,dx$$
$$= \pi\int_0^{\pi} \frac{1}{1+\sin x} \times \frac{1-\sin x}{1-\sin x}\,dx$$
$$= \pi\int_0^{\pi} \frac{1-\sin x}{\cos^2 x}\,dx$$
$$= \pi\int_0^{\pi} \left[\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}\right]dx$$
$$= \pi\int_0^{\pi} (\sec^2 x - \sec x \tan x)\,dx$$
$$= \pi[\tan x - \sec x]_0^{\pi}$$
$$= \pi[(\tan\pi - \sec\pi) - (\tan 0 - \sec 0)] \quad \left[\begin{array}{l}\because \sec 0^\circ = 1, \sec\pi = -1,\\ \tan 0^\circ = 0, \tan\pi = 0\end{array}\right]$$
$$= \pi[(0 + 1) - (0 - 1)]$$
$\Rightarrow \quad 2I = 2\pi$

$\therefore \quad I = \pi$

S99. Let
$$I = \int_0^{\pi} \frac{x\tan x}{\sec x + \tan x}\,dx \quad \text{... (i)}$$
$$\Rightarrow \quad I = \int_0^{\pi} \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)}\,dx \quad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$\Rightarrow \quad I = \int_0^{\pi} \frac{-(\pi - x)\tan x}{-\sec x - \tan x} dx$

$\Rightarrow \quad I = \int_0^{\pi} \frac{(\pi - x)\tan x}{\sec x + \tan x} dx \quad \ldots \text{(ii)}$

Adding Eqs. (i) and (ii), we get

$$2I = \pi \int_0^{\pi} \frac{\tan x}{(\sec x + \tan x)} dx$$

$\Rightarrow \quad 2I = \pi \int_0^{\pi} \frac{\tan x(\sec x - \tan x)}{(\sec^2 x - \tan^2 x)} dx$

[$\because$ Multiply numerator and denominator by $\sec x - \tan x$]

$\Rightarrow \quad 2I = \pi \int_0^{\pi} \sec x \tan x dx - \pi \int_0^{\pi} \tan^2 x dx \quad [\because \sec^2 x - \tan^2 x = 1]$

$$= \pi[\sec x]_0^{\pi} - \pi \int_0^{\pi} (\sec^2 x - 1) dx$$

$$= \pi[\sec x]_0^{\pi} - \pi[\tan x - x]_0^{\pi}$$

$$= \pi[\sec \pi - \sec 0] - \pi[\tan \pi - \pi - \tan 0 + 0]$$

$$= \pi[-1 - 1] - \pi[0 - \pi]$$

$$= -2\pi + \pi^2 = \pi(\pi - 2)$$

$\Rightarrow \quad I = \frac{\pi}{2}(\pi - 2) = \pi\left(\frac{\pi}{2} - 1\right)$

S100. Given Integral is $\int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx$

Given integral can be written as $I = \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16(1 + \sin 2x - 1)} dx$

$\Rightarrow \quad I = \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16[1 - (1 - \sin 2x)]} dx$

$\Rightarrow \quad I = \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16\left[1 - (\cos^2 x + \sin^2 x - 2 \sin x \cos x)\right]} dx$

$\left[\because 1 = \cos^2 x + \sin^2 x \text{ and } \sin 2x = 2 \sin x \cos x\right]$

$\Rightarrow \quad I = \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16\left[1 - (\cos^2 x + \sin^2 x - 2 \sin x \cos x)\right]} dx$

$\Rightarrow \quad I = \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16\left[1 - (\cos x - \sin x)^2\right]} dx$

Now, we put cos x – sin x = t

$\Rightarrow$ (– sin x – cos x) dx = dt $\Rightarrow$ (sin x + cos x) dx = – dt

When x = 0, t = cos 0 – sin 0 = 1

and when $x = \frac{\pi}{4}$, $t = \cos\frac{\pi}{4} - \sin\frac{\pi}{4}$

$$= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$$

$\therefore$ We get, $I = \int_1^0 \frac{-dt}{9+16(1-t^2)}$ $\Rightarrow$ $I = \int_1^0 \frac{-dt}{9+16-16t^2}$

$$= \int_1^0 \frac{-dt}{25-16t^2} = \int_0^1 \frac{dt}{(5)^2-(4t)^2}$$

$$= \frac{1}{2\times5\times4}\left[\log\left|\frac{5+4t}{5-4t}\right|\right]_0^1 \qquad \left[\because \int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)\right]$$

$$= \frac{1}{40}\left[\log\left|\frac{5+4}{5-4}\right| - \log\left|\frac{5}{5}\right|\right]$$

$$= \frac{1}{40}\left[\log\left(\frac{9}{1}\right) - \log\left(\frac{5}{5}\right)\right]$$

$$= \frac{1}{40}(\log 9 - \log 1)$$

or $I = \frac{1}{40}\log(3)^2 = \frac{2}{40}\log 3$

or $I = \frac{1}{20}\log 3$

S101. Let $I = \int_{-a}^{a}\sqrt{\frac{a-x}{a+x}}dx$

First we rationalize above function

$\therefore$ $I = \int_{-a}^{a}\sqrt{\frac{a-x}{a+x}\times\frac{a-x}{a-x}}dx$

$$= \int_{-a}^{a}\sqrt{\frac{(a-x)^2}{a^2-x^2}}dx = \int_{-a}^{a}\frac{a-x}{\sqrt{a^2-x^2}}dx$$

$\therefore$ $I = a\int_{-a}^{a}\frac{dx}{\sqrt{a^2-x^2}} - \int_{-a}^{a}\frac{x}{\sqrt{a^2-x^2}}dx$

So, we get $I = I_1 - I_2$ ... (i)

Now, $I_2 = \int_{-a}^{a}\frac{x}{\sqrt{a^2-x^2}}dx$

Let $f(x) = \frac{x}{\sqrt{a^2-x^2}}$

$$\Rightarrow \quad f(-x) = \frac{-x}{\sqrt{a^2-(-x)^2}} = \frac{-x}{\sqrt{a^2-x^2}} = -f(x)$$

$$\therefore \quad f(-x) = -f(x)$$

$\Rightarrow$ $f(x)$ is an odd function.

$$\therefore \quad I_2 = \int_{-a}^{a} \frac{x}{\sqrt{a^2-x^2}}dx = 0 \qquad \left[\because \int_{-a}^{a} f(x)dx = 0, \text{ if } f(x) \text{ is an odd function}\right]$$

Now, $$I_1 = a\int_{-a}^{a} \frac{dx}{\sqrt{a^2-x^2}} = a\left[\sin^{-1}\frac{x}{a}\right]_{-a}^{a} \qquad \left[\because \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C\right]$$

$$\Rightarrow \quad I_1 = a\left[\sin^{-1}\frac{a}{a} - \sin^{-1}\left(\frac{-a}{a}\right)\right]$$

$$= a\left[\sin^{-1}(1) - \sin^{-1}(-1)\right]$$

$$= a\left[\sin^{-1}\sin\left(\frac{\pi}{2}\right) - \sin^{-1}\sin\left(-\frac{\pi}{2}\right)\right] \qquad \left[\because 1 = \sin\frac{\pi}{2} \text{ and } -1 = \sin\left(-\frac{\pi}{2}\right)\right]$$

$$= a\left[\frac{\pi}{2} + \frac{\pi}{2}\right]$$

$$\therefore \quad I_1 = a\pi$$

Putting value of $I_1 = a\pi$ and $I_2 = 0$ in Eq. (i), we get

$$I = I_1 - I_2 \text{ or } I = a\pi - 0$$

Hence, $I = a\pi$

S102. Let $$I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$$

Given integral can be written as

$$I = \int_{\pi/6}^{\pi/3} \frac{1}{1+\frac{\sqrt{\sin x}}{\sqrt{\cos x}}}dx$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx \qquad \ldots \text{(i)}$$

We know that $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)dx$

Applying this property in Eq. (i), we get,

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}+\sqrt{\sin\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}}dx$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}+\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}dx$$

$\therefore$ $$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx \quad \text{... (ii)}$$

$$\left[\cos\left(\frac{\pi}{2}-x\right)=\sin x \text{ and } \sin\left(\frac{\pi}{2}-x\right)=\cos x\right]$$

Now, adding Eqs. (i) and (ii), we get

$$2I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$$

$\Rightarrow$ $$2I = \int_{\pi/6}^{\pi/3} 1dx$$

$\Rightarrow$ $$= [x]_{\pi/6}^{\pi/3} \quad \left[\because \int 1\,dx = x + C\right]$$

$\Rightarrow$ $$2I = \frac{\pi}{3}-\frac{\pi}{6}$$

$\Rightarrow$ $$2I = \frac{\pi}{6} \Rightarrow I = \frac{\pi}{12}$$

S103. When $-1 < x < \frac{3}{2} \Rightarrow -\pi < \pi x < \frac{3\pi}{2}$

For $-1 < x < 0$, $-\pi < \pi x < 0$

$\Rightarrow \quad \sin \pi x < 0 \Rightarrow x \sin \pi x > 0 \quad [\because -1 < x < 0]$

$\Rightarrow \quad |x \sin \pi x| = x \sin \pi x$

For $0 < x < 1$, $0 < \pi x < \pi$

$\Rightarrow \quad \sin \pi x > 0 \Rightarrow x \sin \pi x > 0 \quad [\because x > 0]$

$\Rightarrow \quad |x \sin \pi x| = x \sin \pi x$

For $1 < x < \frac{3}{2}$, $\pi < \pi x < \frac{3\pi}{2}$

$\Rightarrow \quad \sin \pi x < 0 \Rightarrow x \sin \pi x < 0 \quad [\because x > 0]$

$\Rightarrow \quad |x \sin \pi x| = -x \sin \pi x$

Thus, $$|x \sin \pi x| = \begin{cases} x\sin \pi x, & \text{if } -1 < x < 1 \\ -x \sin \pi x, & \text{if } 1 < x < \frac{3}{2} \end{cases}$$

$$\therefore \int_{-1}^{\frac{3}{2}} |x \sin \pi x|\, dx = \int_{-1}^{1} |x \sin \pi x|\, dx + \int_{1}^{\frac{3}{2}} |x \sin \pi x|\, dx$$

$$= \int_{-1}^{1} x \sin \pi x\, dx - \int_{1}^{\frac{3}{2}} x \sin \pi x\, dx$$

$$= \left[-x \frac{\cos \pi x}{\pi}\right]_{-1}^{1} - \int_{-1}^{1} (1)\left(\frac{-\cos \pi x}{\pi}\right) dx - \left[x\left(\frac{-\cos \pi x}{\pi}\right)\right]_{1}^{\frac{3}{2}} + \left[\int_{1}^{\frac{3}{2}} (1)\left(\frac{-\cos \pi x}{\pi}\right) dx\right]$$

$$= \left[\frac{-x \cos \pi x}{\pi} + \frac{\sin \pi x}{\pi^2}\right]_{-1}^{1} - \left[\frac{-x \cos \pi x}{\pi} - \frac{\sin \pi x}{\pi^2}\right]_{1}^{\frac{3}{2}}$$

$$= \left[\left(-\frac{\cos \pi}{\pi} + 0\right) - \left(-\frac{\cos(-\pi)}{\pi} + 0\right)\right] - \left[\left(0 - \frac{1}{\pi^2} \sin \frac{3\pi}{2}\right) - \left(-\frac{\cos \pi}{\pi} + 0\right)\right]$$

$$= \frac{2}{\pi} - \left[\frac{-1}{\pi^2} - \frac{1}{\pi}\right] = \frac{3}{\pi} + \frac{1}{\pi^2} = \frac{3\pi + 1}{\pi^2}$$

S104. We have, $I = \int_{-\pi}^{\pi} \frac{2x(1 + \sin x)}{1 + \cos^2 x}\, dx$

$\Rightarrow$ $I = \int_{-\pi}^{\pi} \frac{2x}{1 + \cos^2 x}\, dx + \int_{-\pi}^{\pi} \frac{2x \sin x}{1 + \cos^2 x}\, dx$

$\Rightarrow$ $I = I_1 + I_2$, where $I_1 = \int_{-\pi}^{\pi} \frac{2x}{1 + \cos^2 x}\, dx$

and $I_2 = \int_{-\pi}^{\pi} \frac{2x \sin x}{1 + \cos^2 x}\, dx$

Since $f(x) = \frac{2x}{1 + \cos^2 x}$ is an odd function and $g(x) = \frac{2x \sin x}{1 + \cos^2 x}$ is an even function.

$\therefore$ $I_1 = 0$ and $I_2 = 2\int_{0}^{\pi} \frac{2x \sin x}{1 + \cos^2 x}\, dx$

Now, $I_2 = 4\int_{0}^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, dx$ … (i)

$\Rightarrow$ $I_2 = 4\int_{0}^{\pi} \frac{(\pi - x) \sin(\pi - x)}{1 + \cos^2(\pi - x)}\, dx$

$$\Rightarrow \qquad I_2 = 4\int_0^{\pi}\frac{(\pi - x)\sin x}{1+\cos^2 x}\,dx \qquad \ldots \text{(ii)}$$

Adding (i) and (ii), we get

$$\Rightarrow \qquad 2I_2 = 4\pi\int_0^{\pi}\frac{1}{1+\cos^2 x}\sin x\,dx$$

$$\Rightarrow \qquad 2I_2 = -4\pi\int_1^{-1}\frac{1}{1+t^2}\,dt \qquad \text{where} \quad t = \cos x$$

$$dt = -\sin x\,dx$$

$$\Rightarrow \qquad 2I_2 = -4\pi\left[\tan^{-1} t\right]_1^{-1}$$

$$\Rightarrow \qquad 2I_2 = -4\pi\left\{-\frac{\pi}{4}-\frac{\pi}{4}\right\} = 2\pi^2$$

$$\Rightarrow \qquad I_2 = \pi^2$$

Hence, $I = 0 + \pi^2 = \pi^2$

S105. Let $I = \int_0^{\pi/2} 2\sin x\cos x\tan^{-1}(\sin x)dx$

Put $\sin x = t \Rightarrow \cos x\,dx = dt$

If $x = 0, t = \sin 0 = 0$

and if $x = \frac{\pi}{2}, t = \sin\frac{\pi}{2} = 1$

$\therefore$ We can write

$$I = 2\int_0^1 t\times\tan^{-1} t\,dt$$

Applying integration by parts taking $\tan^{-1} t$ as 1st function and t as 2nd function.

$$I = 2\left[\frac{t^2}{2}\times\tan^{-1} t\right]_0^1 - 2\int_0^1\frac{1}{1+t^2}\times\frac{t^2}{2}dt \qquad \left[\because \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}\right]$$

$$\Rightarrow \qquad I = 2\left[\frac{t^2}{2}\times\tan^{-1} t\right]_0^1 - \int_0^1\frac{t^2}{1+t^2}dt$$

$$\Rightarrow \qquad I = 2\times\frac{1}{2}\times\tan^{-1} 1 - \int_0^1\frac{1+t^2-1}{1+t^2}dt$$

$$= 1\times\frac{\pi}{4} - \int_0^1\left(\frac{1+t^2}{1+t^2}-\frac{1}{1+t^2}\right)dt$$

$$\Rightarrow \qquad I = \frac{\pi}{4} - [t - \tan^{-1} t]_0^1$$

$$= \frac{\pi}{4} - 1 + \tan^{-1} 1$$

$$= \frac{\pi}{4} - 1 + \frac{\pi}{4} = \frac{2\pi}{4} - 1 \qquad \left[\because 1 = \tan\frac{\pi}{4}\right]$$

$$I = \left(\frac{\pi}{2} - 1\right)$$

S106. Let $I = \int_0^{\pi} \frac{x\,dx}{a^2\cos^2 x + b^2\sin^2 x}$ ... (i)

$\Rightarrow$ $I = \int_0^{\pi} \frac{(\pi - x)\,dx}{a^2\cos^2(\pi - x) + b^2\sin^2(\pi - x)}$ $\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

$$I = \int_0^{\pi} \frac{(\pi - x)}{a^2\cos^2 x + b^2\sin^2 x}dx \qquad \text{... (ii)}$$

Adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi} \frac{(x + \pi - x)dx}{a^2\cos^2 x + b^2\sin^2 x}$$

$$2I = \pi\int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$$

Now, we know that

$$\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx, \qquad \text{if } f(2a - x) = f(x)$$

$\therefore$ $2I = 2\pi\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$

Divide numerator and denominator by $\cos^2 x$, we get

$$2I = 2\pi\int_0^{\pi/2} \frac{\sec^2 x\,dx}{a^2 + b^2\tan^2 x}$$

Put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

When $x = 0$, $t = 0$ and when $x = \frac{\pi}{2}$, $t = \infty$

$\therefore$ $I = \pi\int_0^{\infty} \frac{dt}{a^2 + b^2t^2}$

$$= \pi\int_0^{\infty} \frac{dt}{a^2 + (bt)^2}$$

$$= \frac{\pi}{b^2}\int_0^{\infty} \frac{dt}{\left(\frac{a}{b}\right)^2 + (t)^2}$$

$$I = \frac{\pi}{ab}\left[\tan^{-1}\frac{bt}{a}\right]_0^{\infty} \qquad \left[\because \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$I = \frac{\pi}{ab}[\tan^{-1}\infty - \tan^{-1}0]$$

$$I = \frac{\pi}{ab}\left[\frac{\pi}{2} - 0\right] \left[\because \tan^{-1}\infty = \tan^{-1}\tan\frac{\pi}{2} = \frac{\pi}{2} \text{ and } \tan^{-1}0 = \tan^{-1}\tan 0 = 0\right]$$

Hence, $$I = \frac{\pi^2}{2ab}$$

S107. Let $$I = \int_0^{2\pi} \frac{x\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}\,dx \qquad \ldots \text{(i)}$$

Then, $$I = \int_0^{2\pi} \frac{(2\pi - x)\sin^{2n}(2\pi - x)}{\sin^{2n}(2\pi - x) + \cos^{2n}(2\pi - x)}\,dx$$

$$\Rightarrow \quad I = \int_0^{2\pi} \frac{(2\pi - x)\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}\,dx \qquad \ldots \text{(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{2\pi} \left(\frac{x\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x} + \frac{(2\pi - x)\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}\right)dx$$

$$\Rightarrow \quad 2I = \int_0^{2\pi} \frac{2\pi\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}\,dx$$

$$\Rightarrow \quad I = \pi\int_0^{2\pi} \frac{\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}\,dx$$

$$\Rightarrow \quad I = 2\pi\int_0^{\pi} \frac{\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}\,dx$$

$$\Rightarrow \quad I = 4\pi\int_0^{\pi/2} \frac{\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}\,dx \qquad \ldots \text{(iii)}$$

$$\Rightarrow \quad I = 4\pi\int_0^{\pi/2} \frac{\sin^{2n}\left(\frac{\pi}{2} - x\right)}{\sin^{2n}\left(\frac{\pi}{2} - x\right) + \cos^{2n}\left(\frac{\pi}{2} - x\right)}\,dx$$

$$\Rightarrow \quad I = 4\pi\int_0^{\pi/2} \frac{\cos^{2n}x}{\cos^{2n}x + \sin^{2n}x}\,dx \qquad \ldots \text{(iv)}$$

Adding (iii) and (iv), we get

$$2I = 4\pi\int_0^{\pi/2} 1.\,dx = 4\pi \times \frac{\pi}{2} \quad \Rightarrow \quad I = \pi^2$$

S108. Let $$I = \int_0^{\pi/2} f(\sin 2x)\sin x\, dx \qquad \ldots \text{(i)}$$

Then, $$I = \int_0^{\pi/2} f\left\{\sin 2\left(\frac{\pi}{2} - x\right)\right\} \sin\left(\frac{\pi}{2} - x\right) dx$$

$\Rightarrow$ $$I = \int_0^{\pi/2} f\{\sin(\pi - 2x)\} \cos x\, dx$$

$\Rightarrow$ $$I = \int_0^{\pi/2} f\{\sin 2x\} \cos x\, dx \qquad \ldots \text{(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} f(\sin 2x)\cdot(\sin x + \cos x)\, dx$$

$\Rightarrow$ $$2I = 2\int_0^{\pi/4} f(\sin 2x)\cdot(\sin x + \cos x)\, dx$$

$\Rightarrow$ $$2I = 2\sqrt{2}\int_0^{\pi/4} f(\sin 2x)\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right) dx$$

$\Rightarrow$ $$2I = 2\sqrt{2}\int_0^{\pi/4} f(\sin 2x)\sin\left(x + \frac{\pi}{4}\right) dx$$

$\Rightarrow$ $$2I = 2\sqrt{2}\int_0^{\pi/4} f\left\{\sin 2\left(\frac{\pi}{4} - x\right)\right\} \sin\left(\frac{\pi}{4} - x + \frac{\pi}{4}\right) dx$$

$\Rightarrow$ $$2I = 2\sqrt{2}\int_0^{\pi/4} f\left\{\sin\left(\frac{\pi}{2} - 2x\right)\right\} \sin\left(\frac{\pi}{2} - x\right) dx$$

$\Rightarrow$ $$2I = 2\sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x\, dx$$

$\therefore$ $$I = \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x\, dx$$

Hence, $$\int_0^{\pi/2} f(\cos 2x)\cos x\, dx = \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x\, dx$$

S109. Let $I = \int_0^{\pi/2} \frac{x\sin x\cos x}{\sin^4 x + \cos^4 x}\, dx \qquad \ldots \text{(i)}$

Using $\int_0^a f(x)dx = \int_0^a f(a - x)dx$, we get

$$I = \int_0^{\pi/2} \frac{\left[\left(\frac{\pi}{2}-x\right)\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)\right]}{\sin^4\left(\frac{\pi}{2}-x\right)+\cos^4\left(\frac{\pi}{2}-x\right)}dx$$

$\Rightarrow$ $$I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2}-x\right)\cos x \sin x}{\cos^4 x + \sin^4 x}dx \quad \left[\because \sin\left(\frac{\pi}{2}-x\right)=\cos x \text{ and } \cos\left(\frac{\pi}{2}-x\right)=\sin x\right]$$

$\Rightarrow$ $$I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x}dx - \int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x}dx \quad \dots \text{(ii)}$$

Adding Eqs. (i) and (ii), we get

$$2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x}dx$$

$$2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin x \cos x \, dx}{\sin^4 x + (1-\sin^2 x)^2} \quad \left[\because \cos^2 x = 1 - \sin^2 x\right]$$

Now, put $\sin^2 x = t$

$\Rightarrow$ $2 \sin x \cos x \, dx = dt$

$\Rightarrow$ $\sin x \cos x \, dx = \frac{dt}{2}$

Also, when $x = 0$, $t = \sin^2 0 = 0$

and when $x = \frac{\pi}{2}, t = \sin^2 \frac{\pi}{2} = 1$

$\therefore$ We have,

$$2I = \frac{\pi}{4}\int_0^1 \frac{dt}{t^2 + (1-t)^2}$$

$\Rightarrow$ $$I = \frac{\pi}{8}\int_0^1 \frac{dt}{t^2 + (1-t)^2}$$

$$= \frac{\pi}{8}\int_0^1 \frac{dt}{t^2 + 1 + t^2 - 2t}$$

$$= \frac{\pi}{8}\int_0^1 \frac{dt}{2t^2 - 2t + 1}$$

$$= \frac{\pi}{8 \times 2}\int_0^1 \frac{1}{t^2 - t + \frac{1}{2}}dt$$

$$= \frac{\pi}{16}\int_0^1 \frac{1}{t^2 - t + \frac{1}{2} + \frac{1}{4} - \frac{1}{4}}\,dt$$

$$\Rightarrow \quad I = \frac{\pi}{16}\int_0^1 \frac{1}{\left(t-\frac{1}{2}\right)^2 - \frac{1}{4} + \frac{1}{2}}\,dt \qquad \left[\because t^2 - t + \frac{1}{4} = \left(t-\frac{1}{2}\right)^2\right]$$

$$I = \frac{\pi}{16}\int_0^1 \frac{1}{\left(t-\frac{1}{2}\right)^2 + \frac{1}{4}}\,dt$$

$$I = \frac{\pi}{16}\int_0^1 \frac{1}{\left(t-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}\,dt$$

$$I = \frac{\pi}{16}\cdot\frac{1}{\frac{1}{2}}\tan^{-1}\left|\frac{t-\frac{1}{2}}{\frac{1}{2}}\right|_0^1 \qquad \left[\because \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C\right]$$

$$= \frac{\pi}{8}[\tan^{-1}1 - \tan^{-1}(-1)]$$

$$= \frac{\pi}{8}\left[\tan^{-1}\tan\frac{\pi}{4} - \tan^{-1}\tan\left(-\frac{\pi}{4}\right)\right]$$

$$= \frac{\pi}{8}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] \qquad \left[\because 1 = \tan\frac{\pi}{4} \text{ and } -1 = \tan\left(-\frac{\pi}{4}\right)\right]$$

$$\Rightarrow \quad I = \frac{\pi^2}{16}$$

S110. $$I = \int_0^1 x(\tan^{-1}x)^2\,dx.$$

Put $\tan^{-1}x = \theta$ *i.e.*, $x = \tan\theta$ so that $dx = \sec^2\theta\,d\theta$.

when $x = 0$, $\tan\theta = 0 \Rightarrow \theta = 0$

when $x = 1$, $\tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

$$\therefore \quad I = \int_0^{\frac{\pi}{4}} \tan\theta\cdot\theta^2.\sec^2\theta\,d\theta = \int_0^{\frac{\pi}{4}} \theta^2(\tan\theta\sec^2\theta)\,d\theta$$

$$= \left[\theta^2 \frac{\tan^2\theta}{2}\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} 2\theta.\frac{\tan^2\theta}{2}\,d\theta \qquad \text{[Integrating by parts]}$$

$$= \left(\frac{\pi^2}{16}\cdot\frac{1}{2} - 0\right) - \int_0^{\frac{\pi}{4}} \theta\cdot\tan^2\theta\,d\theta$$

$$= \frac{\pi^2}{32} - I_1 \qquad \text{...(i)}$$

Now $$I_1 = \int_0^{\frac{\pi}{4}} \theta\cdot\tan^2\theta\,d\theta = \int_0^{\frac{\pi}{4}} \theta(\sec^2\theta - 1)d\theta$$

$$= \int_0^{\frac{\pi}{4}} \theta\cdot\sec^2\theta\,d\theta - \int_0^{\frac{\pi}{4}} \theta\,d\theta$$

$$= [\theta.\tan\theta]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} (1)\tan\theta\,d\theta - \left[\frac{\theta^2}{2}\right]_0^{\frac{\pi}{4}} \qquad \text{[Integrating by parts]}$$

$$= \left(\frac{\pi}{4} - 0\right) - [\log|\sec\theta|]_0^{\frac{\pi}{4}} - \left[\frac{\pi^2}{32} - 0\right]$$

$$= \frac{\pi}{4} - \left[\log\sqrt{2} - 0\right] - \frac{\pi^2}{32} = \frac{\pi}{4} - \frac{1}{2}\log 2 - \frac{\pi^2}{32}.$$

Putting in (i), $$I = \frac{\pi^2}{32} - \left(\frac{\pi}{4} - \frac{1}{2}\log 2 - \frac{\pi^2}{32}\right)$$

$$= \left(\frac{\pi^2}{32} + \frac{\pi^2}{32}\right) - \frac{\pi}{4} + \frac{1}{2}\log 2$$

$$= \frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2}\log 2 = \frac{\pi}{4}\left(\frac{\pi}{4} - 1\right) + \frac{1}{2}\log 2$$

S111. Let $$I = \int_0^{\pi/2} \frac{x}{\sin x + \cos x}\,dx \qquad \text{... (i)}$$

$$I = \int_0^{\pi/2} \frac{\frac{\pi}{2} - x}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)}\,dx \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)dx\right]$$

$$I = \int_0^{\pi/2} \frac{\frac{\pi}{2} - x}{\cos x + \sin x}\,dx \qquad \text{... (ii)}$$

Adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{x + \frac{\pi}{2} - x}{\sin x + \cos x} dx$$

$$\Rightarrow \quad 2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$$

Dividing numerator and denominator by $\sqrt{2}$, we get

$$2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{\frac{1}{\sqrt{2}} dx}{\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x} \qquad \left[\because \cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}\right]$$

$$= \frac{\pi}{2} \cdot \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{dx}{\cos\frac{\pi}{4} \sin x + \sin\frac{\pi}{4} \cos x}$$

$$= \frac{\pi}{2\sqrt{2}} \int_0^{\pi/2} \frac{dx}{\sin\left(x + \frac{\pi}{4}\right)} \qquad [\because \sin x \cos y + \cos x \sin y = \sin(x + y)]$$

$$= \frac{\pi}{2\sqrt{2}} \int_0^{\pi/2} \operatorname{cosec}\left(x + \frac{\pi}{4}\right) dx \qquad \left[\because \frac{1}{\sin\theta} = \operatorname{cosec}\theta\right]$$

$$= \frac{\pi}{2\sqrt{2}} \left[\log\left|\operatorname{cosec}\left(x + \frac{\pi}{4}\right) - \cot\left(x + \frac{\pi}{4}\right)\right|\right]_0^{\pi/2} \qquad \left[\because \int \operatorname{cosec} x dx = \log|\operatorname{cosec} x - \cot x|\right]$$

$$= \frac{\pi}{2\sqrt{2}} \left[\log\left|\operatorname{cosec}\left(\frac{\pi}{2} + \frac{\pi}{4}\right) - \cot\left(\frac{\pi}{2} + \frac{\pi}{4}\right)\right| - \log\left|\operatorname{cosec}\frac{\pi}{4} - \cot\frac{\pi}{4}\right|\right]$$

$$= \frac{\pi}{2\sqrt{2}} \left[\log\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \log\left|\operatorname{cosec}\frac{\pi}{4} - \cot\frac{\pi}{4}\right|\right]$$

$$\left[\because \operatorname{cosec}\left(\frac{\pi}{2} + x\right) = \sec x, \cot\left(\frac{\pi}{2} + x\right) = -\tan x\right]$$

$$= \frac{\pi}{2\sqrt{2}} \left[\log|\sqrt{2} + 1| - \log|\sqrt{2} - 1|\right]$$

$$= \frac{\pi}{2\sqrt{2}} \log\left|\frac{\sqrt{2} + 1}{\sqrt{2} - 1}\right| \qquad \left[\because \log m - \log n = \log\frac{m}{n}\right]$$

$$= \frac{\pi}{2\sqrt{2}} \log\left|\frac{\sqrt{2} + 1}{\sqrt{2} - 1} \times \frac{\sqrt{2} + 1}{\sqrt{2} + 1}\right|$$

$$= \frac{\pi}{2\sqrt{2}} \log\left|\frac{(\sqrt{2} + 1)^2}{2 - 1}\right|$$

$$= \frac{\pi}{2\sqrt{2}} \cdot 2 \log|\sqrt{2} + 1|$$

$$= \frac{\pi}{\sqrt{2}} \log\left|\sqrt{2}+1\right|$$

$\therefore$ $$I = \frac{\pi}{2\sqrt{2}} \log\left|\sqrt{2}+1\right|$$

S112. Let $$I = \int_0^a \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx$$

Put $$x = a\tan^2 t$$

$\Rightarrow$ $$dx = 2a\tan t\sec^2 t\,dt$$

Now, when $x = 0$, $t = 0$ and at $x = a$, $t = \frac{\pi}{4}$

$\therefore$ $$I = \int_0^{\pi/4} \sin^{-1}\sqrt{\frac{a\tan^2 t}{a+a\tan^2 t}}\;2a\tan t\sec^2 t\,dt$$

$$= \int_0^{\pi/4} \sin^{-1}\sqrt{\frac{a\tan^2 t}{a\sec^2 t}}\cdot 2a\tan t\sec^2 t\,dt \qquad \left[\because 1+\tan^2 x = \sec^2 x\right]$$

$$= \int_0^{\pi/4} \sin^{-1}\left(\frac{\sin t}{\cos t}\cdot\cos t\right)2a\tan t\sec^2 t\,dt$$

$$= 2a\int_0^{\pi/4} \sin^{-1}(\sin t)\tan t\sec^2 t\,dt$$

$$= 2a\int_0^{\pi/4} t\tan t\sec^2 t\,dt$$

Now, taking t as 1st function and tan t sec^2 t *as* 2nd function and applying integration by parts, we get

$$= 2a\left[t\int_0^{\pi/4}\tan t\sec^2 t\,dt - \int_0^{\pi/4}\left\{\frac{d}{dt}(t)\int_0^{\pi/4}\tan t\sec^2 t\,dt\right\}dt\right]$$

$\Rightarrow$ $$I = 2a\left[\left[t\frac{\tan^2 t}{2}\right]_0^{\pi/4} - \int_0^{\pi/4} 1\times\frac{\tan^2 t}{2}\,dt\right. \qquad \left[\begin{array}{l}\because \int\tan t\sec^2 t\,dt \text{ put } \tan t = z \Rightarrow \sec^2 t\,dt = dz\\ \therefore \int z\,dz = \frac{z^2}{2} = \frac{\tan^2 t}{2} \quad = dz\end{array}\right]$$

$\Rightarrow$ $$= 2a\left[\frac{\pi}{4}\frac{\tan^2\cdot\left(\frac{\pi}{4}\right)}{2} - 0 - \frac{1}{2}\int_0^{\pi/4}\tan^2 t\,dt\right]$$

$$= 2a\left[\frac{\pi}{8} - \frac{1}{2}\int_0^{\pi/4}(\sec^2 t - 1)\,dt\right]$$

$$= \frac{\pi a}{4} - a[\tan t - t]_0^{\pi/4} = \frac{\pi a}{4} - a\left(\tan\frac{\pi}{4} - \frac{\pi}{4} - 0\right)$$

$$= \frac{\pi a}{4} - a\left(1 - \frac{\pi}{4}\right) = \frac{\pi a}{4} - a + \frac{\pi a}{4}$$

$$= \frac{2\pi a}{4} - a = \frac{\pi a}{2} - a = a\left(\frac{\pi - 2}{2}\right)$$

S113. Let $I = \int_0^{\pi/2}(2\log\sin x - \log\sin 2x)dx$

$$I = \int_0^{\pi/2}(2\log\sin x - \log 2\sin x\cos x)dx \qquad [\because \sin 2x = 2\sin x\cos x]$$

$$= \int_0^{\pi/2}[2\log\sin x - (\log 2 + \log\sin x + \log\cos x)]dx$$

$$I = \int_0^{\pi/2}(2\log\sin x - \log 2 - \log\sin x - \log\cos x)dx$$

$$[\because \log(mnp) = \log m + \log n + \log p]$$

$$I = \int_0^{\pi/2}(\log\sin x - \log 2 - \log\cos x)dx$$

$$I = \int_0^{\pi/2}\log\sin x\,dx - \int_0^{\pi/2}\log 2\,dx - \int_0^{\pi/2}\log\cos x\,dx$$

$$I = \int_0^{\pi/2}\log\sin\left(\frac{\pi}{2} - x\right)dx - \log 2[x]_0^{\pi/2} - \int_0^{\pi/2}\log\cos x dx$$

$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx \text{ use in 1st integral}\right]$$

$$I = \int_0^{\pi/2}\log\cos x dx - \log 2\left[\frac{\pi}{2} - 0\right] - \int_0^{\pi/2}\log\cos x dx$$

$\therefore$ $$I = -\frac{\pi}{2}\log 2$$

S114.

$$I = \int_0^{\frac{\pi}{2}}\log\left(\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right)dx = \int_0^{\frac{\pi}{2}}\log\left(\frac{\sin^2 x + \cos^2 x}{\sin x\cos x}\right)dx$$

$$= \int_0^{\frac{\pi}{2}}\log\left(\frac{1}{\sin x\,\cos x}\right)dx = \int_0^{\frac{\pi}{2}}(\log 1 - \log\sin x - \log\cos x)\,dx$$

$$= \int_0^{\frac{\pi}{2}}(0 - \log\sin x - \log\cos x)\,dx = -\int_0^{\frac{\pi}{2}}\log\sin\left(\frac{\pi}{2} - x\right)dx - \int_0^{\frac{\pi}{2}}\log\cos x\,dx$$

$$= -2\int_0^{\frac{\pi}{2}}\log\cos x\,dx \qquad \ldots \text{(i)}$$

Let $\qquad I_1 = \int_0^{\pi/2} \log\cos x\, dx \qquad$ ... (ii)

$$= \int_0^{\pi/2} \log\cos\left(\frac{\pi}{2} - x\right) dx$$

$\Rightarrow \qquad I_1 = \int_0^{\pi/2} \log\sin x\, dx \qquad$ ... (iii)

Adding Eqs. (ii) and (iii), we get

$$2I_1 = \int_0^{\pi/2} (\log\cos x + \log\sin x)\, dx = \int_0^{\pi/2} \log\sin x\cos x \times \frac{2}{2} dx$$

$$= \int_0^{\pi/2} \log\sin 2x dx - \left[\int_0^{\pi/2} \log 2 dx\right]$$

Put $2x = t$ in first integral

$\therefore \qquad dx = \dfrac{dt}{2}$

When $\qquad x = 0 \Rightarrow t = 0$

and $\qquad x = \dfrac{\pi}{2} \Rightarrow t = \pi$

$\Rightarrow \qquad 2I_1 = \dfrac{1}{2}\int_0^{\pi} \log\sin t\, dt - \dfrac{\pi}{2}\log 2$

$\Rightarrow \qquad 2I_1 = \dfrac{2}{2}\int_0^{\pi/2} \log\sin t\, dt - \dfrac{\pi}{2}\log 2 \qquad \left\{\because \int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx, \text{ if } f(2a-x) = f(x) \atop = 0 \qquad , \text{if } f(2a-x) = -f(x)\right\}$

$\Rightarrow \qquad 2I_1 = \int_0^{\pi/2} \log\sin x\, dx - \dfrac{\pi}{2}\log 2 \qquad \left[\because \int_0^{a} f(x)dx = \int_0^{a} f(t)dt\right]$

$\Rightarrow \qquad 2I_1 = I_1 - \left(\dfrac{\pi}{2}\right)\log 2$

$\Rightarrow \qquad I_1 = -\dfrac{\pi}{2}\log 2$

From (i)

$$I = -2\left(-\frac{\pi}{2}\log 2\right)$$

$$= \pi \log 2.$$

S115. Let $\qquad I = \int_0^{\pi} \log(1 + \cos x)\, dx = \int_0^{\pi} \log\left(2\cos^2\frac{x}{2}\right) dx$

$$= \int_0^{\pi} \left[\log 2 + \log\left(\cos^2\frac{x}{2}\right)\right] dx$$

$$= \log 2 \int_0^{\pi} 1 .\, dx + 2 \int_0^{\pi} \log \cos \frac{x}{2}\, dx$$

$$= \log 2\, [x]_0^{\pi} + 2 I_1 = \pi \log 2 + 2\, I_1 \qquad \dots \text{(i)}$$

Now, $\qquad I_1 = \int_0^{\pi} \log \cos \frac{x}{2} dx$

Put $\frac{x}{2} = t$ so that $dx = 2dt$

When $x = 0$, $t = 0$. When $x = \pi$, $t = \frac{\pi}{2}$

$\therefore \qquad I_1 = \int_0^{\frac{\pi}{2}} \log \cos t (2dt)$

$$= 2 \int_0^{\frac{\pi}{2}} \log \cos t\ dt \qquad \dots \text{(ii)}$$

Let $\qquad I_2 = \int_0^{\frac{\pi}{2}} \log \cos t\ dt \qquad \dots$ (iii)

$$= \int_0^{\pi/2} \log \cos\left(\frac{\pi}{2} - t\right) dt$$

$\Rightarrow \qquad I_2 = \int_0^{\pi/2} \log \sin t\, dt \qquad \dots$ (iv)

Adding Eqs. (iii) and (iv), we get

$$2I_2 = \int_0^{\pi/2} (\log \cos t + \log \sin t)\, dt$$

$$= \int_0^{\pi/2} \log \sin t \cos t \times \frac{2}{2} dt$$

$$= \int_0^{\pi/2} \log \sin 2t dt - \left[\int_0^{\pi/2} \log 2 dt\right]$$

Put $2t = u$ in first integral

$\therefore \qquad dt = \frac{du}{2}$

When $\qquad t = 0 \;\Rightarrow\; u = 0$

and $\qquad t = \frac{\pi}{2} \;\Rightarrow\; u = \pi$

$\Rightarrow \qquad 2I_2 = \frac{1}{2} \int_0^{\pi} \log \sin u\ du - \frac{\pi}{2} \log 2$

$$\Rightarrow \qquad 2I_2 = \frac{2}{2}\int_0^{\pi/2} \log \sin u \, du - \frac{\pi}{2}\log 2$$

$$\left\{ \because \int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx, \text{ if } f(2a-x) = f(x) \atop \qquad\qquad = \ 0 \qquad ,\text{if } f(2a-x) = -f(x) \right\}$$

$$\Rightarrow \qquad 2I_2 = \int_0^{\pi/2} \log \sin t \, dt - \frac{\pi}{2}\log 2 \qquad \left[\because \int_0^a f(x)dx = \int_0^a f(t)dt\right]$$

$$\Rightarrow \qquad 2I_2 = I_2 - \left(\frac{\pi}{2}\right)\log 2$$

$$\Rightarrow \qquad I_2 = -\frac{\pi}{2}\log 2$$

From (ii),

$$I_1 = 2\left(-\frac{\pi}{2}\log 2\right)$$

$$= -\pi \log 2$$

$\therefore$ From (i),

$$I = \pi \log 2 - 2\pi \log 2 = -\pi \log 2$$

S116.

Let $$I = \int_0^{\frac{\pi}{4}} \left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{\tan x + 1}{\sqrt{\tan x}}\, dx$$

Put $\sqrt{\tan x} = t$ *i.e.*, $\tan x = t^2$

So that $\sec^2 x \, dx = 2t\, dt \Rightarrow (1 + \tan^2 x)\, dx = 2t\, dt$

$$\Rightarrow \qquad (1 + t^4)\, dx = 2t\, dt \Rightarrow dx = \frac{2t}{1+t^4}dt$$

When $x = 0$, $t = 0$. When $x = \frac{\pi}{4}$, $t = 1$

$$\therefore \qquad I = \int_0^1 \frac{t^2+1}{t} \cdot \frac{2t}{1+t^4}dt = 2\int_0^1 \frac{t^2+1}{t^4+1}dt$$

$$= 2\int_0^1 \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}\,dt \qquad \text{[Dividing Num. \& Den. by } t^2\text{]}$$

Put $t - \frac{1}{t} = y$ so that $\left(1+\frac{1}{t^2}\right)dt = dy$

Also $t^2 - 2 + \frac{1}{t^2} = y^2 \Rightarrow t^2 + \frac{1}{t^2} = y^2 + 2$

$\therefore$ $$I = 2\int_{t=0}^{1} \frac{dy}{y^2+2}$$

$$= \frac{2}{\sqrt{2}}\left[\tan^{-1}\frac{y}{\sqrt{2}}\right]_{t=0}^{1}$$

$$= \sqrt{2}\left[\tan^{-1}\frac{1}{\sqrt{2}}\left(t-\frac{1}{t}\right)\right]_0^1$$

$$= \sqrt{2}\left[\tan^{-1}(0) - \tan^{-1}(-\infty)\right]$$

$$= \sqrt{2}\left[\tan^{-1}(\infty)\right]$$

$$= \sqrt{2}\,.\frac{\pi}{2}$$

S117. Let $$I = \int_0^{\pi} \frac{x}{1+\sin\alpha\sin x}\,dx \quad \text{... (i)}$$

$\therefore$ $$I = \int_0^{\pi} \frac{(\pi-x)\,dx}{1+\sin\alpha\sin(\pi-x)} \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi} \frac{(\pi-x)\,dx}{1+\sin\alpha\sin x} \quad \text{... (ii)}$$

Adding (i) and (ii), $$2I = \int_0^{\pi} \frac{\pi\,dx}{1+\sin\alpha\sin x} = 2\pi\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin\alpha\sin x}$$

$$\left[\therefore \int_0^{2a} f(x)dx = 2\int_0^a f(x)dx, \text{ if } f(2a-x) = f(x)\right]$$

$\Rightarrow$ $$I = \pi\int_0^{\frac{\pi}{2}} \frac{dx}{1+2\sin\alpha\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$= \pi\int_0^{\frac{\pi}{2}} \frac{\sec^2\frac{x}{2}dx}{1+\tan^2\frac{x}{2}+2\sin\alpha\tan\frac{x}{2}}$$

Put $\tan\frac{x}{2} = t$ so that $\frac{1}{2}\sec^2\frac{x}{2}dx = dt$ *i.e.* $\sec^2\frac{x}{2}dx = 2dt$

When $x = 0$, $t = 0$. When $x = \frac{\pi}{2}$, $t = \tan\frac{\pi}{4} = 1$

$$\therefore \quad I = \pi\int_0^1 \frac{2dt}{1+t^2+2\sin\alpha.t}$$

$$= 2\pi\int_0^1 \frac{dt}{(t+\sin\alpha)^2+\cos^2\alpha}$$

$$= \frac{2\pi}{\cos\alpha}\left[\tan^{-1}\left(\frac{t+\sin\alpha}{\cos\alpha}\right)\right]_0^1 \qquad \left[\therefore \int\frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}+c\right]$$

$$= \frac{2\pi}{\cos\alpha}\left[\tan^{-1}\left(\frac{1+\sin\alpha}{\cos\alpha}\right)-\tan^{-1}\left(\frac{\sin\alpha}{\cos\alpha}\right)\right]$$

$$= \frac{2\pi}{\cos\alpha}\left[\tan^{-1}\left(\frac{\left(\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}\right)^2}{\left(\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}\right)}\right)-\tan^{-1}(\tan\alpha)\right]$$

$$= \frac{2\pi}{\cos\alpha}\left[\tan^{-1}\left(\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}\right)-\alpha\right]$$

$$= \frac{2\pi}{\cos\alpha}\left[\tan^{-1}\left(\frac{1+\tan\frac{\alpha}{2}}{1-\tan\frac{\alpha}{2}}\right)-\alpha\right]$$

$$= \frac{2\pi}{\cos\alpha}\left[\tan^{-1}\left(\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)\right)-\alpha\right]$$

$$= \frac{2\pi}{\cos\alpha}\left[\frac{\pi}{4}+\frac{\alpha}{2}-\alpha\right] = \frac{2\pi}{\cos\alpha}\left[\frac{\pi}{4}-\frac{\alpha}{2}\right]$$

$$= \pi\left(\frac{\pi}{2}-\alpha\right)\sec\alpha$$

S118 Let $$I = \int_0^\pi \frac{x}{1+\sin^2 x}dx \qquad \text{... (i)}$$

$$\therefore \quad I = \int_0^\pi \frac{\pi-x}{1+\sin^2(\pi-x)}dx = \int_0^\pi \frac{\pi-x}{1+\sin^2 x}dx \qquad \left[\therefore \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \pi\int_0^\pi \frac{1}{1+\sin^2 x}dx - \int_0^\pi \frac{x}{1+\sin^2 x}dx$$

$$= \pi\int_0^\pi \frac{1}{1+\sin^2 x}dx - I \qquad \text{[Using (i)]}$$

$$\Rightarrow \qquad 2I = \pi \int_0^{\pi} \frac{1}{1+\sin^2 x} dx$$

$$\Rightarrow \qquad I = \frac{\pi}{2} \int_0^{\pi} \frac{1}{1+\sin^2 x} dx$$

$$\Rightarrow \qquad I = \frac{\pi}{2} \cdot 2 \int_0^{\frac{\pi}{2}} \frac{1}{1+\sin^2 x} dx = \pi \int_0^{\pi/2} \frac{1}{\sin^2 x + \cos^2 x + \sin^2 x} dx$$

$$\left[\therefore \int_0^{2a} f(x)\, dx = 2 \int_0^{a} f(x)\, dx, \text{ if } \quad f(2a-x) = f(x) \right]$$

$$= \pi \int_0^{\frac{\pi}{2}} \frac{1}{2\sin^2 x + \cos^2 x} dx$$

$$= \pi \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{2\tan^2 x + 1} dx \qquad \text{[Dividing Num. and Denom. by } \cos^2 x]$$

Put $\tan x = t$ so that $\sec^2 x\, dx = dt$

When $x = 0$, $t = \tan 0 = 0$. When $x = \frac{\pi}{2}$, $t = \tan\frac{\pi}{2} \to \infty$

$$= \frac{\pi}{2} \int_0^{\infty} \frac{dt}{t^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \frac{\pi}{2} \cdot \frac{1}{\frac{1}{\sqrt{2}}} \left[\tan^{-1} \frac{t}{\frac{1}{\sqrt{2}}} \right]_0^{\infty}$$

$$= \frac{\pi}{\sqrt{2}} \cdot \left[\tan^{-1}\left(\sqrt{2}t\right) \right]_0^{\infty}$$

$$= \frac{\pi}{\sqrt{2}} \left[\tan^{-1}(\infty) - \tan^{-1}(0) \right]$$

$$= \frac{\pi}{\sqrt{2}} \left[\frac{\pi}{2} - 0 \right] = \frac{\pi^2}{2\sqrt{2}}$$

S119.

$$I = \int_0^{\pi} \frac{x}{4 - \cos^2 x} dx = \int_0^{\pi} \frac{(\pi - x)}{4 - \cos^2(\pi - x)} dx \qquad \left[\because \int_0^{a} f(x)\, dx = \int_0^{a} f(a-x)\, dx \right]$$

$$= \int_0^{\pi} \frac{(\pi - x)\, dx}{4 - \cos^2 x} = \pi \int_0^{\pi} \frac{dx}{4 - \cos^2 x} - \int_0^{\pi} \frac{x}{4 - \cos^2 x} dx$$

$$I = \pi \int_0^{\pi} \frac{dx}{4 - \cos^2 x} - I \qquad \left[\because \int_0^{2a} f(x) dx = 2\int_0^{a} f(x) dx, \text{ if } f(2a-x) = f(x) \right]$$

$$\Rightarrow \qquad 2I = 2\pi \int_0^{\frac{\pi}{2}} \frac{dx}{4 - \cos^2 x} \qquad \left[\because \cos^2(\pi - x) = \cos^2 x \right]$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{4\sec^2 x - 1} dx$$ [Dividing Num. and Denom. by $\cos^2 x$]

$$= 2\pi \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{4(1+\tan^2 x)-1} dx = 2\pi \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{3+4\tan^2 x} dx$$

Put $\tan x = t$ so that $\sec^2 x\, dx = dt$

When $x = 0$, $t = 0$. When $x = \frac{\pi}{2}$, $t \to \infty$

$$\therefore \qquad 2I = 2\pi \int_0^{\infty} \frac{dt}{3+4t^2} = \frac{2\pi}{4} \int_0^{\infty} \frac{dt}{\left(\frac{\sqrt{3}}{2}\right)^2 + t^2}$$

$$= \frac{\pi}{2}\left[\frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1} \frac{t}{\frac{\sqrt{3}}{2}}\right]_0^{\infty}$$ $\left[\because \int_0^{\pi} \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]$

$$= \frac{\pi}{2} \cdot \frac{2}{\sqrt{3}} \left[\tan^{-1} \frac{2t}{\sqrt{3}}\right]_0^{\infty}$$

$$= \frac{\pi}{\sqrt{3}} \left[\tan^{-1}(\infty) - \tan^{-1}(0)\right]$$

$$= \frac{\pi}{\sqrt{3}} \left[\frac{\pi}{2} - 0\right] = \frac{\pi^2}{2\sqrt{3}}$$

Hence $\qquad I = \frac{\pi^2}{4\sqrt{3}}$.

S120. Let $\qquad I = \int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$

Let $\qquad \sin x - \cos x = t$

$\therefore \qquad (\cos x + \sin x)\, dx = dt$

Where $\qquad x \to \frac{\pi}{6}, \quad t = \sin\frac{\pi}{6} - \cos\frac{\pi}{6}$

$$= \frac{1}{2} - \frac{\sqrt{3}}{2} = \frac{1-\sqrt{3}}{2}$$

Where $\qquad x \to \frac{\pi}{3}, t = \sin\frac{\pi}{3} - \cos\frac{\pi}{3}$

$$= \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3}-1}{2}$$

Also, $\qquad (\sin x - \cos x)^2 = t^2$

$\Rightarrow \quad \sin^2 x + \cos^2 x - 2 \sin x \cos x = t^2$

$1 - \sin 2x = t^2$

$\Rightarrow \quad \sin 2x = 1 - t^2$

$$\therefore \quad I = \int_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}} = [\sin^{-1} t]_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}}$$

$$= \sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right) - \sin^{-1}\left(\frac{1-\sqrt{3}}{2}\right)$$

$$= \sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right) + \sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right) \qquad [\because \sin^{-1}(-x) = -\sin^{-1} x]$$

$$= 2\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$$