Date: 18/10/2021

Q1. The amount of pollution content added in air in a city due to x-diesel vehicles is given by $P(x) = 0.005x^3 + 0.02x^2 + 30x$. Find the marginal increase in pollution content when 3 diesel vehicles are added.

Q2. The money to be spent for the welfare of the employees of a firm is proportional to the rate of change of its total revenue (marginal revenue). If the total revenue (in Rs.) received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$, find the marginal revenue, when $x = 5$.

Q3. The total cost $C(x)$ associated with the production of x units of an item is given by $C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000$. Find the marginal cost when 17 units are produced.

Q4. Find the rate of change of the volume of a sphere with respect to it's surface area when the radius is 2 cm.

Q5. A balloon which always remain spherical, has a variable diameter $\frac{3}{2}(2x+3)$. Determine the rate of change of volume with respect to x.

Q6. Find the rate of change of the area of a circular disc with respect to it's circumference when the radius is 3 cm.

Q7. A balloon, which always remains spherical, has a variable radius. Find the rate at which it's volume is increasing with respect to it's radius when the radius is 7 cm.

Q8. Find the rate of change of the area of a circle with respect to it's radius. How fast is the area changing with respect to the radius when the radius is 3 cm?

Q9. A stone is dropped into a quiet take and waves move in circles at a speed of 4 cm/sec. At the instant when the radius of the circular wave is 10 cm, how fast is the enclosed area increasing?

Q10. A stone is dropped into a quiet lake and waves move in a circle at a speed of 3.5 cm/sec. At the instant when the radius of the circular wave is 7.5 cm, how fast is the enclosed area increasing?

Q11. The radius of a circle is increasing uniformly at the rate of 4 cm/sec. Find the rate at which the area of the circle is increasing when the radius is 8 cm.

Q12. The total revenue received from the sale of x units of a product is given by $R(x) = 13x^2 + 26x + 15$. Find the marginal revenue when $x = 7$.

Q13. An edge of a variable cube is increasing at the rate of 3 cm per second. How fast is the volume of the cube increasing when the edge is 10 cm long?

Q14. The side of a square is increasing at the rate of 0.2 cm/sec. Find the rate of increase of the perimeter of the square.

Q15. The side of a square sheet is increasing at the rate of 4 cm per minute. At what rate is the area increasing when the side is 8 cm long?

Q16. A circular disc of radius 3 cm is being heated. Due to expansion, its radius increases at the rate of 0.05 cm/sec. Find the rate at which its area is increasing when radius is 3.2 cm.

Q17. The radius of a circle is increasing at the rate of 0.7 cm/sec. What is the rate of increase of its circumference?

Q18. The radius of an air bubble is increasing at the rate of 0.5 cm/sec. At what rate is the volume of the bubble increasing when the radius is 1 cm?

Q19. The radius of a spherical soap bubble is increasing at the rate of 0.2 cm/sec. Find the rate of increase of its surface area, when the radius is 7 cm.

Q20. The radius of a balloon is increasing at the rate of 10 cm/sec. At what rate is the surface area of the balloon increasing when the radius is 15 cm?

Q21. If $y = 7x - x^3$ and x increases at the rate of 4 units per second, how fast is the slope of the curve changing when $x = 2$?

Q22. Find an angle θ, which increase twice as fast as its sine.

Q23. A particle moves along the curve $y = x^2 + 2x$. At what point(s) on the curve are the x and y coordinates of the particle changing at the same rate?

Q24. An edge of a variable cube is increasing at the rate of 10 cm/sec. How fast the volume of the cube is increasing when the edge is 5 cm long?

Q25. A balloon which always remains spherical, is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon is increasing when the radius is 15 cm.

Q26. The radius of a cylinder is increasing at the rate of 2 cm/sec. and its altitude is decreasing at the rate of 3 cm/sec. Find the rate of change of volume when radius is 3 cm and altitude 5 cm.

Q27. The length x of a rectangle is decreasing at the rate of 2 cm/sec and the width y is increasing at the rate of 2 cm/sec. When $x = 12$ cm and $y = 5$ cm, find the rate of change of (i) the perimeter and (ii) the area of the rectangle.

Q28. The length x of a rectangle is decreasing at the rate of 5 cm/min and the width y is increasing at the rate of 4 cm/min. When $x = 8$ cm and $y = 6$ cm, find the rate of change of (i) the perimeter (ii) area of rectangle.

Q29. A particle moves along the curve, $6y = x^3 + 2$. Find the point on the curve at which the y-coordinate is changing 8 times as fast as the x-coordinate.

Q30. The volume of a cube is increasing at a rate of 7 cm^3/sec. How fast is the surface area increasing when the length of an edge is 12 cm?

Q31. The volume of a cube is increasing at a constant rate. Prove that the increase in surface area varies inversely as the length of the edge of the cube.

Q32. A ladder 13 m long leans against a wall. The foot of the ladder is pulled along the ground away from the wall, at the rate of 1.5 m/sec. How fast is the angle θ between the ladder and the ground is changing when the foot of the ladder is 12 m away from the wall.

Q33. The surface area of a spherical bubble is increasing at the rate of 2 cm^2/s. When the radius of the bubble is 6 cm, at which rate is the volume of the bubble increasing?

Q34. A man is walking at the rate of 6.5 km/hr towards the foot of the tower having 120 m high. At what rate is he approaching the top of the tower when he is 50 m away from the tower?

Q35. A kite is 120m high and 130m of string is out. If the kite is moving away horizontally at the rate of 52 m/sec, find the rate at which the string is being paid out.

Q36. A water tank has the slope of an inverted right circular cone with it's axis vertical and vertex lower most. It's semi-vertical angle is $\tan^{-1}(0.5)$. Water is poured into it at a constant rate of 5 cubic metre per hour. Find the rate at which the level of the water is rising at the instant when the depth of water in the tank is 4 m.

Q37. A man 2 metres high, walks at a uniform speed of 6 metres per minute away from a lamp post, 5 metres high. Find the rate at which the length of his shadow increases.

Q38. An air force plane is ascending vertically at the rate of 100 km/h. If the radius of the earth is r km, how fast is the area of the earth, visible from the plane, increasing at 3 minutes after it started ascending? Given that the visible area A at height h is given by $A = 2\pi r^2 \frac{h}{r+h}$.

Q39. A ladder 5 m long is leaning against a wall. Bottom of ladder is pulled along the ground away from wall at the rate of 2 m/s. How fast is the height on the wall decreasing when the foot of ladder is 4 m away from the wall?

Q40. A man is moving away from a tower 41.6 m high at the rate of 2m/sec. Find the rate at which the angle of elevation of the top of tower is changing, when he is at a distance of 30 m from the foot of the tower. Assume that the eye level of the man is 1.6 m above from the ground.

Q41. The two equal sides of an isosceles triangle with fixed base b are decreasing at the rate of 3 cm/sec. How fast is the area decreasing when the two equal sides are equal to the base?

Q42. Sand is pouring from the pipe at the rate of 12 cm^3/s. The falling sand forms a cone on a ground in such a way that the height of cone is always one-sixth of radius of the base. How fast is the height of sand cone increasing when the height is 4 cm?

Q43. Water is dripping out from a conical funnel of semi-vertical angle $\frac{\pi}{4}$ at the uniform rate of 2 cm^2/sec in it's surface area through a tiny hole at the vertex in the bottom. When the slant height of the water is 4 cm, find the rate of decrease of the slant height of the water.

Q44. Water is dripping out from a conical funnel at a uniform rate of 4 cm^3/sec through a tiny hole at the vertex in the bottom. When the slant height of the water is 3 cm, find the rate of decrease of the slant height of the water-cone. Given that the vertical angle of the funnel is 120°.

Q45. An inverted cone has a depth of 10 cm and a base of radius 5 cm. Water is poured into it at the rate of 3/2 c.c. per minute. Find the rate at which the level of water in the cone is rising when the depth is 4 cm.

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S1. Given $P(x) = 0.005x^3 + 0.02x^2 + 30x$

On differentiating w.r.t. x, we get

$$P'(x) = 3 \times 0.005x^2 + 2(0.02)x + 30$$

At $x = 3$

$$P'(3) = 3 \times 0.005 \times 9 + 2(0.02)(3) + 30$$

$$= 0.135 + 0.12 + 30$$

$$= 30.255$$

S2. We know that, marginal revenue is the rate of change of total revenue with respect to the number of units sold.

$\therefore$ Marginal Revenue (MR) = $\dfrac{dR}{dx}$

$$= \frac{d}{dx}(3x^2 + 36x + 5) = 6x + 36$$

When, $x = 5$,

$$MR = 6(5) + 36 = 30 + 36 = 66$$

Hence, the required marginal revenue is Rs. 66.

S3. We have,

$$C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000$$

$$\Rightarrow \quad \frac{d}{dx}C(x) = 0.021x^2 - 0.006x + 15$$

$$\therefore \quad \left(\frac{d}{dx}C(x)\right)_{x=17} = 0.021 \times 17^2 - 0.006 \times 17 + 15$$

$$= 6.069 - 0.102 + 15 = 20.967$$

Hence Marginal cost = Rs. 20.967

S4.

$$V = \frac{4}{3}\pi r^3 \quad \text{and} \quad S = 4\pi r^2$$

$$\Rightarrow \quad \frac{dV}{dr} = 4\pi r^2 \quad \text{and} \quad \frac{dS}{dr} = 8\pi r$$

$\therefore$ $$\frac{dV}{dS} = \frac{dV/dr}{dS/dr} = \frac{4\pi r^2}{8\pi r} = \frac{r}{2} \Rightarrow \left(\frac{dV}{dS}\right)_{r=2} = \frac{2}{2} = 1 \text{ cm}$$

S5. Let V be the volume of the balloon. Then

$$V = \frac{4\pi}{3}\left\{\frac{3}{4}(2x+3)\right\}^3 = \frac{9\pi}{16}\cdot(2x+3)^3$$

$\Rightarrow$ $$\frac{dv}{dx} = \frac{9\pi}{16}\cdot 3(2x+3)^2 \frac{d}{dx}(2x+3)$$

$\Rightarrow$ $$\frac{dv}{dx} = \frac{27\pi}{8}(2x+3)^2.$$

S6. We have,

$$A = \pi r^2, \quad C = 2\pi r$$

$\Rightarrow$ $$\frac{dA}{dr} = 2\pi r \quad \text{and} \quad \frac{dC}{dr} = 2\pi$$

$\therefore$ $$\frac{dA}{dC} = \frac{dA/dr}{dC/dr} = \frac{2\pi r}{2\pi} = r$$

$\Rightarrow$ $$\left(\frac{dA}{dC}\right)_{r=3} = 3 \text{ cm}$$

S7. Let x be the radius and V be the volume of the balloon. Then,

$$V = \frac{4}{3}\pi x^3$$

$\Rightarrow$ $$\frac{dV}{dx} = 4\pi x^2$$

$\Rightarrow$ $$\left(\frac{dV}{dx}\right)_{x=7} = 4\pi(7)^2 = 196\pi \text{ cm}^2$$

Hence the volume is increasing w.r.t. it's radius at the rate of $196\,\pi$ cm^2, when the radius is 7 cm.

S8. Let A be the area of the circle. Then,

$$A = \pi r^2$$

$\Rightarrow$ $$\frac{dA}{dr} = 2\pi r$$

$\therefore$ $$\left(\frac{dA}{dr}\right)_{r=3} = (2\pi \times 3) \text{ cm} = 6\pi \text{ cm}.$$

S9. Let r be the radius of the circular region and A be its area. Then,

$$A = \pi r^2$$

$$\Rightarrow \quad \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

It is given that $\frac{dr}{dt}$ = 4 cm/sec

$$\therefore \quad \frac{dA}{dt} = 8\pi r$$

When $r = 10$ cm, $\frac{dA}{dt} = 80\pi$ cm^2/sec

S10. Let r be the radius and A be the area of the circular wave at any time t. Then,

$$A = \pi r^2 \text{ and } \frac{dr}{dt} = 3.5 \text{ cm/sec.} \qquad \text{[Given]}$$

Now, $A = \pi r^2$

$$\Rightarrow \quad \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$\Rightarrow \quad \frac{dA}{dt} = 2\pi r\,(3.5) = 7\pi r \qquad \left[\because \frac{dr}{dt} = 3.5 \text{ cm/sec}\right]$$

$$\Rightarrow \quad \left(\frac{dA}{dt}\right)_{r=7.5} = 7\pi\,(7.5) = 52.5\pi \text{ cm}^2\text{/sec.}$$

S11. Let r be the radius and A be the area of a circle at any time t. Then,

$$A = \pi r^2 \text{ and } \frac{dr}{dt} = 4 \text{ cm/sec} \qquad \text{[Given]}$$

Now, $A = \pi r^2$

$$\Rightarrow \quad \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$\Rightarrow \quad \left(\frac{dA}{dt}\right)_{r=8} = 2\pi \times 8 \times 4 \text{ cm}^2\text{/sec} = 64\pi \text{ cm}^2\text{/sec.}$$

S12. We have, $R(x) = 13x^2 + 26x + 15$

$$\Rightarrow \quad \frac{dR(x)}{dx} = 26x + 26$$

$$\Rightarrow \quad \left(\frac{dR(x)}{dx}\right)_{x=7} = 26 \times 7 + 26 = 208$$

Hence, marginal revenue = Rs. 208.

S13. We have,

$$V = x^3 \quad \text{and} \quad \frac{dx}{dt} = 3,$$

Now,

$$V = x^3$$

$\Rightarrow$ $$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

$\Rightarrow$ $$= \frac{dV}{dt} = 9x^2$$

$\Rightarrow$ $$\left(\frac{dV}{dt}\right)_{x=10} = 9 \times (10)^2 = 900 \text{ cm}^3/\text{sec}$$

S14. We have,

$$P = 4x \quad \text{and} \quad \frac{dx}{dt} = 0.2$$

$$P = 4x$$

$\Rightarrow$ $$\frac{dP}{dt} = 4\frac{dx}{dt}$$

$\Rightarrow$ $$\frac{dP}{dt} = 4 \times 0.2 = 0.8 \text{ cm/sec}$$

S15. We have,

$$A = x^2 \quad \text{and} \quad \frac{dx}{dt} = 4 \text{ cm}$$

Now,

$$A = x^2$$

$\Rightarrow$ $$\frac{dA}{dt} = 2x\frac{dx}{dt}$$

$\Rightarrow$ $$\frac{dA}{dt} = 8x$$

$\Rightarrow$ $$\left(\frac{dA}{dt}\right)_{x=8} = 64 \text{ cm}^2/\text{min}$$

S16. Let r be the radius and A be the area of the disc at any time t. Then $A = \pi r^2$.

It is given that $\frac{dr}{dt} = 0.05$ cm/sec.

Now, $A = \pi r^2$

$\Rightarrow$ $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

When $r = 3.2$ cm

$\Rightarrow$ $\frac{dA}{dt} = 2\pi \times 3.2 \times 0.05 = 0.320\pi \text{ cm}^2/\text{sec}.$

S17. Let r be the radius and C the circumference of the circle. Then,

$$C = 2\pi r$$

It is given that $\frac{dr}{dt} = 0.7$ cm/sec.

Now $C = 2\pi r$

$\Rightarrow$ $\frac{dC}{dt} = 2\pi \frac{dr}{dt}$

$\Rightarrow$ $\frac{dC}{dt} = 2\pi \times 0.7 \text{ cm/sec} = 1.4\pi \text{ cm/sec}.$

S18. We have, $V = \frac{4}{3}\pi r^3$ and $\frac{dr}{dt} = 0.5$

$\Rightarrow$ $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$\Rightarrow$ $\left(\frac{dV}{dt}\right)_{r=1} = 4\pi (1)^2 (0.5) = 2\pi \text{ cm}^3/\text{sec}.$

S19. We have,

$$S = 4\pi r^2 \text{ and } \frac{dr}{dt} = 0.2$$

Now $S = 4\pi r^2$

$\Rightarrow$ $\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$

$\Rightarrow$ $\frac{dS}{dt} = 8\pi r (0.2) = 1.6\pi r$

$\Rightarrow$ $\left(\frac{dS}{dt}\right)_{r=7} = 1.6\pi \times 7 = 11.2\pi \text{ cm}^2/\text{sec}$

S20. Let r be the radius and S be the surface area of the balloon at any time t. Then,

$$S = 4\pi r^2 \text{ and } \frac{dr}{dt} = 10 \text{ cm/sec.} \qquad \text{[Given]}$$

Now, $S = 4\pi r^2$

$\Rightarrow$ $$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$$

$\Rightarrow$ $$\frac{dS}{dt} = 80\pi r \qquad \left[\because \frac{dr}{dt} = 10\,\text{cm/sec}\right]$$

$\Rightarrow$ $$\left(\frac{dS}{dt}\right)_{r=15} = 80\pi\,(15) = 1200\,\pi\ \text{cm}^2/\text{sec}$$

S21. We have

$$m = \text{Slope of the curve} = \frac{dy}{dx} = 7 - 3x^2$$

Now,

$$m = 7 - 3x^2$$

$\Rightarrow$ $$\frac{dm}{dt} = -6x\frac{dx}{dt}$$

$\Rightarrow$ $$\frac{dm}{dt} = -6x.(4) \Rightarrow \frac{dm}{dt} = -24x \Rightarrow \left(\frac{dm}{dt}\right)_{x=2} = -48 \quad \left[\because \frac{dx}{dt} = 4 \text{ (given)}\right]$$

S22. We have,

$$\frac{d\theta}{dt} = 2\frac{d}{dt}(\sin\theta) \Rightarrow \frac{d\theta}{dt} = 2\cos\theta\frac{d\theta}{dt}$$

$\Rightarrow$ $$\cos\theta = \frac{1}{2} \Rightarrow \theta = \pi/3.$$

S23. We have,

$$y = x^2 + 2x$$

$\Rightarrow$ $$\frac{dy}{dt} = (2x+2)\frac{dx}{dt}$$

$\Rightarrow$ $$1 = 2x + 2 \qquad \left[\therefore \frac{dy}{dx} = \frac{dx}{dt} \text{ (given)}\right]$$

$\Rightarrow$ $$x = -\frac{1}{2}$$

put $x = -\frac{1}{2}$ in $y = x^2 + 2x$

$\Rightarrow$ $$y = \left(-\frac{1}{2}\right)^2 + 2\times\left(-\frac{1}{2}\right) = \frac{1}{4} - 1 = -\frac{3}{4}$$

$\therefore$ Point is $\left(-\frac{1}{2}, -\frac{3}{4}\right)$.

S24. Let x be the length of the edge of the cube and V be its volume at any time t. Then,

$$V = x^3 \text{ and } \frac{dx}{dt} = 10 \text{ cm/sec} \qquad \text{[Given]}$$

Now, $V = x^3$

$\Rightarrow$ $$\frac{dV}{dt} = 3x^2\frac{dx}{dt}$$

$\Rightarrow$ $$\frac{dV}{dt} = (3x^2)(10) \qquad \left[\because \frac{dx}{dt} = 10\right]$$

$\Rightarrow$ $$\left(\frac{dV}{dt}\right) = 30x^2$$

$\Rightarrow$ $$\left(\frac{dV}{dt}\right)_{x=5} = 30(5)^2 = 750\,\text{cm}^3/\text{sec}$$

Thus, the volume of the cube is increasing at the rate of 750 cm^3/sec when the edge is 5 cm long.

S25. Let r be the radius and V be the volume of the balloon. Then,

$$V = \frac{4}{3}\pi r^3. \text{ It is given that } \frac{dV}{dt} = 900 \text{ cm}^3/\text{sec}$$

Now, $V = \frac{4}{3}\pi r^3$

$\Rightarrow$ $$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

$\Rightarrow$ $$900 = 4\pi \times 15^2 \frac{dr}{dt} \qquad \left[\because \frac{dv}{dt} = 900 \text{ and } r = 15\right]$$

$\Rightarrow$ $$\frac{dr}{dt} = \frac{1}{\pi} \text{ cm/sec.}$$

S26. We have,

$$V = \pi r^2 h, \frac{dr}{dt} = 2 \quad \text{and} \quad \frac{dh}{dt} = -3$$

$\therefore$ $V = \pi r^2 h$

$\Rightarrow$ $$\frac{dV}{dt} = \pi\left\{2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right\} \qquad \text{[Differentiating both side w.r.t. } t\text{]}$$

$\Rightarrow$ $$\frac{dV}{dt} = \pi[4rh - 3r^2]$$

When $r = 3$, $h = 5$, we obtain

$$\frac{dV}{dt} = \pi(60 - 27) = 33\pi \text{ cm}^3/\text{sec}.$$

S27. Let P be the perimeter and A be the area of the rectangle at any time t. Then,

$$P = 2(x + y) \text{ and } A = xy.$$

It is given that

$$\frac{dx}{dt} = -2 \text{ cm/sec and } \frac{dy}{dt} = 2 \text{ cm/sec}.$$

(i) We have, $P = 2(x + y)$

Differentiating both side w.r.t. t

$$\Rightarrow \quad \frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$$

$$= 2(-2 + 2) = 0 \text{ cm/sec}$$ *i.e.*, the perimeter remains constant.

(ii) We have, $A = xy$

Differentiating both side w.r.t. t

$$\Rightarrow \quad \frac{dA}{dt} = \left(\frac{dx}{dt}\right)y + x\left(\frac{dy}{dt}\right)$$

$$\Rightarrow \quad \frac{dA}{dt} = -2 \times 5 + 12 \times 2 \qquad [\because x = 12 \text{ cm and } y = 5 \text{ cm(given)}]$$

$$\Rightarrow \quad \frac{dA}{dt} = 14 \text{ cm}^2/\text{sec}.$$

S28. Given the length x of a rectangle is decreasing at the rate of 5 cm/min.

$$\therefore \quad \frac{dx}{dt} = -5 \text{ cm/min} \qquad \dots \text{(i)}$$

Also, the breadth y of rectangle is increasing at the rate of 4 cm/min.

$$\therefore \quad \frac{dy}{dt} = 4 \text{ cm/min}. \qquad \dots \text{(ii)}$$

To find,

(i) Rate of change of perimeter *i.e.*, $\frac{dP}{dt}$

(ii) Rate of change of area *i.e.*, $\frac{dA}{dt}$

where, P denotes the perimeter and A the area of rectangle.

(i) We know that $P = 2(x + y)$

Differentiating both sides w.r.t. t, we get

$$\frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$$

$\therefore$ $\frac{dP}{dt} = 2(-5 + 4) = 2(-1) = -2$ cm/min [By using Eqs. (i) and (ii)]

$\therefore$ $\frac{dP}{dt} = -2$ cm/min

(ii) We know that area of rectangle $A = xy$

Differentiating both sides w.r.t. t, we get

$$\frac{dA}{dt} = x \cdot \frac{dy}{dt} + y \cdot \frac{dx}{dt} \qquad \left[\because \frac{d}{dx}(xy) = x\frac{dy}{dt} + y\frac{dx}{dt}\right]$$

Now, we are given that $x = 8$ cm, $y = 6$ cm

$$\frac{dx}{dt} = -5 \text{ cm/min}, \quad \frac{dy}{dt} = 4\text{cm/min}.$$

$\therefore$ $\frac{dA}{dt} = (8 \times 4) + \{6 \times (-5)\}$

$= 32 - 30 = 2$ cm^2/min

$\therefore$ $\frac{dA}{dt} = 2$ cm^2/min

Note: If the rate of change is decreasing, we take –ve sign and if the rate of change is increasing, we take +ve sign.

S29. Let the required point be $P(x, y)$. It is given that

Rate of change of y-coordinate = 8 (Rate of change of x-coordinate)

$\Rightarrow$ $\frac{dy}{dt} = 8\frac{dx}{dt}$ ...(i)

Now, $6y = x^3 + 2$

$\Rightarrow$ $6\frac{dy}{dt} = 3x^2\frac{dx}{dt}$ [Differentiating both side w.r.t. t]

$\Rightarrow$ $6\left(8\frac{dx}{dt}\right) = 3x^2\frac{dx}{dt}$ [Using (i)]

$\Rightarrow$ $3x^2 = 48$

$\Rightarrow \quad x^2 = 16$

$\Rightarrow \quad x = \pm 4$

Now, $\quad x = 4$

$\Rightarrow \quad 6y = 4^3 + 2 = 66$

$\Rightarrow \quad y = 11$

and, $\quad x = -4$

$\Rightarrow \quad 6y = (-4)^3 + 2 = -62$

$\Rightarrow \quad y = -\dfrac{31}{3}$

So, the required point are $(-4, -31/3)$ and $(4, 11)$.

S30. Let x be the length of an edge of the cube, V be the volume and S be the surface area at any time t. Then,

$$V = x^3 \text{ and } S = 6x^2$$

Also, $\quad \dfrac{dV}{dt} = 7 \text{ cm}^3/\text{sec}$ [Given]

$\Rightarrow \quad \dfrac{d}{dt}(x^3) = 7$

$\Rightarrow \quad 3x^2 \dfrac{dx}{dt} = 7$

$\Rightarrow \quad \dfrac{dx}{dt} = \dfrac{7}{3x^2}$ [Differentiating both side w.r.t. t]

Now, $\quad S = 6x^2$

$\Rightarrow \quad \dfrac{dS}{dt} = 12x\dfrac{dx}{dt}$

$\Rightarrow \quad \dfrac{dS}{dt} = 12x \times \dfrac{7}{3x^2}$ $\left[\because \dfrac{dx}{dt} = \dfrac{7}{3x^2}\right]$

$\Rightarrow \quad \dfrac{dS}{dt} = \dfrac{28}{x}$

$\Rightarrow \quad \left(\dfrac{dS}{dt}\right)_{x=12} = \dfrac{28}{12} \text{ cm}^2/\text{sec} = \dfrac{7}{3} \text{ cm}^2/\text{sec.}$

S31. Let x be the length of each edge of the cube, S be its surface area and V be its volume at any time t.

Then, $S = 6x^2$ and $V = x^3$. It is the given that $\dfrac{dV}{dt} = k$ (constant)

Now, $V = x^3$

Differentiating both side w.r.t. t

$$\Rightarrow \quad \frac{dV}{dt} = 3x^2 \frac{dx}{dt} \quad \text{... (i)}$$

$$\Rightarrow \quad k = 3x^2 \frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{k}{3x^2}$$

and, $S = 6x^2$

Differentiating both side w.r.t. t

$$\Rightarrow \quad \frac{dS}{dt} = 12x \cdot \frac{dx}{dt}$$

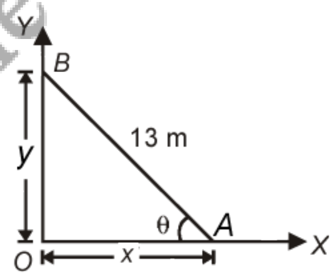
$$\Rightarrow \quad \frac{ds}{dt} = 12x\left(\frac{k}{3x^2}\right) \quad \text{[Using (i)]}$$

$$\Rightarrow \quad \frac{dS}{dt} = \frac{4k}{x} \quad \Rightarrow \quad \frac{dS}{dt} \propto \frac{1}{x}.$$

Hence, the rate of increase in surface area varies inversely as the length of the edge of the cube.

S32. Let the bottom of the ladder be at a distance x m. From the wall and the top be at a height y from the ground. Then $x^2 + y^2 = 13^2$ and $\tan\theta = \frac{y}{x}$, $\frac{dx}{dt} = 1.5$.

Differentiating both equation both side w.r.t. t,



$$\therefore \quad 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \text{ and } \sec^2\theta\frac{d\theta}{dt} = \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2}$$

$$\Rightarrow \quad 3x + 2y\frac{dy}{dt} = 0 \text{ and } \sec^2\theta\frac{d\theta}{dt} = \frac{x\frac{dy}{dt} - \frac{3y}{2}}{x^2}$$

$$\Rightarrow \quad \frac{dy}{dt} = -\frac{3x}{2y} \text{ and } \sec^2\theta\frac{d\theta}{dt} = \frac{x\left(-\frac{3x}{2y}\right) - \frac{3y}{2}}{x^2} \quad \left[\because \frac{dx}{dt} = 1.5\right]$$

$$\Rightarrow \quad \frac{d\theta}{dt} = -\frac{3}{2}\frac{(x^2+y^2)}{x^2 y \sec^2\theta} = -\frac{3}{2}\frac{(x^2+y^2)}{x^2 y(1+\tan^2\theta)} = -\frac{3}{2}\frac{(x^2+y^2)}{x^2 y\left(1+\frac{y^2}{x^2}\right)}$$

$\Rightarrow \quad \frac{d\theta}{dt} = -\frac{3}{2y}$

When $\quad x = 12, \quad x^2 + y^2 = 13^2$

$\Rightarrow \quad y = 5$

$\therefore \quad \frac{d\theta}{dt} = -\frac{3}{10}.$

S33. We have $\quad S = 4\pi r^2$

$\Rightarrow \quad \frac{dS}{dt} = 8\pi r \frac{dr}{dt}$

$\Rightarrow \quad 2 = 8\pi r \frac{dr}{dt} \qquad \left[\because \frac{ds}{dt} = 2cm^2/\text{sec (given)}\right]$

$\Rightarrow \quad \frac{dr}{dt} = \frac{2}{8\pi r} \qquad \text{... (i)}$

Now, $\quad V = \frac{4}{3}\pi r^3$

$\Rightarrow \quad \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \qquad$ [Differentiating both side w.r.t. t]

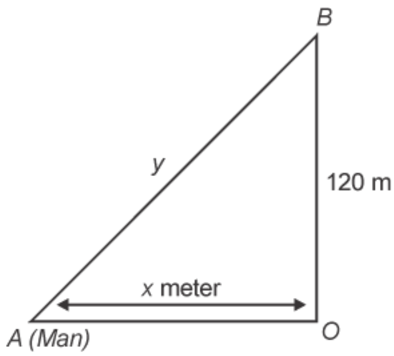
$\Rightarrow \quad \frac{dV}{dt} = 4\pi r^2 \times \frac{2}{8\pi r} \qquad$ [Using (i)]

$\Rightarrow \quad \frac{dV}{dt} = r$

Hence, $\quad \left(\frac{dV}{dt}\right)_{r=6} = 6 \text{ cm}^3/\text{sec}.$

S34. Let at any time t, the man be at distance of x and y metres from the foot and top of the tower respectively. Then,

$y^2 = x^2 + (120)^2 \qquad \text{... (i)}$



Differentiating both side w.r.t. t

$\Rightarrow \quad 2y\frac{dy}{dt} = 2x\frac{dx}{dt}$

$\Rightarrow \quad \frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt}$

We are given that $\frac{dx}{dt}$ = –6.5 km/hr (negative sign due to decreasing x). Therefore,

$\frac{dy}{dt} = -\frac{6.5x}{y} \qquad \text{... (ii)}$

Putting $x = 50$ in Eq. (i), we get

$$y = \sqrt{50^2 + 120^2} = 130$$

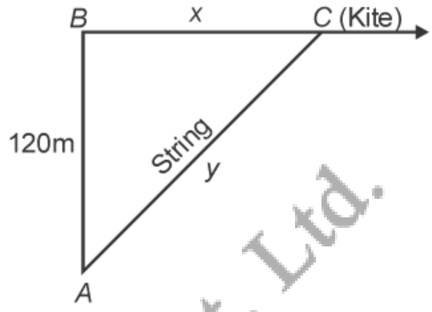
Putting $x = 50$, $y = 130$ in Eq. (ii), we get

$$\frac{dy}{dt} = -\frac{6.5 \times 50}{130} = -2.5 \text{ km/hr}$$

Thus, the man is approaching the top of the tower at the rate of 2.5 km/hr.

S35. We have,

$$y^2 = x^2 + (120)^2$$



Differentiating both side w.r.t. t

$$\Rightarrow \quad 2y\frac{dy}{dt} = 2x\frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt}$$

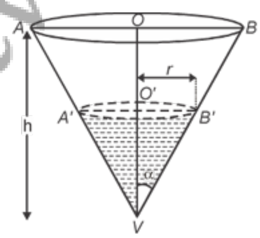
$$\Rightarrow \quad \frac{dy}{dt} = 52\left(\frac{x}{y}\right) \qquad \left[\because \frac{dx}{dt} = 52 \text{ m/sec}\right]$$

Putting $y = 130$ in $y^2 - x^2 = (120)^2 \Rightarrow 130^2 - 120^2 = x^2 \Rightarrow x^2 = 50^2$

we get $x = 50$.

$$\therefore \quad \frac{dy}{dt} = \frac{52 \times 50}{130} = 20 \text{ m/sec}.$$

S36. Let α be the semi-vertical angle of the water tank in the form of cone. Then,



$$\tan\alpha = 0.5 = \frac{1}{2}$$

$$\Rightarrow \quad \frac{r}{h} = \frac{1}{2}$$

$$\Rightarrow \quad r = \frac{h}{2}$$

Let $VA'B'$ be the water cone of volume V. Then,

$$\frac{dV}{dt} = 5\text{m}^3/\text{hr} \qquad \text{[Given]}$$

We have to find $\frac{dh}{dt}$ when $h = 4$ m.

We have,

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$$

Differentiating both side w.r.t. t

$\Rightarrow$ $$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$$

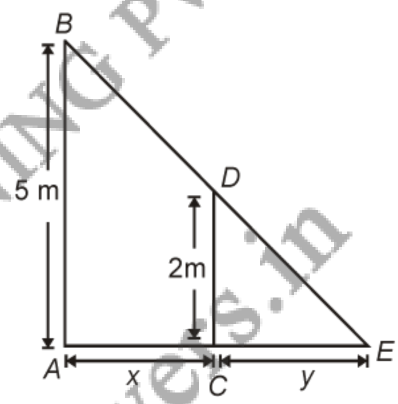
$\Rightarrow$ $$5 = \frac{\pi}{4}\times 4^2\times\frac{dh}{dt}$$

$\Rightarrow$ $$\frac{dh}{dt} = \frac{5}{4\pi} = \frac{5}{4}\times\frac{7}{22}\ \text{m/h} = \frac{35}{88}\ \text{m/h}.$$

Thus, the rate of change of water level is $\frac{35}{88}$ m/h.

S37. $$\frac{dx}{dt} = 6 \text{ metres/minute} \qquad \text{[Given]} \quad \text{... (i)}$$

Clearly, triangle ABE and CDE are similar.



$\therefore$ $$\frac{AB}{CD} = \frac{AE}{CE}$$

$\Rightarrow$ $$\frac{5}{2} = \frac{x+y}{y}$$

$\Rightarrow$ $$\frac{5}{2} = 1+\frac{x}{y}$$

$\Rightarrow$ $$\frac{3}{2} = \frac{x}{y} \Rightarrow 3y = 2x$$

Differentiating both side w.r.t. t

$\Rightarrow$ $$3\frac{dy}{dt} = 2\frac{dx}{dt}$$

$\Rightarrow$ $$3\frac{dy}{dt} = 2(6)$$

$\Rightarrow$ $$\frac{dy}{dt} = 4$$

Thus, the shadow increases at the rate of 4 metres/minute.

S38. It is given that the plane is ascending vertically at the constant rate of 100 km/h.

$\therefore$ $$\frac{dh}{dt} = 100 \text{ km/h}$$

$\therefore$ Height of the plane after 3 minutes = $100 \times \frac{3}{60} = 5$ km. [Using $h = vt$]

Now, $A = 2\pi r^2 \frac{h}{r+h}$

Differentiating both side w.r.t. t

$$\Rightarrow \quad \frac{dA}{dt} = 2\pi r^2 \frac{d}{dt}\left(\frac{h}{r+h}\right) = 2\pi r^2 \left\{\frac{(r+h)\frac{dh}{dt} - h\frac{d}{dt}(r+h)}{(r+h)^2}\right\}$$

$$\Rightarrow \quad \frac{dA}{dt} = 2\pi r^2 \left\{\frac{(r+h)\frac{dh}{dt} - h\frac{dh}{dt}}{(r+h)^2}\right\} = \frac{2\pi r^3}{(r+h)^2}\frac{dh}{dt}$$

$$\Rightarrow \quad \frac{dA}{dt} = \frac{2\pi r^3}{(r+h)^2} \times 100 = \frac{200\pi r^3}{(r+h)^2} \qquad \left[\because \frac{dh}{dt} = 100 \text{ km/h}\right]$$

To find $\frac{dA}{dt}$ when $t = 3$ minutes.

Also at $t = 3$, we have $h = 5$ km.

$$\therefore \quad \left(\frac{dA}{dt}\right) = \frac{200\pi r^3}{(r+5)^2}.$$

S39. The ladder when leaning against the wall forms a right angled triangle with length of ladder equal to the hypotenuse of the triangle and base and altitude equal to the distance of foot from, wall and distance of top from ground.

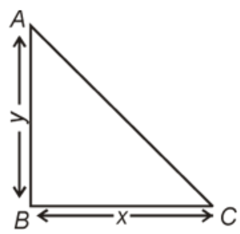
Let AC be the ladder.

Let $BC = x$ and height $AB = y$

To find $\frac{dy}{dt}$, where $x = 4$m

From the right angled ΔABC, by Pythagoras theorem, we have

$$(AB)^2 + (BC)^2 = (AC)^2$$



$$\Rightarrow \quad x^2 + y^2 = 25 \quad \ldots \text{(i)}$$

$$\Rightarrow \quad (4)^2 + y^2 = 25$$

$$\Rightarrow \quad 16 + y^2 = 25$$

$$\Rightarrow \quad y^2 = 9$$

$$\therefore \quad y = 3$$

Differentiating Eq. (i) w.r.t. t on both sides, we get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$\Rightarrow \qquad x\dfrac{dx}{dt} + y\dfrac{dy}{dt} = 0$

Substituting the values of x, y and $\dfrac{dx}{dt}$ above, we have

$$(4 \times 2) + 3\,\frac{dy}{dt} = 0$$

$\Rightarrow \qquad 8 + 3 \times \dfrac{dy}{dt} = 0$

$\therefore \qquad \dfrac{dy}{dt} = \dfrac{-8}{3}\,\text{m/s}$

Hence, height on the wall is decreasing at the rate of $\dfrac{8}{3}$ m/s.

Note: In a rate of change of quantity +ve sign shows that it is increasing and –ve sign shows that it is decreasing.

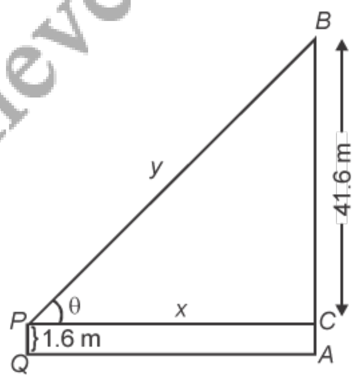
S40. Let AB be the tower. Let at any time t, the man be at a distance of x metres from the tower AB and let θ be the angle of elevation at the time. Then,

$$\tan\theta = \frac{BC}{PC}$$

$\Rightarrow \qquad \tan\theta = \dfrac{40}{x}$

$\Rightarrow \qquad x = 40\cot\theta$ ... (i)

Differentiating both side w.r.t. t



$\Rightarrow \qquad \dfrac{dx}{dt} = -40\,\text{cosec}^2\theta\,\dfrac{d\theta}{dt}$

We are given that $\dfrac{dx}{dt} = 2$ m/sec.

$\therefore \qquad 2 = -40\,\text{cosec}^2\theta\,\dfrac{d\theta}{dt}$

$\Rightarrow \qquad \dfrac{d\theta}{dt} = -\dfrac{1}{20\,\text{cosec}^2\theta}$ ... (ii)

When $x = 30$, we have

$\cot\theta = \dfrac{30}{40} = \dfrac{3}{4}$ [Putting $x = 30$ in Eq. (i)]

$\therefore \qquad \text{cosec}^2\theta = 1 + \cot^2\theta = 1 + \dfrac{9}{16} = \dfrac{25}{16}$

Substituting $\text{cosec}^2\theta = \frac{25}{16}$ in Eq. (ii), we get

$$\frac{d\theta}{dt} = \frac{-1}{20 \times \frac{25}{16}} = -\frac{4}{125} \text{ radian/sec.}$$

Thus the angle of elevation of the top of tower is changing at the rate of (4/125) radian/sec.

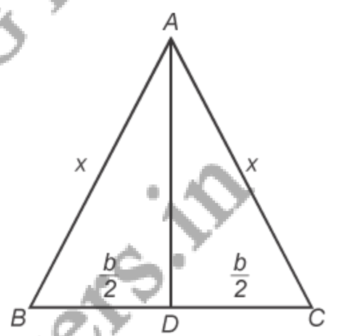
S41. Let at any time t, the length of each equal side be x cm and area of the triangle be A. Then,

$$A = \frac{1}{2} BC \times AD$$

$$\Rightarrow \quad A = \frac{1}{2} \times b \times \sqrt{x^2 - \frac{b^2}{4}}$$

$$\Rightarrow \quad A = \frac{b}{4}\sqrt{4x^2 - b^2}$$

Differentiating both side w.r.t. t



$$\Rightarrow \quad \frac{dA}{dt} = \frac{b}{4}\frac{d}{dt}\sqrt{4x^2 - b^2}$$

$$\Rightarrow \quad \frac{dA}{dt} = \frac{b}{4} \times \frac{1}{2\sqrt{4x^2 - b^2}} \frac{d}{dt}(4x^2 - b^2)$$

$$\Rightarrow \quad \frac{dA}{dt} = \frac{b}{8\sqrt{4x^2 - b^2}} \times 8x \frac{dx}{dt}$$

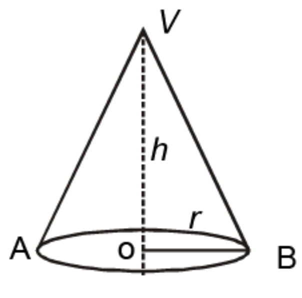
$$\Rightarrow \quad \frac{dA}{dt} = \frac{bx}{\sqrt{4x^2 - b^2}} \frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dA}{dt} = \frac{3bx}{\sqrt{4x^2 - b^2}} \qquad \left[\because \frac{dx}{dt} = 3 \text{ cm/sec (given)}\right]$$

$$\Rightarrow \quad \left(\frac{dA}{dt}\right)_{x=b} = \frac{3b^2}{\sqrt{4b^2 - b^2}} = \sqrt{3}\, b \text{ cm}^2\text{/sec.}$$

S42. Let V be the volume of cone, h be the height and r be the radius of base of the cone. Given that,

$$\frac{dV}{dt} = 12 \text{ cm}^3\text{/s} \qquad \ldots \text{(i)}$$



Also, height of cone = $\frac{1}{6}$ (radius of base of cone)

$$h = \frac{1}{6} r \qquad \ldots \text{(ii)}$$

To find $\frac{dh}{dt}$

We know that, volume of cone is given by

$$V = \frac{1}{3}\pi r^2 h \qquad \text{... (iii)}$$

Putting $r = 6h$ from Eq. (ii) in Eq. (iii), we get

$$V = \frac{1}{3}\pi(6h)^2 \cdot h$$

$$\Rightarrow \qquad V = \frac{\pi}{3}\cdot 36h^3$$

$$\text{or} \qquad V = 12\pi h^3$$

Differentiating above equation w.r.t. t, we get

$$\frac{dV}{dt} = 12\pi \times 3h^2 \cdot \frac{dh}{dt}$$

$$\Rightarrow \qquad \frac{dV}{dt} = 36\pi h^2 \cdot \frac{dh}{dt}$$

Put $\frac{dV}{dt}$ = 12 cm^3/s, h = 4 cm, we get

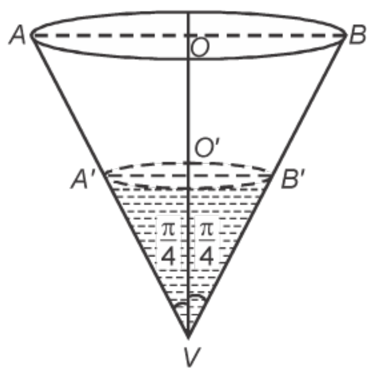
$$12 = 36\pi \times 16 \times \frac{dh}{dt}$$

$$\Rightarrow \qquad \frac{dh}{dt} = \frac{12}{36\pi \times 16}$$

$$\therefore \qquad \frac{dh}{dt} = \frac{1}{48\pi}\text{ cm/s.}$$

S43. Let VAB be a conical funnel of semi-vertical angle $\frac{\pi}{4}$. At any time t the water in the cone also forms a cone. Let r be it's radius, l be the slant height and S be the surface area. Then, VA' = l, $O'A' = r$ and $\angle A'VO' = \frac{\pi}{4}$.

In $\Delta VO'A'$, we have



$$\cos\frac{\pi}{4} = \frac{VO'}{VA'} = \frac{VO'}{l} \text{ and } \sin\frac{\pi}{4} = \frac{O'A'}{VA'} = \frac{O'A'}{l}.$$

$$\Rightarrow \qquad VO' = l\cos\frac{\pi}{4} \text{ and } O'A' = l\sin\frac{\pi}{4}.$$

The surface area S of the conical funnel is given by

$$S = \pi(O'A')\ (VA') \qquad [\text{Using } S = \pi r l]$$

$\Rightarrow \quad S = \pi l \sin\frac{\pi}{4} . l = \pi l^2 \sin\frac{\pi}{4} = \frac{\pi l^2}{\sqrt{2}}$

Differentiating both side w.r.t. t

$$\Rightarrow \quad \frac{dS}{dt} = \frac{2\pi l}{\sqrt{2}}\frac{dl}{dt}$$

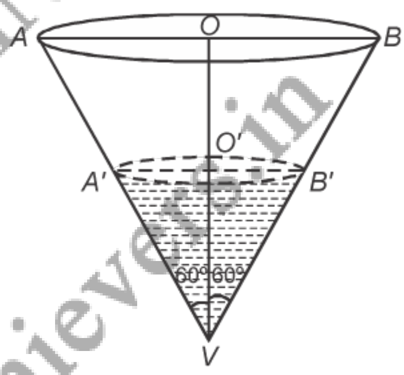
$$\Rightarrow \quad -2 = \frac{2\pi l}{\sqrt{2}}\frac{dl}{dt} \qquad \left[\because \frac{dS}{dt} = -2\,\text{cm}^2/\text{sec}\right]$$

$$\Rightarrow \quad \frac{dl}{dt} = \frac{\sqrt{2}}{\pi l}$$

$$\Rightarrow \quad \left(\frac{dl}{dt}\right)_{l=4} = -\frac{\sqrt{2}}{4\pi}\ \text{cm/sec.}$$

Thus, the rate of decrease of the slant height is $\frac{\sqrt{2}}{4\pi}$ cm/sec.

S44. Let at any time t, V be the volume of the water in cone *i.e.*, the volume of the water-cone $VA'B'$, and let l be the slant height. Then,



$$O'A' = l \sin 60° = \frac{l\sqrt{3}}{2}$$

and $$VO' = l \cos 60° = \frac{l}{2}.$$

$$V = \frac{1}{3}\pi(O'A')^2 VO'$$

$$\therefore \quad V = \frac{1}{3}\pi\left(\frac{l\sqrt{3}}{2}\right)^2\left(\frac{l}{2}\right) = \frac{\pi l^3}{8}$$

Differentiating both side w.r.t. t

$$\Rightarrow \quad \frac{dV}{dt} = \frac{3\pi l^2}{8}\frac{dl}{dt} \qquad \dots \text{(i)}$$

We are given that $\frac{dV}{dt} = -4\ \text{cm}^3/\text{sec}$ (negative sign due to decrease V).

$$\therefore \quad -4 = \frac{3\pi}{8} l^2 \frac{dl}{dt}$$

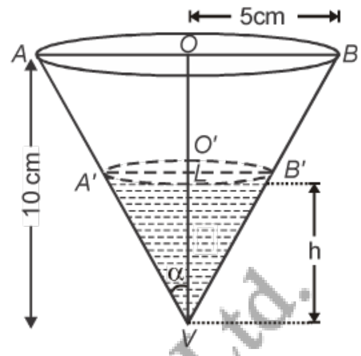
$$\Rightarrow \quad \frac{dl}{dt} = -\frac{32}{3\pi l^2}$$

When $l = 3$, we have

$$\frac{dl}{dt} = \frac{-32}{3\pi(3)^2} = \frac{-32}{27\pi} \text{ cm/sec}$$

Thus, the slant height of the water-cone is decreasing at the rate of $\frac{32}{27\pi}$ cm/sec.

S45. Let α be the semi-vertical angle of the cone VAB whose height VO is 10 cm and radius OB = 5 cm. Then,



$$\tan\alpha = \frac{5}{10} = \frac{1}{2}$$

Let V be the volume of the water in the cone *i.e.*, the volume of the cone $VA'B'$ after time t minutes and h be the height of water. Then,

$$V = \frac{1}{3}\pi(O'B')^2(VO')$$

$\Rightarrow$ $$V = \frac{1}{3}\pi h^3 \tan^2\alpha \qquad \left[\because \tan\alpha = \frac{O'B'}{VO'} = \frac{O'B'}{h} \Rightarrow O'B' = h\tan\alpha\right]$$

$\Rightarrow$ $$V = \frac{\pi}{12}h^3 \qquad \left[\because \tan\alpha = \frac{1}{2}\right]$$

Differentiating both side w.r.t. t

$\Rightarrow$ $$\frac{dV}{dt} = \frac{\pi}{12}3h^2\frac{dh}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$$

$\Rightarrow$ $$\frac{3}{2} = \frac{\pi h^2}{4}\frac{dh}{dt} \qquad \left[\because \frac{dV}{dt} = \frac{3}{2}\text{cm}^3\text{/minutes (given)}\right]$$

$\Rightarrow$ $$\frac{dh}{dt} = \frac{6}{\pi h^2}$$

$\Rightarrow$ $$\left(\frac{dh}{dt}\right)_{h=4} = \frac{6}{\pi(4)^2} = \frac{3}{8\pi} \text{ cm/min.}$$

Date: 18/10/2021

Q1. Show that the function $f(x) = \frac{3}{x} + 7$ is decreasing for $x \in R$ $(x \neq 0)$.

Q2. Show that $f(x) = (x - 1)\, e^x + 1$ is an increasing function for all $x > 0$

Q3. Prove that the function f given by $f(x) = \log \sin x$ is strictly increasing on $\left(0, \frac{\pi}{2}\right)$ and strictly decreasing on $\left(\frac{\pi}{2}, \pi\right)$.

Q4. Show that the function $f(x) = x^3 - 3x^2 + 3x$, $x \in R$ is increasing on R.

Q5. Determine the value of x for which $f(x) = \frac{x-2}{x+1}$, $x \neq -1$ is increasing or decreasing.

Q6. Find the least value of a such that the function $x^2 + ax + 1$ is increasing on $[1, 2]$.

Q7. Prove that the function f given by $f(x) = x^3 - 3x^2 + 4x$ is strictly increasing on R.

Q8. Find the values of a for which $f(x) = x^3 - ax$ is an increasing function on R.

Q9. Find the values of b for which the function $f(x) = \sin x - bx + c$ is decreasing function on R.

Q10. Find the intervals in which the following function is strictly increasing or decreasing. $f(x) = 10 - 6x - 2x^2$

Q11. Let I be any interval disjoint from $(-1, 1)$. Prove that the function f given by $f(x) = x + \frac{1}{x}$ is strictly increasing on I.

Q12. Determine the intervals in which the function $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$ is decreasing or increasing.

Q13. For which values of x, the function $f(x) = \frac{x}{x^2 + 1}$ is increasing and for which values of x, it is decreasing.

Q14. Find the intervals for which $f(x) = x^4 - 2x^2$ is increasing or decreasing.

Q15. Find the intervals in which $f(x) = \frac{x}{2} + \frac{2}{x}$, $x \neq 0$ is increasing or decreasing.

Q16. Find the intervals in which $f(x) = (x + 1)^3 (x - 3)^3$ is increasing or decreasing.

Q17. Find the intervals in which $f(x) = -x^2 - 2x + 15$ is increasing or decreasing.

Q18. Determine the values of x for which $f(x) = x^x$, $x > 0$ is increasing or decreasing.

Q19. Determine the values of x for which $f(x) = -2x^3 - 9x^2 - 12x + 1$ is increasing or decreasing.

Q20. Find the intervals in which the function f given by $f(x) = x^3 + \frac{1}{x^3}$, $x \neq 0$ is (a) increasing (b) decreasing.

Q21. Find the intervals in which the function $f(x) = x^4 - \frac{x^3}{3}$ is increasing or decreasing.

Q22. Determine the values of x for which $f(x) = \{x(x-2)\}^2$ is increasing or decreasing.

Q23. Determine the values of x for which $f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5}x + 11$ is increasing or decreasing.

Q24. Find the intervals in which $f(x) = x^3 - 12x^2 + 36x + 17$ is increasing or decreasing function.

Q25. Find the intervals in which the function $f(x) = (x-1)(x-2)^2$ is increasing or decreasing.

Q26. Find the intervals in which the function $f(x) = (x-1)^3(x-2)^2$ is (a) increasing (b) decreasing.

Q27. Find the intervals in which the function $f(x) = 2x^3 + 9x^2 + 12x + 20$ is (*a*) increasing (*b*) decreasing.

Q28. Find the intervals in which the following function is (*a*) increasing (*b*) decreasing.

$$f(x) = 2x^3 - 9x^2 + 12x + 15$$

Q29. Find the intervals in which the function $f(x) = 2x^3 - 15x^2 + 36x + 17$ is increasing or decreasing.

Q30. Find the intervals in which $f(x) = \frac{x}{\log x}$ is increasing or decreasing.

Q31. Find the intervals in which $f(x) = 2\log(x-2) - x^2 + 4x + 1$ is increasing.

Q32. Find the intervals in which the function f given by $f(x) = \frac{4\sin x - 2x - x\cos x}{2 + \cos x}$, $0 \le x \le 2\pi$ is (i) increasing (ii) decreasing

Q33. Separate the interval $[0, \pi/2]$ into sub-intervals in which $f(x) = \sin^4 x + \cos^4 x$ is increasing or decreasing.

Q34. Show that $y = \log(1 + x) - \frac{2x}{2+x}$, $x > -1$ is an increasing function of x, throughout its domain.

Q35. Separate the interval $[0, \pi/2]$ in sub intervals in which $f(x) = \sin 3x$ is increasing or decreasing.

Q36. Show that $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function on the interval $[0, \pi/4]$.

Q37. Find the intervals in which the function $f(x) = \sin x + \cos x$, $0 \le x \le 2\pi$ is strictly increasing or strictly decreasing.

Q38. Prove that the function f given by $f(x) = \log \cos x$ is strictly increasing on $(-\pi/2, 0)$ and strictly decreasing on $(0, \pi/2)$.

Q39. Show that the function $f(x) = \tan x - 4x$ is decreasing function on $[-\pi/3, \pi/3]$.

Q40. Show that $f(x) = \cos(2x + \pi/4)$ is an increasing function on $[3\pi/8, 7\pi/8]$

Q41. Prove that $y = \frac{4\sin\theta}{2+\cos\theta} - \theta$ is an increasing function in $\left[0, \frac{\pi}{2}\right]$.

Q42. Prove that the function $f(x) = x^2 - x + 1$ is neither increasing nor decreasing on $(-1, 1)$.

Q43. Find the values of x for which $f(x) = [x(x-2)]^2$ is an increasing function. Also, find the points on the curve where the tangent is parallel to X-axis.

Q44. Find the intervals in which the following function $f(x) = 20 - 9x + 6x^2 - x^3$ is

(a) strictly increasing (b) strictly decreasing

Q45. Find the interval in which the following function is increasing or decreasing.

$$f(x) = x^3 - 6x^2 + 9x + 15$$

Q46. Find the interval in which the following function is increasing or decreasing.

$$f(x) = 5x^3 - 15x^2 - 120x + 3$$

Q47. For the function $f(x) = 6 + 12x + 3x^2 - 2x^3$, find (a) the interval (s) where it is increasing; (b) the interval (s) where it is decreasing.

Q48. Find the intervals in which the following function f: (a) strictly increasing (b) strictly decreasing

$$f(x) = 2x^3 - 3x^2 - 36x + 7$$

Q49. Find the intervals in which the following function f: (a) strictly increasing (b) strictly decreasing

$$f(x) = x^3 - 6x^2 - 36x + 2$$

Q50. Find the intervals in which the function f given by $f(x) = \sin x - \cos x$, $0 \le x \le 2\pi$, is strictly increasing and strictly decreasing.

Q51. For the function $f(x) = 2x^3 - 8x^2 + 10x + 5$, find the interval ($s$):

(a) in which $f(x)$ is increasing (b) in which $f(x)$ is decreasing.

Q52. Find the intervals in which the following function f is an increasing or decreasing

$$f(x) = 2x^3 - 24x + 5$$

Q53. Find the intervals in which the following function f is an increasing or decreasing

$$f(x) = 2x^3 - 6x^2 - 48x + 17$$

Q54 On which the following intervals is the function $f(x) = x^{100} + \sin x - 1$ strictly increasing?

(i) $(0, \pi/2)$ (ii) $(\pi/2, \pi)$ (iii) $(0, 1)$ (iv) $(-1, 1)$

Q55. Find the values of k for which $f(x) = kx^3 - 9kx^2 + 9x + 3$ is strictly increasing on R.

Q56. Find the value of 'a' for which the function $f(x) = (a+2)x^3 - 3ax^2 + 9ax - 1$ decreases for all real values of x.

Date: 18/10/2021

S1. We have,

$$f(x) = \frac{3}{x} + 7$$

$\Rightarrow$ $$f'(x) = -\frac{3}{x^2}$$

Now, $x \in R,\ x \neq 0$

$\Rightarrow$ $$\frac{1}{x^2} > 0$$

$\Rightarrow$ $$-\frac{3}{x^2} < 0$$

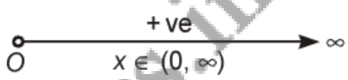
$\Rightarrow$ $$f'(x) < 0$$

Hence, $f(x)$ is decreasing for $x \in R,\ x \neq 0$.

S2. We have,

$$f(x) = (x - 1)\, e^x + 1$$

Differentiating w.r.t. "x", we get



$$f'(x) = (x - 1)\, e^x + e^x (1) + 0$$

$\Rightarrow$ $$f'(x) = xe^x - e^x + e^x = xe^x > 0 \quad \text{for all} \quad x > 0.$$

[$\because$ when $x > 0$, $e^x > 0$ and therefore, $x e^x > 0$]

$\Rightarrow$ $$f'(x) > 0 \quad \text{for all} \quad x > 0.$$

Hence, $f(x)$ is an increasing function for all $x > 0$ *i.e.,* $(0, \infty)$.

S3. We have, $f(x) = \log \sin x$

$\Rightarrow$ $$f'(x) = \frac{1}{\sin x} \cos x = \cot x$$

When, $0 < x < \frac{\pi}{2}$, $f'(x)$ is + ve

$\Rightarrow$ $f(x)$ is increasing.

When, $\frac{\pi}{2} < x < \pi$, $f'(x)$ is – ve

$\Rightarrow$ $f(x)$ is decreasing.

Hence, f is increasing on $\left(0, \frac{\pi}{2}\right)$ and decreasing on $\left(\frac{\pi}{2}, \pi\right)$.

S4. Given that $y = x^3 - 3x^2 + 3x$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 3x^2 - 6x + 3 \quad \Rightarrow \quad \frac{dy}{dx} = 3(x^2 - 2x + 1)$$

$$\Rightarrow \quad \frac{dy}{dx} = 3(x-1)^2$$

Now, $\quad 3(x-1)^2 \geq 0, \ \forall \ x \in R$

$\therefore$ We get that $\frac{dy}{dx} \geq 0, \forall \ x \in R$

Hence, the given function is increasing on R.

S5. We have,

$$f(x) = \frac{x-2}{x+1}, x \neq -1$$

$$\Rightarrow \quad f'(x) = \frac{(x+1)\cdot 1 - (x-2)\cdot 1}{(x+1)^2} = \frac{3}{(x+1)^2}, x \neq -1$$

Clearly, $\quad f'(x) = \frac{3}{(x+1)^2} \geqslant 0 \quad [\because \ (x+1)^2 > 0, x \neq -1]$

So, $f(x)$ is increasing on $R - \{-1\}$.

S6. We have,

$$f(x) = x^2 + ax + 1$$

$$f'(x) = 2x + a$$

Since $f(x)$ is an increasing function on $[1, 2]$. Therefore,

$$f'(x) \geqslant 0 \quad \text{for all } x \in [1, 2]$$

$$\Rightarrow \quad 2x + a \geqslant 0 \quad \text{for all } x \in [1, 2]$$

$$\Rightarrow \quad a \geqslant -2x \quad \text{for all } x \in [1, 2]$$

but maximum value of $-2x$ in $x \in [1, 2]$ is -2.

Thus, the least value of a is -2.

S7. We have,

$$f(x) = x^3 - 3x^2 + 4x$$

$$\Rightarrow \quad f'(x) = 3x^2 - 6x + 4$$

$$= 3(x^2 - 2x + 1) + 1$$

$\Rightarrow \quad f'(x) = 3(x-1)^2 + 1 > 0$ for all $x \in R$.

Hence, $f(x)$ is strictly increasing on R.

S8. $f(x)$ is increasing on R

$\Rightarrow \quad f'(x) \geq 0 \quad$ for all $x \in R$

$\Rightarrow \quad 3x^2 - a \geq 0 \quad$ for all $x \in R$

$\Rightarrow \quad a \leqslant 3x^2 \quad$ for all $x \in R$

But, the least value of $3x^2$ is 0.

Therefore, $\quad a \leqslant 0$.

S9. $f(x)$ is decreasing on R

$\Rightarrow \quad f'(x) \leqslant 0 \quad$ for all $x \in R$

$\Rightarrow \quad \cos x - b \leqslant 0 \quad$ for all $x \in R$

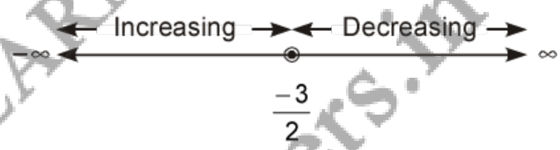
$\Rightarrow \quad b \geqslant \cos x \quad$ for all $x \in R$

But, the maximum value of $\cos x$ is 1.

Therefore, $\quad b \geqslant 1$.

S10. We have,

$$f(x) = 10 - 6x - 2x^2$$



$\Rightarrow \quad f'(x) = -6 - 4x = -2(3 + 2x)$

Now, $f(x)$ is increasing if $f'(x) > 0$

i.e., $\quad -6 - 4x > 0$

i.e., $\quad -4x > 6 \quad \Rightarrow \quad 4x < -6 \quad \Rightarrow \quad x < -\frac{3}{2}$

and $f'(x)$ is decreasing if $f'(x) < 0$ *i.e.*, if $-6 - 4x < 0$.

i.e., $\quad -4x < 6 \quad \Rightarrow \quad 4x > -6 \quad \Rightarrow \quad x > -\frac{3}{2}$

Hence, $f(x)$ is increasing for $x < -\frac{3}{2}$ *i.e.*, in the interval. $\left(-\infty, \frac{-3}{2}\right)$

and decreasing for $x > \frac{-3}{2}$ *i.e.*, $\left(\frac{-3}{2}, \infty\right)$.

S11. We have,

$$f(x) = x + \frac{1}{x}$$

$\Rightarrow \quad f'(x) = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}$

$\Rightarrow \quad x^2 - 1 > 0 \quad \Rightarrow \quad \frac{x^2 - 1}{x^2} > 0 \qquad [\because\ x^2 \geq 1 > 0]$

$\Rightarrow \quad f'(x) > 0$

Thus, $f'(x) > 0$ for all $x \notin I$.

Now, $x \in I \Rightarrow x \notin (-1, 1) \Rightarrow x < -1$ or $x > 1 \Rightarrow x^2 > 1$.

Hence, $f(x)$ is increasing on I.

S12. We have,

$$f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$$

$\Rightarrow \quad f'(x) = 4x^3 - 24x^2 + 44x - 24 = 4(x^3 - 6x^2 + 11x - 6)$

$\Rightarrow \quad f'(x) = 4(x - 1)(x^2 - 5x + 6)$

For $f(x)$ to be increasing, we must have

$$f'(x) \geqslant 0$$

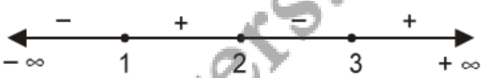
$\Rightarrow \quad 4(x - 1)(x^2 - 5x + 6) \geqslant 0$

$\Rightarrow \quad (x - 1)(x^2 - 5x + 6) \geqslant 0 \qquad [\because\ 4 > 0]$

$\Rightarrow \quad (x - 1)(x - 2)(x - 3) \geqslant 0$

$\Rightarrow \quad 1 \leqslant x \leqslant 2$ or $3 \leqslant x < \infty$

$\Rightarrow \quad x \in [1, 2] \cup [3, \infty)$



So, $f(x)$ is increasing on $[1, 2] \cup [3, \infty)$

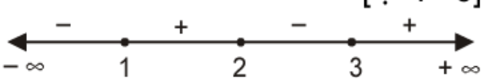
For $f(x)$ to be decreasing, we must have

$$f'(x) \leqslant 0$$

$\Rightarrow \quad 4(x - 1)(x^2 - 5x + 6) \leqslant 0$

$\Rightarrow \quad (x - 1)(x^2 - 5x + 6) \leqslant 0 \qquad [\because\ 4 > 0]$

$\Rightarrow \quad (x - 1)(x - 2)(x - 3) \leqslant 0$



$\Rightarrow \quad x \leqslant 1$ or $2 \leqslant x \leqslant 3$

$\Rightarrow \quad x \in (-\infty, 1] \cup [2, 3]$

So, $f(x)$ is decreasing on $(-\infty, 1] \cup [2, 3]$.

S13. We have,

$$f(x) = \frac{x}{x^2 + 1}$$

$\Rightarrow \quad f'(x) = \dfrac{(x^2+1)\cdot 1 - x(2x+0)}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2}.$

For $f(x)$ to be increasing, we must have

$$f'(x) \geqslant 0$$

$$\Rightarrow \quad \frac{1-x^2}{(x^2+1)^2} \geqslant 0$$

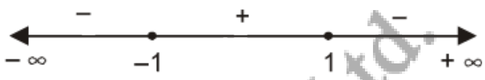
$$\Rightarrow \quad 1-x^2 \geqslant 0 \qquad [\because\ (x^2+1)^2 > 0]$$

$$\Rightarrow \quad -(x^2-1) \geqslant 0$$

$$\Rightarrow \quad x^2 - 1 \leqslant 0$$

$$\Rightarrow \quad (x-1)(x+1) \leqslant 0$$

$$\Rightarrow \quad -1 \leqslant x \leqslant 1$$



So, $f(x)$ is increasing on $[-1, 1]$.

For $f(x)$ to be decreasing, we must have

$$f'(x) \leqslant 0$$

$$\Rightarrow \quad \frac{1-x^2}{(x^2+1)^2} \leqslant 0$$

$$\Rightarrow \quad 1-x^2 \leqslant 0 \qquad [\because\ (x^2+1)^2 > 0]$$

$$\Rightarrow \quad -(x^2-1) \leqslant 0$$

$$\Rightarrow \quad (x-1)(x+1) \geqslant 0$$

$$\Rightarrow \quad x \leqslant -1 \text{ or } x \geqslant 1$$

So, $f(x)$ is decreasing on $(-\infty, -1] \cup [1, \infty)$.

S14. We have,

$$f(x) = x^4 - 2x^2$$

$$\Rightarrow \quad f'(x) = 4x^3 - 4x = 4x(x^2-1)$$

For $f(x)$ to be increasing, we must have

$$f'(x) \geqslant 0$$

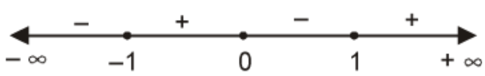
$$\Rightarrow \quad 4x(x^2-1) \geqslant 0$$

$$\Rightarrow \quad x(x^2-1) \geqslant 0 \qquad [\because\ 4 > 0]$$

$$\Rightarrow \quad x(x-1)(x+1) \geqslant 0$$

$$\Rightarrow \quad -1 \leqslant x \leqslant 0 \text{ or } x \geqslant 1$$

$$\Rightarrow \quad x \in [-1, 0] \cup [1, \infty)$$



So, $f(x)$ is increasing on $[-1, 0] \cup [1, \infty)$

For $f(x)$ to be decreasing, we must have

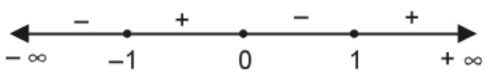
$$f'(x) \leqslant 0$$

$$\Rightarrow \quad 4x(x^2-1) \leqslant 0$$

$$\Rightarrow \quad x(x^2-1) \leqslant 0 \qquad [\because\ 4>0]$$

$$\Rightarrow \quad x(x-1)(x+1) \leqslant 0$$

$$\Rightarrow \quad x \leqslant -1 \text{ or } 0 \leqslant x \leqslant 1$$



$$\Rightarrow \quad x \in (\infty, -1] \cup [0, 1]$$

So, $f(x)$ is decreasing on $(-\infty, -1] \cup [0, 1]$.

S15. We have,

$$f(x) = \frac{x}{2} + \frac{2}{x}$$

$$\Rightarrow \quad f'(x) = \frac{1}{2} - \frac{2}{x^2} = \frac{x^2-4}{2x^2}$$

For $f(x)$ to be increasing, we must have

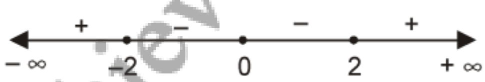
$$f'(x) \geqslant 0$$

$$\Rightarrow \quad \frac{x^2-4}{2x^2} \geqslant 0$$

$$\Rightarrow \quad x^2 - 4 \geqslant 0 \qquad [\because\ 2x^2 > 0 \text{ and } x \neq 0]$$

$$\Rightarrow \quad (x-2)(x+2) \geqslant 0$$

$$\Rightarrow \quad x \leqslant -2 \text{ or } x \geqslant 2 \text{ and } x \neq 0$$



$$\Rightarrow \quad x \in (-\infty, -2] \cup [2, \infty)$$

So, $f(x)$ is increasing on $(-\infty, -2] \cup [2, \infty)$

For $f(x)$ to be decreasing, we must have

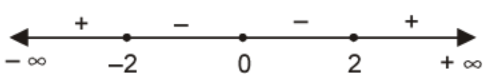
$$f'(x) \leqslant 0$$

$$\Rightarrow \quad \frac{x^2-4}{2x^2} \leqslant 0$$

$$\Rightarrow \quad x^2 - 4 \leqslant 0 \qquad [2x^2 > 0 \text{ and } x \neq 0]$$

$$\Rightarrow \quad (x-2)(x+2) \leqslant 0$$

$$\Rightarrow \quad x \in [-2, 2] - \{0\}.$$



So, $f(x)$ is decreasing on $[-2, 0) \cup (0, 2]$.

S16. We have,

$$f(x) = (x+1)^3 (x-3)^3$$

$$\Rightarrow \quad f'(x) = (x-3)^3 \cdot 3(x+1)^2 \frac{d}{dx}(x+1) + (x+1)^3 \cdot 3(x-3)^2 \frac{d}{dx}(x-3)$$

$$\Rightarrow \quad f'(x) = 3(x+1)^2 (x-3)^3 + 3(x+1)^3 (x-3)^2$$

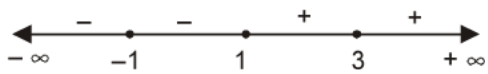
$$\Rightarrow \quad f'(x) = 3(x+1)^2 (x-3)^2 (x+1+x-3)$$

$$\Rightarrow \quad f'(x) = 6(x+1)^2 (x-3)^2 (x-1)$$

For $f(x)$ to be increasing, we must have

$$f'(x) \geqslant 0$$

$$\Rightarrow \quad 6(x+1)^2 (x-3)^2 (x-1) \geqslant 0$$



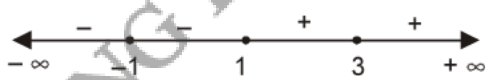
$$\Rightarrow \quad x \in [1, \infty)$$

So, $f(x)$ is increasing on $[1, \infty)$.

For $f(x)$ to be decreasing, we must have

$$f'(x) \leqslant 0$$

$$\Rightarrow \quad 6(x+1)^2 (x-3)^2 (x-1) \leqslant 0$$



$$\Rightarrow \quad x \in (-\infty, 1]$$

So, $f(x)$ is decreasing on $(-\infty, 1]$.

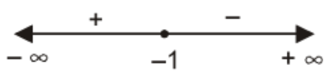
S17. We have,

$$f(x) = -x^2 - 2x + 15$$

$$\Rightarrow \quad f'(x) = -2x - 2 = -2(x+1)$$

For $f(x)$ to be increasing, we must have

$$f'(x) \geqslant 0$$



$$\Rightarrow \quad -2(x+1) \geqslant 0$$

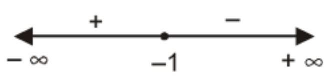
$$\Rightarrow \quad x + 1 \leqslant 0$$

$$\Rightarrow \quad x \in (-\infty, -1]$$

Thus, $f(x)$ is increasing on the interval $(-\infty, -1)$.

For $f(x)$ to be decreasing, we must have

$$f'(x) \leqslant 0$$



$$\Rightarrow \quad -2(x+1) \leqslant 0$$

$$\Rightarrow \quad x + 1 \geqslant 0$$

$$\Rightarrow \quad x \in [-1, \infty)$$

So, $f(x)$ is decreasing on $(-1, \infty)$.

S18. Given, $f(x) = x^x$ is defined for $x > 0$.

Now, $$f(x) = x^x$$
$$\log f(x) = x \log x$$

Differentiating w.r.t. x both side

$$\Rightarrow \quad \frac{1}{f(x)} \cdot f'(x) = x \cdot \frac{1}{x} + \log x \cdot 1$$

$$\Rightarrow \quad f'(x) = f(x)\,(1 + \log_e x)$$

$$\Rightarrow \quad f'(x) = x^x\,(1 + \log_e x)$$

For $f(x)$ to be increasing, we must have

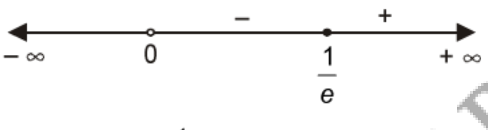
$$\Rightarrow \quad f'(x) \geq 0$$

$$\Rightarrow \quad x^x\,(1 + \log_e x) \geq 0 \qquad [\because x^x > 0 \text{ for } x > 0]$$

$$\Rightarrow \quad 1 + \log_e x \geq 0$$

$$\Rightarrow \quad 1 + \log_e x \geq 0$$

$$\Rightarrow \quad \log_e x \geq -1 \qquad \left[\begin{array}{l}\because \log_a x \geq N \Rightarrow x \geq a^N \text{ for } a > 1 \\ \text{Here, } e > 1. \text{ So, } \log_e x \geq -1 \Rightarrow x \geq e^{-1}\end{array}\right]$$



$$\Rightarrow \quad x \geq e^{-1}$$

$$\Rightarrow \quad x \in \left(\frac{1}{e}, \infty\right)$$

Thus, $f(x)$ is increasing on $\left(\frac{1}{e}, \infty\right)$

For $f(x)$ to be decreasing, we must have

$$f'(x) \leq 0$$

$$\Rightarrow \quad x^x(1 + \log_e x) \leq 0$$

$$\Rightarrow \quad 1 + \log_e x \leq 0 \qquad [\because x^x > 0 \text{ for } x > 0]$$

$$\Rightarrow \quad \log_e x \leq -1$$

$$\Rightarrow \quad x \leq e^{-1}$$

$$\Rightarrow \quad x \in (0, 1/e)$$

Thus, $f(x)$ is decreasing on $(0, 1/e)$.

Hence, $f(x)$ is increasing on $(1/e, \infty)$ and decreasing on $(0, 1/e)$.

S19. We have,

$$f(x) = -2x^3 - 9x^2 - 12x + 1$$

$$\Rightarrow \quad f'(x) = -6x^2 - 18x - 12$$

$$= -6(x^2 + 3x + 2)$$

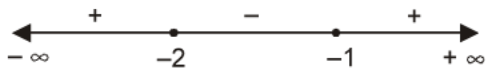
$$= -6(x + 1)(x + 2)$$

For $f(x)$ to be increasing, we must have

$$f'(x) \geqslant 0$$

$$\Rightarrow \quad -6(x + 1)(x + 2) \geqslant 0$$

$$\Rightarrow \quad (x + 1)(x + 2) \leqslant 0$$



$$\Rightarrow \quad x \in [-2, -1]$$

For $f(x)$ to be decreasing, we must have

$$f'(x) \leqslant 0$$

$$\Rightarrow \quad -6(x + 1)(x + 2) \leqslant 0$$

$$\Rightarrow \quad (x + 1)(x + 2) \geqslant 0$$

$$\Rightarrow \quad x \leqslant -2 \quad \text{or} \quad x \geqslant -1$$

$$\Rightarrow \quad x \in (-\infty, -2] \cup [-1, \infty)$$

Hence, $f(x)$ is increasing on $[-2, -1]$ and decreasing on $(-\infty, -2] \cup [-1, \infty)$.

S20. The given function is

$$f(x) = x^3 + \frac{1}{x^3}, x \neq 0$$

Differentiating w.r.t. x, we get

$$f'(x) = 3x^2 - \frac{3}{x^4}$$

Put $\quad f'(x) = 0$

$$\Rightarrow \quad 3x^2 - \frac{3}{x^4} = 0$$

$$\Rightarrow \quad \frac{3x^6 - 3}{x^4} = 0$$

$$\Rightarrow \quad 3x^6 - 3 = 0$$

$$\Rightarrow \quad 3x^6 = 3$$

$$\Rightarrow \quad x^6 = 1$$

$\Rightarrow \qquad x = \pm 1 \qquad \left[\begin{array}{l}\because x^6 = (1)^6 \text{ and } x^6 = (-1)^6 \\ \Rightarrow x = 1 \text{ and } x = -1\end{array}\right]$

Now, we find intervals in which $f(x)$ is increasing or decreasing.

Interval	$f'(x) = \frac{3x^6 - 3}{x^4}$	Sign of $f'x$
$x < -1$	$\frac{(+)}{(+)}$	+ve
$-1 < x < 1$,	$\frac{(-)}{(+)}$	-ve
$x \neq 0$		
$x > 1$	$\frac{(+)}{(+)}$	+ve

Now, we know that a function $f(x)$ is increasing when $f'(x) \geq 0$ and it is said to be decreasing when $f'(x) \leq 0$. So, $f(x)$ is increasing on intervals $(-\infty, -1)$ and $(1, \infty)$ and it is decreasing on $(-1, 1) - \{0\}$.

S21. We have,

$\Rightarrow \qquad f(x) = x^4 - \frac{x^3}{3}$

$\Rightarrow \qquad f'(x) = 4x^3 - x^2 = x^2(4x - 1)$

For $f(x)$ to be increasing, we must have

$f'(x) \geqslant 0$

$\Rightarrow \qquad x^2(4x - 1) \geqslant 0 \qquad [\because x^2 \geqslant 0]$

$\Rightarrow \qquad x \in \left[\frac{1}{4}, \infty\right)$

So, $f(x)$ is increasing on $[1/4, \infty)$.

For $f(x)$ to be decreasing, we must have

$f'(x) \leqslant 0$

$\Rightarrow \qquad x^2(4x - 1) \leqslant 0 \qquad [\because x^2 > 0]$

$\Rightarrow \qquad x \in \left(-\infty, \frac{1}{4}\right]$

So, $f(x)$ is decreasing on $\left(-\infty, \frac{1}{4}\right]$.

S22. We have

$f(x) = x^2(x - 2)^2$

$\Rightarrow \qquad f'(x) = 2x(x - 2)^2 + 2x^2(x - 2) = 2x(x - 2)(2x - 2)$

$\Rightarrow \qquad f'(x) = 4x(x - 2)(x - 1)$

For $f(x)$ to be increasing, we must have

$$f'(x) \geqslant 0$$

$\Rightarrow$ $4x(x-2)(x-1) \geqslant 0$

$\Rightarrow$ $x(x-2)(x-1) \geqslant 0$

$\Rightarrow$ $x \in [0, 1] \cup [2, \infty)$

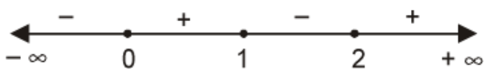
So, $f(x)$ is increasing on $[0, 1] \cup [2, \infty)$

For $f(x)$ to be decreasing, we must have

$$f'(x) \leqslant 0$$

$\Rightarrow$ $4x(x-2)(x-1) \leqslant 0$

$\Rightarrow$ $x(x-2)(x-1) \leqslant 0$



$\Rightarrow$ $x \in (-\infty, 0] \cup [1, 2]$

So, $f(x)$ is decreasing on $(-\infty, 0] \cup [1, 2]$

S23. We have,

$$f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5}x + 11$$

$\Rightarrow$ $f'(x) = \frac{6}{5}x^3 - \frac{12}{5}x^2 - 6x + \frac{36}{5}$

$\Rightarrow$ $f'(x) = \frac{6}{5}(x^3 - 2x^2 - 5x + 6)$

$\Rightarrow$ $= \frac{6}{5}(x-1)(x^2 - x - 6)$

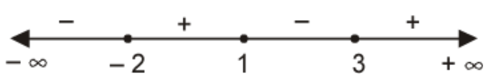
$\Rightarrow$ $= \frac{6}{5}(x-1)(x-3)(x+2)$

For $f(x)$ to be increasing, we must have

$$f'(x) \geqslant 0$$

$\Rightarrow$ $\frac{6}{5}(x-1)(x-3)(x+2) \geqslant 0$

$\Rightarrow$ $(x-1)(x-3)(x+2) \geqslant 0$



$\Rightarrow$ $x \in [-2, 1] \cup [3, \infty)$

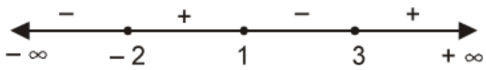
So, $f(x)$ is increasing on $[-2, 1] \cup [3, \infty)$.

For $f(x)$ to be decreasing, we must have

$$f'(x) \leqslant 0$$

$\Rightarrow$ $\frac{6}{5}(x-1)(x-3)(x+2) \leqslant 0$

$\Rightarrow \qquad (x-1)(x-3)(x+2) \leqslant 0$



$\Rightarrow \qquad x \in (-\infty, -2] \cup [1, 3]$

S24. The given function is

$$f(x) = x^3 - 12x^2 + 36x + 17$$

Differentiating both sides, w.r.t. x, we get

$$f'(x) = 3x^2 - 24x + 36$$

Putting $f'(x) = 0$

$\therefore \qquad 3x^2 - 24x + 36 = 0$

$\Rightarrow \qquad 3(x^2 - 8x + 12) = 0$

$\Rightarrow \qquad 3(x-2)(x-6) = 0$

$\Rightarrow \qquad x - 2 = 0 \text{ and } x - 6 = 0$

$\therefore \qquad x = 2 \text{ or } 6$

Now, we find intervals in which the function $f(x)$ is increasing or decreasing.

Interval	**$f'(x) = 3(x-2)(x-6)$**	**Sign of $f'x$**
$x < 2$	(+) (–) (–)	+ve
$2 < x < 6$	(+) (+) (–)	–ve
$x > 6$	(+) (+) (+)	+ve

We know that a function $f(x)$ is said to be an increasing function, if $f'(x) \geq 0$ and a decreasing function, if $f'(x) \leq 0$. So, $f(x)$ is increasing on intervals $(-\infty, 2)$ and $(6, \infty)$ and it is decreasing on $(2, 6)$.

S25. The given function is $f(x) = (x-1)(x-2)^2$

Differentiating both sides, w.r.t. x using product rule, we get

$$f'(x) = (x-1) \times \frac{d}{dx}(x-2)^2 + (x-2)^2 \times \frac{d}{dx}(x-1)$$

$\Rightarrow \qquad f'(x) = (x-1)\, 2(x-2) + (x-2)^2 \cdot 1$

$\therefore \qquad f'(x) = 2(x-1)(x-2) + (x-2)^2$

$\qquad = (x-2)[2x - 2 + x - 2]$

$\qquad f'(x) = (x-2)(3x-4)$

Putting $\qquad f'(x) = 0$

$\therefore \qquad (x-2)(3x-4) = 0$

$\Rightarrow \quad x - 2 = 0 \quad \text{or} \quad 3x - 4 = 0$

$\therefore \quad x = \frac{4}{3} \text{ or } 2$

Now, we find intervals in which $f(x)$ is increasing or decreasing.

Interval	$f'(x) = (x - 2)(3x - 4)$	Sign of $f'x$
$x < \frac{4}{3}$	(–) (–)	+ve
$\frac{4}{3} < x < 2$	(–) (+)	–ve
$x > 2$	(+) (+)	+ve

We know that a function $f(x)$ is said to be an increasing function, if $f'(x) \geq 0$ and a decreasing function, when $f'(x) \leq 0$. So, $f(x)$ is increasing on $\left(-\infty, \frac{4}{3}\right)$ and $(2, \infty)$ and decreasing on $\left(\frac{4}{3}, 2\right)$.

S26. Given that, $\quad f(x) = (x - 1)^3 (x - 2)^2$

Differentiating both sides w.r.t. x, using product rule, we get

$$f'(x) = (x-1)^3 \cdot \frac{d}{dx}(x-2)^2 + (x-2)^2 \cdot \frac{d}{dx}(x-1)^3$$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\Rightarrow \quad f'(x) = (x-1)^3 \cdot 2(x-2) + (x-2)^2 \cdot 3(x-1)^2$$
$$= (x-1)^2 (x-2) [2(x-1) + 3(x-2)]$$
$$= (x-1)^2 (x-2) (2x - 2 + 3x - 6)$$

$$\Rightarrow \quad f'(x) = (x-1)^2 (x-2) (5x-8)$$

Now, we put $\quad f'(x) = 0$

So, $\quad (x-1)^2 (x-2)(5x-8) = 0$

Either $\quad (x-1)^2 = 0$ or $x - 2 = 0$

or $\quad 5x - 8 = 0$

$\therefore$ We get, $\quad x = 1, \frac{8}{5}, 2$

Now, we find intervals and check in which interval $f(x)$ is increasing or decreasing.

Interval	$f'(x) = (x-1)^2$ $(x-2)(5x-8)$	Sign of $f'(x)$
$x < 1$	(+) (–) (–)	+ve
$1 < x < \frac{8}{5}$	(+) (–) (–)	+ve
$\frac{8}{5} < x < 2$	(+) (–) (+)	–ve
$x > 2$	(+) (+) (+)	+ve

We know that a function $f(x)$ is said to be an increasing function, if $f'(x) \geq 0$ and decreasing, if $f'(x) \leq 0$. So, given function $f(x)$ is

(a) increasing on intervals $(-\infty, 1)$, $\left(1, \frac{8}{5}\right)$ and $(2, \infty)$

(b) decreasing on $\left(\frac{8}{5}, 2\right)$.

S27. The given function is

$$f(x) = 2x^3 + 9x^2 + 12x + 20$$

Differentiating w.r.t. x, we get

$$f'(x) = 6x^2 + 18x + 12$$

Putting $f'(x) = 0$

$\therefore \quad 6x^2 + 18x + 12 = 0$

$\Rightarrow \quad 6(x^2 + 3x + 2) = 0$

$\Rightarrow \quad 6(x+1)(x+2) = 0$

$\Rightarrow \quad (x+1)(x+2) = 0$

$\Rightarrow \quad x + 1 = 0$

or $\quad x + 2 = 0$

$\therefore \quad x = -2, -1$

Now, we find intervals and check in which interval $f(x)$ is increasing or decreasing.

Interval	$f'(x) = 6(x+1)(x+2)$	Sign of $f'(x)$
$x < -2$	(+) (–) (–)	+ve
$-2 < x < -1$	(+)(–) (+)	–ve
$x > -1$	(+) (+) (+)	+ve

We know that a function $f(x)$ is said to be an increasing function, if $f'(x) \geq 0$ and decreasing, if $f'(x) \leq 0$. So, given function is

(a) increasing on intervals $(-\infty, -2)$ and $(-1, \infty)$

(b) decreasing on interval $(-2, -1)$.

S28. The given function is

$$f(x) = 2x^3 - 9x^2 + 12x + 15$$

Differentiating both sides, w.r.t. x, we get

$$f'(x) = 6x^2 - 18x + 12$$

Putting $f'(x) = 0$

$\therefore \quad 6x^2 - 18x + 12 = 0$

$\Rightarrow \quad 6(x^2 - 3x + 2) = 0$

$\Rightarrow \quad 6(x-1)(x-2) = 0$

$\Rightarrow \quad x - 1 = 0$ and $x - 2 = 0$

$\therefore \quad x = 1$ or 2

Now, we find intervals and check in which interval $f(x)$ is increasing or decreasing.

Interval	**$f'(x) = 6(x-1)(x-2)$**	**Sign of $f'x$**
$x < 1$	(+) (–) (–)	+ve
$1 < x < 2$	(+) (+) (–)	–ve
$x > 2$	(+) (+) (+)	+ve

We know that a function $f(x)$ is said to be an increasing function when $f'(x) \geq 0$ and it is said to be a decreasing function when $f'(x) \leq 0$. So, $f(x)$ is

(a) increasing on intervals $(-\infty, 1)$ and $(2, \infty)$

(b) decreasing on $(1, 2)$.

S29. The given function is

$$f(x) = 2x^3 - 15x^2 + 36x + 17$$

Differentiating w.r.t. x, we get

$$f'(x) = 6x^2 - 30x + 36$$

Putting $f'(x) = 0$

$\therefore \quad 6x^2 - 30x + 36 = 0$

$\Rightarrow \quad 6(x^2 - 5x + 6) = 0$

$\Rightarrow \quad 6(x-2)(x-3) = 0$

$\Rightarrow \qquad x - 2 = 0 \text{ or } x - 3 = 0$

$\therefore \qquad x = 2 \text{ or } 3$

Now, we find the intervals and check in which interval $f(x)$ is increasing or decreasing.

Interval	$f'(x) = 6(x - 2)(x - 3)$	Sign of $f'(x)$
$x < 2$	(+) (–) (–)	+ve
$2 < x < 3$	(+) (+) (–)	–ve
$x > 3$	(+) (+) (+)	+ve

We know that $f(x)$ is said to be an increasing function when $f'(x) \geq 0$ and decreasing function when $f'(x) \leq 0$. So, $f(x)$ is increasing on $(-\infty, 2)$ and $(3, \infty)$ and it is decreasing on $(2, 3)$.

S30. The domain of $f(x)$ is the set of all positive real numbers other than unity *i.e.*, $(0, 1) \cup (1, \infty)$

Now, $\qquad f(x) = \dfrac{x}{\log x}$

$$\Rightarrow \qquad f'(x) = \frac{\log x \cdot 1 - x \cdot \dfrac{1}{x}}{(\log x)^2}$$

$$\Rightarrow \qquad f'(x) = \frac{\log x - 1}{(\log x)^2}$$

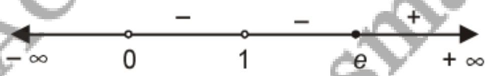
For $f(x)$ to be increasing, we must have

$$f'(x) \geqslant 0$$

$$\Rightarrow \qquad \frac{\log x - 1}{(\log x)^2} \geqslant 0$$

$\Rightarrow \qquad \log x - 1 \geqslant 0 \qquad [\because (\log x)^2 > 0 \text{ for } x > 0, x \neq 1]$

$\Rightarrow \qquad \log x \geqslant 1$



$\Rightarrow \qquad x \geqslant e^1 \qquad \left[\begin{array}{l}\because \log_a x \geqslant N \Rightarrow x \geqslant a^N \text{for } a > 1 \\ \text{Here}, e > 1 \therefore \log_e x \geqslant 1 \Rightarrow x \geqslant e^1\end{array}\right]$

$\Rightarrow \qquad x \in [e, \infty)$

So, $f(x)$ is increasing on $[e, \infty)$

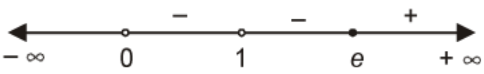
For $f(x)$ to be decreasing, we must have

$$f'(x) \leqslant 0$$

$\Rightarrow$ $$\frac{\log x - 1}{(\log x)^2} \leqslant 0$$

$\Rightarrow$ $$\log x - 1 \leqslant 0 \qquad [\because \ (\log x)^2 \geqslant 0 \text{ for } x \geqslant 0, x \neq 1]$$

$\Rightarrow$ $$\log x \leqslant 1$$



$\Rightarrow$ $$x \leqslant e^1 \qquad \left[\begin{array}{l}\because \log_a x \geqslant N \Rightarrow x \geqslant a^N \text{for } a > 1 \\ \text{Here}, e > 1 \therefore \log_e x \geqslant 1 \Rightarrow x \geqslant e^1\end{array}\right]$$

$\Rightarrow$ $$x \in (0, e] - \{1\} \qquad [\because \ f(x) \text{ is defined for } x \geqslant 0, x \neq 1]$$

So, $f(x)$ is decreasing on $(0, e] - \{1\}$.

S31. $f(x)$ is defined for all $x > 2$

Now, $$f(x) = 2 \log (x - 2) - x^2 + 4x + 1$$

$\Rightarrow$ $$f'(x) = \frac{2}{x-2} - 2x + 4$$

$\Rightarrow$ $$f'(x) = \frac{2 - 2x(x-2) + 4(x-2)}{x-2} = \frac{-2x^2 + 8x - 6}{x-2}$$

$\Rightarrow$ $$f'(x) = \frac{-2(x^2 - 4x + 3)}{x-2} = \frac{-2(x-1)(x-3)}{x-2}$$

For $f(x)$ to be increasing, we must have

$$f'(x) \geqslant 0$$

$\Rightarrow$ $$\frac{-2(x-1)(x-3)}{x-2} \geqslant 0$$

$\Rightarrow$ $$\frac{(x-1)(x-3)}{x-2} \leqslant 0$$

$\Rightarrow$ $$x - 3 \leqslant 0 \qquad [\because \ f(x) \text{ is defined for } x > 2 \Rightarrow x - 1 > 0, x - 2 > 0)$$

$\Rightarrow$ $$x \leqslant 3$$

$\Rightarrow$ $$x \in (2, 3]$$

So, $f(x)$ is increasing on $(2, 3]$.

S32. We have,

$$f(x) = \frac{4 \sin x - 2x - x \cos x}{2 + \cos x}$$

$\Rightarrow \quad f'(x) = \dfrac{(2+\cos x)(4\cos x - 2 - \cos x + x\sin x) + (4\sin x - 2x - x\cos x)\sin x}{(2+\cos x)^2}$

$$= \frac{6\cos x + 2x\sin x - 4 + 3\cos^2 x + x\sin x\cos x + 4\sin^2 x - 2x\sin x - x\sin x\cos x - 2\cos x}{(2+\cos x)^2}$$

$$= \frac{4\cos x - 4 + 3 + 1 - \cos^2 x}{(2+\cos x)^2}$$

$\Rightarrow \quad f'(x) = \dfrac{\cos x(4-\cos x)}{(2+\cos x)^2}$

(i) For $f(x)$ to be increasing, we must have

$$f'(x) \geqslant 0$$

$\Rightarrow \quad \dfrac{\cos x(4-\cos x)}{(2+\cos x)^2} \geqslant 0$

$\Rightarrow \quad \cos x \geqslant 0 \qquad \left[\because \dfrac{4-\cos x}{(2+\cos x)^2} > 0\right]$

$\Rightarrow \quad x \in \left[0, \dfrac{\pi}{2}\right] \cup \left[\dfrac{3\pi}{2}, 2\pi\right]$

Hence, $f(x)$ is increasing on $\left[0, \dfrac{\pi}{2}\right] \cup \left[\dfrac{3\pi}{2}, 2\pi\right]$

(ii) For $f(x)$ to be decreasing, we must have

$$f'(x) \leqslant 0$$

$\Rightarrow \quad \dfrac{\cos x(4-\cos x)}{(2+\cos x)^2} \leqslant 0$

$\Rightarrow \quad \cos x \leqslant 0 \qquad \left[\because \dfrac{4-\cos x}{(2+\cos x)^2} > 0\right]$

$\Rightarrow \quad x \in \left[\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right]$

Hence, $f(x)$ is decreasing on $\left[\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right]$.

S33. We have,

$$f(x) = \sin^4 x + \cos^4 x$$

$\Rightarrow \quad f'(x) = 4\sin^3 x\cos x - 4\cos^3 x\sin x$

$\Rightarrow \quad f'(x) = -4 \sin x \cos x (\cos^2 x - \sin^2 x)$

$\Rightarrow \quad f'(x) = -2(2\sin x \cos x)(\cos^2 x - \sin^2 x)$

$\Rightarrow \quad f'(x) = -2 \sin 2x \cos 2x$

$\Rightarrow \quad f'(x) = -\sin 4x$

We have, $0 \leqslant x \leqslant \pi/2 \Rightarrow 0 \leqslant 4x \leqslant 2\pi$.

Case I: When $0 \leqslant 4x \leqslant \pi$

In this case, we have

$$\sin 4x > 0$$

$\Rightarrow \quad -\sin 4x \leqslant 0$

$\Rightarrow \quad f'(x) \leqslant 0$

$\therefore \quad f'(x) \leqslant 0 \qquad$ [for $0 \leqslant 4x \leqslant \pi$ or $0 \leqslant x \leqslant \pi/4$]

So, $f(x)$ is decreasing on $[0, \pi/4]$.

Case II: When, $\pi \leqslant 4x \leqslant 2\pi$

In this case, we have

$$\sin 4x \leqslant 0$$

$\Rightarrow \quad -\sin 4x \geqslant 0$

$\Rightarrow \quad f'(x) \geqslant 0$

$\therefore \quad f'(x) \geqslant 0 \qquad \left[\text{for } \pi \leqslant 4x \leqslant 2\pi \text{ or } \frac{\pi}{4} \leqslant x \leqslant \frac{\pi}{2}\right]$

So, $f(x)$ is increasing on $[\pi/4, \pi/2]$.

S34. The given function $y = \log(1 + x) - \dfrac{2x}{2+x}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{1+x}(1) - \frac{(2+x)\cdot 2 - 2x \cdot 1}{(2+x)^2}$$

$$= \frac{1}{1+x} - \frac{4+2x-2x}{(2+x)^2}$$

$$= \frac{(2+x)^2 - 4(1+x)}{(1+x)(2+x)^2}$$

$$= \frac{(4+x^2+4x-4-4x)}{(1+x)(2+x)^2}$$

$$= \frac{x^2}{(1+x)(2+x)^2} \qquad \ldots \text{(i)}$$

Now, x^2, $(2 + x)^2$ are always positive, also $1 + x > 0$ for $x > -1$.

From Eq. (i), $\frac{dy}{dx} > 0$ for $x > -1$. Hence, function increases for $x > -1$.

S35. We have,

$$f(x) = \sin 3x$$

$$\Rightarrow \quad f'(x) = 3\cos 3x$$

Now, $\quad 0 \leqslant x \leqslant \pi/2$

$\Rightarrow \quad 0 \leqslant 3x \leqslant 3\pi/2.$

Since cosine function is positive in first quadrant and negative in the second and third quadrants. Therefore, we consider the following cases.

Case I: When $\quad 0 \leqslant 3x \leqslant \pi/2$

In this case, we have

$$0 \leqslant 3x \leqslant \pi/2$$

$\Rightarrow \quad \cos 3x \geqslant 0$

$\Rightarrow \quad 3\cos 3x \geqslant 0$

$f'(x) \geqslant 0$

for $\quad 0 \leqslant 3x \leqslant \pi/2$

i.e., $\quad 0 \leqslant x \leqslant \pi/6$

So, $f(x)$ is increasing on $[0, \pi/6]$

Case II: When $\quad \pi/2 \leqslant 3x \leqslant 3\pi/2$

In this case, we have

$$\pi/2 \leqslant 3x \leqslant 3\pi/2$$

$\Rightarrow \quad 3\cos 3x \leqslant 0$

$\Rightarrow \quad \cos 3x \leqslant 0$

$f'(x) \leqslant 0$

for $\quad \pi/2 \leqslant 3x \leqslant 3\pi/2$

i.e., $\quad \pi/6 \leqslant x \leqslant \pi/2$

So, $f(x)$ is decreasing on $[\pi/6, \pi/2]$

Hence, $f(x)$ is increasing on $[0, \pi/6]$ and decreasing on $[\pi/6, \pi/2]$

S36. We have

$$f(x) = \tan^{-1}(\sin x + \cos x)$$

$$\Rightarrow \quad f'(x) = \frac{1}{1+(\sin x+\cos x)^2} \times \frac{d}{dx}(\sin x + \cos x)$$

$$= \frac{1}{1+(\sin+\cos x)^2} \times (\cos x - \sin x)$$

$$\Rightarrow \quad f'(x) = \frac{\cos x(1-\tan x)}{1+(\sin x+\cos x)^2}$$

for increasing

$$\Rightarrow \quad f'(x) = \frac{\cos x(1-\tan x)}{1+(\sin x+\cos x)^2} > 0$$

$$\Rightarrow \quad \cos x > 0, \qquad \left[\begin{array}{c} \dfrac{1}{1+(\sin x+\cos x)^2} \text{ and } 1 - \tan x > 0 \\ \because \ \tan x < 1 \text{ for } 0 < x < \pi/4 \end{array}\right]$$

$\Rightarrow \quad f'(x) > 0$

Thus, $\quad f'(x) > 0 \quad$ for all $\quad x \in [0, \pi/4]$

Hence, $f(x)$ is increasing on $[0, \pi/4]$.

S37. The given function is

$$f(x) = \sin x + \cos x, \quad 0 \le x \le 2\pi$$

Differentiating both sides, w.r.t. x, we get

$$f'(x) = \cos x - \sin x$$

Putting $f'(x) = 0$, we get

$$\cos x - \sin x = 0$$

$$\Rightarrow \quad \cos x = \sin x$$

Dividing both sides by $\cos x$, we get

$$1 = \tan x$$

$$\Rightarrow \quad \tan x = \tan\frac{\pi}{4} \text{ or } \tan x = \tan\frac{5\pi}{4} \qquad \left[\because \ \tan\frac{\pi}{4} = 1 \ \text{ and } \ \tan\frac{5\pi}{4} = 1\right]$$

$$\therefore \quad x = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}$$

Now, we form intervals and check in which interval $f(x)$ is strictly increasing or strictly decreasing.

Interval	$f'(x) = \cos x - \sin x$	Sign of $f'(x)$
$x < \frac{\pi}{4}$	$\cos\frac{\pi}{6} - \sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3}-1}{2}$	+ve
$\frac{\pi}{4} < x < \frac{5\pi}{4}$	$\cos\frac{\pi}{2} - \sin\frac{\pi}{2} = 0 - 1 = -1$	–ve
$x > \frac{5\pi}{4}$	$\cos\frac{3\pi}{2} - \sin\frac{3\pi}{2} = 0 - (-1) = 1$	+ve

Since, $f'(x) > 0$ at $x < \frac{\pi}{4}$ and $x > \frac{5\pi}{4}$, so $f(x)$ is strictly increasing in the intervals $\left(0, \frac{\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, 2\pi\right)$ while $f'(x) < 0$ at $\frac{\pi}{4} < x < \frac{5\pi}{4}$, so $f(x)$ is strictly decreasing in the interval $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$.

S38. We have,

$$f(x) = \log \cos x$$

$$\Rightarrow \quad f'(x) = \frac{1}{\cos x}(-\sin x)$$

$$= -\tan x$$

Now, $\quad x \in (-\pi/2, 0)$

$$\Rightarrow \quad \tan x < 0$$

$$\Rightarrow \quad -\tan x > 0$$

$$\Rightarrow \quad f'(x) > 0$$

So, $f(x)$ is strictly increasing on $(-\pi/2, 0)$

and, $\quad x \in (0, \pi/2)$

$$\Rightarrow \quad \tan x > 0$$

$$\Rightarrow \quad -\tan x < 0$$

$$\Rightarrow \quad f'(x) < 0$$

So, $f(x)$ is strictly decreasing on $(0, \pi/2)$.

S39. We have,

$$f(x) = \tan x - 4x$$

$$\Rightarrow \quad f'(x) = \sec^2 x - 4 = \frac{1 - 4\cos^2 x}{\cos^2 x}$$

$\Rightarrow \quad f'(x) = \dfrac{4}{\cos^2 x}\left(\dfrac{1}{4} - \cos^2 x\right)$

$= 4\sec^2 x\left(\dfrac{1}{2} - \cos x\right)\left(\dfrac{1}{2} + \cos x\right)$

Now, $\quad -\dfrac{\pi}{3} \leqslant x \leqslant \dfrac{\pi}{3}$

$\Rightarrow \quad \cos x \geqslant \dfrac{1}{2}$

$\Rightarrow \quad \dfrac{1}{2} - \cos x \leqslant 0$

$\Rightarrow \quad 4\sec^2 x\left(\dfrac{1}{2} - \cos x\right)\left(\dfrac{1}{2} + \cos x\right) \leqslant 0$

$\Rightarrow \quad f'(x) \leqslant 0$

So, $f(x)$ is decreasing on $[-\pi/3, \pi/3]$

S40. We have,

$f(x) = \cos(2x + \pi/4)$

$\Rightarrow \quad f'(x) = -2\sin(2x + \pi/4)$

Now,

$x \in [3\pi/8, 7\pi/8]$

$\Rightarrow \quad 3\pi/8 \leqslant x \leqslant 7\pi/8$

$\Rightarrow \quad 3\pi/4 \leqslant 2x \leqslant 7\pi/4$

$\Rightarrow \quad \pi/4 + 3\pi/4 \leqslant 2x + \pi/4 \leqslant 7\pi/4 + \pi/4$

$\Rightarrow \quad \pi \leqslant 2x + \pi/4 \leqslant 2\pi$

$\Rightarrow \quad \sin(2x + \pi/4) \leqslant 0 \quad$ [$\because$ since function is neg. in third and fourth quadrant]

$\Rightarrow \quad -2\sin(2x + \pi/4) \geqslant 0$

$\Rightarrow \quad f'(x) \geqslant 0$

Hence, $f(x)$ is increasing on $[3\pi/8, 7\pi/8]$.

S41. Given function is

$$y = \frac{4\sin\theta}{2 + \cos\theta} - \theta \qquad \ldots \text{(i)}$$

$\therefore$ Differentiating Eq. (i) both sides w.r.t. θ, we get

$$\frac{dy}{d\theta} = \frac{\begin{bmatrix}(2+\cos\theta)\times\frac{d}{d\theta}(4\sin\theta)\\ -4\sin\theta\times\frac{d}{d\theta}(2+\cos\theta)\end{bmatrix}}{(2+\cos\theta)^2} - 1$$

$$\Rightarrow \quad \frac{dy}{d\theta} = \frac{(2+\cos\theta)(4\cos\theta)-4\sin\theta(0-\sin\theta)}{(2+\cos\theta)^2} - 1$$

$$\Rightarrow \quad \frac{dy}{d\theta} = \frac{\left[8\cos\theta+4\cos^2\theta+4\sin^2\theta-(2+\cos\theta)^2\right]}{(2+\cos\theta)^2}$$

$$\Rightarrow \quad \frac{dy}{d\theta} = \frac{\left[8\cos\theta+4(\cos^2\theta+\sin^2\theta)-(4+\cos^2\theta+4\cos\theta)\right]}{(2+\cos\theta)^2}$$

$$\Rightarrow \quad \frac{dy}{d\theta} = \frac{8\cos\theta+4-4-\cos^2\theta-4\cos\theta}{(2+\cos\theta)^2} \qquad [\because \sin^2\theta+\cos^2\theta=1]$$

$$\Rightarrow \quad \frac{dy}{d\theta} = \frac{4\cos\theta-\cos^2\theta}{(2+\cos\theta)^2}$$

$$\Rightarrow \quad \frac{dy}{d\theta} = \frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2}$$

Now, as $\cos\theta \geq 0$ for all $\theta \in \left[0, \frac{\pi}{2}\right]$

Also, $(2 + \cos\theta)^2$ being a perfect square is always positive for all $\theta \in \left[0, \frac{\pi}{2}\right]$.

Again, for $\theta \in \left[0, \frac{\pi}{2}\right]$, we know that $-1 \leq \cos\theta \leq 1$

$\Rightarrow \quad 4 - \cos\theta > 0$ for all $\theta \in \left[0, \frac{\pi}{2}\right]$

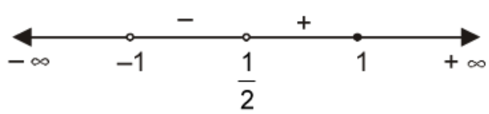
Hence, we conclude that $\frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2} \geq 0$ for all $\theta \in \left[0, \frac{\pi}{2}\right]$

$\Rightarrow \quad \frac{dy}{d\theta} \geq 0$ for all $\theta \in \left[0, \frac{\pi}{2}\right]$

Hence, y is an increasing function in $\left[0, \frac{\pi}{2}\right]$.

S42. We have,

$$f(x) = x^2 - x + 1$$



$$\Rightarrow \quad f'(x) = (2x - 1)$$

Now, $\quad -1 < x < 1/2$

$\Rightarrow \quad (x - 1/2) < 0$

$\Rightarrow \quad (2x - 1) < 0$

$\Rightarrow \quad f'(x) < 0$

and, $\quad 1/2 < x < 1$

$\Rightarrow \quad (x - 1/2) > 0$

$\Rightarrow \quad (2x - 1) > 0$

$\Rightarrow \quad f'(x) > 0$

Thus, $f'(x)$ does not have the same sign throughout the interval $(-1, 1)$.

Hence, $f(x)$ is neither increasing nor decreasing on $(-1, 1)$.

S43. The given function is $f(x) = [x(x-2)]^2$.

or $\quad f(x) = (x^2 - 2x)^2$

We have to find the values of x for which $f(x)$ is an increasing function

So, $\quad f'(x) = 2(x^2 - 2x)(2x - 2)$

Putting $\quad f'(x) = 0$

$\Rightarrow \quad 2(x^2 - 2x)(2x - 2) = 0$

$\Rightarrow \quad 4x(x - 2)(x - 1) = 0$

$\Rightarrow \quad 4x = 0 \quad \text{or} \quad x - 2 = 0 \quad \text{or} \quad x - 1 = 0$

$\Rightarrow \quad x = 0, 1, 2 \qquad \text{... (i)}$

Now, we find the intervals in which $f(x)$ is an increasing function.

Interval	$f'(x) = 4x(x-1)(x-2)$	**Sign of** $f'(x)$
$x < 0$	(–) (–) (–)	– ve
$0 < x < 1$	(+) (–) (–)	+ ve
$1 < x < 2$	(+) (+) (–)	– ve
$x > 2$	(+) (+) (+)	+ ve

From the above table we can see that the given function $f(x)$ is an increasing function when $0 < x < 1$ and when $x > 2$. Because at these values of x, $f'(x)$ is positive and we know that $f(x)$ is said to be an increasing function whenever $f'(x) \geq 0$.

Also, we have to find the points on the given curve where the tangent is parallel to X-axis. We know that when a tangent is parallel to X-axis, then

$$\frac{dy}{dx} = 0$$

where $y = [x(x-2)]^2$

$\therefore$ From Eq. (i),

$$f'(x) = 0 \Rightarrow x = 0, 1, 2$$

when $x = 0$, $y = [0(-2)]^2 = 0$

when $x = 1$, $y = [1(-1)]^2 = 1$

when $x = 2$, $y = [(2(0)]^2 = 0$

Hence, the tangent is parallel to X-axis at the points (0, 0), (1, 1) and (2, 0).

S44. The given function is

$$f(x) = 20 - 9x + 6x^2 - x^3$$

Differentiating w.r.t. x, we get

$$f'(x) = -9 + 12x - 3x^2$$

Putting $f'(x) = 0$

$\therefore$ $-9 + 12x - 3x^2 = 0$

$\Rightarrow$ $-3(x^2 - 4x + 3) = 0$

$\Rightarrow$ $-3(x-1)(x-3) = 0$

$\Rightarrow$ $(x-1)(x-3) = 0$

$\Rightarrow$ $x - 1 = 0$ or $x - 3 = 0$

$\therefore$ $x = 1$ or 3

Now, we find intervals in which $f(x)$ is strictly increasing and the intervals in which it is strictly decreasing.

Interval	$f'(x) = -3(x-1)(x-3)$	Sign of $f'(x)$
$x < 1$	(–) (–) (–)	–ve
$1 < x < 3$	(–) (+) (–)	+ve
$x > 3$	(–) (+) (+)	–ve

Now, we know that a function $f(x)$ is said to be strictly increasing when $f'(x) > 0$ and it is said to be strictly decreasing, if $f'(x) < 0$. So, the $f(x)$ is

(a) strictly increasing on the interval (1, 3)

(b) strictly decreasing on the intervals $(-\infty, 1)$ and $(3, \infty)$

S45. We have,

$$f(x) = x^3 - 6x^2 + 9x + 15$$

$\Rightarrow$ $f'(x) = 3x^2 - 12x + 9$

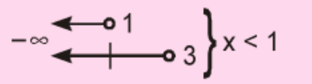
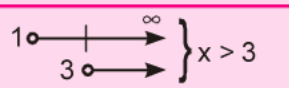
$$= 3(x^2 - 4x + 3)$$

$$= 3(x-1)(x-3) \quad \Rightarrow \quad x = 1 \text{ or } x = 3$$

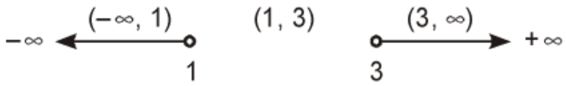
(a) **When $f(x)$ is increasing:**

$$\Rightarrow \quad f'(x) > 0$$

$$\Rightarrow \quad 3(x-1)(x-3) > 0$$

Sign. of $3(x-1)$	Sign. of $(x-3)$	Sign. of $f'(x)$	Graphical intervals	Intervals	Intervals
– ve $x < 1$	– ve $x < 3$	+ ve			$(-\infty, 1)$
+ ve $1 < x$	+ ve $3 < x$	+ ve			$(3, \infty)$ $(x > 3)$

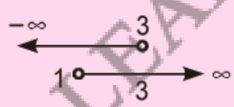
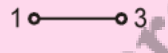
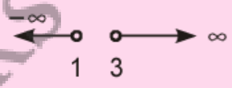
$$\Rightarrow \quad x > 3 \text{ or } x < 1 \quad \Rightarrow \quad x \in (-\infty, 1) \cup (3, \infty)$$

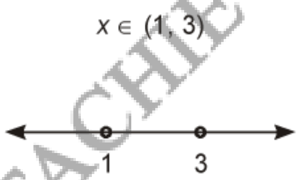


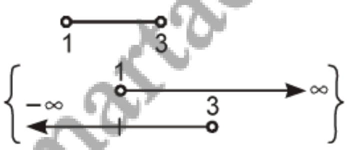
The required interval is $(-\infty, 1) \cup (3, \infty)$.

(b) **When $f(x)$ is decreasing:**

$$\Rightarrow \quad f'(x) < 0 \quad \Rightarrow \quad 3(x-1)(x-3) < 0$$

Sign. of $3(x-1)$	Sign. of $(x-3)$	Sign. of $f'(x)$	Graphical intervals	Intervals	Intervals
+ ve $1 < x$	– ve $x < 3$	– ve			$(1, 3)$
– ve $x < 1$	+ ve $3 < x$	– ve			Null set





Hence, f will be decreasing in $(1, 3)$.

S46. We have,

$$f(x) = 5x^3 - 15x^2 - 120x + 3$$

$$\Rightarrow \quad f'(x) = 15x^2 - 30x - 120$$

$$\Rightarrow \quad f'(x) = 15(x^2 - 2x - 8)$$

$$= 15(x-4)(x+2) \quad \Rightarrow \quad x = 4, -2$$

(a) **When $f(x)$ is increasing function:**

$$f'(x) > 0$$

$\Rightarrow \quad f'(x) > 0 \quad \Rightarrow \quad (x-4)(x+2) > 0$

$\Rightarrow \quad x + 2 > 0$ and $x - 4 > 0 \quad \Rightarrow \quad x > -2$ and $x > 4$

or $\quad x + 2 < 0$ and $x - 4 < 0 \quad \Rightarrow \quad x < -2$ and $x < 4$

Sign. of 15 (x – 4)	**Sign. of (x + 2)**	**Sign. of f ′(x)**	**Graphical intervals**	**Intervals**	**Intervals**
– ve $x < 4$	– ve $x < -2$	+ ve	$\{-\infty \leftarrow 4,\ -\infty \leftarrow -2\}$	$-\infty \leftarrow -2$	$(-\infty, -2)$
+ ve $x > 4$	+ ve $x > -2$	+ ve	$4 \rightarrow \infty$, $-2 \rightarrow \infty$	$4 \rightarrow \infty$	$(4, -\infty)$

$\Rightarrow \quad x \in (-\infty, -2) \cup (4, \infty)$

$-\infty \leftarrow -2 \qquad 4 \rightarrow \infty$

The required interval is $(-\infty, -2) \cup (4, \infty)$.

(b) **When $f(x)$ is decreasing function:**

$\Rightarrow \quad f'(x) < 0 \quad \Rightarrow \quad 15(x-4)(x+2) < 0$

Sign. of 15(x-4)	**Sign. of (x +2)**	**Sign. of f ′(x)**	**Graphical intervals**	**Intervals**	**Intervals**
– ve $x < 4$	+ve $-2 < x$	– ve	$-\infty \leftarrow 4$, $-2 \rightarrow \infty$	$-2 \text{ — } 4$	$(-2, 4)$
+ ve $x > 4$	– ve $x < -2$	– ve	$-\infty \leftarrow -2$, $4 \rightarrow \infty$		Null set

Also, f (x) will decreasing when $x \in (-2, 4)$.

$-2 \text{ — } 4$

The required interval is $(-2, 4)$.

S47. Here, $\quad f(x) = 6 + 12x + 3x^2 - 2x^3$

$\Rightarrow \quad f'(x) = 12 + 6x - 6x^2$

$= 6(2 + x - x^2)$

$= 6(2 - x)(1 + x)$

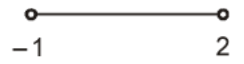
(a) **For $f(x)$ to be increasing, $f'(x) > 0$**

$6(2-x)(1+x) > 0$

Either $\quad 2 - x > 0$ and $1 + x > 0$

or $\quad 2 - x < 0$ and $1 + x < 0$

Sign. of $6(2-x)$	Sign. of $(1+x)$	Sign. of $f'(x)$	Graphical intervals	Intervals	Intervals
+ ve $x<2$	+ ve $x>-1$	+ ve	$-\infty \leftarrow 2$; $-1 \rightarrow \infty$	-1 —— 2	$(-1, 2)$
− ve $2<x$	− ve $x<-1$	+ ve	$2 \rightarrow \infty$; $-\infty \rightarrow -1$		Null set



$\Rightarrow \quad x \in (-1, 2)$

The required interval is $(-1, 2)$.

(b) **For $f(x)$ to be decreasing, $f'(x) < 0$**

$\Rightarrow \quad 6(2-x)(1+x) < 0$

Either $\quad 2-x>0 \quad$ and $\quad 1+x<0$

or $\quad 2-x<0 \quad$ and $\quad 1+x>0$

Sign. of $6(2-x)$	Sign. of $(1+x)$	Sign. of $f'(x)$	Graphical intervals	Intervals	Intervals
+ ve $x<2$	− ve $x<-1$	− ve	$-\infty \leftarrow 2$; $-\infty \leftarrow -1$	$-\infty \leftarrow -1$	$(-\infty, -1)$
− ve $x>2$	+ ve $x>-1$	− ve	$2 \rightarrow \infty$; $-1 \rightarrow \infty$	$2 \rightarrow \infty$	$(2, \infty)$

$= \quad x < -1 \quad$ or $\quad x > 2$

$= \quad x \in (-\infty, -1) \cup (2, \infty)$

The required interval is $(-\infty, -1) \cup (2, \infty)$.

S48. Here, $\quad f(x) = 2x^3 - 3x^2 - 36x + 7$

$\therefore \quad f'(x) = 6x^2 - 6x - 36$

$= 6(x^2 - x - 6)$

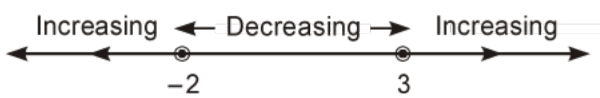
$= 6(x-3)(x+2)$

$\Rightarrow \quad f'(x) = 0$ at $x = 3$ and $x = -2$

$f'(x) = 6(x-3)(x+2)$

If $x < -2$ then, $\quad f'(x) = (+)(-)(-) = +\text{ve}$

So, $f(x)$ in increasing function for $x < -2$.



If $x > 3$ then, $\quad f'(x) = 6(x-3)(x+2)$

$= (+)(+)(+) = +\text{ ve}$

So, $f(x)$ in increasing function for $x > 3$.

If $-2 < x < 3$ then, $f'(x) = 6(x - 3)(x + 2)$

$= (+)(-)(+) = -\text{ve}$

So, $f(x)$ in decreasing function for $-2 < x < 3$.

Hence, $f(x)$ is increasing in the interval $(-\infty, -2) \cup (3, \infty)$ and decreasing in the interval $(-2, 3)$.

S49. We have,

$$f(x) = x^3 - 6x^2 - 36x + 2 \quad \ldots \text{(i)}$$

Differentiating Eq. (i) w.r.t. "x", we get

$$f'(x) = 3x^2 - 12x - 36$$

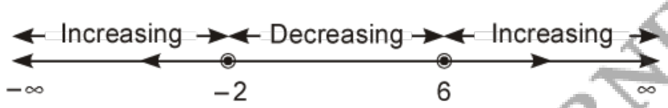
$$f'(x) = 3(x^2 - 4x - 12)$$

$$\Rightarrow \quad f'(x) = 3(x - 6)(x + 2)$$

Putting $f'(x) = 0$ we get $3(x - 6)(x + 2) = 0$

$\Rightarrow$ either $x - 6 = 0$ or $x + 2 = 0$ $\Rightarrow$ $x = 6$ or $x = -2$

So, Intervals are $(-\infty, -2)$, $(-2, 6)$, $(6, \infty)$.



(i) When $x \in (-\infty, -2)$, then

$$f'(x) = 3(x - 6)(x + 2)$$

$= (+\text{ve})(-\text{ve})(-\text{ve}) = +\text{ve} \Rightarrow f'(x) > 0.$

Thus $f(x)$ is increasing in the interval $(-\infty, -2)$.

(ii) When $x \in (-2, 6)$, then

$$f'(x) = 3(x - 6)(x + 2)$$

$= (+\text{ve})(-\text{ve})(+\text{ve}) = -\text{ve} \Rightarrow f'(x) < 0.$

Thus $f(x)$ is decreasing in the interval $(-2, 6)$.

(iii) When $x \in (6, \infty)$, then

$$f'(x) = 3(x - 6)(x + 2)$$

$= (+\text{ve})(+\text{ve})(+\text{ve}) = (+\text{ve}) \Rightarrow f'(x) > 0.$

Thus $f(x)$ is increasing in the interval $(6, \infty)$.

Hence, $f(x)$ is increasing in the interval $(-\infty, -2) \cup (6, \infty)$ and decreasing in the interval $(-2, 6)$.

S50. The given function is

$$f(x) = \sin x - \cos x, \quad 0 \le x \le 2\pi$$

Differentiating w.r.t. x, we get

$$f'(x) = \cos x + \sin x$$

Putting $\quad f'(x) = 0$

$\therefore \quad \cos x + \sin x = 0$

$\Rightarrow \quad \sin x = -\cos x$

Divide both sides by $\cos x$, we get

$$\tan x = -1$$

For $\quad x \in [0, 2\pi]$

$$\tan x = \tan\frac{3\pi}{4} \quad \Rightarrow \quad x = \frac{3\pi}{4}$$

$$\tan x = \tan\frac{7\pi}{4} \quad \Rightarrow \quad x = \frac{7\pi}{4}$$

$\therefore \quad x = \frac{3\pi}{4}, \frac{7\pi}{4}$

[$\because$ $\tan\theta$ is -1 in 2nd quadrant and 4th quadrant as in 2nd quadrant $\tan(\pi - \theta) = -\tan\theta$ and in 4th quadrant $\tan(2\pi - \theta) = -\tan\theta$]

Now, we find the intervals in which $f(x)$ is strictly increasing or strictly decreasing.

Interval	$f'(x) = \cos x + \sin x$	Sign of $f'(x)$
$x < \frac{3\pi}{4}$	$f'\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} + \sin\frac{\pi}{2} = 0 + 1 = 1$	+ve
$\frac{3\pi}{4} < x < \frac{7\pi}{4}$	$f'\left(\frac{5\pi}{6}\right) = \cos\frac{5\pi}{6} + \sin\frac{5\pi}{6}$ $= \cos\left(\pi - \frac{\pi}{6}\right) + \sin\left(\pi - \frac{\pi}{6}\right)$ $= -\cos\frac{\pi}{6} + \sin\frac{\pi}{6}$ $= -\frac{\sqrt{3}}{2} + \frac{1}{2} = \frac{-\sqrt{3}+1}{2}$	-ve
$\frac{7\pi}{4} < x < 2\pi$	$f'\left(11\frac{\pi}{6}\right) = \cos\left(11\frac{\pi}{6}\right) + \sin\left(11\frac{\pi}{6}\right)$ $= \cos\left(2\pi - \frac{\pi}{6}\right) + \sin\left(2\pi - \frac{\pi}{6}\right)$ $= \cos\frac{\pi}{6} - \sin\frac{\pi}{6}$ $= \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3}-1}{2}$	+ ve

We know that a function $f(x)$ is said to be strictly increasing in an interval when $f'(x) > 0$ and it is said to be strictly decreasing when $f'(x) < 0$. So, the given function $f(x)$ is strictly increasing in intervals $\left(0, \frac{3\pi}{4}\right)$ and $\left(\frac{7\pi}{4}, 2\pi\right)$ and it is strictly decreasing in the interval $\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$.

S51. We have,

$$f(x) = 2x^3 - 8x^2 + 10x + 5 \quad \text{... (i)}$$

Differentiating Eq. (i) w.r.t. "x", we get

$$f'(x) = 6x^2 - 16x + 10$$

$$= 2(3x^2 - 8x + 5)$$

$\Rightarrow$ $$f'(x) = 2(3x-5)(x-1) \quad \text{... (ii)}$$

Putting $f'(x) = 0$, we get $x = 5/3$ or $x = 1$

So intervals are $(-\infty, 1), \left(1, \frac{5}{3}\right), \left(\frac{5}{3}, \infty\right)$.

(i) **For $x \in (-\infty, 1)$**

$$f'(x) = 2(3x-5)(x-1)$$

$$= (+\,\text{ve})(-\,\text{ve})(-\,\text{ve}) = +\,\text{ve}$$

$\Rightarrow \quad f'(x) > 0 \quad \Rightarrow \quad f(x)$ is increasing on $x \in (-\infty, 1)$.

(ii) **For $x \in \left(1, \frac{5}{3}\right)$**

$$f'(x) = 2(3x-5)(x-1)$$
$$= (+\,\text{ve})(-\,\text{ve})(+\,\text{ve}) = -\,\text{ve}$$

$\Rightarrow \quad f'(x) < 0 \quad \Rightarrow \quad f(x)$ is decreasing function on $\left(1, \frac{5}{3}\right)$.

(iii) **For $x \in \left(\frac{5}{3}, \infty\right)$**

$$f'(x) = 2(3x-5)(x-1)$$
$$= (+\,\text{ve})(-\,\text{ve})(-\,\text{ve}) = +\,\text{ve}$$

$\Rightarrow \quad f'(x) > 0 \quad \Rightarrow \quad f(x)$ is increasing in $\left(\frac{5}{3}, \infty\right)$.

Hence

(a) $f(x)$ is increasing in $(-\infty, 1) \cup \left(\frac{5}{3}, \infty\right)$

(b) and decreasing in $\left(1, \frac{5}{3}\right)$.

S52. We have,

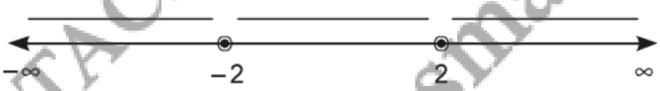
$$f(x) = 2x^3 - 24x + 5 \qquad \dots \text{(i)}$$

Differentiating Eq. (i) w.r.t. "x", we get

$$f'(x) = 6x^2 - 24 = 6(x^2 - 4) = 6(x-2)(x+2)$$

Critical points are 2 and -2.

There are three intervals $(-\infty, -2)$, $(-2, 2)$ and $(2, \infty)$.



(a) When $x \in (-\infty, -2)$, then

$$f'(x) = 6(x-2)(x+2) = (+)(-)(-) = (+)$$

Since $f'(x)$ is + ve, so $f(x)$ is increasing in $(-\infty, -2)$.

(b) When $x \in (-2, 2)$, then

$$f'(x) = 6(x-2)(x+2) = (+)(-)(+) = -\,\text{ve}$$

Since $f'(x)$ is $-$ ve, so $f(x)$ is decreasing in $(-2, 2)$.

(c) When $x \in (2, \infty)$, then

$$f'(x) = 6(x-2)(x+2) = (+)(+)(+) = +ve$$

Since, $f'(x)$ is +ve, so $f(x)$ is increasing in $(2, \infty)$.

Hence, $f(x)$ is increasing in the interval $(-\infty, -2) \cup (2, \infty)$ and decreasing in the interval $(-2, 2)$.

S53. We have,

$$f(x) = 2x^3 - 6x^2 - 48x + 17$$
$$f'(x) = 6x^2 - 12x - 48$$
$$f'(x) = 6(x^2 - 2x - 8)$$
$$= 6(x-4)(x+2)$$

If $f'(x) = 0 \Rightarrow 6(x-4)(x+2) = 0 \Rightarrow x = 4, -2$

The three intervals are $(-\infty, -2)$, $(-2, 4)$, $(4, \infty)$

(i) When $x \in (-\infty, -2)$

$$f'(x) = 6(x-4)(x+2) = (+)(-)(-) = +ve$$

Since, $f'(x)$ is +ve, so $f'(x)$ is increasing in $(-\infty, -2)$.

(ii) When $x \in (-2, 4)$, then

$$f'(x) = 6(x-4)(x+2) = (+)(-)(+) = -ve$$

Since $f'(x)$ is $-$ve, so $f(x)$ is decreasing in $(-2, 4)$.

(iii) When $x \in (4, \infty)$, then

$$f'(x) = 6(x-4)(x+2) = (+)(+)(+) = +ve$$

Since, $f'(x)$ is positive, so $f(x)$ is increasing in $(4, \infty)$

Hence, $f(x)$ is increasing in the interval $(-\infty, -2) \cup (4, \infty)$ and decreasing in the interval $(-2, 4)$.

S54 We have,

$$f(x) = x^{100} + \sin x - 1$$

$\Rightarrow$ $$f'(x) = 100x^{99} + \cos x$$

(i) $$x \in (0, \pi/2)$$

$\Rightarrow$ $x^{99} > 0$ and $\cos x > 0$

$\Rightarrow$ $100x^{99} + \cos x > 0$

$\Rightarrow$ $f'(x) > 0$

Thus, $f(x)$ is strictly increasing on $(0, \pi/2)$.

(ii) $x \in (\pi/2, \pi)$

$\Rightarrow \quad x^{99} > 1 \qquad \left[\because \pi/2 < x < \pi \Rightarrow \frac{22}{14} < x < \frac{22}{7}\right]$

$\Rightarrow \quad 100x^{99} + \cos x > 100 - 1 = 99 > 0$

$\Rightarrow \quad 100x^{99} + \cos x > 0$

$\Rightarrow \quad f'(x) > 0$

Thus, $f(x)$ is strictly increasing on $(\pi/2, \pi)$.

(iii) $x \in (0, 1)$

$\Rightarrow \quad x^{99} > 0$

$\Rightarrow \quad 100x^{99} > 0$

Also, $x \in (0, 1)$

$\Rightarrow$ x lies between 0 and 1 radian

$\Rightarrow$ x lies between 0° and 57° $\qquad [\because \; 1^{rad} = 57°]$

$\Rightarrow$ x lies in first quadrant

$\Rightarrow \quad \cos x > 0$

$\Rightarrow \quad 100x^{99} + \cos x > 0$

$\Rightarrow \quad f'(x) > 0$

Thus, $f(x)$ is strictly increasing on (0, 1).

(iv) As $f'(x) > 0$ for $0 < x < 1$.

But, $f'(x)$ can be positive as well as negative when $-1 < x < 0$

So, $f'(x)$ can be positive as well as negative for $x \in (-1, 1)$.

Hence, $f(x)$ is neither increasing nor decreasing on $(-1, 1)$.

S55. It is given that $f(x)$ is increasing of R. Therefore,

$f'(x) > 0$ for all $x \in R$

$\Rightarrow \quad 3kx^2 - 18kx + 9 > 0$ for all $x \in R$

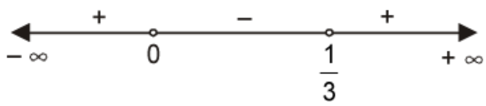
$\Rightarrow \quad kx^2 - 6kx + 3 > 0$ for all $x \in R$

$\Rightarrow \quad k > 0$ and $36k^2 - 12k < 0$ [$\because$ $ax^2 + bx + c > 0$ for all $x \in R$ $a > 0$ & Disc < 0]

$\Rightarrow \quad k > 0$ and $12k(3k - 1) < 0$

$\Rightarrow \quad k > 0$ and $k(3k - 1) < 0$

$\Rightarrow \quad 3k - 1 < 0 \qquad [\because \; k > 0]$



$\Rightarrow \qquad k < \dfrac{1}{3}$

$\Rightarrow \qquad k \in (0, 1/3)$

Hence, $f(x)$ is strictly increasing on R, if $k \in (0, 1/3)$.

S56. We have,

$$f(x) = (a + 2)\, x^3 - 3ax^2 + 9ax - 1$$

$$\Rightarrow \qquad f'(x) = 3\,(a + 2)x^2 - 6ax + 9a$$

Since $f(x)$ is decreasing for all real values of x. Therefore,

$$f'(x) < 0 \text{ for all } x \in R$$

$$\Rightarrow \qquad 3\,(a + 2)x^2 - 6ax + 9a < 0 \text{ for all } x \in R$$

$$\Rightarrow \qquad (a + 2)\,x^2 - 2ax + 3a < 0 \text{ for all } x \in R$$

$$\Rightarrow \qquad a + 2 < 0 \text{ and } 4a^2 - 4 \times (a + 2) \times 3a < 0 \qquad \left[\begin{array}{l} \because ax^2 + bx + c < 0 \text{ for all } x \in R \\ \Rightarrow a < 0 \textit{ and Disc} < 0 \end{array}\right]$$

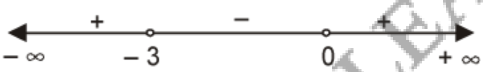
$$\Rightarrow \qquad a < -2 \text{ and } a^2 - 3a^2 - 6a < 0$$

$$\Rightarrow \qquad a < -2 \text{ and } -2a^2 - 6a < 0$$

$$\Rightarrow \qquad a < -2 \text{ and } -2a\,(a + 3) < 0$$

Now, $\qquad -2a\,(a + 3) < 0$

$$\Rightarrow \qquad a\,(a + 3) > 0$$



$$\Rightarrow \qquad a < -3 \text{ or } a > 0$$

$$\Rightarrow \qquad a \in (-\infty, -3) \cup (0, \infty)$$

Hence, $f(x)$ is strictly decreases for all $x \in R$, if $a \in (-\infty, -3) \cup (0, \infty)$.

Date: 18/10/2021

Q1. Find the slope of tangent to the curve $f(x) = 3x^2 + 4x$ at point, where x-coordinate is – 2.

Q2. Find the slope of tangent to the curve $y = 3x^2 - 4x$ at point whose x-coordinate is 2.

Q3. Find the slope of the normal to the curve $x = 1 - a \sin \theta$, $y = b \cos^2 \theta$ at $\theta = \frac{\pi}{2}$.

Q4. Find the slope of the tangent of the curve $y = 3x^4 - 4x$ at $x = 1$.

Q5. Find the slope of tangent to the curve $y = 3x^2 - 6$ at point on it whose x-coordinate is 2.

Q6. Prove that the tangents to the curve $y = x^2 - 5x + 6$ at the points (2, 0) and (3, 0) are at right angles.

Q7. Show that the tangents to the curve $y = 2x^3 - 3$ at the points where $x = 2$ and $x = -2$ are parallel.

Q8. Find the slope of the tangent and the normal to the curve $x^2 + 3y + y^2 = 5$ at (1, 1).

Q9. The slope of the curve $2y^2 = ax^2 + b$ at (1, – 1) is – 1. Find a, b.

Q10. Find the slope of the normal to the curve $x = a \cos^3 \theta$, $y = a \sin^3 \theta$ at $\theta = \frac{\pi}{4}$.

Q11. Find the equation of the tangent to the curve $y = -5x^2 + 6x + 7$ at the point $\left(\frac{1}{2}, \frac{35}{4}\right)$.

Q12. Find the points on curve $y = x^3 - 11x + 5$ at which equation of tangent is $y = x - 11$.

Q13. Find the points on the curve $y = (x - 3)^2$, where the tangent is parallel to the line joining (4, 1) and (3, 0).

Q14. Find the points on the curve $x^2 + y^2 - 2x - 3 = 0$ at which tangent is parallel to X-axis.

Q15. Find the points on the curve $y = 2x^2 - 6x - 4$ at which the tangent is parallel to the x-axis.

Q16. Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to y-coordinate of the point.

Q17. Find the equation of the normal to the curve $y = 2x^2 + 3 \sin x$ at $x = 0$.

Q18. Find the equation of tangent to the curve $y = \frac{x-7}{x^2 - 5x + 6}$ at the point, where it cuts the X-axis.

Q19. Find the equation of the tangent to the curve $y = (x^3 - 1)(x - 2)$ at the points where the curve cuts the x-axis.

Q20. Find the equation of tangent to the curve $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at point $x = 1$, $y = 0$.

Q21. Find the equation of the tangent and the normal to the curve $y = x^4 - bx^3 + 13x^2 - 10x + 5$ at the point (0, 5).

Q22. Find the equation of the tangent and the normal to the curve $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at the point ($x = 1$).

Q23. Find the equation of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

Q24. Find the equation of the tangent line to the curve $y = x^2 - 2x + 7$ which is parallel to the line $2x - y + 9 = 0$.

Q25. Find the equation of the normal to the curve $ay^2 = x^3$ at the point (am^2, am^3).

Q26. Find the equation of tangent to the curve $y = 2x^2 + 7$, which is parallel to the line $4x - y + 3 = 0$.

Q27. Prove that $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ touches the straight line $\frac{x}{a} + \frac{y}{b} = 2$ for all $n \in N$, at the point (a, b).

Q28. Find the equation of the tangent line to the curve $y = x^2 - 2x + 7$ which is perpendicular to the line $5y - 15x = 13$.

Q29. Show that the line $\frac{x}{a} + \frac{y}{b} = 1$ touches the curve $y = be^{-\frac{x}{a}}$ at the point where it crosses the y-axis.

Q30. Find the equations of all lines having slope 2 and that are tangent to the curve $y = \frac{1}{x-3}, x \neq 3$.

Q31. At what points will the tangent of the curve $y = 2x^3 - 15x^2 + 36x - 21$ be parallel to X-axis? Also, find the equations of tangents to the curve.

Q32. Find the points on the curve $\frac{x^2}{4} + \frac{y^2}{25} = 1$ at which the tangents are parallel to the (i) x-axis, (ii) y-axis.

Q33. Find the points on the curve $y = x^3 - 2x^2 - x$ at which the tangent lines are parallel to the line $y = 3x - 2$.

Q34. Find the point on the curve $4x^2 + 9y^2 = 1$, where the tangents are perpendicular to the line $2y + x = 0$.

Q35. Find the points on the curve $9y^2 = x^3$ where normal to the curve makes equal intercepts with the axes.

Q36. Find the equation of tangent to the curve $x = a \sin^3 t$ and $y = a \cos^3 t$ at the point, where $t = \frac{\pi}{4}$.

Q37. Find the equation of tangent to the curve $4x^2 + 9y^2 = 36$ at the point $(3 \cos \theta, 2 \sin \theta)$.

Q38. Find the equation of tangent to curve $x = \sin 3t$, $y = \cos 2t$ at $t = \frac{\pi}{4}$.

Q39. Find the equation of tangent and normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1).

Q40. Find the equation of the tangent and the normal to the curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_0, y_0).

Q41. Find the equation of the normal to the curve $y = (1 + x)^y + \sin^{-1}(\sin^2 x)$ at $x = 0$.

Q42. Find the equations of tangents to the curve $y = (x^2 - 1)(x - 2)$ at the points, where the curve cuts the X-axis.

Q43. Find the equation of the tangent and the normal to the curve $16x^2 + 9y^2 = 144$ at (x_1, y_1) where $x_1 = 2$ and $y_1 > 0$.

Q44. Find the equation of tangent to the curve $x^2 + 3y = 3$, which is parallel to line $y - 4x + 5 = 0$.

Q45. Find the equation of the tangent line to the curve $y = \sqrt{5x - 3} - 2$ which is parallel to the line $4x - 2y + 3 = 0$.

Q46. Find the equation of tangent to the curve $y = \sqrt{3x - 2}$ which is parallel to the line $4x - 2y + 5 = 0$.

Q47. Find the equations of all lines of slope zero and that are tangent to the curve $y = \dfrac{1}{x^2 - 2x + 3}$.

Q48. Find the equation(s) of normal(s) to the curve $3x^2 - y^2 = 8$ which is (are) parallel to the line $x + 3y = 4$.

Q49. Find the equations of the normal to the curve $y = x^3 + 2x + 6$ which are parallel to line $x + 14y + 4 = 0$.

Q50. Find all the tangents to the curve $y = \cos(x + y)$, $-2\pi \le x \le 2\pi$ that are parallel to the line $x + 2y = 0$.

Q51. Find the angle between the parabolas $y^2 = 4ax$ and $x^2 = 4by$ at their point of intersection other than the origin.

Q52. Find the angle of intersection of the $y^2 = 4x$ and $x^2 = 4y$.

Q53. Find the angle of intersection of the curves $xy = 6$ and $x^2y = 12$.

Q54. Find the coordinates of the points on the curve $y = x^2 + 3x + 4$, at which the tangent passes through the origin.

Q55. For the curve $y = 4x^3 - 2x^5$ find all points at which the tangent passes through the origin.

Q56. Find the equations of all lines of slope –1 that are tangents to the curve $y = \dfrac{1}{x - 1}$, $x \neq 1$.

Q57. Show that the curves $xy = a^2$ and $x^2 + y^2 = 2a^2$ touch each other.

Q58. Show that the curves

$$ax^2 + by^2 = 1$$

and

$$a'x^2 + b'y^2 = 1$$

should intersect orthogonally if $\dfrac{1}{a} - \dfrac{1}{b} = \dfrac{1}{a'} - \dfrac{1}{b'}$.

Q59. Show that the angle between the tangent at any point P and the line joining P to the origin O is the same at all points on the curve

$$\log(x^2 + y^2) = k \tan^{-1}\left(\frac{y}{x}\right).$$

Q60. Show that the curves $x = y^2$ and $xy = k$ cut at right angles, if $8k^2 = 1$.

Q61. Find the equation of tangent and normal to the curve $x = 1 - \cos\theta$, $y = \theta - \sin\theta$ at $\theta = \frac{\pi}{4}$.

Q62. Find the equation of the normal to the curve $x^2 = 4y$ which passes through the point (1, 2).

Q63. Show that the normal at any point θ to the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ is at a constant distance from the origin.

Q64. The curve $y = ax^3 + bx^2 + cx + 5$ touches the x-axis at $P(-2, 0)$ and cuts the y-axis at the point Q where it's gradient is 3. Find the equation of the curve completely.

Q65. Determine the quadratic curve $y = f(x)$ if it touches the line $y = x$ at the point $x = 1$ and passes through the point (–1, 0).

Q66. Find the equations of tangents to the curve $3x^2 - y^2 = 8$, which passes through the point $\left(\frac{4}{3}, 0\right)$

Q67. Find the equation of the normal at a point on the curve $x^2 = 4y$ which passes through the point (1, 2). Also find the equation of the corresponding tangent.

Date: 18/10/2021

S1. Given that $y = 3x^2 + 4x$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 6x + 4$$

Slope of tangent = $\frac{dy}{dx}$ at $x = -2 = 6(-2) + 4 = -12 + 4 = -8$.

S2. Given that $y = 3x^2 - 4x$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 6x - 4$$

Slope of tangent = $\left[\frac{dy}{dx}\right]$ at $x = 2 = 6(2) - 4 = 12 - 4 = 8$.

S3. We have, $x = 1 - a\sin\theta$ and $y = b\cos^2\theta$

$$\Rightarrow \quad \frac{dx}{d\theta} = -a\cos\theta \text{ and } \frac{dy}{d\theta} = -2b\cos\theta\sin\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-2b\sin\theta\cos\theta}{-a\cos\theta} = \frac{2b}{a}\sin\theta$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{2}} = \frac{2b}{a}\sin\frac{\pi}{2} = \frac{2b}{a}$$

Hence, $\left(\text{Slope of the normal at } \theta = \frac{\pi}{2}\right) = -\frac{1}{\left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{2}}} = -\frac{a}{2b}$

S4. Given that $y = 3x^4 - 4x$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 12x^3 - 4$$

Slope of tangent = $\left[\frac{dy}{dx}\right]$ at $x = 1 = 12(1)^3 - 4 = 12 - 4 = 8$.

S5. Given that $y = 3x^2 - 6$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 6x$$

Slope of tangent = $\frac{dy}{dx}$ at $x = 2 = 6(2) = 12$.

S6. The equation of the curve is $y = x^2 - 5x + 6$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 2x - 5$$

Now, m_1 = Slope of the tangent at (2, 0) = $\left(\frac{dy}{dx}\right)_{(2,\,0)} = 2 \times 2 - 5 = -1$

and, m_2 = Slope of the tangent at (3, 0) = $\left(\frac{dy}{dx}\right)_{(3,\,0)} = 2 \times 3 - 5 = 1$

Clearly, $m_1 m_2 = -1 \times 1 = -1$

Thus, the tangents to the given curve at (2, 0) and (3, 0) are at right angles.

S7. The equation of the curve is $y = 2x^3 - 3$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 6x^2$$

Now, m_1 = Slope of the tangent at (x = 2) = $\left(\frac{dy}{dx}\right)_{x = 2} = 6 \times (2)^2 = 24$

and, m_2 = Slope of the tangent at (x = –2) = $\left(\frac{dy}{dx}\right)_{x = -2} = 6\,(-2)^2 = 24$

Clearly, $m_1 = m_2$

Thus, the tangents to the given curve at the points where $x = 2$ and $x = -2$ are parallel.

S8. The equation of the curve is $x^2 + 3y + y^2 = 5$

Differentiating w.r.t. x, we get

$$2x + 3\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \qquad \frac{dy}{dx} = -\frac{2x}{2y+3}$$

$$\Rightarrow \qquad \left(\frac{dy}{dx}\right)_{(1,1)} = -\frac{2}{2+3} = -\frac{2}{5}$$

$\therefore$ Slope of the tangent at (1, 1) = $\left(\frac{dy}{dx}\right)_{(1,1)} = -\frac{2}{5}$

and Slope of the normal at (1, 1) = $\frac{-1}{\left(\frac{dy}{dx}\right)_{(1,1)}} = \frac{-1}{\frac{-2}{5}} = \frac{5}{2}$

S9. The equation of the curve is $2y^2 = ax^2 + b$ ... (i)

Differentiating w.r.t. x, we get

$$4y\frac{dy}{dx} = 2ax$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{ax}{2y}$$

$$\Rightarrow \qquad \left(\frac{dy}{dx}\right)_{(1,-1)} = \frac{-a}{2}$$

It is given that the slope of the tangent at (1, – 1) is – 1. Therefore,

$$-\frac{a}{2} = -1 \quad \Rightarrow \quad a = 2$$

Since the point (1, – 1) lies on (i). Therefore,

$$2(-1)^2 = a(1)^2 + b \quad \Rightarrow \quad a + b = 2$$

Putting $a = 2$ in $a + b = 2$, we obtain $b = 0$

Hence, $a = 2$ and $b = 0$.

S10. We have, $x = a\cos^3\theta,\ y = a\sin^3\theta$

$$\Rightarrow \qquad \frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$$

and $\qquad \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$

Now, $\qquad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta$$

$\therefore$ Slope of the normal at any point on the curve $= -\dfrac{1}{\dfrac{dy}{dx}} = \dfrac{-1}{-\tan\theta} = \cot\theta$

Hence, $\left(\text{Slope of the normal at } \theta = \dfrac{\pi}{4}\right) = \cot\dfrac{\pi}{4} = 1.$

S11. The equation of the given curve is

$$y = -5x^2 + 6x + 7$$

$$\Rightarrow \quad \frac{dy}{dx} = -10x + 6$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{\left(\frac{1}{2}, \frac{35}{4}\right)} = -\frac{10}{2} + 6 = 1$$

The required equation of the tangent at $\left(\dfrac{1}{2}, \dfrac{35}{4}\right)$ is

$$y - \frac{35}{4} = \left(\frac{dy}{dx}\right)_{\left(\frac{1}{2}, \frac{35}{4}\right)}\left(x - \frac{1}{2}\right)$$

$$\Rightarrow \quad y - \frac{35}{4} = 1\left(x - \frac{1}{2}\right)$$

$$\Rightarrow \quad y = x + \frac{33}{4}.$$

S12. Given equation of curve is $\quad y = x^3 - 11x + 5 \quad$ ... (i)

Slope of the tangent to the given curve is $\dfrac{dy}{dx}$.

$$\therefore \quad \frac{dy}{dx} = 3x^2 - 11$$

Also, slope of the tangent $y = x - 11$ is 1.

$$\therefore \quad \frac{d}{dx}(x^3 - 11x + 5) = 1$$

$$\therefore \quad 3x^2 - 11 = 1$$

$$\Rightarrow \quad 3x^2 = 12$$

$$\Rightarrow \quad x^2 = 4$$

$$\therefore \quad x = \pm 2$$

$\therefore$ From Eq. (i),

At $x = 2, y = (2)^3 - 11(2) + 5 = 8 - 22 + 5 = -9$

At $x = -2, y = (-2)^3 - 11(-2) + 5$

$= -8 + 22 + 5 = 19$

Hence, the required points on the curve are (2, –9) and (–2, 19).

S13. Let the required point be $P(x_1, y_1)$. The equation of the given curve is

$$y = (x-3)^2 \quad \text{... (i)}$$

$$\Rightarrow \quad \frac{dy}{dx} = 2(x-3)$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 2(x_1 - 3)$$

Since the tangent at P is parallel to the line joining (4, 1) and (3, 0). Therefore,

Slope of the tangent at P = Slope of the line joining (4, 1) and (3, 0)

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{0-1}{3-4}$$

$$\Rightarrow \quad 2(x_1 - 3) = 1$$

$$\Rightarrow \quad x_1 = \frac{7}{2}$$

Since the point $P(x_1, y_1)$ lies on (i). Therefore,

$$y_1 = (x_1 - 3)^2$$

$$\therefore \quad x_1 = \frac{7}{2}$$

$$\Rightarrow \quad y_1 = \left(\frac{7}{2} - 3\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

Thus, the required point is $\left(\frac{7}{2}, \frac{1}{4}\right)$.

S14. Given equation of curve is

$$x^2 + y^2 - 2x - 3 = 0 \quad \text{... (i)}$$

We know that when a tangent to the curve is parallel to X-axis, then $\frac{dy}{dx} = 0$.

So, we find $\frac{dy}{dx}$ from Eq. (i) and equate it to zero. Now, differentiating both sides of Eq. (i) w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} - 2 = 0$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = 2 - 2x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2-2x}{2y}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1-x}{y}$$

Putting $\frac{dy}{dx} = 0$, we get

$$1 - x = 0$$

$$\Rightarrow \quad x = 1$$

Now, putting $x = 1$ in Eq. (i), we get

$$1 + y^2 - 2 - 3 = 0$$

$$\Rightarrow \quad y^2 - 4 = 0$$

$$\Rightarrow \quad y^2 = 4$$

$$\Rightarrow \quad y = \pm 2$$

Hence, required points are (1, 2) and (1, –2).

S15. Let the required point be $P(x_1, y_1)$. The given curve is

$$y = 2x^2 - 6x - 4 \quad \ldots \text{(i)}$$

$$\Rightarrow \quad \frac{dy}{dx} = 4x - 6$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 4x_1 - 6$$

Since the tangent at (x_1, y_1) is parallel to x-axis. Therefore,

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$$

$$\Rightarrow \quad 4x_1 - 6 = 0$$

$$\Rightarrow \quad x_1 = \frac{3}{2}$$

Since, (x_1, y_1) lies on curve (i). Therefore, $y_1 = 2x_1^2 - 6x_1 - 4$

Now, $x_1 = \dfrac{3}{2}$

$\Rightarrow$ $y_1 = 2\left(\dfrac{3}{2}\right)^2 - 6\left(\dfrac{3}{2}\right) - 4$

$= \dfrac{9}{2} - 9 - 4 = -\dfrac{17}{2}$

So, the required point is $\left(\dfrac{3}{2}, \dfrac{-17}{2}\right)$.

S16. Given that, $y = x^3$ ... (i)

We know that, slope of tangent = $\dfrac{dy}{dx}$

$\therefore$ Slope of tangent = $\dfrac{dy}{dx} = 3x^2$

Now, given that slope of tangent = y-coordinate

$$\frac{dy}{dx} = y$$

$\Rightarrow$ $3x^2 = y$ $\left[\because \dfrac{dy}{dx} = 3x^2\right]$

$\Rightarrow$ $3x^2 = x^3$ [$\because$ $y = x^3$, Given]

$\Rightarrow$ $3x^2 - x^3 = 0$

$\Rightarrow$ $x^2(3 - x) = 0$

$\therefore$ $x = 0, 3$

Now, putting $x = 0, 3$ in Eq. (i), we get

$y = (0)^3 = 0$ [At $x = 0$]

and $y = (3)^3 = 27$ [At $x = 3$]

Hence, the required points are (0, 0) and (3, 27).

S17. The equation of the given curve is $y = 2x^2 + 3 \sin x$ ... (i)

Putting, $x = 0$ in (i), we get $y = 0$

So, the point of contact is (0, 0).

Now, $y = 2x^2 + 3 \sin x$

$\Rightarrow$ $\dfrac{dy}{dx} = 4x + 3 \cos x$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(0,\,0)} = 4 \times 0 + 3\cos 0° = 3$$

So, the equation of the normal at (0, 0) is

$$y - 0 = -\frac{1}{3}(x - 0) \qquad \left[\text{Using : } y - y_1 = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1,\,y_1)}}(x - x_1)\right]$$

$$\Rightarrow \quad x + 3y = 0$$

S18. The given equation of curve is

$$y = \frac{x-7}{x^2-5x+6} \qquad \text{... (i)}$$

Differentiating both sides, w.r.t. x using quotient rule, we get

$$\frac{dy}{dx} = \frac{(x^2-5x+6)\cdot 1-(x-7)(2x-5)}{(x^2-5x+6)^2} \qquad \left[\because \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\left[(x^2-5x+6)-y(x^2-5x+6)(2x-5)\right]}{(x^2-5x+6)^2} \qquad \text{... (ii)}$$

$$\left[\begin{array}{l}\because \text{given, } y = \dfrac{x-7}{x^2-5x+6} \\ \therefore (x-7) = y(x^2-5x+6)\end{array}\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1-(2x-5)y}{x^2-5x+6} \qquad \text{... (iii)}$$

[Dividing numerator and denominator of Eq. (ii) by $x^2 - 5x + 6$]

As, it is given that given curve cuts X-axis, so its y-coordinate is zero.

$\therefore$ Put $y = 0$ in Eq. (i), we get

$$\frac{x-7}{x^2-5x+6} = 0$$

$$\Rightarrow \quad x = 7$$

$\therefore$ Curve passes through the point (7, 0). Also, from Eq. (iii), we get

$$\text{Slope of tangent} = m = \left[\frac{dy}{dx}\right]_{(7,\,0)}$$

$$= \frac{1-0}{49-35+6} = \frac{1}{20}$$

Hence, the required equation of tangent passing through the point (7, 0) having slope $\frac{1}{20}$ is

$$y - 0 = \frac{1}{20}(x - 7) \quad \text{or} \quad 20y = x - 7$$

or $\quad x - 20y = 7.$

S19. The equation of the curve is $y = (x^3 - 1)(x - 2)$ ... (i)

It cuts x-axis at $y = 0$. So, putting $y = 0$ in (i), we get

$$(x^3 - 1)(x - 2) = 0$$

$$\Rightarrow \quad (x - 1)(x - 2)(x^2 + x + 1) = 0$$

$$\Rightarrow \quad x - 1 = 0, x - 2 = 0 \qquad [\because x^2 + x + 1 \neq 0]$$

$$\Rightarrow \quad x = 1, 2$$

Thus the points of intersection of curve (i) with x-axis are (1, 0) and (2, 0).

Now, $\quad y = (x^3 - 1)(x - 2)$

$$\Rightarrow \quad \frac{dy}{dx} = 3x^2(x - 2) + (x^3 - 1)$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(1, 0)} = 3(1)^2(1 - 2) + 1^3 - 1$$

$$= -3 + 0 = -3$$

and $\quad \left(\frac{dy}{dx}\right)_{(2, 0)} = 3(2)^2(2 - 2) + 2^3 - 1$

$$= 0 + 8 - 1 = 7$$

The equation of the tangents at (1, 0) and (2, 0) are respectively

$$y - 0 = -3(x - 1)$$

and $\quad y - 0 = 7(x - 2)$

$$\Rightarrow \quad y + 3x - 3 = 0$$

and $\quad 7x - y - 14 = 0.$

S20. Given equation of curve is

$$y = x^4 - 6x^3 + 13x^2 - 10x + 5$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$$

We know that slope of a tangent

$$m = \left[\frac{dy}{dx}\right]_{(x_1, y_1)}$$

$\therefore \quad m = \left[\frac{dy}{dx}\right]_{x=1} = 4 - 18 + 26 - 10 = 2$

$\therefore$ Equation of tangent is given by

$$y - y_1 = m(x - x_1)$$

Here, $x_1 = 1$, $y_1 = 0$, $m = 2$

$\therefore \quad y - 0 = 2(x - 1)$

$$y = 2x - 2$$

Hence, $\quad 2x - y = 2.$

S21. The equation of the curve is $y = x^4 - bx^3 + 13x^2 - 10x + 5$.

$\therefore \quad \frac{dy}{dx} = 4x^3 - 3bx^2 + 26x - 10$

$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(0,\,5)} = 4(0)^3 - 3b(0) + 26(0) - 10$

$$= 0 - 0 + 0 - 10 = -10$$

The equation of tangent at (0, 5) is

$$y - 5 = \left(\frac{dy}{dx}\right)_{(0,\,5)} (x - 0)$$

or $\quad y - 5 = -10\,(x - 0) \quad$ or $\quad 10x + y - 5 = 0$

The equation of the normal at (0, 5) is

$$y - 5 = -\frac{1}{\left(\frac{dy}{dx}\right)_{(0,\,5)}}(x - 0)$$

or $\quad y - 5 = \frac{1}{10}(x - 0) \quad$ or $\quad x - 10y + 50 = 0$

S22. The equation of the curve is $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ ...(i)

$\therefore \quad \frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$

$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{x=1} = 4 - 18 + 26 - 10 = 2$

Putting $x = 1$ in (i), we get $y = (1)^4 - 6(1)^3 + 13(1)^2 - 10(1) + 5$

$$= 1 - 6 + 13 - 10 + 5 = 3$$

The equation of the tangent at (1, 3) is

$$y - 3 = \left(\frac{dy}{dx}\right)_{x=1}(x - 1)$$

or $$y - 3 = 2(x - 1) \quad \text{or} \quad 2x - y + 1 = 0$$

The equation of the normal at (1, 3) is

$$y - 3 = -\frac{1}{\left(\frac{dy}{dx}\right)_{x=1}}(x - 1)$$

or $$y - 3 = -\frac{1}{2}(x - 1) \quad \text{or} \quad x + 2y - 7 = 0.$$

S23. The equation of the given curve is $y^2 = 4ax$ ... (i)

Differentiating (i) w.r.t. x, we get

$$2y\frac{dy}{dx} = 4a$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2a}{y}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(at^2,\, 2at)} = \frac{2a}{2at} = \frac{1}{t}$$

So, the equation of the tangent at $(at^2, 2at)$ is

$$y - 2at = \left(\frac{dy}{dx}\right)_{(at^2,\, 2at)}(x - at^2)$$

$$\Rightarrow \quad y - 2at = \frac{1}{t}(x - at^2)$$

$$\Rightarrow \quad ty = x + at^2$$

The equation of the normal at $(at^2, 2at)$ is

$$\Rightarrow \quad y - 2at = -\frac{1}{\left(\frac{dy}{dx}\right)_{(at^2,\, 2at)}}(x - at^2)$$

$$\Rightarrow \quad y - 2at = -\frac{1}{\frac{1}{t}}(x - at^2)$$

$$\Rightarrow \quad y - 2at = -t(x - at^2)$$

$$\Rightarrow \quad y + tx = 2at + at^3.$$

S24. The equation of the curve is $y = x^2 - 2x + 7$

$$\therefore \quad \frac{dy}{dx} = 2x - 2$$

Let $P(x_1, y_1)$ be a point on $y = x^2 - 2x + 7$ such that tangent at P is parallel to the line $2x - y + 9 = 0$

$$\therefore \quad \left(\frac{dy}{dx}\right)_P = 2$$

$$\Rightarrow \quad 2x_1 - 2 = 2$$

$$\Rightarrow \quad x_1 = 2$$

Since $P(x_1, y_1)$ lies on $y = x^2 - 2x + 7$. Therefore

$$y_1 = x_1^2 - 2x_1 + 7$$

$$\Rightarrow \quad y_1 = 4 - 4 + 7 = 7$$

Hence, required point is (2, 7).

The equation of the tangent is

$$y - 7 = \left(\frac{dy}{dx}\right)_P (x - 2)$$

or, $\quad y - 7 = 2(x - 2) \quad$ or $\quad 2x - y + 3 = 0.$

S25. We have, $\quad ay^2 = x^3$

Differentiating with respect to x, we get

$$2ay\frac{dy}{dx} = 3x^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3x^2}{2ay}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(am^2,\, am^3)} = \frac{3a^2m^4}{2a^2m^3} = \frac{3m}{2}$$

The equation of the normal at (am^2, am^3) is

$$y - am^3 = \frac{1}{\left(\frac{dy}{dx}\right)_{(am^2,\, am^3)}} (x - am^2)$$

$$y - am^3 = -\frac{2}{3m}(x - am^2)$$

or $\quad 2x + 3my - am^2(2 + 3m^2) = 0$

S26. Let the point of contact of the required tangent line be (x_1, y_1)

The equation of the given curve is $y = 2x^2 + 7$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 4x$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 4x_1$$

Since the line $4x - y + 3 = 0$ is parallel to the tangent line at (x_1, y_1)

$\therefore$ Slope of the tangent at (x_1, y_1) = (Slope of the line $4x - y + 3 = 0$)

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-4}{-1} \qquad \left[\because \text{Slope} = -\frac{\text{Coeff. of } x}{\text{Coeff. of } y}\right]$$

$$\Rightarrow \quad 4x_1 = 4$$

$$\Rightarrow \quad x_1 = 1$$

Now, (x_1, y_1) lies on $y = 2x^2 + 7$

$$\therefore \quad y_1 = 2x_1^2 + 7$$

$$\Rightarrow \quad y_1 = 2 + 7 = 9 \qquad [\because x_1 = 1]$$

So, the coordinates of the point of contact are (1, 9).

Hence, the required equation of the tangent line is

$$y - 9 = 4(x - 1)$$

$$\Rightarrow \quad 4x - y + 5 = 0$$

S27. We have, $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$

Differentiating both sides w.r.t. to x, we get

$$n\left(\frac{x}{a}\right)^{n-1}\frac{1}{a} + n\left(\frac{y}{b}\right)^{n-1}\frac{1}{b}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-b}{a}\left(\frac{x}{a}\right)^{n-1}\left(\frac{b}{y}\right)^{n-1} \quad \Rightarrow \quad \left(\frac{dy}{dx}\right)_{(a, b)} = \frac{-b}{a}$$

The equation of the tangent at (a, b) is

$$y - b = \frac{-b}{a}(x - a)$$

$$\Rightarrow \quad ay - ab = -bx + ab$$

$\Rightarrow \quad bx + ay = 2ab$

$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} = 2$

Hence, $\frac{x}{a} + \frac{y}{b} = 2$ touches the given curve at (a, b) for all $n \in N$.

S28. The equation of the curve is $y = x^2 - 2x + 7$

$\therefore \quad \frac{dy}{dx} = 2x - 2$

If the tangent at $P(x_1, y_1)$ is perpendicular to the line $5y - 15x = 13$. Then,

$$\left(\frac{dy}{dx}\right)_P \times 3 = -1$$

$\Rightarrow \quad (2x_1 - 2) \times 3 = -1$

$\Rightarrow \quad x_1 = \frac{5}{6}$

Since $P(x_1, y_1)$ lies on $y = x^2 - 2x + 7$

$\therefore \quad y_1 = x_1^2 - 2x_1 + 7$

$\Rightarrow \quad y_1 = \frac{25}{36} - \frac{5}{3} + 7 = \frac{217}{36}$

The equation of the tangent at $P\left(\frac{5}{6}, \frac{217}{36}\right)$ is

$$y - \frac{217}{36} = -\frac{1}{3}\left(x - \frac{5}{6}\right) \qquad \left[\because \left(\frac{dy}{dx}\right)_P = -\frac{1}{3}\right]$$

$$36y - 217 = -12x + 10$$

or, $\quad 12x + 36y - 227 = 0.$

S29. The equation of the given curve is $y = be^{-\frac{x}{a}}$ ... (i)

It crosses y-axis at the point, where $x = 0$

Putting $x = 0$ in (i), we get $y = be^0 = b$

So, the point of contact is $(0, b)$.

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = be^{-\frac{x}{a}} \frac{d}{dx}\left(-\frac{x}{a}\right)$$

$\Rightarrow \qquad \frac{dy}{dx} = \frac{-b}{a} e^{\frac{-x}{a}}$

$\Rightarrow \qquad \left(\frac{dy}{dx}\right)_{(0,\, b)} = \frac{-b}{a} e^{0} = -\frac{b}{a}$

The equation of the tangent at (0, *b*) is

$$y - b = -\frac{b}{a}(x - 0) \qquad \left[\text{Using} : y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} (x - x_1)\right]$$

$\Rightarrow \qquad ay - ab = -bx$

$\Rightarrow \qquad bx + ay = ab$

$\Rightarrow \qquad \frac{x}{a} + \frac{y}{b} = 1$

Hence, $\frac{x}{a} + \frac{y}{b} = 1$ touches the curve $y = be^{-\frac{x}{a}}$ at the point where it crosses the *y* axis.

S30. Let (x_1, y_1) be the point of contact of a line of slope 2 which touches the curve $y = \frac{1}{x-3}, x \neq 3$.

Now, $\qquad y = \frac{1}{x-3}$

$\Rightarrow \qquad \frac{dy}{dx} = -\frac{1}{(x-3)^2}$

$\Rightarrow \qquad \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} = -\frac{1}{(x_1-3)^2}$

But, $\qquad \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} = 2$

$\therefore \qquad -\frac{1}{(x_1-3)^2} = 2$

$\Rightarrow \qquad 2(x_1 - 3)^2 = -1,$

which is not possible at LHS is positive and RHS is negative

Hence, there is no tangent line of slope 2 to the given curve.

S31. We know that when a tangent is parallel to *X*-axis.

Then, $\qquad \frac{dy}{dx} = 0$

$\therefore$ Differentiating the given equation $y = 2x^3 - 15x^2 + 36x - 21$ w.r.t. x, we get

$$\frac{dy}{dx} = 6x^2 - 30x + 36$$

Put $\frac{dy}{dx} = 0$, we get

$$6x^2 - 30x + 36 = 0$$

$\Rightarrow$ $6(x^2 - 5x + 6) = 0$

$\Rightarrow$ $x^2 - 5x + 6 = 0$

$\Rightarrow$ $(x-2)(x-3) = 0$

or $x = 2$ or 3

Now, when $x = 2$, we get

$$y = 2(2)^3 - 15(2)^2 + 36(2) - 21$$

$\Rightarrow$ $y = 16 - 60 + 72 - 21$

$$= 88 - 81 = 7$$

Also, when $x = 3$, we get

$$y = 2(3)^3 - 15(3)^2 + 36(3) - 21$$

$$= 54 - 135 + 108 - 21$$

$$y = 162 - 156 = 6$$

Hence, the tangent passes through the points (2, 7) and (3, 6).

Now, we find the equation of tangents to the given curve.

$\therefore$ Slope of tangent at point (2, 7) is

$$m = \left[\frac{dy}{dx}\right]_{(2,7)} = 6(2)^2 - 30(2) + 36$$

$$= 24 - 60 + 36 = 0$$

and slope of tangent at point (3, 6) is m

$$m = \left(\frac{dy}{dx}\right)_{(3,6)} = 6(3)^2 - 30(3) + 36$$

$$= 54 - 90 + 36 = 0$$

$\therefore$ Equation of tangent at point (2, 7) having slope 0 is

$$y - 7 = 0(x - 2)$$

or $y - 7 = 0$

$$y = 7$$

and equation of tangent at point (3, 6) having slope 0 is

$$y - 6 = 0(x - 3)$$

or $$y - 6 = 0$$

$$y = 6$$

Hence, equation of tangents are $y = 7$ and $y = 6$.

S32. Let $P(x_1, y_1)$ be a point on the curve $\frac{x^2}{4} + \frac{y^2}{25} = 1$. Then,

$$\frac{x_1^2}{4} + \frac{y_1^2}{25} = 1 \quad \text{... (i)}$$

Now, $$\frac{x^2}{4} + \frac{y^2}{25} = 1$$

$$\Rightarrow \quad \frac{x}{2} + \frac{2y}{25}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{25}{4}\frac{x}{y}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_P = -\frac{25}{4}\frac{x_1}{y_1}$$

(i) If tangent at P is parallel to x-axis, then

$$\left(\frac{dy}{dx}\right)_P = 0$$

$$\Rightarrow \quad -\frac{25}{4}\frac{x_1}{y_1} = 0$$

$$\Rightarrow \quad x_1 = 0$$

Putting $x_1 = 0$ in (i), we get

$$\frac{y_1^2}{25} = 1$$

$$\Rightarrow \quad y_1 = \pm 5$$

Hence, required points are (0, 5) and (0, –5).

(ii) If the tangent at P is parallel to y-axis, then

$$\frac{1}{\left(\frac{dy}{dx}\right)_P} = 0$$

$$\Rightarrow \quad -\frac{4}{25}\frac{y_1}{x_1} = 0$$

$\Rightarrow \qquad y_1 = 0$

Putting $y_1 = 0$ in (i), we get

$$\frac{x_1^2}{4} = 1$$

$\Rightarrow \qquad x_1 = \pm 2$

Hence, required points are $(\pm 2, 0)$.

S33. Let $P(x_1, y_1)$ be the required point. The given curve is

$$y = x^3 - 2x^2 - x \qquad \dots \text{(i)}$$

$$\Rightarrow \qquad \frac{dy}{dx} = 3x^2 - 4x - 1$$

$$\Rightarrow \qquad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 3x_1^2 - 4x_1 - 1$$

Since the tangent at (x_1, y_1) is parallel to the line $y = 3x - 2$.

$\therefore$ Slope of the tangent at (x_1, y_1) = Slope of the line $y = 3x - 2$

$$\Rightarrow \qquad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 3$$

$$\Rightarrow \qquad 3x_1^2 - 4x_1 - 1 = 3$$

$$\Rightarrow \qquad 3x_1^2 - 4x_1 - 4 = 0$$

$$\Rightarrow \qquad (x_1 - 2)(3x_1 + 2) = 0$$

$$\Rightarrow \qquad x_1 = 2, -\frac{2}{3}$$

Since, (x_1, y_1) lies on curve Eq. (i). Therefore, $y_1 = x_1^3 - 2x_1^2 - x_1$

Now, $\qquad x_1 = 2$

$\Rightarrow \qquad y_1 = 2^3 - 2(2)^2 - 2$

$\qquad = 8 - 8 - 2 = -2$

$$x_1 = -\frac{2}{3}$$

$$\Rightarrow \qquad y_1 = \left(-\frac{2}{3}\right)^3 - 2\left(-\frac{2}{3}\right)^2 + \frac{2}{3}$$

$$= -\frac{8}{27} - \frac{8}{9} + \frac{2}{3} = \frac{-8-24+18}{27} = \frac{-14}{27}$$

Thus, required points are (2, – 2) and $\left(\frac{-2}{3}, \frac{-14}{27}\right)$.

S34. Let the required point be $P(x_1, y_1)$. The equation of the given curve is

$$4x^2 + 9y^2 = 1 \quad \text{... (i)}$$

$$\Rightarrow \quad 8x + 18y\frac{dy}{dx} = 0 \quad \text{[Differentiating w.r.t. } x\text{]}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{4x}{9y}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-4x_1}{9y_1}$$

Since tangent at (x_1, y_1) is perpendicular to the line $2y + x = 0$. Therefore,

Slope of the tangent at (x_1, y_1) × Slope of the line = – 1

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} \times \left(-\frac{1}{2}\right) = -1$$

$$\Rightarrow \quad \frac{-4x_1}{9y_1} \times \left(-\frac{1}{2}\right) = -1$$

$$\Rightarrow \quad y_1 = \frac{-2x_1}{9} \quad \text{... (ii)}$$

Since $P(x_1, y_1)$ lies on the curve (i). Therefore,

$$4x_1^2 + 9y_1^2 = 1$$

$$\Rightarrow \quad 4x_1^2 + 9\left(\frac{-2x_1}{9}\right)^2 = 1 \quad \text{[Using (ii)]}$$

$$\Rightarrow \quad 4x_1^2 + \frac{4x_1^2}{9} = 1$$

$$\Rightarrow \quad x_1^2 = \frac{9}{40}$$

$$\Rightarrow \quad x_1 = \pm\frac{3}{2\sqrt{10}}$$

Now, $$x_1 = \frac{3}{2\sqrt{10}}$$

$$\Rightarrow \quad y_1 = \frac{-2}{9}\left(\frac{3}{2\sqrt{10}}\right) = -\frac{1}{3\sqrt{10}}$$

and $$x_1 = -\frac{3}{2\sqrt{10}}$$

$$\Rightarrow \quad y_1 = \frac{-2}{9}\left(-\frac{3}{2\sqrt{10}}\right) = \frac{1}{3\sqrt{10}}$$

Hence, the required points are $\left(\frac{3}{2\sqrt{10}}, \frac{-1}{3\sqrt{10}}\right)$ and $\left(\frac{-3}{2\sqrt{10}}, \frac{1}{3\sqrt{10}}\right)$.

S35. Let the required point be (x_1, y_1).

The equation of the curve is $9y^2 = x^3$.

Since (x_1, y_1) lies on the curve. Therefore,

$$9y_1^2 = x_1^3 \quad \text{... (i)}$$

Now, $9y^2 = x^3$

$$\Rightarrow \quad 18y\frac{dy}{dx} = 3x^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2}{6y}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{x_1^2}{6y_1}$$

Since the normal to the curve at (x_1, y_1) make equal intercepts with the coordinate axes. Therefore

Slope of the normal = ± 1

$$\Rightarrow \quad -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}} = \pm 1$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \pm 1$$

$$\Rightarrow \quad \frac{x_1^2}{6y_1} = \pm 1$$

$$\Rightarrow \quad x_1^2 = \pm 6\, y_1$$

$$\Rightarrow \quad x_1^4 = 36y_1^2 = 36\left(\frac{x_1^3}{9}\right) \quad \text{[Using (i)}$$

$$\Rightarrow \quad x_1^4 = 4x_1^3$$

$$\Rightarrow \quad x_1^3\ (x_1 - 4) = 0$$

$$\Rightarrow \quad x_1 = 0, 4$$

Putting $x_1 = 0$ in (i), we get

$$9y_1^2 = 0$$

$$\Rightarrow \quad y_1 = 0$$

Putting $x_1 = 4$ in (i), we get

$$9y_1^2 = 4^3$$

$$\Rightarrow \qquad y_1 = \pm\frac{8}{3}$$

But, the line making equal intercepts with the coordinate axes cannot pass through the origin.

Hence, the required points are $\left(4, \frac{8}{3}\right)$ and $\left(4, -\frac{8}{3}\right)$.

S36. The given curve is $x = a \sin^3 t$ and $y = a \cos^3 t$

First, we find $\dfrac{dy}{dx} = \dfrac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$.

Now, $x = a \sin^3 t$

$\therefore \quad \dfrac{dx}{dt} = a \cdot 3 \sin^2 t \cos t$ [Differentiate w.r.t. t]

$= 3a \sin^2 t \cos t$

and $y = a \cos^3 t$

$\therefore \quad \dfrac{dy}{dt} = a \cdot 3 \cos^2 t\,(- \sin t)$ [Differentiate w.r.t. t]

$= - 3a \cos^2 t \sin t$

$$\therefore \quad \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-3a\cos^2 t\sin t}{3a\sin^2 t\cos t} = -\cot t$$

Now, slope of tangent, $(m) = \left[\dfrac{dy}{dx}\right]_{t=\frac{\pi}{4}} = -\cot\dfrac{\pi}{4} = -1$ $\left[\because \ \cot\dfrac{\pi}{4} = 1\right]$

Also, $x_1 = a\sin^3\dfrac{\pi}{4} = a\left(\dfrac{1}{\sqrt{2}}\right)^3 = \dfrac{a}{2\sqrt{2}}$

$y_1 = a\cos^3\dfrac{\pi}{4} = a\left(\dfrac{1}{\sqrt{2}}\right)^3 = \dfrac{a}{2\sqrt{2}}$

Hence, equation of tangent is

$$y - y_1 = m(x - x_1)$$

Here, $x_1 = \dfrac{a}{2\sqrt{2}}$,

$y_1 = \dfrac{a}{2\sqrt{2}}$, $m = -1$

$\therefore \qquad y - \dfrac{a}{2\sqrt{2}} = -1\left(x - \dfrac{a}{2\sqrt{2}}\right)$

$\Rightarrow \qquad 2\sqrt{2}y - a = -2\sqrt{2}x + a$

$\Rightarrow \qquad 2\sqrt{2}x + 2\sqrt{2y} = 2a$

Here, the required equation is

$$\sqrt{2}x + \sqrt{2}y = a.$$

S37. Given equation of the curve is

$$4x^2 + 9y^2 = 36$$

Differentiating both sides w.r.t. x, we get

$$8x + 18y\frac{dy}{dx} = 0$$

$\Rightarrow \qquad 18y\dfrac{dy}{dx} = -8x$

$\Rightarrow \qquad \dfrac{dy}{dx} = -\dfrac{8x}{18y}$

$\Rightarrow \qquad \dfrac{dy}{dx} = -\dfrac{4x}{9y} \qquad$ ...(i)

But given that tangent passes through the point $(3\cos\theta, 2\sin\theta)$

$\therefore$ Putting $x = 3\cos\theta$, $y = 2\sin\theta$ in Eq. (i), we get

$$\frac{dy}{dx} = \frac{-12\cos\theta}{18\sin\theta}$$

or $\qquad \dfrac{dy}{dx} = \dfrac{-2\cos\theta}{3\sin\theta}$

$\therefore$ Slope of the tangent, $\qquad m = \dfrac{-2\cos\theta}{3\sin\theta} \qquad \left\{\because \text{ We know that } m = \left[\dfrac{dy}{dx}\right]_{(x_1,\, y_1)}\right\}$

Now, equation of tangent at the point $(3\cos\theta, 2\sin\theta)$ having slope

$$m = \frac{-2\cos\theta}{3\sin\theta} \text{ is}$$

$$y - y_1 = m(x - x_1)$$

$\Rightarrow \qquad y - 2\sin\theta = \dfrac{-2\cos\theta}{3\sin\theta}(x - 3\cos\theta)$

$\Rightarrow \quad 3y \sin\theta - 6\sin^2\theta = -2x\cos\theta + 6\cos^2\theta$

$\Rightarrow \quad 2x\cos\theta + 3y\sin\theta - 6(\sin^2\theta + \cos^2\theta) = 0$

$\Rightarrow \quad 2x\cos\theta + 3y\sin\theta - 6 = 0 \qquad [\because \sin^2\theta + \cos^2\theta = 1]$

S38. We know that the equation of tangent at the point (x_1, y_1) is

$$y - y_1 = m(x - x_1), \text{ where } m = \left[\frac{dy}{dx}\right]_{(x_1,\, y_1)} \qquad \text{... (i)}$$

First, we find $\frac{dy}{dx}$.

Now, given $\qquad x = \sin 3t \qquad$ ... (ii)

$$\therefore \quad \frac{dx}{dt} = 3\cos 3t \qquad \text{[Differentiate w.r.t. } t\text{]}$$

and $\qquad y = \cos 2t \qquad$ ... (iii)

$$\therefore \quad \frac{dy}{dx} = -2\sin 2t \qquad \text{[Differentiate w.r.t. } t\text{]}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-2\sin 2t}{3\cos 3t}$$

Putting $t = \frac{\pi}{4}$, we get

$$m = \left[\frac{dy}{dx}\right]_{t=\frac{\pi}{4}} = \frac{-2\sin\frac{\pi}{2}}{3\cos\frac{3\pi}{4}} = \frac{-2}{-\frac{3}{\sqrt{2}}} = \frac{2\sqrt{2}}{3} \qquad \left[\because \sin\frac{\pi}{2} = 1 \text{ and } \cos\frac{3\pi}{4} = \cos\left(\pi - \frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}\right]$$

Also, to find (x_1, y_1), we put $t = \frac{\pi}{4}$ in Eqs. (ii) and (iii),

$$\therefore \quad x_1 = \sin\frac{3\pi}{4} = \sin\left(\pi - \frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

and $\qquad y_1 = \cos\frac{\pi}{2} = 0$

$$\therefore \quad (x_1, y_1) = \left(\frac{1}{\sqrt{2}}, 0\right)$$

So, putting $(x_1, y_1) = \left(\frac{1}{\sqrt{2}}, 0\right)$ and $m = \frac{2\sqrt{2}}{3}$ in Eq. (i), we get

$$y - 0 = \frac{2\sqrt{2}}{3}\left(x - \frac{1}{\sqrt{2}}\right)$$

$\Rightarrow$ $$3y = 2\sqrt{2}x - \frac{2}{3}$$

Required equation of tangent is

$$6\sqrt{2}x - 9y - 2 = 0.$$

S39. We have, $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{... (i)}$$

Since $P(x_1, y_1)$ lies on the curve (i). Therefore,

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1 \quad \text{... (ii)}$$

Differentiating (i) with respect to x, we get

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$\Rightarrow$ $$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y}$$

$\Rightarrow$ $$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{b^2 x_1}{a^2 y_1}$$

The equation of the tangent at $P(x_1, y_1)$ is

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}(x - x_1)$$

$\Rightarrow$ $$y - y_1 = -\frac{b^2 x_1}{a^2 y_1}(x - x_1)$$

$\Rightarrow$ $$\frac{yy_1 - y_1^2}{b^2} = -\left(\frac{xx_1 - x_1^2}{a^2}\right)$$

$\Rightarrow$ $$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$$

$\Rightarrow$ $$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \quad \text{[Using (ii)]}$$

The equation of normal at $P(x_1, y_1)$ is

$$y - y_1 = -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}(x - x_1)$$

$\Rightarrow$ $$y - y_1 = \frac{a^2 y_1}{b^2 x_1}(x - x_1)$$

$\Rightarrow \quad \dfrac{b^2(y-y_1)}{y_1} = \dfrac{a^2(x-x_1)}{x_1}$

$\Rightarrow \quad \dfrac{b^2 y}{y_1} - b^2 = \dfrac{a^2 x}{x_1} - a^2$

$\Rightarrow \quad \dfrac{a^2 x}{x_1} - \dfrac{b^2 y}{y_1} = a^2 - b^2.$

S40. The equation of the curve is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ ... (i)

Point $P(x_0, y_0)$ lies on (i). Therefore,

$$\frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1 \quad \text{... (ii)}$$

Differentiating (i) with respect to x, we get

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{b^2 x}{a^2 y}$

$\Rightarrow \quad \left(\dfrac{dy}{dx}\right)_P = \dfrac{b^2 x_0}{a^2 y_0}$

The equation of the tangent at $P(x_0, y_0)$ is

$\Rightarrow \quad y - y_0 = \dfrac{b^2 x_0}{a^2 y_0}(x - x_0)$

$\Rightarrow \quad \dfrac{yy_0 - y_0^2}{b^2} = \dfrac{xx_0 - x_0^2}{a^2}$

$\Rightarrow \quad \dfrac{xx_0}{a^2} - \dfrac{yy_0}{b^2} = \dfrac{x_0^2}{a^2} - \dfrac{y_0^2}{b^2}$

$\Rightarrow \quad \dfrac{xx_0}{a^2} - \dfrac{yy_0}{b^2} = 1$ [Using (ii)]

The equation of the normal $P(x_0, y_0)$ is

$$y - y_0 = -\frac{a^2 y_0}{b^2 x_0}(x - x_0)$$

$\Rightarrow \quad \dfrac{b^2}{y_0}(y - y_0) = -\dfrac{a^2}{x_0}(x - x_0)$

$\Rightarrow \quad \dfrac{a^2 x}{x_0} + \dfrac{b^2 y}{y_0} = a^2 + b^2.$

S41. We have, $$y = (1 + x)^y + \sin^{-1}(\sin^2 x) \quad \text{... (i)}$$

Putting $x = 0$, we get

$$y = (1 + 0)^y + \sin^{-1}(\sin^2 0)$$

$$\Rightarrow \quad y = 1$$

Thus, we have to write the equation of the normal to (i) at $P(0, 1)$.

$$y - 1 = \frac{1}{\left(\frac{dy}{dx}\right)_P}(x - 0)$$

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = e^{y \log(1+x)} \cdot \frac{d}{dx}\{y \log(1+x)\} + \frac{1}{\sqrt{1-\sin^4 x}} \frac{d}{dx}(\sin^2 x)$$

$$\Rightarrow \quad \frac{dy}{dx} = (1+x)^y \left\{\frac{dy}{dx} \cdot \log(1+x) + \frac{y}{1+x}\right\} + \frac{2\sin x \cos x}{|\cos x|\sqrt{1+\sin^2 x}}$$

Putting, $x = 0$ and $y = 1$, we obtain

$$\left(\frac{dy}{dx}\right)_P = \left\{\left(\frac{dy}{dx}\right)_P \times 0 + 1\right\} + 0$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_P = 1$$

Hence, the equation of the normal at $P(0, 1)$ is

$$y - 1 = \frac{-1}{\left(\frac{dy}{dx}\right)_P}(x - 0)$$

$$y - 1 = -1(x - 0) \quad \text{or} \quad x + y = 1.$$

S42. Given equation of the curve is

$$y = (x^2 - 1)(x - 2) \quad \text{... (i)}$$

Since, the curve cuts the X-axis, so y-coordinate in its equation is zero.

$$\therefore \quad (x^2 - 1)(x - 2) = 0$$

$$\Rightarrow \quad x^2 = 1 \text{ or } x = 2$$

$$\Rightarrow \quad x = \pm 1 \text{ or } 2$$

$$\text{or} \quad x = -1, 1, 2$$

$\therefore$ The given curve cuts the X-axis at points $(-1, 0)$, $(1, 0)$, $(2, 0)$.

Now, differentiating Eq. (i) w.r.t. x, we get

$$\frac{dy}{dx} = (x^2 - 1) \cdot 1 + (x - 2) \cdot 2x$$

$\Rightarrow \qquad \dfrac{dy}{dx} = x^2 - 1 + 2x^2 - 4x$

$\Rightarrow \qquad \dfrac{dy}{dx} = 3x^2 - 4x - 1$

We know that slope of a tangent is $\dfrac{dy}{dx}$.

$\therefore \quad m$ = slope of tangent of $3x^2 - 4x - 1$ at points $(-1, 0)$ or $(1, 0)$ or $(2, 0)$.

So, we get three values of m, *i.e.*,

$$m_1 = \left[\frac{dy}{dx}\right]_{(-1,\,0)} = 3(-1)^2 - 4(-1) - 1$$
$$= 3 + 4 - 1 = 6$$

or
$$m_2 = \left[\frac{dy}{dx}\right]_{(1,\,0)} = 3(1)^2 - 4(1) - 1$$
$$= 3 - 4 - 1 = -2$$

or
$$m_3 = \left[\frac{dy}{dx}\right]_{(2,\,0)} = 3(2)^2 - 4(2) - 1$$
$$= 12 - 8 - 1 = 3$$

We know that equation of tangent at the point (x_1, y_1) is given by $y - y_1 = m(x - x_1)$.

$\therefore \quad$ We get three equations of tangents.

Equation of tangent at point $(-1, 0)$ having slope $(m_1) = 6$.

$$y - 0 = 6(x + 1)$$
$\Rightarrow \qquad y = 6x + 6$

$\Rightarrow \qquad 6x - y = -6$

Equation of tangent at point $(1, 0)$ having slope $m_2 = -2$

$$y - 0 = -2(x - 1)$$
$\Rightarrow \qquad y = -2x + 2 \quad$ or $\quad 2x + y = 2$

and equation of tangent at point $(2, 0)$ having slope $m_3 = 3$

$$y - 0 = 3(x - 2)$$
$$y = 3x - 6$$
$\Rightarrow \qquad 3x - y = 6.$

S43. The equation of the given curve is $16x^2 + 9y^2 = 144$ … (i)

Since (x_1, y_1) lies on (i). Therefore,

$$16x_1^2 + 9y_1^2 = 144$$

$\Rightarrow \quad 16(2)^2 + 9y_1^2 = 144$

$\Rightarrow \quad 9y_1^2 = 144 - 64$

$\Rightarrow \quad y_1^2 = \dfrac{80}{9}$

$\Rightarrow \quad y_1 = \dfrac{4\sqrt{5}}{3} \qquad [\because y_1 > 0]$

So, coordinates of the given points are $\left(2, \dfrac{4\sqrt{5}}{3}\right)$.

Now, $\quad 16x^2 + 9y^2 = 144$

$\Rightarrow \quad 32x + 18y\dfrac{dy}{dx} = 0 \qquad$ [Differentiating w.r.t. x]

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{-16x}{9y}$

$\Rightarrow \quad \left(\dfrac{dy}{dx}\right)_{\left(2, \frac{4\sqrt{5}}{3}\right)} = -\dfrac{16 \times 2}{9 \times \dfrac{4\sqrt{5}}{3}} = -\dfrac{8}{3\sqrt{5}}$

The equation of the tangent at $\left(2, \dfrac{4\sqrt{5}}{3}\right)$ is

$$y - \frac{4\sqrt{5}}{3} = \left(\frac{dy}{dx}\right)_{\left(2, \frac{4\sqrt{5}}{3}\right)}(x-2)$$

$\Rightarrow \quad y - \dfrac{4\sqrt{5}}{3} = -\dfrac{8}{3\sqrt{5}}(x-2)$

$\Rightarrow \quad 3\sqrt{5}\,y - 20 = -8x + 16$

$\Rightarrow \quad 8x + 3\sqrt{5}y - 36 = 0$

The equation of the normal at $\left(2, \dfrac{4\sqrt{5}}{3}\right)$ is

$$y - \frac{4\sqrt{5}}{3} = -\frac{1}{\left(\frac{dy}{dx}\right)_{\left(2, \frac{4\sqrt{5}}{3}\right)}}(x-2)$$

$\Rightarrow \quad y - \frac{4\sqrt{5}}{3} = \frac{-1}{\frac{-8}{3\sqrt{5}}}(x-2)$

$\Rightarrow \quad y - \frac{4\sqrt{5}}{3} = \frac{3\sqrt{5}}{8}(x-2)$

$\Rightarrow \quad 24y - 32\sqrt{5} = 9\sqrt{5}x - 18\sqrt{5}$

$\Rightarrow \quad 9\sqrt{5}x - 24y + 14\sqrt{5} = 0$

S44. The given equation of the curve is

$$x^2 + 3y = 3 \qquad \dots \text{(i)}$$

So, differentiating Eq. (i) w.r.t. x, we get

$$2x + 3\frac{dy}{dx} = 0$$

$\Rightarrow \quad \frac{dy}{dx} = -\frac{2x}{3}$

$\therefore$ Slope (m) of tangent = $-\frac{2x}{3}$.

Also, slope of the given line whose equation is

$$y - 4x + 5 = 0 \text{ is } 4.$$

$\because$ The tangent is parallel to the given line

$\therefore$ Slope of tangent = Slope of line

$\Rightarrow \quad -\frac{2x}{3} = 4$

$\Rightarrow \quad -2x = 12$

$\Rightarrow \quad x = -6$

Putting $x = -6$ in Eq. (i), we get

$$(-6)^2 + 3y = 3$$

$\Rightarrow \quad 3y = 3 - 36$

$\Rightarrow \quad 3y = -33$

$\Rightarrow \quad y = -11$

$\therefore$ The tangent is passing through point $(-6, -11)$ and it has slope 4.

$\therefore$ Equation of tangent is

$$y + 11 = 4(x + 6)$$

$\Rightarrow$ $y + 11 = 4x + 24$

or $4x - y = -13.$

S45. Let the point of contact of the tangent line parallel to the given line be $P(x_1, y_1)$.

The equation of the curve is $y = \sqrt{5x-3} - 2$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{5}{2\sqrt{5x-3}}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{5}{2\sqrt{5x_1-3}}$$

Since the tangent at (x_1, y_1) is parallel to the line $4x - 2y + 3 = 0$. Therefore

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \text{(Slope of the line } 4x - 2y + 3 = 0)$$

$$\Rightarrow \quad \frac{5}{2\sqrt{5x_1-3}} = \frac{4}{2}$$

$$\Rightarrow \quad 4\sqrt{5x_1-3} = 5$$

$$\Rightarrow \quad 16(5x_1 - 3) = 25$$

$$\Rightarrow \quad 80x_1 = 25 + 48$$

$$x_1 = \frac{73}{80}$$

Since (x_1, y_1) lies on $y = \sqrt{5x-3} - 2$. Therefore,

$$y_1 = \sqrt{5x_1-3} - 2$$

$$\Rightarrow \quad y_1 = \sqrt{5 \times \frac{73}{80} - 3} - 2 \qquad \left[\because x_1 = \frac{73}{80}\right]$$

$$\Rightarrow \quad y_1 = \sqrt{\frac{73-48}{16}} - 2$$

$$\Rightarrow \quad y_1 = \frac{5}{4} - 2$$

$$y_1 = -\frac{3}{4}$$

So, the coordinates of the point of contact are $\left(\frac{73}{80}, -\frac{3}{4}\right)$

Hence, the required equation of the tangent is

$$y - \left(-\frac{3}{4}\right) = 2\left(x - \frac{73}{80}\right) \qquad \left[\because \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 2\right]$$

$$\Rightarrow \quad \frac{4y+3}{4} = \frac{80x-73}{40}$$

$$\Rightarrow \quad 40y + 30 = 80x - 73$$

$$\Rightarrow \quad 80x - 40y - 103 = 0.$$

S46. The given equation of the curve is

$$y = \sqrt{3x-2} \qquad \ldots \text{(i)}$$

Differentiating Eq. (i) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{3}{2\sqrt{3x-2}}$$

$\therefore$ Slope of tangent = $\dfrac{3}{2\sqrt{3x-2}}$.

Also, slope of the line $4x - 2y + 5 = 0$ is $-\left(\dfrac{4}{-2}\right) = 2$.

$\because$ The tangent is parallel to the line

$\therefore$ Slope of tangent = Slope of line

$$\Rightarrow \quad \frac{3}{2\sqrt{3x-2}} = 2$$

$$\Rightarrow \quad 3 = 4\sqrt{3x-2}$$

Squaring both sides, we get

$$9 = 16(3x-2)$$

$$\Rightarrow \quad 9 = 48x - 32$$

$$\Rightarrow \quad 41 = 48x \text{ or } x = \frac{41}{48}$$

Putting $x = \dfrac{41}{48}$ in Eq. (i), we get

$$y = \sqrt{3\left(\frac{41}{48}\right) - 2} = \sqrt{\frac{41}{16} - 2} = \sqrt{\frac{41-32}{16}}$$

$$= \sqrt{\frac{9}{16}} = \frac{3}{4}$$

$\therefore$ The tangent passes through the point $\left(\dfrac{41}{48}, \dfrac{3}{4}\right)$ and its has slope 2.

$\therefore$ Equation of tangent is

$$y - y_1 = m(x - x_1)$$

Here, $x_1 = \dfrac{41}{48}$, $y_1 = \dfrac{3}{4}$ and $m = 2$

$\therefore$ We get, the required equation of tangent is

$$y - \frac{3}{4} = 2\left(x - \frac{41}{48}\right)$$

$\Rightarrow$ $$\frac{4y-3}{4} = 2\left(\frac{48x-41}{48}\right)$$

$\Rightarrow$ $$\frac{4y-3}{4} = \frac{48x-41}{24}$$

$\Rightarrow$ $$4y - 3 = \frac{48x-41}{6}$$

$\Rightarrow$ $$24y - 18 = 48x - 41$$

$\Rightarrow$ $$48x - 24y = 23$$

S47. Let (x_1, y_1) be the point of contact of a line of slope zero which touches the curve

$$y = \frac{1}{x^2 - 2x + 3}$$

Now, $$y = \frac{1}{x^2 - 2x + 3}$$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{-2(x-1)}{(x^2 - 2x + 3)^2}$$

$\Rightarrow$ $$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-2(x_1 - 1)}{(x_1^2 - 2x_1 + 3)^2}$$

It is given that $$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$$

$\therefore$ $$\frac{-2(x_1 - 1)}{(x_1^2 - 2x_1 + 3)^2} = 0$$

$\Rightarrow$ $$x_1 = 1$$

Since (x_1, y_1) lies on $y = \dfrac{1}{x^2 - 2x + 3}$

$\therefore$ $$y_1 = \frac{1}{x_1^2 - 2x_1 + 3}$$

$\Rightarrow$ $$y_1 = \frac{1}{1 - 2 + 3} = \frac{1}{2}$$

Hence, the equation of the tangent is

$$y - \frac{1}{2} = 0\,(x - 1)$$

$\Rightarrow$ $$y = \frac{1}{2}.$$

S48. Let the required normal be drawn at the point (x_1, y_1). The equation of the given curve is

$$3x^2 - y^2 = 8 \qquad \text{... (i)}$$

Differentiating both sides w.r.t. x, we get

$$6x - 2y\frac{dy}{dx} = 0$$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{3x}{y}$$

$\Rightarrow$ $$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{3x_1}{y_1}$$

Since the normal at (x_1, y_1) is parallel to the line $x + 3y = 4$. Therefore,

Slope of the normal at (x_1, y_1) = (Slope of the line $x + 3y = 4$)

$\Rightarrow$ $$\frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}} = \frac{-1}{3}$$

$\Rightarrow$ $$\frac{-y_1}{3x_1} = -\frac{1}{3}$$

$\Rightarrow$ $$y_1 = x_1 \qquad \text{... (ii)}$$

Since (x_1, y_1) lies on (i). Therefore,

$$3x_1^2 - y_1^2 = 8 \qquad \text{... (iii)}$$

Solving (ii) and (iii), we get

$$3x_1^2 - y_1^2 = 8$$

$\Rightarrow$ $$x_1^2 = 4$$

$\Rightarrow$ $$x_1 = \pm 2$$

Now, $x_1 = 2 \Rightarrow y_1 = 2$ [Using (ii)]

and $x_1 = -2 \Rightarrow y_1 = -2$ [Using (ii)]

Thus, the coordinates of the point are (2, 2) and (–2, –2). The equation of the normal at (2, 2) is

$$y - 2 = -\frac{1}{3}(x-2) \qquad \left[\because -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1,y_1)}} = -\frac{1}{3}\right]$$

$\Rightarrow \quad x + 3y - 8 = 0$

The equation of the normal at (–2, –2) is

$$y - (-2) = -\left(\frac{1}{3}\right)(x-(-2))$$

$\Rightarrow \quad x + 3y + 8 = 0.$

S49. We have $y = x^3 + 2x + 6$ ... (i)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 3x^2 + 2$$

$\therefore$ Slope of normal $= \dfrac{-1}{\left(\frac{dy}{dx}\right)} = \dfrac{-1}{3x^2+2}$

Also, slope of the line

$$x + 14y + 4 = 0 \text{ is } -\frac{1}{14}. \qquad \left[\because \text{Slope of the line } Ax + By + C = 0 \text{ is } -\frac{A}{B.}\right]$$

$\therefore \quad \dfrac{-1}{3x^2+2} = \dfrac{-1}{14}$ [$\because$ When two lines are parallel, their slopes are equal]

$\Rightarrow \quad 3x^2 + 2 = 14$

$\Rightarrow \quad 3x^2 = 12$

$\Rightarrow \quad x^2 = 4$

$\Rightarrow \quad x = \pm 2$

$\therefore$ From Eq. (i), at $x = 2$

$$y = (2)^3 + 2(2) + 6$$
$$= 8 + 4 + 6 = 18$$

and at $x = -2$,

$$y = (-2)^3 + 2(-2) + 6$$
$$= -8 - 4 + 6 = -6$$

$\therefore$ Normal passes through (2, 18) and (–2, –6).

Also, slope of normal = $\frac{-1}{14}$.

Hence, equation of normal at point (2, 18) is

$$y - y_1 = \text{slope of normal } (x - x_1)$$

$$\therefore \quad y - 18 = \frac{-1}{14}(x - 2)$$

$$\Rightarrow \quad 14y - 252 = -x + 2$$

$$\text{or} \quad x + 14y = 254$$

and equation of normal at point (– 2, – 6) is

$$y + 6 = \frac{-1}{14}(x + 2)$$

$$\Rightarrow \quad 14y + 84 = -x - 2$$

$$\text{or} \quad x + 14y = -86$$

Hence, the two equations of normal are $x + 14y = 254$ and $x + 14y = -86$.

S50. Let the point of contact of one of the tangents be (x_1, y_1). Then (x_1, y_1) lies on $y = \cos(x + y)$

$$\therefore \quad y_1 = \cos(x_1 + y_1) \quad \text{... (i)}$$

Since the tangents are parallel to the line $x + 2y = 0$. Therefore,

Slope of the tangent at (x_1, y_1) = (Slope of the line $x + 2y = 0$)

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-1}{2}$$

The equation of the curve is

$$y = \cos(x + y)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = -\sin(x + y)\cdot\left(1 + \frac{dy}{dx}\right)$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\sin(x_1 + y_1)\cdot\left\{1 + \left(\frac{dy}{dx}\right)_{(x_1, y_1)}\right\}$$

$$\Rightarrow \quad -\frac{1}{2} = -\sin(x_1 + y_1)\left(1 - \frac{1}{2}\right)$$

$$\Rightarrow \quad \sin(x_1 + y_1) = 1 \quad \text{... (ii)}$$

Squaring Eq. (i) and Eq. (ii) and then adding, we get

$$\cos^2(x_1 + y_1) + \sin^2(x_1 + y_1) = y_1^2 + 1$$

$$\Rightarrow \quad 1 = y_1^2 + 1$$

$\Rightarrow \qquad y_1 = 0$

Putting $y_1 = 0$ in (i) and (ii), we get

$$\cos x_1 = 0 \text{ and } \sin x_1 = 1$$

$$\Rightarrow \qquad x_1 = \frac{\pi}{2}, -\frac{3\pi}{2} \qquad [\because -2\pi \le x_1 \le 2\pi]$$

Hence, the points of contact are $\left(\frac{\pi}{2}, 0\right)$ and $\left(-\frac{3\pi}{2}, 0\right)$.

The slope of the tangent is $\left(-\frac{1}{2}\right)$. Therefore, equations of tangents are

$$y - 0 = -\frac{1}{2}\left(x - \frac{\pi}{2}\right)$$

and $$y - 0 = -\frac{1}{2}\left(x + \frac{3\pi}{2}\right)$$

$\Rightarrow \qquad 2x + 4y - \pi = 0$

and $\qquad 2x + 4y + 3\pi = 0.$

S51. We have, $\qquad y^2 = 4ax$ and $x^2 = 4by$

$$\Rightarrow \qquad \left(\frac{x^2}{4b}\right)^2 = 4ax \qquad \left[\because x^2 = 4by \Rightarrow y = \frac{x^2}{4b}\right]$$

$\Rightarrow \qquad x^4 - 64\, ab^2 x = 0$

$\Rightarrow \qquad x\,(x^3 - 64\, ab^2) = 0$

$\Rightarrow \qquad x = 0,\ x^3 = 64ab^2$

$\Rightarrow \qquad x = 0,\ x = 4a^{\frac{1}{3}}b^{\frac{2}{3}}$

Putting $x = 0$ and $x = 4a^{\frac{1}{3}}b^{\frac{2}{3}}$ in $y = \frac{x^2}{4b}$,

we get, $\qquad y = 0$ and $y = 4a^{\frac{2}{3}}b^{\frac{1}{3}}$

Thus, the two curves intersect at $P\left(4a^{\frac{1}{3}}b^{\frac{2}{3}}, 4a^{\frac{2}{3}}b^{\frac{1}{3}}\right)$ other than the origin (0, 0).

Now, $\qquad y^2 = 4ax$ and $x^2 = 4by$

$$\Rightarrow \qquad 2y\frac{dy}{dx} = 4a \quad \text{and} \quad 2x = 4b\frac{dy}{dx}$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{2a}{y} \quad \text{and} \quad \frac{dy}{dx} = \frac{x}{2b}$$

$$\Rightarrow \qquad m_1 = \left(\frac{dy}{dx}\right)_P = \frac{2a}{4a^{\frac{2}{3}}b^{\frac{1}{3}}} = \frac{1}{2}\left(\frac{a}{b}\right)^{\frac{1}{3}}$$

and $$m_2 = \left(\frac{dy}{dx}\right)_P = = \frac{4a^{\frac{1}{3}}b^{\frac{2}{3}}}{2b} = 2\left(\frac{a}{b}\right)^{\frac{1}{3}}$$

Let θ be the angle between the tangents to the parabolas $y^2 = 4ax$ and $x^2 = 4by$ at P. Then,

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$\Rightarrow \quad \tan\theta = \left|\frac{\frac{1}{2}\left(\frac{a}{b}\right)^{\frac{1}{3}} - 2\left(\frac{a}{b}\right)^{\frac{1}{3}}}{1 + \frac{1}{2}\left(\frac{a}{b}\right)^{\frac{1}{3}} \times 2\left(\frac{a}{b}\right)^{\frac{1}{3}}}\right| = \left|\frac{-\frac{3}{2}\left(\frac{a}{b}\right)^{\frac{1}{3}}}{1 + \left(\frac{a}{b}\right)^{\frac{2}{3}}}\right|$$

$$\Rightarrow \quad \tan\theta = \left|\frac{3a^{\frac{1}{3}}\ b^{\frac{1}{3}}}{2\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)}\right|$$

$$\Rightarrow \quad \theta = \tan^{-1}\left\{\frac{3(ab)^{\frac{1}{3}}}{2\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)}\right\}.$$

S52. The equations of the two curves are

$$y^2 = 4x \quad \text{... (i)}$$

and $$x^2 = 4y \quad \text{... (ii)}$$

From (i), we obtain $x = \frac{y^2}{4}$. Putting $x = \frac{y^2}{4}$ in (ii) we get

$$\left(\frac{y^2}{4}\right)^2 = 4y \Rightarrow y^4 - 64y = 0 \Rightarrow y(y^3 - 64) = 0 \Rightarrow y = 0,\ y = 4$$

From (i), when $y = 0$, we get $x = 0$ and when $y = 4$, we get $x = 4$.

Thus the two curves intersect at (0, 0) and (4, 4).

Differentiating (i) w.r.t. x, we get

$$2y\frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{2}{y} \quad \text{... (iii)}$$

Differentiating (ii) w.r.t. x, we get

$$2x = 4.\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x}{2} \quad \text{... (iv)}$$

Angle of Intersection at (0, 0):

From (iii), we have

$$m_1 = \left(\frac{dy}{dx}\right)_{(0,0)} = \infty$$

Therefore, the tangent to (i) curve at (0, 0) is parallel to *y*-axis.

From (iv), we have $$m_2 = \left(\frac{dy}{dx}\right)_{(0,0)} = 0$$

Therefore, the tangent to (ii) curve at (0, 0) is parallel to *x*-axis.

Hence, the angle between the tangents to two curves at (0, 0) is a right angle. Consequently, the two curves intersect at right angle at (0, 0).

Angle of Intersection at (4, 4): From (iii), we have

$$m_1 = \left(\frac{dy}{dx}\right)_{(4,4)} = \frac{2}{4} = \frac{1}{2}$$

From (iv), we have $$m_2 = \left(\frac{dy}{dx}\right)_{(4,4)} = \frac{4}{2} = 2$$

Let θ be the angle of intersection of the two curves. Then,

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\frac{2 - \left(\frac{1}{2}\right)}{1 + 2\times\left(\frac{1}{2}\right)}\right| = \left|\frac{\frac{3}{2}}{\frac{2}{1}}\right| = \frac{3}{4}$$

$$\Rightarrow \qquad \theta = \tan^{-1}\left(\frac{3}{4}\right).$$

S53. The equations of the two curves are

$$xy = 6 \qquad \text{... (i)}$$

and $$x^2y = 12 \qquad \text{... (ii)}$$

From (i), we obtain $y = \frac{6}{x}$. Putting this value of *y* in (ii), we obtain

$$x^2\left(\frac{6}{x}\right) = 12$$

$$\Rightarrow \qquad 6x = 12$$

$$\Rightarrow \qquad x = 2$$

Putting $x = 2$ in (i) or (ii), we get $y = 3$.

Thus, the two curves intersect at $P(2, 3)$.

Differentiating (i) w.r.t. *x*, we get

$$x\frac{dy}{dx} + y = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{y}{x}$$

$$\Rightarrow \quad m_1 = \left(\frac{dy}{dx}\right)_{(2,\,3)} = -\frac{3}{2}$$

Differentiating (ii) w.r.t. x, we get

$$x^2\frac{dy}{dx} + 2xy = 0 \;\Rightarrow\; \frac{dy}{dx} = -\frac{2y}{x} \;\Rightarrow\; m_2 = \left(\frac{dy}{dx}\right)_{(2,\,3)} = -3$$

Let θ be the angle of intersection of (i) and (ii), then

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{-\left(\frac{3}{2}\right) + 3}{1 + \left(-\frac{3}{2}\right)(-3)} = \frac{-\frac{3}{2} + 3}{1 + \frac{9}{2}} = \frac{\frac{3}{2}}{\frac{11}{2}} = \frac{3}{11}$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{3}{11}\right).$$

S54. Let $P(x_1, y_1)$ be a point on the given curve such that the tangent at P passes through the origin.

Since $P(x_1, y_1)$ lies on $y = x^2 + 3x + 4$

$$\therefore \quad y_1 = x_1^2 + 3x_1 + 4 \qquad \text{... (i)}$$

Now, $\quad y = x^2 + 3x + 4$

$$\Rightarrow \quad \frac{dy}{dx} = 2x + 3$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_P = 2x_1 + 3$$

The equation of the tangent at $P(x_1, y_1)$ is

$$y - y_1 = (2x_1 + 3)(x - x_1)$$

It passes through the origin *i.e.*, (0, 0)

$$\therefore \quad 0 - y_1 = (2x_1 + 3)(0 - x_1)$$

$$\Rightarrow \quad y_1 = 2x_1^2 + 3x_1 \qquad \text{... (ii)}$$

Subtracting (ii) from (i), we get

$$-x_1^2 + 4 = 0$$

$\Rightarrow$ $$x_1 = \pm 2$$

From (i), we obtain that

$$x_1 = 2$$

$\Rightarrow$ $$y_1 = 4 + 6 + 4 = 14$$

$$x_1 = -2$$

$\Rightarrow$ $$y_1 = 4 - 6 + 4 = 2$$

Hence, the required points are (2, 14) and (–2, 2).

S55. Let (x_1, y_1) be the required point on $y = 4x^3 - 2x^5$. Then,

$$y_1 = 4x_1^3 - 2x_1^5 \quad \text{... (i)}$$

The equation of the given curve is $y = 4x^3 - 2x^5$. Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 12x^2 - 10x^4$$

$\Rightarrow$ $$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 12x_1^2 - 10x_1^4$$

So, the equation of the tangent at (x_1, y_1) is

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$$

$\Rightarrow$ $$y - y_1 = \left(12x_1^2 - 10x_1^4\right)(x - x_1)$$

This passes through the origin. Therefore,

$$0 - y_1 = \left(12x_1^2 - 10x_1^4\right)(0 - x_1)$$

$\Rightarrow$ $$y_1 = 12x_1^3 - 10x_1^5 \quad \text{... (ii)}$$

Subtracting (ii) from (i), we get

$$0 = -8x_1^3 + 8x_1^5$$

$\Rightarrow$ $$8x_1^3\left(x_1^2 - 1\right) = 0$$

$\Rightarrow$ $$x_1 = 0 \quad \text{or} \quad x_1 = \pm 1$$

When $$x_1 = 0$$

$\Rightarrow$ $$y_1 = 0 \quad \text{[Using (ii)]}$$

When $x_1 = 1$

$\Rightarrow$ $y_1 = 12 - 10 = 2$ [Using (ii)]

When $x_1 = -1$

$\Rightarrow$ $y_1 = -12 + 10 = -2$ [Using (ii)]

Hence, the required points are (0, 0), (1, 2) and (– 1, – 2).

S56. Let (x_1, y_1) be the point of contact of a line of slope – 1 which touches the curve $y = \frac{1}{x-1}$.

Now, $$y = \frac{1}{x-1}$$

$\Rightarrow$ $$\frac{dy}{dx} = -\frac{1}{(x-1)^2}$$

But, $$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -1$$

$\therefore$ $$-\frac{1}{(x_1-1)^2} = -1$$

$\Rightarrow$ $(x_1 - 1)^2 = 1$

$\Rightarrow$ $x_1 - 1 = \pm 1$

$\Rightarrow$ $x_1 = 0, \quad x_1 = 2$

Since (x_1, y_1) lies on the curve $y = \frac{1}{x-1}$. Therefore,

$$y_1 = \frac{1}{x_1 - 1} \qquad \text{... (i)}$$

Now, $x_1 = 0$

$\Rightarrow$ $y_1 = \frac{1}{-1} = -1$ [Using (i)]

and, $x_1 = 2$

$\Rightarrow$ $y_1 = \frac{1}{2-1} = 1$ [Using (i)]]

Thus, the coordinates of the points of contact are (0, – 1) and (2, 1).

The equations of the tangent at (0, – 1) and (2, 1) are respectively.

$$y + 1 = -1\,(x - 0) \quad \text{and} \quad (y - 1) = -1(x - 2)$$

$\Rightarrow$ $x + y + 1 = 0$ and $x + y - 3 = 0$.

S57. The given curves are

$$xy = a^2 \quad \ldots \text{(i)}$$

$$x^2 + y^2 = 2a^2 \quad \ldots \text{(ii)}$$

Substituting the value of y obtained from (i) in equation (ii), we get

$$x^2 + \frac{a^4}{x^2} = 2a^2$$

$$\Rightarrow \quad x^4 - 2a^2x^2 + a^4 = 0$$

$$\Rightarrow \quad (x^2 - a^2)^2 = 0$$

$$\Rightarrow \quad x = \pm a$$

From (i), we have $y = a$ for $x = a$

and $y = -a$ for $x = -a$

Thus, the two curves intersect at $P(a, a)$ and $Q(-a, -a)$

Now, $xy = a^2$

$$\Rightarrow \quad x\frac{dy}{dx} + y = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{y}{x}$$

and, $x^2 + y^2 = a^2$

$$\Rightarrow \quad 2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}$$

At $P(a, a)$, we have $\left(\frac{dy}{dx}\right)_{C_1} = -\frac{a}{a} = -1$

$$\left(\frac{dy}{dx}\right)_{C_2} = -\frac{a}{a} = -1$$

Clearly, $\left(\frac{dy}{dx}\right)_{C_1} = \left(\frac{dy}{dx}\right)_{C_2}$ at P

So, the two curves touch each other at P.

Similarly we can show that these two curves touch each other at Q also.

S58. Let (x_1, y_1) be the point of intersection of the given curves. Then,

$$ax^2 + by^2 = 1 \quad \ldots \text{(i)}$$

$$a'x^2 + b'y^2 = 1 \quad \ldots \text{(ii)}$$

Differentiating (i) w.r.t. x, we get

$$2ax + 2by\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{ax}{by}$$

$$\Rightarrow \quad m_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$$

$$= -\frac{ax_1}{by_1} \quad \text{... (iii)}$$

Differentiating (ii) w.r.t. x, we obtain

$$2a'x + 2b'y\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{a'x}{b'y}$$

$$\Rightarrow \quad m_2 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{a'x_1}{b'y_1} \quad \text{... (iv)}$$

The two curves will intersect orthogonally, if

$$m_1 m_2 = -1$$

$$\Rightarrow \quad -\frac{ax_1}{by_1} \times \left(-\frac{a'x_1}{b'y_1}\right) = -1$$

$$\Rightarrow \quad aa'x_1^2 = -bb'y_1^2 \quad \text{... (v)}$$

Subtracting (ii) from (i), we obtain

$$(a - a')x^2 = -(b - b')y^2$$

$$(a - a')x_1^2 = -(b - b')y_1^2$$ (since curve passes through point(x_1, y_1) ... (vi)

Dividing (vi) by (v), we get

$$\frac{a - a'}{aa'} = \frac{b - b'}{bb'}$$

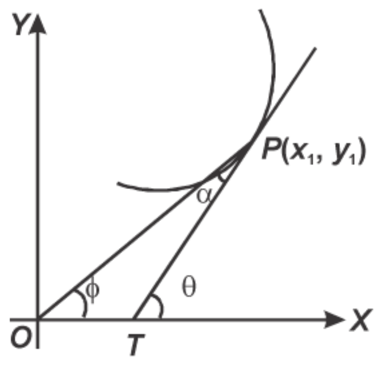
$$\Rightarrow \quad \frac{1}{a'} - \frac{1}{a} = \frac{1}{b'} - \frac{1}{b}$$

$$\Rightarrow \quad \frac{1}{a} - \frac{1}{b} = \frac{1}{a'} - \frac{1}{b'}.$$

S59. We have, $\log(x^2 + y^2) = k\tan^{-1}\left(\frac{y}{x}\right)$

Differentiating w.r.t. x, we get

$$\frac{1}{x^2+y^2}\left\{2x+2y\frac{dy}{dx}\right\} = k\frac{1}{1+\frac{y^2}{x^2}}\left\{\frac{x\frac{dy}{dx}-y}{x^2}\right\}$$



$$\Rightarrow \quad 2\left\{x+y\frac{dy}{dx}\right\} = k\left\{x\frac{dy}{dx}-y\right\}$$

$$\Rightarrow \quad 2x + ky = (kx - 2y)\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2x+ky}{kx-2y}$$

Let the coordinates of P be (x_1, y_1). Then,

$$\left(\frac{dy}{dx}\right)_P = \frac{2x_1+ky_1}{kx_1-2y_1}$$

If the tangent at P makes an angle θ with x-axis, then

$$\tan\theta = \frac{2x_1+ky_1}{kx_1-2y_1}$$

Suppose OP makes an angle ϕ with x-axis. Then,

$$\tan\phi = \text{Slope of } OP = \frac{y_1}{x_1}$$

Let α be the angle between OP and PT. Then

$$\theta = \alpha + \phi$$

$$\Rightarrow \quad \alpha = \theta - \phi$$

$$\tan\alpha = \tan(\theta - \phi)$$

$$\Rightarrow \quad \tan\alpha = \frac{\tan\theta - \tan\phi}{1+\tan\theta\tan\phi}$$

$$\Rightarrow \quad \tan\alpha = \frac{\frac{2x_1+ky_1}{kx_1-2y_1}-\frac{y_1}{x_1}}{1+\frac{2x_1+ky_1}{kx_1-2y_1}\times\frac{y_1}{x_1}}$$

$$\Rightarrow \quad \tan\alpha = \frac{2x_1^2+kx_1y_1-kx_1y_1+2y_1^2}{kx_1^2-2x_1y_1+2x_1y_1+ky_1^2} = \frac{2}{k}$$

$$\Rightarrow \quad \alpha = \tan^{-1}\left(\frac{2}{k}\right) = \text{Constant.}$$

S60. The given curves are $x = y^2$ ... (i)

and $xy = k$ ... (ii)

From (i), we obtain $x = y^2$. Putting this value of x in (ii), we obtain

$$y^3 = k$$

$$\Rightarrow \quad y = k^{\frac{1}{3}}$$

Putting $y = k^{\frac{1}{3}}$ in (i), we get $x = k^{\frac{2}{3}}$

So, the two curves intersect at the point $P\left(k^{\frac{2}{3}}, k^{\frac{1}{3}}\right)$

Differentiating (i) w.r.t. x, we get

$$1 = 2y\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2y}$$

$$\Rightarrow \quad m_1 = \left(\frac{dy}{dx}\right)_P = \frac{1}{2k^{\frac{1}{3}}}$$

Differentiating (ii) w.r.t. x, we get

$$1\cdot y + x\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{y}{x}$$

$$\Rightarrow \quad m_2 = \left(\frac{dy}{dx}\right)_P = -\frac{k^{\frac{1}{3}}}{k^{\frac{2}{3}}} = -\frac{1}{k^{\frac{1}{3}}}$$

For the curves (i) and (ii) to cut at right angles at P, we must have

$$m_1 m_2 = -1$$

$$\Rightarrow \quad \frac{1}{2k^{\frac{1}{3}}}\times\left(-\frac{1}{k^{\frac{1}{3}}}\right) = -1$$

$$\Rightarrow \quad 2k^{\frac{2}{3}} = 1$$

$$\Rightarrow \quad \left(2k^{\frac{2}{3}}\right)^3 = 1^3$$

$$\Rightarrow \quad 8k^2 = 1.$$

S61. Given curve is $x = 1 - \cos\theta$ and $y = \theta - \sin\theta$. To find equation of tangent and normal, first we find $\frac{dy}{dx}$ by using the formula.

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}$$

$\therefore$ $$\frac{dx}{d\theta} = \frac{d}{d\theta}(1-\cos\theta) = \sin\theta$$

and $$\frac{dy}{d\theta} = \frac{d}{d\theta}(\theta - \sin\theta) = 1-\cos\theta$$

$\therefore$ $$\frac{dy}{dx} = \frac{1-\cos\theta}{\sin\theta}$$

$\therefore$ $$\left[\frac{dy}{dx}\right]_{\theta=\frac{\pi}{4}} = \frac{1-\cos\frac{\pi}{4}}{\sin\frac{\pi}{4}} = \frac{1-\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = \sqrt{2}-1$$

At $$\theta = \frac{\pi}{4},\ x_1 = 1-\cos\frac{\pi}{4} = 1-\frac{1}{\sqrt{2}} = \frac{\sqrt{2}-1}{\sqrt{2}}$$

and $$y_1 = \frac{\pi}{4} - \sin\frac{\pi}{4} = \frac{\pi}{4} - \frac{1}{\sqrt{2}}$$

Now, we know that equation of tangent is given by

$$y - y_1 = m(x - x_1)$$

$\therefore$ $$y - \left(\frac{\pi}{4} - \frac{1}{\sqrt{2}}\right) = (\sqrt{2}-1)\left[x - \left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)\right]$$

$\Rightarrow$ $$y - \frac{\pi}{4} + \frac{1}{\sqrt{2}} = x(\sqrt{2}-1) - \frac{(\sqrt{2}-1)^2}{\sqrt{2}}$$

$\Rightarrow$ $$y - \frac{\pi}{4} + \frac{1}{\sqrt{2}} = x(\sqrt{2}-1) - \frac{(2+1-2\sqrt{2})}{\sqrt{2}}$$

$\Rightarrow$ $$\left(y - \frac{\pi}{4} + \frac{1}{\sqrt{2}}\right) = x(\sqrt{2}-1) - \frac{(3-2\sqrt{2})}{\sqrt{2}}$$

or $$x(\sqrt{2}-1) - y = \frac{3-2\sqrt{2}}{\sqrt{2}} - \frac{\pi}{4} + \frac{1}{\sqrt{2}}$$

Hence, the equation of tangent is

$$x(\sqrt{2}-1)-y = \frac{12-8\sqrt{2}-\sqrt{2}\pi+4}{4\sqrt{2}}$$

$$\Rightarrow \quad x(8-4\sqrt{2})-4\sqrt{2}y = (16-\sqrt{2}\pi-8\sqrt{2})$$

Also, the equation of normal is given by

$$y-y_1 = -\frac{1}{m}(x-x_1)$$

$$\Rightarrow \quad y-\left(\frac{\pi}{4}-\frac{1}{\sqrt{2}}\right) = \frac{-1}{\sqrt{2}-1}\left(x-\frac{\sqrt{2}-1}{\sqrt{2}}\right)$$

$$\Rightarrow \quad y(\sqrt{2}-1)-\left(\frac{\sqrt{2}\pi-4}{4\sqrt{2}}\right)(\sqrt{2}-1) = -x+\frac{\sqrt{2}-1}{\sqrt{2}}$$

$$\Rightarrow \quad y(\sqrt{2}-1)-\left(\frac{2\pi-\sqrt{2}\pi-4\sqrt{2}+4}{4\sqrt{2}}\right) = \frac{-\sqrt{2}x+\sqrt{2}-1}{\sqrt{2}}$$

$$\Rightarrow \quad 4\sqrt{2}y(\sqrt{2}-1)-2\pi+\sqrt{2}\pi+4\sqrt{2}-4 = -4\sqrt{2}x+4\sqrt{2}-4$$

$$\Rightarrow \quad 4\sqrt{2}x+4\sqrt{2}y(\sqrt{2}-1) = 2\pi-\sqrt{2}\pi$$

$$\Rightarrow \quad 4\sqrt{2}x+y(8-4\sqrt{2}) = 2\pi-\sqrt{2}\pi$$

$$\Rightarrow \quad 4\sqrt{2}x+(8-4\sqrt{2})y = \pi(2-\sqrt{2})$$

S62. Suppose the normal at $P(x_1, y_1)$ on the parabola $x^2 = 4y$ passes through the point (1, 2).

We have, $\quad x^2 = 4y$

$$\Rightarrow \quad 2x = 4\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x}{2}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_P = \frac{x_1}{2}$$

The equation of the normal at $P(x_1, y_1)$ is

$$y-y_1 = \frac{-1}{\left(\frac{dy}{dx}\right)_P}(x-x_1)$$

$\Rightarrow$ $$y - y_1 = -\frac{2}{x_1}(x - x_1) \quad \text{... (i)}$$

It passes through (1, 2).

$\therefore$ $$2 - y_1 = -\frac{2}{x_1}(1 - x_1)$$

$\Rightarrow$ $$2 - y_1 = -\frac{2}{x_1} + 2$$

$\Rightarrow$ $$x_1 y_1 = 2 \quad \text{... (ii)}$$

Also, $P(x_1, y_1)$ lies on $x^2 = 4y$

$\therefore$ $$x_1^2 = 4y_1 \quad \text{... (iii)}$$

Eliminating y_1 between (ii) and (iii), we have

$$\frac{x_1^3}{4} = 2 \;\Rightarrow\; x_1^3 = 8 \;\Rightarrow\; x_1 = 2$$

Putting $x_1 = 2$ in (ii), we get $y_1 = 1$

Putting the values of x_1 and y_1 in (i), we get

$$y - 1 = -1(x - 2)$$

$\Rightarrow$ $$x + y - 3 = 0$$

This is the required equation of the normal.

S63. We have, $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$

$\Rightarrow$ $$\frac{dx}{d\theta} = a(-\sin\theta + \theta\cos\theta + \sin\theta)$$

and $$\frac{dy}{d\theta} = a(\cos\theta - \cos\theta + \theta\sin\theta)$$

$\Rightarrow$ $$\frac{dx}{d\theta} = a\theta\cos\theta \quad \text{and} \quad \frac{dy}{d\theta} = a\theta\sin\theta$$

$\therefore$ $$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$$

The equation of the normal at any point θ is

$\Rightarrow$ $$y - a(\sin\theta - \theta\cos\theta) = -\frac{1}{\left(\frac{dy}{dx}\right)}\{x - a(\cos\theta + \theta\sin\theta)\}$$

$\Rightarrow \quad y - a(\sin\theta - \theta\cos\theta) = -\dfrac{\cos\theta}{\sin\theta}\{x - a(\cos\theta + \theta\sin\theta)\}$

$\Rightarrow \quad y\sin\theta - a\sin^2\theta + a\theta\sin\theta\cos\theta = -x\cos\theta + a\cos^2\theta + a\theta\sin\theta\cos\theta$

$\Rightarrow \quad x\cos\theta + y\sin\theta = a(\cos^2\theta + \sin^2\theta)$

$\Rightarrow \quad x\cos\theta + y\sin\theta = a$ ... (i)

Now, Length of the perpendicular from the origin (0, 0) to (i)

$$= \left|\frac{0\cos\theta + 0\sin\theta - a}{\sqrt{\cos^2\theta + \sin^2\theta}}\right| = a = \text{Constant}$$

Hence, the normal at any point θ to the given curve is at a constant distance from the origin.

S64. We have, $y = ax^3 + bx^2 + cx + 5$

$\Rightarrow \quad \dfrac{dy}{dx} = 3ax^2 + 2bx + c$

Since the curve $y = ax^3 + bx^2 + cx + 5$ touches the x-axis at $P(-2, 0)$. This means that the curve passes through $P(-2, 0)$ and x-axis is the tangent at $P(-2, 0)$.

$\therefore \quad 0 = -8a + 4b - 2c + 5$

$\Rightarrow \quad 8a - 4b + 2c = 5$ ... (i)

and, $\left(\dfrac{dy}{dx}\right)_P = 0$

$\Rightarrow \quad 3a(-2)^2 + 2b \times (-2) + c = 0$

$\Rightarrow \quad 12a - 4b + c = 0$ ... (ii)

The curve $y = ax^3 + bx^2 + cx + 5$ meets y-axis at Q.

Putting $x = 0$ in $y = ax^3 + bx^2 + cx + 5$, we get $y = 5$

Thus, the coordinates of Q are (0, 5).

It is given that the gradient of the curve at Q is 3.

$\therefore \quad \left(\dfrac{dy}{dx}\right)_Q = 3$

$\Rightarrow \quad 3a \times 0 + 2b \times 0 + c = 3$

$\Rightarrow \quad c = 3$

Putting $c = 3$ in (i) and (ii), we get

$$8a - 4b = -1 \quad \text{and} \quad 12a - 4b = -3$$

Solving these two equations, we get $a = -\dfrac{1}{2}$ and $b = -\dfrac{3}{4}$

Substituting the values of a, b and c in the equation of the curve, we obtain

$$y = -\frac{1}{2}x^3 - \frac{3}{4}x^2 + 3x + 5$$

As the equation of the curve.

S65. Let the required quadratic curve be

$$y = ax^2 + bx + c \qquad \dots \text{(i)}$$

$$\frac{dy}{dx} = 2ax + b$$

It passes through (–1, 0). Therefore,

$$0 = a - b + c \qquad \dots\text{(ii)}$$

Since the line $y = x$ touches (i) at $x = 1$. Therefore,

(Slope of the tangent at $x = 1$) = (Slope of the line $y = x$)

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{x=1} = 1$$

$$\Rightarrow \quad 2a + b = 1 \qquad \dots \text{(iii)}$$

Putting, $x = 1$ in $y = x$, we get $y = 1$

Thus, the curve (i) passes through (1, 1)

$$\Rightarrow \quad 1 = a + b + c \qquad \dots \text{(iv)}$$

Solving (ii), (iii) and (iv), we get

$$a = \frac{1}{4}, \; b = \frac{1}{2} \text{ and } c = \frac{1}{4}$$

Substituting these values in (i), we get

$$y = \frac{x^2}{4} + \frac{x}{2} + \frac{1}{4}$$ is the required quadratic curve.

S66. Given equation of curve is

$$3x^2 - y^2 = 8 \qquad \dots \text{(i)}$$

On differentiation w.r.t. x, we get

$$6x - 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3x}{y}$$

Equation of tangent at point (h, k) is

$$y - k = \left.\frac{dy}{dx}\right|_{(h,k)} (x - h)$$

$$\Rightarrow \quad y - k = \frac{3h}{k}(x - h) \quad \text{... (ii)}$$

Since, it is passes through the point $\left(\frac{4}{3}, 0\right)$

$$\therefore \quad 0 - k = \frac{3h}{k}\left(\frac{4}{3} - h\right)$$

$$\Rightarrow \quad -k^2 = 3h\,\frac{(4 - 3h)}{3}$$

$$\Rightarrow \quad 3h^2 - k^2 - 4h = 0 \quad \text{... (iii)}$$

Also the point (h, k) lies on the Eq. (i)

$$\therefore \quad 3h^2 - k^2 = 8 \quad \text{... (iv)}$$

Now, from Eqs.(iii) and (iv), we get

$$4h = 8 \Rightarrow h = 2$$

On putting $h = 2$ in Eqs. (iv), we get

$$3(2)^2 - k^2 = 8$$

$$\Rightarrow \quad k^2 = 4$$

$$\Rightarrow \quad k = \pm 2$$

Now, putting the value of h and k in Eq. (ii), we get

$$y - (\pm 2) = \frac{3(2)}{\pm 2}(x - 2)$$

$$\Rightarrow \quad y \mp 2 = \pm 3(x - 2)$$

$$\Rightarrow \quad y = \pm 3x \mp 6 \pm 2$$

It will gives four possible equations but in there of them only $y = -3x + 4$ and $y = 3x - 4$ satisfies the point $\left(\frac{4}{3}, 0\right)$.

Hence the equation of tangents are $y = -3x + 4$ and $y = 3x - 4$

S67. Differentiating $x^2 = 4y$ w.r.t. x we get

$$2x = 4\,\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x}{2}$$

Let (h, k) be the coordinates of the points of contact of the normal to the curve $x^2 = 4y$. Now slope of the tangent at (h, k) is given by $\left[\frac{dy}{dx}\right]_{(h,k)} = \frac{h}{2}$

Hence, slope of the normal at $(h, k) = \frac{-2}{h}$ $[\because m_1m_2 = -1]$

Therefore, the equation of normal at (h, k) is

$$y - k = \frac{-2}{h}(x - h) \quad \text{... (i)}$$

[$\because$ Equation of normal in slope form is $y - y_1 = -\frac{1}{m}(x - x_1)$]

Since, it passes through the point (1, 2), we have

$$2 - k = \frac{-2}{h}(1 - h)$$

$$\Rightarrow \quad k = 2 + \frac{2}{h}(1 - h) \quad \text{... (ii)}$$

Since, (h, k) also lies on the curve $x^2 = 4y$, we have

$$h^2 = 4k \quad \text{... (iii)}$$

On solving Eqs.(ii) and (iii), we have $h = 2$ and $k = 1$. Substituting the values of h and k in Eq. (i), we get the required equation of normal

$$y - 1 = \frac{-2}{2}(x - 2)$$

$$\Rightarrow \quad x + y = 3$$

Now, equation of tangent at (h, k) is

$$y - k = \frac{h}{2}(x - h)$$

Put $h = 2$ and $k = 1$, we get

$$y - 1 = \frac{2}{2}(x - 2)$$

$$\Rightarrow \quad y - 1 = x - 2$$

$$\Rightarrow \quad y = x - 1.$$

Date: 18/10/2021

Q1. Find the approximate value of $f(2.01)$, where $f(x) = 4x^2 + 5x + 2$.

Q2. Find the approximate value of $f(5.001)$, where $f(x) = x^3 - 7x^2 + 15$.

Q3. If the radius of a sphere is measured as 9 m with an error of 0.03m, find the approximate error in calculating it's surface area.

Q4. Find the approximate change in the volume V of a cube of side x meters caused by increasing the side by 2%.

Q5. If $y = x^4 - 10$ and if x changes from 2 to 1.99, what is the approximate change in y? Also find the changed value of y.

Q6. If $y = \sin x$ and x changes from $\frac{\pi}{2}$ to $\frac{22}{14}$, what is the approximate change in y?

Q7. The radius of a sphere shrinks from 10 to 9.8 cm. Find approximately the decrease in its volume.

Q8. If the radius of a sphere is measured as 7 m with an error of 0.02 m, find the approximate error in calculating it's volume.

Q9. A circular metal plate expands under heating so that it's radius increases by 2%. Find the approximate increase in the area of the plate if the radius of the plate before heating is 10 cm.

Q10. Using differentials, find the approximate value of $(0.999)^{\frac{1}{10}}$ upto 3 places of decimal.

Q11. If the radius of a sphere is measured as 9 cm with an error of 0.03 cm, then find the approximate error in calculating its volume.

Q12. If the radius of sphere is measured as 9 cm with an error of 0.03 cm, then find the approximate error in calculating it's surface area.

Q13. Find the approximate change in the surface area of a cube of side x metres caused by decreasing the side by 1%.

Q14. Find the percentage error in calculating the volume of a cubical box if an error of 1% is made in measuring the lengths of edges of the cube.

Q15. The time T of a complete oscillation of a simple pendulum of length l is given by the equation $T = 2\pi\sqrt{\frac{l}{g}}$, where g is constant. What is the percentage error in T when l is increased by 1%?

Q16. Use differentials to approximate $\sqrt{25.2}$.

Q17. Use differentials to find the approximate value of $\sqrt{0.037}$.

Q18. Use differentials to approximate the cube root of 127.

Q19. Using differentials, find approximate value of $\sqrt{49.5}$.

Q20. Using differentials, find the approximate value of $\sqrt{0.6}$ upto 3 places of decimal.

Q21. Using differentials, find the approximate value of $(255)^{\frac{1}{4}}$ upto 3 places of decimal.

Q22. Using differentials, find the approximate value of $(82)^{\frac{1}{4}}$ upto 3 places of decimal.

Q23. Using differentials, find the approximate value of $(81.5)^{\frac{1}{4}}$ upto 3 places of decimal.

Q24. Find the approximate value of $f(3.02)$, where $f(x) = 3x^2 + 5x + 3$.

Q25. Using differentials, find the approximate value of $(32.15)^{\frac{1}{5}}$ upto 3 places of decimal.

Q26. Using differentials, find the approximate value of $(3.968)^{\frac{3}{2}}$ upto 3 places of decimal.

Q27. Using differentials, find the approximate value of $(26.57)^{\frac{1}{3}}$ upto 3 places of decimal.

Q28. If $f(x) = 3x^2 + 15x + 5$, find the approximate value of $f(3.02)$ using differentials.

Q29. Use differentials to find the approximate value of $\log_e(4.01)$, having given that $\log_e 4 = 1.3863$.

Q30. Using differentials find the approximate value of tan 46°, if it is being given that 1° = 0.01745 radians.

Date: 18/10/2021

S1. We have, $f(x) = 4x^2 + 5x + 2$

Let $x = 2$ and $x + \Delta x = 2.01$ then $\Delta x = 0.01$

$\therefore$ $f(2) = 4 \times 4 + 5 \times 2 + 2 = 28$

and $\dfrac{df}{dx} = 8x + 5$

$\Rightarrow$ $\left(\dfrac{df}{dx}\right)_{x=2} = 8 \times 2 + 5 = 21$

Let $f(2.01) = f(2) + \Delta f.$ ... (i)

Then, $\Delta f = \dfrac{df}{dx}\Delta x$

$\Rightarrow$ $\Delta f = 21 \times 0.01 = 0.21$

$\therefore$ $f(2.01) = 28 + 0.21 = 28.21$ [Using Eq. (i)]

S2. We have, $f(x) = x^3 - 7x^2 + 15$

Let $x = 5$ and $x + \Delta x = 5.001$. Then, $\Delta x = 0.001$

$\therefore$ $f(5) = 5^3 - 7 \times 5^2 + 15 = 125 - 175 + 15 = -35$

$\Rightarrow$ $\dfrac{df}{dx} = 3x^2 - 14x$

Also $\left(\dfrac{df}{dx}\right)_{x=5} = 3 \times 5^2 - 14 \times 5 = 5$ ...(i)

Let $f(5.001) = f(5) + \Delta f.$

Then, $\Delta f = \dfrac{df}{dx}\Delta x$

$\Rightarrow$ $\Delta f = 5 \times 0.001 = 0.005$

Hence, $f(5.001) = (125 - 175 + 15) + 0.005$ [Using Eq. (i)]

$= -35 + 0.005 = -34.995$

S3. Let r be the radius and S be the surface area of the sphere. Then,

$S = 4\pi r^2$

$\Rightarrow$ $\dfrac{dS}{dr} = 8\pi r$

It is given that $r = 9$ and $\Delta r = 0.03$. Therefore, error ΔS in S is given by

$$\Delta S = \left(\frac{dS}{dr}\right)_{r=9} \Delta r = 8\pi \times 9 \times 0.03 = 2.16\ \pi m^2$$

Hence, approximate error in calculating surface area is $2.16\ \pi m^2$.

S4. Let Δx be the change in x and ΔV be the corresponding change in V. It is given that $\frac{\Delta x}{x} \times 100 = 2$.

We have, $V = x^3$

$\Rightarrow$ $$\frac{dV}{dx} = 3x^2$$

$\therefore$ $$\Delta V = \frac{dV}{dx} \Delta x$$

$\Rightarrow$ $$\Delta V = 3x^2\ \Delta x$$

$\Rightarrow$ $$\Delta V = 3x^2 \times \frac{2x}{100} \qquad \left[\because \frac{\Delta x}{x} \times 100 = 2 \Rightarrow \Delta x = \frac{2x}{100}\right]$$

$\Rightarrow$ $$\Delta V = 0.06\ x^3\ m^3$$

Thus, the approximate change in volume is $0.06\ x^3\ m^3$.

S5. Let $x = 2$, $x + \Delta x = 1.99$. Then, $\Delta x = 1.99 - 2 = -0.01$

Let, $dx = \Delta x = -0.01$

We have, $y = x^4 - 10$

$\Rightarrow$ $$\frac{dy}{dx} = 4x^3$$

$\Rightarrow$ $$\left(\frac{dy}{dx}\right)_{x=2} = 4(2)^3 = 32$$

$\therefore$ $$\Delta y = \frac{dy}{dx} \Delta x$$

$\Rightarrow$ $$\Delta y = 32\ (-0.01) = -0.32$$

$\Rightarrow$ $$\Delta y = -0.32 \text{ approximately} \qquad [\because\ \Delta y \cong dy]$$

So, approximate change in y

$$\Delta y = -0.32$$

When $x = 2$, we have, $y = 2^4 - 10 = 6$

So, changed value of $y = y + \Delta y = 6 + (-0.32) = 5.68$

S6. Take $x = \frac{\pi}{2}$, $x + \Delta x = \frac{22}{14}$

$\therefore$ $dx = \Delta x = \frac{22}{14} - \frac{\pi}{2}$

Now, $y = \sin x$

$\Rightarrow$ $\frac{dy}{dx} = \cos x$

$\Rightarrow$ $\left(\frac{dy}{dx}\right)_{x=\frac{\pi}{2}} = \cos\frac{\pi}{2} = 0$

$\therefore$ $\Delta y = \frac{dy}{dx}\Delta x = 0\left(\frac{22}{14} - \frac{\pi}{2}\right) = 0$

$\Rightarrow$ $\Delta y = 0$

So approximate change in y = Δy = 0.

S7. Let x be the radius and y be the volume. Then,

$$y = \frac{4}{3}\pi x^3$$

Let $x = 10$, $x + \Delta x = 9.8$. Then $\Delta x = -0.2$

Now, $y = \frac{4}{3}\pi x^3$

$\Rightarrow$ $\frac{dy}{dx} = 4\pi x^2$

$\Rightarrow$ $\left(\frac{dy}{dx}\right)_{x=10} = 400\pi$

$\therefore$ $\Delta y = \frac{dy}{dx}\Delta x$

$\Rightarrow$ $dy = 400\pi\ (-0.2) = -80\pi\ \text{Cm}^3$

$\therefore$ Approximate change in volume = -80π Cm^3.

S8. Let r be the radius and V be the volume of the sphere. Then,

$$V = \frac{4}{3}\pi r^3$$

$\Rightarrow$ $\frac{dV}{dr} = 4\pi r^2$

Let Δr be the error in r and the corresponding error in V be ΔV. Then,

$$\Delta V = \frac{dV}{dr}\Delta r = 4\pi r^2 \Delta r$$

It is given that $r = 7$ and $\Delta r = 0.02$

$\therefore$ $$\Delta V = 4\pi \times 7^2 \times 0.02 = 3.92\pi\ m^3$$

Hence, approximate error in calculating volume is $3.92\ \pi\ m^3$.

S9. Let at any time, x be the radius and y be the area of the plate. Then, $y = \pi x^2$.

Let Δx be the change in the radius and let Δy be the corresponding change in the area of the plate. Then,

$$\frac{\Delta x}{x} \times 100 = 2 \text{ (Given)}$$

When $x = 10$, $$\frac{\Delta x}{x} \times 100 = 2$$

$\Rightarrow$ $$\frac{\Delta x}{10} \times 100 = 2$$

$\Rightarrow$ $$\Delta x = \frac{2}{10} \qquad \{ dx \approxeq \Delta x \}$$

Now, $$y = \pi x^2$$

$\Rightarrow$ $$\frac{dy}{dx} = 2\pi x$$

$\Rightarrow$ $$\left(\frac{dy}{dx}\right)_{x=10} = 20\pi$$

$\therefore$ $$\Delta y = \frac{dy}{dx}\Delta x$$

$\Rightarrow$ $$\Delta y = 20\pi \times \frac{2}{10} = 4\pi$$

$\Rightarrow$ $$\Delta y = 4\pi\ \text{cm}^2 \qquad [\because dy \cong \Delta y]$$

Hence, the approximate change in the area of the plate is $4\pi\ \text{cm}^2$.

S10. Let $y = x^{\frac{1}{10}}$ $$\frac{dy}{dx} = \frac{1}{10x^{\frac{9}{10}}}$$

Let $x = 1$ and $x + \Delta x = 0.999$

$\therefore$ $$\Delta x = (x + \Delta x) - x$$
$$= 0.999 - 1 = -0.001$$

$$\Delta y = (x + \Delta x)^{\frac{1}{10}} - (x)^{\frac{1}{10}}$$

$$= (0.999)^{\frac{1}{10}} - (1)^{\frac{1}{10}}$$

$$= (0.999)^{\frac{1}{10}} - 1$$

$\therefore \qquad (0.999)^{\frac{1}{10}} = 1 + \Delta y$ ... (i)

Since dy is approximately equal to Δy and is given by

$$\Delta y = \frac{dy}{dx} \cdot \Delta x$$

$$= \frac{1}{10x^{\frac{9}{10}}} \times (-0.001)$$

$$= \frac{(-0.001)}{10 \times (1)^{\frac{9}{10}}}$$

$$= \frac{(-0.001)}{10} = -0.0001$$

$\therefore \qquad (0.999)^{\frac{1}{10}} = 1 - 0.0001 = 0.9999$ [Using Eq. (i)]

Thus approximate value of $(0.999)^{\frac{1}{10}}$ is 0.9999.

S11. Let r be the radius of the sphere and Δr be the error in measuring the radius. Then,

$$r = 9 \text{ cm and } \Delta r = 0.03 \text{ cm}$$

Let V be the volume of the sphere. Then,

$$V = \frac{4}{3}\pi r^3$$

$$\Rightarrow \qquad \frac{dV}{dr} = 4\pi r^2$$

$$\Rightarrow \qquad \left(\frac{dV}{dr}\right)_{r=9} = 4\pi \times 9^2$$

$$= 324\,\pi$$

Let ΔV be the error in V due to error Δr in r. Then,

$$\Delta V = \frac{dV}{dr}\Delta r$$

$$\Rightarrow \qquad \Delta V = 324\pi \times 0.03$$

$$= 9.72\,\pi \text{ cm}^3$$

Thus, the approximate error in calculating the volume is 9.72π cm^3.

S12. Let S be the surface area, r be the radius of the sphere. Now, given that $r = 9$ cm

Let dr = approximate error in radius r

and dS = approximate error in surface area.

Now, we know that surface area of sphere is given by

$$S = 4\pi r^2$$

Differentiating both sides w.r.t. r, we get

$$\frac{dS}{dr} = 4\pi \times 2r = 8\pi r$$

$\Rightarrow$ $dS = 8\pi r \cdot dr$

$\therefore$ $dS = 8\pi \times 9 \times 0.03$ $[\because\ r = 9 \text{ cm and } dr = 0.03 \text{ cm}]$

$\Rightarrow$ $dS = 72 \times 0.03\pi$

$\therefore$ $dS = 2.16\pi \text{ cm}^2$

Hence, approximate error in surface area = 2.16π cm^2.

S13. Let S be the surface area of the cube of edge length x metres. Then,

$$S = 6x^2$$

$\Rightarrow$ $$\frac{dS}{dx} = 12x$$

Let Δx be the decrease in its edge and let the corresponding decrease in S be ΔS. Then

$$\Delta S = \frac{dS}{dx}\Delta x$$

$\Rightarrow$ $\Delta S = 12x\ \Delta x$

$\Rightarrow$ $$\frac{\Delta S}{S} = \frac{12x\ \Delta x}{6x^2}$$

$\Rightarrow$ $$\frac{\Delta S}{S} = 2\frac{\Delta x}{x}$$

$\therefore$ $$\frac{\Delta S}{S} \times 100 = \frac{2\Delta x}{x} \times 100$$

$$= 2 \times 1 = 2 \qquad \left[\because \frac{\Delta x}{x} \times 100 = 1\right]$$

Thus, the approximate change in surface area is 2%.

S14. Let x be the length of an edge of the cube and y be its volume. Then, y = x^3. Let Δx. be the error in x and Δy be the corresponding error in y. Then,

$$\frac{\Delta x}{x} \times 100 = 1 \text{ (given)} \qquad \text{... (i)}$$

Now, $y = x^3$

$\Rightarrow \quad \frac{dy}{dx} = 3x^2$

$\Rightarrow \quad \Delta y = \frac{dy}{dx} \Delta x$

$\Rightarrow \quad \Delta y = 3x^2 \cdot \Delta x$

$\Rightarrow \quad \frac{\Delta y}{y} = \frac{3x^2}{y} \cdot \Delta x$

$\Rightarrow \quad \frac{\Delta y}{y} = \frac{3x^2}{x^3} \Delta x$

$\Rightarrow \quad \frac{\Delta y}{y} = 3\frac{\Delta x}{x}$

$\Rightarrow \quad \frac{\Delta y}{y} \times 100 = 3\left(\frac{\Delta x}{x} \times 100\right) = 3$ From (i)

$\Rightarrow \quad \frac{\Delta y}{y} \times 100 = 3$

So, there is 3% error in calculating the volume of the cube.

S15. Let Δl be the change in l and ΔT be the corresponding error in T. Then,

$\frac{\Delta l}{l} \times 100 = 1$ (given) ...(i)

Now, $T = 2\pi\sqrt{\frac{l}{g}}$

taking log both side

$\Rightarrow \quad \log T = \log 2\pi + \left(\frac{1}{2}\right)\log l - \left(\frac{1}{2}\right)\log g$

Differentiating w.r.t. l both side

$\Rightarrow \quad \frac{1}{T}\frac{dT}{dl} = \frac{1}{2} \cdot \frac{1}{l}$

$\Rightarrow \quad \frac{dT}{T} = \frac{1}{2}\frac{dl}{l}$

$\Rightarrow \quad \frac{dT}{T} \times 100 = \frac{1}{2}\left(\frac{dl}{l} \times 100\right)$

$\Rightarrow \quad \frac{dT}{T} \times 100 = \frac{1}{2}$ [Using (i)]

So, there is $\left(\frac{1}{2}\right)$% error in calculating the time period T.

S16. Consider the function $y = f(x) = \sqrt{x}$

Let, $x = 25$ and $x + \Delta x = 25.2$. Then,

$$\Delta x = 25.2 - 25 = 0.2$$

For $x = 25$, we have $\quad y = \sqrt{25} = 5 \quad$ [Putting $x = 25$ in $y = \sqrt{x}$]

Now, $\quad y = \sqrt{x}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{x=25} = \frac{1}{2(5)} = \frac{1}{10}$$

$$\therefore \quad \Delta y = \frac{dy}{dx}\Delta x$$

$$\Rightarrow \quad \Delta y = \frac{1}{10}(0.2) = 0.02$$

$$\Rightarrow \quad \Delta y = 0.02$$

Hence, $\quad \sqrt{25.2} = y + \Delta y = 5 + 0.02 = 5.02$

S17. Let $y = f(x) = \sqrt{x}$, $x = 0.040$ and $x + \Delta x = 0.037$. Then,

$$\Delta x = 0.037 - 0.040 = -0.003$$

For $\quad x = 0.040$, we have

$$y = \sqrt{0.040} = 0.2 \quad \left[\text{Putting } x = 0.040 \text{ in } y = \sqrt{x}\right]$$

Now, $\quad y = \sqrt{x}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{x=0.040} = \frac{1}{2\sqrt{0.040}} = \frac{1}{0.4}$$

$$\therefore \quad \Delta y = \frac{dy}{dx}\Delta x$$

$$\Rightarrow \quad \Delta y = \frac{1}{0.4}(-0.003) = -\frac{3}{400}$$

$$\Rightarrow \quad \Delta y = -\frac{3}{400}$$

Hence, $\sqrt{0.037} = y + \Delta y = 0.2 - \frac{3}{400}$

$= 0.2 - 0.0075 = 0.1925$

S18. Since we have to find the approximate value of the cube root of 127. So, we consider the function $y = f(x) = x^{\frac{1}{3}}$.

Let $x = 125$ and $x + \Delta x = 127$. Then,

$$\Delta x = 127 - 125 = 2$$

For $x = 125$, we have $y = (125)^{\frac{1}{3}} = 5$ $\left[\text{Putting } x = 125 \text{ in } y = x^{\frac{1}{3}}\right]$

Now, $y = x^{\frac{1}{3}}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{3x^{\frac{2}{3}}}$

$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{x=125} = \frac{1}{3(125)^{\frac{2}{3}}}$

$= \frac{1}{3(5^3)^{\frac{2}{3}}} = \frac{1}{75}$

$\therefore \quad \Delta y = \frac{dy}{dx}\Delta x$

$\Rightarrow \quad \Delta y = \frac{1}{75}(2) = \frac{2}{75}$

$\Rightarrow \quad \Delta y = \frac{2}{75}$

Hence, $(127)^{\frac{1}{3}} = y + \Delta y = 5 + \frac{2}{75} = 5.026$

S19. Let $x = 49$, $\Delta x = 0.5$

$y = \sqrt{x}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

$\Rightarrow \quad \left[\frac{dy}{dx}\right]_{x=49} = \frac{1}{2\sqrt{49}} = \frac{1}{2\times 7} = \frac{1}{14}$

We know that, $\Delta y = \frac{dy}{dx} \cdot \Delta x$

$$\therefore \quad \Delta y = \frac{1}{14} \times 0.5 = \frac{5}{140} = \frac{1}{28}$$

$$\therefore \quad \sqrt{49.5} = y + \Delta y = \sqrt{49} + \frac{1}{28} = 7 + \frac{1}{28}$$

$$= \frac{196+1}{28} = \frac{197}{28} \qquad [\because y = \sqrt{x} = \sqrt{49} = 7]$$

$$= 7.035.$$

S20. Let $y = \sqrt{x}$ $\qquad \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

Let $x = 0.64$ and $x + \Delta x = 0.6$

$$\therefore \quad \Delta x = (x + \Delta x) - x$$

$$= 0.6 - 0.64 = -0.04$$

$$\Delta y = \sqrt{x + \Delta x} - \sqrt{x}$$

$$= \sqrt{0.6} - \sqrt{0.64} = \sqrt{0.6} - 0.8$$

$$\therefore \quad \sqrt{0.6} = 0.8 + \Delta y \qquad \ldots \text{(i)}$$

Since dy is approximately equal to Δy and is given by

$$\Delta y = \frac{dy}{dx} \cdot \Delta x$$

$$\Delta y = \frac{1}{2\sqrt{x}} \times (-0.04)$$

$$= \frac{-0.04}{2\sqrt{0.64}} = \frac{-0.04}{1.6} = -0.025$$

$$\therefore \quad \sqrt{0.6} = 0.8 - 0.025 = 0.775 \qquad \text{[Using Eq. (i)]}$$

Thus approximate value of $\sqrt{0.6}$ is 0.775.

S21. Let $y = x^{\frac{1}{4}}$

$$\frac{dy}{dx} = \frac{1}{4x^{\frac{3}{4}}}$$

Let $x = 256$ and $x + \Delta x = 255$

$$\therefore \quad \Delta x = (x + \Delta x) - x$$

$$= 255 - 256 = -1$$

$$\Delta y = (x + \Delta x)^{\frac{1}{4}} - (x)^{\frac{1}{4}}$$

$$= (255)^{\frac{1}{4}} - (256)^{\frac{1}{4}}$$

$$= (255)^{\frac{1}{4}} - 4$$

$\therefore$ $$(255)^{\frac{1}{4}} = 4 + \Delta y \quad \text{... (i)}$$

Since dy is approximately equal to Δy and is given by

$$\Delta y = \frac{dy}{dx} \cdot \Delta x = \frac{1}{4x^{\frac{3}{4}}} \times (-1)$$

$$= \frac{-1}{4 \times (256)^{\frac{3}{4}}}$$

$$= \frac{-1}{4 \times 64} = -0.004$$

$\therefore$ $$(255)^{\frac{1}{4}} = 4 - 0.004 = 3.996 \quad \text{[Using Eq. (i)]}$$

Thus approximate value of $(255)^{\frac{1}{4}}$ is 3.996.

S22. Let $y = x^{\frac{1}{4}}$ $$\frac{dy}{dx} = \frac{1}{4x^{\frac{3}{4}}}$$

Let $x = 81$ and $x + \Delta x = 82$

$\therefore$ $$\Delta x = (x + \Delta x) - x$$

$$= 82 - 81 = 1$$

$$\Delta y = (x + \Delta x)^{\frac{1}{4}} - (x)^{\frac{1}{4}}$$

$$= (81)^{\frac{1}{4}} - (82)^{\frac{1}{4}}$$

$$= (82)^{\frac{1}{4}} - 3$$

$\therefore$ $$(82)^{\frac{1}{4}} = 3 + \Delta y \quad \text{... (i)}$$

Since dy is approximately equal to Δy and is given by

$$\Delta y = \frac{dy}{dx} \cdot \Delta x$$

$$= \frac{1}{4x^{\frac{3}{4}}} \times 1$$

$$= \frac{1}{4 \times (81)^{\frac{3}{4}}}$$

$$= \frac{1}{4 \times 27} = 0.009$$

$\therefore \quad (82)^{\frac{1}{4}} = 3 + 0.009 = 3.009$ [Using Eq. (i)]

Thus approximate value of $(82)^{\frac{1}{4}}$ is 3.009.

S23. Let $y = x^{\frac{1}{4}}$ $\quad \frac{dy}{dx} = \frac{1}{4x^{\frac{3}{4}}}$

Let $x = 81$ and $x + \Delta x = 81.5$

$$\therefore \quad \Delta x = (x + \Delta x) - x$$

$$= 81.5 - 81 = 0.5$$

$$\Delta y = (x + \Delta x)^{\frac{1}{4}} - (x)^{\frac{1}{4}}$$

$$= (81.5)^{\frac{1}{4}} - (81)^{\frac{1}{4}}$$

$$= (81.5)^{\frac{1}{4}} - 3$$

$\therefore \quad (81.5)^{\frac{1}{4}} = 3 + \Delta y$ ... (i)

Since dy is approximately equal to Δy and is given by

$$\Delta y = \frac{dy}{dx} \cdot \Delta x$$

$$= \frac{1}{4x^{\frac{3}{4}}} \times (0.5)$$

$$= \frac{0.5}{4 \times (81)^{\frac{3}{4}}}$$

$$= \frac{0.5}{4 \times 27} = 0.0046$$

$\therefore \quad (81.5)^{\frac{1}{4}} = 3 + 0.0046 = 3.0046$ [Using Eq. (i)]

Thus approximate value of $(81.5)^{\frac{1}{4}}$ is 3.0046.

S24. Let $y = f(x)$, $x = 3$ and $x + \Delta x = 3.02$. Then, $\Delta x = 0.02$

For $x = 3$, we have $\quad y = f(3) = 3 \times 3^2 + 5 \times 3 + 3 = 45$

Now, $\quad y = f(x)$

$\Rightarrow \quad y = 3x^2 + 5x + 3$

$\Rightarrow \quad \dfrac{dy}{dx} = 6x + 5$

$\Rightarrow \quad \left(\dfrac{dy}{dx}\right)_{x=3} = 6 \times 3 + 5 = 23$

Let Δy be the change in y due to change of Δx in x. Then,

$$\Delta y = \frac{dy}{dx}\Delta x$$

$\Rightarrow \quad \Delta y = 23 \times 0.02 = 0.46$

$\therefore \quad f(3.02) = y + \Delta y$

$$= 45 + 0.46 = 45.46$$

S25. Let $y = x^{\frac{1}{5}}$ $\quad \dfrac{dy}{dx} = \dfrac{1}{5x^{\frac{4}{5}}}$

Let $x = 32$ and $x + \Delta x = 32.15$

$\therefore \quad \Delta x = (x + \Delta x) - x = 32.15 - 32 = 0.15$

$$\Delta y = (x + \Delta x)^{\frac{1}{5}} - (x)^{\frac{1}{5}}$$

$$= (32.15)^{\frac{1}{5}} - (32)^{\frac{1}{5}}$$

$$= (32.15)^{\frac{1}{5}} - 2$$

$\therefore \quad (32.15)^{\frac{1}{5}} = 2 + \Delta y \quad \ldots \text{(i)}$

Since dy is approximately equal to Δy and is given by

$$\Delta y = \frac{dy}{dx} \cdot \Delta x$$

$$= \frac{1}{5x^{\frac{4}{5}}} \cdot 0.15$$

$$= \frac{0.15}{5\,(32)^{\frac{4}{5}}}$$

$$= \frac{0.15}{5 \times 16} = 0.0019$$

$\therefore \quad (32.15)^{\frac{1}{5}} = 2 + 0.0019 = 2.0019$ [Using Eq. (i)]

Thus approximate value of $(32.15)^{\frac{1}{5}}$ is 2.0019.

S26. Let $y = x^{\frac{3}{2}}$ $\quad \frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}}$

Let $x = 4$ and $x + \Delta x = 3.968$

$$\therefore \quad \Delta x = (x + \Delta x) - x$$

$$= 3.968 - 4 = -0.032$$

$$\Delta y = (x + \Delta x)^{\frac{3}{2}} - (x)^{\frac{3}{2}}$$

$$= (3.968)^{\frac{3}{2}} - (4)^{\frac{3}{2}}$$

$$= (3.968)^{\frac{3}{2}} - 8$$

$$\therefore \quad (3.968)^{\frac{3}{2}} = 8 + \Delta y \qquad \text{... (i)}$$

Since dy is approximately equal to Δy and is given by

$$\Delta y = \frac{dy}{dx} \cdot \Delta x$$

$$= \frac{3}{2}x^{\frac{1}{2}} \cdot (-0.032)$$

$$= \frac{3}{2} \times (4)^{\frac{1}{2}} \times (-0.032)$$

$$= \frac{3}{2} \times 2 \times (-0.032) = -0.096$$

$\therefore \quad (3.968)^{\frac{3}{2}} = 8 - 0.096 = 7.904$ [Using Eq. (i)]

Thus approximate value of $(3.968)^{\frac{3}{2}}$ is 7.904.

S27. Let $y = x^{\frac{1}{3}}$ $\quad \frac{dy}{dx} = \frac{1}{3x^{\frac{2}{3}}}$

Let $x = 27$ and $x + \Delta x = 26.57$

$$\therefore \quad \Delta x = (x + \Delta x) - x$$

$$= 26.57 - 27 = -0.43$$

$$\Delta y = (x + \Delta x)^{\frac{1}{3}} - (x)^{\frac{1}{3}}$$

$$= (26.57)^{\frac{1}{3}} - (27)^{\frac{1}{3}}$$

$$= (26.57)^{\frac{1}{3}} - 3$$

$\therefore \qquad (26.57)^{\frac{1}{3}} = 3 + \Delta y$ ... (i)

Since dy is approximately equal to Δy and is given by

$$\Delta y = \frac{dy}{dx} \cdot \Delta x$$

$$= \frac{1}{3x^{\frac{2}{3}}} \times (-0.43)$$

$$= \frac{-0.43}{3 \times (27)^{\frac{2}{3}}}$$

$$= \frac{-0.43}{3 \times 9} = -0.016$$

$\therefore \qquad (26.57)^{\frac{1}{3}} = 3 - 0.016 = 2.984$ [Using Eq. (i)]

Thus approximate value of $(26.57)^{\frac{1}{3}}$ is 2.984.

S28. Given function is

$$f(x) = 3x^2 + 15x + 5$$

Let $x = 3$

and $\Delta x = 0.02$, so that

$$f(x + \Delta x) = f(3.02)$$

Using $f(x + \Delta x) = f(x) + \Delta x \cdot f'(x)$

$\Rightarrow \qquad f(x + \Delta x) = f(x) + \Delta x \cdot (6x + 15)$ $\left[\begin{array}{l} \because f(x) = 3x^2 + 15x + 5 \\ \therefore f'(x) = 6x + 15 \end{array}\right]$

$\Rightarrow \qquad f(x + \Delta x) = (3x^2 + 15x + 5) + \Delta x \cdot (6x + 15)$

Putting $x = 3$ and $\Delta x = 0.02$, we get

$$f(3.02) = [3(3)^2 + 15(3) + 5] + (0.02)\,[6(3) + 15]$$

$$= (27 + 45 + 5) + (0.02)\,(18 + 15)$$

$$= 77 + 0.02(33)$$

$$= 77 + 0.66 = 77.66$$

Hence, $f(3.02) = 77.66$

S29. Let $y = f(x) = \log_e x$, $x = 4$ and $x + \Delta x = 4.01$. Then, $\Delta x = 0.01$

For $x = 4$, we have $y = f(4) = \log_e 4$

$= 1.3863$

Now, $y = \log_e x$

$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x}$

$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{x=4} = \frac{1}{4}$

$\therefore \quad \Delta y = \frac{dy}{dx}\,\Delta x$

$\Rightarrow \quad \Delta y = \frac{1}{4} \times 0.01 = 0.0025$

$\Rightarrow \quad \Delta y = 0.0025$

Hence, $\log_e(4.01) = y + \Delta y$

$= 1.3863 + 0.0025 = 1.3888$

S30.

Let $y = f(x) = \tan x$, $x = 45° = \left(\frac{\pi}{4}\right)^c$ and $x + \Delta x = 46°$.

Then, $\Delta x = 1° = 0.01745$ radians.

For, $x = \frac{\pi}{4}$, we have $y = f\left(\frac{\pi}{4}\right) = \tan\frac{\pi}{4} = 1$

Now, $y = \tan x$

$\Rightarrow \quad \frac{dy}{dx} = \sec^2 x$

$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{x=\frac{\pi}{4}} = \sec^2\frac{\pi}{4} = 2$

$\therefore \quad \Delta y = \frac{dy}{dx}\Delta x$

$\Rightarrow \quad \Delta y = 2\,(0.01745) = 0.03490$

$\Rightarrow \quad \Delta y = 0.03490$

Hence, $\tan 46° = y + \Delta y$

$= 1 + 0.03490 = 1.03490.$

Date: 18/10/2021

Q1. Find all the points of local maxima and local minima and the corresponding local maximum and local minimum values for the function.

$$f(x) = -(x-1)^3(x+1)^2$$

Q2. Find all the points of local maxima and minima of the function.

$$f(x) = x^3 - 6x^2 + 9x + 15$$

Q3. Find the local maxima or local minima, if any of $f(x) = \frac{1}{x^2+5}$

Q4. Find the local maxima or local minima, if any of $f(x) = x^3 - 3x$.

Q5. Find the local maxima or local minima, if any, of the following function, using the first derivative test only.

$$f(x) = x\sqrt{1-x}, \qquad x \le 1$$

Q6. Find the local maxima or local minima, if any of the following function.

$$f(x) = \frac{x}{2} + \frac{2}{x}, \qquad x > 0$$

Q7. Find the local maxima or local minima, if any of the following function.

$$f(x) = 3x^{\frac{5}{3}} - 5x, \qquad x > 0$$

Q8. Find the local maxima or local minima, if any of the following function.

$$f(x) = 2x^3 - 3x^2 - 12x + 4$$

Q9. Find the local maxima or local minima, if any of the following function.

$$f(x) = -x^3 + 12x^2 - 5$$

Q10. Find the local maxima or local minima, if any of the following function.

$$f(x) = x^4 - 62x^2 + 120x + 9$$

Q11. Find the local maxima or local minima, if any of the following function.

$$f(x) = (x-1)(x+2)^2$$

Q12. Find the local maxima or local minima, if any of the following function.

$$f(x) = (x-1)^3(x+1)^2$$

Q13. Find the local maxima or local minima, if any of the following function.

$$f(x) = (x+1)(x+2)^{1/3}, \qquad x \ge -2$$

Q14. Find the local maxima or local minima, if any of the following function.

$$f(x) = \frac{x}{(x-1)(x-4)}, \qquad 1 < x < 4$$

Q15. Find the local maxima or local minima, if any of the following function.

$$f(x) = x\sqrt{32-x^2}, \qquad -5 \le x \le 5$$

Q16. Find all the points of local maxima and minima and the corresponding maximum and minimum values of the function

$$f(x) = 2x^3 - 21x^2 + 36x - 20$$

Q17. Find the value of x for which the function $(x-2)^3(x-3)^2$ is a maximum or minimum. Also find the point of inflexion.

Q18. Find the points of local maxima or local minima and corresponding local maximum and local minimum values of following function. Also find the points of inflexion, if any.

$$y = 9x^3 - 45x^2 + 48x + 11$$

Q19. Find all the points of local maxima and minima and the corresponding maximum and minimum values of the function.

$$f(x) = -\frac{3}{4}x^4 - 8x^3 - \frac{45}{2}x^2 + 105$$

Q20. Find the points of local maxima or local minima and corresponding local maximum and local minimum values of following function. Also find the points of inflexion, if any.

$$y = x + \sqrt{1-x}, \quad x \le 1$$

Q21. Find the points of local maxima and local minima, if any find also the local maximum and local minimum values:

$$f(x) = 2\sin x - x, \ \frac{-\pi}{2} \le x \le \frac{\pi}{2}$$

Q22. Find the points of local maxima or local minima, if any, find also the local maximum or local minimum values:

$$f(x) = \sin 2x, \text{ where } 0 < x < \pi$$

Q23. Find the points of local maxima and local minima, if any find also the local maximum and local minimum values:

$$f(x) = \sin 2x - x, \quad \text{where } -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

Q24. Find the points of local maxima or local minima, if any, also find the local maximum or local minimum values:

$$f(x) = \sin x - \cos x, \text{ where } 0 < x < 2\pi$$

Q25. Find the points of local maxima or local minima, if any find also the local maximum or local minimum values:

$$f(x) = \sin x + \cos x, \text{ where } 0 < x < \frac{\pi}{2}$$

Q26. Find the points of local maxima or local minima and corresponding local maximum and local minimum values of following function. Also find the points of inflexion, if any.

$$y = 3\sin^2 x + 4\cos^2 x.$$

Q27. Find the points of local maxima and local minima, if any find also the local maximum and local minimum values:

$$f(x) = \sin^4 x + \cos^4 x, \ 0 < x < \frac{\pi}{2}.$$

Q28. Find the points of local maxima and local minima, if any find also the local maximum and local minimum values:

$$f(x) = \sin x + \frac{1}{2}\cos 2x, \text{ where } 0 \le x \le \frac{\pi}{2}$$

Q29. Show that $\frac{\log x}{x}$ has a maximum value at $x = e$ and maximum value is $\frac{1}{e}$.

Q30. Show that the maximum value of $\left(\frac{1}{x}\right)^x$ is $e^{1/e}$.

Q31. If $f(x) = a \log |x| + bx^2 + x$ has extreme values at $x = -1$ and at $x = 2$, then find a and b.

Q32. Find the maximum profit that a company can make, if the profit function is given by $P(x) = 41 + 24x - 18x^2$.

Q33. Find the points at which the function f given by $f(x) = (x - 2)^4(x + 1)^3$ has:

(i) local maxima (ii) local minima (iii) point of inflexion.

Q34. Find all the points of local maxima and minima and the correspondidng maximum and minimum values of the function given below:

$$f(x) = -\frac{3}{4}x^4 + 2x^3 + \frac{9}{2}x^2 + 100$$

Q35. Find the points of local maxima or local minima and corresponding local maximum and local minimum values of following function. Also find the points of inflexion, if any.

$$y = x^3 - 2ax^2 + a^2x, \quad a > 0, \quad x \in R$$

Q36. Find the maximum value of $f(x) = (x - 1)(x - 2)(x - 3)$.

Q37. If $y = \frac{ax - b}{(x-1)(x-4)}$ has a turning point $P(2, -1)$, find the values of a and b and show that y is maximum at P.

Q38. Show that $\sin^p \theta \cos^q \theta$ attains a maximum, when $\theta = \tan^{-1}\sqrt{\frac{p}{q}}$.

Q39. Find the maximum and minimum values of $y = \tan x - 2x$ where $0 < x < \pi$.

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S1. We have, $y = f(x) = -(x-1)^3(x+1)^2$

Then, $\dfrac{dy}{dx} = -3(x-1)^2(x+1)^2 - 2(x+1)(x-1)^3$

$\Rightarrow \quad \dfrac{dy}{dx} = -(x-1)^2(x+1)[3(x+1)+2(x-1)]$

$\Rightarrow \quad \dfrac{dy}{dx} = -(x-1)^2(x+1)(5x+1)$

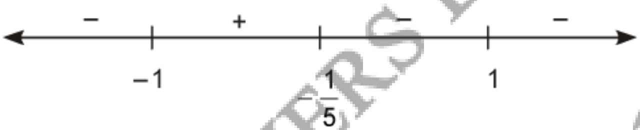
For local maxima or local minima we have

$\therefore \quad \dfrac{dy}{dx} = 0$

$\Rightarrow \quad -(x-1)^2(x+1)(5x+1) = 0 \quad \Rightarrow \quad x = 1$ or -1 or $x = -\dfrac{1}{5}$.

The change in sign of $\dfrac{dy}{dx}$ for different values of x are shown below:

Clearly, $\dfrac{dy}{dx}$ does not change its sign as x passes through 1. So $x = 1$ is neither a point of local maxima nor a point of local minima. In fact, $x = 1$ is a point of inflexion.



Clearly, $\dfrac{dy}{dx}$ change sign from negative to positive as x passes through -1.

So, $x = -1$ is a point of local minima. The local minimum value of $f(x)$ at $x = -1$ is

$f(-1) = (-2)^3(-1+1)^2 = 0.$

It is evident from Fig. that $\dfrac{dy}{dx}$ changes sign from positive to negative as x passes through $-\dfrac{1}{5}$.

So, $x = -\dfrac{1}{5}$ is a point of local maxima.

The local maximum value of $f(x)$ at $x = -\dfrac{1}{5}$ is

$$f\left(\frac{-1}{5}\right) = -\left(-\frac{1}{5}-1\right)^3\left(-\frac{1}{5}+1\right)^2 = \frac{3456}{3125}.$$

S2. We have,

$$y = f(x) = x^3 - 6x^2 + 9x + 15$$

Differentiating both sides w.r.t. 'x', we get

$$\frac{dy}{dx} = 3x^2 - 12x + 9$$

$$= 3(x^2 - 4x + 3)$$

$$= 3(x - 1)(x - 3)$$

For local maxima or local minima we have $\frac{dy}{dx} = 0$

$\therefore \quad 3(x^2 - 4x + 3) = 0$

$\Rightarrow \quad (x - 1)(x - 3) = 0 \quad \Rightarrow \quad x = 1$ or 3.

Case I: When $x = 1$.

In this case, when x is slightly less than 1 then, $f'(x) = 3(x - 1)(x - 3)$ is positive and when x is slightly more than 1, then $f'(x)$ is negative.

Thus, $f'(x)$ changes sign from positive to negative.

So, $x = 1$ is a point of local maximum.

Case II: When $x = 3$.

In this case, when x is slightly less than 3 then, $f'(x) = 3(x - 1)(x - 3)$ is negative and when x is slightly greater than 3, then $f'(x)$ is positive.

Thus, $f'(x)$ changes sign from negative to positive.

So, $x = 3$ is a point of local minima.

S3. We have,

$$y = f(x) = \frac{1}{x^2 + 5}$$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 + 5)^{-1}$$

$$= -(x^2 + 5)^{-2}\frac{d}{dx}(x^2 + 5)$$

$$= -\frac{2x}{(x^2 + 5)^2}$$

For local maxima or local minima we have $\frac{dy}{dx} = 0$

$$-\frac{2x}{(x^2+5)^2} = 0 \quad \Rightarrow \quad 2x = 0 \quad \Rightarrow \quad x = 0.$$

When $x = 0$.

In this case, when x is slightly less than 0, then $\frac{dy}{dx} = \frac{-2x}{(x^2+5)^2}$ is positive.

When x is slightly more than 0, then $\frac{dy}{dx} = \frac{-2x}{(x^2+5)^2}$ is negative.

Thus, $\frac{dy}{dx}$ changes sign from positive to negative.

So, $x = 0$ is a point of local maxima.

Also, local maximum value is

$$f(0) = \frac{1}{(0^2+5)} = \frac{1}{5}.$$

S4. We have,

$$y = f(x) = x^3 - 3x$$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = 3x^2 - 3 = 3(x^2 - 1) = 3(x + 1)(x - 1)$$

For local maximum or local minimum, we have $\frac{dy}{dx} = 0$.

$\Rightarrow \quad 3(x + 1)(x - 1) = 0$

$\Rightarrow \quad x = 1, -1$

Case I: When $x = 1$.

In this case, when x is slightly less than 1, then $\frac{dy}{dx} = 3(x + 1)(x - 1)$ is negative and when x is slightly more than 1, then $\frac{dy}{dx}$ is positive.

Thus, $\frac{dy}{dx}$ changes sign from negative to positive.

So, $x = 1$ is a point of local minima.

Also, local minimum value $= f(1) = (1)^3 - 3(1) = 1 - 3 = -2$.

Case II: When $x = -1$.

In this case, when x is slightly less than – 1, Thus, $\frac{dy}{dx} = 3(x+1)(x-1)$ is positive and when x is slightly more than – 1, then $\frac{dy}{dx}$ is negative. Thus, $\frac{dy}{dx}$ changes sign from positive to negative.

So, $x = 1$ is a point of local maxima.

Also, local maximum value $= f(-1) = (-1)^3 - 3(-1) = -1 + 3 = 2$.

S5. We have, $y = f(x) = x\sqrt{1-x}$

Differentiating both sides w.r.t. 'x', we get

$$\frac{dy}{dx} = x \cdot \frac{d}{dx}\sqrt{1-x} + \sqrt{1-x} \cdot \frac{d}{dx}x$$

$$= x\frac{(-1)}{2\sqrt{1-x}} + \sqrt{1-x}$$

$$= \frac{-x+2-2x}{2\sqrt{1-x}} = \frac{2-3x}{2\sqrt{1-x}}$$

For local maxima or local minima we have $\frac{dy}{dx} = 0$

$$\Rightarrow \quad \frac{2-3x}{2\sqrt{1-x}} = 0 \quad \Rightarrow \quad 2-3x = 0 \quad \text{or} \quad x = \frac{2}{3}.$$

When x is slightly less than $\frac{2}{3}$, then $\frac{dy}{dx} = \frac{2-3x}{2\sqrt{1-x}}$ is positive and when x is slightly more than $\frac{2}{3}$, then $\frac{dy}{dx}$ is negative.

Thus, $\frac{dy}{dx}$ change sign from positive to negative.

So, $x = \frac{2}{3}$ is a point of local maxima.

Also, local maximum value,

$$y\left(\text{at } x = \frac{2}{3}\right) = \frac{2}{3}\sqrt{1-\frac{2}{3}} = \frac{2}{3}\sqrt{\frac{3-2}{3}} = \frac{2}{3}\frac{1}{\sqrt{3}} = \frac{2\sqrt{3}}{9}.$$

S6. Let $y = f(x) = \frac{x}{2} + \frac{2}{x}$

Differentiating w.r.t. "x", we get

$$\frac{dy}{dx} = \frac{1}{2} + \left(\frac{-2}{x^2}\right)$$

For local maxima or local minima $\frac{dy}{dx} = 0$.

$\Rightarrow \qquad \frac{1}{2} - \frac{2}{x^2} = 0$

$\Rightarrow \qquad \frac{2}{x^2} = \frac{1}{2} \quad \Rightarrow \quad x^2 = 4 \quad \Rightarrow \quad x = \pm 2$

Since it is given $x > 0$.

So we will take $x = 2$ only.

If x is slightly less than 2 then $\frac{dy}{dx}$ is negative.

If x is slightly more than 2 then $\frac{dy}{dx}$ is positive.

Thus, $\frac{dy}{dx}$ changes sign from negative to positive.

So, $x = 2$ is point of local minima and local minimum value is

$$f(2) = \frac{2}{2} + \frac{2}{2} = 1 + 1 = 2.$$

S7. Let $\qquad y = f(x) = 3x^{\frac{5}{3}} - 5x$

Differentiating w.r.t. "x", we get

$$\frac{dy}{dx} = 3 \times \frac{5}{3} x^{\frac{2}{3}} - 5 = 5x^{\frac{2}{3}} - 5$$

For local maxima or local minima $\frac{dy}{dx} = 0 \quad \Rightarrow \quad 5\left(x^{\frac{2}{3}} - 1\right) = 0$.

$\Rightarrow \qquad x^{\frac{2}{3}} = 1 \quad \Rightarrow \quad x = 1$

In this case, if x is slightly less than 1 then $\frac{dy}{dx}$ is negative.

If x is slightly more than 1 then $\frac{dy}{dx}$ is positive.

Thus, $\frac{dy}{dx}$ changes sign from negative to positive.

So, $x = 1$ is point of local minima and local minimum value is

$$f(1) = 3(1)^{5/3} - 5(1) = 3 - 5 = -2.$$

S8. Let $\qquad y = f(x) = 2x^3 - 3x^2 - 12x + 4$

Differentiating w.r.t. "x", we get

$$\frac{dy}{dx} = 6x^2 - 6x - 12$$

For local maxima or local minima $\frac{dy}{dx} = 0$.

$\Rightarrow \quad 6(x^2 - x - 2) = 0 \quad \Rightarrow \quad x^2 - 2x + x - 2 = 0$

$\Rightarrow \quad x(x-2) + 1(x-2) = 0 \quad \Rightarrow \quad (x+1)(x-2) = 0$

So, $\quad x = -1, 2.$

Case I: When $x = -1$.

If x is slightly less than -1 then $\frac{dy}{dx}$ is positive.

If x is slightly more than -1 then $\frac{dy}{dx}$ is negative.

Thus, $\frac{dy}{dx}$ changes sign from positive to negative.

So, $x = -1$ is the point of local maxima and local maximum value is

$$f(-1) = -2 - 3 + 12 + 4 = 11.$$

Case II: When $x = 2$.

If x is slightly less than 2 then $\frac{dy}{dx}$ is negative.

If x is slightly more than 2 then $\frac{dy}{dx}$ is positive.

Thus, $\frac{dy}{dx}$ changes sign from negative to positive.

So, $x = 2$ is the point of local minima and local minimum value is

$$f(2) = 16 - 12 - 24 + 4 = -16.$$

S9. Let $\quad y = f(x) = -x^3 + 12x^2 - 5$

Differentiating w.r.t. "x", we get

$$\frac{dy}{dx} = -3x^2 + 24x$$

For local maxima or local minima $\frac{dy}{dx} = 0$.

$\Rightarrow \quad -3x^2 + 24x = 0 \quad \Rightarrow \quad -3x(x-8) = 0$

$\Rightarrow \quad x = 0, 8$

Case I: When $x = 0$.

If x is slightly less than 0 then $\frac{dy}{dx}$ is negative.

If x is slightly more than 0 then $\frac{dy}{dx}$ is positive.

Thus, $\frac{dy}{dx}$ changes sign from negative to positive.

So, $x = 0$ is the point of local minima and local minimum value is

$$f(0) = -5.$$

Case II: When $x = 8$.

If x is slightly less than 8 then $\frac{dy}{dx}$ is positive.

If x is slightly more than 8 then $\frac{dy}{dx}$ is negative.

Thus, $\frac{dy}{dx}$ changes sign from positive to negative.

So, $x = 8$ is the point of local maxima and local maximum value is

$$f(8) = -(8)^3 + 12(8)^2 - 5$$
$$= -512 + 768 - 5 = 251.$$

S10. Let $\qquad y = f(x) = x^4 - 62x^2 + 120x + 9$

Differentiating w.r.t. "x", we get

$$\frac{dy}{dx} = 4x^3 - 124x + 120$$

For local maxima or local minima we have $\frac{dy}{dx} = 0$.

$\Rightarrow \quad 4x^3 - 124x + 120 = 0 \quad \Rightarrow \quad 4[x^3 - 31x + 30] = 0$

$\Rightarrow \quad (x-1)(x-5)(x+6) = 0$

$\Rightarrow \quad x = 1, 5, -6.$

Case I: When $x = -6$.

If x is slightly less than -6 then $\frac{dy}{dx}$ is negative.

If x is slightly more than -6 then $\frac{dy}{dx}$ is positive.

Thus, $\frac{dy}{dx}$ changes sign from negative to positive.

So, $x = -6$ is the point of local minima and local minimum value is

$$f(-6) = 1296 - 2232 - 720 + 9 = -1647.$$

Case II: When $x = 1$.

If x is slightly less than 1 then $\frac{dy}{dx}$ is positive.

If x is slightly more than 1 then $\frac{dy}{dx}$ is negative.

Thus, $\frac{dy}{dx}$ changes sign from positive to negative.

So, $x = 1$ is the point of local maxima and local maximum value is

$$f(1) = 1 - 62 + 120 + 9 = 68.$$

Case III: When $x = 5$.

If x is slightly less than 5 then $\frac{dy}{dx}$ is negative.

If x is slightly more than 5 then $\frac{dy}{dx}$ is positive.

Thus, $\frac{dy}{dx}$ changes sign from negative to positive.

So, $x = 5$ is the point of local minima and local minimum value is

$$f(5) = 625 - 1550 + 600 + 9 = -316.$$

S11. Let $y = f(x) = (x-1)(x+2)^2$

Differentiating w.r.t. "x", we get

$$\frac{dy}{dx} = (x - 1)(2)(x + 2) + (x + 2)^2$$

$$= (x + 2)(2x - 2 + x + 2)$$

$$= (x + 2)(3x)$$

For local maxima or local minima $\frac{dy}{dx} = 0$.

$\Rightarrow \quad 3x(x + 2) = 0 \quad \Rightarrow \quad x = 0, -2$

Case I: When $x = 0$.

If x is slightly less than 0 then $\frac{dy}{dx}$ is negative.

If x is slightly more than 0 then $\frac{dy}{dx}$ is positive.

Thus, $\frac{dy}{dx}$ changes sign from negative to positive.

So, $x = 0$ is the point of local minima and local minimum value is

$$f(0) = (-1)(4) = -4.$$

Case II: When $x = -2$.

If x is slightly less than -2 then $\frac{dy}{dx}$ is positive.

If x is slightly more than -2 then $\frac{dy}{dx}$ is negative.

Thus, $\frac{dy}{dx}$ changes sign from positive to negative.

So, $x = -2$ is the point of local maxima and local maximum value is

$$f(-2) = (-2-1)(-2+2)^2 = 0.$$

S12. Let $y = f(x) = (x-1)^3 (x+1)^2$

Differentiating w.r.t., "x" we get

$$\frac{dy}{dx} = (x-1)^3 2(x+1) + (x+1)^2 3(x-1)^2$$

$$= (x+1)(x-1)^2[2(x-1) + 3(x+1)]$$

$$= (x+1)(x-1)^2[2x - 2 + 3x + 3]$$

$$= (x+1)(5x+1)(x-1)^2.$$

For local maximum or local minimum $\frac{dy}{dx} = 0$.

$\Rightarrow$ $(x+1)(5x+1)(x-1)^2 = 0$ $\Rightarrow$ $x = -1, -\frac{1}{5}, 1$

Since, $(x-1)^2$ is always positive so sign of $\frac{dy}{dx}$ depends upon the sign of $(x+1)(5x+1)$.

Case I: When $x = -1$.

If x is slightly less than -1 then $\frac{dy}{dx}$ is positive.

If x is slightly more than -1 then $\frac{dy}{dx}$ is negative.

Thus, $\frac{dy}{dx}$ changes sign from positive to negative.

So, $x = -1$ is the point of local maxima and local maximum value is.

$$f(-1) = (-1-1)^3(-1+1)^2 = 0$$

Case II: When $x = -\frac{1}{5}$.

If x is slightly less than $-\frac{1}{5}$ then $\frac{dy}{dx}$ is negative.

If x is slightly more than $-\frac{1}{5}$ then $\frac{dy}{dx}$ is positive.

Thus, $\frac{dy}{dx}$ changes sign from negative to positive.

So, $x = -\frac{1}{5}$ is the point of local minima and local minimum value is.

$$f\left(-\frac{1}{5}\right) = \left(-\frac{1}{5}-1\right)^3\left(-\frac{1}{5}+1\right)^2 = \left(-\frac{6}{5}\right)^3\left(\frac{4}{5}\right)^2 = \frac{-3456}{3125}.$$

Case III: When $x = 1$.

There is no change in the sign of $\frac{dy}{dx}$.

Therefore, $x = 1$ is the point of inflexion.

S13. Let $y = f(x) = (x + 1)(x + 2)^{1/3}$

Differentiating w.r.t., "x" we get

$$\frac{dy}{dx} = \frac{1}{3}(x+1)(x+2)^{-\frac{2}{3}} + (x+2)^{\frac{1}{3}}.$$

For local maximum or local minimum $\frac{dy}{dx} = 0$.

$$\Rightarrow \quad \frac{1}{3}(x+1)(x+2)^{-\frac{2}{3}} + (x+2)^{\frac{1}{3}} = 0$$

$$\Rightarrow \quad (x+2)^{-\frac{2}{3}}\left[\frac{1}{3}(x+1)+(x+2)\right] = 0$$

$$\Rightarrow \quad (x+2)^{-\frac{2}{3}}\left[\frac{4}{3}x+\frac{7}{3}\right] = 0 \quad \Rightarrow x = -2,\ -\frac{7}{4}$$

Case I: When $x = -2$.

But it is given that $x \geq -2$.

So we cannot take slightly less than part. Thus, we cannot take the point $x = -2$.

Case II: When $x = -\frac{7}{4}$.

If x is slightly less than $-\frac{7}{4}$ then $\frac{dy}{dx}$ is negative.

If x is slightly more than $-\frac{7}{4}$ then $\frac{dy}{dx}$ is positive.

Thus, $\frac{dy}{dx}$ changes sign from negative to positive.

So, $x = -\frac{7}{4}$ is the point of local minima and local minimum value is.

$$f\left(-\frac{7}{4}\right) = \left(-\frac{7}{4}+1\right)\left(-\frac{7}{4}+2\right)^{1/3} = \left(\frac{-3}{4}\right)\left(\frac{1}{4}\right)^{1/3} = \frac{-3}{4\sqrt[3]{4}}.$$

S14. Let,

$$y = f(x) = \frac{x}{(x-1)(x-4)}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(x-1)(x-4) - x[(x-1)+(x-4)]}{(x-1)^2(x-4)^2}$$

$$= \frac{x^2 - 5x + 4 - (2x^2 - 5x)}{(x-1)^2(x-4)^2}$$

$$= \frac{x^2 - 5x + 4 - 2x^2 + 5x}{(x-1)^2(x-4)^2}$$

$$= \frac{-x^2 + 4}{(x-1)^2(x-4)^2}$$

For local maximum or local minimum $\frac{dy}{dx} = 0$.

$$\Rightarrow \quad \frac{-x^2+4}{(x-1)^2(x-4)^2} = 0 \quad \Rightarrow \quad -x^2 + 4 = 0 \quad \Rightarrow \quad x = \pm 2$$

Since it is given that $1 < x < 4$.

So we will take $x = 2$ only.

If x is slightly less than 2 then $\frac{dy}{dx}$ is positive.

If x is slightly more than 2 then $\frac{dy}{dx}$ is negative.

Thus, $\frac{dy}{dx}$ changes sign from positive to negative.

So, $x = 2$ is the point of local maxima and local maximum value is.

$$f(2) = \frac{2}{(2-1)(2-4)} = \frac{2}{(1)(-2)} = -1.$$

S15. Let, $y = f(x) = x\sqrt{32 - x^2} = \sqrt{32x^2 - x^4}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{2}(32x^2 - x^4)^{-1/2}(64x - 4x^3)$$

$$= \frac{x(16 - x^2) \times 4}{x\sqrt{32 - x^2} \times 2} = \frac{2(16 - x^2)}{\sqrt{32 - x^2}}$$

For local maximum or local minimum $\frac{dy}{dx} = 0 \Rightarrow \frac{2(16 - x^2)}{\sqrt{32 - x^2}} = 0$

$\Rightarrow \quad 16 - x^2 = 0 \quad \Rightarrow \quad x = \pm 4$

Case I: When $x = -4$.

If x is slightly less than -4 then $\frac{dy}{dx}$ is negative.

If x is slightly more than -4 then $\frac{dy}{dx}$ is positive.

Thus, $\frac{dy}{dx}$ changes sign from negative to positive.

So, $x = -4$ is the point of local minima.

Thus, local minimum value is

$$f(-4) = -4\sqrt{32 - 16} = -4\sqrt{16} = -4 \times 4 = -16$$

Case II: When $x = 4$.

If x is slightly less than 4 then $\frac{dy}{dx}$ is positive.

If x is slightly more than 4 then $\frac{dy}{dx}$ is negative.

Thus, $\frac{dy}{dx}$ changes sign from positive to negative.

So, $x = 4$ is the point of local maxima and local maximum value is.

$$f(4) = 4\sqrt{32 - 16} = 4 \times 4 = 16.$$

S16. We have,

$$f(x) = 2x^3 - 21x^2 + 36x - 20$$

$$\Rightarrow \quad f'(x) = 6x^2 - 42x + 36$$

For local maximum or local minimum, we have

$$f'(x) = 0$$

$$\Rightarrow \quad 6x^2 - 42x + 36 = 0$$

$$\Rightarrow \quad 6(x-1)(x-6) = 0$$

$$\Rightarrow \quad x = 1, 6$$

Thus, $x = 1$ and $x = 6$ are the possible points of local maxima and minima.

Now we test the function at each of these points.

We have, $f''(x) = 12x - 42$

At $x = 1$

$$f''(1) = 12 - 42 = -30 < 0.$$

So, $x = 1$ is a point of local maximum.

The local maximum value is $f(1) = 2 - 21 + 36 - 20 = -3$

At $x = 6$

$$f''(6) = 12(6) - 42 = 30 > 0$$

So, $x = 6$ is a point of local minimum.

The local minimum value is $f(6) = 2(6)^3 - 21(6)^2 + 36 \times 6 - 20 = -128$.

S17. We have, $y = (x-2)^3 . (x-3)^2$

$$\frac{dy}{dx} = (x-2)^3 . 2(x-3) + (x-3)^2 . 3(x-2)^2$$

$$\frac{dy}{dx} = (x-2)^2 (x-3)[2x - 4 + 3x - 9]$$

$$\Rightarrow \quad \frac{dy}{dx} = (x-2)^2 (x-3)(5x-13) \quad \ldots \text{(i)}$$

For maxima or minima, $\frac{dy}{dx} = 0$

$$\therefore \quad (x-2)^2 (x-3)(5x-13) = 0 \quad \Rightarrow \quad x = 2, 3, \frac{13}{5}.$$

Again differentiating Eq. (i), we get

$$\frac{d^2y}{dx^2} = 2(x-2)(x-3)(5x-13) + (x-2)^2(5x-13) + (x-2)^2(x-3) . 5$$

$$= (x-2)[(10x^2 - 56x + 78) + (5x^2 - 23x + 26) + (5x^2 - 25x + 30)]$$

$$= (x-2)(20x^2 - 104x + 134)$$

(i) **When $x = 2$**, $\frac{d^2y}{dx^2} = 0$ and $\frac{d^3y}{dx^3} \neq 0$

The given function has a point of inflexion at $x = 2$.

(ii) **When $x = 3$**, $\frac{d^2y}{dx^2}$ = (3 – 2) (180 – 312 + 134) = + 2 = + ve

The given function is minimum at $x = 3$.

The minimum value of the function = $f(3) = (3-2)^3(3-3)^2 = 0$

(iii) **When $x = \frac{13}{5}$**, $\frac{d^2y}{dx^2} = \left[\frac{13}{5} - 2\right] \times \left[20 \times \frac{169}{25} - 104 \times \frac{13}{5} + 134\right]$

$$= \frac{3}{5} \times \frac{1}{5}(676 - 1352 + 670) = \frac{3}{25}(-6) = -\frac{18}{25} = -\text{ve}$$

The given function has maxima at $x = \frac{13}{5}$.

The maximum value of the function $= f\left(\frac{13}{5}\right) = \left[\frac{13}{5} - 2\right]^3 \times \left[\frac{13}{5} - 3\right]^2$

$$= \frac{27}{125} \times \frac{4}{25} = \frac{108}{3125}.$$

S18. We have,

$$f(x) = y = 9x^3 - 45x^2 + 48x + 11$$

Differentiating w.r.t. "x", we get

$$\frac{dy}{dx} = 27x^2 - 90x + 48 \qquad \text{... (i)}$$

For local maxima or local minima $\frac{dy}{dx} = 0$.

$\Rightarrow \quad 27x^2 - 90x + 48 = 0$

$\Rightarrow \quad 9x^2 - 30x + 16 = 0$

$\Rightarrow \quad 9x^2 - 24x - 6x + 16 = 0$

$\Rightarrow 3x(3x - 8) - 2(3x - 8) = 0$

$\Rightarrow \quad (3x - 2)(3x - 8) = 0 \quad \Rightarrow \quad x = \frac{2}{3}, \frac{8}{3}$

Again differentiating Eq. (i) w.r.t. "x", we get

$$\frac{d^2y}{dx^2} = 54x - 90$$

At point $x = \frac{2}{3}$, $\frac{d^2y}{dx^2} = 54 \times \frac{2}{3} - 90 < 0$

So, at point $x = \frac{2}{3}$ it is local maximum & local maximum value is

$$f\left(\frac{2}{3}\right) = 9 \times \frac{8}{27} - 45 \times \frac{4}{9} + 48 \times \frac{2}{3} + 11 = \frac{77}{3}$$

At point $x = \frac{8}{3}$, $\frac{d^2y}{dx^2} = \frac{54 \times 8}{3} - 90 > 0$

So, at point $x = \frac{8}{3}$ it is local minimum and local minimum value is

$$f\left(\frac{8}{3}\right) = 9 \times \frac{512}{27} - 45 \times \frac{64}{9} + 48 \times \frac{8}{3} + 11 = -\frac{31}{3}.$$

S19. We have,

$$f(x) = -\frac{3}{4}x^4 - 8x^3 - \frac{45}{2}x^2 + 105$$

$\Rightarrow$ $f'(x) = -3x^3 - 24x^2 - 45x = -3x(x^2 + 8x + 15)$

For local maximum and local minimum, we must have

$f'(x) = 0$

$\Rightarrow$ $-3x(x^2 + 8x + 15) = 0$

$\Rightarrow$ $-3x(x + 3)(x + 5) = 0$

$\Rightarrow$ $x = 0, -3, -5$

Thus, $x = 0$, $x = -3$ and $x = -5$ are the possible points of local maxima and minima.

Now we test the function at each of these points.

We have, $f''(x) = -9x^2 - 48x - 45$

At $x = 0$

$f''(x) = 0 - 45 < 0$

So, $x = 0$ is a point of local maximum.

The local maximum value of $f(x)$ at $x = 0$ is $f(0) = 105$

At $x = -3$

$f''(-3) = -9(-3)^2 - 48(-3) - 45 = 18 > 0$

So, $x = -3$ is a point of local minimum.

The local minimum value of $f(x)$ at $x = -3$ is

$$f(-3) = -\frac{3}{4}(-3)^4 - 8(-3)^3 - \frac{45}{2}(-3)^2 + 105 = \frac{231}{4}$$

At $x = -5$:

$$f''(-5) = -9(-5)^2 - 48(-5) - 45 = -30 < 0$$

So, $x = -5$ is a point of local maximum.

The local maximum value of $f(x)$ at $x = -5$ is

$$f(-5) = -\frac{3}{4}(-5)^4 - 8(-5)^3 - \frac{45}{2}(-5)^2 + 105 = \frac{295}{4}.$$

S20. We have,

$$y = f(x) = x + \sqrt{1-x}$$

Differentiating w.r.t. "x", we get

$$\frac{dy}{dx} = 1 + \frac{1}{2}(1-x)^{-1/2} \times (-1) = 1 - \frac{1}{2\sqrt{1-x}} \quad \text{... (i)}$$

Now for local maxima or local minima $\frac{dy}{dx} = 0 \quad \Rightarrow \quad 1 - \frac{1}{2\sqrt{1-x}} = 0$

$$\Rightarrow \qquad 1 = \frac{1}{2\sqrt{1-x}}$$

$$\Rightarrow \qquad 2\sqrt{1-x} = 1 \qquad \Rightarrow \qquad \sqrt{1-x} = \frac{1}{2}$$

$$\Rightarrow \qquad 1 - x = \frac{1}{4} \qquad \Rightarrow \qquad x = \frac{3}{4}$$

Again differentiating Eq. (i) w.r.t. "x", we get

$$\frac{d^2y}{dx^2} = \frac{1}{4}(1-x)^{-3/2}(-1)$$

At $x = \frac{3}{4}$, $\quad \left(\frac{d^2y}{dx^2}\right)_{x=\frac{3}{4}} < 0$

So, $x = \frac{3}{4}$ is the point of local maxima & local maximum value is

$$f\left(\frac{3}{4}\right) = \frac{3}{4} + \sqrt{1 - \frac{3}{4}} = \frac{3}{4} + \sqrt{\frac{1}{4}} = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}.$$

S21. We have,

$$f(x) = 2\sin x - x, \quad \frac{-\pi}{2} \le x \le \frac{\pi}{2}$$

$$\Rightarrow \qquad f'(x) = 2\cos x - 1.$$

For local maximum or local minimum, we have

$$f'(x) = 0$$

$$\Rightarrow \quad 2\cos x - 1 = 0$$

$$\Rightarrow \quad \cos x = 1/2$$

$$\Rightarrow \quad x = \pm\pi/3 \qquad [\because \ -\pi/2 \le x \le \pi/2]$$

Thus, $x = \pm\pi/3$ are possible points of local maximum or local minimum.

Now $\quad f''(x) = -2\sin x$

At $\quad x = -\pi/3.$

$$f''(x) = -2\sin(-\pi/3) = 2\sin\pi/3$$

$$= 2\sqrt{3}/2 = \sqrt{3} > 0$$

So, $x = -\pi/3$ is a point of local minimum.

The local minimum value is

$$f(-\pi/3) = 2\sin(-\pi/3) - (-\pi/3)$$

$$= -\sqrt{3} + \pi/3.$$

At $x = \pi/3$

$$f''(x) = -2\sin\pi/3 = -\sqrt{3} < 0.$$

So, $x = \pi/3$, is a point of local maximum.

The local maximum value of $f(x)$ is

$$f(\pi/3) = 2\sin\pi/3 - \pi/3$$

$$= \sqrt{3} - \pi/3.$$

S22. We have,

$$f(x) = \sin 2x, \quad \text{where } 0 < x < \pi$$

$$\Rightarrow \quad f'(x) = 2\cos 2x.$$

For local maximum or local minimum, we have

$$f'(x) = 0$$

$$\Rightarrow \quad 2\cos 2x = 0$$

$$\Rightarrow \quad \cos 2x = 0$$

$$\Rightarrow \quad 2x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad [\because 0 < x < \pi \therefore 0 < 2x < 2\pi]$$

$$\Rightarrow \quad x = \frac{\pi}{4}, \frac{3\pi}{4}$$

Thus, $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$ are possible points of local maximum or local minimum.

Now we test the function at these points.

We have, $f''(x) = -4 \sin 2x$

At $x = \pi/4$: We have,

$$f''\left(\frac{\pi}{4}\right) = -4\sin\frac{\pi}{2} = -4 < 0$$

So, $x = \frac{\pi}{4}$ is a point of local maximum.

The local maximum value of $f(x)$ is $f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{2} = 1.$

At $x = 3\pi/4$: We have,

$$f''\left(\frac{3\pi}{4}\right) = -4\sin\frac{3\pi}{2} = 4 > 0$$

So, $x = \frac{3\pi}{4}$, is a point of local minimum.

The local minimum value of $f(x)$ is $f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{2} = -1.$

S23. We have $\quad f(x) = \sin 2x - x$

$\Rightarrow \quad f'(x) = 2\cos 2x - 1$

For local maximum or minimum, we must have

$$f'(x) = 0$$

$\Rightarrow \quad 2\cos 2x - 1 = 0$

$\Rightarrow \quad \cos 2x = \frac{1}{2}$

$\Rightarrow \quad 2x = -\frac{\pi}{3}$ or, $2x = \frac{\pi}{3}$ $\quad \left[\because -\frac{\pi}{2} \le x \le \frac{\pi}{2} \therefore -\pi \le 2x \le \pi\right]$

$\Rightarrow \quad x = -\frac{\pi}{6}$ or, $x = \frac{\pi}{6}$

Thus, $x = -\frac{\pi}{6}$ and $x = \frac{\pi}{6}$ are possible points of local maxima or minima.

Now, we test the function at each of these points.

We have, $f''(x) = -4 \sin 2x$.

At $x = -\pi/6$: We have,

$$f''\left(-\frac{\pi}{6}\right) = -4\sin\left(-\frac{\pi}{3}\right) = 4 \times \frac{\sqrt{3}}{2} = 2\sqrt{3} > 0$$

So, $x = -\frac{\pi}{6}$ is a point of local minimum.

The local minimum value is $f\left(-\frac{\pi}{6}\right) = \sin\left(-\frac{\pi}{3}\right) + \frac{\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{\pi}{6}$

At $x = \pi/6$: We have,

$$f''\left(\frac{\pi}{6}\right) = -4\sin\frac{\pi}{3} = -4\left(\frac{\sqrt{3}}{2}\right) = -2\sqrt{3} < 0$$

So, $x = \frac{\pi}{6}$ is a point of local maximum.

The local maximum value is $f\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{3} - \frac{\pi}{6} = \frac{\sqrt{3}}{2} - \frac{\pi}{6}$.

S24. We have,

$$f(x) = \sin x - \cos x, \text{ where } 0 < x < 2\pi$$
$$f'(x) = \cos x + \sin x$$

For local maximum or minimum, we have

$$f'(x) = 0$$

$\Rightarrow \quad \cos x + \sin x = 0$

$\Rightarrow \quad \sin x = -\cos x$

$\Rightarrow \quad \tan x = -1$

$\Rightarrow \quad x = \frac{3\pi}{4}$ or, $x = \frac{7\pi}{4}$ $\qquad [\because 0 < x < 2\pi]$

Thus, $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$ are possible points of local maximum or local minimum.

Now we test the function at each of these points.

We have, $f''(x) = -\sin x + \cos x$

At $x = 3\pi/4$: We have

$$f''\left(\frac{3\pi}{4}\right) = -\sin\frac{3\pi}{4} + \cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}} < 0.$$

So, $x = \frac{3\pi}{4}$ is the point of local maximum.

The local maximum value is

$$f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{4} - \cos\frac{3\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

At $x = 7\pi/4$: We have,

$$f''\left(\frac{7\pi}{4}\right) = -\sin\frac{7\pi}{4} + \cos\frac{7\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} > 0$$

So, the function attains a local minimum at $x = \frac{7\pi}{4}$.

The local minimum value is

$$f\left(\frac{7\pi}{4}\right) = \sin\frac{7\pi}{4} - \cos\frac{7\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}.$$

S25. We have,

$$f(x) = \sin x + \cos x, \text{ where } 0 < x < \frac{\pi}{2}$$

$$f'(x) = \cos x - \sin x$$

For local maximum or minimum, we have

$$f'(x) = 0$$

$$\Rightarrow \quad \cos x - \sin x = 0$$

$$\Rightarrow \quad \cos x = \sin x$$

$$\Rightarrow \quad \tan x = 1$$

$$\Rightarrow \quad x = \frac{\pi}{4} \qquad \left[\because\ 0 < x < \frac{\pi}{2}\right]$$

Thus, $x = \frac{\pi}{4}$ is a point of local maximum or minimum.

Now $\quad f''(x) = -\sin x - \cos x$

$$\Rightarrow \quad f''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} - \cos\frac{\pi}{4}$$

$$= \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = \frac{-2}{\sqrt{2}} = -\sqrt{2} < 0$$

So, $x = \frac{\pi}{4}$ is a point of local maximum.

The local maximum value is

$$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

S26. We have,

$$y = f(x) = 3\sin^2 x + 4\cos^2 x$$

Differentiating w.r.t. "x", we get

$$\frac{dy}{dx} = 6\sin x\cos x + 8\cos x(-\sin x)$$

$$= -2\sin x\cos x = -\sin 2x \quad \text{... (i)}$$

For local maxima or local minima $\frac{dy}{dx} = 0$.

$\Rightarrow \quad -\sin 2x = 0$

$\Rightarrow \quad \sin 2x = \sin 0 \quad \text{or} \quad \sin \pi$

$\Rightarrow \quad x = 0, \frac{\pi}{2}.$

Again differentiating Eq. (i) w.r.t. "x", we get

$$\frac{d^2y}{dx^2} = -2\cos 2x$$

At $x = 0$, $\quad \left(\frac{d^2y}{dx^2}\right)_{x=0} < 0$

So, $x = 0$ is the point of local maxima & local maximum value is

$$f(0) = 3\sin^2(0) + 4\cos^2(0) = 4$$

At $x = \frac{\pi}{2}$, $\quad \left(\frac{d^2y}{dx^2}\right)_{x=\frac{\pi}{2}} > 0$

So, $x = \frac{\pi}{2}$ is the point of local minima & local minimum value is

$$f\left(\frac{\pi}{2}\right) = 3\sin^2\frac{\pi}{2} + 4\cos^2\frac{\pi}{2} = 3.$$

S27. We have,

$$f(x) = \sin^4 x + \cos^4 x, \text{ where } 0 < x < \frac{\pi}{2}.$$

$\therefore \quad f'(x) = 4\sin^3 x\cos x - 4\cos^3 x\sin x$

$\Rightarrow \quad f'(x) = -4 \cos x \sin x (\cos^2 x - \sin^2 x)$

$\Rightarrow \quad f'(x) = -2 \sin 2x \cos 2x = -\sin 4x.$

For local maximum or local minimum, we have

$$f'(x) = 0$$

$\Rightarrow \quad -\sin 4x = 0$

$\Rightarrow \quad 4x = \pi \qquad \left[\because \quad 0 < x < \frac{\pi}{2} \quad \therefore \quad 0 < 4x < 2\pi\right]$

$\Rightarrow \quad x = \pi/4$

Now, $\quad f''(x) = -4 \cos 4x$

$\Rightarrow \quad f''\left(\frac{\pi}{4}\right) = -4 \cos \pi = (-4)(-1) = 4 > 0$

So, $x = \frac{\pi}{4}$ is a point of local minimum and the local minimum value is

$$f\left(\frac{\pi}{4}\right) = \sin^4 \frac{\pi}{4} + \cos^4 \frac{\pi}{4} = \left(\frac{1}{\sqrt{2}}\right)^4 + \left(\frac{1}{\sqrt{2}}\right)^4 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

S28. We have,

$$f(x) = \sin x + \frac{1}{2} \cos 2x, \text{ where } 0 \le x \le \frac{\pi}{2}$$

$\Rightarrow \quad f'(x) = \cos x - \sin 2x.$

For local maximum or minimum, we must have

$$f'(x) = 0$$

$\Rightarrow \quad \cos x - \sin 2x = 0$

$\Rightarrow \quad \cos x - 2 \sin x \cos x = 0$

$\Rightarrow \quad \cos x (1 - 2 \sin x) = 0$

$\Rightarrow \quad \cos x = 0 \text{ or } 1 - 2 \sin x = 0$

$\Rightarrow \quad \cos x = 0 \text{ or, } \sin x = \frac{1}{2}$

$\Rightarrow \quad x = \frac{\pi}{2} \text{ or } x = \frac{\pi}{6} \qquad \left[\because \quad 0 \le x \le \frac{\pi}{2}\right]$

Thus, $x = \frac{\pi}{6}$ and $x = \frac{\pi}{2}$ are possible points of local maxima and local minima.

Now we test the function at each of these points.

We have, $f''(x) = -\sin x - 2 \cos 2x$

At $x = \pi/6$, we have,

$$f''\left(\frac{\pi}{6}\right) = -\sin\frac{\pi}{6} - 2\cos\frac{\pi}{3} = -\frac{1}{2} - 2\times\frac{1}{2} = \frac{-3}{2} < 0$$

So, $x = \frac{\pi}{6}$ is the point of local maximum.

The local maximum value is

$$f\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{6} + \frac{1}{2}\cos\frac{\pi}{3} = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}\right) = \frac{3}{4}$$

At $x = \pi/2$: We have,

$$f''\left(\frac{\pi}{2}\right) = -\sin\frac{\pi}{2} - 2\cos\pi = -1 + 2 = 1 > 0$$

So, $x = \frac{\pi}{2}$ is the point of local minimum.

The local minimum value is

$$f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} + \frac{1}{2}\cos\pi = 1 - \frac{1}{2} = \frac{1}{2}.$$

S29. We have,

$$f(x) = \frac{\log x}{x}$$

$$f'(x) = \frac{x\frac{1}{x} - \log x\cdot 1}{x^2}$$

$$\Rightarrow \quad f'(x) = \frac{1-\log x}{x^2}$$

For local maximum or minimum, we must have

$$f'(x) = 0$$

$$\Rightarrow \quad \frac{1-\log x}{x^2} = 0$$

$$\Rightarrow \quad \log x = 1$$

$$\Rightarrow \quad x = e.$$

Now, $f'(x) = \frac{1-\log x}{x^2} = x^{-2}(1 - \log x)$

$$\Rightarrow \quad f''(x) = (1-\log x)\frac{d}{dx}x^{-2} + x^{-2}\frac{d}{dx}(1-\log x)$$

$$\Rightarrow \quad f''(x) = -2x^{-3}(1 - \log x) - x^{-3}$$

$$= -x^{-3}(3 - 2\log x)$$

$$\Rightarrow \quad f''(e) = -e^{-3}(3 - 2\log e) = \frac{-1}{e^3} < 0$$

Hence, $f(x)$ has a local maximum value at $x = e$.

$\therefore$ Maximum value = $\dfrac{\log e}{e} = \dfrac{1}{e}$.

S30. Let $y = \left(\dfrac{1}{x}\right)^x = x^{-x}$

$\Rightarrow \log y = -x \log x.$

Differentiating both side w.r.t. x

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = -\left(x \cdot \frac{1}{x} + \log x \cdot 1\right)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = -(1 + \log x)$$

$$\Rightarrow \frac{dy}{dx} = -y(1 + \log x)$$

For maximum and minimum, we must have

$$\therefore \frac{dy}{dx} = 0$$

$$\Rightarrow -y(1 + \log x) = 0$$

$$\Rightarrow 1 + \log x = 0$$

$$\Rightarrow \log x = -1$$

$$\Rightarrow x = e^{-1} = \frac{1}{e} \qquad \left[\because \log_e A = B \Rightarrow A = e^B\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{dy}{dx}(1+\log x) - \frac{y}{x} = y(1+\log x)^2 - \frac{y}{x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = x^{-x}(1+\log x)^2 - \frac{x^{-x}}{x} = x^{-x}(1+\log x)^2 - x^{-x-1}$$

Also,

$$\left(\frac{d^2y}{dx^2}\right)_{x=1/e} = \left(\frac{1}{e}\right)^{-1/e}\left(1+\log\frac{1}{e}\right)^2 - \left(\frac{1}{e}\right)^{(-1/e)-1}$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)_{x=1/e} = (e^{-1})^{-1/e}(1-\log e)^2 - (e^{-1})^{-1/e-1}$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)_{x=1/e} = e^{1/e}(1-1)^2 - e^{1/e+1} = -e^{1/e+1} < 0$$

So $x = 1/e$ is a point of local maximum. The local maximum value of y is given by $y = (e)^{1/e}$.

S31. Clearly, domain $(f) = R - [0]$

We have,

$$f(x) = a \log |x| + bx^2 + x$$

$$\Rightarrow \qquad f'(x) = \frac{a}{x} + 2bx + 1$$

Since $f(x)$ has extreme values at $x = -1$ and $x = 2$. Therefore,

$$f'(-1) = 0 \quad \text{and} \quad f'(2) = 0$$

$$\Rightarrow \qquad -a - 2b + 1 = 0 \quad \text{and} \quad \frac{a}{2} + 4b + 1 = 0$$

$$\Rightarrow \qquad a + 2b = 1 \quad \text{and} \quad a + 8b = -2$$

Solving these equations, we get

$$a = 2 \text{ and } b = -\frac{1}{2}.$$

S32. We have

$$P(x) = 41 + 24x - 18x^2$$

$$\Rightarrow \qquad \frac{dP(x)}{dx} = 24 - 36x \text{ and } \frac{d^2P(x)}{dx^2} = -36$$

For maximum or minimum, we must have

$$\frac{dP(x)}{dx} = 0 \Rightarrow 24 - 36x = 0 \Rightarrow x = 2/3$$

Also,

$$\left(\frac{d^2P(x)}{dx^2}\right)_{x=2/3} = -36 < 0.$$

So, profit is maximum when $x = 2/3$.

$$\text{Maximum profit} = (\text{Value of } P(x) \text{ at } x = 2/3)$$
$$= 41 + 24(2/3) - 18(2/3)^2$$
$$= 41 + 16 - 8 = 49.$$

S33. We have,

$$y = f(x) = (x - 2)^4 (x + 1)^3$$

Differentiating w.r.t. "x", we get

$$\frac{dy}{dx} = 4(x - 2)^3 (x + 1)^3 + 3(x - 2)^4 (x + 1)^2$$

$$= (x-2)^3 (x+1)^2 \{4(x+1) + 3(x-2)\}$$
$$= (x-2)^3 (x+1)^2 (7x-2)$$

For local maxima or local minima, we have $\frac{dy}{dx} = 0$

$$\Rightarrow \quad (x-2)^3 (x+1)^2 (7x-2) = 0$$

$$\Rightarrow \quad x = 2, -1, \frac{2}{7}$$

Since, $(x-2)^2 (x+1)^2$ is always positive. So, sign of $\frac{dy}{dx}$ depends upon the sign of $(x-2)(7x-2)$.

Case I: When $x = \frac{2}{7}$.

In this case, when x is slightly less than $\frac{2}{7}$ then, $(x-2)(7x-2)$ is positive.

When x is slightly more than $\frac{2}{7}$, then $(x-2)(7x-2)$ is negative.

Thus, $\frac{dy}{dx}$ changes sign from positive to negative.

So, $x = \frac{2}{7}$ is point of local maxima.

Case II: When $x = 2$.

In this case, when x is slightly less than 2 then, $(x-2)(7x-2)$ is negative.

When x is slightly more than 2, then $(x-2)(7x-2)$ is positive.

Thus, $\frac{dy}{dx}$ changes sign from negative to positive.

So, $x = 2$ is point of local minima .

Case III: When $x = -1$.

In this case, there is no change in the sign of $\frac{dy}{dx}$.

So, $x = -1$ is a point of inflexion.

S34. We have,

$$y = f(x) = -\frac{3}{4}x^4 + 2x^3 + \frac{9}{2}x^2 + 100$$

Differentiating w.r.t. "x", we get

$$\frac{dy}{dx} = -\frac{3}{4} \times 4x^3 + 6x^2 + \frac{9}{2} \times 2x = -3x^3 + 6x^2 + 9x \qquad \text{... (i)}$$

For local maxima or local minima $\frac{dy}{dx} = 0$.

$\Rightarrow \quad -3x[x^2 - 2x - 3] = 0$

$\Rightarrow \quad -3x(x^2 - 3x + x - 3) = 0$

$\Rightarrow \quad -3x(x - 3)(x + 1) = 0$

$\Rightarrow \quad x = 0, -1, 3.$

Again differentiating Eq. (i) w.r.t. "x", we get

$$\frac{d^2y}{dx^2} = -9x^2 + 12x + 9$$

Now at point $x = 0$, $\left(\frac{d^2y}{dx^2}\right)_{x=0} = 0 + 0 + 9 > 0$

So, $x = 0$ is the point of local minima & local minimum value is

$$f(0) = 0 + 0 + 0 + 100 = 100$$

At the point $x = -1$, $\left(\frac{d^2y}{dx^2}\right)_{x=-1} = -9(-1)^2 + 12(-1) + 9 = -9 - 12 + 9 < 0$

So, $x = -1$ is the point of local maxima & local maximum value is

$$f(-1) = -\frac{3}{4}(-1)^4 + 2(-1)^3 + \frac{9}{2}(-1)^2 + 100$$

$$= -\frac{3}{4} - 2 + \frac{9}{2} + 100 = \frac{407}{4}$$

At point $x = 3, \left(\frac{d^2y}{dx^2}\right)_{x=3} = -9(9)^2 + 12(3) + 9 < 0$

So, $x = 3$ is the point of local maxima & local maximum value is

$$f(3) = -\frac{3}{4} \times 81 + 2(27) + \frac{9}{2} \times 9 + 100 = \frac{535}{4}.$$

S35. We have,

$$y = f(x) = x^3 - 2ax^2 + a^2x$$

Differentiating w.r.t. "x", we get

$$\frac{dy}{dx} = 3x^2 - 4ax + a^2 \quad \ldots (i)$$

For local maxima or local minima $\frac{dy}{dx} = 0$.

$\Rightarrow \quad 3x^2 - 4ax + a^2 = 0$

$\Rightarrow \quad 3x^2 - 3ax - ax + a^2 = 0$

$\Rightarrow \quad 3x(x - a) - a(x - a) = 0$

$\Rightarrow \quad (3x - a)(x - a) = 0 \quad \Rightarrow \quad x = \frac{a}{3}, a$

Again differentiating Eq. (i) w.r.t. "x", we get

$$\frac{d^2y}{dx^2} = 6x - 4a$$

At $x = a$, $\left(\frac{d^2y}{dx^2}\right)_{x=a} = 6a - 4a = 2a > 0$

So, the given function $f(x)$ has minima at $x = a$ & local minimum value is

$$f(a) = a^3 - 2a^3 + a^3 = 0$$

At $x = \frac{a}{3}$, $\left(\frac{d^2y}{dx^2}\right)_{x=\frac{a}{3}} = -2a < 0$

So, the function $f(x)$ has maxima at $x = \frac{a}{3}$ & maximum value is

$$f\left(\frac{a}{3}\right) = \frac{a^3}{27} - 2 \times a \times \frac{a^2}{9} + a^2 \times \frac{a}{3} = a^3\left[\frac{1}{27} - \frac{2}{9} + \frac{1}{3}\right] = \frac{4a^3}{27}.$$

S36. We have,

$$y = f(x) = (x - 1)(x - 2)(x - 3)$$

Differentiating w.r.t. "x", we get

$$\frac{dy}{dx} = (x - 1)(x - 2) + (x - 2)(x - 3) + (x - 1)(x - 3)$$

$$= 3x^2 - 12x + 11 \qquad \text{... (i)}$$

For local maxima or local minima $\frac{dy}{dx} = 0$.

$\Rightarrow \quad 3x^2 - 12x + 11 = 0$

$\Rightarrow \quad x = \frac{12 \pm \sqrt{144 - 132}}{6} = \frac{6 \pm \sqrt{3}}{3}$

$\Rightarrow \quad x = \frac{6 + \sqrt{3}}{3}, \frac{6 - \sqrt{3}}{3}.$

Again differentiating Eq. (i) w.r.t. "x", we get

$$\frac{d^2y}{dx^2} = 6x - 12$$

At point $x = \dfrac{6+\sqrt{3}}{3}$,

$$\left(\frac{d^2y}{dx^2}\right)_{x=\frac{6+\sqrt{3}}{3}} = \frac{6(6+\sqrt{3})}{3} - 12 > 0$$

Hence, $x = \dfrac{6+\sqrt{3}}{3}$ is the point of local minima & local minimum value is

$$f\left(\frac{6+\sqrt{3}}{3}\right) = \left(\frac{6+\sqrt{3}}{3} - 1\right)\left(\frac{6+\sqrt{3}}{3} - 2\right)\left(\frac{6+\sqrt{3}}{3} - 3\right)$$

$$= \frac{(3+\sqrt{3})(\sqrt{3})(\sqrt{3}-3)}{27} = \frac{\sqrt{3}}{27}\left((\sqrt{3})^2 - (3)^2\right)$$

$$= \frac{\sqrt{3}}{27}(3-9) = \frac{-6\sqrt{3}}{27} = \frac{-2\sqrt{3}}{9}$$

At point $x = \dfrac{6-\sqrt{3}}{3}$,

$$\left(\frac{d^2y}{dx^2}\right)_{x=\frac{6-\sqrt{3}}{3}} = \frac{6(6-\sqrt{3})}{3} - 12 < 0$$

So, $x = \dfrac{6-\sqrt{3}}{3}$ is the point of local maxima & local maximum value is

$$f\left(\frac{6-\sqrt{3}}{3}\right) = \left(\frac{6-\sqrt{3}}{3} - 1\right)\left(\frac{6-\sqrt{3}}{3} - 2\right)\left(\frac{6-\sqrt{3}}{3} - 3\right)$$

$$= \frac{(3-\sqrt{3})}{27}(-\sqrt{3})(-3-\sqrt{3}) = \frac{(9-3)(\sqrt{3})}{27} = \frac{2\sqrt{3}}{9}.$$

Hence, the maximum value of $f(x)$ is $\dfrac{2\sqrt{3}}{9}$.

S37.

$$y = \frac{ax-b}{(x-1)(x-4)} = \frac{ax-b}{x^2-5x+4} \qquad \dots \text{(i)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(x^2-5x+4)\frac{d}{dx}(ax-b) - (ax-b)\frac{d}{dx}(x^2-5x-4)}{(x^2-5x+4)^2} \qquad \dots \text{(ii)}$$

$$= \frac{(x^2-5x+4)a - (ax-b)(2x-5)}{(x^2-5x+4)^2}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_P = \frac{(4-10+4)a - (2a-b)(4-5)}{(4-10+4)^2} = \frac{-2a+2a-b}{4} = -\frac{b}{4}$$

Since P is a turning point of the curve (i). Therefore

$$\left(\frac{dy}{dx}\right)_P = \Rightarrow \quad -\frac{b}{4} = 0 \quad \Rightarrow \quad b = 0 \qquad \text{... (iii)}$$

Since $P(2, -1)$ lies on $y = \dfrac{ax - b}{(x-1)(x-4)}$. Therefore,

$$-1 = \frac{2a - b}{(2-1)(2-4)}$$

$$-1 = \frac{2a-b}{-2}$$

$$2a - b = 2 \qquad \text{... (iv)}$$

From Eq. (iii) and Eq. (iv), we get $a = 1$, $b = 0$.

Substituting the values of a and b in (ii), we get

$$\frac{dy}{dx} = \frac{(x^2 - 5x + 4) - x(2x - 5)}{(x^2 - 5x + 4)^2} = \frac{-x^2 + 4}{(x^2 - 5x + 4)^2}$$

$$\Rightarrow \qquad \frac{d^2y}{dx^2} = \frac{(x^2 - 5x + 4)^2 \frac{d}{dx}(4 - x^2) - (-x^2 + 4)\frac{d}{dx}(x^2 - 5x + 4)^2}{(x^2 - 5x + 4)^4}$$

$$= \frac{(x^2 - 5x + 4)^2(-2x) - 2(-x^2 + 4)(x^2 - 5x + 4)(2x - 5)}{(x^2 - 5x + 4)^4}$$

or $$\frac{d^2y}{dx^2} = \frac{-2x(x^2 - 5x + 4) + 2(x^2 - 4)(2x - 5)}{(x^2 - 5x + 4)^3}$$

$$\left(\frac{d^2y}{dx^2}\right)_{(2,-1)} = \frac{-4(4 - 10 + 4) + 2(4 - 4)(4 - 5)}{(4 - 10 + 4)^3}$$

$$= \frac{8 + 0}{-8} = -1 < 0$$

Now, $\left(\dfrac{dy}{dx}\right)_{(2,-1)} = 0$ and $\dfrac{d^2y}{dx^2} = -1 < 0$

So, y is maximum at P when $a = 1$ and $b = 0$.

S38. Let $y = \sin^p\theta\cos^q\theta$. Then,

$$\frac{dy}{d\theta} = p\sin^{p-1}\theta\cos\theta\cos^q\theta + \sin^p\theta\, q\cos^{q-1}\theta(-\sin\theta)$$

$$\Rightarrow \qquad \frac{dy}{d\theta} = p\sin^{p-1}\theta\cos^{q+1}\theta - q\sin^{p+1}\theta\cos^{q-1}\theta$$

$$\Rightarrow \quad \frac{dy}{d\theta} = \sin^{p-1}\theta\cos^{q-1}\theta(p\cos^2\theta - q\sin^2\theta)$$

$$\Rightarrow \quad \frac{dy}{d\theta} = \sin^p\theta\cos^q\theta\left(\frac{p\cos^2\theta - q\sin^2\theta}{\sin\theta\cos\theta}\right)$$

$$\Rightarrow \quad \frac{dy}{d\theta} = \sin^p\theta\cos^q\theta(p\cot\theta - q\tan\theta)$$

For maximum or minimum, we must have

$$\frac{dy}{d\theta} = 0$$

$$\Rightarrow \quad \sin^p\theta\cos^q\theta(p\cot\theta - q\tan\theta) = 0$$

$$\Rightarrow \quad \sin^p\theta = 0 \text{ or } \cos^q\theta = 0 \text{ or } p\cot\theta - q\tan\theta = 0$$

$$\Rightarrow \quad \sin^p\theta = 0 \text{ or, } \cos^q\theta = 0 \text{ or, } \tan\theta = \sqrt{\frac{p}{q}}$$

$$\Rightarrow \quad \theta = 0 \text{ or, } \theta = \pi/2 \text{ or, } \theta = \tan^{-1}\sqrt{\frac{p}{q}} = \alpha \text{ (say)}$$

Now, $$\frac{dy}{d\theta} = \sin^p\theta\cos^q\theta(p\cot\theta - q\tan\theta) = y(p\cot\theta - q\tan\theta)$$

$$\Rightarrow \quad \frac{d^2y}{d\theta^2} = \frac{dy}{d\theta}(p\cot\theta - q\tan\theta) + y(-p\,\text{cosec}^2\theta - q\sec^2\theta)$$

$$\Rightarrow \quad \left(\frac{d^2y}{d\theta^2}\right)_{\theta=\alpha} = \left(\frac{dy}{d\theta}\right)_{\theta=\alpha}\left(p\sqrt{\frac{q}{p}} - q\sqrt{\frac{p}{q}}\right) + \sin^p\theta\cos^q\theta[-p\,\text{cosec}^2\theta - q\sec^2\theta]$$

$$\Rightarrow \quad \frac{d^2y}{d\theta^2} = \left(\frac{dy}{d\theta}\right)_{\theta=\alpha}\left(\sqrt{pq} - \sqrt{pq}\right) - \sin^p\theta\cos^q\theta[p\,\text{cosec}^2\theta + q\sec^2\theta]$$

$$\Rightarrow \quad \left(\frac{d^2y}{d\theta^2}\right)_{\theta=\alpha} = 0 - \sin^p\theta\cos^q\theta(p\,\text{cosec}^2\theta + q\sec^2\theta) < 0$$

Hence, y is maximum when $\theta = \alpha = \tan^{-1}\sqrt{\frac{p}{q}}$.

S39. We have,

$$y = f(x) = \tan x - 2x$$

Differentiating w.r.t. "x", we get

$$\frac{dy}{dx} = \sec^2 x - 2 \qquad \text{... (i)}$$

For local maxima or local minima $\frac{dy}{dx} = 0$.

$\Rightarrow \qquad \sec^2 x - 2 = 0$

$\Rightarrow \qquad \sec^2 x = 2 \qquad \Rightarrow \qquad \sec x = \pm\sqrt{2}$

$\Rightarrow \qquad x = \frac{\pi}{4}, \frac{3\pi}{4}.$

Again differentiating Eq. (i) w.r.t. "x", we get

$$\frac{d^2y}{dx^2} = 2 \sec x (\sec x \tan x) = 2 \sec^2 x \tan x$$

Now at point $x = \frac{\pi}{4}$,

$$\left(\frac{d^2y}{dx^2}\right)_{x=\frac{\pi}{4}} > 0$$

So, $x = \frac{\pi}{4}$ is the point of local minima & local minimum value is

$$f\left(\frac{\pi}{4}\right) = \tan\frac{\pi}{4} - 2 \times \frac{\pi}{4} = 1 - \frac{\pi}{2}.$$

At point $x = \frac{3\pi}{4}$,

$$\left(\frac{d^2y}{dx^2}\right)_{x=\frac{3\pi}{4}} = 2\sec^2\left(\frac{3\pi}{4}\right)\tan\frac{3\pi}{4} < 0.$$

So, $x = \frac{3\pi}{4}$ is the point of local maxima & local maximum value is

$$f\left(\frac{3\pi}{4}\right) = \tan\frac{3\pi}{4} - 2 \times \frac{3\pi}{4} = -1 - \frac{3\pi}{2}.$$

Date: 18/10/2021

Q1. If the function $x^4 - 62x^2 + ax + 9$ attains the maximum value at x = 1 on the interval [0,2]. Find the value of *a*.

Q2. Find the maximum and minimum values of $f(x) = 2x^3 - 24x + 107$ in the interval [1, 3].

Q3. If the function $x^3 - 31x^2 + ax + 5$ attains its maximum value at x = 2 on the interval [0, 3]. Find the value of *a*.

Q4. Find the absolute maximum value and the absolute minimum value of the following functions in the given intervals.

$$f(x) = (x-1)^2 + 3 \text{ in } [-3, 1]$$

Q5. Find the absolute maximum value and the absolute minimum value of the following functions in the given intervals.

$$f(x) = 4x - \frac{x^2}{2} \quad \text{in} \quad [-2, 4.5]$$

Q6. Find the absolute maximum value and the absolute minimum value of the following functions in the given intervals.

$$f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25 \text{ in } [0, 3]$$

Q7. Find the absolute maximum and minimum values of a function *f* given by

$$f(x) = 2x^3 - 15x^2 + 36x + 1 \text{ on the interval } [1, 5]$$

Q8. Find absolute maximum and minimum values of a function *f* given by

$$f(x) = 12x^{4/3} - 6x^{1/3}, \quad x \in [-1, 1]$$

Q9. Find the absolute maximum value and the absolute minimum value of the following function in the given interval:

$$f(x) = (x-2)\sqrt{x-1} \quad \text{in} \quad [1, 9]$$

Q10. Find the absolute minimum value and the absolute maximum value of the function $f(x) = (x-1)^{\frac{1}{3}}(x-2)$, in the interval [1, 9].

Q11. Find the absolute maximum value and the absolute minimum value of the following function in the given interval:

$$f(x) = x^3 - 12x^2 + 36x + 17 \quad \text{in} \quad [1, 10]$$

Q12. Find the absolute maximum value and the absolute minimum value of the following function in the given interval.

$$f(x) = (1/2 - x)^2 + x^3 \text{ in } [-2, 2.5]$$

Q13. Find the maximum and minimum values of $f(x) = x + \sin 2x$ in the interval $[0, 2\pi]$.

Q14. Find the absolute maximum value and the absolute minimum value of the following functions in the given intervals.

$$f(x) = \sin x + \cos x \text{ in } [0, \pi]$$

Q15. Find the maximum value of $2x^3 - 24x + 107$ in the interval $[-3, -1]$.

Q16. Show that $f(x) = \sin x\,(1 + \cos x)$ is maximum at $x = \frac{\pi}{3}$ in the interval $[0, \pi]$.

Q17. Find the maximum and minimum values of $f(x) = x^{50} - x^{20}$ in the interval $[0, 1]$.

Q18. Find both the maximum and minimum value of $3x^4 - 8x^3 + 12x^2 - 48x + 1$ on the interval $[1, 4]$.

Q19. Find the absolute maximum and minimum values of the function f given by

$$f(x) = \cos^2 x + \sin x,\ x \in [0, \pi]$$

Q20. Find the absolute maximum value and the absolute minimum value of the following function in the given interval:

$$f(x) = \sin x + \frac{1}{2}\cos 2x \quad \text{in} \quad \left[0, \frac{\pi}{2}\right]$$

Date: 18/10/2021

S1. Let $f(x) = x^4 - 62x^2 + ax + 9$. Then,

$$f'(x) = 4x^3 - 124x + a.$$

It is given that $f(x)$ attains its maximum value at $x = 1$.

$\therefore \quad f'(1) = 0$

$\Rightarrow \quad 4 - 124 + a = 0$

$\Rightarrow \quad a = 120.$

S2. We have, $f(x) = 2x^3 - 24x + 107$

$\Rightarrow \quad f'(x) = 6x^2 - 24$

Now, $f'(x) = 0$

$\Rightarrow \quad 6x^2 - 24 = 0$

$\Rightarrow \quad x = \pm 2$

But, $x = -2 \notin [1, 3].$

So $x = 2$ is the only stationary point.

Now, $f(1) = 2 - 24 + 107 = 85,$

$f(2) = 2(2)^3 - 24(2) + 107 = 16 - 48 + 107 = 75$

and, $f(3) = 2(3)^3 - 24 \times 3 + 107 = 54 - 72 + 107 = 89$

Hence, the maximum value of $f(x)$ is 89 which attains at $x = 3$ and the minimum value is 75 which is attained at $x = 2$.

S3. We have,

$$y = f(x) = x^3 - 31x^2 + ax + 5$$

Differentiating w.r.t. "x", we get

$$\frac{dy}{dx} = 3x^2 - 62x + a$$

Since it is given that at $x = 2$ it attains maximum value

So, $\frac{dy}{dx} = 0 \quad \text{at} \quad x = 2$

$3(4) - 62 \times 2 + a = 0$

$12 - 124 + a = 0$

$a = 112.$

S4. We have,

$$f(x) = (x-1)^2 + 3 \text{ in } [-3, 1]$$

$\therefore$ $f'(x) = 2(x-1)$

$f'(x) = 0$

$\Rightarrow$ $(x-1) = 0$

$\Rightarrow$ $x = 1.$

Now, $f(-3) = (-3-1)^2 + 3$

$= 16 + 3 = 19$

and $f(1) = (1-1)^2 + 3$

$= 3$

$\therefore$ Absolute maximum is 19 at $x = -3$, Absolute minimum is 3 at $x = 1$.

S5. We have,

$$f(x) = 4x - \frac{x^2}{2} \text{ in } [-2, 4, 5]$$

$\Rightarrow$ $f'(x) = 4 - x$

put $f'(x) = 0$

$\Rightarrow$ $x = 4$

Now, $f(-2) = 4(-2) - \frac{(2)^2}{2}$

$= -8 - 2 = -10,$

$f(4.5) = 4(4.5) - \frac{(4.5)^2}{2}$

$= 18 - 10.125 = 7.875$

and $f(4) = 4(4) - \frac{(4)^2}{2}$

$= 16 - 8 = 8.$

$\therefore$ Absolute maximum is 8 at $x = 4$.
Absolute minimum is -10 at $x = -2$

S6. We have,

$$f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25 \qquad x \in [0, 3]$$

$\Rightarrow$ $f'(x) = 12x^3 - 24x^2 + 24x - 48$

$\therefore$ $f'(x) = 0$

$\Rightarrow$ $12x^3 - 24x^2 + 24x - 48 = 0$

$\Rightarrow$ $12(x^3 - 2x^2 + 2x - 4) = 0$

$\Rightarrow$ $x^2(x-2) + 2(x-2) = 0$

$\Rightarrow \quad (x-2)(x^2+2) = 0$

$\Rightarrow \quad x = 2 \quad [\because x^2 + 2 \neq 0]$

Now, $f(0) = 25,$

$f(2) = 3(2)^4 - 8(2)^3 + 12(2)^2 - 48(2) + 25$

$= 48 - 64 + 48 - 96 + 25 = -39$

and $f(3) = 3(3)^4 - 8(3)^3 + 12(3)^2 - 48(3) + 25$

$= 243 - 216 + 108 - 144 + 24 = 16$

$\therefore$ Absolute maximum is 25 at $x = 0$

Absolute minimum $= -39$ at $x = 2$.

S7. We have,

$f(x) = 2x^3 - 15x^2 + 36x + 1, \; x \in [1, 5]$

$\therefore \quad f'(x) = 6x^2 - 30x + 36$

$= 6(x^2 - 5x + 6)$

$f'(x) = 0$

$\Rightarrow \quad 6(x^2 - 5x + 6) = 0$

$\Rightarrow \quad x = 2, 3$

Now, $f(1) = 2 - 15 + 36 + 1 = 24$

$f(2) = 2(2)^3 - 15(2)^2 + 36(2) + 1$

$= 16 - 60 + 72 + 1 = 29$

$f(3) = 2(3)^3 - 15(3)^2 + 36(3) + 1$

$= 54 - 135 + 108 + 1 = 28$

$f(5) = 2(5)^3 - 15(5)^2 + 36(5) + 1$

$= 250 - 375 + 180 + 1 = 56$

$\therefore$ Absolute maximum $= 56$ at $x = 5$

Absolute minimum $= 24$ at $x = 1$.

S8. $f(x) = 12x^{4/3} - 6x^{1/3}, \quad x \in [-1, 1]$

$f'(x) = 16x^{1/3} - 2x^{-2/3}$

$= \dfrac{2(8x-1)}{x^{2/3}}$

$\therefore \quad f'(x) = 0$

$\Rightarrow \quad \dfrac{2(8x-1)}{x^{2/3}} = 0$

$\Rightarrow \quad x = \dfrac{1}{8}$

Now, $f(-1) = 12(-1)^{4/3} - 6\,(-1)^{1/3}$

$= 12 + 6 = 18$

$$f\left(\frac{1}{8}\right) = 12\left(\frac{1}{8}\right)^{4/3} - 6\left(\frac{1}{8}\right)^{1/3}$$

$$= 12\left(\frac{1}{16}\right) - 6\left(\frac{1}{2}\right)$$

$$= \frac{3}{4} - 3 = \frac{3-12}{4} = -\frac{9}{4}$$

$$f(1) = 12(1)^{4/3} - 6\,(1)^{1/3}$$

$$= 12 - 6 = 6$$

$\therefore$ Absolute maximum = 18 at $x = -1$

Absolute minimum = $-\frac{9}{4}$ at $x = \frac{1}{8}$

S9. We have,

$$y = f(x) = (x-2)\sqrt{x-1} \qquad \dots (i)$$

Differentiating w.r.t. "x", we get

$$\frac{dy}{dx} = (x-2)\frac{1}{2}(x-1)^{-1/2} + \sqrt{x-1} = (x-1)^{-1/2}\left[\frac{x-2}{2} + (x-1)\right]$$

$$= (x-1)^{-1/2}\left[\frac{x-2+2x-2}{2}\right] = (x-1)^{-1/2}\left[\frac{3x-4}{2}\right)$$

For maxima or minima $\frac{dy}{dx} = 0$.

$$(x-1)^{-1/2}\left(\frac{3x-4}{2}\right) = 0$$

$\Rightarrow$ $x = \frac{4}{3} \in [1, 9]$

Putting $x = 1, \frac{4}{3}, 9$ in Eq. (i), we get

$$f(1) = 0$$

$$f\left(\frac{4}{3}\right) = \left(\frac{4}{3} - 2\right)\sqrt{\frac{4}{3} - 1} = \left(\frac{-2}{3}\right)\sqrt{\frac{1}{3}}$$

$$f(9) = (9-2)\sqrt{9-1} = 14\sqrt{2}$$

$\therefore$ The function $f(x)$ has maximum value $14\sqrt{2}$ which attains at $x = 9$ & minimum value is $\frac{-2}{3\sqrt{3}}$ which attains at $x = \frac{4}{3}$.

S10. We have,

$$f(x) = (x-1)^{\frac{1}{3}}(x-2) \quad \text{... (i)}$$

Differentiating Eq. (i) w.r.t. "x", we get

$$\Rightarrow \quad f'(x) = (x-1)^{\frac{1}{3}} \cdot 1 + (x-2) \cdot \frac{1}{3}(x-1)^{-\frac{2}{3}}$$

$$= (x-1)^{\frac{1}{3}} + \frac{(x-2)}{3(x-1)^{\frac{2}{3}}} = \frac{4x-5}{3(x-1)^{2/3}}$$

For maxima or minima, we must have $f'(x) = 0$

$$\Rightarrow \quad \frac{4x-5}{3(x-1)^{2/3}} = 0 \quad \Rightarrow \quad 4x-5 = 0 \quad \Rightarrow \quad x = \frac{5}{4}.$$

Now, putting $x = 1$, $\frac{5}{4}$ and 9 in Eq. (i), we get

$$f(1) = (1-1)^{\frac{1}{3}}(1-2) = 0$$

$$f\left(\frac{5}{4}\right) = \left(\frac{5}{4}-1\right)^{1/3}\left(\frac{5}{4}-2\right) = \frac{-3}{4^{\frac{4}{3}}}$$

$$f(9) = (9-1)^{\frac{1}{3}}(9-2) = 14.$$

Thus, the maximum value of $f(x)$ is 14 which is attained at $x = 9$ and the minimum value is $\frac{-3}{4^{\frac{4}{3}}}$, which is attained at $x = \frac{5}{4}$.

S11. We have,

$$y = f(x) = x^3 - 12x^2 + 36x + 17 \quad \text{... (i)}$$

Differentiating w.r.t. "x", we get

$$\frac{dy}{dx} = 3x^2 - 24x + 36 \quad \text{... (ii)}$$

For local maxima or local minima $\frac{dy}{dx} = 0$.

$$3(x^2 - 8x + 12) = 0$$

$$\Rightarrow \quad 3(x-2)(x-6) = 0 \quad \Rightarrow \quad x = 2, 6$$

As $2 \in [1, 10]$ & $6 \in [1, 10]$

So 2 and 6 are the stationary point.

Now putting $x = 1, 2, 6, 10$ in Eq. (i), we have

$$f(1) = 1 - 12 + 36 + 17 = 42$$

$$f(2) = 8 - 48 + 72 + 17 = 49$$

$$f(6) = 216 - 432 + 216 + 17 = 17$$

$$f(10) = 1000 - 1200 + 360 + 17 = 177$$

Thus, the maximum value of $f(x)$ is 177 which attains at $x = 10$ & minimum value of $f(x)$ is 17 which attains at $x = 6$.

S12. We have,

$$f(x) = (1/2 - x)^2 + x^3 \text{ in } [-2, 2.5]$$

$$f'(x) = 2\left(\frac{1}{2} - x\right)(-1) + 3x^2$$

$$= -1 + 2x + 3x^2$$

For maximum or minimum we must have

$$f'(x) = 0$$

$$\Rightarrow \quad 3x^2 + 2x - 1 = 0$$

$$\Rightarrow \quad (3x - 1)(x + 1) = 0$$

$$\Rightarrow \quad x = 1/3, -1$$

Now, $$f(-2) = \left(\frac{1}{2} + 2\right)^2 + (-2)^3 = \frac{25}{4} - 8 = -\frac{7}{4}$$

$$f\left(\frac{1}{3}\right) = \left(\frac{1}{2} - \frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 = \frac{1}{36} + \frac{1}{27} = \frac{7}{108},$$

$$f(-1) = \left(\frac{1}{2} + 1\right)^2 + (-1)^3 = \frac{9}{4} - 1 = \frac{5}{4}$$

and $$f(2.5) = \left(\frac{1}{2} - 2.5\right)^2 + (2.5)^3 = 4 + \frac{125}{8} = \frac{32 + 125}{8} = \frac{157}{8}$$

Thus, the absolute maximum $= \frac{157}{8}$ at $x = 2.5$ and,

the absolute minimum $= -\frac{7}{4}$ at $x = -2$

S13. We have,

$$f(x) = x + \sin 2x$$

$$\Rightarrow \quad f'(x) = 1 + 2\cos 2x$$

For stationary points, we have

$$f'(x) = 0$$

$$\Rightarrow \quad 1 + 2\cos 2x = 0$$

$$\Rightarrow \quad \cos 2x = -\frac{1}{2}$$

$$\Rightarrow \quad 2x = \frac{2\pi}{3} \text{ or, } 2x = \frac{4\pi}{3} \qquad [\because 0 \le x \le 2\pi \quad \therefore 0 \le 2x \le 4\pi]$$

$$\Rightarrow \quad x = \frac{\pi}{3} \text{ or, } x = \frac{2\pi}{3}$$

Now, $f(0) = 0 + \sin 0 = 0$

$$f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} + \sin\frac{2\pi}{3} = \frac{\pi}{3} + \frac{\sqrt{3}}{2},$$

$$f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + \sin\frac{4\pi}{3} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

and $f(2\pi) = 2\pi + \sin 4\pi = 2\pi + 0 = 2\pi$

Thus, the maximum value of $f(x)$ is 2π and the minimum value is 0. In the interval $[0, 2\pi]$

S14. We have,

$$f(x) = \sin x + \cos x, \text{ where } x \in [0, \pi]$$

$$\Rightarrow \quad f'(x) = \cos x - \sin x$$

For maximum or minimum, we must have

$$f'(x) = 0$$

$$\Rightarrow \quad \cos x - \sin x = 0$$

$$\Rightarrow \quad \cos x = \sin x$$

$$\Rightarrow \quad \tan x = 1$$

$$\Rightarrow \quad x = \pi/4$$

Now, $f(0) = \sin 0 + \cos 0 = 0 + 1$

$$f(\pi/4) = \sin \pi/4 + \cos \pi/4$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

and $f(\pi) = \sin \pi + \cos \pi = 0 - 1 = -1.$

So, absolute maximum = $\sqrt{2}$ and, absolute minimum = -1.

S15. Let $f(x) = 2x^3 - 24x + 107$. Then,

$$f'(x) = 6x^2 - 24 = 6(x-2)(x+2)$$

$$\therefore \quad f'(x) = 0$$

$$\Rightarrow \quad 6(x-2)(x+2) = 0$$

$$\Rightarrow \quad x = -2, 2$$

Because $f(x)$ is defined on $[-3, -1]$

then $\quad f'(x) = 0$ at $x = -2$.

Now, $\quad f(-3) = 2(-3)^3 - 24(-3) + 107$

$$= -54 + 72 + 107 = 125$$

$$f(-2) = 2(-2)^3 - 24(-2) + 107$$

$$= -16 + 48 + 107 = 139$$

$$f(-1) = (-1)^3 - 24(-1) + 107$$

$$= -2 + 24 + 107 = 129$$

$\therefore$ The absolute maximum is 139 at $x = -2$.

S16. We have,

$$\Rightarrow \quad f(x) = \sin x(1 + \cos x)$$

$$\Rightarrow \quad f'(x) = (1+\cos x)\frac{d}{dx}(\sin x) + \sin x\,\frac{d}{dx}(1+\cos x)$$

$$\Rightarrow \quad f'(x) = \cos x(1 + \cos x) - \sin^2 x$$

$$\Rightarrow \quad f'(x) = \cos x + \cos^2 x - (1 - \cos^2 x)$$

$$\Rightarrow \quad f'(x) = 2\cos^2 x + \cos x - 1 = (2\cos x - 1)(\cos x + 1)$$

For stationary values, we have,

$$f'(x) = 0$$

$$\Rightarrow \quad (2\cos x - 1)(\cos x + 1) = 0$$

$$\Rightarrow \quad \cos x = \frac{1}{2} \text{ or, } \cos x = -1$$

$$\Rightarrow \quad x = \frac{\pi}{3} \text{ or, } x = \pi$$

Now, $\quad f(0) = \sin 0°(1 + \cos 0°)$

$$= 0(1 + 1) = 0$$

$$f\left(\frac{\pi}{3}\right) = \sin\frac{\pi}{3}\left(1+\cos\frac{\pi}{3}\right)$$

$$= \frac{\sqrt{3}}{2}\left(1+\frac{1}{2}\right) = \frac{3\sqrt{3}}{4}$$

and $f(\pi) = \sin \pi\ (1 + \cos \pi).$

$$= 0\ (1 - 1) = 0.$$

Hence, $f(x)$ attains the maximum value $\frac{3\sqrt{3}}{4}$ at $x = \pi/3$ in the interval $[0, \pi]$

S17. Let $f(x) = x^{50} - x^{20}.$

Then, $f'(x) = 50x^{49} - 20x^{19},$

For maximum or minimum we must have

$$f'(x) = 0$$

$\Rightarrow$ $50x^{49} - 20x^{19} = 0$

$\Rightarrow$ $x^{19}(50x^{30} - 20) = 0$

$\Rightarrow$ $x = 0$ or $50x^{30} = 20$

$\Rightarrow$ $x = 0$ or $x = \left(\frac{2}{5}\right)^{1/30}$

Now, $f(0) = 0,$

$$f\left(\frac{2}{5}\right)^{1/30} = \left(\frac{2}{5}\right)^{50/30} - \left(\frac{2}{5}\right)^{20/30}$$

$$= \left(\frac{2}{5}\right)^{2/3}\left(\frac{2}{5}-1\right) = -\frac{3}{5}\left(\frac{2}{5}\right)^{2/3}$$

and, $f(1) = 1 - 1 = 0$

Thus, the maximum value of $f(x)$ is 0 and the minimum value of $f(x)$ is $-\frac{3}{5}\left(\frac{2}{5}\right)^{2/3}$ in the interval [0, 1]

S18. Let $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 1.$

Then, $f'(x) = 12x^3 - 24x^2 + 24x - 48$

Now, $f'(x) = 0$

$\Rightarrow$ $12x^3 - 24x^2 + 24x - 48 = 0$

$\Rightarrow$ $x^3 - 2x^2 + 2x - 4 = 0$

$\Rightarrow$ $x^2(x - 2) + 2(x - 2) = 0$

$\Rightarrow \quad (x-2)(x^2+2) = 0$

$\Rightarrow \quad x = 2 \qquad [\because x^2 + 2 \neq 0]$

Now, $f(2) = 3(2)^4 - 8(2)^3 + 12(2)^2 - 48(2) + 1$

$= 48 - 64 + 48 - 96 + 1 = -63$

$f(1) = 3 - 8 + 12 - 48 + 1 = -40$

$f(4) = 3(4)^4 - 8(4)^3 + 12(4)^2 - 48(4) + 1$

$= 4(192 - 128 + 48 - 48) + 1$

$= 4(64) + 1$

$= 256 + 1 = 257.$

So, the minimum and maximum values of $f(x)$ on [1, 4] are -63 and 257 respectively.

S19. We have,

$f(x) = \cos^2 x + \sin x,\ x \in [0, \pi]$

$\Rightarrow \quad f'(x) = -2\sin x \cos x + \cos x$

$\Rightarrow \quad f'(x) = \cos x\,(1 - 2\sin x)$

$\therefore \quad f'(x) = 0$

$\Rightarrow \quad \cos x = 0,\ \sin x = \dfrac{1}{2}$

$\Rightarrow \quad x = \dfrac{\pi}{2}, \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

Now, $f(0) = \cos^2 0° + \sin 0°$

$= 1 + 0 = 1,$

$f(\pi) = \cos^2\pi + \sin\pi$

$= (-1)^2 + 0 = 1,$

$f\left(\dfrac{\pi}{2}\right) = \cos^2\dfrac{\pi}{2} + \sin\dfrac{\pi}{2}$

$= 0 + 1 = 1$

$f\left(\dfrac{\pi}{6}\right) = \cos^2\dfrac{\pi}{6} + \sin\dfrac{\pi}{6}$

$= \left(\dfrac{\sqrt{3}}{2}\right)^2 + \dfrac{1}{2}$

$= \dfrac{3}{4} + \dfrac{1}{2} = \dfrac{5}{4}$

and $$f\left(\frac{5\pi}{6}\right) = \cos^2\frac{5\pi}{6} + \sin\frac{5\pi}{6}$$

$$= \left(\frac{-\sqrt{3}}{2}\right)^2 + \frac{1}{2}$$

$$= \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$$

$\therefore$ Absolute maximum = $\frac{5}{4}$ at $\frac{\pi}{6}, \frac{5\pi}{6}$

Absolute minimum = 1, at $x = 0, \frac{\pi}{2}, \pi$

S20. We have,

$$y = f(x) = \sin x + \frac{1}{2}\cos 2x \quad \text{... (i)}$$

Differentiating w.r.t. "x", we get

$$\frac{dy}{dx} = \cos x + \frac{1}{2}(\sin 2x)(-2) = \cos x - \sin 2x$$

For maxima or minima $\frac{dy}{dx}$ = 0.

$$\cos x - \sin 2x = 0$$

$$\sin 2x = \cos x$$

$$2\sin x \cos x = \cos x$$

$$2\sin x = 1$$

$$\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$$

But $\frac{5\pi}{6} \notin \left[0, \frac{\pi}{2}\right]$ so $\frac{\pi}{6}$ is the only stationary point.

Putting $x = 0, \frac{\pi}{6}, \frac{\pi}{2}$ in Eq. (i), we get

$$f(0) = \sin 0 + \frac{1}{2}\cos 0 = \frac{1}{2}$$

$$f\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{6} + \frac{1}{2}\cos\frac{\pi}{3} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) + \frac{1}{2}\cos\pi = 1 - \frac{1}{2} = \frac{1}{2}$$

Thus, the maximum value of $f(x)$ is $\frac{3}{4}$ which attains at $x = \frac{\pi}{6}$ and minimum value of $f(x)$ is $\frac{1}{2}$ which attains at $x = 0, \frac{\pi}{2}$.

Date: 18/10/2021

Q1. Show that the rectangle of maximum perimeter which can be inscribed in a circle be of radius a is a square of side $\sqrt{2}a$.

Q2. Show that of all the rectangles with a given perimeter, the square has the largest area.

Q3. Show that of all rectangles of given area, the square has the smallest perimeter.

Q4. A point on the hypotenuse of a right angle triangle is at distance a and b from the sides of the triangle. Show that the minimum length of the hypotenuse is $\left(a^{2/3}+b^{2/3}\right)^{3/2}$.

Q5. If the sum of the lengths of the hypotenuse and a side of a right angled triangle is given, show that the area of the triangle is maximum when the angle between them is $\frac{\pi}{3}$.

Q6. Prove that the area of a right angle triangled of given hypotenuse is maximum when the triangle is isosceles.

Q7. Find two positive numbers whose sum is 14 and the sum of whose square is minimum.

Q8. Amongst all pairs of positive numbers with product 256, find those whose sum is the least.

Q9. Find two positive numbers x and y such that their sum is 35 and the product $x^2 y^5$ is maximum.

Q10. Find two positive numbers x *and* y such that $x + y = 60$ and xy^3 is maximum.

Q11. Find the area of the greatest rectangle that can be inscribed in an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Q12. Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$, with it's vertex at one end of the major axis.

Q13. Let AP and BQ be two vertical poles at points A and B respectively. If $AP = 16$ m, $BQ = 22$ m and $AB = 20$ m, then find the distance of a point R on AB from the point A such that $RP^2 + RQ^2$ is minimum.

Q14. Show that the triangle of maximum area that can be inscribed in a given circle is an equilateral triangle.

Q15. If the length of three sides of a trapezium other than the base is 10 cm each, find the area of the trapezium when it is maximum.

Q16. An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the total surface area is least when depth of the tank is half of it's width.

Q17. Show that the surface area of a closed cuboid with square base and given volume is minimum, when it is a cube.

Q18. An open box with a square base is to be made out of a given quantity of cardboard of area C^2 square units. Show that the maximum volume of box is $\frac{C^3}{6\sqrt{3}}$ cu units.

Q19. A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m^3. If building of tank cost Rs. 70 per sq m for the base and Rs. 45 per sq. m for sides. What is the cost of least expensive tank?

Q20. A square piece of tin of side 24 cm is to be made into a box without top by cutting a square from each corner and folding up the flaps to form a box. What should be the side of the square to be cut off so that the volume of the box is maximum ? Also, find this maximum volume.

Q21. Show that the semi-vertical angle of a right circular cone of given surface area and maximum volume is $\sin^{-1}\left(\frac{1}{3}\right)$.

Q22. Given the sum of the perimeter of a square and a circle is constant, show that the sum of their areas is least when the side of the square is equal to diameter of the circle.

Q23. A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off squares from each corners and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum.

Q24. A square piece of tin of side 18 cm is to be made into a box without top by cutting a square from each corner and folding up the flaps to form a box. What should be the side of the square to be cut off so that the volume of the box is maximum ? Also, find this maximum volume.

Q25. A wire of length 36 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the lengths of the two pieces, so that the combined area of the square and the circle is minimum ?

Q26. A wire of length 28m is to be cut into two pieces. One of the two pieces is to be made into a square and the other into a circle. What should be the lengths of two pieces so that the combined area of circle and square is minimum?

Q27. Show that the height of a closed right circular cylinder of given surface area and maximum volume is equal to diameter of the base.

Q28. Of all the closed cylindrical cans (right circular), which enclose a given volume of 100 cm^3, find the dimensions of the cans which has the minimum surface area ?

Q29. A window has the shape of a rectangle surmounted by an equilateral triangle. If the perimeter of the window is 12 m, find the dimensions of the rectangle that will produce the largest area of the window.

Q30. A window is in the form of a rectangle surmounted by a semi-circular opening. The total perimeter of the window is 10m. Find the dimensions of the rectangle so as to admit maximum light through the whole opening.

Q31. Show that the right circular cylinder of given volume, open at the top, has minimum total surface area when, it's height is equal to the radius of the base.

Q32. Show that the right circular cylinder, open at the top and of given surface area and maximum volume is such that its height is equal to the radius of the base.

Q33. Show that the height of the closed right circular cylinder, of given volume and minimum total surface area, is equal to its diameter.

Q34. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius (R) is $\frac{2R}{\sqrt{3}}$. Also, find the maximum volume.

Q35. Find the volume of the largest right circular cylinder that can be inscribed in a sphere of radius r cm.

Q36. Show that the height of the cylinder of maximum volume that can be inscribed in a cone of height h is $\frac{1}{3}h$.

Q37. Show that the right circular cone of least curved surface area and given volume has an altitude equal to $\sqrt{2}$ times the radius of the base.

Q38. Show that the semi-vertical angle of a right circular cylinder cone of maximum volume and given slant height is $\tan^{-1}\sqrt{2}$.

Q39. Show that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.

Q40. Prove that radius of right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

Q41. Show that the volume of the greatest cylinder which can be inscribed in a cone of height h and semi-vertical angle 30° is $\frac{4}{81}\pi h^3$.

Q42. Show that the volume of the greatest cylinder can be inscribed in a cone of height h and semi-vertical angle α is $\frac{4}{27}\pi h^3 \tan^2\alpha$.

Q43. Find the shortest distance of the point (0, c) from the parabola $y = x^2$, where $0 \le c \le 5$.

Q44. Find the shortest distance between the line $y - x = 1$ and the curve $x = y^2$.

Q45. Find the point on the curve $y^2 = 2x$ which is at a minimum distance from the point (1, 4).

Q46. A jet of an enemy is flying along the curve $y = x^2 + 2$. A soldier is placed at the point (3, 2). What is the nearest distance between the soldier and the jet?

Q47. Find the point on the curve $y^2 = 4x$ which is nearest to the point(2, 1).

Q48. Find the point on the curve $y^2 = 4x$ which is nearest to the point (2, – 8).

Q49. A straight line is drawn through a given point P (1, 4). Determine the least value of the sum of the intercepts on the coordinate axes.

Q50. A manufacturer can sell x items at a price of Rs. $\left(5 - \frac{x}{100}\right)$ each. The cost price of x items is Rs. $\left(\frac{x}{5} + 500\right)$. Find the number of items he should sell to reach maximum profit.

Q51. The cost function of a firm is given by $C = 300x - 10x^2 + \frac{1}{3}x^3$

where x is output. Find the output at which

(i) the average cost is minimum. (ii) the average cost is equal to the marginal cost

Q52. The cost function of a firm is given by $C = 200x - \frac{20}{3}x^2 + \frac{2}{9}x^3$

where x is output. Find the output at which the

(i) average cost is minimum. (ii) marginal cost is minimum.

(iii) marginal cost is equal to average cost.

Q53. The combined resistance R of two resistors R_1 and R_2 (R_1, $R_2 > 0$) is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

If $R_1 + R_2 = C$ (a constant), show that the maximum resistance R is obtained by choosing $R_1 = R_2$.

Q54. A beam of length l is supported at one end. If W is the uniform load per unit length, the bending moment M at a distance x from the end (supporting) is given by

$$M = \frac{1}{2}lx - \frac{1}{2}Wx^2$$

Find the point on the beam at which the bending moment has the maximum value.

Q55. If in a submarine cable the range of signaling varies as $x^2 \log_e\left(\frac{1}{x}\right)$, where x is the ratio of the radius of the case to that of the cable, find the value of x for which range of signaling is maximum.

Q56. Assuming that the petrol burnt (per hour) in driving a motor boat varies as the cube of its velocity. Show that the most economical speed going against a current of C km per hour is $\frac{3}{2}$ C km per hour.

Date: 18/10/2021

S1. Let *ABCD* be a rectangle in a given circle of radius *a* with centre at *O*. Let $AB = 2x$ and $AD = 2y$ be the sides of the rectangle. Then,

$$AM^2 + OM^2 = OA^2 \qquad [\because \Delta\, OAM \text{ is a right angled triangle}]$$

$$\Rightarrow \quad x^2 + y^2 = a^2$$

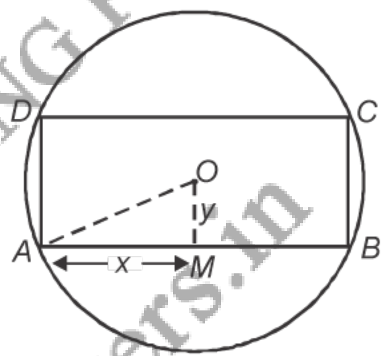
$$\Rightarrow \quad y = \sqrt{a^2 - x^2} \qquad \text{... (i)}$$

Let *P* be the perimeter of the rectangle *ABCD*. Then,

$$P = 4x + 4y$$

$$\Rightarrow \quad P = 4x + 4\sqrt{a^2 - x^2} \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad \frac{dP}{dx} = 4 - \frac{4x}{\sqrt{a^2 - x^2}}$$



For maximum or minimum, values of *P*, we have

$$\frac{dP}{dx} = 0$$

$$\Rightarrow \quad 4 - \frac{4x}{\sqrt{a^2 - x^2}} = 0$$

$$\Rightarrow \quad 4 = \frac{4x}{\sqrt{a^2 - x^2}}$$

$$\Rightarrow \quad \sqrt{a^2 - x^2} = x$$

$$\Rightarrow \quad a^2 - x^2 = x^2$$

$$\Rightarrow \quad 2x^2 = a^2$$

$$\Rightarrow \quad x = \frac{a}{\sqrt{2}}$$

Now,

$$\frac{d^2P}{dx^2} = \frac{-4\left\{\sqrt{a^2 - x^2}\cdot 1 - \frac{x(-x)}{\sqrt{a^2 - x^2}}\right\}}{(\sqrt{a^2 - x^2})^2}$$

$$= -4\left[\frac{a^2 - x^2 + x^2}{(a^2 - x^2)^{3/2}}\right] = \frac{-4a^2}{(a^2 - x^2)^{3/2}}$$

$$\therefore \quad \left(\frac{d^2P}{dx^2}\right)_{x=a/\sqrt{2}} = \frac{-4a^2}{\left(a^2-\frac{a^2}{2}\right)^{3/2}} = \frac{-4a^2}{\left(\frac{a^2}{2}\right)^{3/2}} = \frac{-8\sqrt{2}}{a} < 0$$

Thus, P is maximum when $x = \frac{a}{\sqrt{2}}$.

Putting $x = \frac{a}{\sqrt{2}}$ in (i), we obtain $y = \frac{a}{\sqrt{2}}$. Therefore, $2x = 2y = a\sqrt{2} \Rightarrow 2x = 2y$.

Hence, P is maximum when the rectangle is square of side $2x = \frac{2a}{\sqrt{2}} = \sqrt{2}a$.

S2. Let x and y be the lengths of two sides of a rectangle. Let P denotes its perimeter, A be the area of rectangle. Given that

$$P = 2(x + y) \qquad [\because \text{ Perimeter of rectangle } = 2(l + b)]$$

$$P = 2x + 2y \qquad \ldots(i)$$

$$\Rightarrow \quad y = \frac{P-2x}{2}$$

Also, we know that area of rectangle is given by

$$A = xy$$

$$\Rightarrow \quad A = x\left(\frac{P-2x}{2}\right) \qquad [\text{By using Eq. (i)}]$$

$$\Rightarrow \quad A = \frac{Px-2x^2}{2}$$

Differentiating w.r.t. x, we get $\frac{dA}{dx} = \frac{P-4x}{2}$

Now, for maxima and minima, put $\frac{dA}{dx} = 0$

$$\therefore \quad \frac{P-4x}{2} = 0$$

$$\Rightarrow \quad P - 4x = 0$$

$$\Rightarrow \quad P = 4x$$

$$\Rightarrow \quad 2x + 2y = 4x \qquad [\text{From Eq. (i)}]$$

$$\Rightarrow \quad 2y = 2x$$

$$\Rightarrow \quad x = y$$

$\because \quad x = y$, so the rectangle is a square.

Also, $$\frac{d^2A}{dx^2} = \frac{d}{dx}\left(\frac{P-4x}{2}\right) = -\frac{4}{2} = -2 < 0$$

$\because$ $$\frac{d^2A}{dx^2} < 0 \Rightarrow A \text{ is maximum.}$$

Hence, area is maximum, when rectangle is a square.

S3. Let x and y be the lengths of sides of a rectangle. Let A denotes its area and P be the perimeter.

Now, given that $A = xy$ [$\because$ Area of rectangle = $l \times b$]

$\Rightarrow$ $y = \frac{A}{x}$ ... (i)

Also, $P = 2(x + y)$ [$\because$ Perimeter of rectangle = $2(l + b)$]

$\Rightarrow$ $P = 2\left(x + \frac{A}{x}\right)$ $\left[\because y = \frac{A}{x} \text{ By eq. (i)}\right]$

Differentiating w.r.t. x, we get

$$\frac{dP}{dx} = 2\left(1 - \frac{A}{x^2}\right)$$

Now, for maxima and minima, put $\frac{dP}{dx} = 0$

$\therefore$ $2\left(1 - \frac{A}{x^2}\right) = 0$

$\Rightarrow$ $1 = \frac{A}{x^2}$

or $A = x^2$

$\Rightarrow$ $xy = x^2$ [$\because$ Given that $A = xy$]

$\Rightarrow$ $x = y$

Also, $$\frac{d^2P}{dx^2} = \frac{d}{dx}\left[2\left(1 - \frac{A}{x^2}\right)\right] = 2\left(\frac{2A}{x^3}\right) = \frac{4A}{x^3} > 0$$

and x and A being the side and area of rectangle can never be negative. So P is a minimum.

Hence, perimeter of rectangle is minimum when rectangle is a square.

S4. Let AOB be a right triangle with hypotenuse AB such that a point P on AB is at distances a and b from OA and OB respectively, *i.e.*, $PL = a$ and $PM = b$

Let $\angle OAB = \theta$. Then,

$$AP = a \operatorname{cosec} \theta \quad \text{and} \quad BP = b \sec \theta$$

Let l be the length of the hypotenuse AB. Then,

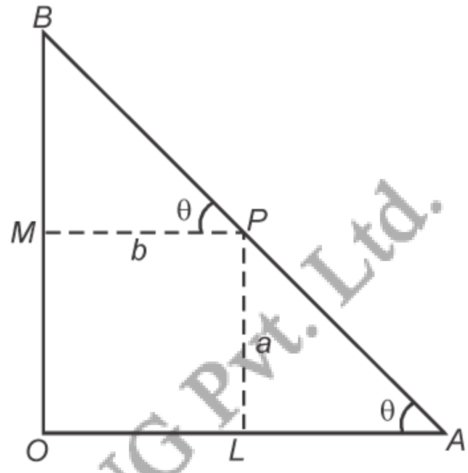
$$l = AP + BP$$

$$\Rightarrow \quad l = a \operatorname{cosec} \theta + b \sec \theta$$

$$\Rightarrow \quad \frac{dl}{d\theta} = -a \operatorname{cosec} \theta \cot \theta + b \sec \theta \tan \theta$$

and, $$\frac{d^2 l}{d\theta^2} = a \operatorname{cosec}^3 \theta + a \operatorname{cosec} \theta \cot^2 \theta + b \sec^3 \theta + b \sec \theta \tan^2 \theta$$

For maximum or minimum, we must have



$$\frac{dl}{d\theta} = 0$$

$$\Rightarrow \quad -a \operatorname{cosec} \theta \cot \theta + b \sec \theta \tan \theta = 0$$

$$\Rightarrow \quad -\frac{a \cos\theta}{\sin^2 \theta} + \frac{b \sin\theta}{\cos^2 \theta} = 0$$

$$\Rightarrow \quad \tan^3\theta = \frac{a}{b}$$

$$\Rightarrow \quad \tan\theta = \left(\frac{a}{b}\right)^{1/3}$$

$$\Rightarrow \quad \sin\theta = \frac{a^{1/3}}{\sqrt{a^{2/3} + b^{2/3}}}$$

and $$\cos\theta = \frac{b^{1/3}}{\sqrt{a^{2/3} + b^{2/3}}}$$

Clearly, $\frac{d^2 l}{d\theta^2} > 0$ for $\tan\theta = \left(\frac{a}{b}\right)^{1/3}$ $(\because$ cosce θ, $\cot\theta$, $\sec\theta$, $\tan\theta$ all are + *ve*)

Thus, l is minimum when $\tan\theta = \left(\frac{a}{b}\right)^{1/3}$

The minimum value of l is given by

$$l = a \operatorname{cosec}\theta + b \sec\theta = a\sqrt{1 + \cot^2\theta} + b\sqrt{1 + \tan^2\theta}$$

$$\Rightarrow \quad l = a\sqrt{1 + \left(\frac{b}{a}\right)^{2/3}} + b\sqrt{1 + \left(\frac{a}{b}\right)^{2/3}}$$

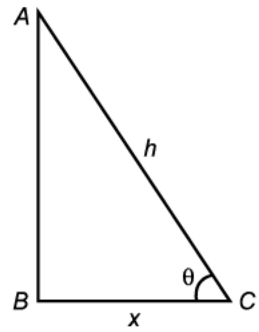
$$\Rightarrow \quad l = \frac{a\sqrt{a^{2/3} + b^{2/3}}}{a^{1/3}} + \frac{b\sqrt{b^{2/3} + a^{2/3}}}{b^{1/3}}$$

$$= a^{2/3}\sqrt{a^{2/3} + b^{2/3}} + b^{2/3}\sqrt{a^{2/3} + b^{2/3}}$$

$$= \left(a^{2/3} + b^{2/3}\right)\sqrt{a^{2/3} + b^{2/3}}$$

$$= \left(a^{2/3} + b^{2/3}\right)^{3/2}.$$

S5. Let ABC be the right angled triangle with $BC = x$ and $AC = h$. Now given that



$$h + x = a \quad \text{... (i)}$$

where, a = constant

Let A denotes the area of triangle. Then,

$$A = \frac{1}{2} BC \times AB$$

$$\Rightarrow \quad A = \frac{1}{2} x \cdot \sqrt{h^2 - x^2}$$

$$\left[\because \text{ In right } \Delta ABC,\ AB^2 = AC^2 - BC^2 = h^2 - x^2 \therefore AB = \sqrt{h^2 - x^2}\right]$$

$$\therefore \quad A = \frac{1}{2} x\sqrt{h^2 - x^2}$$

Squaring both sides, we get

$$A^2 = \frac{x^2}{4}\left(h^2 - x^2\right)$$

$$\Rightarrow \quad A^2 = \frac{x^2}{4}\left[(a - x)^2 - x^2\right] \quad [\because h = a - x \text{ by Eq. (i)}]$$

$$\Rightarrow \quad A^2 = \frac{x^2}{4}\left[a^2 + x^2 - 2ax - x^2\right]$$

Let $$Z = A^2 = \frac{a^2x^2 - 2ax^3}{4}$$

Differentiating both sides w.r.t. x, we get

$$\frac{dZ}{dx} = \frac{1}{4}\left(2a^2x - 6ax^2\right) \quad \text{... (ii)}$$

Now, for maxima and minima, put $\frac{dZ}{dx} = 0$

$$\therefore \quad \frac{1}{4}\left(2a^2x - 6ax^2\right) = 0$$

$$\therefore \quad 2a^2x = 6ax^2$$

$$\Rightarrow \quad 2a = 6x$$

or $$x = \frac{a}{3}$$

Also, differentiating Eq. (ii) w.r.t. x, we get

$$\frac{d^2z}{dx^2} = \frac{1}{4}\left(2a^2 - 12ax\right)$$

$$\left(\frac{d^2z}{dx^2}\right) at \left(x = \frac{a}{3}\right) = \frac{1}{4}\left(2a^2 - 12a\cdot\frac{a}{3}\right)$$

$$= \frac{1}{4}(2a^2 - 4a^2) = -\frac{1}{2}a^2 < 0$$

$\Rightarrow$ Z is maximum $\Rightarrow$ A is maximum.

Also, in the given right ΔABC, we have

$$\cos\theta = \frac{BC}{AC} = \frac{x}{h} = \frac{x}{a-x} \qquad [\because h = a - x]$$

$$\therefore \qquad \cos\theta = \frac{\frac{a}{3}}{a - \frac{a}{3}}$$

$$= \frac{\frac{a}{3}}{\frac{2a}{3}} = \frac{a}{3}\times\frac{3}{2a} = \frac{1}{2}$$

$$\therefore \qquad \cos\theta = \frac{1}{2}$$

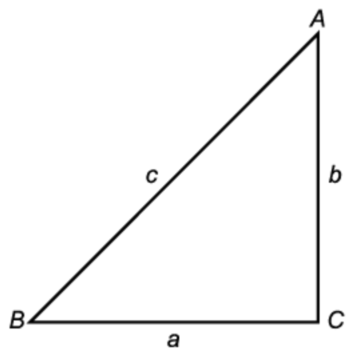
$$\Rightarrow \qquad \cos\theta = \cos\frac{\pi}{3}$$

$$\text{or} \qquad \theta = \frac{\pi}{3}$$

Hence, area of triangle is maximum, when $\theta = \frac{\pi}{3}$.

S6. Let a and b be the side of right angled triangle.

From ΔABC, we have



$$c^2 = a^2 + b^2$$

$$\text{Area of } \Delta ABC(A) = \frac{1}{2}a\cdot b$$

$$= \frac{1}{2}a\sqrt{c^2 - a^2} \qquad \left[\because b = \sqrt{c^2 - a^2}\right]$$

Differentiating w.r.t. a, we get

$$\frac{dA}{da} = \frac{1}{2}\cdot 1\cdot \sqrt{c^2-a^2} + \frac{1}{2}a\frac{1}{2}\frac{(-2a)}{\sqrt{c^2-a^2}}$$

$$= \frac{1}{2}\left[\sqrt{c^2-a^2} - \frac{a^2}{\sqrt{c^2-a^2}}\right]$$

For maxima and minima $\frac{dA}{da} = 0$

$$\therefore \quad \frac{1}{2}\left[\sqrt{c^2-a^2} - \frac{a^2}{\sqrt{c^2-a^2}}\right] = 0$$

$$\Rightarrow \quad c^2 - a^2 - a^2 = 0$$

$$c^2 = 2a^2$$

$$\Rightarrow \quad a = \frac{c}{\sqrt{2}}$$

Also

$$\frac{d^2A}{da^2} = \frac{1}{2}\left[\frac{-2a}{2\sqrt{c^2-a^2}} - \frac{\sqrt{(c^2-a^2)}2a - a^2\frac{1}{2}\times\frac{-2a}{\sqrt{c^2-a^2}}}{(c^2-a^2)}\right]$$

$$= \frac{1}{2}\left[\frac{-a}{\sqrt{c^2-a^2}} - \frac{2a(c^2-a^2)+a^3}{(c^2-a^2)^{3/2}}\right]$$

$$= \frac{1}{2}\left[\frac{-a}{\sqrt{c^2-a^2}} - \frac{2ac^2-a^3}{(c^2-a^2)^{3/2}}\right]$$

$$= \frac{1}{2}\left[\frac{-a(c^2-a^2)-a(2c^2-a^2)}{(c^2-a^2)^{3/2}}\right]$$

$$= -\frac{1}{2}a\left[\frac{3c^2-2a^2}{(c^2-a^2)^{\frac{3}{2}}}\right]$$

$$\frac{d^2A}{da^2}\text{ at } a = \frac{c}{\sqrt{2}} = -\frac{1}{2}\frac{c}{\sqrt{2}}\left(\frac{3c^2-c^2}{\left(c^2-\frac{c^2}{2}\right)^{3/2}}\right) = \frac{-c^3}{\sqrt{2}\left(\frac{c^2}{2}\right)^{3/2}} < 0$$

$\therefore$ Area of ΔABC is maximum at $b = \sqrt{c^2-a^2} = \sqrt{2a^2-a^2} = a.$

S7. Let the numbers be x and y. Then,

$$x + y = 14 \qquad \text{... (i)}$$

Let $S = x^2 + y^2$ Then,

$$S = x^2 + (14 - x)^2 \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad S = x^2 + x^2 - 28x + 196$$

$$\Rightarrow \quad S = 2x^2 - 28x + 196$$

$\Rightarrow \quad \frac{dS}{dx} = 4x - 28$ and $\frac{d^2S}{dx^2} = 4$

For maximum or minimum, values of S, we must have

$$\frac{dS}{dx} = 0$$

$\Rightarrow \quad 4x - 28 = 0$

$\Rightarrow \quad x = 7$

Now, $\quad \frac{d^2S}{dx^2} = 4 > 0$

Thus, S is minimum when $x = 7$.

Putting $x = 7$ in (i), we obtain $y = 7$.

Hence, the required numbers are both equal to 7.

S8. Let the required numbers be x and y. Then,

$$xy = 256 \quad \text{(given)} \qquad \ldots\text{(i)}$$

Let $\quad S = x + y$. Then,

$$S = x + \frac{256}{x} \qquad \text{[Using (i)]}$$

$\Rightarrow \quad \frac{ds}{dx} = 1 - \frac{256}{x^2}$ and $\frac{d^2s}{dx^2} = \frac{512}{x^3}$

For maximum or minimum values of S, we must have

$$\frac{ds}{dx} = 0$$

$\Rightarrow \quad 1 - \frac{256}{x^2} = 0$

$\Rightarrow \quad x^2 = 256$

$\Rightarrow \quad x = 16$

Now, $\quad \left(\frac{d^2s}{dx^2}\right)_{x=16} = \frac{512}{16^3} = \frac{1}{8} > 0$

Thus, S is minimum when $x = 16$.

Putting $x = 16$ in (i) we get $y = 16$.

Hence, the required numbers are both equal to 16.

S9. Let $P = x^2 y^5$. It is given that

$$x + y = 35$$

$\Rightarrow \quad x = 35 - y \qquad \ldots$ (i)

Putting $\quad x = 35 - y$ in P, we get

$$P = (35 - y)^2 y^5$$

$\Rightarrow \quad \frac{dP}{dy} = -2(35-y)y^5 + 5(35-y)^2y^4$

$\Rightarrow \quad \frac{dP}{dy} = (35-y)\, y^4 \{-2y + 5(35-y)\}$

$\Rightarrow \quad \frac{dP}{dy} = y^4\,(35-y)\,(175-7y)$

$= 7y^4\,(35-y)\,(25-y)$

For maximum or minimum values of P, we must have

$\therefore \quad \frac{dP}{dy} = 0$

$\Rightarrow \quad 7y^4(35-y)\,(25-y) = 0$

$\Rightarrow \quad y = 0, 25, 35$

But, $y = 0$ and $y = 35$ are not possible. So, $y = 25$.

$$\frac{d^2P}{dy^2} = (35-y)(25-y)\frac{d}{dy}7y^4 + 7y^4\frac{d}{dy}(25-y)(35-y)$$

Now, $\frac{d^2P}{dy^2} = 28y^3\,(35-y)\,(25-y) - 7y^4\,(25-y) - 7y^4\,(35-y)$

$$\left(\frac{d^2p}{dy^2}\right)_{y=25} = 28(25)^3\,(35-25)\,(25-25) - 7(25)^4\,(25-25) - 7(25)^4\,(35-25)$$

$= 0 - 0 - 7(25)^4\,10 = -7\,(25)^4\,(10) < 0$

Thus P has maximum when $y = 25$.

From (i), when $y = 25$, we have $x = 35 - 25 = 10$.

Hence, $x = 10$, and $y = 25$.

S10. Let $P = xy^3$

It is given that

$x + y = 60$

$\Rightarrow \quad x = 60 - y$

$\therefore \quad P = (60-y)\,y^3 = 60y^3 - y^4$

$\Rightarrow \quad \frac{dP}{dy} = 180y^2 - 4y^3$ and $\frac{d^2P}{dy^2} = 360y - 12y^2$

For maximum or minimum values of P, we must have

$\frac{dP}{dy} = 0.$

$\Rightarrow \quad 180y^2 - 4y^3 = 0$

$\Rightarrow \quad 4y^2 (45 - y) = 0$

$\Rightarrow \quad y = 0 \text{ or } 45 \qquad [\because y = 0 \text{ is not possible}]$

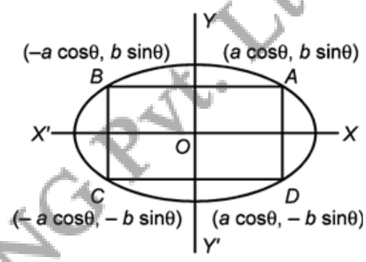
Now, $\left(\frac{d^2P}{dy^2}\right)_{y=45} = 360 \times 45 - 12(45)^2 = 12 \times 45 (30 - 45) = -8100 < 0$

So, P is maximum when $y = 45$.

When $y = 45$, $x + y = 60 \Rightarrow x = 15$.

Hence, the numbers are $x = 15$ and $y = 45$.

S11. Let A ($a \cos\theta$, $b \sin\theta$) be the parametric coordinate of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where θ be the eccentric angle.



Here, length of $AB = a\cos\theta + a\cos\theta$

$= 2a\cos\theta$

and length of $AD = b\sin\theta + b\sin\theta$

$= 2b\sin\theta$

Now,

area of rectangle $ABCD$ (Say (Z)) = $(2a\cos\theta)(2b\sin\theta)$

$Z = 2ab\,(2\sin\theta \times \cos\theta)$

$Z = 2ab\sin 2\theta \qquad \text{...(i)}$

Differentiating (i) w.r.t. θ

$$\frac{dZ}{d\theta} = 2ab \times 2\cos 2\theta \qquad \text{... (ii)}$$

for maxima and minima $\frac{dZ}{d\theta} = 0$

$4ab\cos 2\theta = 0$

$\cos 2\theta = 0$

$2\theta = \pi/2$

$\theta = \pi/4$

Differentiating (ii) again w.r.t. θ

$$\frac{d^2Z}{d\theta^2} = -8\,ab\sin 2\theta$$

$$\frac{d^2Z}{d\theta^2} \text{ at } (\theta = \pi/4) = -8\,ab\sin\left(\frac{2\pi}{4}\right)$$

$$= -8\,ab < 0$$

Area of Rectangle is maximum when $\theta = \pi/4$ and maximum area = $2ab\sin\left(2\dfrac{\pi}{4}\right)$ = 2ab

Hence, area of greatest rectangle is equal to $2ab$ when eccentric angle of an ellipse is 45°.

S12. Given equation of ellipse is $\dfrac{x^2}{25}+\dfrac{y^2}{16}=1$

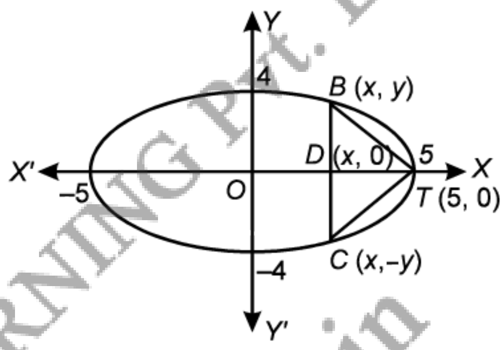
Here, $a = 5$, $b = 4$

$\therefore\ a > b$

So, major axis is along X-axis.

Let ΔBTC is the isosceles triangle which is inscribed in the ellipse. Let $OD = x$, $BC = 2y$ and $TD = 5 - x$.

Let A denotes the area of triangle. Then, we have



$$A = \frac{1}{2}\text{base} \times \text{height}$$

$$= \frac{1}{2}BC \times TD$$

$\Rightarrow$ $$A = \frac{1}{2} \times 2y\,(5 - x)$$

$\Rightarrow$ $$A = y\,(5 - x)$$

Squaring both sides, we get

$$A^2 = y^2(5 - x)^2 \qquad \text{...(i)}$$

Now, $$\frac{x^2}{25}+\frac{y^2}{16} = 1 \;\Rightarrow\; \frac{y^2}{16} = 1 - \frac{x^2}{25}$$

$\Rightarrow$ $$y^2 = \frac{16}{25}(25 - x^2)$$

$\therefore$ From Eq. (i), we get $$A^2 = \frac{16}{25}(25 - x^2)\,(5 - x)^2$$

Let $$A^2 = z$$

$\therefore$ $$z = \frac{16}{25}(25 - x^2)\,(5 - x)^2$$

Differentiating w.r.t. x, we get

$$\frac{dz}{dx} = \frac{16}{25}\left[(25 - x^2)\,2(5 - x)\,(-1) + (5 - x)^2\,(-2x)\right]$$

$$= \frac{16}{25}(-2)(5-x)[(5+x)(5-x)+x(5-x)]$$

$$= \frac{16}{25}(-2)(5-x)^2(2x+5)$$

Now, for maxima and minima, put $\frac{dz}{dx} = 0$

$\therefore \quad -\frac{32}{25}(5-x)^2(2x+5) = 0$

$\Rightarrow \quad x = 5 \text{ or } -\frac{5}{2}$

Now, when $x = 5$, then $Z = \frac{16}{25}(25-25)(5-5)^2 = 0$ which is not possible. So $x = 5$ is rejected.

$\therefore \quad x = -\frac{5}{2}$

Now, $\quad \frac{d^2z}{dx^2} = \frac{d}{dx}\left(\frac{dz}{dx}\right)$

$$= \frac{d}{dx}\left[-\frac{32}{25}(5-x)^2(2x+5)\right]$$

$$= -\frac{32}{25}\left[(5-x)^2 \cdot 2 - (2x+5)\,2\,(5-x)\right]$$

$$= -\frac{32}{25} \times 2(5-x)[5-x-2x-5]$$

$$= -\frac{64}{25}(5-x)(-3x)$$

$$= \frac{192x}{25}(5-x)$$

$\because \quad \left[\frac{d^2z}{dx^2}\right]_{x=-\frac{5}{2}} < 0 \Rightarrow z$ is maximum.

$\therefore$ Area A is maximum, when $x = -\frac{5}{2}$ and $y = 12$.

Also, the maximum area $\quad z = A^2 = \frac{16}{25}\left(25 - \frac{25}{4}\right)\left[5+\frac{5}{2}\right]^2$

$$= \frac{16}{25} \times \frac{75}{4} \times \frac{225}{4} = 3 \times 225$$

Hence, the maximum area, $A = \sqrt{3 \times 225}$

$= 15\sqrt{3}$ sq. units

☛ ***Note** If A is maximum/minimum, then A^2 is maximum/minimum.*

S13. Let R be a point on AB such that $AR = x$ m. Then, $RB = (20 - x)$ m

In Δ's RAP and RBQ, we have

$$PR^2 = x^2 + 16^2 \qquad ...(i)$$

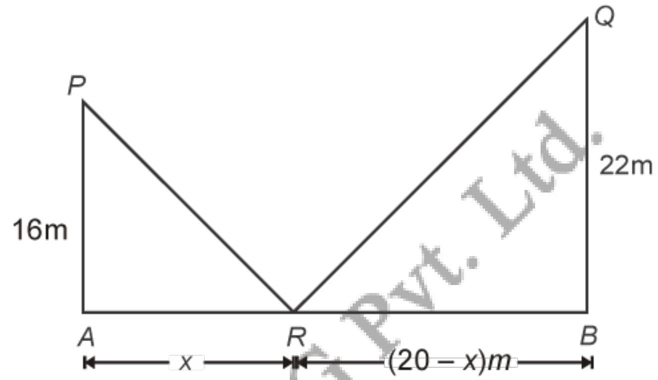
$$RQ^2 = 22^2 + (20 - x)^2 \qquad ...(ii)$$

$\therefore \quad PR^2 + RQ^2 = x^2 + 16^2 + 22^2 + (20 - x)^2 = 2x^2 - 40x + 1140$

Let $Z = PR^2 + RQ^2$. Then,

$$Z = 2x^2 - 40x + 1140.$$

$$\frac{dZ}{dx} = 4x - 40 \text{ and } \frac{d^2Z}{dx^2} = 4$$



For maximum or minimum, we must have

$$\frac{dZ}{dx} = 0 \Rightarrow 4x - 40 = 0 \Rightarrow x = 10.$$

Clearly, $\frac{d^2Z}{dx^2} = 4 > 0$ for all x

$\therefore$ Z is minimum when $x = 10$.

Thus, the distance of R from point A = 10 m.

S14. Let ABC be a triangle inscribed in a given circle with centre O and radius r.

The area of the triangle will be maximum if its vertex A opposite to the base BC is at a maximum distance from the base BC. This is possible only when A lies on the diameter perpendicular to BC. Thus, $AD \perp BC$. So, triangle ABC must be an isosceles triangle.

Let $OD = x$.

In right triangle ODB, we have

$$OB^2 = OD^2 + BD^2$$

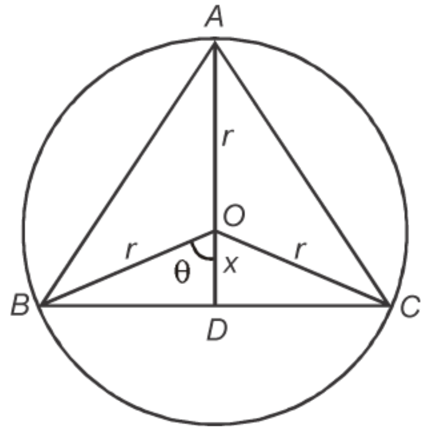
$$\Rightarrow \quad r^2 = x^2 + BD^2$$

$$\Rightarrow \quad BD = \sqrt{r^2 - x^2}$$

$$\Rightarrow \quad BC = 2BD = 2\sqrt{r^2 - x^2}.$$

Also, $AD = AO + OD = r + x.$

Let A denote the area of ΔABC. Then,



$$A = \frac{1}{2}(BC \times AD)$$

$$\Rightarrow \quad A = \frac{1}{2} \times 2\sqrt{r^2 - x^2} \times (r + x)$$

$$\Rightarrow \quad A = (r + x)\sqrt{r^2 - x^2}$$

$$\Rightarrow \quad \frac{dA}{dx} = \sqrt{r^2 - x^2} - \frac{2x(r + x)}{2\sqrt{r^2 - x^2}}$$

$$\Rightarrow \quad \frac{dA}{dx} = \frac{r^2 - x^2 - rx - x^2}{\sqrt{r^2 - x^2}}$$

$$\Rightarrow \quad \frac{dA}{dx} = \frac{r^2 - rx - 2x^2}{\sqrt{r^2 - x^2}}$$

For maximum or minimum values of A, we must have

$$\Rightarrow \quad \frac{dA}{dx} = 0$$

$$\Rightarrow \quad \frac{r^2 - rx - 2x^2}{\sqrt{r^2 - x^2}} = 0$$

$$\Rightarrow \quad (r - 2x)(r + x) = 0$$

$$\Rightarrow \quad r - 2x = 0$$

$$\Rightarrow \quad x = \frac{r}{2} \qquad [\because r + x \neq 0]$$

Now, $$\frac{dA}{dx} = \frac{r^2 - rx - 2x^2}{\sqrt{r^2 - x^2}} = (r^2 - rx - 2x^2)(r^2 - x^2)^{-\frac{1}{2}}$$

$$\Rightarrow \quad \frac{d^2A}{dx^2} = (r^2 - x^2)^{-1/2}\frac{d}{dx}(r^2 - rx - 2x^2) + (r^2 - rx - 2x^2)\frac{d}{dx}(r^2 - x^2)^{-1/2}$$

$$\Rightarrow \quad \frac{d^2A}{dx^2} = \frac{(-r - 4x)}{\sqrt{r^2 - x^2}} + \frac{(r^2 - rx - 2x^2)x}{(r^2 - x^2)^{3/2}}$$

$$\Rightarrow \quad \left(\frac{d^2A}{dx^2}\right)\Bigg|_{x=\frac{r}{2}} = \frac{-r - 4\left(\frac{r}{2}\right)}{\sqrt{r^2 - \frac{r^2}{4}}} + \frac{\left(r^2 - r\cdot\frac{r}{2} - 2\left(\frac{r}{2}\right)^2\right)\frac{r}{2}}{\left(r^2 - \frac{r^2}{4}\right)^{3/2}}$$

$$= \frac{-3r}{\frac{\sqrt{3}r}{2}} + 0 = -2\sqrt{3} < 0$$

Thus, A is maximum when $x = \frac{r}{2}$.

$\therefore \qquad BD = \sqrt{r^2 - x^2}$

$\Rightarrow \qquad BD = \frac{\sqrt{3}r}{2}$

In ΔODB, we have

$$\tan\theta = \frac{BD}{OD}$$

$$\Rightarrow \qquad \tan\theta = \frac{\frac{\sqrt{3}r}{2}}{\frac{r}{2}} = \sqrt{3} \quad \Rightarrow \quad \theta = 60°$$

$\therefore \quad \angle BAC = \theta = 60°$

But, $AB = AC$. Therefore, $\angle B = \angle C$

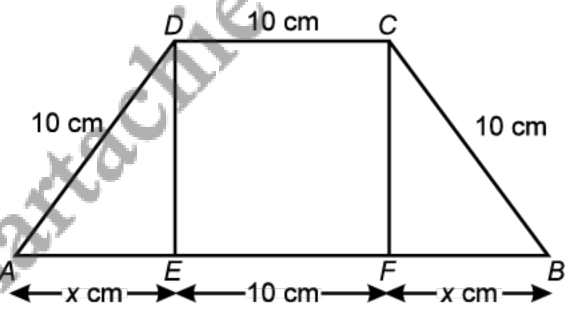
Thus, we have $\angle A = \angle B = \angle C = 60°$

Hence, A is maximum when ΔABC is equilateral.

S15. Let $ABCD$ be the trapezium. Such that $AD = BC = CD = 10$cm.

Let $AE = BF = x$ cm.

Draw $DE \perp AB$ and $CF \perp AB$.



Now, in ΔDEA and ΔCFB, using Pythagoras theorem, we get

$$DE = CF = \sqrt{100 - x^2}$$

Let A denotes the area of trapezium. Then,

$$A = \frac{1}{2}(10 + 10 + 2x)\left(\sqrt{100 - x^2}\right)$$

$$\left[\text{Area of trapezium} = \frac{1}{2}\text{(Sum of parallel sides)} \times \text{(Distance between them)}\right]$$

$$\Rightarrow \qquad A = \frac{1}{2}(20 + 2x)\left(\sqrt{100 - x^2}\right)$$

or $\qquad A = (10 + x)\sqrt{100 - x^2}$

Differentiating w.r.t. x, we get

$$\frac{dA}{dx} = \sqrt{100-x^2}\cdot 1 - \frac{2x(10+x)}{2\sqrt{100-x^2}}$$

$$= \frac{100-x^2-10x-x^2}{\sqrt{100-x^2}}$$

$\therefore$ $$\frac{dA}{dx} = \frac{100-10x-2x^2}{\sqrt{100-x^2}}$$

Now, for maxima and minima, put $\frac{dA}{dx} = 0$

$\therefore$ $$\frac{100-10x-2x^2}{\sqrt{100-x^2}} = 0$$

$\Rightarrow$ $100 - 10x - 2x^2 = 0$

$\Rightarrow$ $-2(x^2 + 5x - 50) = 0$

$\Rightarrow$ $(x + 10)(x - 5) = 0$

Either $x + 10 = 0$

or $x - 5 = 0$

$\therefore$ $x = -10$ or 5

$\because x > 0 \Rightarrow x = -10$ is rejected.

$\therefore$ $x = 5$

Also, $$\frac{d^2A}{dx^2} = \frac{d}{dx}\left(\frac{dA}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{100-10x-2x^2}{\sqrt{100-x^2}}\right)$$

$$= \frac{\left[\sqrt{100-x^2}(-10-4x) - (100-10x-2x^2)\left(\frac{-2x}{2\sqrt{100-x^2}}\right)\right]}{(100-x^2)}$$

$$= \frac{\left[(100-x^2)(-10-4x) + (100x-10x^2-2x^3)\right]}{(100-x^2)^{\frac{3}{2}}}$$

$$\frac{d^2A}{dx^2} = \frac{2x^3-300x-1000}{(100-x^2)^{\frac{3}{2}}}$$

Putting $x = 5$, we get

$$\left[\frac{d^2A}{dx^2}\right]_{x=5} = \frac{2(5)^3 - 300(5) - 1000}{(100-25)^{\frac{3}{2}}}$$

$$= \frac{250 - 1500 - 1000}{(75)\sqrt{75}}$$

$$= -\frac{2250}{(75)\sqrt{75}} = -\frac{30}{\sqrt{75}} < 0$$

$\because \frac{d^2A}{dx^2} < 0 \Rightarrow A$ is maximum.

$\therefore$ Area of trapezium is maximum, when x = 5.

Also, maximum area of trapezium is given by

$$A = (10 + x)\sqrt{100 - x^2}$$

Put $x = 5$, we get

$$A = (10+5)\times\sqrt{100-25}$$

$$= 15\times\sqrt{75}$$

$$= 15\sqrt{25\times 3} = 75\sqrt{3} \text{ cm}^2$$

Hence, maximum area of trapezium = $75\sqrt{3}$ cm²

S16. Let the length, breadth and depth of the open tank be x, x and y respectively. Length and breadth are same because given that the tank has a square base. Let V denotes its volume and S be it's surface area. Now, given that

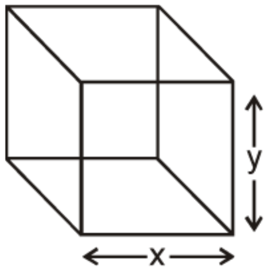
$$V = x^2 y \qquad \text{... (i)}$$

Also, we know that the total surface area of the open tank is given by

$$S = x^2 + 4xy \qquad \text{... (ii)}$$

[$\because$ S = area of square base + area of the four walls]

Putting $y = \frac{V}{x^2}$ from Eq. (i) in Eq. (ii), we get



$$S = x^2 + 4x\cdot\frac{V}{x^2}$$

or $$S = x^2 + \frac{4V}{x}$$

Differentiating w.r.t. x, we get $$\frac{dS}{dx} = 2x - \frac{4V}{x^2}$$

Now, for maxima and minima, put $\frac{dS}{dx} = 0$

$\therefore \quad 2x - \frac{4V}{x^2} = 0$

$\Rightarrow \quad \frac{4V}{x^2} = 2x$

$\Rightarrow \quad 4V = 2x^3$

$\Rightarrow \quad 4x^2y = 2x^3$ [$\because V = x^2y$ From Eq. (i)]

$\Rightarrow \quad 2y = x \quad \text{or} \quad y = \frac{x}{2}$

$\therefore$ Depth of tank is half of its width.

Also $\frac{d^2S}{dx^2} = \frac{d}{dx}\left(\frac{dS}{dx}\right) = \frac{d}{dx}\left(2x - \frac{4V}{x^2}\right)$

$= 2 + \frac{8V}{x^3} = 2 + \frac{8 \cdot x^2 y}{x^3} = 2 + \frac{8y}{x}$ [From Eq. (i)]

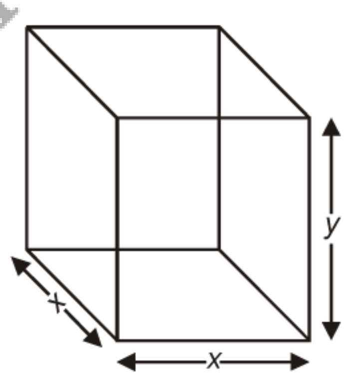
$\left(\frac{d^2S}{dx^2}\right) at\ y = \frac{x}{2} = 2 + 8.\frac{x}{2}.\frac{1}{x} = 2 + 4 = 6 > 0$

$\because \quad \frac{d^2S}{dx^2} > 0,$

$\therefore \quad S$ is minimum.

Hence, total surface area of the tank is least, when depth is half of its width.

S17. Let V be the fixed volume of a closed cuboid with length x, breadth x and height y. Let S be the surface area of the cuboid. Then,



$V = x^2y$ ... (i)

and, $S = 2(x^2 + xy + xy) = 2x^2 + 4xy$ ... (ii)

Now, $S = 2x^2 + 4xy$

$\Rightarrow \quad S = 2x^2 + 4x\frac{V}{x^2}$ $\left[\because y = \frac{V}{x^2}\right]$

$\Rightarrow \quad S = 2x^2 + \frac{4V}{x}$

$\frac{dS}{dx} = 4x - \frac{4V}{x^2}$ ...(iii)

For maximum or minimum, we must have

$$\frac{dS}{dx} = 0$$

$$\Rightarrow \quad 4x - \frac{4V}{x^2} = 0$$

$$\Rightarrow \quad V = x^3 \Rightarrow x^2y = x^3 \qquad [\because V = x^2 y]$$

$$\Rightarrow \quad x = y.$$

Differentiating (iii) w.r.t. x, we get

$$\frac{d^2S}{dx^2} = 4 + \frac{8V}{x^3} = 4 + \frac{8x^2y}{x^3} = 4 + \frac{8y}{x}$$

$$\left.\frac{d^2S}{dx^2}\right|_{y=x} = 4 + 8 = 12 > 0.$$

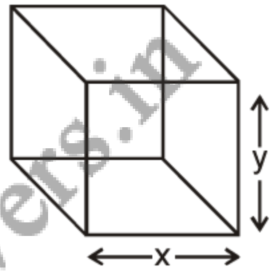
Hence, S is minimum when length = x, breadth = x and height = x *i.e.*, when it is a cube.

S18. Let the dimensions of the box be x and y. Let V denotes its volume and S be its total surface area.

Now, $S = x^2 + 4xy$

[$\because$ S = Area of square base + Area of the four walls]

Given that $x^2 + 4xy = C^2$



or $$y = \frac{C^2 - x^2}{4x} \quad \text{... (i)}$$

Also, volume of the box is given as

$$V = x^2y$$

$$\Rightarrow \quad V = x^2\left(\frac{C^2 - x^2}{4x}\right) \qquad \text{[From Eq. (i)]}$$

or $$V = \frac{xC^2 - x^3}{4}$$

Differentiating w.r.t. x, we get

$$\frac{dV}{dx} = \frac{C^2 - 3x^2}{4}$$

For maxima and minima, put $\frac{dV}{dx} = 0$

$$\therefore \quad \frac{C^2 - 3x^2}{4} = 0$$

$$\Rightarrow \quad C^2 = 3x^2$$

$\Rightarrow \qquad x = \frac{C}{\sqrt{3}}$

Also, $\qquad \frac{d^2V}{dx^2} = \frac{d}{dx}\left(\frac{C^2 - 3x^2}{4}\right)$

$$= \frac{-6x}{4} = \frac{-3x}{2}$$

$$\left(\frac{d^2V}{dx^2}\right) at \left(x = \frac{C}{\sqrt{3}}\right) = \frac{-3C}{2\sqrt{3}} < 0$$

$\Rightarrow$ V is Maximum

$\therefore \qquad V_{max} = \frac{xC^2 - x^3}{4} = \frac{1}{4}[xC^2 - x^3]$

$$= \frac{1}{4}\left[\frac{C}{\sqrt{3}} \cdot C^2 - \left(\frac{C}{\sqrt{3}}\right)^3\right] \qquad \left[\text{Put } x = \frac{C}{\sqrt{3}}\right]$$

$$= \frac{1}{4}\left[\frac{C^3}{\sqrt{3}} - \frac{C^3}{3\sqrt{3}}\right]$$

$$= \frac{1}{4}\left[\frac{3C^3 - C^3}{3\sqrt{3}}\right]$$

$$= \frac{1}{4} \times \frac{2C^3}{3\sqrt{3}} = \frac{C^3}{6\sqrt{3}}$$

Hence, maximum volume is $\frac{C^3}{6\sqrt{3}}$ cu. units.

S19. Let x m be the length, y m be the breadth and h = 2 m be the depth of the tank. Let Rs. H be the total expensive for building the tank. Now, given that h = 2m and volume of tank = 8m^3

Also, Area of the base of the tank = length $\times$ breadth = x y m^2

and the area of the four sides = 2 (length + breadth) $\times$ height

$= 2(x + y) \times 2 = 4\,(x + y)$ m^2

$\therefore$ Total cost, $\qquad H = 70 \times xy + 45 \times 4(x + y)$

or $H = 70xy + 180(x + y)$ ... (i)

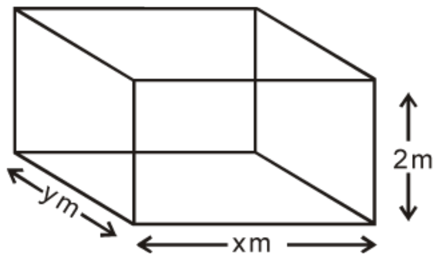
$\because$ Volume of tank = $8m^3$

$\therefore$ $l \times b \times h = 8$

$\Rightarrow$ $x \times y \times 2 = 8$

$\Rightarrow$ $2xy = 8$

$\Rightarrow$ $y = \frac{4}{x}$ ... (ii)



Putting value of y from eq. (ii) in eq. (i), we get

$$H = 70x.\frac{4}{x} + 180\left(x + \frac{4}{x}\right) = 280 + 180\left(x + \frac{4}{x}\right) \quad \text{... (iii)}$$

Differentiating w.r.t. x, we get $\frac{dH}{dx} = 180\left(1 - \frac{4}{x^2}\right)$

For maxima and minima, put $\frac{dH}{dx} = 0$

$\therefore$ $180\left(1 - \frac{4}{x^2}\right) = 0$

$\Rightarrow$ $1 - \frac{4}{x^2} = 0$

$\Rightarrow$ $\frac{4}{x^2} = 1$

$\Rightarrow$ $x^2 = 4$

$\Rightarrow$ $x = 2$

Also, $$\frac{d^2H}{dx^2} = \frac{d}{dx}\left(\frac{dH}{dx}\right) = \frac{d}{dx}\left[180\left(1 - \frac{4}{x^2}\right)\right] = \frac{8}{x^3} \times 180$$

$\therefore$ $$\left[\frac{d^2H}{dx^2}\right]_{x=2} = \frac{8}{2^3} \times 180 = 180 > 0$$

$\because$ $\frac{d^2H}{dx^2} > 0$

$\Rightarrow$ H is least at $x = 2$.

Also, the least cost = $280 + 180\left(2 + \frac{4}{2}\right)$

[Put $x = 2$ in Eq. (iii) to get least cost H]

$= 280 + 180 \times 4 = 280 + 720 = \text{Rs. } 1000$

Hence, the cost of least expensive tank is Rs. 1000.

S20. Let x cm be the length of a side of the square which is cut-off from each corner of the plate. Then, sides of the box as shown in figure are $24 - 2x$, $24 - 2x$ and x.

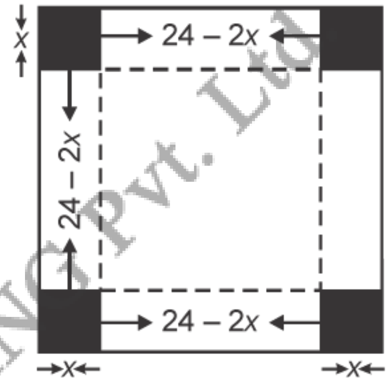
Let V be the volume of the box. Then,

$$V = (24 - 2x)^2 x = (576 - 96x + 4x^2)x$$
$$= 4x^3 - 96x^2 + 576x$$

$$\Rightarrow \quad \frac{dV}{dx} = 12x^2 - 192x + 576$$

$$\text{and} \quad \frac{d^2V}{dx^2} = 24x - 192$$

For maximum or minimum values of V, we must have



$$\frac{dV}{dx} = 0$$
$$\Rightarrow \quad 12x^2 - 192x + 576 = 0$$
$$\Rightarrow \quad x^2 - 16x + 48 = 0$$
$$\Rightarrow \quad (x - 12)(x - 4) = 0$$
$$\Rightarrow \quad x = 12, 4$$

But, $x = 12$ is not possible.

Therefore, $x = 4$.

$$\text{Now,} \quad \left(\frac{d^2V}{dx^2}\right)_{x=4} = 24 \times 4 - 192 = 96 - 192 < 0.$$

Thus, V is maximum when $x = 4$.

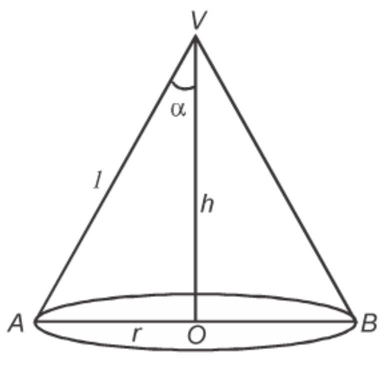
Hence, the volume of the box is maximum when the side of the square is 4 cm.

The maximum volume is, $V = (24 - 8)^2 \times 4 = 256 \times 4 = 1024$ cm^3.

S21. Let r be radius, l be the slant height and h be the height of the cone of given surface area S. Then,

$$S = \pi r^2 + r\pi l$$

$$\Rightarrow \quad l = \frac{S - \pi r^2}{\pi r} \qquad \dots \text{(i)}$$



Let V be the volume of the cone. Then,

$$V = \frac{1}{3}\pi r^2 h$$

$$\Rightarrow \quad V^2 = \frac{1}{9}\pi^2 r^4 h^2 = \frac{1}{9}\pi^2 r^4 (l^2 - r^2) \qquad \left[\because l^2 = h^2 + r^2\right]$$

$$\Rightarrow \quad V^2 = \frac{\pi^2}{9} r^4 \left[\left(\frac{S - \pi r^2}{\pi r}\right)^2 - r^2\right] \qquad \text{[Using Eq. (i)]}$$

$\Rightarrow \qquad V^2 = \dfrac{\pi^2 r^4}{9}\left[\dfrac{(S-\pi r^2)^2 - \pi^2 r^4}{\pi^2 r^2}\right]$

$\Rightarrow \qquad V^2 = \dfrac{1}{9}S(Sr^2 - 2\pi r^4)$

Let $Z = V^2$. Then, V is maximum or minimum according as Z is maximum or minimum.

$\therefore \qquad Z = \dfrac{1}{9}S(Sr^2 - 2\pi r^4)$

$\Rightarrow \qquad \dfrac{dZ}{dr} = \dfrac{1}{9}S(2Sr - 8\pi r^3)$ ... (ii)

For maximum or minimum, we must have

$\dfrac{dZ}{dr} = 0$

$\Rightarrow \quad 2Sr - 8\pi r^3 = 0$

$\Rightarrow \qquad S = 4\pi r^2$ ... (iii)

Differentiating (ii) w.r.t. r, we get

$\dfrac{d^2Z}{dr^2} = \dfrac{S}{9}(2S - 24\pi r^2)$

$\Rightarrow \qquad \dfrac{d^2Z}{dr^2} = \dfrac{S}{9}\left[2S - 24\pi\dfrac{S}{4\pi}\right] = \dfrac{S}{9}(2S - 6S)$ $\left[\text{Putting } r^2 = \dfrac{S}{4\pi} \text{ from (iii)}\right]$

$\Rightarrow \qquad \dfrac{d^2Z}{dr^2} = -\dfrac{4S^2}{9} < 0$

So, Z is the maximum when $S = 4\pi r^2$

Hence, V is maximum when $S = 4\pi r^2$

Now, $\qquad S = 4\pi r^2$

$\Rightarrow \qquad \pi r l + \pi r^2 = 4\pi r^2$

$\Rightarrow \qquad \pi r l = 3\pi r^2$

$\Rightarrow \qquad l = 3r.$

$\therefore \qquad \sin\alpha = \dfrac{r}{l} = \dfrac{r}{3r} = \dfrac{1}{3}$

Hence, V is maximum when $\alpha = \sin^{-1}\left(\dfrac{1}{3}\right)$.

S22. Let r be the radius of circle and x be the side of a square. We are given that

Perimeter of square + Circumference of circle = k

i.e., $\qquad 4x + 2\pi r = k$

$\Rightarrow \qquad x = \dfrac{k - 2\pi r}{4} \qquad \text{...(i)}$

Let A denotes the sum of their areas.

$\therefore \qquad A = x^2 + \pi r^2 \qquad \text{...(ii)}$

$\left[\because \text{Area of a square} = (\text{side})^2 \text{ and area of circle} = \pi r^2\right]$

Putting value of x from Eq. (i) in Eq. (ii), we get

$$A = \left(\frac{k - 2\pi r}{4}\right)^2 + \pi r^2$$

Differentiating w.r.t. r, we get $\quad \dfrac{dA}{dr} = 2\left(\dfrac{k - 2\pi r}{4}\right)\left(-\dfrac{2\pi}{4}\right) + 2\pi r = -\dfrac{\pi}{4}(k - 2\pi r) + 2\pi r$

Now, for maxima and minima, put $\dfrac{dA}{dr} = 0$

$\therefore \qquad -\dfrac{\pi}{4}(k - 2\pi r) + 2\pi r = 0$

$\Rightarrow \qquad -\dfrac{\pi}{4}(k - 2\pi r) = -2\pi r$

$\Rightarrow \qquad k - 2\pi r = 8r$

$\Rightarrow \qquad 2\pi r + 8r = k$

or $\qquad r = \dfrac{k}{2\pi + 8} \qquad \text{...(iii)}$

Now, $\qquad \dfrac{d^2A}{dr^2} = \dfrac{d}{dr}\left(\dfrac{dA}{dr}\right)$

$$= \frac{d}{dr}\left[2\pi r - \frac{\pi}{4}(k - 2\pi r)\right] = 2\pi + \frac{2\pi^2}{4} > 0$$

$\because \dfrac{d^2A}{dr^2} > 0 \Rightarrow A$ is maximum.

Now, from Eq. (iii), we get $\qquad r = \dfrac{k}{2\pi + 8}$

$\Rightarrow \qquad 2r\pi + 8r = k$

$\Rightarrow \qquad 2r\pi + 8r = 4x + 2\pi r \qquad [\because k = 4x + 2\pi r]$

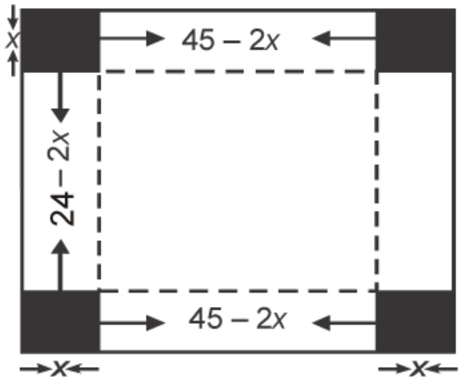
$\qquad 8r = 4x \quad \text{or} \quad x = 2r$

i.e., Side of square = Diameter of the circle

Hence, sum of area of a circle and a square is least, when side of square is equal to diameter of circle.

S23. Let the length of the each side of the square which is cut from each corner of the tin sheet be x. By folding up the flaps, a cuboidal box is formed whose length, breadth and height are $45 - 2x$, $24 - 2x$ and x respectively. Then its volume V is given by

$$V = (45 - 2x)(24 - 2x)x$$
$$= (1080 - 138x + 4x^2)x$$
$$= 1080x - 138x^2 + 4x^3$$



$\Rightarrow$ $$\frac{dV}{dx} = 1080 - 276x + 12x^2$$

and $$\frac{d^2V}{dx^2} = -276 + 24x$$

For maximum value of V, we must have

$$\frac{dV}{dx} = 0$$

$\Rightarrow$ $$1080 - 276x + 12x^2 = 0$$

$\Rightarrow$ $$x^2 - 23x + 90 = 0$$

$\Rightarrow$ $$x = 5, 18.$$

But $x = 18$ is not possible. Therefore, $x = 5$

When $x = 5$

$$\frac{d^2V}{dx^2} = -276 + 120 = -156 < 0$$

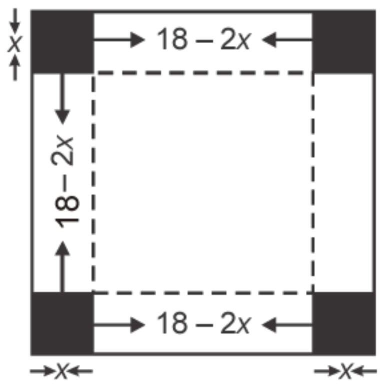
So, V is maximum when $x = 5$ *i.e.* the length of each side of the rectangular sheet to be cut is 5 cm.

$\therefore$ Maximum volume = (45 – 10) (24 – 10) 5 = 2450 cm^3.

S24. Let the length of the each side of the square which is cut from each corner of the tin sheet be x. By folding up the flaps, a cuboidal box is formed whose length, breadth and height are $18 - 2x$, $18 - 2x$ and x respectively. Then its volume V is given by

$$V = (18 - 2x)(18 - 2x)x = (324 - 72x + 4x^2)x$$
$$= 324x - 72x^2 + 4x^3$$

$\Rightarrow$ $$\frac{dV}{dx} = 324 - 144x + 12x^2$$



and $$\frac{d^2V}{dx^2} = -144 + 24x$$

For maximum value of V, we must have

$$\frac{dV}{dx} = 0$$

$\Rightarrow$ $$324 - 144x + 12x^2 = 0$$

$\Rightarrow$ $$x^2 - 12x + 27 = 0$$

$\Rightarrow$ $$x = 3, 9.$$

But $x = 9$ is not possible. Therefore, $x = 3$

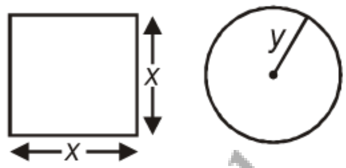
When $x = 3$

$$\frac{d^2V}{dx^2} = -144 + 72 = -72 < 0$$

So, V is maximum when $x = 3$ *i.e.* the length of each side of the square to be cut is 3 cm.

$\therefore$ Maximum volume $= (18 - 6)^2 \times 3 = 432 \text{ cm}^3$.

S25. Let x metres be the length of a side of the square and y metres be the radius of the circle. Then, we have



$$4x + 2\pi y = 36$$

$$\Rightarrow \quad 2x + \pi y = 18 \qquad \text{... (i)}$$

Let A be the combined area of the square and the circle. Then,

$$A = x^2 + \pi y^2 \qquad \text{...(ii)}$$

$$\Rightarrow \quad A = x^2 + \pi\left(\frac{18 - 2x}{\pi}\right)^2 \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad A = x^2 + \frac{1}{\pi}(18 - 2x)^2$$

$$\Rightarrow \quad \frac{dA}{dx} = 2x + \frac{2}{\pi}(18-2x)(-2) = 2x - \frac{4}{\pi}(18 - 2x)$$

and, $$\frac{d^2A}{dx^2} = 2 - \frac{4}{\pi}(-2) = 2 + \frac{8}{\pi}$$

For maximum or minimum A, we must have

$$\frac{dA}{dx} = 0$$

$$\Rightarrow \quad 2x - \frac{4}{\pi}(18 - 2x) = 0$$

$$\Rightarrow \quad 2x = \frac{4}{\pi}(18 - 2x)$$

$$\Rightarrow \quad 2\pi x + 8x = 72$$

$$\Rightarrow \quad x(\pi + 4) = 36$$

$$\Rightarrow \quad x = \frac{36}{\pi + 4}$$

As, $$\left(\frac{d^2A}{dx^2}\right)_{(x=36/\pi+4)} = 2 + \frac{8}{\pi} > 0$$

Thus, A is minimum when $x = \dfrac{36}{\pi+4}$

Putting the value of x in (i), we obtain

$$\pi y = 18 - \frac{2 \times 36}{\pi + 4}$$

$$\Rightarrow \quad \pi y = \frac{18\pi + \cancel{72} - \cancel{72}}{\pi + 4}$$

$$\Rightarrow \quad \pi y = \frac{18\pi}{\pi + 4} \quad \Rightarrow \quad y = \frac{18}{\pi + 4}$$

So, Length of square part = $4x = 4 \times \dfrac{36}{\pi+4} = \dfrac{144}{\pi+4}$ m

and Length of circular part = $2\pi y = 2\pi \times \dfrac{18}{\pi+4} = \dfrac{36\pi}{\pi+4}$ m.

S26. Let x, be the side of the square and r be the radius of circular part. Then,

Length of square part = Perimeter of square

$= 4 \times \text{Side} = 4x$

and length of circular part = circumference of circle = $2\pi r$

$\therefore$ Given that, $4x + 2\pi r = 28$

or $2x + \pi r = 14$

or $$x = \frac{14 - \pi r}{2} \quad \dots \text{(i)}$$

Let A denotes the combined area of circle and square.

Then, $A = \pi r^2 + x^2$

$$\Rightarrow \quad A = \pi r^2 + \left(\frac{14 - \pi r}{2}\right)^2 \quad \left[\because x = \frac{14 - \pi r}{2} \text{ by eq. (i)}\right]$$

Differentiating w.r.t. r, we get

$$\frac{dA}{dr} = 2\pi r + 2\left(\frac{14 - \pi r}{2}\right)\left(-\frac{\pi}{2}\right)$$

$$= 2\pi r - \left(\frac{14\pi - \pi^2 r}{2}\right)$$

Now, for maxima and minima, put $\dfrac{dA}{dr} = 0$

$\therefore \qquad 2\pi r - \left(\frac{14\pi - \pi^2 r}{2}\right) = 0$

$\Rightarrow \qquad 2\pi r = \frac{14\pi - \pi^2 r}{2}$

$\Rightarrow \qquad 4\pi r + \pi^2 r = 14\pi$

$\Rightarrow \qquad \pi r(4 + \pi) = 14\pi$

$\Rightarrow \qquad r = \frac{14}{\pi + 4}$

Again, $\qquad \frac{d^2A}{dr^2} = \frac{d}{dr}\left(\frac{dA}{dr}\right)$

$$= \frac{d}{dr}\left[2\pi r - \left(\frac{14\pi - \pi^2 r}{2}\right)\right]$$

$\Rightarrow \qquad \frac{d^2A}{dr^2} = 2\pi + \frac{\pi^2}{2} > 0$

$\because \frac{d^2A}{dr^2} > 0 \Rightarrow A$ is minimum.

Now, putting $r = \frac{14}{\pi + 4}$ in Eq. (i), we get

$$x = \frac{14 - \pi\left(\frac{14}{\pi + 4}\right)}{2}$$

$$= \frac{14\pi + 56 - 14\pi}{2(\pi + 4)} = \frac{28}{\pi + 4}$$

$\therefore$ We get, $\qquad x = \frac{28}{\pi + 4}$

and $\qquad r = \frac{14}{\pi + 4}$

$\therefore \qquad$ Length of circular part $= 2\pi r = 2\pi \times \frac{14}{\pi + 4} = \frac{28\pi}{\pi + 4}$

and $\qquad$ Length of square part $= 4x = 4 \times \frac{28}{\pi + 4} = \frac{112}{\pi + 4}$.

S27. Let S be the surface area, V be the volume, h be the height and r be the radius of base of the right circular cylinder. We know that

$$S = 2\pi r^2 + 2\pi rh \quad \text{... (i)}$$

$$\Rightarrow \quad h = \frac{S - 2\pi r^2}{2\pi r} \quad \text{... (ii)}$$

Also, volume of right circular cylinder is given by

$$V = \pi r^2 h$$

$$\therefore \quad V = \pi r^2 \left(\frac{S - 2\pi r^2}{2\pi r}\right) \quad \text{[By Eq. (ii)]}$$

$$\text{or} \quad V = \frac{rS - 2\pi r^3}{2}$$

Differentiating w.r.t. r, we get

$$\frac{dV}{dr} = \frac{S - 6\pi r^2}{2}$$

For maxima and minima, put $\frac{dV}{dr} = 0$

$$\therefore \quad \frac{S - 6\pi r^2}{2} = 0$$

$$\Rightarrow \quad S = 6\pi r^2$$

$$\therefore \quad \text{From Eq. (i),} \quad 2\pi r^2 + 2\pi rh = 6\pi r^2$$

$$\Rightarrow \quad 4\pi r^2 = 2\pi rh$$

$$\Rightarrow \quad h = 2r$$

$\therefore$ Height = Diameter of the base

$$\text{Also} \quad \frac{d^2V}{dr^2} = \frac{d}{dr}\left(\frac{dV}{dr}\right)$$

$$= \frac{d}{dr}\left(\frac{S - 6\pi r^2}{2}\right)$$

$$= -6\pi r < 0.$$

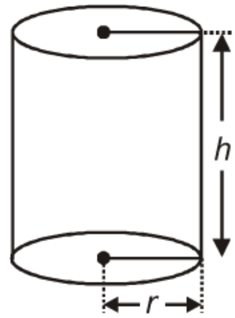
$\Rightarrow$ V is maximum.

Hence, V is maximum at $h = 2r$.

S28. Let r be the radius and h be the height of the closed cylindrical can of volume 100 cm^3. Then,

$$\pi r^2 h = 100 \quad \Rightarrow \quad h = \frac{100}{\pi r^2} \quad \text{... (i)}$$

Let S be the surface area of the can. Then,



$$S = 2\pi rh + 2\pi r^2$$

$$\Rightarrow \qquad S = \frac{200}{r} + 2\pi r^2$$

Now for maxima and minima $\frac{dS}{dr} = 0$

$$\Rightarrow \qquad -\frac{200}{r^2} + 4\pi r = 0$$

$$\Rightarrow \qquad 4\pi r^3 = 200$$

$$\Rightarrow \qquad r = \left(\frac{50}{\pi}\right)^{1/3}$$

$$\frac{d^2S}{dr^2} = \frac{400}{r^3} + 4\pi$$

$$\therefore \left(\frac{d^2s}{dr^2}\right) \text{at} \left(r = \left(\frac{50}{\pi}\right)^{1/3}\right) = \frac{400}{\frac{50}{\pi}} + 4\pi$$

$$= 8\pi + 4\pi = 12\pi > 0$$

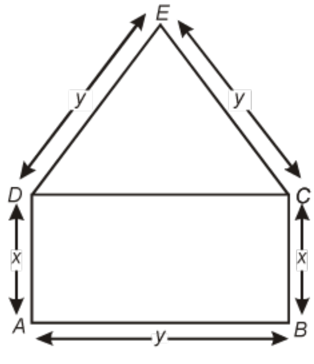
Hence, S is minimum when $r = \left(\frac{50}{\pi}\right)^{1/3}$.

Putting value of r in eq. (i), we get

$$h = 2\left(\frac{50}{\pi}\right)^{1/3} \text{c.m}$$

S29. Let $ABCD$ be the rectangle which is surmounted by an equilateral ΔEDC.

Now, given that



$$\text{Perimeter of window} = 12 \text{ m}$$

$$\Rightarrow \qquad 2x + 3y + y = 12$$

$$\Rightarrow \qquad 2x + 4y = 12$$

$$\Rightarrow \qquad x + 2y = 6$$

$$\Rightarrow \qquad x = 6 - 2y \qquad \dots \text{(i)}$$

Let A denotes the combined area of the window. Then,

$$A = xy + \frac{\sqrt{3}}{4}y^2$$

[$\because$ Combined area = Area of rectangle + Area of equilateral triangle]

$\Rightarrow \qquad A = y(6-2y)+\frac{\sqrt{3}}{4}y^2 \qquad [\because x = 6-2y \text{ by eq. (i)}]$

$\Rightarrow \qquad A = 6y-2y^2+\frac{\sqrt{3}}{4}y^2$

Differentiating w.r.t. y, we get $\quad \frac{dA}{dy} = 6-4y+\frac{2\sqrt{3}}{4}y$

Now, for maxima and minima, put $\frac{dA}{dy} = 0$

$\therefore \qquad 6-4y+\frac{2\sqrt{3}}{4}y = 0$

$\Rightarrow \qquad 6-4y+\frac{\sqrt{3}}{2}y = 0$

$\Rightarrow \qquad y\left(\frac{\sqrt{3}}{2}-4\right) = -6$

$\Rightarrow \qquad y\left(\frac{\sqrt{3}-8}{2}\right) = -6$

$\Rightarrow \qquad y = \frac{12}{8-\sqrt{3}}$

Now, $\qquad \frac{d^2A}{dy^2} = \frac{d}{dy}\left(\frac{dA}{dy}\right) = \frac{d}{dy}\left(6-4y+\frac{2\sqrt{3}}{4}y\right)$

$$= -4+\frac{2\sqrt{3}}{4} = -4+\frac{\sqrt{3}}{2} = \frac{-8+\sqrt{3}}{2} < 0$$

$\therefore$ A is maximum.

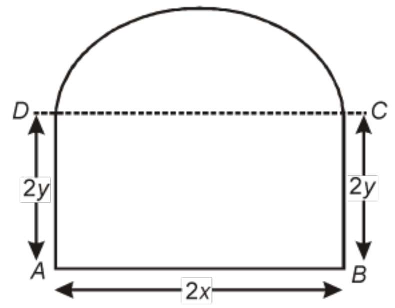
Now putting value of $\quad y = \frac{12}{8-\sqrt{3}}$ in Eq. (i), we get

$$x = 6-2\left(\frac{12}{8-\sqrt{3}}\right) = \frac{48-6\sqrt{3}-24}{8-\sqrt{3}} \text{ or } x = \frac{24-6\sqrt{3}}{8-\sqrt{3}}$$

Hence, the area of the window is largest and the dimensions of the window are

$$x = \frac{24-6\sqrt{3}}{8-\sqrt{3}} \text{ and } y = \frac{12}{8-\sqrt{3}}.$$

S30. Let the dimensions of the window are $2x$ and $2y$ *i.e.*, $AB = 2x$ and $BC = 2y$. Let P denotes the perimeter of the window and A denotes its area.



Given that $P = 10$m

To find dimensions of the window so that maximum light pass through the whole opening.

From the figure, we can see that perimeter of the window is given by

$$P = 2x + 2y + 2y + \pi x$$

[$\because$ In a semi-circle $2x$ is the diameter, so radius is x]

or $$P = 2x + 4y + \pi x$$

$\Rightarrow$ $$2x + 4y + \pi x = 10 \qquad [\because P = 10\text{m}]$$

$\Rightarrow$ $$y = \frac{10 - \pi x - 2x}{4} \qquad \text{...(i)}$$

Also, area of the whole window is given by

$$A = (2x)(2y) + \frac{\pi x^2}{2}$$

[Area of window = Area of rectangle + Area of semi-circle]

or $$A = 4xy + \frac{\pi x^2}{2} \qquad \text{...(ii)}$$

Putting value of y from Eq. (i) in Eq. (ii), we get

$$A = 4x\left(\frac{10 - \pi x - 2x}{4}\right) + \frac{\pi x^2}{2}$$

$\Rightarrow$ $$A = 10x - \pi x^2 - 2x^2 + \frac{\pi x^2}{2}$$

or $$A = 10x - 2x^2 - \frac{\pi x^2}{2}$$

Differentiating w.r.t. x, we get

$$\frac{dA}{dx} = 10 - 4x - \frac{2\pi x}{2}$$
$$= 10 - 4x - \pi x$$

For maxima and minima, put $\frac{dA}{dx} = 0$

$\therefore$ $$10 - 4x - \pi x = 0$$

$\Rightarrow \quad 4x + \pi x = 10$

$\therefore \quad x = \dfrac{10}{\pi + 4}$

Putting value of x in Eq. (i), we get

$$y = \frac{10 - \pi x - 2x}{4}$$

$$\Rightarrow \quad y = \frac{10 - \pi\left(\frac{10}{\pi+4}\right) - 2\left(\frac{10}{\pi+4}\right)}{4}$$

$$\Rightarrow \quad y = \frac{10 - \frac{10\pi}{\pi+4} - \frac{20}{\pi+4}}{4}$$

$$\Rightarrow \quad y = \frac{10\pi + 40 - 10\pi - 20}{4(\pi + 4)} = \frac{20}{4(\pi+4)} = \frac{5}{\pi+4}$$

$\therefore$ We get $x = \dfrac{10}{\pi + 4}$ and $y = \dfrac{5}{\pi + 4}$

Now, $$\frac{d^2A}{dx^2} = \frac{d}{dx}\left(\frac{dA}{dx}\right)$$

$$= \frac{d}{dx}(10 - 4x - \pi x) = -4 - \pi < 0$$

$\because \dfrac{d^2A}{dx^2} < 0$, $\therefore$ A is maximum.

So, maximum light passes through the window.

Also, dimensions of window are $2x = \dfrac{20}{\pi + 4}$ and $2y = \dfrac{10}{\pi + 4}$.

S31. Let r be the radius, h be the height, V be the volume and S be the total surface area of right circular cylinder which is open at the top. Now, given that

$$V = \pi r^2 h$$

$$\Rightarrow \quad h = \frac{V}{\pi r^2} \qquad \text{...(i)}$$

Also, we know that total surface area S is given by

$$S = 2\pi rh + \pi r^2$$

[$\because$ Cylinder is open at the top, therefore curved surface area of cylinder + area of base]

$\Rightarrow$ $$S = 2\pi r\left(\frac{V}{\pi r^2}\right) + \pi r^2 \qquad \left[\text{Putting } h = \frac{V}{\pi r^2} \text{ from eq. (i)}\right]$$

$\Rightarrow$ $$S = \frac{2V}{r} + \pi r^2$$

Differentiating w.r.t. r, we get $$\frac{dS}{dr} = -\frac{2V}{r^2} + 2\pi r$$

Now, for maxima and minima, put $\frac{dS}{dr} = 0$

$\therefore$ $$-\frac{2V}{r^2} + 2\pi r = 0$$

$\Rightarrow$ $$-\frac{2V}{r^2} = -2\pi r$$

$\Rightarrow$ $$V = \pi r^3$$

$\Rightarrow$ $$\pi r^2 h = \pi r^3 \qquad [\because V = \pi r^2 h, \text{ Given}]$$

$\Rightarrow$ $$h = r$$

Also, $$\frac{d^2S}{dr^2} = \frac{d}{dr}\left(\frac{dS}{dr}\right) = \frac{d}{dr}\left(\frac{-2V}{r^2} + 2\pi r\right)$$

$\Rightarrow$ $$\frac{d^2S}{dr^2} = \frac{4V}{r^3} + 2\pi$$

Putting $r = h$, we get $$\left[\frac{d^2S}{dr^2}\right]_{r=h} = \frac{4V}{h^3} + 2\pi > 0 \text{ as } h > 0$$

$\because \frac{d^2S}{dr^2} > 0 \Rightarrow S$ is minimum.

Hence, S is minimum, when $h = r$, *i.e.*, when height of cylinder is equal to radius of the base.

S32. Let V be the volume, S be the total surface area of a right circular cylinder which is open at the top. Let r be the radius of base, h be the height. Now, given that

$$S = 2\pi rh + \pi r^2 \qquad [\because \text{ Cylinder is open at top}]$$

$\Rightarrow$ $$h = \frac{S - \pi r^2}{2\pi r} \qquad \dots \text{(i)}$$

Also, volume of cylinder is given by

$$V = \pi r^2 h$$

$\Rightarrow$ $$V = \pi r^2\left(\frac{S - \pi r^2}{2\pi r}\right) \qquad [\text{Using Eq. (i)}]$$

$$\Rightarrow \quad V = \frac{rS - \pi r^3}{2}$$

Differentiating w.r.t. r, we get $\frac{dV}{dr} = \frac{S - 3\pi r^2}{2}$

For maxima and minima, put $\frac{dV}{dr} = 0$

$$\therefore \quad \frac{S - 3\pi r^2}{2} = 0$$

$$\Rightarrow \quad S = 3\pi r^2$$

$$\Rightarrow \quad 2\pi rh + \pi r^2 = 3\pi r^2 \qquad \text{[From Eq. (i)]}$$

$$\Rightarrow \quad 2\pi rh = 2\pi r^2$$

$$\Rightarrow \quad h = r$$

$\therefore$ Height of cylinder = Radius of the base

Also,
$$\frac{d^2V}{dr^2} = \frac{d}{dr}\left(\frac{dV}{dr}\right) = \frac{d}{dr}\left(\frac{S - 3\pi r^2}{2}\right)$$

$$= -\frac{6\pi r}{2} = -3\pi r < 0, \text{ as } r > 0$$

$\because \frac{d^2V}{dr^2} < 0 \Rightarrow V$ is maximum.

Hence, volume of cylinder is maximum, when its height is equal to radius of the base.

Hence proved.

S33. Let r be the radius of base, h be the height, V the volume and S be the total surface area of the closed right circular cylinder. Then, given that

$$V = \pi r^2 h$$

$$\Rightarrow \quad h = \frac{V}{\pi r^2} \qquad \text{...(i)}$$

Now, we know that total surface area of cylinder is given by

$$S = 2\pi r^2 + 2\pi rh$$

$$\Rightarrow \quad S = 2\pi r^2 + 2\pi r\left(\frac{V}{\pi r^2}\right) \qquad \left[\because h = \frac{V}{\pi r^2} \text{ by eq. (i)}\right]$$

$$\Rightarrow \quad S = 2\pi r^2 + \frac{2V}{r}$$

Differentiating w.r.t. r, we get $$\frac{dS}{dr} = 4\pi r - \frac{2V}{r^2}$$

For maxima and minima, put $$\frac{dS}{dr} = 0$$

$\therefore$ $$4\pi r - \frac{2V}{r^2} = 0$$

$\Rightarrow$ $$4\pi r = \frac{2V}{r^2}$$

$\Rightarrow$ $$4\pi r^3 = 2V$$

or $$V = 2\pi r^3$$

$\Rightarrow$ $$\pi r^2 h = 2\pi r^3 \qquad [\because\ V = \pi r^2 h]$$

$\Rightarrow$ $$h = 2r$$

i.e., Height = Diameter of the base

Also, $$\frac{d^2S}{dr^2} = \frac{d}{dr}\left(\frac{dS}{dr}\right)$$

$$= \frac{d}{dr}\left(4\pi r - \frac{2V}{r^2}\right)$$

$$= 4\pi + \frac{4V}{r^3} > 0, \text{ as } r > 0 \text{ and } V > 0$$

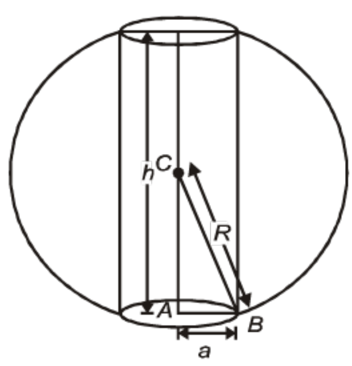
$\because$ $$\frac{d^2S}{dr^2} > 0 \quad \Rightarrow \quad S \text{ is a minimum.}$$

Hence, S is minimum, when height is equal to the diameter of the base.

S34. Let the cylinder inscribed in the given sphere of radius(R) having height(h) and base radius(a).

From ΔABC, $$AB^2 + AC^2 = BC^2$$

$\Rightarrow$ $$a^2 + \left(\frac{h}{2}\right)^2 = R^2$$



$\Rightarrow$ $$a^2 = R^2 - \frac{h^2}{4}$$

Volume of cylinder, $$V = \pi a^2 h$$

$$= \pi h\left(R^2 - \frac{h^2}{4}\right) = \frac{\pi}{4}\left(4R^2 h - h^3\right)$$

Differentiating w.r.t. h, we get

$$\frac{dV}{dh} = \frac{\pi}{4}\left(4R^2 - 3h^{2)}\right) \qquad \ldots \text{(i)}$$

For maxima or minima $\quad \frac{dV}{dh} = 0$

$\Rightarrow \quad \frac{\pi}{4}\left(4R^2 - 3h^{2)}\right) = 0$

$\Rightarrow \quad h^2 = \frac{4}{3}R^2 \quad \Rightarrow \quad h = \frac{2}{\sqrt{3}}R$

Differentiating (i) w.r.t. h, again

and $\quad \frac{d^2V}{dh^2} = \frac{\pi}{4}(-6h) = -\frac{3\pi h}{2}$

$$\left(\frac{d^2V}{dh^2}\right) at \left(h = \frac{2}{\sqrt{3}}R\right) = \frac{-3\pi}{2}\cdot\frac{2}{\sqrt{3}}R = -\sqrt{3}\pi R < 0$$

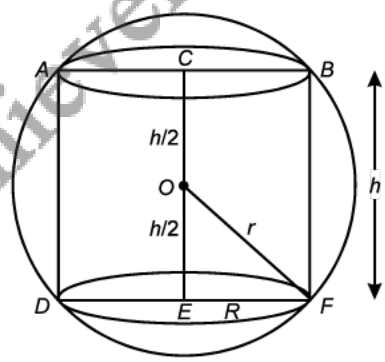
$\Rightarrow$ Volume of cylinder is maximum at $\quad h = \frac{2}{\sqrt{3}}R$

$\therefore \quad$ Maximum volume of cylinder = $\pi h\left(R^2 - \frac{h^2}{4}\right) = \pi\frac{2R}{\sqrt{3}}\left(R^2 - \frac{1}{4}\frac{4}{3}R^2\right)$

$$= \frac{2\pi R}{\sqrt{3}}\frac{\left(3R^2 - R^2\right)}{3} = \frac{4\pi R^3}{3\sqrt{3}} \text{ cu units.}$$

S35. Let r be the radius of sphere. Let h be the height, R the radius of base and V be the volume of the right circular cylinder.

Now, from right ΔOEF in the figure, we have by Pythagoras theorem



$$OE^2 + EF^2 = OF^2$$

$\Rightarrow \quad \left(\frac{h}{2}\right)^2 + R^2 = r^2$

or $\quad R^2 + \frac{h^2}{4} = r^2 \quad \ldots$ (i)

Also, we know that volume of cylinder is given by

$$V = \pi R^2 h$$

$\Rightarrow \quad V = \pi\left(r^2 - \frac{h^2}{4}\right)\cdot h \quad \left[\because R^2 = r^2 - \frac{h^2}{4} \text{ By using eq. (i)}\right]$

$\Rightarrow \quad V = \pi\left(r^2 h - \frac{h^3}{4}\right)$

Differentiating w.r.t. h, we get

$$\frac{dV}{dh} = \pi\left(r^2 - \frac{3h^2}{4}\right)$$

Now for maxima and minima, put $\frac{dV}{dh} = 0$

$\therefore \quad \pi\left(r^2 - \frac{3h^2}{4}\right) = 0$

$\Rightarrow \quad r^2 = \frac{3h^2}{4}$

$\Rightarrow \quad 4r^2 = 3h^2$

or $\quad h^2 = \frac{4r^2}{3}$

or $\quad h = \frac{2r}{\sqrt{3}}$

Putting $h = \frac{2r}{\sqrt{3}}$ in Eq. (i), we get

$$R^2 + \frac{4r^2}{12} = r^2$$

$\Rightarrow \quad R^2 = r^2 - \frac{r^2}{3} = \frac{2r^2}{3}$

$\Rightarrow \quad R = \frac{\sqrt{2}r}{\sqrt{3}} = \sqrt{\frac{2}{3}}r$

Also, $\quad \frac{d^2V}{dh^2} = \frac{d}{dh}\left[\pi\left(r^2 - \frac{3h^2}{4}\right)\right]$

$$= \pi\left(-\frac{6h}{4}\right) = -\frac{3\pi h}{2}$$

$\Rightarrow \quad V$ is maximum as $\frac{d^2V}{dh^2} < 0.$ [as $h > 0$]

Now, putting $R = \sqrt{\frac{2}{3}}r$ and $h = \frac{2r}{\sqrt{3}}$ in $V = \pi R^2 h$,

We get $\quad V = \left(\frac{2}{3}r^2\right)\left(\frac{2r}{\sqrt{3}}\right)\pi$

or $\quad V = \frac{4\pi r^3}{3\sqrt{3}}$ cm^3

Hence, the maximum volume of the right circular cylinder which is inscribed in a sphere of radius r cm is $\dfrac{4\pi r^3}{3\sqrt{3}}$ cm^3.

S36. Let VAB be the given cone of height h and a semi-vertical angle α. Let V denotes the volume of cylinder. From the figure, we get H = Height of cylinder = $OO' = h - VO'$

Now, in right angled $\Delta VO'C$, we get

$$\tan\alpha = \frac{O'C}{VO'} = \frac{r}{VO'}$$

or $$VO' = \frac{r}{\tan\alpha} = r\cot\alpha$$

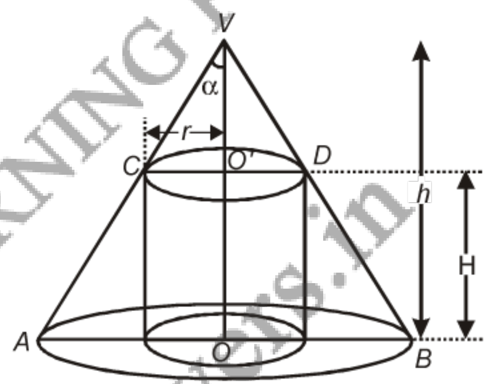
$\therefore$ Height of cylinder = $H = h - VO' = h - r\cot\alpha$

Also, radius of base of cylinder = $O'C = r$

$\therefore$ Volume of cylinder is given by, $V = \pi r^2 H$

$$V = \pi r^2(h - r\cot\alpha) \qquad [\because H = h - r\cot\alpha]$$

$$\Rightarrow \quad V = \pi r^2 h - \pi r^3\cot\alpha$$



Differentiating w.r.t. r, we get $\dfrac{dV}{dr} = 2\pi rh - 3\pi r^2\cot\alpha$

Now, for maxima and minima, put $\dfrac{dV}{dr} = 0$

$\therefore \quad 2\pi rh - 3\pi r^2\cot\alpha = 0$

$\Rightarrow \quad 2\pi rh = 3\pi r^2\cot\alpha$

$\Rightarrow \quad 2h = 3r\cot\alpha$

$\Rightarrow \quad r = \dfrac{2h}{3}\tan\alpha$

Now, $$\frac{d^2V}{dr^2} = \frac{d}{dr}\left(\frac{dV}{dr}\right) = \frac{d}{dr}\left(2\pi rh - 3\pi r^2\cot\alpha\right) = 2\pi h - 6\pi r\cot\alpha$$

$$\therefore \quad \left[\frac{d^2V}{dr^2}\right]_{r=\frac{2h}{3}\tan\alpha} = 2\pi h - 6\pi\cot\alpha\cdot\frac{2h}{3}\tan\alpha$$

$$= 2\pi h - 4\pi h\tan\alpha\cot\alpha$$

$$= 2\pi h - 4\pi h \qquad [\because \tan\alpha\cot\alpha = 1]$$

$$= -2\pi h < 0 \text{ as } h > 0$$

$\because \quad \dfrac{d^2V}{dr^2} < 0 \Rightarrow$ Volume is maximum.

Also, height of cylinder, $H = h - r\cot\alpha$

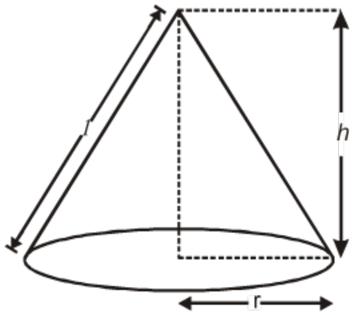
$$= h - \frac{2h}{3}\tan\alpha\cot\alpha \qquad \left[\because r = \frac{2h}{3}\tan\alpha\right]$$

$$= h - \frac{2h}{3} = \frac{h}{3} \qquad [\because \tan\alpha\times\cot\alpha = 1]$$

$\therefore$ Height of cylinder of maximum volume that can be inscribed in a cone of height h is $\frac{1}{3}h$.

S37. Let C denotes the curved surface area, r be the radius of base, h be the height and V be the volume of right circular cone.

To show, $h = \sqrt{2}r$



We know that volume of cone is given by

$$V = \frac{1}{3}\pi r^2 h$$

$\Rightarrow$ $$h = \frac{3V}{\pi r^2} \quad \text{... (i)}$$

Also, the curved surface area of cone is given by $C = \pi r l$, where $l = \sqrt{r^2 + h^2}$ is the slant height of cone.

$\therefore$ $$C = \pi r\sqrt{r^2 + h^2}$$

Squaring both sides, we get $$C^2 = \pi^2 r^2 (r^2 + h^2)$$

or $$C^2 = \pi^2 r^4 + \pi^2 r^2 h^2$$

Let $C^2 = z$

$\therefore$ $$z = \pi^2 r^4 + \pi^2 r^2 h^2 \quad \text{...(ii)}$$

$$z = \pi^2 r^4 + \pi^2 r^2 \left(\frac{3V}{\pi r^2}\right)^2 \quad \text{[From Eq. (i)]}$$

$\Rightarrow$ $$z = \pi^2 r^4 + \pi^2 r^2 \times \frac{9V^2}{\pi^2 r^4}$$

$\Rightarrow$ $$z = \pi^2 r^4 + \frac{9V^2}{r^2}$$

Differentiating both sides w.r.t. r, we get

$$\frac{dz}{dr} = 4\pi^2 r^3 - \frac{18V^2}{r^3}$$

For maxima and minima, put $\frac{dz}{dr} = 0$

$\therefore$ $$4\pi^2 r^3 - \frac{18V^2}{r^3} = 0$$

$\Rightarrow$ $$4\pi^2 r^3 = \frac{18V^2}{r^3}$$

$\Rightarrow \qquad 4\pi^2 r^6 = 18\left(\frac{1}{3}\pi r^2 h\right)^2 \qquad \left[\because V = \frac{1}{3}\pi r^2 h\right]$

$\Rightarrow \qquad = 18 \times \frac{1}{9}\pi^2 r^4 h^2 = 2\pi^2 r^4 h^2$

$\Rightarrow \qquad 2r^2 = h^2$

or $\qquad h = \sqrt{2}r$

Hence, height = $\sqrt{2}$ [radius of base]

Also, $$\frac{d^2z}{dr^2} = \frac{d}{dr}\left(\frac{dz}{dr}\right) = \frac{d}{dr}\left(4\pi^2 r^3 - \frac{18V^2}{r^3}\right)$$

$$= 12\pi^2 r^2 + \frac{54V^2}{r^4}$$

$\therefore$ $$\frac{d^2z}{dr^2} = 12\pi^2 r^2 + \frac{54V^2}{r^4} > 0$$

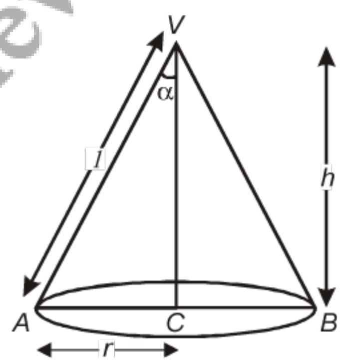
$\Rightarrow$ z is minimum

$\Rightarrow$ C is minimum.

Hence, curved surface area is least, when $h = \sqrt{2}r$.

Hence proved.

S38. Let h be the height, l be the slant height, r be the radius of base of the right circular cone. Let α be the semi-vertical angle of the cone.



Using Pythagoras theorem is ΔVAC,

$$l^2 = r^2 + h^2$$

$\Rightarrow \qquad r^2 = l^2 - h^2 \qquad \dots$ (i)

Let V be the volume of cone which is given by

$$V = \frac{1}{3}\pi r^2 h$$

$\Rightarrow \qquad V = \frac{\pi}{3}\left(l^2 - h^2\right)\cdot h$

or $\qquad V = \frac{\pi}{3}\left(l^2 h - h^3\right)$

Differentiating w.r.t. h, we get $\qquad \frac{dV}{dh} = \frac{\pi}{3}\left(l^2 - 3h^2\right)$

Now, for maxima and minima, put $\frac{dV}{dh} = 0$

$$\therefore \quad \frac{\pi}{3}(l^2 - 3h^2) = 0$$

$$\Rightarrow \quad l^2 = 3h^2$$

$$\Rightarrow \quad r^2 + h^2 = 3h^2 \qquad [\because l^2 = r^2 + h^2]$$

$$\Rightarrow \quad 2h^2 = r^2$$

$$\text{or} \quad r = \sqrt{2}h \qquad \text{... (ii)}$$

Now, in right angle ΔCVA in the figure, we have

$$\tan\alpha = \frac{AC}{VC}$$

$$\therefore \quad \tan\alpha = \frac{r}{h} \qquad [\because AC = r \text{ and } VC = h]$$

$$\Rightarrow \quad \tan\alpha = \frac{\sqrt{2}h}{h} \qquad \left[\because r = \sqrt{2}h \text{ By Eq. (ii)}\right]$$

$$\Rightarrow \quad \tan\alpha = \sqrt{2}$$

$$\text{or} \quad \alpha = \tan^{-1}\sqrt{2}$$

Also,
$$\frac{d^2V}{dh^2} = \frac{d}{dh}\left[\frac{\pi}{3}(l^2 - 3h^2)\right]$$

$$= \frac{\pi}{3}(-6h) = -2\pi h < 0 \qquad [\because h > 0]$$

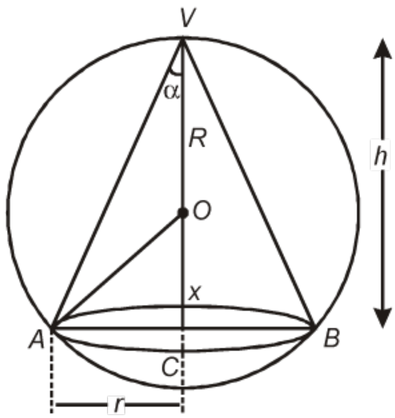
$\therefore$ V is maximum.

Hence, the volume is maximum, when $\alpha = \tan^{-1}\sqrt{2}$.

S39. Let R be the radius of sphere. Let r be the radius of base of cone and h be the height of cone.

Then, from the figure $h = R + x$ ... (i)

Let V denotes the volume of cone. Now, in right ΔOCA in the figure, we get



$$OA^2 = OC^2 + AC^2 \quad \text{[By Pythagoras theorem]}$$

$$\Rightarrow \quad R^2 = x^2 + r^2$$

$$\Rightarrow \quad r^2 = R^2 - x^2 \qquad \text{... (ii)}$$

Also, we know that volume of cone is given by

$$V = \frac{1}{3}\pi r^2 h$$

$$\Rightarrow \quad V = \frac{1}{3}\pi(R^2 - x^2)(R + x)$$

$$\left[\because h = R + x \text{ By eq. (i) and } r^2 = R^2 - x^2 \text{ By eq. (ii)}\right]$$

$\Rightarrow$ $$V = \frac{\pi}{3}\left(R^3 + R^2x - x^2R - x^3\right)$$

Differentiating w.r.t. x, we get

$$\frac{dV}{dx} = \frac{\pi}{3}\left(R^2 - 2xR - 3x^2\right)$$

$\Rightarrow$ $$\frac{dV}{dx} = \frac{\pi}{3}\left[R^2 - 3xR + xR - 3x^2\right] = \frac{\pi}{3}\left[R(R-3x) + x(R-3x)\right]$$

$\therefore$ $$\frac{dV}{dx} = \frac{\pi}{3}(R+x)(R-3x)$$

Now, for maxima and minima, put $\frac{dV}{dx} = 0$

$\therefore$ $\frac{\pi}{3}(R+x)(R-3x) = 0$

Either $R + x = 0$ or $R - 3x = 0$

Now, $R + x = h$ which is height of cone. As h can never be zero. So, $R + x = 0$ is rejected.

$\therefore$ $$R - 3x = 0$$

$$3x = R$$

or $$x = \frac{R}{3}$$

Also, $$\frac{d^2V}{dx^2} = \frac{d}{dx}\left(\frac{dV}{dx}\right) = \frac{d}{dx}\left[\frac{\pi}{3}\left(R^2 - 2xR - 3x^2\right)\right]$$

$$\frac{d^2V}{dx^2} = \frac{\pi}{3}(-2R - 6x)$$

$\Rightarrow$ $$\left[\frac{d^2V}{dx^2}\right]_{x=\frac{R}{3}} = \frac{\pi}{3}\left[-2R - \frac{6R}{3}\right] = \frac{\pi}{3}(-4R) = -\frac{4\pi R}{3} < 0$$

$\because$ $\frac{d^2V}{dx^2} < 0 \Rightarrow V$ is maximum.

Now, we have to show that

Volume of largest cone = $\frac{8}{27}$ [Volume of sphere]

Now, volume of cone is

$$V = \frac{\pi}{3}(R^2 - x^2)(R + x)$$

Putting x = $\frac{R}{3}$, we get

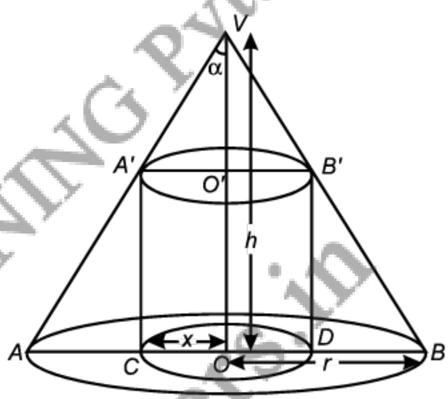
$$V = \frac{\pi}{3}\left[R^2 - \frac{R^2}{9}\right]\left[R + \frac{R}{3}\right] = \frac{\pi}{3}\left(\frac{8R^2}{9}\right)\left(\frac{4R}{3}\right) = \frac{32\pi R^3}{81} = \frac{8}{27}\left(\frac{4}{3}\pi R^3\right)$$

$$= \frac{8}{27} \text{ [volume of sphere]} \qquad \left[\because \text{Volume of sphere} = \frac{4}{3}\pi R^3\right]$$

Hence,volume of largest cone = $\frac{8}{27}$ [volume of sphere]

Hence proved.

S40. Let *VAB* be the cone of radius of base *r* and height *h*. Let radius of base of the inscribed cylinder be *x*.



Now, we observe that $\Delta VOB \sim \Delta B'DB$

$\Rightarrow$ $$\frac{VO}{B'D} = \frac{OB}{DB}$$

$\Rightarrow$ $$\frac{h}{B'D} = \frac{r}{r-x}$$

$\Rightarrow$ $$B'D = \frac{h(r-x)}{r}$$

Let *C* be the curved surface area of cylinder. Then,

$$C = 2\pi\ (OC)\ (B'D)$$

$\Rightarrow$ $$C = \frac{2\pi\ xh(r-x)}{r} = \frac{2\pi h}{r}(rx - x^2)$$

Differentiating w.r.t. *x*, we get

$$\frac{dC}{dx} = \frac{2\pi h}{r}\ (r - 2x)$$

For maxima and minima, put $\frac{dC}{dx} = 0$

$\therefore$ $$\frac{2\pi h}{r}\ (r - 2x) = 0$$

$\Rightarrow$ $$r - 2x = 0$$

$\Rightarrow$ $$r = 2x \quad \text{or} \quad x = \frac{r}{2}$$

Hence, radius of cylinder is half of that of cone.

Also, $$\frac{d^2C}{dx^2} = \frac{d}{dx}\left[\frac{2\pi h(r-2x)}{r}\right]$$

$$= \frac{2\pi h}{r}(-2)$$

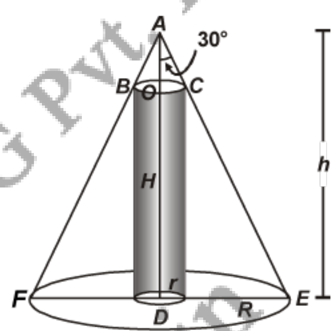
$$= \frac{-4\pi h}{r} < 0 \text{ as } h, r > 0$$

$\therefore \frac{d^2C}{dx^2} < 0, \Rightarrow C$ is maximum or greatest.

Hence, C is greatest at $x = \frac{r}{2}$.

S41. We have given h is the height of cone.

Let r be the radius of cylinder



Then, in ΔADE $\quad \cot 30° = \frac{h}{R}$

$$R = \frac{h}{\cot 30°}$$

In ΔAOC $\quad \cot 30° = \frac{AO}{r}$

$$AO = r\cot 30° = r\sqrt{3}$$

Height of cylinder = $H = h - r\sqrt{3}$

Volume of cylinder = $V = \pi r^2 H = \pi r^2(h - \sqrt{3}r) = \pi r^2 h - \pi r^3\sqrt{3}$.

Differentiating w.r.t. "r", we get

$$\frac{dV}{dr} = 2\pi rh - 3\sqrt{3}\pi r^2 \quad \ldots \text{(i)}$$

For maxima or minima,

$$\frac{dV}{dr} = 0$$

$\Rightarrow \quad \pi r(2h - 3\sqrt{3}r) = 0 \quad \Rightarrow \quad 2h - 3\sqrt{3}r = 0 \quad \Rightarrow \quad r = \frac{2}{3\sqrt{3}}h$

Now, again differentiating (i), we get

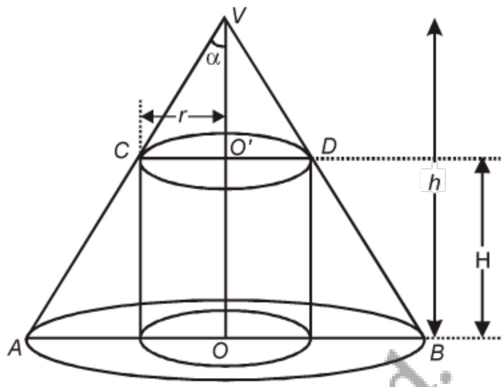
$$\frac{d^2V}{dr^2} = 2\pi h - 6\sqrt{3}\pi r > 0 \quad \text{at} \quad r = \frac{2}{3\sqrt{3}}h$$

Hence for the $r = \frac{2}{3\sqrt{3}}h$ and height = $h - \frac{2}{3\sqrt{3}}h\cdot\sqrt{3} = \frac{1}{3}h$ the cylinder has maximum volume.

$$\text{Volume of cylinder} = \pi\left(\frac{2}{3\sqrt{3}}h\right)^2 \frac{1}{3}h = \frac{4}{81}\pi h^3.$$

S42. Let VAB be the given cone of height h and semi-vertical angle α. Let V denotes the volume of the cylinder. From the figure, we get

$$H = \text{Height of cylinder} = OO' = h - VO'$$



Now, in right angled $\Delta VO'C$, we get

$$\tan\alpha = \frac{O'C}{VO'}$$

$$= \frac{r}{VO'} \quad [\because O'C = r]$$

$$\Rightarrow \quad VO' = \frac{r}{\tan\alpha}$$

or $\quad VO' = r\cot\alpha$

$\therefore$ Height of cylinder, $\quad H = OO' = h - VO' = h - r\cot\alpha$

Let radius of base of cylinder $= O'C = r$

$\therefore$ Volume of cylinder is given by

$$V = \pi r^2 H \qquad [\because H = h - r\cot\alpha]$$

or $\quad V = \pi r^2 (h - r\cot\alpha) \qquad \text{...(i)}$

or $\quad V = \pi r^2 h - \pi r^3 \cot\alpha$

Differentiating w.r.t. r, we get

$$\frac{dV}{dr} = 2\pi rh - 3\pi r^2 \cot\alpha$$

Now, for maxima and minima, put $\frac{dV}{dr} = 0$

$\therefore \quad 2\pi rh - 3\pi r^2 \cot\alpha = 0$

$\Rightarrow \quad 2\pi rh = 3\pi r^2 \cot\alpha$

$\Rightarrow \quad 2h = 3r\cot\alpha$

$\Rightarrow \quad r = \frac{2h}{3}\tan\alpha$

Also, $\quad \frac{d^2V}{dr^2} = \frac{d}{dr}\left(\frac{dV}{dr}\right)$

$$= \frac{d}{dr}\left(2\pi rh - 3\pi r^2\cot\alpha\right)$$

$$\frac{d^2V}{dr^2} = 2\pi h - 6\pi r\cot\alpha$$

$\therefore \qquad \left[\dfrac{d^2V}{dr^2}\right]_{r = \frac{2h}{3}\tan\alpha} = 2\pi h - 6\pi \cot\alpha\left(\dfrac{2h}{3}\tan\alpha\right)$

$= 2\pi h - 4\pi h \tan\alpha \cot\alpha$

$= 2\pi h - 4\pi h$

$= -2\pi h < 0$ as $h > 0$

$\because \qquad \dfrac{d^2V}{dr^2} < 0 \Rightarrow V$ is maximum.

Now, we find the maximum value. From Eq. (i), we get

$$V = \pi r^2 (h - r\cot\alpha)$$

Putting $r = \dfrac{2h}{3}\tan\alpha$, we get

$$V = \pi \times \frac{4h^2}{9}\tan^2\alpha\left[h - \frac{2h}{3}\tan\alpha\cot\alpha\right]$$

$$= \frac{4\pi h^2}{9}\tan^2\alpha\left(h - \frac{2h}{3}\right)$$

$\Rightarrow \qquad V = \dfrac{4\pi h^2}{9}\left(\dfrac{h}{3}\right)\tan^2\alpha$

$\Rightarrow \qquad V = \dfrac{4\pi h^3}{27}\tan^2\alpha$

Hence, the volume of largest cylinder $= \dfrac{4\pi h^3}{27}\tan^2\alpha$.

Hence proved.

S43. Let $P(x, y)$ be any point on the parabola and $Q(0, c)$ be the given point. Then,

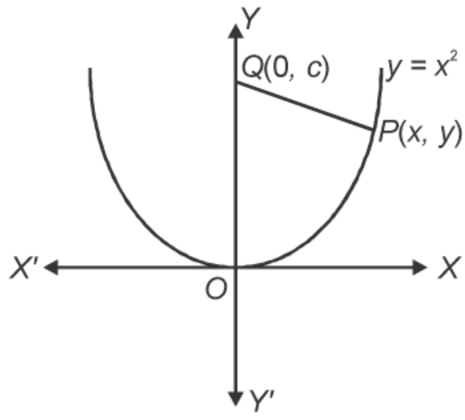
$$PQ^2 = x^2 + (y - c)^2$$

$\Rightarrow \qquad PQ^2 = x^2 + (x^2 - c)^2 = x^2 + x^4 - 2cx^2 + c^2$

$\Rightarrow \qquad PQ^2 = x^4 - x^2(2c - 1) + c^2$

Clearly, PQ will be minimum when PQ^2 is minimum. Let $Z = PQ^2$ Then,

$$Z = x^4 - x^2(2c-1) + c^2$$



$$\Rightarrow \quad \frac{dZ}{dx} = 4x^3 - 2x(2c-1)$$

and $$\frac{d^2Z}{dx^2} = 12x^2 - 2(2c-1)$$

For maximum or minimum value of Z, we must have

$$\frac{dZ}{dx} = 0$$

$$\Rightarrow \quad 4x^3 - 2x(2c-1) = 0$$

$$\Rightarrow \quad 2x\{2x^2 - (2c-1)\} = 0$$

$$\Rightarrow \quad x = 0, x = \pm\sqrt{\frac{2c-1}{2}}$$

$$\Rightarrow \quad x = 0, x = \pm\alpha, \text{ where } \alpha = \sqrt{\frac{2c-1}{2}}$$

Now, $$\left(\frac{d^2Z}{dx^2}\right)_{x=\pm\alpha} = 12x^2 - 4\left(\frac{2c-1}{2}\right)$$

$$= 12\alpha^2 - 4\alpha^2 = 8\alpha^2 > 0$$

So, Z is minimum at $x = \pm\alpha$.

Hence, PQ is minimum at $x = \pm\alpha$.

The minimum value of PQ is given by

$$PQ^2 = \alpha^2 + (\alpha^2 - c)^2$$

$$\Rightarrow \quad PQ^2 = \frac{2c-1}{2} + \left(\frac{2c-1}{2} - c\right)^2 = \frac{2c-1}{2} + \frac{1}{4} = \frac{4c-1}{4}$$

$$\Rightarrow \quad PQ = \frac{\sqrt{4c-1}}{2}$$

Hence, the minimum distance is $\frac{\sqrt{4c-1}}{2}$

S44. Let $P(t^2, t)$ be any point on the curve $x = y^2$. The distance S of P from the given line is

$$S = \left|\frac{t - t^2 - 1}{\sqrt{2}}\right| \qquad t^2 - t + 1 > 0 \text{ for all } t \in R$$

$\Rightarrow$ $$S = \left|\frac{t^2 - t + 1}{\sqrt{2}}\right|$$

$\Rightarrow$ $$S = \frac{t^2 - t + 1}{\sqrt{2}}$$

$\Rightarrow$ $$S = \frac{f(t)}{\sqrt{2}}, \text{ when } f(t) = t^2 - t + 1$$

Clearly, S will be maximum or minimum according as $f(t)$ is maximum or minimum.

Now, $$f(t) = t^2 - t + 1$$

$\Rightarrow$ $$f'(t) = 2t - 1 \quad \text{and} \quad f''(t) = 2$$

For maximum or minimum, we must have

$$f'(t) = 0$$

$\Rightarrow$ $$2t - 1 = 0$$

$\Rightarrow$ $$t = \frac{1}{2}$$

For $t = \frac{1}{2}$, we have

$$f''(t) > 0$$

Hence $f(t)$ is minimum when $t = \frac{1}{2}$.

So, S is the minimum when $t = \frac{1}{2}$.

The minimum value of S is given by

$$S = \frac{\left(\frac{1}{2}\right)^2 - \frac{1}{2} + 1}{\sqrt{2}} = \frac{\frac{5}{4} - \frac{1}{2}}{\sqrt{2}} = \frac{6}{8\sqrt{2}} = \frac{3\sqrt{2}}{8}.$$

S45. The given equation of curve is $y^2 = 2x$ and the given point is $Q(1, 4)$. Let $P(x, y)$ be the point, which is at a minimum distance from point $Q(1, 4)$. Now, distance between points P and Q is given by

$$PQ = \sqrt{(1 - x)^2 + (4 - y)^2}$$

[By using distance formula $S = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$]

$\Rightarrow$ $$PQ = \sqrt{1 + x^2 - 2x + 16 + y^2 - 8y} = \sqrt{x^2 + y^2 - 2x - 8y + 17}$$

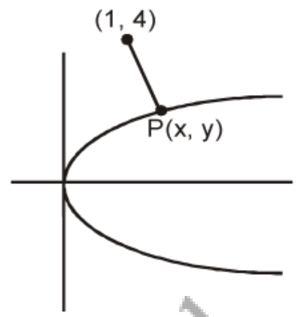
Squaring both sides, we get

$$PQ^2 = x^2 + y^2 - 2x - 8y + 17$$

$\Rightarrow$ $$PQ^2 = \left(\frac{y^2}{2}\right)^2 + y^2 - 2\left(\frac{y^2}{2}\right) - 8y + 17 \qquad \left[\because y^2 = 2x \text{ is given } \Rightarrow x = \frac{y^2}{2}\right]$$

$\therefore$ $$PQ^2 = \frac{y^4}{4} + y^2 - y^2 - 8y + 17$$

or $$PQ^2 = \frac{y^4}{4} - 8y + 17$$



Let $PQ^2 = Z$

$\therefore$ $$Z = \frac{y^4}{4} - 8y + 17$$

Differentiating w.r.t. y, we get

$$\frac{dZ}{dy} = \frac{4y^3}{4} - 8 = y^3 - 8$$

For maxima and minima, put $\frac{dZ}{dy} = 0$

$\therefore$ $y^3 - 8 = 0$

$\Rightarrow$ $y^3 = 8$

$\Rightarrow$ $y = 2$

Also, $$\frac{d^2Z}{dy^2} = \frac{d}{dy}\left(y^3 - 8\right) = 3y^2$$

Putting $y = 2$, we get

$$\left[\frac{d^2Z}{dy^2}\right]_{y=2} = 3(2)^2 = 12 > 0$$

$\because$ $$\frac{d^2Z}{dy^2} > 0$$

$\therefore$ Z is minimum and therefore, PQ is also minimum on $Z = PQ^2$.

Putting $y = 2$ in the given equation *i.e.*, $y^2 = 2x$, we get

$(2)^2 = 2x$

$\Rightarrow$ $4 = 2x$

$\Rightarrow$ $x = 2$

Hence, the point which is at a minimum distance from point (1, 4) is P (2, 2).

S46. Let $P(x, y)$ be the position of jet and the soldier is placed at A (3, 2). Then, the distance between the soldier and jet is given by

$$AP = \sqrt{(x-3)^2 + (y-2)^2} = \sqrt{(x-3)^2 + x^4}$$

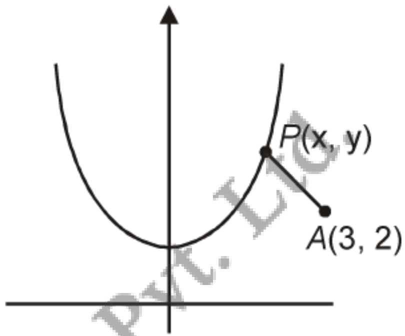
Let $Z = AP^2$. Then,

$$Z = (x-3)^2 + x^4$$

Clearly AP is maximum or minimum according as Z is maximum or minimum.

Now, $Z = (x-3)^2 + x^4$

$$\Rightarrow \quad \frac{dZ}{dx} = 2(x-3) + 4x^3 \text{ and } \frac{d^2Z}{dx^2} = 12x^2 + 2$$



For maximum or minimum, we have

$$\frac{dZ}{dx} = 0$$

$$\Rightarrow \quad 2(x-3) + 4x^3 = 0$$

$$\Rightarrow \quad 2x^3 + x - 3 = 0$$

$$\Rightarrow \quad (x-1)(2x^2 + 2x + 3) = 0$$

$x = 1$ $\quad$ [$\because$ $2x^2 + 2x + 3 = 0$ gives imaginary values of x]

Clearly, $$\left(\frac{d^2Z}{dx^2}\right)_{x=1} = 12 + 2 = 14 > 0$$

Thus, Z is minimum when $x = 1$. Putting $x = 1$ in $y = x^2+2$, we obtain $y = 3$.

Hence, AP is minimum when jet is at the point (1, 3) on the curve.

The nearest distance is given by $AP = \sqrt{(1-3)^2 + 1^4} = \sqrt{4+1} = \sqrt{5}$.

S47. Let $P(x, y)$ be a point on $y^2 = 4x$ and $A(2, 1)$ be the given point. Then,

$$AP^2 = (x-2)^2 + (y-1)^2 = \left(\frac{y^2}{4} - 2\right)^2 + (y-1)^2 \qquad \left[\because y^2 = 4x, \quad \therefore x = y^2/4\right]$$

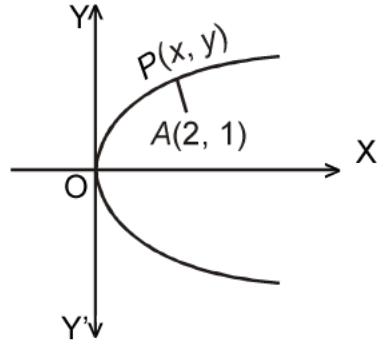
Let Z = AP^2. Then, Z is maximum or minimum according as AP is maximum or minimum.

Now, $$Z = \left(\frac{y^2}{4} - 2\right)^2 + (y-1)^2$$

$$\Rightarrow \quad \frac{dZ}{dy} = 2\left(\frac{y^2}{4} - 2\right)\frac{2y}{4} + 2(y-1)$$

$$= \frac{y^3}{4} - 2y + 2y - 2 \qquad = \frac{y^3}{4} - 2$$

and $$\frac{d^2Z}{dy^2} = \frac{3y^2}{4}$$



For maximum or minimum, we have

$$\frac{dZ}{dy} = 0$$

$$\Rightarrow \quad \frac{y^3}{4} - 2 = 0$$

$$\Rightarrow \quad y^3 = 8$$

$$\Rightarrow \quad y = 2$$

Now, $$\left(\frac{d^2Z}{dy^2}\right)_{y=2} = \frac{3(2)^2}{4} = 3 > 0.$$

Thus, Z is minimum when $y = 2$.

Putting $y = 2$ in $y^2 = 4x$.

$$\therefore \quad 4 = 4x \quad \Rightarrow \quad x = 1$$

Hence the point (1, 2) on $y^2 = 4x$ is nearest to the point (2, 1).

S48. Let any point on the parabola $y^2 = 4x$ be $P(t^2, 2t)$. (here $a = 1$)

Let Q be $(2, -8)$

Now PQ will be smallest if PQ^2 is so.

Let $$Z = PQ^2 = (t^2 - 2)^2 + (2t + 8)^2$$

$$= (t^4 - 4t^2 + 4) + (4t^2 + 32t + 64) = t^4 + 32t + 68$$

$$\Rightarrow \quad \frac{dZ}{dt} = 4t^3 + 32 \qquad ...(i)$$

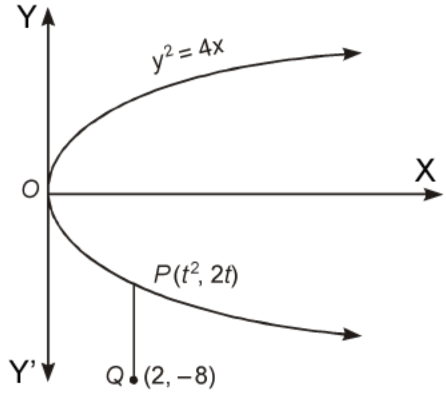
For maxima or minima, $\frac{dZ}{dt} = 0 \quad \Rightarrow \quad 4(t^3 + 8) = 0$

$$\Rightarrow \quad t^3 + 2^3 = 0 \quad \Rightarrow \quad t = -2.$$

(The other two values of t are imaginary)

Differentiating Eq. (i) w.r.t. "t", we get

$$\frac{d^2Z}{dt^2} = 12t^2$$



For $t = -2$, $\frac{d^2Z}{dt^2} = 12 \times 4 = 48 > 0$

$\therefore$ Z is minimum when $t = -2$.

$\Rightarrow$ The point P nearest to Q is $P(4, -4)$.

S49. The equation of a line passing through $P(1, 4)$ is

$$y - 4 = m(x - 1), \qquad \text{... (i)}$$

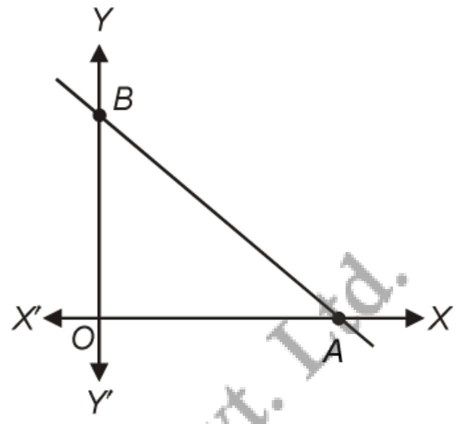
where $m < 0$.

Intercept on x axis is OA,

put $y = 0$

$$0 - 4 = mx - m$$

$$\Rightarrow \quad x = \frac{m-4}{m}$$



$$\therefore \quad OA = \frac{m-4}{m}$$

For Intercept on y axis (OB) put $x = 0$ in Eq. (i)

$$y - 4 = m(0 - 1)$$

$$y = 4 - m$$

Let S be the sum of the intercepts. Then,

$$S = OA + OB$$

$$S = \frac{m-4}{m} - (m - 4)$$

$$= 1 - \frac{4}{m} - m + 4$$

$$= -m + 5 - \frac{4}{m}$$

$$\Rightarrow \quad \frac{dS}{dm} = -1 + \frac{4}{m^2}$$

and $$\frac{d^2S}{dm^2} = -\frac{8}{m^3}$$

For S to be maximum or minimum, we must have

$$\frac{dS}{dm} = 0$$

$$\Rightarrow \quad -1 + \frac{4}{m^2} = 0$$

$$\Rightarrow \quad m^2 = 4$$

$$\Rightarrow \quad m = -2 \qquad [\because m < 0]$$

For $m = -2$, $\dfrac{d^2S}{dm^2} = \dfrac{-8}{(-2)^3} = 1 > 0$

So, S is minimum when $m = -2$

For $m = -2$. The sum of intercepts is given by

$$S = 2 + 5 + 2 = 9.$$

S50. Given that the manufacturer sells x items at price of Rs. $\left(5-\frac{x}{100}\right)$ each.

$\therefore$ Total revenue obtained = Rs. $\left[x\left(5-\frac{x}{100}\right)\right]$ = Rs. $\left(5x-\frac{x^2}{100}\right)$

Also, cost price of x items = Rs. $\left(\frac{x}{5}+500\right)$

Now, we know that Profit = Revenue – Cost

Let $P(x)$ be the profit function. Then, we have

$$P = \left(5x-\frac{x^2}{100}\right)-\left(\frac{x}{5}+500\right)$$

or
$$P = \frac{-x^2}{100}+\frac{24x}{5}-500$$

Differentiating w.r.t. x, we get

$$\frac{dP}{dx} = \frac{-2x}{100}+\frac{24}{5}$$

For maxima and minima, put $\frac{dP}{dx} = 0$

$$\therefore \quad \frac{-2x}{100}+\frac{24}{5} = 0$$

$$\Rightarrow \quad \frac{-2x}{100} = -\frac{24}{5}$$

$$\Rightarrow \quad 10x = 2400$$

$$\Rightarrow \quad x = 240$$

Also,
$$\frac{d^2P}{dx^2} = \frac{d}{dx}\left(\frac{dP}{dx}\right) = \frac{d}{dx}\left(-\frac{2x}{100}+\frac{24}{5}\right)$$

$$= -\frac{2}{100} = -\frac{1}{50} < 0$$

$$\because \quad \frac{d^2P}{dx^2} < 0 \Rightarrow P \text{ is maximum.}$$

$\therefore$ Number of items sold to have maximum profit is 240.

S51. We have,

$$C = 300x-10x^2+\frac{1}{3}x^3$$

(i) Average cost $A = \frac{C}{x} = 300-10x+\frac{1}{3}x^2$ ... (i)

Differentiating Eq. (i) w.r.t "x", we get

$$\Rightarrow \qquad \frac{d}{dx}(A) = -10 + \frac{2}{3}x \qquad \text{... (ii)}$$

For maxima or minima

$$\frac{d}{dx}(A) = 0 \quad \Rightarrow \quad -10 + \frac{2}{3}x = 0$$

$$\Rightarrow \qquad x = 15.$$

Differentiating Eq. (ii) w.r.t "x", we get

$$\frac{d^2}{dx^2}(A) = \frac{2}{3} > 0 \text{ when } x = 15.$$

$\therefore$ A is minima when output is 15.

(ii) Marginal cost $M = \frac{dC}{dx} = 300 - 20x + x^2$ $\left[\text{Marginal cost} = \frac{dC}{dx}\right]$

Now $A = M$

$$\Rightarrow \qquad 300 - 10x + \frac{1}{3}x^2 = 300 - 20x + x^2$$

$$\Rightarrow \qquad 10x - \frac{2}{3}x^2 = 0$$

$$\Rightarrow \qquad x\left(10 - \frac{2}{3}x\right) = 0$$

$$\Rightarrow \qquad x = 15 \quad (\text{as } x \neq 0).$$

$\Rightarrow$ Output is 15, when $A = M$.

S52. Here,

$$C = 200x - \frac{20}{3}x^2 + \frac{2}{9}x^3$$

(i) Average cost $A = \frac{C}{x} = 200 - \frac{20}{3}x + \frac{2}{9}x^2$ ... (i)

Differentiating Eq. (i) w.r.t. "x", we get

$$\Rightarrow \qquad \frac{dA}{dx} = -\frac{20}{3} + \frac{4}{9}x \qquad \text{... (ii)}$$

For A to be minimum or maximum

$$\frac{d}{dx}(A) = 0 \quad \Rightarrow \quad -\frac{20}{3}+\frac{4}{9}x=0$$

$$\Rightarrow \quad x = \frac{20}{3}\cdot\frac{9}{4}=15.$$

Differentiating Eq. (ii) w.r.t. "x", we get

$$\frac{d^2(A)}{dx^2} = \frac{4}{9}>0$$

$\Rightarrow$ A is minima when output is 15.

(ii) Marginal cost $\quad M = \frac{dC}{dx}=200-\frac{40}{3}x+\frac{2}{3}x^2 \quad \left[\text{Marginal cost} = \frac{dC}{dx}\right]$

$$\Rightarrow \quad \frac{d}{dx}(M) = -\frac{40}{3}+\frac{4}{3}x \quad \text{... (iii)}$$

For M to be minima or maxima.

$$\frac{d}{dx}(M) = 0 \quad \Rightarrow \quad \frac{-40}{3}+\frac{4}{3}x=0 \quad \Rightarrow \quad x=\frac{40}{3}\times\frac{3}{4}=10$$

Differentiating Eq. (iii) w.r.t. "x", we get

$$\frac{d^2}{dx^2}(M) = \frac{4}{3}>0$$

$\therefore$ M is minimum when output is 10.

(iii) $M = A$.

$$200-\frac{40}{3}x+\frac{2}{3}x^2 = 200-\frac{20}{3}x+\frac{2}{9}x^2$$

$$\Rightarrow \left(\frac{2}{3}-\frac{2}{9}\right)x^2+\left(\frac{20}{3}-\frac{40}{3}\right)x = 0$$

$$\Rightarrow \quad 4x^2 - 60x = 0$$

$$\Rightarrow \quad 4x(x-15) = 0$$

$$\Rightarrow \quad x = 15 \quad (\because x \neq 0)$$

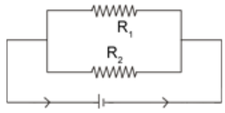
$\therefore$ $M = A$, when output is 15.

S53. We have,

$$\frac{1}{R} = \frac{1}{R_1}+\frac{1}{R_2} \text{ and } R_1 + R_2 = C$$

$$\Rightarrow \quad \frac{1}{R} = \frac{R_1+R_2}{R_1R_2}=\frac{C}{R_1R_2}=\frac{C}{R_1(C-R_1)} \quad [\because R_2 = C-R_1]$$

$$\Rightarrow \qquad R = \frac{R_1C - R_1^2}{C} = R_1 - \frac{R_1^2}{C}$$



$$\Rightarrow \qquad \frac{dR}{dR_1} = 1 - \frac{2R_1}{C} \text{ and } \frac{d^2R}{dR_1^2} = -\frac{2}{C}$$

For maximum or minimum, we must have

$$\frac{dR}{dR_1} = 0 \quad \Rightarrow \quad 1 - \frac{2R_1}{C} = 0 \quad \Rightarrow \quad R_1 = \frac{C}{2}$$

Now, $\qquad \dfrac{d^2R}{dR_1^2} = -\dfrac{2}{C} < 0$ for all values of R_1

Thus, R is maximum when $R_1 = C/2$

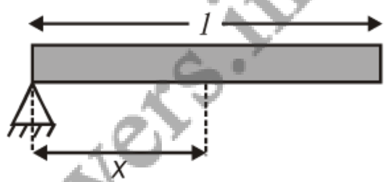
Putting $R_1 = \dfrac{C}{2}$ in $R_1 + R_2 = C$, we get

$$R_2 = C - \frac{C}{2} = \frac{C}{2}$$

Hence, R is maximum when $R_1 = R_2 = C/2$

S54. We have,

$$M = \frac{lx}{2} - \frac{Wx^2}{2}$$



$$\Rightarrow \qquad \frac{dM}{dx} = \frac{l}{2} - Wx$$

For maximum or minimum, we have

$$\frac{dM}{dx} = 0$$

$$\Rightarrow \qquad \frac{l}{2} - Wx = 0$$

$$\Rightarrow \qquad x = \frac{l}{2W}$$

Now, $\qquad \dfrac{d^2M}{dx^2} = -W < 0$ for all values of x

Thus, M is maximum when $x = \dfrac{l}{2W}$

Hence, the required point is at a distance of $\dfrac{l}{2W}$ from the supporting end.

S55.

$$\text{Range} \propto x^2 \log_e \frac{1}{x}$$

$$\text{Range} = kx^2 \log_e \frac{1}{x} \quad \text{or} \quad R = -kx^2 \log_e x$$

Differentiating w.r.t. "x", we get

$$\frac{dR}{dx} = -kx^2\left(\frac{1}{x}\right) - k(2x)\log_e x$$

$$= -kx - 2kx \log_e x \qquad \text{... (i)}$$

For maximum range, $\frac{dR}{dx} = 0$

$\Rightarrow \quad 0 = -kx - 2kx \log_e x \quad \Rightarrow \quad 0 = -1 - 2 \log_e x$

$\Rightarrow \quad \log_e x = -\frac{1}{2} \quad \Rightarrow \quad x = e^{-1/2} = \frac{1}{\sqrt{e}}$

Differentiating Eq. (i) w.r.t. "x", we get

$$\frac{d^2R}{dx^2} = -k - 2kx\left(\frac{1}{x}\right) - 2k(1)\log_e x$$

$$= -3k - 2k \log_e x \qquad \text{... (ii)}$$

At $x = \frac{1}{\sqrt{e}}$, $\quad \frac{d^2R}{dx^2} = -3k - 2k\left(-\frac{1}{2}\right)$

$$= -3k + k = -2k = -\text{ve}$$

Hence, Range of signaling is maximum at $x = \frac{1}{\sqrt{e}}$.

S56. Petrol burnt per hour $\propto V^3$,

Petrol burnt in t hour $\propto V^3 \cdot t$ ... (i)

Let the petrol burnt in t hour be Q.

$\Rightarrow \quad Q \propto V^3 \cdot t \Rightarrow Q = kV^3 t$ ...(i)

Let the distance travelled in t hours be S.

Actual speed $= V - C$

$$\text{Time} = \frac{\text{Distance}}{\text{Velocity}} \quad \Rightarrow \quad t = \frac{S}{V - C}$$

Putting the value of t in Eq. (i), we get

$$\Rightarrow \quad Q = kV^3 \frac{S}{V - C} \qquad \text{... (ii)}$$

Differentiating Eq. (ii) w.r.t. "V", we get

$$\frac{dQ}{dV} = kS\frac{(V-C)\,3V^2 - V^3 \cdot 1}{(V-C)^2} \qquad \text{... (iii)}$$

For Q to be minimum, $\frac{dQ}{dV} = 0$.

$$\Rightarrow \quad kS\frac{(V-C)\,3V^2 - V^3}{(V-C)^2} = 0$$

$$\Rightarrow \quad 3(V-C)\,V^2 - V^3 = 0 \qquad \Rightarrow \quad 3(V-C) - V = 0$$

$$\Rightarrow \quad 3V - V = 3C \qquad \Rightarrow \quad 2V = 3C$$

$$\Rightarrow \quad V = \frac{3C}{2}$$

Differentiating Eq. (iii) w.r.t. "V", we get

Since $\frac{d^2Q}{dV^2}$ at $V = \frac{3C}{2}$ is + ve.

Hence, the most economical speed going against a current of C km/h is $\frac{3}{2}C$ km/h.