Date: 18/10/2021

Q1. For what value of k, the following function is continuous at $x = 0$?

$$f(x) = \begin{cases} \dfrac{1-\cos 4x}{8x^2} & \text{if } x \neq 0 \\ k & \text{if } x = 0 \end{cases}$$

Q2. Determine the value of constant k for which the function

$$f(x) = \begin{cases} \dfrac{x^2-9}{x-3}, & x \neq 3 \\ k, & x = 3 \end{cases}$$ is continuous at $x = 3$.

Q3. Discuss the continuity of the function

$$f(x) = \begin{cases} \dfrac{x^2-x-6}{x^2-2x-3} & \text{if } x \neq 3 \\ \dfrac{5}{8} & \text{if } x = 3 \end{cases}$$ at the point $x = 3$.

Q4. Find the values of a for which function

$$f(x) = \begin{cases} \dfrac{\sin^2 ax}{x^2}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$ is continuous at $x = 0$.

Q5. Determine the value of k for which function

$$f(x) = \begin{cases} \dfrac{\sin 2x}{5x}, & x \neq 0 \\ k, & x = 0 \end{cases}$$ is continuous at $x = 0$.

Q6. Find all points of discontinuity of f, where f is defined by:

$$f(x) = \begin{cases} 2x+3 & \text{if } x \leq 2 \\ 2x-3 & \text{if } x > 2 \end{cases}$$

Q7. Discuss the continuity of the function:

$$f(x) = \begin{cases} 5x-1 & \text{if } x < 0 \\ 5x+1 & \text{if } x \geq 0 \end{cases}$$ at x = 0.

Q8. Discuss the continuity of this function at the point $x = 0, 1$

$$f(x) = \begin{cases} -x & \text{if } x \leq 0 \\ x & \text{if } 0 < x \leq 1 \\ 2-x & \text{if } 1 < x \leq 2 \end{cases}$$

Q9. Discuss the continuity of the function:

$$f(x) = \begin{cases} \frac{|x|}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad \text{at } x = 0.$$

Q10. Show that the function $f(x) = 5x - |x|$ is continuous at $x = 0$.

Q11. Discuss the continuity of the function:

$$f(x) = \begin{cases} x + 5 & \text{if } x \leq 1 \\ x - 5 & \text{if } x > 1 \end{cases} \quad \text{at } x = 1.$$

Q12. Discuss the continuity of function:

$$f(x) = \begin{cases} x^3 - 3 & \text{if } x \leq 2 \\ x^2 + 1 & \text{if } x > 2 \end{cases} \quad \text{at } x = 2.$$

Q13. Discuss the continuity of the function:

$$f(x) = \begin{cases} x + 1 & \text{if } x \geq 1 \\ x^2 + 1 & \text{if } x < 1 \end{cases} \quad \text{at } x = 1.$$

Q14. Discuss the continuity of the function:

$$f(x) = \begin{cases} x^{10} - 1 & \text{if } x \leq 1 \\ x^2 & \text{if } x > 1 \end{cases} \quad \text{at } x = 1.$$

Q15. Discuss the continuity of the function:

$$f(x) = \begin{cases} \frac{x}{|x|} & \text{if } x < 0 \\ -1 & \text{if } x \geq 0 \end{cases} \quad \text{at } x = 0.$$

Q16. Discuss the continuity of the function:

$$f(x) = \begin{cases} 2x & \text{if } x < 0 \\ 0 & \text{if } 0 \leq x \leq 1 \\ 4x & \text{if } x > 1 \end{cases}$$

Q17. Find all points of discontinuity of f where

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x < 0 \\ x + 1 & \text{if } x \geq 0 \end{cases}$$

Q18. Discuss the continuity of the function:

$$f(x) = \begin{cases} -2 & \text{if } x \leq -1 \\ 2x & \text{if } -1 < x \leq 1 \\ 2 & \text{if } x > 1 \end{cases}$$

Q19. Show that $f(x) = \begin{cases} 5x - 4, & \text{when } 0 < x \leq 1 \\ 4x^3 - 3x, & \text{when } 1 < x < 2 \end{cases}$ is continuous at $x = 1$.

Q20. Find all points of discontinuity of f, where f is defined as follows:

$$f(x) = \begin{cases} |x| + 3, & x \le -3 \\ -2x, & -3 < x < 3 \\ 6x + 2, & x \ge 3 \end{cases}$$

Q21. Show that the function $f(x)$ defined by

$$f(x) = \begin{cases} \dfrac{\sin x}{x} + \cos x, & x > 0 \\ 2, & x = 0 \\ \dfrac{4(1 - \sqrt{1 - x})}{x}, & x < 0 \end{cases}$$ is continuous at $x = 0$.

Q22. Discuss the continuity of the function:

$$f(x) = \begin{cases} x^2 \sin\dfrac{1}{x} & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$$ at x = 0.

Q23. If the following function $f(x)$ is continuous at $x = 0$, find the value of k:

$$f(x) = \begin{cases} \dfrac{1 - \cos 2x}{2x^2}, & x \ne 0 \\ k, & x = 0 \end{cases}$$

Q24. Discuss the continuity of the function:

$$f(x) = \begin{cases} \sin x - \cos x & \text{if } x \ne 0 \\ -1 & \text{if } x = 0 \end{cases}$$ at x = 0.

Q25. If $f(x) = \begin{cases} kx^2 & \text{if } x \le 2 \\ 3 & \text{if } x > 2 \end{cases}$ is continuous at $x = 2$. Then find the value of k.

Q26. If $f(x) = \begin{cases} kx + 1 & \text{if } x \le 5 \\ 3x - 5 & \text{if } x > 5 \end{cases}$ is continuous at $x = 5$. Then find the value of k.

Q27. Find the value of k so that the function f defined by

$$f(x) = \begin{cases} kx + 1, \text{ if } x \le \pi \\ \cos x, \text{ if } x > \pi \end{cases}$$ is continuous at $x = \pi$.

Q28. For what values of λ, is the function

$$f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if } x \le 0 \\ 4x + 1, & \text{if } x > 0 \end{cases}$$ continuous at $x = 0$?

Q29. If the function

$$f(x) = \begin{cases} 6ax + 3b & \text{if } x > 1 \\ ax - 2b & \text{if } x < 1 \\ 15 & \text{if } x = 1 \end{cases}$$ is continuous at $x = 1$, find the value of a and b.

Q30. If the function " f " defined by

$$f(x) = \begin{cases} 2x-1, & x<2 \\ a, & x=2 \\ x+1, & x>2 \end{cases}$$

is continuous at $x = 2$ find the value of a. Also discuss the continuity of $f(x)$ at $x = 3$.

Q31. Find the value of k, for which

$$f(x) = \begin{cases} \dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \text{if } -1 \le x < 0 \\ \dfrac{2x+1}{x-1}, & \text{if } 0 \le x < 1 \end{cases}$$

is continuous at x = 0.

Q32. Find the value of k so that the function f defined by

$$f(x) = \begin{cases} \dfrac{k\cos x}{\pi - 2x}, & \text{if } x \ne \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases}$$

is continuous at $x = \pi$.

Q33. For what value of k is the following function continuous at $x = 2$?

$$f(x) = \begin{cases} 2x+1; & x<2 \\ k; & x=2 \\ 3x-1; & x>2 \end{cases}$$

Q34. Discuss the continuity of function:

$$f(x) = \begin{cases} \dfrac{\cos x}{\frac{\pi}{2} - x}, & x \ne \dfrac{\pi}{2} \\ 1, & x = \dfrac{\pi}{2} \end{cases}$$

at $x = \dfrac{\pi}{2}$.

Q35. Discuss the continuity of function:

$$f(x) = \begin{cases} \dfrac{\sin x}{x} + \cos x, & x \ne 0 \\ 2, & x = 0 \end{cases}$$

at x = 0.

Q36. Discuss the continuity of function:

$$f(x) = \begin{cases} x\sin\dfrac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

at x = 0.

Q37. Discuss the continuity of function:

$$f(x) = \begin{cases} \dfrac{\sin 3x}{x}, & x \ne 0 \\ 1, & x = 0 \end{cases}$$

at x = 0.

Q38. Discuss the continuity of function:

$$f(x) = \begin{cases} 2x-1, & x<0 \\ 2x+1, & x \ge 0 \end{cases}$$

at x = 0.

Q39. Discuss the continuity of function:

$$f(x) = \begin{cases} 3x-2 & , \quad x \le 0 \\ x+1 & , \quad x > 0 \end{cases} \quad \text{at } x = 0.$$

Q40. Discuss the continuity of function:

$$f(x) = \begin{cases} 1+x^2 & , \quad x \le 1 \\ 2-x^2 & , \quad x > 1 \end{cases} \quad \text{at } x = 1.$$

Q41. Discuss the continuity of function:

$$f(x) = \begin{cases} 2-x & , \quad x < 2 \\ 2+x & , \quad x \ge 2 \end{cases} \quad \text{at } x = 2.$$

Q42. Discuss the continuity of function:

$$f(x) = \begin{cases} \dfrac{e^{1/x}-1}{e^{1/x}+1} & , \quad x \ne 0 \\ 0 & , \quad x = 0 \end{cases} \quad \text{at } x = 0.$$

Q43. Discuss the continuity of function:

$$f(x) = \begin{cases} \dfrac{x-|x|}{2} & , \quad x \ne 0 \\ 2 & , \quad x = 0 \end{cases} \quad \text{at } x = 0.$$

Q44. Discuss the continuity of function:

$$f(x) = \begin{cases} \dfrac{|x-a|}{x-a} & , \quad x \ne a \\ 1 & , \quad x = a \end{cases} \quad \text{at } x = a.$$

Q45. Discuss the continuity of function:

$$f(x) = \begin{cases} \dfrac{1-\cos x}{x^2} & , \quad x \ne 0 \\ 1 & , \quad x = 0 \end{cases} \quad \text{at } x = 0.$$

Q46. Discuss the continuity of function:

$$f(x) = \begin{cases} \dfrac{2|x|+x^2}{x} & , \quad x \ne 0 \\ 0 & , \quad x = 0 \end{cases} \quad \text{at } x = 0.$$

Q47. Discuss the continuity of function:

$$f(x) = \begin{cases} \dfrac{1}{2}-x & , \quad x < \dfrac{1}{2} \\ 1 & , \quad x = \dfrac{1}{2} \\ \dfrac{3}{2}-x & , \quad x > \dfrac{1}{2} \end{cases} \quad \text{at} \quad x = \dfrac{1}{2}$$

Q48. Discuss the continuity of function:

$$f(x) = \begin{cases} x & , 0 \le x < \frac{1}{2} \\ 1/2 & , x = \frac{1}{2} \\ 1 - x & , \frac{1}{2} < x \le 1 \end{cases} \quad \text{at} \quad x = \frac{1}{2}$$

Q49. Discuss continuity of function:

$$f(x) = \begin{cases} x & , x > 0 \\ 1 & , x = 0 \\ -x & , x < 0 \end{cases} \quad \text{at} \quad x = 0$$

Q50. Discuss the continuity of function:

$$f(x) = \begin{cases} \frac{|x^2 - 1|}{x - 1} & , x \ne 1 \\ 2 & , x = 1 \end{cases} \quad \text{at} \quad x = 1$$

Q51. Discuss the continuity of function:

$$f(x) = \begin{cases} e^{1/x} & , x \ne 0 \\ 1 & , x = 0 \end{cases} \quad \text{at } x = 0$$

Q52. Find the value of constant λ so that function

$$f(x) = \begin{cases} \frac{x^2 - 2x - 3}{x + 1} & , x \ne -1 \\ \lambda & , x = -1 \end{cases}$$ is continuous at $x = -1$.

Q53. Determine the value of constant *k* so that the function

$$f(x) = \begin{cases} \frac{x^2 - 3x + 2}{x - 1} & , x \ne 1 \\ k & , x = 1 \end{cases}$$ is continuous at $x = 1$.

Q54. Discuss the continuity of function:

$$f(x) = \begin{cases} \frac{\sin 3x}{\tan 2x} & , x < 0 \\ \frac{3}{2} & , x = 0 \\ \frac{\log(1 + 3x)}{e^{2x} - 1} & , x > 0 \end{cases} \quad \text{at} \quad x = 0$$

Q55.
If function $f(x) = \begin{cases} \frac{1 - \cos kx}{x \sin x} & , x \ne 0 \\ \frac{1}{2} & , x = 0 \end{cases}$ **is continuous at** $x = 0$**, find** k**.**

Q56.
If function $f(x) = \begin{cases} (x - 1)\tan\frac{\pi x}{2} & , x \ne 1 \\ k & , x = 1 \end{cases}$ **is continuous at** $x = 1$**, find** k**.**

Q57. Find the value of k for which function

$$f(x) = \begin{cases} \dfrac{1-\cos 2kx}{x^2}, & x \neq 0 \\ 8, & x = 0 \end{cases}$$ is continuous at $x = 0$.

Q58. If $f(x) = \dfrac{2x+3\sin x}{3x+2\sin x}$ is continuous at $x = 0$, find $f(0)$.

Q59. If $f(x) = \dfrac{1-\cos 7(x-\pi)}{5(x-\pi)^2}$ is continuous at $x = \pi$, find $f(\pi)$.

Q60.
If $f(x) = \begin{cases} \dfrac{\cos^2 x - \sin^2 x - 1}{\sqrt{x^2+1}-1}, & x \neq 0 \\ k, & x = 0 \end{cases}$, is continuous at x = 0, find k.

Q61. Find the value of constant k so that function

$$f(x) = \begin{cases} \dfrac{\log(1+ax)-\log(1-bx)}{x}, & x \neq 0 \\ k, & x = 0 \end{cases}$$ is continuous at $x = 0$.

Q62. If function

$$f(x) = \begin{cases} 3ax+b, & x > 1 \\ 11, & x = 1 \\ 5ax-b, & x < 1 \end{cases}$$ is continuous at $x = 1$, find a, b.

Q63 find the value of $f\left(\dfrac{\pi}{4}\right)$ so that $f(x)$ is continuous in $\left[0, \dfrac{\pi}{2}\right]$. Where

$$f(x) = \frac{\tan\left(\dfrac{\pi}{4}-x\right)}{\cot 2x} \text{ for } x \neq \frac{\pi}{4},$$

Q64. If below function $f(x)$ is continuous at $x = 0$, find a.

$$f(x) = \begin{cases} \dfrac{1-\cos 4x}{x^2}, & x < 0 \\ a, & x = 0 \\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}, & x > 0 \end{cases}$$

Q65. Find the value of a so that function

$$f(x) = \begin{cases} ax+5, & x \leq 2 \\ x-1, & x > 2 \end{cases}$$ is continuous at $x = 2$, find a.

Q66. Find the relationship between a and b so that

$$f(x) = \begin{cases} ax+1, & x \leq 3 \\ bx+3, & x > 3 \end{cases}$$ is continuous at $x = 3$.

Q67. If below function $f(x)$ is continuous at $x = 4$, find a, b.

$$f(x) = \begin{cases} \dfrac{x-4}{|x-4|}+a, & x<4 \\ a+b, & x=4 \\ \dfrac{x-4}{|x-4|}+b, & x>4 \end{cases}$$

Q68. Find the value of a and b so that function

$$f(x) = \begin{cases} 1, & x\le 3 \\ ax+b, & 3<x<5 \\ 7, & x\ge 5 \end{cases}$$ is continuous at $x = 3$ and $x = 5$.

Q69. Find the value of a for which function

$$f(x) = \begin{cases} a\sin\dfrac{\pi}{2}(x+1), & x\le 0 \\ \dfrac{\tan x-\sin x}{x^3}, & x>0 \end{cases}$$ is continuous at $x = 0$.

Q70. If below function $f(x)$ is continuous at $x = 0$, find a, b.

$$f(x) = \begin{cases} \dfrac{1-\sin^3 x}{3\cos^2 x}, & x<\dfrac{\pi}{2} \\ a, & x=\dfrac{\pi}{2} \\ \dfrac{b(1-\sin x)}{(\pi-2x)^2}, & x>\dfrac{\pi}{2} \end{cases}$$

Q71. Determine the value of a, b, c for which function

$$f(x) = \begin{cases} \dfrac{\sin(a+1)x+\sin x}{x}, & x<0 \\ c, & x=0 \\ \dfrac{\sqrt{x+bx^2}-\sqrt{x}}{bx^{3/2}}, & x>0 \end{cases}$$ is continuous at $x = 0$

Q72. If below function $f(x)$ is continuous at $x = 0$, find k.

$$f(x) = \begin{cases} \dfrac{1-\cos 2x}{2x^2}, & x<0 \\ k, & x=0 \\ \dfrac{x}{|x|}, & x>0 \end{cases}$$

Q73. If below function $f(x)$ is continuous at $x = \dfrac{\pi}{2}$, find k.

$$f(x) = \begin{cases} \dfrac{k\cos x}{\pi-2x}, & x<\dfrac{\pi}{2} \\ 3, & x=\dfrac{\pi}{2} \\ \dfrac{2\tan 2x}{2x-\pi}, & x>\dfrac{\pi}{2} \end{cases}$$

Q74. Find the values of *a* and *b* such that the function defined as follows is continuous:

$$f(x) = \begin{cases} x+2; & x \le 2 \\ ax+b; & 2 < x < 5 \\ 3x-2; & x \ge 5 \end{cases}$$

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S1. Here,

$$f(x) = \begin{cases} \dfrac{1-\cos 4x}{8x^2} & \text{if } x \neq 0 \\ k & \text{if } x = 0 \end{cases}$$

$\therefore$ $$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{1-\cos 4x}{8x^2} = \lim_{x\to 0}\frac{2\sin^2 2x}{8x^2} = \lim_{x\to 0}\left(\frac{\sin 2x}{2x}\right)^2 = (1)^2 = 1$$

Here $f(0) = k$

Since $f(x)$ is continuous at $x = 0$

$$\lim_{x\to 0} f(x) = f(0) \Rightarrow 1 = k \Rightarrow k = 1.$$

S2. As $f(x)$ is continuous at $x = 3$

$\therefore$ $$f(3) = \lim_{x\to 3} f(x)$$

$$= \lim_{x\to 3}\frac{x^2-9}{x-3}$$

$$= \lim_{x\to 3}\frac{(x-3)(x+3)}{x-3}$$

$$= \lim_{x\to 3}(x+3) = 6$$

$\therefore$ $f(3) = k = 6$ hence $k = 6$.

S3. Here,

$$f(x) = \begin{cases} \dfrac{x^2-x-6}{x^2-2x-3} & \text{if } x \neq 3 \\ \dfrac{5}{8} & \text{if } x = 3 \end{cases}$$

$\therefore$ $$\lim_{x\to 3} f(x) = \lim_{x\to 3}\frac{x^2-x-6}{x^2-2x-3}\left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 3}\frac{(x-3)(x+2)}{(x-3)(x+1)} = \lim_{x\to 3}\frac{x+2}{x+1} = \frac{3+2}{3+1} = \frac{5}{4}$$

But $f(3) = \dfrac{5}{8}$

$\therefore \quad \lim_{x \to 3} f(x) \neq f(3)$

Thus $f(x)$ is not continuous at $x = 3$

S4. As $f(x)$ is continuous at $x = 0$

$$\therefore \quad f(0) = \lim_{x \to 0} f(x)$$

$$= \lim_{x \to 0} \frac{\sin^2 ax}{x^2}$$

$$= \lim_{x \to 0} a^2 \left(\frac{\sin ax}{ax}\right)^2$$

$$= a^2$$

But $\quad f(0) = 1$

$\Rightarrow \quad a^2 = 1$

$\Rightarrow \quad a = \pm 1$

S5. As function $f(x)$ is continuous at $x = 0$

$$\therefore \quad f(0) = \lim_{x \to 0} f(x)$$

$$= \lim_{x \to 0} \frac{\sin 2x}{5x}$$

$$= \lim_{x \to 0} \frac{2 \sin 2x}{2x} \cdot \frac{1}{5}$$

$$= \frac{2}{5}$$

$\therefore \quad f(0) = k = \frac{2}{5} \qquad$ hence $k = \frac{2}{5}$.

S6. Here,

$$f(x) = \begin{cases} 2x + 3 & \text{if } x \le 2 \\ 2x - 3 & \text{if } x > 2 \end{cases}$$

$$\text{L.H.L.} = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x + 3)$$

Put $\quad x = 2 - h$ as $x \to 2$, $h \to 0$

$$\therefore \quad \lim_{h \to 0} [2(2 - h) + 3] = \lim_{h \to 0} (7 - 2h) = 7 - 2 \times 0 = 7$$

$$\text{R.H.L.} = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (2x - 3)$$

Put $x = 2 + h$ as $x \to 2,\ h \to 0$

$\therefore$ $\lim_{h\to 0}[2(2+h)-3] = \lim_{h\to 0}(1+2h) = 1 + 2\times 0 = 1$

$\therefore$ L.H.L. $\neq$ R.H.L.

So $f(x)$ is discontinuous at $x = 2$.

S7. Here,

$$f(x) = \begin{cases} 5x - 1 \text{ if } x < 0 \\ 5x + 1 \text{ if } x \geq 0 \end{cases}$$

At $x < 0$, $f(x) = 5x - 1$, which is a polynomial function.

So $f(x)$ is continuous at all $x < 0$.

At $x > 0$, $f(x) = 5x + 1$, which is a polynomial function.

So $f(x)$ is continuous at all $x > 0$.

At $x = 0$

$f(0) = 5 \times 0 + 1 = 0 + 1 = 1$

L.H.L. $= \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(5x - 1)$

Put $x = 0 - h$ as $x \to 0,\ h \to 0$

$\therefore$ $\lim_{h\to 0}[5(0-h)-1] = \lim_{h\to 0}(-5h-1) = -5\times 0 - 1 = -1$

R.H.L. $= \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(5x + 1)$

Put $x = 0 + h$ as $x \to 0,\ h \to 0$

$\therefore$ $\lim_{h\to 0}[5(0+h)+1] = \lim_{h\to 0}(5h+1) = 5\times 0 + 1 = 1$

$\therefore$ L.H.L. $\neq$ R.H.L.

So $f(x)$ is discontinuous at $x = 0$

Thus $f(x)$ is continuous on $R - \{0\}$.

S8. Here,

$$f(x) = \begin{cases} -x & \text{if } x \leq 0 \\ x & \text{if } 0 < x \leq 1 \\ 2 - x & \text{if } 1 < x \leq 2 \end{cases}$$

At $x = 0$

$$\text{L.H.L.} = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-x)$$

Put $x = 0 - h$ as $x \to 0$, $h \to 0$

$\therefore$ $$\lim_{h \to 0} -(0 - h) = \lim_{h \to 0} (h) = 0$$

$$\text{R.H.L.} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x$$

Put $x = 0 + h$ as $x \to 0$, $h \to 0$

$\therefore$ $$\lim_{h \to 0} (0 + h) = \lim_{h \to 0} h = 0$$

Also $f(0) = 0$

$\therefore$ L.H.L. = R.H.L. = $f(0)$

Thus $f(x)$ is continuous at $x = 0$

At $x = 1$

$$\text{L.H.L.} = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x$$

Put $x = 1 - h$ as $x \to 1$, $h \to 0$

$\therefore$ $$\lim_{h \to 0} (1 - h) = 1 - 0 = 1$$

$$\text{R.H.L.} = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2 - x)$$

Put $x = 1 + h$ as $x \to 1$, $h \to 0$

$\therefore$ $$\lim_{h \to 0} [2 - (1 + h)] = \lim_{h \to 0} (1 - h) = 1 - 0 = 1$$

Also $f(1) = 1$

$\therefore$ L.H.L. = R.H.L. = $f(1)$

Thus $f(x)$ is continuous at $x = 1$.

S9. Here,

$$f(x) = \begin{cases} \dfrac{|x|}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$$\text{L.H.L.} = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{|x|}{x}$$

Put $x = 0 - h$ as $x \to 0$, $h \to 0$

$\therefore \quad \lim_{h\to 0}\frac{|0-h|}{0-h} = \lim_{h\to 0}\frac{|-h|}{-h} = \frac{h}{-h} = -1$

$$\text{R.H.L.} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\frac{|x|}{x}$$

Put $\quad x = 0 + h$ as $x \to 0$, $h \to 0$

$\therefore \quad \lim_{h\to 0}\frac{|0+h|}{0+h} = \lim_{h\to 0}\frac{h}{h} = 1$

$\therefore \quad \text{L.H.L.} \neq \text{R.H.L.}$

So $f(x)$ is discontinuous at $x = 0$.

S10. Here $f(x) = 5x - |x|$.

$\therefore \quad f(x) = \begin{cases} 5x - x & \text{if } x \geq 0 \\ 5x - (-x) & \text{if } x < 0 \end{cases}$

$\therefore \quad f(x) = \begin{cases} 4x & \text{if } x \geq 0 \\ 6x & \text{if } x < 0 \end{cases}$

$\text{L.H.L.} = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 6x \qquad [\because f(x) = 6x]$

Put $\quad x = 0 - h$ as $x \to 0$, $h \to 0$

$\therefore \quad \lim_{h\to 0} 6(0-h) = \lim_{h\to 0}(-6h) = -6 \times 0 = 0$

$\text{R.H.L.} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 4x \qquad [\because f(x) = 4x]$

Put $\quad x = 0 + h$ as $x \to 0$, $h \to 0$

$$= \lim_{h\to 0} 4(0+h)$$

$$= \lim_{h\to 0} 4h = 4 \times 0 = 0.$$

and $\quad f(0) = 4 \times 0 = 0$

$\therefore \quad \text{L.H.L.} = \text{R.H.L.} = f(0)$

Thus $f(x)$ is continuous at $x = 0$.

S11. Here,

$$f(x) = \begin{cases} x + 5 & \text{if } x \leq 1 \\ x - 5 & \text{if } x > 1 \end{cases}$$

$$\text{L.H.L.} = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x+5)$$

Put $\quad x = 1 - h$ as $x \to 1$, $h \to 0$

$\therefore$ $$\lim_{h\to 0}[1-h+5] = \lim_{h\to 0}(6-h) = 6-0 = 6$$

$$\text{R.H.L.} = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(x-5)$$

Put $x = 1 + h$ as $x \to 1$, $h \to 0$

$\therefore$ $$\lim_{h\to 0}[(1+h-5) = \lim_{h\to 0}(h-4) = 0-4 = -4$$

$\therefore$ L.H.L. $\neq$ R.H.L.

Thus $f(x)$ is discontinuous at $x = 1$.

S12. Here,

$$f(x) = \begin{cases} x^3 - 3 & \text{if } x \le 2 \\ x^2 + 1 & \text{if } x > 2 \end{cases}$$

$$\text{L.H.L.} = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(x^3 - 3)$$

Put $x = 2 - h$ as $x \to 2$, $h \to 0$

$\therefore$ $$\lim_{h\to 0}[(2-h)^3 - 3] = \lim_{h\to 0}[8 - h^3 - 12h + 6h^2 - 3]$$

$$= \lim_{h\to 0}(5 - h^3 - 12h + 6h^2) = 5$$

$$\text{R.H.L.} = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(x^2 + 1)$$

Put $x = 2 + h$ as $x \to 2$, $h \to 0$

$\therefore$ $$\lim_{h\to 0}[(2+h)^2 + 1] = \lim_{h\to 0}(4 + h^2 + 4h + 1)$$

$$= \lim_{h\to 0}(5 + h^2 + 4h) = 5$$

$$f(2) = (2)^3 - 3 = 8 - 3 = 5$$

$\therefore$ L.H.L. = R.H.L. = $f(2)$

Thus $f(x)$ is continuous at $x = 2$. There is no point of discontinuity for this function $f(x)$.

S13. Here,

$$f(x) = \begin{cases} x + 1 & \text{if } x \ge 1 \\ x^2 + 1 & \text{if } x < 1 \end{cases}$$

$$\text{L.H.L.} = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x^2 + 1)$$

Put $x = 1 - h$ as $x \to 1$, $h \to 0$

$\therefore$ $$\lim_{h\to 0}[(1-h)^2 + 1] = \lim_{h\to 0}[1 + h^2 - 2h + 1]$$

$$= \lim_{h\to 0} [2 + h^2 - 2h] = 2 + 0 - 0 = 2$$

$$\text{R.H.L.} = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x+1)$$

Put $x = 1 + h$ as $x \to 1$, $h \to 0$

$\therefore$ $$\lim_{h\to 0}(1+h+1) = \lim_{h\to 0}(2+h) = 2+0 = 2$$

$$f(1) = 1 + 1 = 2$$

$\therefore$ L.H.L. = R.H.L. = $f(1)$

Thus $f(x)$ is continuous at $x = 1$. There is no point of discontinuity for this function $f(x)$.

S14. Here,

$$f(x) = \begin{cases} x^{10} - 1 & \text{if } x \le 1 \\ x^2 & \text{if } x > 1 \end{cases}$$

$$\text{L.H.L.} = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^{10} - 1)$$

Put $x = 1 - h$ as $x \to 1$, $h \to 0$

$\therefore$ $$\lim_{h\to 0}[(1-h)^{10} - 1] = (1 - 0)^{10} - 1 = 1 - 1 = 0$$

$$\text{R.H.L.} = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x^2)$$

Put $x = 1 + h$ as $x \to 1^+$, $h \to 0$

$\therefore$ $$\lim_{h\to 0}(1+h)^2 = \lim_{h\to 0} 1 + h^2 + 2h = 1$$

$\therefore$ L.H.L. $\ne$ R.H.L.

Thus $f(x)$ is not continuous at $x = 1$.

S15. Here,

$$f(x) = \begin{cases} \dfrac{x}{|x|} & \text{if } x < 0 \\ -1 & \text{if } x \ge 0 \end{cases}$$

$$\text{L.H.L.} = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{x}{|x|}$$

Put $x = 0 - h$ as $x \to 0$, $h \to 0$

$\therefore$ $$\lim_{h\to 0}\frac{0-h}{|0-h|} = \lim_{h\to 0}\frac{-h}{|-h|} = \frac{-h}{h} = -1$$

$$\text{R.H.L.} = \lim_{x\to 0^+} f(x) = -1$$

$$f(0) = -1$$

$\therefore$ $$\text{L.H.L.} = \text{R.H.L.} = f(0)$$

Thus $f(x)$ is continuous at $x = 0$. There is no point of discontinuity for this function $f(x)$.

S16. Here,

$$f(x) = \begin{cases} 2x & \text{if } x < 0 \\ 0 & \text{if } 0 \le x \le 1 \\ 4x & \text{if } x > 1 \end{cases}$$

At $$x = 0$$

$$\text{L.H.L.} = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(2x)$$

Put $$x = 0 - h \text{ as } x \to 0,\ h \to 0$$

$\therefore$ $$\lim_{h\to 0}[2(0-h)] = \lim_{h\to 0}(-2h) = -2\times 0 = 0$$

$$\text{R.H.L.} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(0) = 0$$

and $$f(0) = 0$$

$\therefore$ $$\text{L.H.L.} = \text{R.H.L.} = f(0)$$

Thus $f(x)$ is continuous at $x = 0$

At $$x = 1$$

$$\text{L.H.L.} = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(0) = 0$$

$$\text{R.H.L.} = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(4x)$$

Put $$x = 1 + h \text{ as } x \to 1,\ h \to 0$$

$\therefore$ $$\lim_{h\to 0}[4(1+h)] = \lim_{h\to 0}(4+4h) = 4+4\times 0 = 4+0 = 4$$

$\therefore$ $$\text{L.H.L.} \ne \text{R.H.L.}$$

Thus $f(x)$ is discontinuous at $x = 1$.

S17. Here,

$$f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x < 0 \\ x+1 & \text{if } x \ge 0 \end{cases}$$

$$\text{L.H.L.} = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{\sin x}{x}$$

Put $x = 0 - h$ as $x \to 0$, $h \to 0$

$$\therefore \quad \lim_{h \to 0} \frac{\sin(0-h)}{(0-h)} = \lim_{h \to 0} \frac{-\sin h}{-h} = \lim_{h \to 0} \frac{\sin h}{h} = 1$$

$$\text{R.H.L.} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x+1)$$

Put $x = 0 + h$ as $x \to 0$, $h \to 0$

$$\therefore \quad \lim_{h \to 0} (0 + h + 1) = 0 + 1 = 1$$

$$f(0) = 0 + 1 = 1$$

$$\therefore \quad \text{L.H.L.} = \text{R.H.L.} = f(0)$$

Thus $f(x)$ is continuous at $x = 0$.

Thus, there is no point of discontinuity for this function $f(x)$.

S18. Here,

$$f(x) = \begin{cases} -2 & \text{if } x \le -1 \\ 2x & \text{if } -1 < x \le 1 \\ 2 & \text{if } x > 1 \end{cases}$$

At $x = -1$

$$\text{L.H.L.} = \lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (-2) = -2$$

$$\text{R.H.L.} = \lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (2x)$$

Put $x = -1 + h$ as $x \to -1$, $h \to 0$

$$\therefore \quad \lim_{h \to 0} 2(-1+h) = \lim_{h \to 0} (-2 + 2h) = -2 + 0 = -2$$

$$f(-1) = -2$$

$$\therefore \quad \text{L.H.L.} = \text{R.H.L.} = f(-1)$$

Thus $f(x)$ is continuous at $x = -1$

At $x = 1$

$$\text{L.H.L.} = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(2x)$$

Put $x = 1 - h$ as $x \to 1$, $h \to 0$

$$\lim_{h\to 0}[2(1-h)] = \lim_{h\to 0}[(2-2h)] = 2 - 2\times 0 = 2 - 0 = 2$$

$$\text{R.H.L.} = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(2) = 2$$

$$f(1) = 2 \times 1 = 2$$

$\therefore$ L.H.L. = R.H.L. = $f(1)$

Thus $f(x)$ is continuous at $x = 1$.

Thus $f(x)$ is continuous at all points.

S19. Given function is

$$f(x) = \begin{cases} 5x - 4, & \text{when } 0 < x \le 1 \\ 4x^3 - 3x, & \text{when } 1 < x < 2 \end{cases}$$

To show that $f(x)$ is continuous at $x = 1$, we need to prove

$$\text{L.H.L.}_{x=1} = \text{R.H.L.}_{x=1} = f(1) \quad \text{... (i)}$$

Now, $\text{L.H.L.} = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(5x-4)$

Put $x = 1 - h$, when $x \to 1$, $h \to 0$

$$= \lim_{h\to 0} 5(1-h) - 4$$

$$= \lim_{h\to 0} 5 - 5h - 4$$

$$= \lim_{h\to 0} 1 - 5h$$

L.H.L. = 1

Now, $\text{R.H.L.} = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(4x^3 - 3x)$

Put $x = 1 + h$

When $x \to 1$, $h \to 0$

$$= \lim_{h\to 0} 4(1+h)^3 - 3(1+h)$$

$$= 4(1)^3 - 3(1)$$

R.H.L. = 4 – 3 = 1

Also, from function $f(1)$= value of $f(x)$ at $x = 1$

$$f(1) = 5(1) - 4 \qquad [\text{put } x = 1 \text{ in } f(x) = 5x - 4]$$

$$= 5 - 4 = 1$$

as $\quad \text{L.H.L.}_{x=1} = \text{R.H.L.}_{x=1} = f(1)$

$\Rightarrow \quad f(x)$ is continuous at $x = 1$ $\qquad$ Using eq.(i)

S20. The given function is

$$f(x) = \begin{cases} |x| + 3, & x \le -3 \\ -2x, & -3 < x < 3 \\ 6x + 2, & x \ge 3 \end{cases}$$

First we verify continuity at $x = -3$ and then at $x = 3$.

Continuity at $x = -3$

$$\text{L.H.L.} = \lim_{x \to (-3)^-} f(x) = \lim_{x \to (-3)^-} |x| + 3$$

Put $\quad x = -3 - h$, when $x \to -3$, $h \to 0$

$$= \lim_{h \to 0} |-3 - h| + 3$$

$$= |-3| + 3 \qquad [\text{Put } h = 0]$$

$$= 3 + 3 = 6 \qquad [\because |-x| = x, \forall x \in R]$$

R.H.L. $\quad = \lim_{x \to (-3)^+} f(x) = \lim_{x \to (-3)^+} -2x$

Put $\quad x = -3 + h$, when $x \to -3$, $h \to 0$

$$= \lim_{h \to 0} -2(-3 + h)$$

$$= \lim_{h \to 0} 6 - 2h$$

$$\text{R.H.L.} = 6 \qquad [\text{Put } h = 0]$$

Also, $f(-3)$ = value of $f(x)$ at $x = -3$

$$= |-3| + 3 \qquad [\because |-x| = x, \forall x \in R]$$

$$= 3 + 3 = 6$$

$\because \quad$ L.H.L. = R.H.L. = $f(-3)$

$\therefore$ $f(x)$ is continuous at $x = -3$.

Continuity at $x = 3$

$$\text{L.H.L.} = \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} -2x$$

Put $x = 3 - h$, when $x \to 3$, $h \to 0$

$$= \lim_{h \to 0} -2(3-h) = \lim_{h \to 0} -6 + 2h$$

L.H.L. $= -6$ [put $h = 0$]

Now, R.H.L. $= \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} 6x + 2$

Put $x = 3 + h$, when $x \to 3$, $h \to 0$

$$= \lim_{h \to 0} 6(3+h) + 2$$

$$= \lim_{h \to 0} 18 + 6h + 2$$

R.H.L. $= 20$ [Put $h = 0$]

As, L.H.L. $\neq$ R.H.L.

$\therefore$ $f(x)$ is not continuous at $x = 3$. So, $x = 3$ is the point of discontinuity of $f(x)$.

S21. To show that the given function is continuous at $x = 0$, we show that

L.H.L.$_{x=0}$ = R.H.L.$_{x=0}$ = $f(0)$

Now, given function is

$$f(x) = \begin{cases} \dfrac{\sin x}{x} + \cos x, & x > 0 \\ 2, & x = 0 \\ \dfrac{4(1-\sqrt{1-x})}{x}, & x < 0 \end{cases}$$

$$\text{L.H.L.} = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{4(1-\sqrt{1-x})}{x}$$

Put $x = 0 - h$, when $x \to 0$, $h \to 0$

$$= \lim_{h \to 0} \frac{4[1-\sqrt{1-(0-h)}]}{0-h}$$

$$= \lim_{h \to 0} \frac{4[1-\sqrt{1+h}]}{-h}$$

$$= \lim_{h \to 0} \frac{4[1-\sqrt{1+h})]}{-h} \times \frac{1+\sqrt{1+h}}{1+\sqrt{1+h}}$$

By rationalising with $(1 + \sqrt{1+h})$, we have

$$= \lim_{h \to 0} \frac{4[1-(1+h)]}{-h[1+\sqrt{1+h}]}$$

$$= \lim_{h\to 0} \frac{-h\times 4}{-h[1+\sqrt{1+h}]}$$

$$= \lim_{h\to 0} \frac{4}{[1+\sqrt{1+h}]} \qquad \text{[Put } h = 0]$$

$$\text{L.H.L.} = \frac{4}{1+\sqrt{1}} = \frac{4}{2} = 2$$

$\therefore$ L.H.L. = 2

Now, R.H.L. = $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\sin x}{x} + \cos x$

Put $x = 0 + h$, when $x \to 0$, $h \to 0$

$$= \lim_{h\to 0^+} \frac{\sinh}{h} + \cos h$$

$$= \lim_{h\to 0} \frac{\sin h}{h} + \lim_{h\to 0} \cos h \qquad [\text{put } h = 0]$$

$$= 1 + \cos 0^o \qquad \left[\because \lim_{h\to 0} \frac{\sin h}{h} = 1\right]$$

$$= 1 + 1 \qquad [\because \cos 0 = 1]$$

$$= 2$$

$\therefore$ R.H.L. = 2

Also, given that at $x = 0$, $f(x) = 2$

$\Rightarrow$ $f(0) = 2$

Since, L.H.L.$_{x=0}$ = R.H.L.$_{x=0}$ = $f(0)$ = 2

$\Rightarrow$ $f(x)$ is continuous at $x = 0$.

S22.

Here $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

$$\text{L.H.L.} = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2 \sin\frac{1}{x}$$

Put $x = 0 - h$ as $x \to 0$, $h \to 0$

$$\therefore \lim_{h\to 0}(0-h)^2 \sin\frac{1}{(0-h)} = \lim_{h\to 0} -h^2 \sin\frac{1}{h} = -\lim_{h\to 0} h^2 \sin\frac{1}{h} = 0 \times (\text{finite number between}(-1 \text{ and } 1)) = 0$$

$$\text{R.H.L.} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 \sin\frac{1}{x}$$

Put $x = 0 + h$ as $x \to 0,\ h \to 0$

$\therefore$ $\lim_{h\to 0}(0+h)^2 \sin\frac{1}{0+h} = \lim_{h\to 0} h^2 \sin\frac{1}{h}$

$$= \lim_{h\to 0}(0+h)^2 \sin\frac{1}{0+h} = \lim_{h\to 0} h^2 \sin\frac{1}{h}$$

$$= 0 \times (\text{finite number between } (-1 \text{ and } 1)) = 0$$

and $f(0) = 0$

$\therefore$ L.H.L. = R.H.L. = $f(0)$

Thus $f(x)$ is continuous at $x = 0$.

S23. The given function is

$$f(x) = \begin{cases} \frac{1-\cos 2x}{2x^2}, & x \neq 0 \\ k, & x = 0 \end{cases}$$

As it is given that $f(x)$ is continuous at $x = 0$.

$\therefore$ $\text{L.H.L.}_{x=0} = \text{R.H.L.}_{x=0} = f(0)$ ... (i)

Now, L.H.L. = $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1-\cos 2x}{2x^2}$

Put $x = 0 - h = -h$, when $x \to 0,\ h \to 0$

$$= \lim_{h\to 0} \frac{1-\cos(-2h)}{2h^2}$$

$$= \lim_{h\to 0} \frac{1-\cos 2h}{2h^2} \qquad [\because \cos(-\theta) = \cos\theta]$$

$$= \lim_{h\to 0} \frac{2\sin^2 h}{2h^2} \qquad [\because 1-\cos 2\theta = 2\sin^2\theta]$$

$$= \lim_{h\to 0} \frac{\sin^2 h}{h^2} = \lim_{h\to 0}\left(\frac{\sin h}{h}\right)^2$$

$$= (1)^2$$

$$= 1 \qquad \left[\because \lim_{h\to 0}\frac{\sin h}{h} = 1\right]$$

$\because$ Given that at $x = 0$, $f(x) = k$, *i.e.*, $f(0) = k$

Now, from Eq. (i), we get

L.H.L. = $f(0)$

$\Rightarrow$ $1 = k$ or $k = 1$

S24. Here,

$$f(x) = \begin{cases} \sin x - \cos x & \text{if } x \neq 0 \\ -1 & \text{if } x = 0 \end{cases}$$

$$\text{L.H.L.} = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (\sin x - \cos x)$$

Put $x = 0 - h$ as $x \to 0$, $h \to 0$

$\therefore$ $\lim_{h\to 0}[\sin(0-h) - \cos(0-h)] = \lim_{h\to 0^+}(-\sin h - \cos h) = 0 - 1 = -1$

$$\text{R.H.L.} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (\sin x - \cos x)$$

Put $x = 0 + h$ as $x \to 0$, $h \to 0$

$\therefore$ $\lim_{h\to 0}[\sin(0+h) - \cos(0+h)] = \lim_{h\to 0}(\sin h - \cos h) = 0 - 1 = -1$

and $f(0) = -1$

$\therefore$ L.H.L. = R.H.L. = $f(0)$

Thus $f(x)$ is continuous at $x = 0$.

S25. Here,

$$f(x) = \begin{cases} kx^2 & \text{if } x \leq 2 \\ 3 & \text{if } x > 2 \end{cases}$$

$$\text{L.H.L.} = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} kx^2$$

Put $x = 2 - h$ as $x \to 2$, $h \to 0$

$\therefore$ $\lim_{h\to 0} k(2-h)^2 = \lim_{h\to 0} k(4 + h^2 - 4h) = 4k$

$$\text{R.H.L.} = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (3) = 3$$

$$f(2) = k \times (2)^2 = 4k$$

Since $f(x)$ is continuous at $x = 2$

$\therefore$ L.H.L. = R.H.L. = $f(2)$

$\Rightarrow$ $4k = 3 \quad \Rightarrow \quad k = \dfrac{3}{4}$.

S26. Here,

$$f(x) = \begin{cases} kx + 1 & \text{if } x \leq 5 \\ 3x - 5 & \text{if } x > 5 \end{cases}$$

$$\text{L.H.L.} = \lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} (kx + 1)$$

Put $x = 5 - h$ as $x \to 5,\ h \to 0$

$\therefore$ $\lim_{h \to 0} [k(5-h)+1] = \lim_{h \to 0} [5k - kh + 1] = 5k + 1$

$$\text{R.H.L.} = \lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} (3x - 5)$$

Put $x = 5 + h$ as $x \to 5,\ h \to 0$

$\therefore$ $\lim_{h \to 0} [3(5+h) - 5] = \lim_{h \to 0} (15 + 3h - 5) = \lim_{h \to 0} (10 + 3h) = 10$

and $f(5) = 5k + 1$

Since $f(x)$ is continuous at $x = 5$

$\therefore$ $\text{L.H.L.} = \text{R.H.L.} = f(5)$

$\Rightarrow$ $5k + 1 = 10 \Rightarrow k = \frac{9}{5}$

S27. Given function is,

$$f(x) = \begin{cases} kx + 1, \text{if } x \le \pi \\ \cos x, \text{if } x > \pi \end{cases}$$

Given that $f(x)$ is continuous at $x = \pi$.

$\therefore$ $\text{L.H.L.}_{x=\pi} = \text{R.H.L.}_{x=\pi} = f(\pi)$ ... (i)

Now, $\text{L.H.L.} = \lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^-} kx + 1$

Put $x = \pi - h$, when $x \to \pi$, $h = 0$

$= \lim_{h \to 0} k(\pi - h) + 1$

$= \lim_{h \to 0} k\pi - kh + 1$

$= k\pi + 1$ [Put $h = 0$]

Now, $\text{R.H.L.} = \lim_{x \to \pi^+} f(x) = \lim_{x \to \pi^+} \cos x$

Put $x = \pi + h$, when $x \to \pi$, $h \to 0$

$= \lim_{h \to 0} \cos(\pi + h) = \lim_{h \to 0} -\cos h = -1$

$\text{R.H.L.} = -1$

Now, from Eq. (i), we get

$$\text{L.H.L.} = \text{R.H.L.}$$

$\Rightarrow$ $k\pi + 1 = -1$

$\Rightarrow$ $k\pi = -2$

or $k = \dfrac{-2}{\pi}$.

S28. Given function is

$$f(x) = \begin{cases} \lambda(x^2 - 2x), \text{ if } x \le 0 \\ 4x + 1, \quad \text{if } x > 0 \end{cases}$$

Given that $f(x)$ is continuous at $x = 0$.

$\therefore$ $\text{L.H.L.}_{x=0} = \text{R.H.L.}_{x=0} = f(0)$ ... (i)

Now, $\text{L.H.L.} = \lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} \lambda(x^2 - 2x)$

Put $x = 0 - h = -h$, when $x \to 0$, $h \to 0$

$= \lim\limits_{h \to 0} \lambda(h^2 + 2h)$

$= \lambda(0)$ [Put $h = 0$]

$= 0$

Now, $\text{R.H.L.} = \lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} 4x + 1$

Put $x = 0 + h = h$, when $x \to 0$, $h \to 0$

$= \lim\limits_{h \to 0} 4h + 1$

$\text{R.H.L.} = 1$ [Put $h = 0$]

$\therefore$ We get L.H.L. = 0 and R.H.L. = 1

$\therefore$ We get L.H.L. $\ne$ R.H.L. [From Eq. (i)]

Hence, we get a contradiction.

$\Rightarrow$ There doesn't exist any real value of λ for which $f(x)$ is continuous at $x = 0$.

S29. Here,

$$f(x) = \begin{cases} 6ax + 3b & \text{if } x > 1 \\ ax - 2b & \text{if } x < 1 \\ 15 & \text{if } x = 1 \end{cases}$$

$$\text{L.H.L.} = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(ax - 2b)$$

Put $x = 1 - h$ as $x \to 1,\ h \to 0$

$\therefore$ $$\lim_{h\to 0}[a(1-h)-2b] = a - 2b$$

$$\text{R.H.L.} = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(6ax + 3b)$$

Put $x = 1 + h$ as $x \to 1,\ h \to 0$

$\therefore$ $$\lim_{h\to 0}[6a(1+h)+3b] = \lim_{h\to 0}(6a + 6ah + 3b)$$

$\therefore$ $$= 6a + 3b$$

Also $f(1) = 15$

Since $f(x)$ is continuous at $x = 1$

$\therefore$ L.H.L. = R.H.L. = $f(1)$

$\Rightarrow$ $a - 2b = 6a + 3b = 15$

$\therefore$ $a - 2b = 15$ ... (i)

and $6a + 3b = 15$ ... (ii)

Solving (i) and (ii), we have

$$a = 5 \text{ and } b = -5$$

S30. The given function is

$$f(x) = \begin{cases} 2x-1, & x < 2 \\ a, & x = 2 \\ x+1, & x > 2 \end{cases}$$

Given that $f(x)$ is continuous at $x = 2$.

$\therefore$ $\text{L.H.L.}_{x=2} = \text{R.H.L.}_{x=2} = f(2)$

Now, $$\text{L.H.L.} = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(2x-1)$$

Put $x = 2 - h$,

When $x \to 2,\ h \to 0$

$$= \lim_{h\to 0} 2(2-h) - 1$$

$$= \lim_{h\to 0} 4 - 2h - 1$$

$$= \lim_{h\to 0} 3 - 2h \qquad [\because h = 0]$$

$$\text{L.H.L.} = 3$$

Also, from the given function, at $x = 2$, $f(x) = a$

$$f(2) = \text{L.H.L.}$$

$\Rightarrow$ $$a = 3$$

Now, $f(x)$ being a polynomial function, so it is continuous at $x = 3$.

Note: Every polynomial is continuous in its domain.

S31. Given function is

$$f(x) = \begin{cases} \dfrac{\sqrt{1+kx} - \sqrt{1-kx}}{x}, & \text{if } -1 \le x < 0 \\ \dfrac{2x+1}{x-1}, & \text{if } 0 \le x < 1 \end{cases} \quad \text{is continuous at } x = 0$$

Now, $$f(0) = \frac{2\cdot 0+1}{0-1} = \frac{1}{-1} = -1$$

$$\text{L.H.L.} = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sqrt{1+kx} - \sqrt{1-kx}}{x}$$

Put x = 0 – h, when x $\to$ 0, h $\to$ 0

$$\text{L.H.L.} = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) \qquad \dots \text{(i)}$$

$$= \lim_{h\to 0} \frac{\sqrt{1-kh} - \sqrt{1+kh}}{-h}$$

$$= \lim_{h\to 0} \frac{\sqrt{1-kh} - \sqrt{1+kh}}{-h\left(\sqrt{1-kh} + \sqrt{1+kh}\right)} \times \left(\sqrt{1-kh} + \sqrt{1+kh}\right)$$

$$= \lim_{h\to 0} \frac{-2kh}{-h(\sqrt{1-kh} + \sqrt{1+kh})}$$

$$= \lim_{h\to 0} \frac{2k}{\sqrt{1-kh} + \sqrt{1+kh}}$$

$$= \frac{2k}{1+1} = \frac{2k}{2} = k$$

$\therefore$ $f(x)$ is continuous at $x = 0$

$$f(0) = \text{L.H.L.}$$

$\Rightarrow$ $$-1 = k$$

$\Rightarrow$ $$k = -1$$

S32. Given function is

$$f(x) = \begin{cases} \dfrac{k\cos x}{\pi - 2x}, & \text{if } x \neq \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases}$$

Given that function is continuous at $x = \dfrac{\pi}{2}$.

$$\therefore \quad L.H.L_{x=\frac{\pi}{2}} = R.H.L_{x=\frac{\pi}{2}} = f\left(\frac{\pi}{2}\right) \quad \text{... (i)}$$

Now, $$\text{L.H.L.} = \lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{x \to \frac{\pi}{2}^-} \frac{k\cos x}{\pi - 2x}$$

Put $$x = \frac{\pi}{2} - h, \text{ when } x \to \frac{\pi}{2}, h \to 0$$

$$= \lim_{h \to 0} \frac{k\cos\left(\frac{\pi}{2} - h\right)}{\pi - 2\left(\frac{\pi}{2} - h\right)}$$

$$= \lim_{h \to 0} \frac{k\sin h}{\pi - \pi + 2h} \qquad \left[\because \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta\right]$$

$$= \lim_{h \to 0} \frac{k\sin h}{2h}$$

$$= \frac{k}{2}\lim_{h \to 0} \frac{\sin h}{h} = \frac{k}{2} \qquad \left[\because \lim_{h \to 0} \frac{\sin h}{h} = 1\right]$$

$$\therefore \quad \text{L.H.L.} = \frac{k}{2}$$

Also, from the given function, we get

$$f\left(\frac{\pi}{2}\right) = 3$$

Now, from Eq. (i), we get

$$\text{L.H.L.} = f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \quad \frac{k}{2} = 3$$

Hence, $k = 6$.

S33. The given function is

$$f(x) = \begin{cases} 2x+1; & x<2 \\ k \;; & x=2 \\ 3x-1; & x>2 \end{cases}$$

Given that $f(x)$ is continuous at $x = 2$.

$\therefore$ $\quad$ L.H.L.$_{x=2}$ = R.H.L.$_{x=2}$ = $f(2)$ $\quad$ ... (i)

Now, $\quad$ L.H.L. = $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 2x+1$

Put $\quad x = 2 - h$, when $x \to 2$, $h \to 0$ $\quad$ [for L.H.L. put $x = 2 - h$]

$$= \lim_{h\to 0} 2(2-h)+1 = \lim_{h\to 0} 4-2h+1$$

$$= \lim_{h\to 0} 5-2h \quad [\because h = 0]$$

L.H.L. = 5

Also, from $f(x)$, we get at

$$x = 2, \quad f(x) = k.$$

$\because$ $\quad f(2) = k$

Also, from Eq. (i), we get

L.H.L. = $f(2)$

$\Rightarrow$ $\quad 5 = k$ or $k = 5$.

S34.

L.H.L. at $x = \frac{\pi}{2}$

Put $\quad x = \frac{\pi}{2} - h$

as $\quad x \to \frac{\pi}{2}$, $h \to 0$

$$\text{L.H.L.} = \lim_{x\to \frac{\pi}{2}^-} f(x) = \lim_{h\to 0} f\left(\frac{\pi}{2}-h\right)$$

$$= \lim_{h\to 0} \frac{\cos\left(\frac{\pi}{2}-h\right)}{\frac{\pi}{2}-\left(\frac{\pi}{2}-h\right)}$$

$$= \lim_{h\to 0} \frac{\sin h}{h} = 1$$

R.H.L. at $x = \frac{\pi}{2}$

Put $\quad x = \frac{\pi}{2} + h$

as $\quad x \to \frac{\pi}{2}$, $h \to 0$

$$\text{R.H.L} = \lim_{x\to \frac{\pi}{2}^+} f(x) = \lim_{h\to 0} f\left(\frac{\pi}{2}+h\right)$$

$$= \lim_{h\to 0} \frac{\cos\left(\frac{\pi}{2}+h\right)}{\frac{\pi}{2}-\left(\frac{\pi}{2}+h\right)}$$

$$= \lim_{h\to 0} \frac{-\sin h}{-h} = 1$$

and $f\left(\frac{\pi}{2}\right)=1$

As $\lim_{x\to\frac{\pi}{2}^-} f(x)=\lim_{x\to\frac{\pi}{2}^+} f(x)=f\left(\frac{\pi}{2}\right)$

$\therefore$ $f(x)$ is continuous at x = $\frac{\pi}{2}$.

S35. L.H.L. at $x = 0$

Put $x = 0 - h$

as $x \to 0$, $h \to 0$

L.H.L. = $\lim_{x\to 0^-} f(x)=\lim_{h\to 0} f(0-h)$

$= \lim_{h\to 0}\left\{\frac{\sin(-h)}{-h}+\cos(-h)\right\}$

$= \lim_{h\to 0}\left\{\frac{\sin h}{h}+\cos h\right\}$

$= 1 + 1 = 2$

R.H.L. at $x = 0$

Put $x = 0 + h$

as $x \to 0$, $h \to 0$

R.H.L = $\lim_{x\to 0^+} f(x)=\lim_{h\to 0} f(0+h)$

$= \lim_{h\to 0}\left\{\frac{\sin h}{h}+\cos h\right\}$

$= 1 + 1$

$= 2$

and $f(0) = 2$

As $\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=f(0)$

$\therefore$ $f(x)$ is continuous at $x = 0$

S36. L.H.L. at $x = 0$

Put $x = 0 - h$

L.H.L. = $\lim_{x\to 0^-} f(x)=\lim_{h\to 0} f(0-h)$

$= \lim_{h\to 0} -h\sin\left(-\frac{1}{h}\right)$

$= \lim_{h\to 0} h\sin\frac{1}{h}$

$= 0 \times$ (finite no. between – 1 and 1)

$= 0$

R.H.L. at $x = 0$

Put $x = 0 + h$

R.H.L = $\lim_{x\to 0^+} f(x)=\lim_{h\to 0} f(0+h)$

$= \lim_{h\to 0} h\sin\frac{1}{h}$

$= 0 \times$ (finite no between –1 and 1)

$= 0$

and $f(0) = 0$

As $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$

$\therefore$ $f(x)$ is continuous at $x = 0$

S37.

L.H.L. at $x = 0$

Put $x = 0 - h$

as $x \to 0$, $h \to 0$

L.H.L. = $\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h)$

$= \lim_{h \to 0} \frac{\sin 3(0-h)}{(0-h)}$

$= \lim_{h \to 0} \frac{-\sin 3h}{-h} \times \frac{3}{3}$

$= 3 \times 1 = 1$

R.H.L. at $x = 0$

Put $x = 0 + h$

as $x \to 0$, $h \to 0$

R.H.L = $\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h)$

$= \lim_{h \to 0} \frac{\sin 3(0+h)}{(0+h)}$

$= \lim_{h \to 0} \frac{\sin 3h}{h} \times \frac{3}{3}$

$= 3 \times 1 = 3$

and $f(0) = 1$

As $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) \neq f(0)$

$\therefore$ $f(x)$ is discontinuous at $x = 0$.

S38.

L.H.L. at $x = 0$

Put $x = 0 - h$

as $x \to 0$, $h \to 0$

L.H.L. = $\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h)$

$= \lim_{h \to 0} 2(0-h) - 1$

$= \lim_{h \to 0} -2h - 1$

$= -2 \times 0 - 1 = -1$

R.H.L. at $x = 0$

Put $x = 0 + h$

as $x \to 0$, $h \to 0$

R.H.L = $\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h)$

$= \lim_{h \to 0} 2(0+h) + 1$

$= \lim_{h \to 0} 2h + 1$

$= 2 \times 0 + 1 = 1$

and $f(0) = 2 \times 0 + 1 = 1$

As $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x) = f(0)$

$\therefore$ $f(x)$ is discontinuous at $x = 0$.

S39.

L.H.L. at $x = 0$

Put $x = 0 - h$

R.H.L. at $x = 0$

Put $x = 0 + h$

as $x \to 0$, $h \to 0$

L.H.L. = $\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h)$

= $\lim_{h \to 0} 3(0-h) - 2$

= $\lim_{h \to 0} -3h - 2$

= – 0 – 2 = – 2

as $x \to 0$, $h \to 0$

R.H.L = $\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h)$

= $\lim_{h \to 0} (0+h) + 1$

= 0 + 1 = 1

and $f(0) = 3 \times 0 - 2 = -2$

As $\lim_{x \to 0^-} f(x) = f(0) \neq \lim_{x \to 0^+} f(x)$

$\therefore$ $f(x)$ is discontinuous at $x = 0$.

S40. L.H.L. at $x = 1$

Put $x = 1 - h$

as $x \to 1$, $h \to 0$

L.H.L. = $\lim_{x \to 1^-} f(x) = \lim_{h \to 0} f(1-h)$

= $\lim_{h \to 0} 1 + (1-h)^2$

= $\lim_{h \to 0} 1 + 1 + h^2 - 2h$

= 1 + 1 + 0 – 0 = 2

R.H.L. at $x = 1$

Put $x = 1 + h$

as $x \to 1$, $h \to 0$

R.H.L = $\lim_{x \to 1^+} f(x) = \lim_{h \to 0} f(1+h)$

= $\lim_{h \to 0} 2 - (1+h)^2$

= $\lim_{h \to 0} 2 - 1 - h^2 - 2h$

= 1 – 0 = 1

and $f(1) = 1 + (1)^2 = 2$

As $\lim_{x \to 1^-} f(x) = f(1) \neq \lim_{x \to 1^+} f(x)$

$\therefore$ $f(x)$ is discontinuous at $x = 1$.

S41. L.H.L. at $x = 2$

Put $x = 2 - h$

as $x \to 2$, $h \to 0$

L.H.L. = $\lim_{x \to 2^-} f(x) = \lim_{h \to 0} f(2-h)$

= $\lim_{h \to 0} 2 - (2-h)$

= $\lim_{h \to 0} h = 0$

R.H.L. at $x = 2$

Put $x = 2 + h$

as $x \to 2$, $h \to 0$

R.H.L = $\lim_{x \to 2^+} f(x) = \lim_{h \to 0} f(2+h)$

= $\lim_{h \to 0} 2 + (2+h)$

= 4 + 0 = 4

and $f(2) = 2 + 2 = 4$

As $\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x) = f(2)$

$\therefore$ $f(x)$ is discontinuous at $x = 2$.

S42.

L.H.L. at $x = 0$

Put $x = 0 - h$

as $x \to 0$, $h \to 0$

L.H.L. = $\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h)$

$= \lim_{h \to 0} \frac{e^{1/-h} - 1}{e^{1/-h} + 1}$

$= \lim_{h \to 0} \frac{\frac{1}{e^{1/h}} - 1}{\frac{1}{e^{1/h}} + 1}$ $\left[\begin{array}{l} \text{As } h \to 0, \frac{1}{h} \to \infty \\ \therefore e^{1/h} \to \infty, \frac{1}{e^{1/h}} \to 0 \end{array}\right]$

$= \frac{0 - 1}{0 + 1} = -1$

R.H.L. at $x = 0$

Put $x = 0 + h$

as $x \to 0$, $h \to 0$

R.H.L = $\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h)$

$= \lim_{h \to 0} \frac{e^{1/h} - 1}{e^{1/h} + 1}$

$= \lim_{h \to 0} \frac{1 - \frac{1}{e^{1/h}}}{1 + \frac{1}{e^{1/h}}}$

$= \frac{1 - 0}{1 + 0} = 1$

and $f(0) = 0$

As $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x) \neq f(0)$

$\therefore$ $f(x)$ is discontinuous at $x = 0$.

S43.

L.H.L. at $x = 0$

Put $x = 0 - h$

as $x \to 0$, $h \to 0$

L.H.L. = $\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h)$

$= \lim_{h \to 0} \frac{(0-h) - |0-h|}{2}$

$= \lim_{h \to 0} \frac{h-h}{2}$

$= 0$

R.H.L. at $x = 0$

Put $x = 0 + h$

as $x \to 0$, $h \to 0$

R.H.L = $\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h)$

$= \lim_{h \to 0} \frac{(0+h) - |0+h|}{2}$

$= \lim_{h \to 0} \frac{h-h}{2}$

$= 0$

and $f(0) = 2$

As $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) \neq f(0)$

$\therefore$ $f(x)$ is discontinuous at $x = 0$.

S44. L.H.L. at $x = a$

Put $x = a - h$

as $x \to a$, $h \to 0$

L.H.L. = $\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h)$

= $\lim_{h \to 0} \frac{|a-h-a|}{a-h-a}$

= $\lim_{h \to 0} \frac{|-h|}{-h}$

= $\lim_{h \to 0} \frac{h}{-h}$ = – 1

R.H.L. at $x = a$

Put $x = a + h$

as $x \to a$, $h \to 0$

R.H.L = $\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h)$

= $\lim_{h \to 0} \frac{|a+h-a|}{a+h-a}$

= $\lim_{h \to 0} \frac{|h|}{h}$

= $\lim_{h \to 0} \frac{h}{h} = 1$

and $f(a) = 1$

As $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x) = f(a)$

$\therefore$ $f(x)$ is discontinuous at $x = a$.

S45. L.H.L. at $x = 0$

Put $x = 0 - h$

as $x \to 0$, $h \to 0$

L.H.L. = $\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h)$

= $\lim_{h \to 0} \frac{1-\cos(-h)}{(-h)^2}$

= $\lim_{h \to 0} \frac{1-\cos h}{h^2} = \lim_{h \to 0} \frac{2\sin^2 h/2}{4h^2/4}$

= $\lim_{h \to 0} \frac{1}{2}\left(\frac{\sin h/2}{h/2}\right)^2 = \frac{1}{2}$

R.H.L. at $x = 0$

Put $x = 0 + h$

as $x \to 0$, $h \to 0$

R.H.L = $\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h)$

= $\lim_{h \to 0} \frac{1-\cos h}{h^2}$

= $\lim_{h \to 0} \frac{2\sin^2 h/2}{4h^2/4}$

= $\lim_{h \to 0} \frac{1}{2}\left(\frac{\sin h/2}{h/2}\right)^2 = \frac{1}{2}$

and $f(0) = 1$

As $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) \neq f(0)$

$\therefore$ $f(x)$ is discontinuous at $x = 0$.

S46. L.H.L. at $x = 0$

Put $x = 0 - h$

R.H.L. at $x = 0$

Put $x = 0 + h$

as $x \to 0$, $h \to 0$

L.H.L. = $\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h)$

$= \lim_{h \to 0} \frac{2|0-h|+(0-h)^2}{0-h}$

$= \lim_{h \to 0} \frac{2|-h|+h^2}{-h}$

$= \lim_{h \to 0} \frac{2h+h^2}{-h}$

$= \lim_{h \to 0} -(2+h)$

$= -2$

as $x \to 0$, $h \to 0$

R.H.L = $\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(0+h)$

$= \lim_{h \to 0} \frac{2|0+h|+(0+h)^2}{0+h}$

$= \lim_{h \to 0} \frac{2|h|+h^2}{h}$

$= \lim_{h \to 0} 2+h$

$= 2$

and $f(0) = 0$

As $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x) \neq f(0)$

$\therefore$ $f(x)$ is discontinuous at $x = 0$.

S47. L.H.L. at $x = \frac{1}{2}$

Put $x = \frac{1}{2} - h$

as $x \to \frac{1}{2}$, $h \to 0$

L.H.L. = $\lim_{x \to \frac{1}{2}^-} f(x) = \lim_{h \to 0} f\left(\frac{1}{2} - h\right)$

$= \lim_{h \to 0} \frac{1}{2} - \left(\frac{1}{2} - h\right)$

$= \lim_{h \to 0} h = 0$

R.H.L. at $x = \frac{1}{2}$

Put $x = \frac{1}{2} + h$

as $x \to \frac{1}{2}$, $h \to 0$

R.H.L = $\lim_{x \to \frac{1}{2}^+} f(x) = \lim_{h \to 0} f\left(\frac{1}{2} + h\right)$

$= \lim_{h \to 0} \frac{3}{2} - \left(\frac{1}{2} + h\right)$

$= \lim_{h \to 0} 1 - h = 1$

and $f\left(\frac{1}{2}\right) = 1$

As $\lim_{x \to \frac{1}{2}^-} f(x) \neq \lim_{x \to \frac{1}{2}^+} f(x) = f\left(\frac{1}{2}\right)$

$\therefore$ $f(x)$ is discontinuous at $x = \frac{1}{2}$.

S48. L.H.L. at $x = \frac{1}{2}$

Put $x = \frac{1}{2} - h$

as $x \to \frac{1}{2}$, $h \to 0$

L.H.L. = $\lim_{x \to \frac{1}{2}^-} f(x) = \lim_{h \to 0} f\left(\frac{1}{2} - h\right)$

$= \lim_{h \to 0} \frac{1}{2} - h$

$= \frac{1}{2}$

R.H.L. at $x = \frac{1}{2}$

Put $x = \frac{1}{2} + h$

as $x \to \frac{1}{2}$, $h \to 0$

R.H.L = $\lim_{x \to \frac{1}{2}^+} f(x) = \lim_{h \to 0} f\left(\frac{1}{2} + h\right)$

$= \lim_{h \to 0} \frac{1}{2} - h$

$= \lim_{h \to 0} \frac{1}{2} - h = \frac{1}{2}$

and $f\left(\frac{1}{2}\right) = \frac{1}{2}$

As $\lim_{x \to \frac{1}{2}^-} f(x) = \lim_{x \to \frac{1}{2}^+} f(x) = f\left(\frac{1}{2}\right)$

$\therefore$ $f(x)$ is continuous at $x = \frac{1}{2}$.

S49. L.H.L. at $x = 0$

Put $x = 1 - h$

as $x \to 0$, $h \to 0$

L.H.L. = $\lim_{x \to 1^-} f(x) = \lim_{h \to 0} f(0 - h)$

$= \lim_{h \to 0} -(0 - h)$

$= \lim_{h \to 0} h = 0$

R.H.L. at $x = 0$

Put $x = 0 + h$

as $x \to 0$, $h \to 0$

R.H.L = $\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h)$

$= \lim_{h \to 0} (0 + h)$

$= \lim_{h \to 0} h = 0$

and $f(0) = 1$

As $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) \neq f(0)$

$\therefore$ $f(x)$ is discontinuous at $x = 0$.

S50. L.H.L. at $x = 1$

Put $x = 1 - h$

R.H.L. at $x = 1$

Put $x = 1 + h$

as $x \to 1$, $h \to 0$

L.H.L. $= \lim_{x \to 1^-} f(x) = \lim_{h \to 0} f(1-h)$

$= \lim_{h \to 0} \frac{|(1-h)^2 - 1|}{1-h-1}$

$= \lim_{h \to 0} \frac{|(1+h^2 - 2h - 1|}{-h}$

$= \lim_{h \to 0} -|h-2|$

$= -|0-2| = -2$

as $x \to 1$, $h \to 0$

R.H.L $= \lim_{x \to 1^+} f(x) = \lim_{h \to 0} f(1+h)$

$= \lim_{h \to 0} \frac{|(1+h)^2 - 1|}{1+h-1}$

$= \lim_{h \to 0} \frac{|(1+h^2 + 2h - 1|}{h}$

$= \lim_{h \to 0} |h+2|$

$= |0+2| = 2$

and $f(1) = 2$

As $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x) = f(1)$

$\therefore$ $f(x)$ is discontinuous at $x = 1$.

S51. L.H.L. at $x = 0$

Put $x = 0 - h$

as $x \to 0$, $h \to 0$

L.H.L. $= \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h)$

$= \lim_{h \to 0} e^{\frac{1}{-h}}$

$= \lim_{h \to 0} \frac{1}{e^{\frac{1}{h}}}$ $\left[\begin{array}{l} \text{As } h \to 0, \frac{1}{h} \to \infty \\ e^{\frac{1}{h}} \to \infty, \frac{1}{e^{1/h}} \to 0 \end{array}\right]$

$= 0$

R.H.L. at $x = 0$

Put $x = 0 + h$

as $x \to 0$, $h \to 0$

R.H.L $= \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h)$

$= \lim_{h \to 0} e^{\frac{1}{h}}$

$= \infty$

and $f(0) = 1$

As $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x) \neq f(0)$

$\therefore$ $f(x)$ is discontinuous at $x = 0$

S52. As $f(x)$ is continuous at $x = -1$

$\therefore$ $f(-1) = \lim_{x \to -1} f(x)$

$= \lim_{x \to -1} \frac{(x^2 - 2x - 3)}{x+1}$

$= \lim_{x \to -1} \frac{(x+1)(x-3)}{x+1}$

$$= \lim_{x \to -1} (x - 3)$$

$$= -1 - 3 = -4$$

$\therefore \quad f(-1) = \lambda = -4 \qquad$ hence $\lambda = -4$

S53. As $f(x)$ is continuous at $x = 1$

$$\therefore \quad f(1) = \lim_{x \to 1} f(x)$$

$$= \lim_{x \to 1} \frac{x^2 - 3x + 2}{x - 1}$$

$$= \lim_{x \to 1} \frac{(x-2)(x-1)}{(x-1)}$$

$$= \lim_{x \to 1} (x - 2)$$

$$= 1 - 2 = -1$$

$\therefore \quad f(1) = k = -1 \qquad$ hence $k = -1$.

S54. L.H.L. at $x = 0$

Put $x = 0 - h$

as $x \to 0$, $h \to 0$

$$\text{L.H.L.} = \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h)$$

$$= \lim_{h \to 0} \frac{\sin 3(0 - h)}{\tan 2(0 - h)}$$

$$= \lim_{h \to 0} \frac{-\sin 3h}{-\tan 2h} \cdot \frac{2h}{3h} \cdot \frac{3h}{2h}$$

$$= \lim_{h \to 0} \frac{\sin 3h}{3h} \cdot \frac{2h}{\tan 2h} \cdot \frac{3h}{2h}$$

$$= \frac{3}{2}$$

R.H.L. at $x = 0$

Put $x = 0 + h$

as $x \to 0$, $h \to 0$

$$\text{R.H.L} = \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h)$$

$$= \lim_{h \to 0} \frac{\log(1 + 3(0 + h))}{e^{2(0 + h)} - 1}$$

$$= \lim_{h \to 0} \frac{\log(1 + 3h)}{e^{2h} - 1} \cdot \frac{2h}{3h} \cdot \frac{3h}{2h}$$

$$= \lim_{h \to 0} \frac{\log(1 + 3h)}{3h} \cdot \frac{2h}{e^{2h} - 1} \cdot \frac{3h}{2h}$$

$$= \frac{3}{2}$$

and $\quad f(0) = \dfrac{3}{2}$

As $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$

$\therefore \quad f(x)$ is continuous at $x = 0$.

S55. As function $f(x)$ is continuous at $x = 0$

$$\therefore \quad f(0) = \lim_{x \to 0} \frac{1 - \cos kx}{x \sin x}$$

$$= \lim_{x \to 0} \frac{2\sin^2 \frac{kx}{2}}{x \sin x} = \lim_{x \to 0} \frac{\sin^2 \frac{kx}{2}}{\frac{k^2 x^2}{4} \cdot 2} \cdot \frac{x}{\sin x} \cdot k^2 = \frac{k^2}{2}$$

But $\quad f(0) = \frac{1}{2}$

$$\Rightarrow \quad \frac{k^2}{2} = \frac{1}{2}$$

$$\Rightarrow \quad k = \pm 1 .$$

S56. As function $f(x)$ is continuous at $x = 1$

$$\therefore \quad f(1) = \lim_{x \to 1} (x - 1) \tan \frac{\pi x}{2}$$

Put $\quad x = 1 + h$

As $\quad x \to 1, h \to 0$

$$= \lim_{h \to 0} (1 + h - 1) \tan \frac{\pi}{2} (1 + h)$$

$$= \lim_{h \to 0} h \tan \left(\frac{\pi}{2} + \frac{\pi h}{2} \right)$$

$$= \lim_{h \to 0} h \cdot \left(-\cot \frac{\pi h}{2} \right)$$

$$= \lim_{h \to 0} h \cdot \left(\frac{-1}{\tan \frac{\pi h}{2}} \right)$$

$$= \lim_{h \to 0} \frac{\pi h}{2} \cdot \left(\frac{-1}{\tan \frac{\pi h}{2}} \right) \cdot \frac{2}{\pi}$$

$$= -\frac{2}{\pi}$$

$$\therefore \quad f(1) = k = -\frac{2}{\pi} \qquad \text{hence} \quad k = -\frac{2}{\pi}.$$

S57. As function $f(x)$ is continuous at $x = 0$

$$\therefore \quad f(0) = \lim_{x \to 0} f(x)$$

$$= \lim_{x \to 0} \frac{1 - \cos 2kx}{x^2}$$

$$= \lim_{x \to 0} \frac{2\sin^2 kx}{x^2}$$

$$= \lim_{x \to 0} 2\left(\frac{\sin kx}{kx}\right)^2 \cdot k^2$$

$$= 2 \cdot (1)^2 \cdot k^2 = 2k^2$$

But $\quad f(0) = 8$

$\Rightarrow \quad 2k^2 = 8$

$\Rightarrow \quad k = \pm 2.$

S58. A function $f(x)$ is continuous at $x = 0$

$$\therefore \quad f(0) = \lim_{x \to 0} f(x)$$

$$= \lim_{x \to 0} \frac{2x + 3\sin x}{3x + 2\sin x}$$

$$= \lim_{x \to 0} \frac{\lim_{x \to 0} \frac{2x}{x} + \frac{3\sin x}{x}}{\lim_{x \to 0} \frac{3x}{x} + \frac{2\sin x}{x}}$$

$$= \lim_{x \to 0} \frac{\lim_{x \to 0} 2 + \frac{3\sin x}{x}}{\lim_{x \to 0} 3 + \frac{2\sin x}{x}}$$

$$= \frac{2 + 3}{3 + 2} = \frac{5}{5} = 1$$

$\therefore \quad f(0) = 1.$

S59. If $f(x)$ is continuous at $x = \pi$

$$\therefore \quad f(\pi) = \lim_{x \to \pi} f(x)$$

$$= \lim_{x \to \pi} \frac{1 - \cos 7(x - \pi)}{5(x - \pi)^2}$$

Put $\quad x = \pi + h$

As $x \to \pi,\ h \to 0$

$$= \lim_{x \to \pi} \frac{1 - \cos 7(\pi + h - \pi)}{5(\pi + h - \pi)^2}$$

$$= \lim_{h \to 0} \frac{1 - \cos 7h}{5h^2}$$

$$= \lim_{h \to 0} \frac{2\sin^2 \frac{7h}{2}}{5h^2}$$

$$= \lim_{h \to 0} \frac{2\sin^2 \frac{7h}{2}}{5 \cdot \frac{49h^2}{4}} \cdot \frac{49}{4}$$

$$= \frac{2}{5} \cdot 1 \cdot \frac{49}{4} = \frac{49}{10}$$

$\therefore$ $f(\pi) = \frac{49}{10}$.

S60. As function $f(x)$ is continuous at $x = 0$

$\therefore$ $f(0) = \lim_{x \to 0} f(x)$

$$= \lim_{x \to 0} \frac{\cos^2 x - \sin^2 x - 1}{\sqrt{x^2 + 1} - 1}$$

$$= \lim_{x \to 0} \frac{(\cos 2x - 1)(\sqrt{x^2 + 1} + 1)}{(\sqrt{x^2 + 1} - 1)(\sqrt{x^2 + 1} + 1)}$$

$$= \lim_{x \to 0} \frac{(\cos 2x - 1)(\sqrt{x^2 + 1} + 1)}{x^2 + 1 - 1}$$

$$= \lim_{x \to 0} \frac{-(1 - \cos 2x)(\sqrt{x^2 + 1} + 1)}{x^2}$$

$$= \lim_{x \to 0} \frac{-2\sin^2 x}{x^2}(\sqrt{x^2 + 1} + 1)$$

$$= -2.1.(\sqrt{0 + 1} + 1) = -4$$

$\therefore$ $f(0) = k = -4$ hence $k = -4$.

S61. As $f(x)$ is continuous at $x = 0$

$\therefore$ $f(0) = \lim_{x \to 0} \frac{\log(1 + ax) - \log(1 - bx)}{x}$

$$= \lim_{x \to 0} \left[\frac{\log(1+ax)}{x} - \frac{\log(1-bx)}{x} \right]$$

$$= \lim_{x \to 0} \left[\frac{a\log(1+ax)}{ax} - \frac{-b\log(1-bx)}{-bx} \right]$$

$$= \lim_{x \to 0} \frac{a\log(1+ax)}{ax} - \lim_{x \to 0} \frac{-b\log(1-bx)}{-bx}$$

$$= a - (-b) \qquad \left[\because \lim_{x \to 0} \frac{\log(1+x)}{x} = 1 \right]$$

$$= a + b$$

$\therefore \quad f(0) = k = a + b$ hence $k = a + b$.

S62. L.H.L. at $x = 1$

Put $x = 1 - h$

as $x \to 1$, $h \to 0$

L.H.L. $= \lim_{x \to 1^-} f(x) = \lim_{h \to 0} f(1-h)$

$= \lim_{h \to 0} 5a(1-h) - b$

$= \lim_{h \to 0} 5a - 5ah - b$

$= 5a - b$

R.H.L. at $x = 1$

Put $x = 1 + h$

as $x \to 1$, $h \to 0$

R.H.L $= \lim_{x \to 1^+} f(x) = \lim_{h \to 0} f(1+h)$

$= \lim_{h \to 0} 3a(1+h) + b$

$= \lim_{h \to 0} 3a + 3ah + b$

$= 3a + b$

and $f(1) = 11$

As $f(x)$ is continuous at $x = 1$

$\therefore \quad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$

$\Rightarrow \quad 5a - b = 3a + b = 11$

$\Rightarrow \quad 8a = 22 \Rightarrow a = \frac{22}{8} = \frac{11}{4}$

and $\quad b = 11 - \frac{33}{4} = \frac{11}{4}$

$\therefore \quad a = b = \frac{11}{4}$.

S63. As function $f(x)$ is continuous at $x = \frac{\pi}{4}$

$\therefore \quad f\left(\frac{\pi}{4}\right) = \lim_{x \to \frac{\pi}{4}} f(x)$

$$= \lim_{x \to \frac{\pi}{4}} \frac{\tan\left(\frac{\pi}{4} - x\right)}{\cot 2x}$$

Put $\quad x = \frac{\pi}{4} - h$

As $\quad x \to \frac{\pi}{4}, \quad h \to 0$

$$= \lim_{h \to 0} \frac{\tan\left\{\frac{\pi}{4} - \left(\frac{\pi}{4} - h\right)\right\}}{\cot 2\left(\frac{\pi}{4} - h\right)}$$

$$= \lim_{h \to 0} \frac{\tan h}{\cot\left(\frac{\pi}{2} - 2h\right)}$$

$$= \lim_{h \to 0} \frac{\tan h}{\tan 2h} \cdot \frac{2h}{h} \cdot \frac{h}{2h}$$

$$= \lim_{h \to 0} \frac{\tan h}{h} \cdot \frac{2h}{\tan 2h} \cdot \frac{h}{2h}$$

$$= 1 \cdot 1 \cdot \frac{1}{2} = \frac{1}{2}$$

$\therefore \quad f\left(\frac{\pi}{4}\right) = \frac{1}{2}.$

S64. L.H.L. at $x = 0$

Put $x = 0 - h$

as $x \to 0,\ h \to 0$

$$\text{L.H.L} = \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h)$$

$$= \lim_{h \to 0} \frac{1 - \cos 4(0 - h)}{(0 - h)^2}$$

$$= \lim_{h \to 0} \frac{1 - \cos 4h}{h^2}$$

$$= \lim_{h \to 0} \frac{2\sin^2 2h}{4h^2} \times 4$$

$$= \lim_{h \to 0} 2 \cdot (1)^2 \cdot 4 = 8$$

R.H.L. at $x = 0$

Put $x = 0 + h$

as $x \to 0,\ h \to 0$

$$\text{R.H.L} = \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h)$$

$$= \lim_{h \to 0} \frac{\sqrt{0 + h}}{\sqrt{16 + \sqrt{0 + h}} - 4}$$

$$= \lim_{h \to 0} \frac{\sqrt{h}\,(\sqrt{16 + \sqrt{h}} + 4)}{(\sqrt{16 + \sqrt{h}} - 4)(\sqrt{16 + \sqrt{h}} + 4)}$$

$$= \lim_{h \to 0} \frac{\sqrt{h}(\sqrt{16 + \sqrt{h}} + 4)}{16 + \sqrt{h} - 16}$$

$$= \lim_{h \to 0} (\sqrt{16 + \sqrt{h}} + 4)$$

$= \sqrt{16+0}+4 = 8$

As $f(x)$ is continuous at $x = 0$

and $f(0) = a$

$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

$\Rightarrow a = 8.$

S65. L.H.L. at $x = 2$

Put $x = 2 - h$

as $x \to 2$, $h \to 0$

L.H.L. $= \lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h)$

$= \lim_{h\to 0} a(2-h) + 5$

$= \lim_{h\to 0} 2a - ah + 5$

$= 2a + 5$

R.H.L. at $x = 2$

Put $x = 2 + h$

as $x \to 2$, $h \to 0$

R.H.L $= \lim_{x\to 2^+} f(x) = \lim_{h\to 0} f(2+h)$

$= \lim_{h\to 0} 2 + h - 1$

$= \lim_{h\to 0} h + 1$

$= 1$

and $f(2) = 2a + 5$

As $f(x)$ is continuous at $x = 2$

$\therefore \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$

$\Rightarrow 2a + 5 = 1$

$\Rightarrow a = -2.$

S66. L.H.L. at $x = 3$

Put $x = 3 - h$

as $x \to 3$, $h \to 0$

L.H.L. $= \lim_{x\to 3^-} f(x) = \lim_{h\to 0} f(3-h)$

$= \lim_{h\to 0} a(3-h) + 1$

$= \lim_{h\to 0} 3a - ah + 1$

$= 3a + 1$

R.H.L. at $x = 3$

Put $x = 3 + h$

as $x \to 3$, $h \to 0$

R.H.L $= \lim_{x\to 3^+} f(x) = \lim_{h\to 0} f(3+h)$

$= \lim_{h\to 0} b(3+h) + 3$

$= \lim_{h\to 0} 3b + bh + 3$

$= 3b + 3$

and $f(3) = 3a + 1$

As $f(x)$ is continuous at $x = 3$

$\therefore \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3)$

$\Rightarrow \quad 3a + 1 = 3b + 3$

$\Rightarrow \quad 3a - 3b = 2.$

S67. L.H.L. at $x = 4$ | R.H.L. at $x = 4$

Put $x = 4 - h$ | Put $x = 4 + h$

as $x \to 4,\ h \to 0$ | as $x \to 4,\ h \to 0$

L.H.L. $= \lim_{x \to 4^-} f(x) = \lim_{h \to 0} f(4 - h)$

$= \lim_{h \to 0} \dfrac{4 - h - 4}{|4 - h - 4|} + a$

$= \lim_{h \to 0} \dfrac{-h}{|-h|} + a$

$= \lim_{h \to 0} -1 + a$

$= a - 1$

R.H.L $= \lim_{x \to 4^+} f(x) = \lim_{h \to 0} f(4 + h)$

$= \lim_{h \to 0} \dfrac{4 + h - 4}{|4 + h - 4|} + b$

$= \lim_{h \to 0} \dfrac{h}{|h|} + b$

$= b + 1$

and $f(4) = a + b$

As $f(x)$ is continuous at $x = 4$

$\therefore \quad \lim_{x \to 4^-} f(x) = \lim_{x \to 4^+} f(x) = f(4)$

$\Rightarrow \quad a - 1 = b + 1 = a + b$

$\Rightarrow \quad a = 1 \quad \text{and} \quad b = -1.$

S68. L.H.L. at $x = 3$ | R.H.L. at $x = 3$

Put $x = 3 - h$ | Put $x = 3 + h$

as $x \to 3,\ h \to 0$ | as $x \to 3,\ h \to 0$

L.H.L. $= \lim_{x \to 3^-} f(x) = \lim_{h \to 0} f(3 - h)$

$= \lim_{h \to 0} 1$

$= 1$

R.H.L $= \lim_{x \to 3^+} f(x) = \lim_{h \to 0} f(3 + h)$

$= \lim_{h \to 0} a(3 + h) + b$

$= 3a + b$

and $f(3) = 1$

As $f(x)$ is continuous at $x = 3$

$\therefore \quad \lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = f(3)$

$\Rightarrow \quad 3a + b = 1 \qquad \ldots \text{(i)}$

L.H.L. at $x = 5$ | R.H.L. at $x = 5$

Put $x = 5 - h$ | Put $x = 5 + h$

as $x \to 5,\ h \to 0$

$$\text{L.H.L.} = \lim_{x \to 5^-} f(x) = \lim_{h \to 0} f(5-h)$$

$$= \lim_{h \to 0} a(5-h) + b$$

$$= \lim_{h \to 0} 5a + b$$

$$= 5a + b$$

as $x \to 5,\ h \to 0$

$$\text{R.H.L} = \lim_{x \to 5^+} f(x) = \lim_{h \to 0} f(5+h)$$

$$= \lim_{h \to 0} 7$$

$$= 7$$

and $f(5) = 7$

As $f(x)$ is continuous at $x = 5$

$$\therefore \quad \lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x) = f(5)$$

$\Rightarrow \quad 5a + b = 7$ ... (ii)

Solving Eq. (i) and (ii),

$$\Rightarrow \quad 5a + b = 7$$

$$\underline{3a + b = 1}$$

$$2a = 6$$

$$a = 3$$

and $b = 7 - 15$

$b = -8$

$\therefore \quad a = 3$ and $b = -8$.

S69. L.H.L. at $x = 0$

Put $x = 0 - h$

as $x \to 0,\ h \to 0$

$$\text{L.H.L.} = \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h)$$

$$= \lim_{h \to 0} a \sin\frac{\pi}{2}(0 - h + 1)$$

$$= \lim_{h \to 0} a \sin\left(\frac{\pi}{2} - \frac{\pi h}{2}\right)$$

$$= \lim_{h \to 0} a \cos\frac{\pi h}{2}$$

$$= a$$

R.H.L. at $x = 0$

Put $x = 0 + h$

as $x \to 0,\ h \to 0$

$$\text{R.H.L} = \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h)$$

$$= \lim_{h \to 0} \frac{\tan(0+h) - \sin(0+h)}{(0+h)^3}$$

$$= \lim_{h \to 0} \frac{\tan h - \sin h}{h^3}$$

$$= \lim_{h \to 0} \frac{\sin h}{\cos h}\frac{(1 - \cos h)}{h^3}$$

$$= \lim_{h \to 0} \frac{\tan h}{h} \cdot \frac{2\sin^2\frac{h}{2}}{4 \cdot \frac{h^2}{4}}$$

$$= 1 \cdot \frac{1}{2} \cdot (1)^2 = \frac{1}{2}$$

and $f(0) = a \sin\frac{\pi}{2} = a$

As $f(x)$ is continuous at $x = 0$

$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

$\Rightarrow a = \frac{1}{2}.$

S70. L.H.L. at $x = \frac{\pi}{2}$

Put $x = \frac{\pi}{2} - h$

as $x \to \frac{\pi}{2}$, $h \to 0$

$$\text{L.H.L.} = \lim_{x\to \frac{\pi}{2}^-} f(x) = \lim_{h\to 0} f\left(\frac{\pi}{2} - h\right)$$

$$= \lim_{h\to 0} \frac{1-\sin^3\left(\frac{\pi}{2}-h\right)}{3\cos^2\left(\frac{\pi}{2}-h\right)}$$

$$= \lim_{h\to 0} \frac{1-\cos^3 h}{3\sin^2 h}$$

$$= \lim_{h\to 0} \frac{(1-\cos h)(1+\cos^2 h+\cos h)}{3(1-\cos^2 h)}$$

$$= \lim_{h\to 0} \frac{(1-\cos h)(1+\cos^2 h+\cos h)}{3(1-\cos h)(1+\cos h)}$$

$$= \lim_{h\to 0} \frac{1+1+1}{3(1+1)} = \frac{1}{2}$$

R.H.L. at $x = \frac{\pi}{2}$

Put $x = \frac{\pi}{2} + h$

as $x \to \frac{\pi}{2}$, $h \to 0$

$$\text{R.H.L} = \lim_{x\to \frac{\pi}{2}^+} f(x) = \lim_{h\to 0} f\left(\frac{\pi}{2} + h\right)$$

$$= \lim_{h\to 0} \frac{b\left(1-\sin\left(\frac{\pi}{2}+h\right)\right)}{\left(\pi - 2\left(\frac{\pi}{2}-h\right)\right)^2}$$

$$= \lim_{h\to 0} \frac{b(1-\cos h)}{4h^2}$$

$$= \lim_{h\to 0} \frac{b\,2\sin^2\frac{h}{2}}{4\times 4\cdot\frac{h^2}{4}}$$

$$= \lim_{h\to 0} \frac{b}{8}$$

$$= \frac{b}{8}$$

and $f\left(\frac{\pi}{2}\right) = a$

As $f(x)$ is continuous at $x = \frac{\pi}{2}$

$$\therefore \quad \lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{x \to \frac{\pi}{2}^+} f(x) = f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \quad \frac{1}{2} = \frac{b}{8} = a$$

$$\Rightarrow \quad a = \frac{1}{2} \quad \text{and} \quad b = 4.$$

S71. L.H.L. at $x = 0$

Put $x = 0 - h$

as $x \to 0, h \to 0$

$$\text{L.H.L.} = \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h)$$

$$= \lim_{h \to 0} \frac{\sin(a+1)(0-h) + \sin(0-h)}{0-h}$$

$$= \lim_{h \to 0} \frac{\sin(a+1)h + \sin h}{h}$$

$$= \lim_{h \to 0} (a+1)\frac{\sin(a+1)h}{(a+1)h} + \lim_{h \to 0} \frac{\sin h}{h}$$

$$= a + 1 + 1 = a + 2$$

R.H.L. at $x = 0$

Put $x = 0 + h$

as $x \to 0, h \to 0$

$$\text{R.H.L} = \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h)$$

$$= \lim_{h \to 0} \frac{\sqrt{(0+h) + b(0+h)^2} - \sqrt{0+h}}{b(0+h)^{3/2}}$$

$$= \lim_{h \to 0} \frac{\sqrt{h + bh^2} - \sqrt{h}}{bh^{3/2}}$$

$$= \lim_{h \to 0} \frac{\sqrt{1+bh} - 1}{bh}$$

$$= \lim_{h \to 0} \frac{(\sqrt{1+bh} - 1)(1 + \sqrt{1+bh})}{(\sqrt{1+bh} + 1)bh}$$

$$= \lim_{h \to 0} \frac{1 + bh - 1}{(\sqrt{1+bh} + 1)bh}$$

$$= \lim_{h \to 0} \frac{1}{(\sqrt{1+bh} + 1)} = \frac{1}{2}$$

and $f(0) = c$

As $f(x)$ is continuous at $x = 0$

$$\therefore \quad \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

$$\Rightarrow \quad a + 2 = \frac{1}{2} = c$$

$$\Rightarrow \quad a = \frac{-3}{2}, \ c = \frac{1}{2} \quad \text{and} \quad b \in R$$

$\therefore$ Value of R.H.L. is remain same for all values of b.

S72. L.H.L. at $x = 0$

Put $x = 0 - h$

R.H.L. at $x = 0$

Put $x = 0 + h$

as $x \to 0, h \to 0$

L.H.L. $= \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h)$

$= \lim_{h \to 0} \dfrac{1-\cos 2(0-h)}{2(0-h)^2}$

$= \lim_{h \to 0} \dfrac{1-\cos 2h}{2h^2}$

$= \lim_{h \to 0} \dfrac{2\sin^2 h}{2h^2}$

$= 1 \cdot (1)^2 = 1$

as $x \to 0, h \to 0$

R.H.L $= \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h)$

$= \lim_{h \to 0} \dfrac{0+h}{|0+h|}$

$= \lim_{h \to 0} \dfrac{h}{|h|}$

$= 1$

and $f(0) = k$

As $f(x)$ is continuous at $x = 0$

$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$

$\Rightarrow k = 1.$

S73. L.H.L. at $x = \dfrac{\pi}{2}$

Put $x = 0 - h$

as $x \to \dfrac{\pi}{2}, h \to 0$

L.H.L. $= \lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{h \to 0} f\left(\dfrac{\pi}{2} - h\right)$

$= \lim_{h \to 0} \dfrac{k\cos\left(\dfrac{\pi}{2} - h\right)}{\pi - 2\left(\dfrac{\pi}{2} - h\right)}$

$= \lim_{h \to 0} \dfrac{k \sin h}{2h}$

$= \dfrac{k}{2}$

R.H.L. at $x = \dfrac{\pi}{2}$

Put $x = \dfrac{\pi}{2} + h$

as $x \to \dfrac{\pi}{2}, h \to 0$

R.H.L $= \lim_{x \to \frac{\pi}{2}^+} f(x) = \lim_{h \to 0} f\left(\dfrac{\pi}{2} + h\right)$

$= \lim_{h \to 0} \dfrac{3\tan 2\left(\dfrac{\pi}{2} + h\right)}{2\left(\dfrac{\pi}{2} + h\right) - \pi}$

$= \lim_{h \to 0} \dfrac{3\tan(\pi + 2h)}{2h}$

$= \dfrac{3\tan 2h}{2h}$

$= 3$

and $f\left(\dfrac{\pi}{2}\right) = 3$

As $f(x)$ is continuous at $x = \frac{\pi}{2}$

$\therefore \quad \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$

$\Rightarrow \quad \frac{k}{2} = 3 \quad \Rightarrow \quad k = 6.$

S74. The given function is

$$f(x) = \begin{cases} x+2; & x \le 2 \\ ax+b; & 2 < x < 5 \\ 3x-2; & x \ge 5 \end{cases}$$

Given that $f(x)$ is continuous at $x = 2$ and $x = 5$.

$\therefore$ By definition of continuity, we get

$$\text{L.H.L.}_{x=2} = \text{R.H.L.}_{x=2} = f(2) \quad \ldots \text{(i)}$$

and $$\text{L.H.L.}_{x=5} = \text{R.H.L.}_{x=5} = f(5) \quad \ldots \text{(ii)}$$

First we find L.H.L. and R.H.L. at $x = 2$

$$\text{L.H.L.} = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x+2)$$

Put $x = 2 - h$, when $x \to 2$, $h \to 0$

$$= \lim_{h \to 0} 2 - h + 2 = \lim_{h \to 0} 4 - h$$

$$\text{L.H.L.} = 4 \quad \text{[Put } h = 0]$$

Now, $$\text{R.H.L.} = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} ax + b$$

Put $x = 2 + h$, when $x \to 2$, $h \to 0$

$$= \lim_{h \to 0} a(2+h) + b$$

$$= \lim_{h \to 0} 2a + ah + b$$

$$\text{R.H.L.} = 2a + b \quad \text{[Put } h = 0]$$

$\therefore$ From Eq. (i), we get

$$\text{L.H.L.} = \text{R.H.L.}$$

$\therefore$ $$2a + b = 4 \quad \ldots \text{(iii)}$$

Now, we find L.H.L. and R.H.L. at $x = 5$

$$\text{L.H.L.} = \lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} ax + b$$

Put $x = 5 - h$, when $x \to 5$, $h \to 0$

$$= \lim_{h \to 0} a(5-h)+b = \lim_{h \to 0} 5a - ah + b$$

L.H.L. $= 5a + b$ [Put $h = 0$]

Now, R.H.L. $= \lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} 3x - 2$

Put $x = 5 + h$, when $x \to 5$, $h \to 0$

$$= \lim_{h \to 0} 3(5+h)-2$$

$$= \lim_{h \to 0} 15 + 3h - 2$$

R.H.L. $= 13$ [Put $h = 0$]

$\therefore$ By using Eq. (ii), we get

L.H.L. = R.H.L.

$\therefore$ $5a + b = 13$ ... (iv)

Subtracting Eq. (iv) from Eq. (iii), we get

$$-3a = -9$$

or $a = 3$

Put $a = 3$ in Eq. (*iv*), we get

$$15 + b = 13$$

or $b = -2$

Hence, $a = 3$ and $b = -2$.

Date: 18/10/2021

Q1. If $f(2) = 4$ and $f'(2) = 1$, then, find

$$\lim_{x \to 2} \frac{xf(2) - 2f(x)}{x - 2}.$$

Q2. If $f(x)$ is differentiable at $x = a$, find

$$\lim_{x \to a} \frac{x^2 f(a) - a^2 f(x)}{x - a}.$$

Q3. For the function f given by $f(x) = x^2 - 6x + 8$, prove that $f'(5) - 3f'(2) = f'(8)$.

Q4. If $f(x) = x^2 + 2x + 7$, find $f'(3)$.

Q5. Show that the function $f(x) = |x - 3|$, $x \in R$, is continuous but not differentiable at $x = 3$

Q6. Show that $f(x) = |x|$ is not differentiable at $x = 0$.

Q7. Find $f'(2)$ and $f'(5)$ when $f(x) = x^2 + 7x + 4$.

Q8. What value of a and b is the function

$f(x) = \begin{cases} x^2 & , x \le c \\ ax + b & , x > c \end{cases}$ is differentiable at $x = c$.

Q9. If $f(x) = \begin{cases} x^2 + 3x + a & \text{, for } x \le 1 \\ bx + 2 & \text{, for } x > 1 \end{cases}$ is everywhere differentiable, find the values of a and b.

Q10. Discuss the differentiability of $f(x) = x\,|x|$ at $x = 0$.

Q11. Discuss the differentiability of

$f(x) = \begin{cases} xe^{-\left(\frac{1}{|x|} + \frac{1}{x}\right)} & , x \ne 0 \\ 0 & , x = 0 \end{cases}$ at x = 0.

Q12. Show that $f(x) = x^2$ is differentiable at $x = 1$ and find $f'(1)$.

Q13. Show that the function

$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & , \text{ if } x \ne 0 \\ 0 & , \text{ if } x = 0 \end{cases}$ is differentiable at $x = 0$ and $f'(0) = 0$.

Q14. Discuss the continuity and differentiability of

$f(x) = \begin{cases} 1 - x & , x < 1 \\ (1 - x)(2 - x) & , 1 \le x \le 2 \\ 3 - x & , x > 2 \end{cases}$ at x = 1, 2.

Q15. Discuss the differentiability of $f(x) = |x - 1| + |x - 2|$.

Q16. Discuss the continuity and differentiability of

$$f(x) = \begin{cases} 3x - 2, & 0 < x \le 1 \\ 2x^2 - x & 1 < x \le 2 \\ 5x - 4, & x > 2 \end{cases} \quad \text{at } x = 1, 2.$$

Q17. If $f(x) = |x|^3$, show that $f''(x)$ exists for all real x and find it.

Date: 18/10/2021

S1. We have,

$$\lim_{x\to 2}\frac{xf(2)-2f(x)}{x-2} = \lim_{x\to 2}\frac{xf(2)-2f(2)+2f(2)-2f(x)}{x-2}$$

$$= \lim_{x\to 2}\frac{(x-2)f(2)-2(f(x)-f(2))}{x-2}$$

$$= \lim_{x\to 2}\frac{(x-2)f(2)}{x-2} - 2\lim_{x\to 2}\frac{f(x)-f(2)}{x-2}$$

$$= f(2) - 2f'(2) \qquad \left[\because f'(2) = \lim_{x\to 2}\frac{f(x)-f(2)}{x-2}\right]$$

$$= 4 - 2\times 1 = 2. \qquad [\because f(2) = 4 \text{ and } f'(2) = 1]$$

S2. Since $f(x)$ is differentiable at $x = a$.

Therefore, $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ exists finitely.

As, $$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = f'(a) \qquad \text{... (i)}$$

Now, $$\lim_{x\to a}\frac{x^2f(a)-a^2f(x)}{x-a} = \lim_{x\to a}\frac{x^2f(a)-a^2f(a)+a^2f(a)-a^2f(x)}{x-a}$$

$$= \lim_{x\to a}\frac{(x^2-a^2)f(a)-a^2(f(x)-f(a))}{x-a}$$

$$= \lim_{x\to a}\left[\frac{(x^2-a^2)f(a)}{x-a} - a^2\left(\frac{f(x)-f(a)}{x-a}\right)\right]$$

$$= \lim_{x\to a}(x+a)f(a) - a^2\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

$$= 2af(a) - a^2f'(a) \qquad \text{[Using (i)]}$$

S3. Clearly, $f(x)$ being a polynomial function, is everywhere differentiable. The derivative $f'(x)$ is given by

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$\Rightarrow \quad f'(x) = \lim_{h\to 0} \frac{\{(x+h)^2 - 6(x+h) + 8\} - \{x^2 - 6x + 8\}}{h}$

$f'(x) = \lim_{h\to 0}\left[\frac{x^2 + h^2 + 2hx - 6x - 6h + 8 - x^2 + 6x - 8}{h}\right]$

$\Rightarrow \quad f'(x) = \lim_{h\to 0} \frac{2hx - 6h + h^2}{h} = \lim_{h\to 0}(2x - 6 + h) = 2x - 6$

$\therefore \quad f'(5) - 3f'(2) = (2 \times 5 - 6) - 3(2 \times 2 - 6) = 4 + 6 = 10$

and $\quad f'(8) = 2 \times 8 - 6 = 10$

Hence, $\quad f'(5) - 3f'(2) = f'(8)$

S4. We know that a polynomial function is everywhere differentiable. Therefore, $f(x)$ is differentiable at $x = 3$.

As $\quad f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$

$\therefore \quad f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h}$

$\Rightarrow \quad f'(3) = \lim_{h\to 0} \frac{\{(3+h)^2 + 2(3+h) + 7\} - \{9 + 6 + 7\}}{h}$

$f'(3) = \lim_{h\to 0}\left[\frac{9 + h^2 + 6h + 6 + 2h + 7 - 22}{h}\right]$

$\Rightarrow \quad f'(3) = \lim_{h\to 0} \frac{8h + h^2}{h} = \lim_{h\to 0}(8 + h) = 8.$

S5. Given, $f(x) = |x - 3|$

Firstly, we check the continuity of the given function $f(x)$ at $x = 3$

$\therefore \quad \text{LHL} = \lim_{x\to 3^-} |x - 3|$ [At $x = 3$]

$= \lim_{h\to 0} |3 - h - 3|$ $\left[\begin{array}{l}\because \text{ Put } x = 3 - h \\ \text{as } x \to 3,\ h \to 0\end{array}\right]$

$= \lim_{h\to 0} |-h| = 0$

$\text{RHL} = \lim_{x\to 3^+} |x - 3|$ [At $x = 3$]

$= \lim_{h\to 0} |3 + h - 3|$ $\left[\begin{array}{l}\because \text{ Put } x = 3 + h \\ \text{as } x \to 3,\ h \to 0\end{array}\right]$

$= \lim_{h\to 0} |h| = 0$

Since, LHL = RHL

$\because$ f is continuous at $x = 3$.

Now we check the differentiability of the given function $f(x)$ at $x = 3$

$$\text{LHD} = f'(3^-) = \lim_{h\to 0}\frac{f(3-h)-f(3)}{-h} \qquad \left[\because Lf'(a) = \lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}\right]$$

$$= \lim_{h\to 0}\frac{|3-h-3|-|3-3|}{-h}$$

$$= \lim_{h\to 0}\frac{|-h|}{-h}$$

$$= \lim_{h\to 0}\frac{h}{-h}$$

$$= -1$$

$$\text{RHD} = f'(3^+) = \lim_{h\to 0}\frac{f(3+h)-f(3)}{h} \qquad \left[\because Rf'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\right]$$

$$= \lim_{h\to 0}\frac{|3+h-3|-|3-3|}{h}$$

$$= \lim_{h\to 0}\frac{|h|}{h}$$

$$= \lim_{h\to 0}\frac{h}{h}$$

$$= 1$$

As LHD $\neq$ RHD $\Rightarrow$ f is not differentiable.

S6. We have, $\quad$ (LHD at $x = 0$) = $\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}$

put $x = 0 - h$, as $x \to 0$, $h \to 0$

$$= \lim_{h\to 0}\frac{f(0-h)-f(0)}{0-h-0}$$

$$= \lim_{h\to 0}\frac{f(-h)-f(0)}{-h}$$

$$= \lim_{h\to 0}\frac{|-h|-|0|}{-h} = \lim_{h\to 0}\frac{|-h|}{-h} = \lim_{h\to 0}\frac{h}{-h} = \lim_{h\to 0} -1 = -1$$

and $\qquad$ (RHD at $x = 0$) = $\lim_{h\to 0^+}\frac{f(x)-f(0)}{x-0}$

put $x = 0 + h$, as $x \to 0$, $h \to 0$

$$= \lim_{h\to0}\frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to0}\frac{f(h)-f(0)}{h} = \lim_{h\to0}\frac{|h|-|0|}{h} = \lim_{h\to0}\frac{h}{h} = \lim_{h\to0}1 = 1$$

$\therefore$ (LHD at $x = 0$) $\neq$ (RHD at $x = 0$)

So, $f(x)$ is not differentiable at $x = 0$.

S7. We know that a polynomial function is everywhere differentiable. Therefore, $f(x)$ is everywhere differentiable. The derivative $f'(x)$ is given by

$$f'(x) = \lim_{h\to0}\frac{f(x+h)-f(x)}{h}$$

$\Rightarrow$ $$f'(x) = \lim_{h\to0}\frac{\{(x+h)^2+7(x+h)+4\}-\{x^2+7x+4\}}{h}$$

$$f'(x) = \lim_{h\to0}\left[\frac{\cancel{x^2}+h^2+2hx+\cancel{7x}+7h+\cancel{4}-\cancel{x^2}-\cancel{7x}-\cancel{4}}{h}\right]$$

$\Rightarrow$ $$f'(x) = \lim_{h\to0}\frac{2hx+7h+h^2}{h} = \lim_{h\to0}(2x+7+h) = 2x+7$$

$\Rightarrow$ $f'(2) = 2\times2+7 = 11$

and $f'(5) = 2\times5+7 = 17.$

S8. It is given that $f(x)$ is differentiable at $x = c$ and every differentiable function is continuous. So, $f(x)$ is continuous at $x = c$.

$\therefore$ $$\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c)$$

$\Rightarrow$ $$\lim_{x\to c^-} x^2 = \lim_{x\to c^+}(ax+b) = c^2$$ [Using def. of $f(x)$]

$\Rightarrow$ $$c^2 = ac + b$$ ...(i)

Now, $f(x)$ is differentiable at $x = c$.

$\Rightarrow$ (LHD at $x = c$) = (RHD at $x = c$)

$\Rightarrow$ $$\lim_{x\to c^-}\frac{f(x)-f(c)}{x-c} = \lim_{x\to c^+}\frac{f(x)-f(c)}{x-c}$$

$\Rightarrow$ $$\lim_{x\to c}\frac{x^2-c^2}{x-c} = \lim_{x\to c}\frac{(ax+b)-c^2}{x-c}$$ [Using def. of $f(x)$]

$\Rightarrow$ $$\lim_{x\to c}\frac{x^2-c^2}{x-c} = \lim_{x\to c}\frac{ax+b-(ac+b)}{x-c}$$ [Using (i)]

$\Rightarrow \quad \lim_{x \to c}(x + c) = \lim_{x \to c} \frac{a(x - c)}{x - c}$

$\Rightarrow \quad \lim_{x \to c}(x + c) = \lim_{x \to c} a$

$\Rightarrow \quad 2c = a$ ...(ii)

From (i) and (ii), we get

$c^2 = 2c(c) + b$

$c^2 = 2c^2 + b$

$\Rightarrow \quad b = -c^2$

Hence, $a = 2c$ and $b = -c^2$.

S9. For $x \le 1$, we have $f(x) = x^2 + 3x + a$ *i.e.*, a polynomial

For $x > 1$, we have $f(x) = bx + 2$ which is also a polynomial

Since a polynomial function is everywhere differentiable. Therefore, $f(x)$ is differentiable for all $x > 1$ and also for all $x < 1$. Thus, we have to use the differentiability of $f(x)$ at $x = 1$.

Now, $f(x)$ is differentiable at $x = 1$

$\Rightarrow \quad f(x)$ is continuous at $x = 1$

$\Rightarrow \quad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$

$\Rightarrow \quad \lim_{x \to 1^-} x^2 + 3x + a = \lim_{x \to 1^+} bx + 2 = 1 + 3 + a$

$\Rightarrow \quad 1 + 3 + a = b + 2$

$\Rightarrow \quad a - b + 2 = 0$ ...(i)

Again, $f(x)$ is differentiable at $x = 1$

$\Rightarrow \quad$ (LHD at $x = 1$) = (RHD at $x = 1$)

$\Rightarrow \quad \lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1}$

$\Rightarrow \quad \lim_{x \to 1} \frac{x^2 + 3x + a - (4 + a)}{x - 1} = \lim_{x \to 1} \frac{(bx + 2) - (4 + a)}{x - 1}$

$\Rightarrow \quad \lim_{x \to 1} \frac{x^2 + 3x - 4}{x - 1} = \lim_{x \to 1} \frac{bx - 2 - a}{x - 1}$

$\Rightarrow \quad \lim_{x \to 1} \frac{(x + 4)(x - 1)}{x - 1} = \lim_{x \to 1} \frac{bx - b}{x - 1}$ [Using (i)]

$\Rightarrow \quad \lim_{x \to 1} (x + 4) = \lim_{x \to 1} b$

$\Rightarrow \qquad 5 = b$

Putting, $b = 5$ in (i), we get $a = 3$.

Hence, $a = 3$ and $b = 5$.

S10. We have,

$$f(x) = x\,|x| = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$$

Now, (LHD at $x = 0$) $= \lim\limits_{x \to 0^-} \dfrac{f(x) - f(0)}{x - 0}$

$= \lim\limits_{x \to 0} \dfrac{-x^2 - 0}{x - 0}$ [Using def. of $f(x)$]

$= \lim\limits_{x \to 0} (-x) = 0$

and (RHD at x = 0) $= \lim\limits_{x \to 0^+} \dfrac{f(x) - f(0)}{x - 0}$

$= \lim\limits_{x \to 0} \dfrac{x^2 - 0}{x - 0}$ [Using def. of $f(x)$]

$= \lim\limits_{x \to 0} x = 0$

$\therefore$ (LHD at $x = 0$) = (RHD at $x = 0$)

So, $f(x)$ is differentiable at $x = 0$.

S11. We have,

$$f(x) = \begin{cases} xe^{-\left(\frac{1}{x} + \frac{1}{x}\right)} = xe^{-2/x}, & x \geq 0 \\ xe^{-\left(-\frac{1}{x} + \frac{1}{x}\right)} = x, & x < 0 \\ 0, & x = 0 \end{cases}$$

Now, (LHD at $x = 0$) $= \lim\limits_{x \to 0^-} \dfrac{f(x) - f(0)}{x - 0}$

$= \lim\limits_{x \to 0} \dfrac{x - 0}{x - 0} = 1$ [$\because$ $f(x) = x$ for $x < 0$ and $f(0) = 0$]

and (RHD at $x = 0$) $= \lim\limits_{x \to 0^+} \dfrac{f(x) - f(0)}{x - 0}$

$= \lim\limits_{x \to 0} \dfrac{xe^{\frac{-2}{x}} - 0}{x}$ [$\because f(x) = xe^{\frac{-2}{x}}$ for $x > 0$ and $f(0) = 0$]

$$= \lim_{x \to 0} e^{\frac{-2}{x}} = 0 \qquad \left[\text{As } x \to 0, \frac{2}{x} \to \infty \Rightarrow e^{\frac{-2}{x}} = \frac{1}{e^{\frac{2}{x}}} \to 0\right]$$

$\therefore$ (LHD at $x = 0$) $\neq$ (RHD at $x = 0$)

So, $f(x)$ is not differentiable at $x = 0$.

S12. We have, (LHD at $x = 1$) $= \lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1}$

put $x = 1 - h$, as $x \to 1$, $h \to 0$

$$= \lim_{h \to 0} \frac{f(1-h) - f(1)}{1 - h - 1}$$

$$= \lim_{h \to 0} \frac{(1-h)^2 - 1^2}{-h} = \lim_{h \to 0} \frac{-2h + h^2}{-h} = \lim_{h \to 0} 2 - h = 2$$

and (RHD at $x = 1$) $= \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1}$

put $x = 1 + h$ as $x \to 1$, $h \to 0$

$$= \lim_{h \to 0} \frac{f(1+h) - f(1)}{1 + h - 1}$$

$$= \lim_{h \to 0} \frac{(1+h)^2 - 1}{h} = \lim_{h \to 0} \frac{2h + h^2}{h} = \lim_{h \to 0} 2 + h = 2$$

$\therefore$ (LHD at $x = 1$) = (RHD at $x = 1$) = 2.

So, $f(x)$ is differentiable at $x = 1$ and $f'(1) = 2$.

S13. We have, (LHD at $x = 0$) $= \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0}$

put $x = 0 - h$, as $x \to 0$, $h \to 0$

$$= \lim_{h \to 0} \frac{f(0-h) - f(0)}{0 - h - 0}$$

$$= \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{(-h)^2 \sin\left(\frac{1}{-h}\right) - 0}{-h}$$

$$= \lim_{h \to 0} h \sin\left(\frac{1}{h}\right)$$

= 0 × (an oscillating number between – 1 and 1) = 0

(RHD at $x = 0$) $= \lim_{h \to 0^+} \frac{f(x) - f(0)}{x - 0}$

put $x = 0 + h$, as $x \to 0$, $h \to 0$

$$= \lim_{h\to 0} \frac{f(0+h) - f(0)}{0+h-0}$$

$$= \lim_{h\to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h}$$

$$= \lim_{h\to 0} h \sin\left(\frac{1}{h}\right)$$

= 0 × (an oscillating number between – 1 and 1) = 0

$\therefore$ (LHD at $x = 0$) = (RHD at $x = 0$)

So, $f(x)$ is differentiable at $x = 0$ and $f'(0) = 0$.

S14. When $x < 1$, we have, $f(x) = 1 - x$

We know that a polynomial function is everywhere continuous and differentiable. So, $f(x)$ is continuous and differentiable for all $x < 1$.

Similarly, $f(x)$ is continuous and differentiable for all $x \in (1, 2)$ and $x > 2$.

Thus, the possible points where we have to check the continuity and differentiability of $f(x)$ are $x = 1$ and $x = 2$.

Continuity at $x = 1$:

We have, $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (1 - x)$ [$\because f(x) = 1 - x$ for $x < 1$]

$\Rightarrow \lim_{x\to 1^-} f(x) = 1 - 1 = 0$

$\Rightarrow \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (1 - x)(2 - x)$ [$\because f(x) = (1 - x)(2 - x)$, for $1 \le x \le 2$]

$\Rightarrow \lim_{x\to 1^+} f(x) = 0$

and, $f(1) = (1 - 1)(2 - 1) = 0$

$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$

So, $f(x)$ is continuous at $x = 1$.

Differentiability at $x = 1$:

We have, (LHD at $x = 1$) = $\lim_{x\to 1^-} \frac{f(x) - f(1)}{x - 1}$

$$= \lim_{x\to1} \frac{(1-x)-0}{x-1} \qquad \text{[Using def. of } f(x)\text{]}$$

$$= -\lim_{x\to1} \frac{x-1}{x-1} = -1$$

$$\text{(RHD at } x = 1) = \lim_{x\to1^+} \frac{f(x)-f(1)}{x-1}$$

$$= \lim_{x\to1} \frac{(1-x)(2-x)-0}{x-1} \qquad \text{[Using def. of } f(x)\text{]}$$

$$= \lim_{x\to1} \frac{(x-1)(x-2)}{x-1}$$

$$= \lim_{x\to1} x-2$$

$$= 1-2 = -1$$

$\therefore$ (LHD at $x = 1$) = (RHD at $x = 1$)

So, $f(x)$ is differentiable at $x = 1$.

Continuity at $x = 2$:

We have, $\lim_{x\to2^-} f(x) = \lim_{x\to2^-} (1-x)(2-x)$ [$\because$ $f(x) = (1-x)(2-x)$ for $1 \le x \le 2$]

$\Rightarrow$ $\lim_{x\to2^-} f(x) = (1-2)(2-2) = 0$

$\Rightarrow$ $\lim_{x\to2^+} f(x) = \lim_{x\to2^+} (3-x)$ [$\because$ $f(x) = 3-x$ for $x > 2$]

$= 3-2 = 1$

$\therefore$ $\lim_{x\to2^-} f(x) \neq \lim_{x\to2^+} f(x)$

So, $f(x)$ is not continuous at $x = 2$.

Differentiability at $x = 2$:

Since $f(x)$ is not continuous at $x = 2$. So, it is not differentiable at $x = 2$.

S15. We have, $f(x) = |x-1| + |x-2|$

$$\Rightarrow \quad f(x) = \begin{cases} -(x-1)-(x-2) \text{ for } x < 1 \\ x-1-(x-2) \text{ for } 1 \le x < 2 \\ (x-1)+(x-2) \text{ for } x \ge 2 \end{cases}$$

$$\Rightarrow \quad f(x) = \begin{cases} -2x+3, \; x < 1 \\ 1 \quad , 1 \le x < 2 \\ 2x-3, \; x \ge 2 \end{cases}$$

When $x < 1$, we have

$f(x) = -2x + 3$ which, being a polynomial function is continuous and differentiable for all $x < 1$.

When $1 \le x < 2$, we have

$f(x) = 1$ which, being a constant function is continuous and differentiable on $(1, 2)$

When $x \ge 2$, we have

$f(x) = 2x - 3$ which, being a polynomial function is differentiable for all $x > 2$.

Thus, the possible points on non-differentiability of $f(x)$ are $x = 1$ and $x = 2$.

Now, $$(\text{LHD at } x = 1) = \lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x \to 1} \frac{(-2x + 3) - 1}{x - 1} \qquad [\because f(x) = -2x + 3 \text{ for } x < 1]$$

$$= \lim_{x \to 1} \frac{-2(x - 1)}{x - 1} = -2$$

$$(\text{RHD at } x = 1) = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x \to 1} \frac{1 - 1}{x - 1} = 0 \qquad [\because f(x) = 1 \text{ for } 1 \le x < 2]$$

$\therefore$ (LHD at $x = 1$) $\ne$ (RHD at $x = 1$)

So, $f(x)$ is not differentiable at $x = 1$.

$$(\text{LHD at } x = 2) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2}$$

$$\Rightarrow (\text{LHD at } x = 2) = \lim_{x \to 2} \frac{1 - (2 \times 2 - 3)}{x - 2} \qquad [\because f(x) = 1 \text{ for } 1 \le x < 2 \text{ and } f(2) = 2 \times 2 - 3]$$

$$\Rightarrow (\text{LHD at } x = 2) = \lim_{x \to 2} \frac{1 - 1}{x - 2} = 0$$

$$(\text{RHD at } x = 2) = \lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2}$$

$$\Rightarrow (\text{RHD at } x = 2) = \lim_{x \to 2} \frac{(2x - 3) - (2 \times 2 - 3)}{x - 2} \qquad [\because f(x) = 2x - 3 \text{ for } x \ge 2]$$

$$\Rightarrow (\text{RHD at } x = 2) = \lim_{x \to 2} \frac{2x - 4}{x - 2}$$

$$= \lim_{x \to 2} \frac{2(x - 2)}{x - 2} = 2$$

$\therefore$ (LHD at $x = 2$) $\ne$ (RHD at $x = 2$)

So, $f(x)$ is not differentiable at $x = 2$.

Remarks: It should be noted that the function $f(x)$ given by

$$f(x) = |x - a_1| + |x - a_2| + |x - a_3| + \ldots + |x - a_n|$$

is not differentiable at $x = a_1, a_2, a_3, \ldots, a_n$.

S16. Firstly, we will check the continuity of the given function at $x = 1, 2$ by L.H.L. = R.H.L. = $f(x)$ and then we will check the differentiability of that function $f(x)$ at these points by L.H.D. = R.H.D. If L.H.D. $\neq$ R.H.D., then function is not differentiable.

The given function is $f(x) = \begin{cases} 3x - 2, & 0 < x \le 1 \\ 2x^2 - x & 1 < x \le 2 \\ 5x - 4, & x > 2 \end{cases}$

First we show the continuity of above function at $x = 1$ and $x = 2$.

Continuity at $x = 1$:

$$\text{L.H.L.} = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (3x - 2)$$

Put $x = 1 - h$, when $x \to 1$, $h \to 0$

$$= \lim_{h \to 0} 3(1 - h) - 2$$

$$= \lim_{h \to 0} 3 - 3h - 2$$

$$= 1 \qquad \text{[Put } h = 0\text{]}$$

Now, $\quad \text{R.H.L.} = \lim_{h \to 1^+} f(x) = \lim_{x \to 1^+} (2x^2 - x)$

Put $x = 1 + h$, when $x \to 1$, $h \to 0$

$$= \lim_{h \to 0} 2(1 + h)^2 - (1 + h)$$

$$= \lim_{h \to 0} 2(1 + h^2 + 2h) - (1 + h)$$

$$= \lim_{h \to 0} 2 + 2h^2 + 4h - 1 - h$$

$$= \lim_{h \to 0} 2h^2 + 3h + 1$$

$$\text{R.H.L.} = 1 \qquad \text{[Put } h = 0\text{]}$$

Also, from the function, we get

$$f(1) = 3(1) - 2 = 3 - 2 = 1$$

Since, L.H.L. = R.H.L. = $f(1)$

$\therefore f(x)$ is continuous at $x = 1$.

Continuity at $x = 2$:

$$\text{L.H.L.} = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x^2 - x)$$

Put $x = 2 - h$, when $x \to 2$, $h \to 0$

$$= \lim_{h \to 0} 2(2-h)^2 - (2-h)$$

$$= \lim_{h \to 0} 2(4 + h^2 - 4h) - (2-h)$$

$$= \lim_{h \to 0} 8 + 2h^2 - 8h - 2 + h$$

L.H.L. = 8 – 2 = 6 [Put $h = 0$]

Now, $$\text{R.H.L.} = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (5x - 4)$$

Put $x = 2 + h$, when $x \to 2$, $h \to 0$

$$= \lim_{h \to 0} 5(2+h) - 4$$

$$= \lim_{h \to 0} 10 + 5h - 4$$

$$= \lim_{h \to 0} 5h + 6$$ [Put $h = 0$]

R.H.L. = 6

Also, from the given function, at $x = 2$

$f(2) = 2(2)^2 - 2$ [for $f(2)$, put $x = 2$ in $f(x) = 2x^2 - x$]

$= 8 - 2 = 6$

$\because$ L.H.L. = R.H.L. = $f(2)$

$\therefore$ $f(x)$ is continuous at $x = 2$

$\therefore$ $f(x)$ is continuous at all indicated points.

Now, we verify differentiability of the given function at $x = 1$ and $x = 2$.

We know that a function $f(x)$ is said to be differentiable at the point $x = a$, if

LHD = RHD at $x = a$ $\begin{bmatrix} \text{LHD} = \text{Left hand derivative} \\ \text{RHD} = \text{Right hand derivative} \end{bmatrix}$

where $$\text{LHD} = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}$$

and $$\text{RHD} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Differentiability at $x = 1$:

$$\text{LHD} = \lim_{h\to 0} \frac{f(1-h)-f(1)}{-h}$$

$$= \lim_{h\to 0} \frac{[3(1-h)-2]-[3-2]}{-h}$$

$$= \lim_{h\to 0} \frac{(1-3h)-(1)}{-h} = \lim_{h\to 0} \frac{-3h}{-h}$$

$\therefore$ LHD = 3

Now, $$\text{RHD} = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0} \frac{[2\,(1+h)^2-(1+h)]-[3-2]}{h}$$

$$= \lim_{h\to 0} \frac{[2+2h^2+4h-1-h]-1}{h}$$

$$= \lim_{h\to 0} \frac{2h^2+3h}{h}$$

$$= \lim_{h\to 0} 2h+3 \qquad \text{[Put } h = 0]$$

RHD = 3

LHD = RHD

$\Rightarrow f(x)$ is differentiable at $x = 1$.

Differentiability at $x = 2$:

$$\text{LHD} = \lim_{h\to 0} \frac{f(2-h)-f(2)}{-h}$$

$$\Rightarrow \text{LHD} = \lim_{h\to 0} \frac{[2(2-h)^2-(2-h)]-[8-2]}{-h}$$

$$= \lim_{h\to 0} \frac{2(4+h^2-4h)-(2-h)-6}{-h}$$

$$= \lim_{h\to 0} \frac{h(2h-7)}{-h}$$

$\therefore$ LHD = 7 [Put $h = 0$]

Now, $$\text{RHD} = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to 0} \frac{[5\,(2+h)-4]-[8-2]}{h}$$

$$= \lim_{h\to 0} \frac{(6+5h)-(6)}{h}$$

$$= \lim_{h \to 0} \frac{5h}{h} \qquad \text{[Put } h = 0\text{]}$$

$\therefore$ RHD = 5

$\because$ LHD $\neq$ RHD

So, $f(x)$ is not differentiable at $x = 2$.

Hence, $f(x)$ is continuous at $x = 1$ and $x = 2$ but not differentiable $x = 2$.

S17. We have,

$$f(x) = |x|^3 = \begin{cases} (-x)^3 = -x^3 & \text{, if } x < 0 \\ x^3 & \text{, if } x \geq 0 \end{cases}$$

for $x < 0$, $f(x) = -x^3$

L.H.D. at x = 0 $\quad f'(x) = \lim_{h \to 0} \frac{f(x-h) - f(x)}{x - h - x}$

$$= \lim_{h \to 0} \frac{-(x-h)^3 - (-x^3)}{-h}$$

$$= \lim_{h \to 0} \frac{(x-h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^3} - h^3 - 3xh(x-h) - \cancel{x^3}}{h}$$

$$= \lim_{h \to 0} \frac{-\cancel{h}(h^2 + 3x(x-h))}{\cancel{h}}$$

$$= -(0 + 3x(x-0)) = -3x^2$$

for $x \geq 0$, $\quad f'(x) = x^3$

R.H.D. at x = 0 $\quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{x + h - x}$

$$= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^3} + h^3 + 3xh(x+h) - \cancel{x^3}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}[h^2 + 3x(x+h)]}{\cancel{h}}$$

$$= 0 + 3x(x+0) = 3x^2$$

Hence, $\quad f'(x) = \begin{cases} -3x^2 & \text{, if } x < 0 \\ 3x^2 & \text{, if } x \geq 0 \end{cases}$

for $x < 0$, $f'(x) = -3x^2$

$$\text{L.H.D. of } f'(x) = \lim_{h\to 0}\frac{f'(x-h)-f'(x)}{x-h-x}$$

$$= \lim_{h\to 0}\frac{-3(x-h)^2-(-3x^2)}{-h}$$

$$= 3\lim_{h\to 0}\frac{(x-h)^2-x^2}{h}$$

$$= 3\lim_{h\to 0}\frac{\cancel{x^2}+h^2-2hx-\cancel{x^2}}{h}$$

for $x > 0$, $f'(x) = 3x^2$

$$\text{R.H.D. of } f'(x) = \lim_{h\to 0}\frac{f'(x+h)-f'(x)}{x+h-x}$$

$$= \lim_{h\to 0}\frac{3(x+h)^2-(3x^2)}{h}$$

$$= 3\lim_{h\to 0}\frac{\cancel{x^2}+h^2+2xh-\cancel{x^2}}{h}$$

$$= 3\lim_{h\to 0}\frac{\cancel{h}(h+2x)}{\cancel{h}} = 3(0+2x) = 6x$$

Hence, $$f''(x) = \begin{cases}-6x, & \text{if } x<0\\ 6x, & \text{if } x\ge 0\end{cases}.$$

Date: 18/10/2021

Q1. If $y = \cos^{-1}\left(\dfrac{1-x^{2n}}{1+x^{2n}}\right)$ find $\dfrac{dy}{dx}$

Q2. If $y = \sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) + \sec^{-1}\left(\dfrac{1+x^2}{1-x^2}\right)$ find $\dfrac{dy}{dx}$

Q3. If $y = \tan^{-1}\sqrt{\dfrac{1+\cos x}{1-\cos x}}$ find $\dfrac{dy}{dx}$

Q4. If $y = \tan^{-1}\sqrt{\dfrac{1-\cos x}{1+\cos x}}$ find $\dfrac{dy}{dx}$

Q5. If $y = \cos^{-1}\left(\dfrac{\cos x + \sin x}{\sqrt{2}}\right)$ find $\dfrac{dy}{dx}$

Q6. If $y = \tan^{-1}\left(\dfrac{\cos x}{1+\sin x}\right)$ find $\dfrac{dy}{dx}$

Q7. If $y = \tan^{-1}\left\{\sqrt{\dfrac{1+\sin x}{1-\sin x}}\right\}$ find $\dfrac{dy}{dx}$

Q8. If $y = \sin^{-1}\left(\sqrt{\dfrac{1}{1+x^2}}\right)$ find $\dfrac{dy}{dx}$

Q9. If $y = \sin^{-1}\left(\dfrac{2^{x+1}}{1+4^x}\right)$ find $\dfrac{dy}{dx}$

Q10. If $y = \sin^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right) + \cos^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right)$ find $\dfrac{dy}{dx}$

Q11. If $y = \sin\left[2\tan^{-1}\left\{\sqrt{\dfrac{1-x}{1+x}}\right\}\right]$ find $\dfrac{dy}{dx}$

Q12. If $y = \tan^{-1}\left\{\sqrt{1+x^2} + x\right\}$ find $\dfrac{dy}{dx}$

Q13. If $y = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) + \sec^{-1}\left(\dfrac{1+x^2}{1-x^2}\right)$ find $\dfrac{dy}{dx}$

Q14. If $y = \sin^{-1}\left\{x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}\right\}$ find $\dfrac{dy}{dx}$

Q15. If $y = \tan^{-1}\left(\dfrac{3a^2x - x^3}{a^3 - 3ax^2}\right)$ find $\dfrac{dy}{dx}$

Q16. If $y = \tan^{-1}\left\{\sqrt{1+x^2} - x\right\}$ find $\dfrac{dy}{dx}$

Q17. If $y = \sin^{-1}\left[\dfrac{2^{x+1}\cdot 3^x}{1+(36)^x}\right]$. Find $\dfrac{dy}{dx}$

Q18. If $y = \sin^{-1}\left[\dfrac{5x + 12\sqrt{1-x^2}}{13}\right]$ find $\dfrac{dy}{dx}$.

Q19. If $y = \cot^{-1}\left[\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$, find $\dfrac{dy}{dx}$.

Q20. If $y = \cos^{-1}\left(\dfrac{3x+4\sqrt{1-x^2}}{5}\right)$, find $\dfrac{dy}{dx}$.

Q21. If $y = \tan^{-1}(\sec x + \tan x)$ find $\dfrac{dy}{dx}$.

Q22. If $y = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$ find $\dfrac{dy}{dx}$

Q23. If $y = \cos^{-1}\left(\dfrac{x}{\sqrt{x^2+a^2}}\right)$ find $\dfrac{dy}{dx}$

Date: 18/10/2021

S1.

$$y = \cos^{-1}\left(\frac{1-x^{2n}}{1+x^{2n}}\right)$$

Put $x^n = \tan\theta$

$$y = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$= \cos^{-1}\cos 2\theta$$

$$= 2\theta = 2\tan^{-1}x^n$$

$\therefore$ $$\frac{dy}{dx} = \frac{2}{1+x^{2n}}\frac{d}{dx}x^n$$

$$= \frac{2nx^{n-1}}{1+x^{2n}}.$$

S2.

$$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)+\sec^{-1}\left(\frac{1+x^2}{1-x^2}\right)$$

$$= \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)+\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$= \frac{\pi}{2} \qquad \left\{\because \sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}\right\}$$

$\therefore$ $$\frac{dy}{dx} = 0.$$

S3.

$$y = \tan^{-1}\sqrt{\frac{1+\cos x}{1-\cos x}}$$

$$= \tan^{-1}\sqrt{\frac{2\cos^2\frac{x}{2}}{2\sin^2\frac{x}{2}}}$$

$$= \tan^{-1}\left(\cot\frac{x}{2}\right)$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{2}-\frac{x}{2}\right)\right\}$$

$$= \frac{\pi}{2} - \frac{x}{2}$$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{2}.$$

S4.

$$y = \tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$$

$$= \tan^{-1}\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}$$

$$= \tan^{-1}\left(\tan\frac{x}{2}\right)$$

$$= \frac{x}{2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}.$$

S5.

$$y = \cos^{-1}\left(\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x\right)$$

$$= \cos^{-1}\left(\cos x\cos\frac{\pi}{4} + \sin x\sin\frac{\pi}{4}\right)$$

$$= \cos^{-1}\left(\cos\left(x - \frac{\pi}{4}\right)\right)$$

$$= x - \frac{\pi}{4}$$

$$\therefore \quad \frac{dy}{dx} = 1.$$

S6.

$$y = \tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$$

$$= \tan^{-1}\left\{\frac{\sin\left(\frac{\pi}{2}+x\right)}{1-\cos\left(\frac{\pi}{2}+x\right)}\right\}$$

$$= \tan^{-1}\left\{\frac{\cancel{2}\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}{\cancel{2}\sin^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}\right\}$$

$$= \tan^{-1}\left\{\cot\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}$$

$$= \tan^{-1}\left\{\tan\frac{\pi}{2}-\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\}$$

$$= \frac{\pi}{4}-\frac{x}{2}$$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{2}.$$

S7.

$$y = \tan^{-1}\left\{\sqrt{\frac{1+\sin x}{1-\sin x}}\right\}$$

$$= \tan^{-1}\left\{\sqrt{\frac{1+\cos\left(\frac{\pi}{2}-x\right)}{1-\cos\left(\frac{\pi}{2}-x\right)}}\right\}$$

$$= \tan^{-1}\left\{\sqrt{\frac{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\sin^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}}\right\}$$

$$= \tan^{-1}\left\{\cot\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\}$$

$$= \tan^{-1}\left\{\tan\frac{\pi}{2}-\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\}$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}$$

$$= \frac{\pi}{4}+\frac{x}{2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}.$$

S8.
$$y = \sin^{-1}\left(\sqrt{\frac{1}{1+x^2}}\right)$$

Put $x = \tan\theta$

$$y = \sin^{-1}\left(\sqrt{\frac{1}{1+\tan^2\theta}}\right)$$

$$= \sin^{-1}\left(\sqrt{\frac{1}{\sec^2\theta}}\right)$$

$$= \sin^{-1}(\cos\theta)$$

$$= \sin^{-1}\left\{\sin\left(\frac{\pi}{2}-\theta\right)\right\}$$

$$= \frac{\pi}{2}-\theta = \frac{\pi}{2}-\tan^{-1}x$$

$\therefore$
$$\frac{dy}{dx} = \frac{-1}{1+x^2}.$$

S9.
$$y = \sin^{-1}\left(\frac{2\cdot 2^x}{1+(2^x)^2}\right)$$

Put $2^x = \tan\theta$

$$y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$= \sin^{-1}(\sin 2\theta)$$

$$= 2\theta = 2\tan^{-1}(2^x)$$

$\therefore$
$$\frac{dy}{dx} = \frac{2}{1+(2^x)^2}\frac{d}{dx}2^x$$

$$= \frac{2\cdot 2^x \log 2}{1+4^x}.$$

S10.
$$y = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)+\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$$

Put $x = \tan\theta$

$$y = \sin^{-1}\left(\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}\right) + \cos^{-1}\left(\frac{1}{\sqrt{1+\tan^2\theta}}\right)$$

$$= \sin^{-1}\left(\frac{\tan\theta}{\sqrt{\sec^2\theta}}\right) + \cos^{-1}\left(\frac{1}{\sqrt{\sec^2\theta}}\right)$$

$$= \sin^{-1}(\tan\theta \,.\, \cos\theta) + \cos^{-1}(\cos\theta)$$

$$= \sin^{-1}\left(\frac{\sin\theta}{\cos\theta}\cdot\cos\theta\right) + \cos^{-1}(\cos\theta)$$

$$= \sin^{-1}(\sin\theta) + \cos^{-1}(\cos\theta)$$

$$= \theta + \theta = 2\theta$$

$$= 2\tan^{-1}x$$

$\therefore$ $$\frac{dy}{dx} = \frac{2}{1+x^2}.$$

S11. $$y = \sin\left[2\tan^{-1}\left\{\sqrt{\frac{1-x}{1+x}}\right\}\right]$$

Put $x = \cos 2\theta$

$$= \sin\left[2\tan^{-1}\left\{\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}\right\}\right]$$

$$= \sin\left[2\tan^{-1}\left\{\sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}}\right\}\right]$$

$$= \sin\left[2\tan^{-1}\tan\theta\right]$$

$$= \sin 2\theta$$

$$= \sqrt{1-\cos^2 2\theta} = \sqrt{1-x^2}$$

$\therefore$ $$\frac{dy}{dx} = -\frac{1}{2\sqrt{1-x^2}}\frac{d}{dx}x^2$$

$$= \frac{-2x}{2\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}.$$

S12. $y = \tan^{-1}\left\{\sqrt{1+x^2} + x\right\}$

Put $x = \cot\theta$

$$y = \tan^{-1}\left\{\sqrt{1+\cot^2\theta} + \cot\theta\right\}$$

$$= \tan^{-1}\{\operatorname{cosec}\theta + \cot\theta\}$$

$$= \tan^{-1}\left\{\frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}\right\}$$

$$= \tan^{-1}\left\{\frac{1+\cos\theta}{\sin\theta}\right\}$$

$$= \tan^{-1}\left\{\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right\}$$

$$= \tan^{-1}\cot\frac{\theta}{2}$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right\}$$

$$= \frac{\pi}{2} - \frac{\theta}{2}$$

$$= \frac{\pi}{2} - \frac{1}{2}\cot^{-1}x$$

$\therefore$ $$\frac{dy}{dx} = 0 - \frac{1}{2}\left(-\frac{1}{1+x^2}\right) = \frac{1}{2(1+x^2)}.$$

S13. $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right) + \sec^{-1}\left(\frac{1+x^2}{1-x^2}\right)$

$$= \sin^{-1}\left(\frac{2x}{1+x^2}\right) + \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

Put $x = \tan\theta$

$$= \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) + \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$= \sin^{-1}(\sin 2\theta) + \cos^{-1}(\cos 2\theta)$$

$$= 2\theta + 2\theta$$

$$= 4\theta = 4\tan^{-1} x$$

$\therefore$ $$\frac{dy}{dx} = \frac{4}{1+x^2}.$$

S14. $$y = \sin^{-1}\left\{x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}\right\}$$

$$= \sin^{-1}\left\{x\sqrt{1-(\sqrt{x})^2} - \sqrt{x}\sqrt{1-x^2}\right\}$$

$$= \sin^{-1} x - \sin^{-1}\sqrt{x}$$

$\therefore$ $$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-(\sqrt{x})^2}}\frac{d}{dx}\sqrt{x}$$

$$= \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x}} \times \frac{1}{2\sqrt{x}}\ .$$

S15. $$y = \tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right)$$

$$= \tan^{-1}\left(\frac{\frac{3x}{a} - \left(\frac{x}{a}\right)^3}{1 - 3\left(\frac{x}{a}\right)^2}\right)$$

Put $$\frac{x}{a} = \tan\theta$$

$$= \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 3\theta)$$

$$= 3\theta = 3\tan^{-1}\left(\frac{x}{a}\right)$$

$\therefore$ $$\frac{dy}{dx} = \frac{3}{1+\frac{x^2}{a^2}} \cdot \frac{d}{dx}\left(\frac{x}{a}\right)$$

$$= \frac{3a^2}{a^2+x^2} \cdot \frac{1}{a} = \frac{3a}{a^2+x^2}.$$

S16. $y = \tan^{-1}\left\{\sqrt{1+x^2} - x\right\}$

Put $x = \cot\theta$

$$y = \tan^{-1}\left\{\sqrt{1+\cot^2\theta} - \cot\theta\right\}$$

$$= \tan^{-1}\{\operatorname{cosec}\theta - \cot\theta\}$$

$$= \tan^{-1}\left\{\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right\}$$

$$= \tan^{-1}\left\{\frac{1-\cos\theta}{\sin\theta}\right\}$$

$$= \tan^{-1}\left\{\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right\}$$

$$= \tan^{-1}\tan\frac{\theta}{2}$$

$$= \frac{1}{2}\theta = \frac{1}{2}\cot^{-1}x$$

$\therefore$ $$\frac{dy}{dx} = -\frac{1}{2(1+x^2)}.$$

S17. $y = \sin^{-1}\left[\frac{2^{x+1}\cdot 3^x}{1+(36)^x}\right] = \sin^{-1}\left[\frac{2\cdot 2^x\cdot 3^x}{1+4^x\cdot 9^x}\right]$

$$= \sin^{-1}\left[\frac{2\cdot 6^x}{1+(6^x)^2}\right]$$

Put $\tan\theta = 6^x \Rightarrow \theta = \tan^{-1}(6^x)$

$\therefore$ $$y = \sin^{-1}\left(\frac{2\cdot\tan\theta}{1+\tan^2\theta}\right)$$

$$= \sin^{-1}(\sin 2\theta) \qquad \left[\because \quad \sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}\right]$$

$$= 2\theta$$

$\Rightarrow$ $$y = 2\tan^{-1}(6^x)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{2}{1+(6^x)^2}\frac{d}{dx}(6^x) = \left[\frac{2\cdot 6^x}{1+(36^x)}\right]\log 6 \qquad \left[\because \quad \frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}\right]$$

$$\Rightarrow \frac{d}{dx}\left[\sin^{-1}\left(\frac{2^{x+1}\cdot 3^x}{1+(36)^x}\right)\right] = \left[\frac{2^{x+1}\cdot 3^x}{1+(36)^x}\right]\log 6.$$

S18. Given that,

$$y = \sin^{-1}\left[\frac{5x+12\sqrt{1-x^2}}{13}\right]$$

Put $x = \sin\theta$

So that $\theta = \sin^{-1} x$

$\therefore$ We get

$$y = \sin^{-1}\left[\frac{5\sin\theta+12\sqrt{1-\sin^2\theta}}{13}\right]$$

$$\Rightarrow \qquad y = \sin^{-1}\left[\frac{5\sin\theta+12\cos\theta}{13}\right] \qquad \left[\because \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta\right]$$

Now, let $\frac{5}{13} = \cos\alpha$ and $\frac{12}{13} = \sin\alpha$ $\qquad \left[\because \cos^2\alpha+\sin^2\alpha = \frac{25}{169}+\frac{144}{169} = \frac{169}{169} = 1\right]$

$$\therefore \qquad y = \sin^{-1}[\sin\theta\cos\alpha+\cos\theta\sin\alpha]$$

$$\Rightarrow \qquad y = \sin^{-1}\sin(\theta+\alpha) \qquad [\because \sin x\cos y + \cos x\sin y = \sin(x+y)]$$

$$\Rightarrow \qquad y = \theta+\alpha \qquad [\because \sin^{-1}\sin x = x]$$

$$\Rightarrow \qquad y = (\sin^{-1} x)+\alpha \qquad [\because \theta = \sin^{-1} x]$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}+0$$

Hence, $\qquad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$

S19. Given that

$$y = \cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$$

Now, we write

$$1 + \sin x = \cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}$$ $\left[\because \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \text{ and } 1 = \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right]$

and $$1 - \sin x = \cos^2\frac{x}{2} + \sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}$$

So, the given function becomes

$$y = \cot^{-1}\left[\frac{\sqrt{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}} + \sqrt{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}}}{\sqrt{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}} - \sqrt{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}}}\right]$$

$$y = \cot^{-1}\left[\frac{\sqrt{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2} + \sqrt{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2}}{\sqrt{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2} - \sqrt{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2}}\right]$$

$$y = \cot^{-1}\left[\frac{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right) + \left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right) - \left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)}\right]$$ $\left[\because a^2 + b^2 + 2ab = (a+b)^2 \text{ and } a^2 + b^2 - 2ab = (a-b)^2 \text{ and here } a = \cos\frac{x}{2}, b = \sin\frac{x}{2}\right]$

$\Rightarrow$ $$y = \cot^{-1}\left[\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right]$$

$\Rightarrow$ $$y = \cot^{-1}\cot\frac{x}{2}$$ $\left[\because \cot\theta = \frac{\cos\theta}{\sin\theta}\right]$

$\Rightarrow$ $$y = \frac{x}{2}$$ $[\because \cot^{-1}(\cot\theta) = \theta]$

Differentiating on both sides w.r.t. x, we get

$\therefore$ $$\frac{dy}{dx} = \frac{1}{2}.$$

S20. Given that, $y = \cos^{-1}\left[\frac{3x + 4\sqrt{1-x^2}}{5}\right]$

Put $x = \sin\theta \Rightarrow \theta = \sin^{-1} x$

$\therefore \quad y = \cos^{-1}\left[\frac{3\sin\theta + 4\sqrt{1-\sin^2\theta}}{5}\right]$

$\Rightarrow \quad y = \cos^{-1}\left[\frac{3\sin\theta + 4\cos\theta}{5}\right] \qquad [\because \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta]$

$\Rightarrow \quad y = \cos^{-1}\left[\frac{3}{5}\sin\theta + \frac{4}{5}\cos\theta\right]$

Now, let $\cos\alpha = \frac{4}{5}$ and $\sin\alpha = \frac{3}{5}$

$\because \quad \sin^2\alpha + \cos^2\alpha = \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2$

$= \frac{9}{25} + \frac{16}{25} = \frac{25}{25} = 1 \qquad [\because \sin^2\theta + \cos^2\theta = 1 \; \forall \theta \in R]$

$\therefore$ We get, $\quad y = \cos^{-1}(\sin\theta\sin\alpha + \cos\theta\cos\alpha)$

$\Rightarrow \quad y = \cos^{-1}\cos(\theta - \alpha) \qquad [\because \cos\theta\cos\alpha + \sin\theta\sin\alpha = \cos(\theta - \alpha)]$

$\Rightarrow \quad y = \theta - \alpha$

$\Rightarrow \quad y = \sin^{-1}x - \alpha \qquad [\because \theta = \sin^{-1}x]$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - 0 \qquad \left[\because \frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}\right]$$

Hence, $\quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$

S21. $\quad y = \tan^{-1}(\sec x + \tan x)$

$$= \tan^{-1}\left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right)$$

$$= \tan^{-1}\left(\frac{1+\sin x}{\cos x}\right)$$

$$= \tan^{-1}\left\{\frac{1+\cos\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right)}\right\}$$

$$= \tan^{-1}\left\{\frac{2\cos^2\left(\frac{\pi}{4} - \frac{x}{2}\right)}{2\sin\left(\frac{\pi}{4} - \frac{x}{2}\right)\cos\left(\frac{\pi}{4} - \frac{x}{2}\right)}\right\}$$

$$= \tan^{-1} \cot\left(\frac{\pi}{4} - \frac{x}{2}\right)$$

$$= \tan^{-1} \tan\left(\frac{\pi}{2} - \left(\frac{\pi}{4} - \frac{x}{2}\right)\right)$$

$$= \tan^{-1} \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$$

$$= \frac{\pi}{4} + \frac{x}{2}$$

$\therefore$ $$\frac{dy}{dx} = \frac{1}{2}.$$

S22. $$y = \tan^{-1}\left(\frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}}\right)$$

Put $x^2 = \cos 2\theta$

$$y = \tan^{-1}\left\{\frac{\sqrt{1+\cos 2\theta} + \sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta} - \sqrt{1-\cos 2\theta}}\right\}$$

$$= \tan^{-1}\left\{\frac{\sqrt{2\cos^2\theta} + \sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta} - \sqrt{2\sin^2\theta}}\right\}$$

$$= \tan^{-1}\left\{\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right\}$$

$$= \tan^{-1}\left\{\frac{1+\tan\theta}{1-\tan\theta}\right\}$$

$$= \tan^{-1}\left\{\frac{\tan\frac{\pi}{4} + \tan\theta}{1 - \tan\frac{\pi}{4}\tan\theta}\right\}$$

$$= \tan^{-1}\tan\left(\frac{\pi}{4} + \theta\right)$$

$$= \frac{\pi}{4} + \theta = \frac{\pi}{4} + \frac{1}{2}\cos^{-1} x^2$$

$\therefore$ $$\frac{dy}{dx} = \frac{1}{2}\frac{-1}{\sqrt{1-x^4}} \cdot \frac{d}{dx}(x^2) = \frac{-x}{\sqrt{1-x^4}}.$$

S23.

$$y = \cos^{-1}\left(\frac{x}{\sqrt{x^2+a^2}}\right)$$

Put $x = a\tan\theta$

$$y = \cos^{-1}\left(\frac{a\tan\theta}{\sqrt{a^2\tan^2\theta + a^2}}\right)$$

$$= \cos^{-1}\left(\frac{a\tan\theta}{\sqrt{a^2\sec^2\theta}}\right)$$

$$= \cos^{-1}\left(\frac{\tan\theta}{\sec\theta}\right)$$

$$= \cos^{-1}\left(\frac{\sin\theta}{\cos\theta\cdot\sec\theta}\right)$$

$$= \cos^{-1}\sin\theta$$

$$= \cos^{-1}\left[\cos\left(\frac{\pi}{2}-\theta\right)\right]$$

$$= \frac{\pi}{2}-\theta = \frac{\pi}{2} - \tan^{-1}\left(\frac{x}{a}\right)$$

$\therefore$

$$\frac{dy}{dx} = 0 - \frac{1}{1+\left(\frac{x}{a}\right)^2}\frac{d}{dx}\left(\frac{x}{a}\right)$$

$$= -\frac{1}{\frac{a^2+x^2}{a^2}}\cdot\frac{1}{a}$$

$$= -\frac{a}{x^2+a^2}.$$

Date: 18/10/2021

Q1. If $y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \ldots\infty}}}$, prove that $(2y-1)\frac{dy}{dx} = \frac{1}{x}$

Q2. If $y = \sqrt{x + \sqrt{x + \sqrt{x + \ldots\infty}}}$, prove that $\frac{dy}{dx} = \frac{1}{2y-1}$

Q3. If $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \ldots\infty}}}$, prove that $\frac{dy}{dx} = \frac{\sec^2 x}{2y-1}$

Q4. If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots. to\ \infty}}}$, prove that $\frac{dy}{dx} = \frac{\cos x}{2y-1}$

Q5. If $y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \ldots\infty}}}$, prove that $\frac{dy}{dx} = \frac{\sin x}{1-2y}$

Q6. If $y = (\sqrt{x})^{(\sqrt{x})^{(\sqrt{x})^{\cdots\infty}}}$, show that $\frac{dy}{dx} = \frac{y^2}{x(2 - y\log x)}$.

Q7. If $y = e^{x+e^{x+\ldots\text{to}\,\infty}}$, show that $\frac{dy}{dx} = \frac{y}{1-y}$

Q8. If $y = a^{x^{a^{x\ldots\infty}}}$, prove that $\frac{dy}{dx} = \frac{y^2 \log y}{x(1 - y\log x \cdot \log y)}$

Q9. If $y = x^{x^{x^{\cdots\infty}}}$, find $\frac{dy}{dx}$

Q10. If $y = x + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \ldots}}}$, prove that $\frac{dy}{dx} = \frac{y}{2y - x}$

Q11. If $y = \cfrac{\sin x}{1 + \cfrac{\cos x}{1 + \cfrac{\sin x}{1 + \cfrac{\cos x}{1 + \ldots \text{to } \infty}}}}$, prove that $\frac{dy}{dx} = \frac{(1+y)\cos x + y\sin x}{1 + 2y + \cos x - \sin x}$

Q12. If $y = (\sin x)^{(\sin x)^{(\sin x)\ldots\infty}}$, prove that $\frac{dy}{dx} = \frac{y^2 \cot x}{1 - y\log \sin x}$

Q13. If $y = (\tan x)^{(\tan x)^{(\tan x)\ldots\infty}}$, prove that $\frac{dy}{dx} = 2$ at $x = \frac{\pi}{4}$

Q14. If $y = (\cos x)^{(\cos x)^{(\cos x)\ldots\infty}}$, prove that $\frac{dy}{dx} = -\frac{y^2 \tan x}{1 - y\log \cos x}$

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S1.

$$y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \ldots \infty}}}$$

$\Rightarrow$ $$y = \sqrt{\log x + y}$$

$\Rightarrow$ $$y^2 = \log x + y$$

Diff. w.r.t. x both side

$\Rightarrow$ $$2y\frac{dy}{dx} = \frac{1}{x} + \frac{dy}{dx}$$

$\Rightarrow$ $$(2y-1)\frac{dy}{dx} = \frac{1}{x}.$$ Proved

S2.

$$y = \sqrt{x + \sqrt{x + \sqrt{x + \ldots \infty}}}$$

$\Rightarrow$ $$y = \sqrt{x + y}$$

$\Rightarrow$ $$y^2 = x + y$$

Diff. w.r.t. x both side

$\Rightarrow$ $$2y\frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$\Rightarrow$ $$\frac{dy}{dx}(2y-1) = 1$$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{1}{(2y-1)}.$$ Proved

S3.

$$y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \ldots \infty}}}$$

$\Rightarrow$ $$y = \sqrt{\tan x + y}$$

$\Rightarrow$ $$y^2 = \tan x + y$$

Diff. w.r.t. x both side

$\Rightarrow$ $$2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$\Rightarrow$ $$(2y-1)\frac{dy}{dx} = \sec^2 x$$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{\sec^2 x}{2y-1}.$$ Proved

S4. The given series may be written as

$$y = \sqrt{\sin x + y}$$

$$y^2 = \sin x + y \qquad \text{[Squaring both sides]}$$

$$2y\frac{dy}{dx} = \cos x + \frac{dy}{dx} \qquad \text{[Differentiating both sides w.r.t. } x\text{]}$$

$$\Rightarrow \quad \frac{dy}{dx}(2y-1) = \cos x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\cos x}{2y-1} \qquad \text{Proved}$$

S5.

$$y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \dots \infty}}}$$

$$\Rightarrow \quad y = \sqrt{\cos x + y}$$

$$\Rightarrow \quad y^2 = \cos x + y$$

Diff. w.r.t. x both side

$$\Rightarrow \quad 2y\frac{dy}{dx} = -\sin x + \frac{dy}{dx}$$

$$\Rightarrow \quad (2y-1)\frac{dy}{dx} = -\sin x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sin x}{1-2y}. \qquad \text{Proved}$$

S6. The given function can be written as

$$y = (\sqrt{x})^y$$

$$\Rightarrow \quad y = x^{y/2}$$

$$\Rightarrow \quad \log y = \frac{y}{2}\log x \qquad \text{[On taking log of both sides]}$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \frac{y}{2}\cdot\frac{1}{x} + \frac{1}{2}\log x\frac{dy}{dx} \qquad \text{[Differentiating both sides w.r.t. } x\text{]}$$

$$\Rightarrow \quad \frac{dy}{dx}\left\{\frac{1}{y} - \frac{1}{2}\log x\right\} = \frac{y}{2x}$$

$$\Rightarrow \quad \frac{dy}{dx}\left\{\frac{2 - y\log x}{2y}\right\} = \frac{y}{2x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y^2}{x(2 - y\log x)} \qquad \text{Proved}$$

S7. The given function may be written as

$$y = e^{x+y}$$

$\Rightarrow$ $\log y = (x + y) . \log e$ [Taking log of both sides]

$\Rightarrow$ $\log y = x + y$ $[\because \log e = 1]$

$\Rightarrow$ $\frac{1}{y}\frac{dy}{dx} = 1 + \frac{dy}{dx}$ [Differentiating w.r.t. x]

$\Rightarrow$ $\frac{dy}{dx}\left(\frac{1}{y} - 1\right) = 1$

$\Rightarrow$ $\frac{dy}{dx} = \frac{y}{1-y}$ Proved

S8. $y = a^{(x^y)}$

$\Rightarrow$ $\log y = x^y \log a$ [Taking log of both sides]

$\Rightarrow$ $\log (\log y) = y \log x + \log (\log a)$ [Taking log again of both sides]

$\Rightarrow$ $\frac{1}{\log y}\frac{d}{dx}(\log y) = \frac{dy}{dx} \cdot \log x + y \cdot \frac{d}{dx}(\log x) + 0$ [Differentiating both sides w.r.t. x]

$\Rightarrow$ $\frac{1}{\log y} \cdot \frac{1}{y}\frac{dy}{dx} = \frac{dy}{dx} \cdot \log x + y \cdot \frac{1}{x}$

$\Rightarrow$ $\frac{dy}{dx}\left\{\frac{1}{y \log y} - \log x\right\} = \frac{y}{x}$

$\Rightarrow$ $\frac{dy}{dx}\left\{\frac{1 - y \log y \cdot \log x}{y \log y}\right\} = \frac{y}{x}$

$\Rightarrow$ $\frac{dy}{dx} = \frac{y^2 \log y}{x(1 - y \log y \cdot \log x)}.$ Proved

S9. Since by deleting a single term from an infinite series, it remain same. Therefore, the given function may be written as

$$y = x^y$$

$\Rightarrow$ $\log y = y \log x$ [$\because$ on taking log of both sides]

$\Rightarrow$ $\frac{1}{y}\frac{dy}{dx} = \frac{dy}{dx} \cdot \log x + y \cdot \frac{d}{dx}(\log x)$ [Differentiating both sides w.r.t. x]

$\Rightarrow$ $\frac{1}{y}\frac{dy}{dx} = \frac{dy}{dx} . \log x + \frac{y}{x}$

$\Rightarrow \quad \dfrac{dy}{dx}\left\{\dfrac{1}{y} - \log x\right\} = \dfrac{y}{x}$

$\Rightarrow \quad \dfrac{dy}{dx}\dfrac{(1 - y\log x)}{y} = \dfrac{y}{x}$

$\Rightarrow \quad \dfrac{dy}{dx}\dfrac{1 - y\log x}{y} = \dfrac{y}{x}$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{y^2}{x(1 - y\log x)}$

S10. We have,

$$y = x + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \ldots}}}$$

$\Rightarrow \quad y = x + \dfrac{1}{y}$

$\Rightarrow \quad y^2 = xy + 1$

$\Rightarrow \quad 2y\dfrac{dy}{dx} = y + x\dfrac{dy}{dx} + 0$ [Differentiating both sides w.r.t. x]

$\Rightarrow \quad \dfrac{dy}{dx}(2y - x) = y$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{y}{2y - x}$ Proved

S11. We have,

$$y = \cfrac{\sin x}{1 + \cfrac{\cos x}{1 + y}}$$

$\Rightarrow \quad y = \dfrac{(1 + y)\sin x}{1 + y + \cos x}$

$\Rightarrow \quad y + y^2 + y\cos x = (1 + y)\sin x$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} + 2y\frac{dy}{dx} + \frac{dy}{dx}\cos x - y\sin x = \frac{dy}{dx}\sin x + (1 + y)\cos x$$

$$\Rightarrow \quad \frac{dy}{dx}\{1+2y+\cos x-\sin x\} = (1 + y) \cos x + y \sin x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(1+y)\cos x + y\sin x}{1+2y+\cos x-\sin x}. \qquad \text{Proved}$$

S12. $$y = (\sin x)^{(\sin x)^{(\sin x)\ldots\infty}}$$

$$\Rightarrow \quad y = (\sin x)^y$$

Taking log both side

$$\Rightarrow \quad \log y = y \log \sin x$$

Diff. w.r.t. x both side

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = y\cdot\frac{1}{\sin x}\cdot\cos x + \log\sin x\,\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}\left[\frac{1}{y}-\log\sin x\right] = y\cot x$$

$$\Rightarrow \quad \frac{dy}{dx}\left[\frac{1-y\log\sin x}{y}\right] = y\cot x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y^2\cot x}{1-y\log\sin x}. \qquad \text{Proved}$$

S13. $$y = (\tan x)^{(\tan x)^{(\tan x)\ldots\infty}}$$

$$\Rightarrow \quad y = (\tan x)^y \qquad \ldots \text{(i)}$$

Taking log both side

$$\Rightarrow \quad \log y = y \log \tan x$$

Diff. w.r.t. x both side

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = y\cdot\frac{1}{\tan x}\cdot\sec^2 x + \log\tan x\cdot\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}\left[\frac{1}{y}-\log\tan x\right] = y\sec x\,\mathrm{cosec}\,x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y^2\sec x\,\mathrm{cosec}\,x}{1-y\log\tan x} \qquad \text{(ii)}$$

Put $x=\frac{\pi}{4}$ in Eq. (i)

$$y = (1)^y$$

$$\Rightarrow \quad y = 1$$

Now, put $y = 1$ and $x = \frac{\pi}{4}$ in Eq. (ii)

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1 \sec\frac{\pi}{4}\operatorname{cosec}\frac{\pi}{4}}{1 - 1\log\tan\frac{\pi}{4}} = \frac{\sqrt{2}\times\sqrt{2}}{1-\log 1}$$

$$\Rightarrow \quad \frac{2}{1-0} = 2.$$

S14. $$y = (\cos x)^{(\cos x)^{(\cos x)\ldots\infty}}$$

$$\Rightarrow \quad y = (\cos x)^y$$

Taking log both side

$$\Rightarrow \quad \log y = y \log \cos x$$

Diff. w.r.t. x both side

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = y\cdot\frac{1}{\cos x}(-\sin x) + \log\cos x\,\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}\left[\frac{1}{y} - \log\cos x\right] = -y\tan x$$

$$\Rightarrow \quad \frac{dy}{dx}\left[\frac{1 - y\log\cos x}{y}\right] = -y\tan x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-y^2\tan x}{1 - y\log\cos x}.$$

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Q1. If $y = (x^x)^x$, find $\frac{dy}{dx}$.

Q2. If $y^x = e^{y-x}$, prove that $\frac{dy}{dx} = \frac{(1+\log y)^2}{\log y}$

Q3. Find $\frac{dy}{dx}$ when $x^y = e^{x-y}$.

Q4. Find the derivative of the function $f(x) = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8)$ and hence find $f'(1)$.

Q5. If $f(x) = \left(\frac{3+x}{1+x}\right)^{2+3x}$, find $f'(0)$

Q6. If $y = \sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}}$, find $\frac{dy}{dx}$.

Q7. If $y = \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$ find $\frac{dy}{dx}$

Q8. If $y = \frac{e^x \cdot \sin x \cdot (x+1)^2}{(2x+1)^3 x^5}$, find $\frac{dy}{dx}$.

Q9. Differentiate $x^x \sin^{-1}\sqrt{x}$ w.r.t. x.

Q10. If $y = x^{x^x}$, find $\frac{dy}{dx}$

Q11. Find $\frac{dy}{dx}$ when $y^x = x^y$.

Q12. If $e^y = y^x$ prove that $\frac{dy}{dx} = \frac{(\log y)^2}{\log y - 1}$

Q13. If $(\cos x)^y = (\tan y)^x$ prove that $\frac{dy}{dx} = \frac{\log \tan y + y \tan x}{\log \cos x - x \sec y \operatorname{cosec} y}$

Q14. If $(\sin y)^x = (\cos y)^x$ prove that $\frac{dy}{dx} = \frac{\log \cos y - \log \sin y}{x \cot y + x \tan y}$

Q15. If $x^y \cdot y^x = 1$ prove that $\frac{dy}{dx} = -\frac{y(y + x \log y)}{x(y \log x + x)}$

Q16. If $y = \sin(x^x)$ prove that $\frac{dy}{dx} = \cos(x^x) \cdot x^x(1 + \log x)$

Q17. If $y = x^x + x^a + a^x + a^a$, where $a > 0$ and a is a fixed number and $x > 0$. Find $\frac{dy}{dx}$

Q18. If $x^m y^n = (x + y)^{m+n}$, find $\frac{dy}{dx}$

Q19. If $(\cos x)^y = (\cos y)^x$ find $\frac{dy}{dx}$.

Q20. If $(\sin x)^y = x + y$ prove that $\frac{dy}{dx} = \frac{1 - (x + y)\, y \cot x}{(x + y) \log \sin x - 1}$

Q21. If $y = x^{\sin x - \cos x} + \dfrac{x^2-1}{x^2+1}$, find $\dfrac{dy}{dx}$.

Q22. Find $\dfrac{dy}{dx}$ when $y = x^{\cot x} + \dfrac{2x^2-3}{x^2+x+2}$

Q23. If $f(x) = (\sin x - \cos x)^{(\sin x - \cos x)}$, $\dfrac{\pi}{4} < x < \dfrac{3\pi}{4}$, then find $\dfrac{dy}{dx}$.

Q24. If $y = (\log x)^{\cos x} + \dfrac{x^2+1}{x^2-1}$, find $\dfrac{dy}{dx}$.

Q25. Differentiate $x^{x\cos x} + \dfrac{x^2+1}{x^2-1}$ with respect to x.

Q26. If $y = x^{\cot x} + (\sin x)^x$, find $\dfrac{dy}{dx}$.

Q27. If $y = (\sin x)^x + \sin^{-1}\sqrt{x}$, find $\dfrac{dy}{dx}$.

Q28. Differentiate $x^x - 2^{\sin x}$ with respect to x.

Q29. If $y = e^x \sin x^3 + (\tan x)^x$ find $\dfrac{dy}{dx}$.

Q30. Find $\dfrac{dy}{dx}$ if $y = (x)^{\cos x} + (\sin x)^{\tan x}$

Q31. Differentiate the following function with respect to x.

$$y = (\log x)^x + x^{\log x}$$

Q32. If $y = (x)^{\cos x} + (\cos x)^{\sin x}$, find $\dfrac{dy}{dx}$

Q33. Differentiate the following function w.r.t. x. $x^{\sin x} + (\sin x)^{\cos x}$.

Q34. If $y = (\log x)^x + (x)^{\cos x}$, find $\dfrac{dy}{dx}$.

Q35. If $y = (x)^{\sin x} + (\log x)^x$, find $\dfrac{dy}{dx}$.

Q36. If $y = (x)^x + (\sin x)^x$, find $\dfrac{dy}{dx}$.

Q37. Find $\dfrac{dy}{dx}$, if $y = (\cos x)^x + (\sin x)^{1/x}$.

Q38. Differentiate $\left(x + \dfrac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$ with respect to x.

Q39. Differentiate $(x\cos x)^x + (x\sin x)^{\frac{1}{x}}$ with respect to x.

Q40. If $x^x + y^x = 1$ prove that $\dfrac{dy}{dx} = -\left\{\dfrac{x^x(1+\log x) + y^x \log y}{x \cdot y^{x-1}}\right\}$

Q41. If $x^y + y^x = 2$, find $\dfrac{dy}{dx}$

Q42. Find $\dfrac{dy}{dx}$, if $y^x + x^y + x^x = a^b$.

Q43. If $x^y + y^x = 4$, find $\dfrac{dy}{dx}$.

Date: 18/10/2021

S1.

$$y = \cos^{-1}\left(\frac{1-x^{2n}}{1+x^{2n}}\right)$$

Put $x^n = \tan\theta$

$$y = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$= \cos^{-1}\cos 2\theta$$

$$= 2\theta = 2\tan^{-1} x^n$$

$\therefore$ $$\frac{dy}{dx} = \frac{2}{1+x^{2n}}\frac{d}{dx}x^n$$

$$= \frac{2nx^{n-1}}{1+x^{2n}}.$$

S2.

$$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right) + \sec^{-1}\left(\frac{1+x^2}{1-x^2}\right)$$

$$= \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right) + \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$= \frac{\pi}{2} \qquad \left\{\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right\}$$

$\therefore$ $$\frac{dy}{dx} = 0.$$

S3.

$$y = \tan^{-1}\sqrt{\frac{1+\cos x}{1-\cos x}}$$

$$= \tan^{-1}\sqrt{\frac{2\cos^2\frac{x}{2}}{2\sin^2\frac{x}{2}}}$$

$$= \tan^{-1}\left(\cot\frac{x}{2}\right)$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{2}-\frac{x}{2}\right)\right\}$$

$$= \frac{\pi}{2} - \frac{x}{2}$$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{2}.$$

S4.

$$y = \tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$$

$$= \tan^{-1}\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}$$

$$= \tan^{-1}\left(\tan\frac{x}{2}\right)$$

$$= \frac{x}{2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}.$$

S5.

$$y = \cos^{-1}\left(\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x\right)$$

$$= \cos^{-1}\left(\cos x \cos\frac{\pi}{4} + \sin x \sin\frac{\pi}{4}\right)$$

$$= \cos^{-1}\left(\cos\left(x - \frac{\pi}{4}\right)\right)$$

$$= x - \frac{\pi}{4}$$

$$\therefore \quad \frac{dy}{dx} = 1.$$

S6.

$$y = \tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$$

$$= \tan^{-1}\left\{\frac{\sin\left(\frac{\pi}{2}+x\right)}{1-\cos\left(\frac{\pi}{2}+x\right)}\right\}$$

$$= \tan^{-1}\left\{\frac{\cancel{2}\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}{\cancel{2}\sin^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}\right\}$$

$$= \tan^{-1}\left\{\cot\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}$$

$$= \tan^{-1}\left\{\tan\frac{\pi}{2}-\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\}$$

$$= \frac{\pi}{4}-\frac{x}{2}$$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{2}.$$

S7.

$$y = \tan^{-1}\left\{\sqrt{\frac{1+\sin x}{1-\sin x}}\right\}$$

$$= \tan^{-1}\left\{\sqrt{\frac{1+\cos\left(\frac{\pi}{2}-x\right)}{1-\cos\left(\frac{\pi}{2}-x\right)}}\right\}$$

$$= \tan^{-1}\left\{\sqrt{\frac{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\sin^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}}\right\}$$

$$= \tan^{-1}\left\{\cot\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\}$$

$$= \tan^{-1}\left\{\tan\frac{\pi}{2}-\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\}$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}$$

$$= \frac{\pi}{4}+\frac{x}{2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}.$$

S8.

$$y = \sin^{-1}\left(\sqrt{\frac{1}{1+x^2}}\right)$$

Put $x = \tan\theta$

$$y = \sin^{-1}\left(\sqrt{\frac{1}{1+\tan^2\theta}}\right)$$

$$= \sin^{-1}\left(\sqrt{\frac{1}{\sec^2\theta}}\right)$$

$$= \sin^{-1}(\cos\theta)$$

$$= \sin^{-1}\left\{\sin\left(\frac{\pi}{2}-\theta\right)\right\}$$

$$= \frac{\pi}{2}-\theta = \frac{\pi}{2}-\tan^{-1}x$$

$\therefore$ $$\frac{dy}{dx} = \frac{-1}{1+x^2}.$$

S9.

$$y = \sin^{-1}\left(\frac{2\cdot 2^x}{1+(2^x)^2}\right)$$

Put $2^x = \tan\theta$

$$y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$= \sin^{-1}(\sin 2\theta)$$

$$= 2\theta = 2\tan^{-1}(2^x)$$

$\therefore$ $$\frac{dy}{dx} = \frac{2}{1+(2^x)^2}\frac{d}{dx}2^x$$

$$= \frac{2\cdot 2^x \log 2}{1+4^x}.$$

S10.

$$y = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)+\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$$

Put $x = \tan\theta$

$$y = \sin^{-1}\left(\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}\right) + \cos^{-1}\left(\frac{1}{\sqrt{1+\tan^2\theta}}\right)$$

$$= \sin^{-1}\left(\frac{\tan\theta}{\sqrt{\sec^2\theta}}\right) + \cos^{-1}\left(\frac{1}{\sqrt{\sec^2\theta}}\right)$$

$$= \sin^{-1}(\tan\theta \,.\, \cos\theta) + \cos^{-1}(\cos\theta)$$

$$= \sin^{-1}\left(\frac{\sin\theta}{\cos\theta}\cdot\cos\theta\right) + \cos^{-1}(\cos\theta)$$

$$= \sin^{-1}(\sin\theta) + \cos^{-1}(\cos\theta)$$

$$= \theta + \theta = 2\theta$$

$$= 2\tan^{-1} x$$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{2}{1+x^2}.$

S11.

$$y = \sin\left[2\tan^{-1}\left\{\sqrt{\frac{1-x}{1+x}}\right\}\right]$$

Put $\quad x = \cos 2\theta$

$$= \sin\left[2\tan^{-1}\left\{\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}\right\}\right]$$

$$= \sin\left[2\tan^{-1}\left\{\sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}}\right\}\right]$$

$$= \sin\left[2\tan^{-1}\tan\theta\right]$$

$$= \sin 2\theta$$

$$= \sqrt{1-\cos^2 2\theta} = \sqrt{1-x^2}$$

$\therefore \quad \dfrac{dy}{dx} = -\dfrac{1}{2\sqrt{1-x^2}}\dfrac{d}{dx}x^2$

$$= \frac{-2x}{2\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}.$$

S12. $$y = \tan^{-1}\left\{\sqrt{1+x^2}+x\right\}$$

Put $x = \cot\theta$

$$y = \tan^{-1}\left\{\sqrt{1+\cot^2\theta}+\cot\theta\right\}$$

$$= \tan^{-1}\{\operatorname{cosec}\theta+\cot\theta\}$$

$$= \tan^{-1}\left\{\frac{1}{\sin\theta}+\frac{\cos\theta}{\sin\theta}\right\}$$

$$= \tan^{-1}\left\{\frac{1+\cos\theta}{\sin\theta}\right\}$$

$$= \tan^{-1}\left\{\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right\}$$

$$= \tan^{-1}\cot\frac{\theta}{2}$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right\}$$

$$= \frac{\pi}{2}-\frac{\theta}{2}$$

$$= \frac{\pi}{2}-\frac{1}{2}\cot^{-1}x$$

$\therefore$ $$\frac{dy}{dx} = 0-\frac{1}{2}\left(-\frac{1}{1+x^2}\right)=\frac{1}{2(1+x^2)}.$$

S13. $$y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)+\sec^{-1}\left(\frac{1+x^2}{1-x^2}\right)$$

$$= \sin^{-1}\left(\frac{2x}{1+x^2}\right)+\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

Put $x = \tan\theta$

$$= \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)+\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$= \sin^{-1}(\sin 2\theta) + \cos^{-1}(\cos 2\theta)$$

$$= 2\theta + 2\theta$$

$$= 4\theta = 4\tan^{-1}x$$

$\therefore$ $$\frac{dy}{dx} = \frac{4}{1+x^2}.$$

S14. $$y = \sin^{-1}\left\{x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}\right\}$$

$$= \sin^{-1}\left\{x\sqrt{1-(\sqrt{x})^2} - \sqrt{x}\sqrt{1-x^2}\right\}$$

$$= \sin^{-1}x - \sin^{-1}\sqrt{x}$$

$\therefore$ $$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-(\sqrt{x})^2}}\frac{d}{dx}\sqrt{x}$$

$$= \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x}} \times \frac{1}{2\sqrt{x}}\ .$$

S15. $$y = \tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right)$$

$$= \tan^{-1}\left(\frac{\frac{3x}{a} - \left(\frac{x}{a}\right)^3}{1 - 3\left(\frac{x}{a}\right)^2}\right)$$

Put $$\frac{x}{a} = \tan\theta$$

$$= \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 3\theta)$$

$$= 3\theta = 3\tan^{-1}\left(\frac{x}{a}\right)$$

$\therefore$ $$\frac{dy}{dx} = \frac{3}{1+\frac{x^2}{a^2}} \cdot \frac{d}{dx}\left(\frac{x}{a}\right)$$

$$= \frac{3a^2}{a^2+x^2} \cdot \frac{1}{a} = \frac{3a}{a^2+x^2}.$$

S16. $$y = \tan^{-1}\left\{\sqrt{1+x^2} - x\right\}$$

Put $$x = \cot\theta$$

$$y = \tan^{-1}\left\{\sqrt{1+\cot^2\theta} - \cot\theta\right\}$$

$$= \tan^{-1}\{\operatorname{cosec}\theta - \cot\theta\}$$

$$= \tan^{-1}\left\{\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right\}$$

$$= \tan^{-1}\left\{\frac{1-\cos\theta}{\sin\theta}\right\}$$

$$= \tan^{-1}\left\{\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right\}$$

$$= \tan^{-1}\tan\frac{\theta}{2}$$

$$= \frac{1}{2}\theta = \frac{1}{2}\cot^{-1}x$$

$\therefore$ $$\frac{dy}{dx} = -\frac{1}{2(1+x^2)}.$$

S17. $$y = \sin^{-1}\left[\frac{2^{x+1}\cdot 3^x}{1+(36)^x}\right] = \sin^{-1}\left[\frac{2\cdot 2^x\cdot 3^x}{1+4^x\cdot 9^x}\right]$$

$$= \sin^{-1}\left[\frac{2\cdot 6^x}{1+(6^x)^2}\right]$$

Put $\tan\theta = 6^x \Rightarrow \theta = \tan^{-1}(6^x)$

$\therefore$ $$y = \sin^{-1}\left(\frac{2\cdot\tan\theta}{1+\tan^2\theta}\right)$$

$$= \sin^{-1}(\sin 2\theta) \qquad \left[\because \quad \sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}\right]$$

$$= 2\theta$$

$\Rightarrow$ $$y = 2\tan^{-1}(6^x)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{2}{1+(6^x)^2}\frac{d}{dx}(6^x) = \left[\frac{2\cdot 6^x}{1+(36^x)}\right]\log 6 \qquad \left[\because \quad \frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}\right]$$

$$\Rightarrow \frac{d}{dx}\left[\sin^{-1}\left(\frac{2^{x+1}\cdot 3^x}{1+(36)^x}\right)\right] = \left[\frac{2^{x+1}\cdot 3^x}{1+(36)^x}\right]\log 6.$$

S18. Given that,

$$y = \sin^{-1}\left[\frac{5x+12\sqrt{1-x^2}}{13}\right]$$

Put $x = \sin\theta$

So that $\theta = \sin^{-1} x$

$\therefore$ We get

$$y = \sin^{-1}\left[\frac{5\sin\theta+12\sqrt{1-\sin^2\theta}}{13}\right]$$

$$\Rightarrow \quad y = \sin^{-1}\left[\frac{5\sin\theta+12\cos\theta}{13}\right] \qquad \left[\because \sqrt{1-\sin^2\theta}=\sqrt{\cos^2\theta}=\cos\theta\right]$$

Now, let $\frac{5}{13}=\cos\alpha$ and $\frac{12}{13}=\sin\alpha$ $\qquad \left[\because \cos^2\alpha+\sin^2\alpha=\frac{25}{169}+\frac{144}{169}=\frac{169}{169}=1\right]$

$$\therefore \quad y = \sin^{-1}[\sin\theta\cos\alpha+\cos\theta\sin\alpha]$$

$$\Rightarrow \quad y = \sin^{-1}\sin(\theta+\alpha) \qquad [\because \sin x\cos y+\cos x\sin y=\sin(x+y)$$

$$\Rightarrow \quad y = \theta+\alpha \qquad [\because \sin^{-1}\sin x = x]$$

$$\Rightarrow \quad y = (\sin^{-1} x)+\alpha \qquad [\because \theta=\sin^{-1} x]$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}+0$$

Hence, $$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

S19. Given that

$$y = \cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$$

Now, we write

$$1 + \sin x = \cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2} \qquad \left[\because \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \text{ and } 1 = \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right]$$

and $$1 - \sin x = \cos^2\frac{x}{2} + \sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}$$

So, the given function becomes

$$y = \cot^{-1}\left[\frac{\sqrt{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}} + \sqrt{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}}}{\sqrt{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}} - \sqrt{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}}}\right]$$

$$y = \cot^{-1}\left[\frac{\sqrt{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2} + \sqrt{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2}}{\sqrt{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2} - \sqrt{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2}}\right]$$

$$y = \cot^{-1}\left[\frac{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right) + \left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right) - \left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)}\right] \qquad \left[\because a^2 + b^2 + 2ab = (a+b)^2 \text{ and } a^2 + b^2 - 2ab = (a-b)^2 \text{ and here } a = \cos\frac{x}{2}, b = \sin\frac{x}{2}\right]$$

$$\Rightarrow \quad y = \cot^{-1}\left[\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right]$$

$$\Rightarrow \quad y = \cot^{-1}\cot\frac{x}{2} \qquad \left[\because \cot\theta = \frac{\cos\theta}{\sin\theta}\right]$$

$$\Rightarrow \quad y = \frac{x}{2} \qquad [\because \cot^{-1}(\cot\theta) = \theta]$$

Differentiating on both sides w.r.t. x, we get

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}.$$

S20. Given that, $y = \cos^{-1}\left[\frac{3x + 4\sqrt{1-x^2}}{5}\right]$

Put $x = \sin\theta \Rightarrow \theta = \sin^{-1} x$

$\therefore \quad y = \cos^{-1}\left[\frac{3\sin\theta + 4\sqrt{1-\sin^2\theta}}{5}\right]$

$\Rightarrow \quad y = \cos^{-1}\left[\frac{3\sin\theta + 4\cos\theta}{5}\right] \quad [\because \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta]$

$\Rightarrow \quad y = \cos^{-1}\left[\frac{3}{5}\sin\theta + \frac{4}{5}\cos\theta\right]$

Now, let $\cos\alpha = \frac{4}{5}$ and $\sin\alpha = \frac{3}{5}$

$\because \quad \sin^2\alpha + \cos^2\alpha = \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2$

$= \frac{9}{25} + \frac{16}{25} = \frac{25}{25} = 1 \quad [\because \sin^2\theta + \cos^2\theta = 1 \, \forall \theta \in R]$

$\therefore$ We get, $\quad y = \cos^{-1}(\sin\theta\sin\alpha + \cos\theta\cos\alpha)$

$\Rightarrow \quad y = \cos^{-1}\cos(\theta - \alpha) \quad [\because \cos\theta\cos\alpha + \sin\theta\sin\alpha = \cos(\theta - \alpha)]$

$\Rightarrow \quad y = \theta - \alpha$

$\Rightarrow \quad y = \sin^{-1}x - \alpha \quad [\because \theta = \sin^{-1}x]$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - 0 \quad \left[\because \frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}\right]$$

Hence, $\quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$.

S21. $\quad y = \tan^{-1}(\sec x + \tan x)$

$$= \tan^{-1}\left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right)$$

$$= \tan^{-1}\left(\frac{1+\sin x}{\cos x}\right)$$

$$= \tan^{-1}\left\{\frac{1+\cos\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right)}\right\}$$

$$= \tan^{-1}\left\{\frac{2\cos^2\left(\frac{\pi}{4} - \frac{x}{2}\right)}{2\sin\left(\frac{\pi}{4} - \frac{x}{2}\right)\cos\left(\frac{\pi}{4} - \frac{x}{2}\right)}\right\}$$

$$= \tan^{-1} \cot\left(\frac{\pi}{4} - \frac{x}{2}\right)$$

$$= \tan^{-1} \tan\left(\frac{\pi}{2} - \left(\frac{\pi}{4} - \frac{x}{2}\right)\right)$$

$$= \tan^{-1} \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$$

$$= \frac{\pi}{4} + \frac{x}{2}$$

$\therefore$ $$\frac{dy}{dx} = \frac{1}{2}.$$

S22. $$y = \tan^{-1}\left(\frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}}\right)$$

Put $x^2 = \cos 2\theta$

$$y = \tan^{-1}\left\{\frac{\sqrt{1+\cos 2\theta} + \sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta} - \sqrt{1-\cos 2\theta}}\right\}$$

$$= \tan^{-1}\left\{\frac{\sqrt{2\cos^2\theta} + \sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta} - \sqrt{2\sin^2\theta}}\right\}$$

$$= \tan^{-1}\left\{\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right\}$$

$$= \tan^{-1}\left\{\frac{1+\tan\theta}{1-\tan\theta}\right\}$$

$$= \tan^{-1}\left\{\frac{\tan\frac{\pi}{4} + \tan\theta}{1 - \tan\frac{\pi}{4}\tan\theta}\right\}$$

$$= \tan^{-1}\tan\left(\frac{\pi}{4} + \theta\right)$$

$$= \frac{\pi}{4} + \theta = \frac{\pi}{4} + \frac{1}{2}\cos^{-1} x^2$$

$\therefore$ $$\frac{dy}{dx} = \frac{1}{2}\frac{-1}{\sqrt{1-x^4}} \cdot \frac{d}{dx}(x^2) = \frac{-x}{\sqrt{1-x^4}}.$$

S23.

$$y = \cos^{-1}\left(\frac{x}{\sqrt{x^2+a^2}}\right)$$

Put $x = a\tan\theta$

$$y = \cos^{-1}\left(\frac{a\tan\theta}{\sqrt{a^2\tan^2\theta + a^2}}\right)$$

$$= \cos^{-1}\left(\frac{a\tan\theta}{\sqrt{a^2\sec^2\theta}}\right)$$

$$= \cos^{-1}\left(\frac{\tan\theta}{\sec\theta}\right)$$

$$= \cos^{-1}\left(\frac{\sin\theta}{\cos\theta\cdot\sec\theta}\right)$$

$$= \cos^{-1}\sin\theta$$

$$= \cos^{-1}\left[\cos\left(\frac{\pi}{2}-\theta\right)\right]$$

$$= \frac{\pi}{2}-\theta = \frac{\pi}{2} - \tan^{-1}\left(\frac{x}{a}\right)$$

$\therefore$
$$\frac{dy}{dx} = 0 - \frac{1}{1+\left(\frac{x}{a}\right)^2}\frac{d}{dx}\left(\frac{x}{a}\right)$$

$$= -\frac{1}{\frac{a^2+x^2}{a^2}}\cdot\frac{1}{a}$$

$$= -\frac{a}{x^2+a^2}.$$

Date: 18/10/2021

Q1. If $x = a\sec^3\theta$ and $y = a\tan^3\theta$, find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{3}$

Q2. If $x = a\sin 2t\,(1 + \cos 2t)$ and $y = b\cos 2t\,(1 - \cos 2t)$, show that

$$\left(\frac{dy}{dx}\right)_{at\, t=\frac{\pi}{4}} = \frac{b}{a}.$$

Q3. If $x = \frac{3at}{1+t^3}, y = \frac{3at^2}{1+t^3}$, find $\frac{dy}{dx}$ at $t = \frac{1}{2}$

Q4. Find $\frac{dy}{dx}$ when $x = e^{\theta}\left(\theta + \frac{1}{\theta}\right), y = e^{-\theta}\left(\theta - \frac{1}{\theta}\right)$

Q5. If $x = a\left(\frac{1+t^2}{1-t^2}\right)$ and $y = \frac{2t}{1-t^2}$, find $\frac{dy}{dx}$.

Q6. If $x = ae^{\theta}(\sin\theta - \cos\theta)$ and $y = ae^{\theta}(\sin\theta + \cos\theta)$, find $\frac{dy}{dx}$

Q7. If $x = \cos^{-1}\frac{1}{\sqrt{1+t^2}}$ and $y = \sin^{-1}\frac{t}{\sqrt{1+t^2}}$, find $\frac{dy}{dx}$

Q8. If $x = 2\cos\theta - \cos 2\theta$ and $y = 2\sin\theta - \sin 2\theta$, prove that $\frac{dy}{dx} = \tan\left(\frac{3\theta}{2}\right)$

Q9. If $x = e^{\cos 2t}$ and $y = e^{\sin 2t}$, prove that $\frac{dy}{dx} = -\frac{y\log x}{x\log y}$

Q10. For a positive constant '*a*' find

$\frac{dy}{dx}$, where $y = a^{t+\frac{1}{t}}$ and $x = \left(t + \frac{1}{t}\right)^a$

Q11. If $x = \sqrt{a^{\sin^{-1}t}}, y = \sqrt{a^{\cos^{-1}t}}, a > 0$ and $-1 < t < 1$, show that $\frac{dy}{dx} = -\frac{y}{x}$

Q12. If $x = \sin^{-1}\left(\frac{2t}{1+t^2}\right)$ and $y = \tan^{-1}\left(\frac{2t}{1-t^2}\right), t > 1$. Prove that $\frac{dy}{dx} = -1$.

Q13. If $x = a\left(\cos t + \frac{1}{2}\log\tan^2\frac{t}{2}\right)$ and $y = a\sin t$, find $\frac{dy}{dx}$

Q14. If $x = a(\theta - \sin\theta)$ and $y = a(1 + \cos\theta)$, find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{3}$.

Q15. If $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}, y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$, find $\frac{dy}{dx}$

Date: 18/10/2021

S1. We have,

$$x = a\sec^3\theta \text{ and } y = a\tan^3\theta$$

Differentiating w.r.t θ, we get

$$\frac{dx}{d\theta} = 3a\sec^2\theta\frac{d}{d\theta}(\sec\theta) \text{ and } \frac{dy}{d\theta} = 3a\tan^2\theta\frac{d}{d\theta}(\tan\theta)$$

$$\Rightarrow \quad \frac{dx}{d\theta} = 3a\sec^3\theta\tan\theta \quad \text{and} \quad \frac{dy}{d\theta} = 3a\tan^2\theta\sec^2\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\tan^2\theta\sec^2\theta}{3a\sec^3\theta\tan\theta} = \frac{\tan\theta}{\sec\theta} = \sin\theta$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{\theta=\pi/3} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

S2. Here, $x = a\sin 2t(1 + \cos 2t)$

$$\Rightarrow \quad \frac{dx}{dt} = a.[2\cos 2t(1+\cos 2t) + \sin 2t(-2\sin 2t)]$$

$$= 2a[\cos 2t + \cos^2 2t - \sin^2 2t]$$

$$= 2a[\cos 2t + \cos 4t]$$

Also $\quad y = b\cos 2t(1 - \cos 2t)$

$$\Rightarrow \quad \frac{dy}{dt} = b[-2\sin 2t(1-\cos 2t) + \cos 2t \,.\, 2\sin 2t]$$

$$= 2b[-\sin 2t + 2\sin 2t\cos 2t]$$

$$= 2b[-\sin 2t + \sin 4t]$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{2b[-\sin 2t + \sin 4t]}{2a[\cos 2t + \cos 4t]}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{t=\frac{\pi}{4}} = \frac{b}{a}\left[\frac{-\sin\frac{\pi}{2} + \sin\pi}{\cos\frac{\pi}{2} + \cos\pi}\right]$$

$$= \frac{b}{a}\left[\frac{-1+0}{0-1}\right] = \frac{b}{a}.$$

S3. Given,

$$x = \frac{3at}{1+t^3} \qquad \dots \text{(i)}$$

$\therefore$ $\log x = \log 3a + \log t - \log (1 + t^3)$ [Taking logarithms]

Differentiating w.r.t. t,

$$\therefore \quad \frac{1}{x}\cdot\frac{dx}{dt} = \frac{1}{t} - \frac{3t^2}{(1+t^3)} \Rightarrow \frac{dx}{dt} = \frac{x}{t}\cdot\frac{1-2t^3}{(1+t^3)}$$

Again $$y = \frac{3at^2}{1+t^3} \qquad \dots \text{(ii)}$$

$\therefore$ $\log y = \log 3a + 2\log t - \log (1 + t^3)$

Differentiating w.r.t. t,

$$\therefore \quad \frac{1}{y}\cdot\frac{dy}{dt} = \frac{2}{t} - \frac{3t^2}{1+t^3} \Rightarrow \frac{dy}{dt} = \frac{y}{t}\cdot\frac{2-t^3}{(1+t^3)}$$

Hence $$\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{y}{t}\cdot\frac{2-t^3}{(1+t^3)}\Big/\frac{x}{t}\cdot\frac{1-2t^3}{(1+t^3)} = \frac{y(2-t^3)}{x(1-2t^3)}$$

$$\frac{dy}{dx} = \frac{t(2-t^3)}{(1-2t^3)} \qquad \left[\because \quad \frac{y}{x} = t \text{ from (ii)} \div \text{(i)}\right]$$

$$\therefore \quad \text{At} \quad t = \frac{1}{2}, \frac{dy}{dx} = \frac{\frac{1}{2}\cdot\left(2-\frac{1}{8}\right)}{1-\frac{1}{4}} = \frac{15}{16}\times\frac{4}{3} = \frac{5}{4}.$$

S4. We have,

$$\frac{dx}{d\theta} = e^{\theta}\left(1-\frac{1}{\theta^2}\right) + \left(\theta+\frac{1}{\theta}\right)e^{\theta}$$

$$= e^{\theta}\left(1-\frac{1}{\theta^2}+\theta+\frac{1}{\theta}\right) = \frac{e^{\theta}(\theta^2-1+\theta^3+\theta)}{\theta^2}$$

and $$\frac{dy}{d\theta} = e^{-\theta}\left(1+\frac{1}{\theta^2}\right) - \left(\theta-\frac{1}{\theta}\right)(e^{-\theta})$$

$$= e^{-\theta}\left(1+\frac{1}{\theta^2}-\theta+\frac{1}{\theta}\right) = \frac{\{e^{-\theta}(\theta^2+1-\theta^3+\theta)\}}{\theta^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} = \frac{\{e^{-\theta}(\theta^2+1-\theta^3+\theta)\}}{\theta^2}\times\frac{\theta^2}{e^{\theta}[\theta^2-1+\theta^3+\theta]}$$

$$= \frac{e^{-2\theta}(\theta^2+1-\theta^3+\theta)}{(\theta^2-1+\theta^3+\theta)}.$$

S5. Here $$x = a\left(\frac{1+t^2}{1-t^2}\right) = a\left(-1+\frac{2}{1-t^2}\right)$$

$\Rightarrow$ $$\frac{dx}{dt} = a\left(0+2\frac{-1}{(1-t^2)^2}\cdot(-2t)\right)$$

$$= \frac{4at}{(1-t^2)^2}$$

and $$y = \frac{2t}{1-t^2}$$

$\Rightarrow$ $$\frac{dy}{dt} = \frac{(1-t^2)\cdot 2-2t(-2t)}{(1-t^2)^2}$$

$$= \frac{2(1+t^2)}{(1-t^2)^2}$$

$\therefore$ $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2(1+t^2)}{(1-t^2)^2}\times\frac{(1-t^2)^2}{4at}$$

$$= \frac{1+t^2}{2at}.$$

S6. $$x = ae^{\theta}(\sin\theta - \cos\theta)$$

Diff. w.r.t. θ

$$\frac{dx}{d\theta} = a[e^{\theta}(\cos\theta + \sin\theta) + (\sin\theta - \cos\theta)e^{\theta}]$$

$$= ae^{\theta}[\cos\theta + \sin\theta + \sin\theta - \cos\theta]$$

$$= 2ae^{\theta}\sin\theta$$

Now, $$y = ae^{\theta}(\sin\theta + \cos\theta)$$

Diff. w.r.t. θ

$$\frac{dy}{d\theta} = a[e^{\theta}(\cos\theta - \sin\theta) + (\sin\theta + \cos\theta)e^{\theta}]$$

$$= ae^{\theta}[\cos\theta - \sin\theta + \sin\theta + \cos\theta]$$

$$= 2ae^{\theta}\cos\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2ae^{\theta}\cos\theta}{2ae^{\theta}\sin\theta}$$

$$= \cot\theta.$$

S7. $x = \cos^{-1}\dfrac{1}{\sqrt{1+t^2}}$ and $y = \sin^{-1}\dfrac{t}{\sqrt{1+t^2}}$

Put $t = \tan\theta$ Put $t = \tan\theta$

$$x = \cos^{-1}\left(\frac{1}{\sqrt{1+\tan^2\theta}}\right) \qquad y = \sin^{-1}\left(\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}\right)$$

$$x = \cos^{-1}\left(\frac{1}{\sqrt{\sec^2\theta}}\right) \qquad y = \sin^{-1}\left(\frac{\tan\theta}{\sqrt{\sec^2\theta}}\right)$$

$$x = \cos^{-1}(\cos\theta) \qquad y = \sin^{-1}(\sin\theta)$$

$$x = \theta = \tan^{-1} t \qquad y = \theta = \tan^{-1} t$$

$$\frac{dx}{dt} = \frac{1}{1+t^2} \qquad \frac{dy}{dt} = \frac{1}{1+t^2}$$

Now, $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{1/(1+t^2)}{1/(1+t^2)} = 1.$

S8. $x = 2\cos\theta - \cos 2\theta$ and $y = 2\sin\theta - \sin 2\theta$

$$\frac{dx}{d\theta} = -2\sin\theta + 2\sin 2\theta \qquad \frac{dy}{d\theta} = 2\cos\theta - 2\cos 2\theta$$

Now, $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{2(\cos\theta - \cos 2\theta)}{2(\sin 2\theta - \sin\theta)}$

$$= \frac{2\sin\left(\frac{2\theta+\theta}{2}\right)\sin\left(\frac{2\theta-\theta}{2}\right)}{2\cos\left(\frac{2\theta+\theta}{2}\right)\sin\left(\frac{2\theta-\theta}{2}\right)}$$

$$= \frac{\sin\frac{3\theta}{2}}{\cos\frac{3\theta}{2}} = \tan\frac{3\theta}{2}.$$ Proved

S9. $x = e^{\cos 2t}$ and $y = e^{\sin 2t}$

$\log x = \cos 2t \log e$ $\log y = \sin 2t \log e$

Diff. w.r.t. t Diff. w.r.t. t

$$\frac{1}{x}\frac{dx}{dt} = -2\sin 2t \qquad \frac{1}{y}\frac{dy}{dt} = 2\cos 2t$$

$$\frac{dx}{dt} = -2x\sin 2t \qquad \frac{dy}{dt} = 2y\cos 2t$$

Now, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2y\cos 2t}{-2x\sin 2t}$

$= -\frac{y\log x}{x\log y}$ Proved [$\because$ $\log x = \cos 2t$ and $\log y = \sin 2t$]

S10. Given $y = a^{t+\frac{1}{t}}$

$\therefore$ $\frac{dy}{dt} = \frac{d}{dt}\left(a^{t+\frac{1}{t}}\right)$

$= a^{t+\frac{1}{t}}\frac{d}{dt}\left(t+\frac{1}{t}\right)\cdot\log a$

$= a^{t+\frac{1}{t}}\left(1-\frac{1}{t^2}\right)\log a$

Again, $x = \left(t+\frac{1}{t}\right)^a$

$\therefore$ $\frac{dx}{dt} = a\left[t+\frac{1}{t}\right]^{a-1}\cdot\frac{d}{dt}\left(t+\frac{1}{t}\right)$

$= a\left[t+\frac{1}{t}\right]^{a-1}\cdot\left(1-\frac{1}{t^2}\right)$

$\because$ $\frac{dx}{dt} \neq 0$. Thus $t \neq \pm 1$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a^{t+\frac{1}{t}}\left(1-\frac{1}{t^2}\right)\log a}{a\left[t+\frac{1}{t}\right]^{a-1}\left(1-\frac{1}{t^2}\right)}$$

$$= \frac{a^{t+\frac{1}{t}}\log a}{a\left(t+\frac{1}{t}\right)^{a-1}}.$$

S11. We have, $x = \sqrt{a^{\sin^{-1}t}}$

$\Rightarrow$ $\frac{dx}{dt} = \frac{1}{2}\left(a^{\sin^{-1}t}\right)^{-1/2}\frac{d}{dt}\left(a^{\sin^{-1}t}\right)$

$\Rightarrow$ $\frac{dx}{dt} = \frac{1}{2}\left(a^{\sin^{-1}t}\right)^{-1/2}\left(a^{\sin^{-1}t}\log_e a\right)\cdot\frac{d}{dt}(\sin^{-1}t)$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{1}{2}\left(a^{\sin^{-1}t}\right)^{1/2}(\log_e a)\times\frac{1}{\sqrt{1-t^2}} = \frac{x\log_e a}{2\sqrt{1-t^2}}$$

and $\quad y = \sqrt{a^{\cos^{-1}t}}$

$$\frac{dy}{dt} = \frac{1}{2}\left(a^{\cos^{-1}t}\right)^{-1/2}\frac{d}{dt}\left(a^{\cos^{-1}t}\right)$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{1}{2}\left(a^{\cos^{-1}t}\right)^{-1/2}\left(a^{\cos^{-1}t}\log_e a\right)\cdot\frac{d}{dt}(\cos^{-1}t)$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{1}{2}\left(a^{\cos^{-1}t}\right)^{\frac{1}{2}}(\log_e a)\left(-\frac{1}{\sqrt{1-t^2}}\right)$$

$$\Rightarrow \quad \frac{dy}{dt} = -\frac{y}{2}\frac{\log_e a}{\sqrt{1-t^2}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-y\log_e a}{2\sqrt{1-t^2}}\times\frac{2\sqrt{1-t^2}}{x\log_e a} = \frac{-y}{x}$$

Alternate method $\quad x^2y^2 = a^{\sin^{-1}t+\cos^{-1}t} \Rightarrow x^2y^2 = a^{\pi/2}$

Differentiating w.r.t x, we get

$$2xy^2 + 2x^2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{y}{x}.$$

S12. Let $t = \tan\theta$. Then,

$$t > 1 \Rightarrow \tan\theta > 1 \Rightarrow \quad \theta > \frac{\pi}{4}$$

Now, $\quad x = \sin^{-1}\left\{\frac{2t}{1+t^2}\right\}$

$$\Rightarrow \quad x = \sin^{-1}\left\{\frac{2\tan\theta}{1+\tan^2\theta}\right\}$$

$$\Rightarrow \quad x = \sin^{-1}(\sin 2\theta)$$

$$\Rightarrow \quad x = \sin^{-1}\{(\sin(\pi-2\theta)\} = \pi - 2\theta = \pi - 2\tan^{-1}t$$

$\Rightarrow \quad \dfrac{dx}{dt} = 0 - \dfrac{2}{1+t^2} = \dfrac{-2}{1+t^2}$

and, $\quad y = \tan^{-1}\left\{\dfrac{2t}{1-t^2}\right\}$

$\Rightarrow \quad y = \tan^{-1}\left\{\dfrac{2\tan\theta}{1-\tan^2\theta}\right\}$

$\Rightarrow \quad y = \tan^{-1}(\tan 2\theta)$

$\Rightarrow \quad y = \tan^{-1}\{-\tan(\pi - 2\theta)\}$

$\Rightarrow \quad y = -\tan^{-1}\{\tan(\pi - 2\theta)\}$

$\Rightarrow \quad y = -(\pi - 2\theta)$

$\Rightarrow \quad y = -\pi + 2\tan^{-1}t$

$\Rightarrow \quad \dfrac{dy}{dt} = 0 + \dfrac{2}{1+t^2}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \dfrac{\dfrac{2}{1+t^2}}{-\dfrac{2}{1+t^2}} = -1.$

S13. $\quad x = a\left(\cos t + \dfrac{1}{2}\log\tan^2\dfrac{t}{2}\right)$

Diff. w.r.t. t,

$$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{2\tan^2\frac{t}{2}}\cdot 2\tan\frac{t}{2}\cdot\sec^2\frac{t}{2}\cdot\frac{1}{2}\right)$$

$$= a\left(-\sin t + \frac{1}{\dfrac{2\sin\frac{t}{2}}{\cos\frac{t}{2}}\cdot\cos^2\frac{t}{2}}\right)$$

$$= a\left(-\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right)$$

$$= a\left(-\sin t + \frac{1}{\sin t}\right) = a\,\frac{(1-\sin^2 t)}{\sin t}$$

$$= a\,\frac{\cos^2 t}{\sin t}$$

Similarly $\quad y = a\sin t$

Diff. w.r.t. t,

$$\frac{dy}{dt} = a\cos t$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\cos t}{\frac{a\cos^2 t}{\sin t}} = \frac{\sin t}{\cos t} = \tan t.$$

S14. Given that $x = a(\theta - \sin\theta)$ ... (i)

and $y = a(1 + \cos\theta)$ ... (ii)

To find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{3}$

We know that

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} \quad \text{... (iii)}$$

Differentiating Eq. (i) and Eq. (ii) w.r.t. θ, we get

$$\therefore \quad \frac{dx}{d\theta} = a(1 - \cos\theta)$$

and $$\frac{dy}{d\theta} = -a\sin\theta$$

$\therefore$ By Eq. (iii), we get

$$\frac{dy}{dx} = \frac{-a\sin\theta}{a(1-\cos\theta)}$$

or $$\frac{dy}{dx} = \frac{-2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{a\times 2\sin^2\frac{\theta}{2}} \quad \left[\because \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} \text{ and } 1-\cos x = 2\sin^2\frac{x}{2}\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = -\cot\frac{\theta}{2}$$

Putting $\theta = \frac{\pi}{3}$, we get

$$\left[\frac{dy}{dx}\right]_{\theta=\frac{\pi}{3}} = -\cot\frac{\pi}{6} = -\sqrt{3}. \quad \left[\because \cot\frac{\pi}{6} = \sqrt{3}\right]$$

S15. We have, $$x = \frac{\sin^3 t}{\sqrt{\cos 2t}}$$

$$\Rightarrow \quad x = \sin^3 t\,(\cos 2t)^{-1/2}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{3\sin^2 t\cos t}{\sqrt{\cos 2t}} + \sin^3 t \times \left(-\frac{1}{2}\right)(\cos 2t)^{-3/2}\frac{d}{dt}(\cos 2t)$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{3\sin^2 t\cos t}{\sqrt{\cos 2t}} + \frac{\sin^3 t\sin 2t}{(\cos 2t)^{3/2}}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{3\sin^2 t\cos t\cos 2t + \sin^3 t\sin 2t}{(\cos 2t)^{3/2}}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{3\sin^2 t\cos t(1-2\sin^2 t) + 2\sin^4 t\cos t}{(\cos 2t)^{3/2}}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{3\sin^2 t\cos t - 4\sin^4 t\cos t}{(\cos 2t)^{3/2}}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{\sin t\cos t(3\sin t - 4\sin^3 t)}{(\cos 2t)^{3/2}}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{\sin 2t\sin 3t}{2(\cos 2t)^{3/2}}$$

Smilarly $\quad y = \dfrac{\cos^3 t}{\sqrt{\cos 2t}}$

$$\Rightarrow \quad y = \cos^3 t\,(\cos\ 2t)^{-1/2}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{-3\cos^2 t\sin t}{\sqrt{\cos 2t}} + \cos^3 t \times \left(-\frac{1}{2}\right)(\cos 2t)^{-3/2}\frac{d}{dt}(\cos 2t)$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{-3\cos^2 t\sin t}{\sqrt{\cos 2t}} + \frac{\cos^3 t\sin 2t}{(\cos 2t)^{3/2}}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{-3\cos^2 t\sin t\cos 2t + \cos^3 t\sin 2t}{(\cos 2t)^{3/2}}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{-3\cos^2 t\sin t(2\cos^2 t - 1) + 2\cos^4 t\sin t}{(\cos 2t)^{3/2}}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{-4\cos^4 t\sin t + 3\cos^2 t\sin t}{(\cos 2t)^{3/2}}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{-\sin t\cos t(4\cos^3 t - 3\cos t)}{(\cos 2t)^{3/2}}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{-\sin 2t\cos 3t}{2(\cos 2t)^{3/2}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\sin 2t\cos 3t}{\sin 2t\sin 3t} = -\cot 3t\,.$$

Date: 18/10/2021

Q1. Differentiate $\log(1 + x^2)$ w.r.t. $\tan^{-1} x$.

Q2. Differentiate $\sin^{-1}(2x\sqrt{1-x^2})$ w.r.t. $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$.

Q3. Differentiate $\tan^{-1}\left(\frac{1+ax}{1-ax}\right)$ w.r.t. $\sqrt{1+a^2x^2}$.

Q4. Differentiate $(\log x)^x$ w.r.t. $\log x$.

Q5. Differentiate x^x with respect to $x \log x$.

Q6. Differentiate $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ w.r.t. $\tan^{-1} x$, $-1 < x < 1$.

Q7. Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ w.r.t. $\tan^{-1} x$, $x \neq 0$

Q8. Differentiate $\tan^{-1}\left(\frac{x-1}{x+1}\right)$ w.r.t. $\sin^{-1}(3x - 4x^3)$.

Q9. Differentiate $(\cos x)^{\sin x}$ w.r.t. $(\sin x)^{\cos x}$.

Q10. Differentiate $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$ w.r.t. $\sec^{-1} x$.

Q11. Differentiate $\tan^{-1}\left(\frac{1+2x}{1-2x}\right)$ w.r.t. $\sqrt{1+4x^2}$

Q12. Differentiate $\cos^{-1}(4x^3 - 3x)$ w.r.t. $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$.

Q13. Differentiate $x^{\sin x}$ w.r.t. $(\cos x)^x$.

Q14. Differentiate $\sin^{-1}(2ax\sqrt{1-a^2x^2})$ w.r.t. $\sqrt{1-a^2x^2}$.

Q15. Differentiate $\tan^{-1}\left\{\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right\}$ w.r.t. $\cos^{-1} x^2$.

Q16. If $x \in \left(\frac{1}{\sqrt{2}}, 1\right)$, differentiate $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ w.r.t. $\cos^{-1}\left(2x\sqrt{1-x^2}\right)$.

Q17. Differentiate $x^{\sin^{-1} x}$ with respect to $\sin^{-1} x$.

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S1. Let $z = \log(1 + x^2)$

and $y = \tan^{-1} x$

$$\frac{dz}{dx} = \frac{1}{1+x^2} \cdot 2x$$

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

$$\therefore \quad \frac{dz}{dy} = \frac{dz/dx}{dy/dx} = \frac{2x/1+x^2}{1/1+x^2} = 2x.$$

S2. $z = \sin^{-1}(2x\sqrt{1-x^2})$ and $y = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

Put $x = \sin\theta$ | Put $x = \sin\theta$

$$z = \sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$$

$$= \sin^{-1}(2\sin\theta\cos\theta)$$

$$= \sin^{-1}\sin 2\theta$$

$$= 2\theta = 2\sin^{-1} x$$

$$\frac{dz}{dx} = \frac{2}{\sqrt{1-x^2}}$$

$$y = \tan^{-1}\left(\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$$

$$= \tan^{-1}(\tan\theta)$$

$$y = \theta = \sin^{-1} x$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \quad \frac{dz}{dy} = \frac{dz/dx}{dy/dx} = \frac{2/\sqrt{1-x^2}}{1/\sqrt{1-x^2}} = 2.$$

S3. Let $z = \tan^{-1}\left(\frac{1+ax}{1-ax}\right)$

$$z = \tan^{-1} 1 + \tan^{-1} ax$$

$$z = \frac{\pi}{4} + \tan^{-1} ax$$

$$\frac{dz}{dx} = \frac{a}{1+a^2x^2}$$

and let $$y = \sqrt{1+a^2x^2}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{1+a^2x^2}}\frac{d}{dx}a^2x^2 = \frac{2a^2x}{2\sqrt{1+a^2x^2}}$$

$\therefore$ $$\frac{dz}{dy} = \frac{a/(1+a^2x^2)}{a^2x/\sqrt{1+a^2x^2}} = \frac{1}{ax\sqrt{1+a^2x^2}}.$$

S4. Let $$z = (\log\ x)^x$$

$$\log z = x \log\log x$$

$$\frac{1}{z}\frac{dz}{dx} = x\cdot\frac{1}{\log x}\cdot\frac{1}{x} + \log\log x\cdot 1$$

$$\frac{dz}{dx} = z\left[\frac{1}{\log x} + \log\log x\right]$$

$$\frac{dz}{dx} = (\log x)^x\left[\frac{1}{\log x} + \log\log x\right]$$

$$\frac{dz}{dx} = (\log x)^x\left[\frac{1+\log x\cdot\log\log x}{\log x}\right]$$

$$\frac{dz}{dx} = (\log x)^{x-1}[1 + \log x \cdot \log\log x]$$

and let, $$y = \log x$$

$$\frac{dy}{dx} = \frac{1}{x}$$

$\therefore$ $$\frac{dz}{dy} = \frac{dz/dx}{dy/dx} = \frac{(\log x)^{x-1}(1+\log x\cdot\log\log x)}{1/x}$$

$$= x(\log)^{x-1}(1 + \log x\cdot\log\log x).$$

S5. Let $u = x^x$ and $v = x\log x$.

Now, $$u = x^x$$

$$\log u = x\log x$$

Differentiate w.r.t. x both side

$\Rightarrow$ $$\frac{1}{u}\frac{du}{dx} = x\cdot\frac{1}{x} + \log x\cdot 1$$

$\Rightarrow$ $$\frac{du}{dx} = u(1+\log x)$$

$\Rightarrow \quad \frac{du}{dx} = x^x (1 + \log x)$

and, $\quad v = x \log x$

$\Rightarrow \quad \frac{dv}{dx} = x \cdot \frac{1}{x} + \log x = (1 + \log x)$

$\Rightarrow \quad \frac{dv}{dx} = (1 + \log x)$

$\therefore \quad \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{x^x (1 + \log x)}{(1 + \log x)} = x^x$

Alternate method

$u = x^x$

$\Rightarrow \quad \log u = x \log x$

$\Rightarrow \quad \log u = v$

$\Rightarrow \quad u = e^v$

$\therefore \quad \frac{du}{dv} = e^v$

$\Rightarrow \quad \frac{du}{dv} = u$

$\Rightarrow \quad \frac{du}{dv} = x^x.$

S6. Let $u = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ and $v = \tan^{-1}x$, $-1 < x < 1$.

Now $\quad u = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Putting $\quad x = \tan\theta$, we have

$u = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta \qquad \left[\because -1 < x < 1 \Rightarrow -\frac{\pi}{4} < \theta < \frac{\pi}{4} \Rightarrow -\frac{\pi}{2} < 2\theta < \frac{\pi}{2}\right]$

$\Rightarrow \quad u = 2\tan^{-1}x$

$\Rightarrow \quad \frac{du}{dx} = \frac{2}{1+x^2}$

and $\quad v = \tan^{-1}x$

$\Rightarrow \qquad \dfrac{dv}{dx} = \dfrac{1}{1+x^2}$

$\therefore \qquad \dfrac{du}{dv} = \dfrac{\dfrac{du}{dx}}{\dfrac{dv}{dx}} = \dfrac{\dfrac{2}{1+x^2}}{\dfrac{1}{1+x^2}} = 2$

S7. Let $u = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ and $v = \tan^{-1} x$

Putting $x = \tan\theta$, we get

Now $\qquad u = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$

Putting $x = \tan\theta$, we get

$\Rightarrow \qquad = \tan^{-1}\left(\dfrac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$

$\Rightarrow \qquad = \tan^{-1}\left(\dfrac{\sec\theta-1}{\tan\theta}\right)$

$\Rightarrow \qquad = \tan^{-1}\left(\dfrac{1-\cos\theta}{\sin\theta}\right)$

$\Rightarrow \qquad u = \tan^{-1}\left(\dfrac{2\sin^2\left(\dfrac{\theta}{2}\right)}{2\sin\left(\dfrac{\theta}{2}\right)\cos\left(\dfrac{\theta}{2}\right)}\right)$

$\Rightarrow \qquad = \tan^{-1}\left(\tan\dfrac{\theta}{2}\right)$

$\Rightarrow \qquad = \dfrac{1}{2}\theta$

$\Rightarrow \qquad = \dfrac{1}{2}\tan^{-1} x$

$\Rightarrow \qquad \dfrac{du}{dx} = \dfrac{1}{2} \times \dfrac{1}{1+x^2}$

and $\qquad v = \tan^{-1} x$

$\Rightarrow \qquad \dfrac{dv}{dx} = \dfrac{1}{1+x^2}$

$\therefore \quad \dfrac{du}{dv} = \dfrac{\frac{du}{dx}}{\frac{dv}{dx}} = \dfrac{1}{2(1+x^2)} \times (1+x^2) = \dfrac{1}{2}.$

S8. $z = \tan^{-1}\left(\dfrac{x-1}{1+x}\right)$ and $y = \sin^{-1}(3x - 4x^3)$

$z = -\tan^{-1}\left(\dfrac{1-x}{1+x}\right)$ $\qquad y = 3\sin^{-1} x$

$z = -[\tan^{-1} 1 - \tan^{-1} x]$ $\qquad \dfrac{dy}{dx} = \dfrac{3}{\sqrt{1-x^2}}$

$z = \tan^{-1} x - \dfrac{\pi}{4}$

$\dfrac{dz}{dx} = \dfrac{1}{1+x^2}$

$\therefore \quad \dfrac{dz}{dy} = \dfrac{dz/dx}{dy/dx} = \dfrac{1/(1+x^2)}{3/\sqrt{1-x^2}} = \dfrac{\sqrt{1-x^2}}{3(1+x^2)}.$

S9. Let $\qquad z = (\cos x)^{\sin x}$

$\log z = \sin x \log \cos x$

$\dfrac{1}{z}\dfrac{dz}{dx} = \sin x \cdot \dfrac{1}{\cos x} \cdot (-\sin x) + \log \cos x \cdot \cos x$

$\dfrac{dz}{dx} = z[\cos x \log \cos x - \sin x \tan x]$

$\dfrac{dz}{dx} = (\cos x)^{\sin x}[\cos x \log \cos x - \sin x \tan x]$

and $\qquad y = (\sin x)^{\cos x}$

$\log y = \cos x \log \sin x$

$\dfrac{1}{y}\dfrac{dy}{dx} = \cos x \cdot \dfrac{1}{\sin x} \cdot \cos x + \log \sin x \cdot (-\sin x)$

$\dfrac{dy}{dx} = y[\cos x \cdot \cot x - \sin x \log \sin x]$

$\dfrac{dz}{dx} = (\sin x)^{\cos x}[\cos x \cot x - \sin x \log \sin x]$

$\therefore \quad \dfrac{dz}{dy} = \dfrac{dz/dx}{dy/dx} = \dfrac{(\cos x)^{\sin x}[\cos x \log \cos x - \sin x \tan x)}{(\sin x)^{\cos x}[\cos x \cot x - \sin x \log \sin x)}.$

S10. Let $z = \tan^{-1}\left(\dfrac{\cos x}{1+\sin x}\right)$ and $y = \sec^{-1} x$

$$z = \tan^{-1}\left(\frac{\sin\left(\frac{\pi}{2}-x\right)}{1+\cos\left(\frac{\pi}{2}-x\right)}\right) \qquad \frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}$$

$$z = \tan^{-1}\left(\frac{2\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right)$$

$$z = \tan^{-1}\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)$$

$$z = \frac{\pi}{4}-\frac{x}{2}$$

$$\frac{dz}{dx} = -\frac{1}{2}$$

$$\therefore \quad \frac{dz}{dy} = \frac{dz/dx}{dy/dx} = \frac{-1/2}{1/(x\sqrt{x^2-1})} = \frac{-x\sqrt{x^2-1}}{2}.$$

S11. Let $u = \tan^{-1}\left(\dfrac{1+2x}{1-2x}\right)$ and $v = \sqrt{1+4x^2}$. Then,

$$u = \tan^{-1} 1 + \tan^{-1} 2x$$

$$\Rightarrow \quad \frac{du}{dx} = \frac{2}{1+4x^2}$$

and $\quad v = \sqrt{1+4x^2}$

$$\frac{dv}{dx} = \frac{1}{2\sqrt{1+4x^2}} \times 8x$$

$$\frac{dv}{dx} = \frac{4x}{\sqrt{1+4x^2}}$$

$$\therefore \quad \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\frac{2}{1+4x^2}}{\frac{4x}{\sqrt{1+4x^2}}} = \frac{1}{2x\sqrt{1+4x^2}}.$$

S12. Let $y = \cos^{-1}(4x^3 - 3x)$ and $z = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

$y = 3\cos^{-1} x$ Put $x = \sin\theta$

$$\frac{dy}{dx} = \frac{-3}{\sqrt{1-x^2}}$$

$$z = \tan^{-1}\left(\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}\right)$$

$$z = \tan^{-1}\left(\frac{\cos\theta}{\sin\theta}\right)$$

$$z = \tan^{-1}\cot\theta$$

$$z = \tan^{-1}\left\{\tan\left(\frac{\pi}{2} - \theta\right)\right\}$$

$$z = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \sin^{-1} x$$

$$\frac{dz}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

$$\therefore \quad \frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{-3/\sqrt{1-x^2}}{-1/\sqrt{1-x^2}} = 3.$$

S13. Let $z = x^{\sin x}$ and $y = (\cos x)^x$

$\log z = \sin x \log x$

$$\frac{1}{z}\frac{dz}{dx} = \sin x \cdot \frac{1}{x} + \log x \cdot \cos x$$

$$\frac{1}{z}\frac{dz}{dx} = \frac{\sin x}{x} + \log x \cdot \cos x$$

$$\frac{dz}{dx} = z\left[\frac{\sin x + x\log x \cdot \cos x}{x}\right]$$

$$\frac{dz}{dx} = x^{\sin x}\left[\frac{\sin x + x\log x \cdot \cos x}{x}\right]$$

$$\frac{dz}{dx} = x^{\sin x - 1}\left[\sin x + x\log x \cos x\right]$$

$\log y = x \log \cos x$

$$\frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{\cos x}(-\sin x) + \log\cos x \cdot 1$$

$$\frac{dy}{dx} = y[-x\tan x + \log\cos x]$$

$$\frac{dy}{dx} = (\cos x)^x[-x\tan x + \log\cos x]$$

Now, $$\frac{dz}{dy} = \frac{dz/dx}{dy/dx} = \frac{x^{\sin x - 1}[\sin x + x\log x\cos x]}{(\cos x)^x[\log\cos x - x\tan x]}.$$

S14. Let $y = \sin^{-1}(2ax\sqrt{1-a^2x^2})$ and $z = \sqrt{1-a^2x^2}$

Put $ax = \sin\theta$

$$y = \sin^{-1}(2\sin\theta\sqrt{1-\sin^2\theta})$$

$$y = \sin^{-1}(2\sin\theta\cos\theta)$$

$$y = \sin^{-1}(\sin 2\theta)$$

$$y = 2\theta = 2\sin^{-1} ax$$

$$\frac{dy}{dx} = \frac{2}{\sqrt{1-a^2x^2}}\frac{d}{dx}ax = \frac{2a}{\sqrt{1-a^2x^2}}$$

$$\frac{dz}{dx} = \frac{1}{2\sqrt{1-a^2x^2}}\frac{d}{dx}(1-a^2x^2)$$

$$= \frac{-2a^2x}{2\sqrt{1-a^2x^2}}$$

$$= \frac{-a^2x}{\sqrt{1-a^2x^2}}$$

Now, $$\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{2a/\sqrt{1-a^2x^2}}{-a^2x/\sqrt{1-a^2x^2}} = \frac{-2}{ax}.$$

S15. Let

$$u = \tan^{-1}\left\{\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right\} \text{ and } v = \cos^{-1} x^2.$$

Now, $$u = \tan^{-1}\left\{\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right\}$$

Putting $x^2 = \cos\theta$, we get

$$u = \tan^{-1}\left\{\frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}\right\}$$

$\Rightarrow$ $$u = \tan^{-1}\left\{\frac{\sqrt{2\cos^2\frac{\theta}{2}}-\sqrt{2\sin^2\frac{\theta}{2}}}{\sqrt{2\cos^2\frac{\theta}{2}}+\sqrt{2\sin^2\frac{\theta}{2}}}\right\}$$

$\Rightarrow$ $$u = \tan^{-1}\left\{\frac{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}\right\}$$

$\Rightarrow \quad u = \tan^{-1}\left\{\dfrac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}\right\}$ $\left[\text{Dividing numerator and denominator by } \cos\frac{\theta}{2}\right]$

$\Rightarrow \quad u = \tan^{-1}\left\{\tan\left(\dfrac{\pi}{4}-\dfrac{\theta}{2}\right)\right\}$

$\Rightarrow \quad u = \dfrac{\pi}{4}-\dfrac{1}{2}\theta$

$\Rightarrow \quad u = \dfrac{\pi}{4}-\dfrac{1}{2}\cos^{-1}x^2$ $[\because x^2 = \cos\theta \therefore \theta = \cos^{-1}x^2]$

$\therefore \quad \dfrac{du}{dx} = -\dfrac{1}{2}\times\dfrac{-2x}{\sqrt{1-x^4}}$

$\Rightarrow \quad \dfrac{du}{dx} = \dfrac{x}{\sqrt{1-x^4}}$

and $\quad v = \cos^{-1} x^2$

$\Rightarrow \quad \dfrac{dv}{dx} = \dfrac{-2x}{\sqrt{1-x^4}}$

So, $\quad \dfrac{du}{dv} = \dfrac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\dfrac{du}{dv} = \dfrac{\frac{x}{\sqrt{1-x^4}}}{\frac{-2x}{\sqrt{1-x^4}}} = -\dfrac{1}{2}$$

S16. Let $u = \tan^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right)$ and $v = \cos^{-1}\left(2x\sqrt{1-x^2}\right)$

Now $\quad u = \tan^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right)$

Put $x = \sin\theta$. Then,

$x \in \left(\dfrac{1}{\sqrt{2}}, 1\right) \Rightarrow \dfrac{1}{\sqrt{2}} < \sin\theta < 1 \Rightarrow \dfrac{\pi}{4} < \theta < \dfrac{\pi}{2}$

$\Rightarrow \quad u = \tan^{-1}\left(\dfrac{\sqrt{1-\sin^2\theta}}{\sin\theta}\right)$

$\Rightarrow \quad u = \tan^{-1}(\cot\theta)$

$\Rightarrow \quad u = \tan^{-1}\left\{\tan\left(\frac{\pi}{2}-\theta\right)\right\}$

$\Rightarrow \quad u = \frac{\pi}{2}-\theta \qquad \left[\because \frac{\pi}{4}<\theta<\frac{\pi}{2} \Rightarrow 0<\frac{\pi}{2}-\theta<\frac{\pi}{4}\right]$

$\Rightarrow \quad u = \frac{\pi}{2}-\sin^{-1}x$

$\Rightarrow \quad \frac{du}{dx} = 0-\frac{1}{\sqrt{1-x^2}} = -\frac{1}{\sqrt{1-x^2}}$

and $\quad v = \cos^{-1}\left(2x\sqrt{1-x^2}\right)$

$\Rightarrow \quad v = \frac{\pi}{2}-\sin^{-1}\left(2x\sqrt{1-x^2}\right)$

$\Rightarrow \quad v = \frac{\pi}{2}-\sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right) \qquad [\because x = \sin\theta]$

$\Rightarrow \quad v = \frac{\pi}{2}-\sin^{-1}(\sin 2\theta)$

$\Rightarrow \quad v = \frac{\pi}{2}-\sin^{-1}\{\sin(\pi-2\theta)\}$

$\Rightarrow \quad v = \frac{\pi}{2}-(\pi-2\theta) \qquad \left[\because \frac{\pi}{4}<\theta<\frac{\pi}{2} \Rightarrow 0<\pi-2\theta<\frac{\pi}{2}\right]$

$\Rightarrow \quad v = -\frac{\pi}{2}+2\theta$

$\Rightarrow \quad v = -\frac{\pi}{2}+2\sin^{-1}x$

$\Rightarrow \quad \frac{dv}{dx} = \frac{2}{\sqrt{1-x^2}}$

$\therefore \quad \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\frac{-1}{\sqrt{1-x^2}}}{\frac{2}{\sqrt{1-x^2}}} = -\frac{1}{2}$

S17. Let $\quad u = x^{\sin^{-1}x}$ and $v = \sin^{-1}x$.

Now $\quad u = x^{\sin^{-1}x}$

$\log u = \sin^{-1}x \cdot \log x$

Differentiate w.r.t. x both side

$$\frac{1}{u}\frac{du}{dx} = \sin^{-1}x\cdot\frac{1}{x}+\log x\cdot\frac{1}{\sqrt{1-x^2}}$$

$\Rightarrow$ $$\frac{du}{dx} = u\left\{\frac{\sin^{-1}x}{x}+\frac{\log x}{\sqrt{1-x^2}}\right\}$$

$\Rightarrow$ $$\frac{du}{dx} = x^{\sin^{-1}x}\left\{\frac{\sin^{-1}x}{x}+\frac{\log x}{\sqrt{1-x^2}}\right\}$$

and, $$v = \sin^{-1}x$$

$\Rightarrow$ $$\frac{dv}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$\therefore$ $$\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{x^{\sin^{-1}x}\left\{\frac{\sin^{-1}x}{x}+\frac{\log x}{\sqrt{1-x^2}}\right\}}{\frac{1}{\sqrt{1-x^2}}}$$

$$\frac{du}{dv} = x^{\sin^{-1}x}\left\{\log x+\frac{\sqrt{1-x^2}}{x}\sin^{-1}x\right\}.$$

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Q1. If $y = \sin \log x$, prove that $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0$.

Q2. If $y = ae^{2x} + be^{-x}$, prove that $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 2y = 0$.

Q3. If $e^y(x + 1) = 1$, show that $\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx}\right)^2$.

Q4. If $y = \tan x + \sec x$, prove that $\dfrac{d^2y}{dx^2} = \dfrac{\cos x}{(1-\sin x)^2}$.

Q5. If $y = x^x$, find $\dfrac{d^2y}{dx^2}$.

Q6. $y = x^x$, prove that $\dfrac{d^2y}{dx^2} - \dfrac{1}{y}\left(\dfrac{dy}{dx}\right)^2 - \dfrac{y}{x} = 0$.

Q7. If $y = Ae^{mx} + Be^{nx}$, show that $\dfrac{d^2y}{dx^2} - (m+n)\dfrac{dy}{dx} + mny = 0$.

Q8. If $x = \tan\left(\dfrac{1}{a}\log y\right)$, show that $(1+x^2)\dfrac{d^2y}{dx^2} + (2x - a)\dfrac{dy}{dx} = 0$.

Q9. If $y = (\tan^{-1} x)^2$, show that $(x^2+1)^2 \dfrac{d^2y}{dx^2} + 2x(x^2+1)\dfrac{dy}{dx} = 2$.

Q10. If $x = a\cos\theta$, $y = b\sin\theta$, show that $\dfrac{d^2y}{dx^2} = \dfrac{-b^4}{a^2y^3}$.

Q11. If $x = a\sin t - b\cos t$ and $y = a\cos t + b\sin t$, prove that $\dfrac{d^2y}{dx^2} = -\left(\dfrac{x^2+y^2}{y^3}\right)$.

Q12. If $x = a(1 - \cos\theta)$, $y = a(\theta + \sin\theta)$ Prove that $\dfrac{d^2y}{dx^2} = -\dfrac{1}{a}$ at $\theta = \dfrac{\pi}{2}$.

Q13. If $x = a(1 - \cos^3\theta)$, $y = a\sin^3\theta$ prove that $\dfrac{d^2y}{dx^2} = \dfrac{32}{27a}$ at $\theta = \dfrac{\pi}{6}$.

Q14. If $x = \cos\theta$, $y = \sin^3\theta$, prove that $y\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = 3\sin^2\theta\,(5\cos^2\theta - 1)$.

Q15. If $y = \sin(\sin x)$, prove that $\dfrac{d^2y}{dx^2} + \tan x\dfrac{dy}{dx} + y\cos^2 x = 0$.

Q16. If $y = (\sin^{-1} x)^2$, prove that $(1 - x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - 2 = 0$.

Q17. If $y = e^{\tan^{-1}x}$ prove that $(1 + x^2)\dfrac{d^2y}{dx^2} + (2x - 1)\dfrac{dy}{dx} = 0$.

Q18. If $y = 500e^{7x} + 600e^{-7x}$, prove that $\dfrac{d^2y}{dx^2} = 49y$.

Q19. If $y = \log(1 + \cos x)$, prove that $\dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} \cdot \dfrac{dy}{dx} = 0$.

Q20. If $x = 2\cos t - \cos 2t$, $y = 2\sin t - \sin 2t$, find $\dfrac{d^2y}{dx^2}$ at $t = \dfrac{\pi}{2}$.

Q21. If $x = a \sin t$ and $y = a(\cos t + \log \tan t/2)$, find $\frac{d^2y}{dx^2}$.

Q22. If $x = a(\cos t + t \sin t)$ and $y = a(\sin t - t \cos t)$, find $\frac{d^2y}{dx^2}$ and $\frac{d^2y}{dt^2}$.

Q23. If $x = a(\theta - \sin\theta)$, $y = a(1 + \cos\theta)$, find $\frac{d^2y}{dx^2}$.

Q24. If $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$, find $\frac{d^2y}{dx^2}$.

Q25. If $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$, find $\frac{d^2y}{dx^2}$.

Q26. If $x = \cos t + \log \tan \frac{t}{2}$ and $y = \sin t$, then find the values of $\frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$.

Q27. If $x = a\cos^3\theta$ and $y = a\sin^3\theta$, then find the value of $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{6}$.

Q28. If $y = a\sin x + b\cos x$, prove that $y^2 + \left(\frac{dy}{dx}\right)^2 = a^2 + b^2$.

Q29. If $\log(\sqrt{1+x^2} - x) = y\sqrt{1+x^2}$, show that $(1+x^2)\frac{dy}{dx} + xy + 1 = 0$.

Q30. If $y = e^x(\sin x + \cos x)$, then show that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$.

Q31. If $y = e^x \sin x$, prove that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$.

Q32. If $y = \left\{x + \sqrt{x^2+1}\right\}^m$, show that $(x^2 + 1)y_2 + xy_1 - m^2y = 0$.

Q33. If $y = 3\cos(\log x) + 4\sin(\log x)$, show that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$.

Q34. If $y = x\log\left(\frac{x}{a+bx}\right)$, prove that $x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$.

Q35. If $y = \log[x + \sqrt{x^2 + a^2}]$, show that $(x^2 + a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$.

Q36. If $y = e^{a\cos^{-1}x}, -1 \le x \le 1$, then show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$.

Q37. If $y = \operatorname{cosec}^{-1} x$, $x > 1$, then show that $x(x^2-1)\frac{d^2y}{dx^2} + (2x^2-1)\frac{dy}{dx} = 0$.

Q38. If $y = (\cot^{-1} x)^2$, then show that $(x^2+1)^2\frac{d^2y}{dx^2} + 2x(x^2+1)\frac{dy}{dx} = 2$.

Q39. If $y = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$, show that $\left(1-x^2\right)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - y = 0$.

Q40. If $x = a\cos\theta + b\sin\theta$ and $y = a\sin\theta - b\cos\theta$, prove that

$y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$.

Q41. If $x = \sin t$, $y = \sin pt$, prove that $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$.

Q42. Find A and B so that $y = A \sin 3x + B \cos 3x$ satisfies the equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 10 \cos 3x.$$

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S1.
$$y = \sin \log x \quad \dots (i)$$
$$\Rightarrow \quad \frac{dy}{dx} = \frac{\cos \log x}{x} \quad \dots (ii)$$
$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{x(-\sin \log x)\cdot \frac{1}{x} - \cos \log x \cdot 1}{x^2}$$
$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} = -(\sin \log x + \cos \log x)$$
$$= -\sin \log x - \frac{x \cos \log x}{x}$$
$$= -y - x\frac{dy}{dx} \quad \text{from (i) and (ii)}$$
$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$$

S2.
$$y = ae^{2x} + be^{-x}$$
$$\Rightarrow \quad \frac{dy}{dx} = 2ae^{2x} - be^{-x}$$
$$\Rightarrow \quad \frac{d^2y}{dx^2} = 4ae^{2x} + be^{-x}$$
Now, $$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 4ae^{2x} + be^{-x} - 2ae^{2x} + be^{-x} - 2ae^{2x} - 2be^{-x}$$
$$= 0$$

S3. We have,
$$e^y(x+1) = 1$$
$$\Rightarrow \quad e^y = \frac{1}{x+1}$$
$$\Rightarrow \quad \log e^y = \log\left(\frac{1}{x+1}\right)$$
$$\Rightarrow \quad y = -\log(x+1)$$
Differentiating w.r.t. x both side

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{x+1}$$

and $$\frac{d^2y}{dx^2} = \frac{1}{(x+1)^2}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$$

S4. We have, $y = \tan x + \sec x$

Differentiating again w.r.t. x both side

$$\therefore \quad \frac{dy}{dx} = \sec^2 x + \sec x \tan x = \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x} = \frac{1+\sin x}{\cos^2 x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1+\sin x}{1-\sin^2 x} = \frac{1}{1-\sin x}$$

Differentiating again w.r.t. x both side

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left\{\frac{1}{1-\sin x}\right\} = \frac{d}{dx}\{(1-\sin x)^{-1}\}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = (-1)(1-\sin x)^{-2}\frac{d}{dx}(1-\sin x)$$

$$\frac{d^2y}{dx^2} = \frac{-1}{(1-\sin x)^2}(-\cos x) = \frac{\cos x}{(1-\sin x)^2}.$$

S5. We have, $y = x^x$

$$\therefore \quad \log y = x \log x$$

Differentiating w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = 1\cdot\log x + x\cdot\frac{1}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = y(1 + \log x) \qquad \dots \text{(i)}$$

Differentiating both sides of (i) *w.r.t.* x, we get

$$\frac{d^2y}{dx^2} = \frac{dy}{dx}(1+\log x) + y\frac{d}{dx}(1+\log x)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{dy}{dx}(1+\log x) + y\cdot\frac{1}{x} = y(1+\log x)^2 + \frac{y}{x} \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = x^x\left\{(1+\log x)^2 + \frac{1}{x}\right\}$$

S6. We have,

$$y = x^x$$

$$\log y = x \log x$$

Differentiating w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{x} + \log x \cdot 1$$

$$\Rightarrow \qquad \frac{dy}{dx} = y(1 + \log x) \qquad \text{... (i)}$$

Differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = y \times \frac{d}{dx}(1+\log x) + \frac{dy}{dx} \times (1+\log x)$$

$$\Rightarrow \qquad \frac{d^2y}{dx^2} = y \times \frac{1}{x} + \frac{dy}{dx} \times (1+\log x)$$

$$\Rightarrow \qquad \frac{d^2y}{dx^2} = \frac{y}{x} + \frac{dy}{dx}\left(\frac{1}{y}\frac{dy}{dx}\right) \qquad \left[\text{From (i), we have } 1+\log x = \frac{1}{y}\frac{dy}{dx}\right]$$

$$\Rightarrow \qquad \frac{d^2y}{dx^2} = \frac{y}{x} + \frac{1}{y}\left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0.$$

S7. To show $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$ ... (i)

Given, $y = Ae^{mx} + Be^{nx}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = Ame^{mx} + Bne^{nx} \qquad \text{... (ii)}$$

Again, differentiating w.r.t. x, we get

$$\Rightarrow \qquad \frac{d^2y}{dx^2} = Am^2e^{mx} + Bn^2e^{nx} \qquad \text{... (iii)}$$

To prove Eq. (i), we put the value of $\frac{dy}{dx}$ from Eq. (ii) and that of $\frac{d^2y}{dx^2}$ from Eq. (iii) along with value of y in L.H.S. of Eq. (i)

$$\therefore \qquad \text{L.H.S.} = \frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny$$

$$= Am^2e^{mx} + Bn^2e^{nx} - (m+n)\cdot(mAe^{mx} + nBe^{nx}) + mn(Ae^{mx} + Be^{nx})$$

$= Am^2e^{mx} + Bn^2e^{nx} - m^2Ae^{mx} - mnBe^{nx} - mnAe^{mx} - n^2Be^{nx} + mnAe^{mx} + mnBe^{nx}$

= 0 = R.H.S.

∴ L.H.S. = R.H.S. **Hence proved.**

S8. Given that $x = \tan\left(\frac{1}{a}\log y\right)$

$\Rightarrow \quad \tan^{-1} x = \frac{1}{a}\log y$

$\Rightarrow \quad a \tan^{-1}x = \log y$

Now, differentiating both sides w.r.t. x, we get

$$a \times \frac{1}{1+x^2} = \frac{1}{y}\cdot\frac{dy}{dx} \qquad \left[\because \frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}\right]$$

$$\Rightarrow \quad (1+x^2)\frac{dy}{dx} = ay \qquad \text{[By cross multiplication]}$$

Differentiating again on both sides w.r.t. x, we get

$$(1+x^2)\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\frac{d}{dx}(1+x^2) = \frac{d}{dx}(ay) \qquad \left[\because \frac{d}{dx}(u\cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}\right]$$

$$\Rightarrow \quad (1+x^2)\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot(2x) = a\cdot\frac{dy}{dx}$$

$$\Rightarrow \quad (1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - a\frac{dy}{dx} = 0$$

$$\Rightarrow \quad (1+x^2)\frac{d^2y}{dx^2} + (2x-a)\frac{dy}{dx} = 0.$$ **Hence proved.**

S9. Given that $y = (\tan^{-1}x)^2$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 2\tan^{-1}x\cdot\frac{1}{1+x^2} \qquad \left[\because \frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2\tan^{-1}x}{1+x^2}$$

$$(1+x^2)\frac{dy}{dx} = 2\tan^{-1}x$$

Differentiating again w.r.t. x on both sides, we get

$$(1+x^2)\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right)+\frac{dy}{dx}\cdot\frac{d}{dx}(1+x^2) = \frac{d}{dx}(2\tan^{-1}x)$$

$$\Rightarrow \quad (1+x^2)\cdot\frac{d^2y}{dx^2}+\frac{dy}{dx}\cdot 2x = \frac{2}{1+x^2} \qquad \left[\because \frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}\right]$$

$$\Rightarrow \quad (1+x^2)^2\frac{d^2y}{dx^2}+2x(1+x^2)\frac{dy}{dx} = 2$$

or $\quad (1+x^2)^2\dfrac{d^2y}{dx^2}+2x(1+x^2)\dfrac{dy}{dx} = 2.$ **Hence proved**.

S10. $\quad x = a\cos\theta \quad$ and $\quad y = b\sin\theta$

$$\frac{dx}{d\theta} = -a\sin\theta \qquad \frac{dy}{d\theta} = b\cos\theta$$

Now $\quad \dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta}=\dfrac{b\cos\theta}{-a\sin\theta}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-b}{a}\cot\theta$$

Again diff. w.r.t. x

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{-b}{a}(-\operatorname{cosec}^2\theta)\frac{d\theta}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{b}{a}\frac{1}{\sin^2\theta}\cdot\frac{1}{-a\sin\theta}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{-b}{a^2}\cdot\frac{1}{\sin^3\theta}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{-b}{a^2}\cdot\frac{b^3}{y^3} \qquad \left[\because \sin\theta=\frac{y}{b}\right]$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{-b^4}{a^2y^3}.$$

S11. $\quad \dfrac{dx}{dt} = a\cos t + b\sin t, \quad \dfrac{dy}{dt} = -a\sin t + b\cos t$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{b\cos t-a\sin t}{a\cos t+b\sin t}$$

Now, $$\frac{d^2y}{dx^2} = \frac{(a\cos t + b\sin t)(-b\sin t - a\cos t)\dfrac{dt}{dx} - (b\cos t - a\sin t)(b\cos t - a\sin t)\dfrac{dt}{dx}}{(a\cos t + b\sin t)^2}$$

$$= -\left[\frac{(a\cos t + b\sin t)^2 + (b\cos t - a\sin t)^2}{(a\cos t + b\sin t)^2}\right]\frac{dt}{dx}$$

$$= -\left[\frac{(a\cos t + b\sin t)^2 + (b\cos t - a\sin t)^2}{(a\cos t + b\sin t)^2}\right]\cdot\frac{1}{(a\cos t + b\sin t)}$$

$$= -\left[\frac{(a\cos t + b\sin t)^2 + (a\sin t - b\cos t)^2}{(a\cos t + b\sin t)^3}\right]$$

$$= -\left[\frac{y^2 + x^2}{y^3}\right]$$

$$\therefore \quad \frac{d^2y}{dx^2} = -\left(\frac{y^2 + x^2}{y^3}\right).$$

S12. $x = a(1 - \cos\theta)$ and $y = a(\theta + \sin\theta)$

$$\frac{dx}{d\theta} = a\sin\theta \qquad \frac{dy}{d\theta} = a(1 + \cos\theta)$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a(1+\cos\theta)}{a\sin\theta}$$

$$= \frac{2\cos^2(\theta/2)}{2\sin(\theta/2)\cdot\cos(\theta/2)} = \cot\frac{\theta}{2}$$

Again diff. w.r.t. x

$$\frac{d^2y}{dx^2} = -\text{cosec}^2\frac{\theta}{2}\cdot\frac{1}{2}\frac{d\theta}{dx}$$

$$= \frac{-1}{2\sin^2\dfrac{\theta}{2}}\cdot\frac{1}{a\sin\theta}$$

$$= \frac{-1}{a(1-\cos\theta)\sin\theta}$$

$$\left.\frac{d^2y}{dx^2}\right|_{\theta = \frac{\pi}{2}} = -\frac{1}{a\left(1-\cos\dfrac{\pi}{2}\right)\sin\dfrac{\pi}{2}} = -\frac{1}{a}.$$

S13. $x = a(1 - \cos^3\theta)$ and $y = a\sin^3\theta$

$$\Rightarrow \quad \frac{dx}{d\theta} = 3a\cos^2\theta\sin\theta \qquad \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{3a\sin^2\theta\cos\theta}{3a\cos^2\theta\sin\theta} = \tan\theta$.

Again diff. w.r.t. x

$$\frac{d^2y}{dx^2} = \sec^2\theta \cdot \frac{d\theta}{dx}$$

$$= \frac{\sec^2\theta}{3a\cos^2\theta\sin\theta} = \frac{1}{3a}\sec^4\theta\ \text{cosec}\ \theta$$

$$\therefore \quad \left.\frac{d^2y}{dx^2}\right|_{\theta = \frac{\pi}{6}} = \frac{1}{3a}\sec^4\frac{\pi}{6}\cdot\text{cosec}\frac{\pi}{6}$$

$$= \frac{1}{3a}\left(\frac{2}{\sqrt{3}}\right)^4 \cdot 2$$

$$= \frac{32}{27a}.$$

S14. $x = \cos\theta$ and $y = \sin^3\theta$

$$\frac{dx}{d\theta} = -\sin\theta \qquad \frac{dy}{d\theta} = 3\sin^2\theta\cos\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3\sin^2\theta\cos\theta}{-\sin\theta}$$

$$= -3\sin\theta\cos\theta$$

$$= -\frac{3}{2}\sin 2\theta$$

Now, $$\frac{d^2y}{dx^2} = \frac{-3}{2}(\cos 2\theta)\, 2\cdot\frac{d\theta}{dx}$$

$$= \frac{-3\cos 2\theta}{-\sin\theta} = \frac{3\cos 2\theta}{\sin\theta}$$

$$\therefore \quad y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = \sin^3\theta\cdot\frac{3\cos 2\theta}{\sin\theta} + \left(\frac{-3}{2}\sin 2\theta\right)^2$$

$$= 3\sin^2\theta\cos 2\theta + 9\sin^2\theta\cos^2\theta$$

$$= 3\sin^2\theta(\cos 2\theta + 3\cos^2\theta)$$

$$= 3\sin^2\theta(2\cos^2\theta - 1 + 3\cos^2\theta)$$

$$= 3\sin^2\theta(5\cos^2\theta - 1).$$

S15. $y = \sin(\sin x)$

$\Rightarrow \quad \dfrac{dy}{dx} = \cos(\sin x)\cdot\cos x$

$\Rightarrow \quad \dfrac{d^2y}{dx^2} = \cos(\sin x)\cdot(-\sin x) + \cos x\cdot(-\sin(\sin x))\cdot\cos x$

$\therefore \quad \dfrac{d^2y}{dx^2} = -[\sin x\cos(\sin x) + \cos^2 x\sin(\sin x)]$

and $\quad \tan x\dfrac{dy}{dx} = \dfrac{\sin x}{\cos x}\cdot\cos(\sin x)\cos x$

$= \sin x\cdot\cos(\sin x)$

and $\quad y\cos^2 x = \sin(\sin x)\cos^2 x$

Now $\quad \dfrac{d^2y}{dx^2} + \tan x\dfrac{dy}{dx} + y\cos^2 x = -\sin x\cos(\sin x) - \cos^2 x\sin(\sin x)$

$+ \sin x\cdot\cos(\sin x) + \sin(\sin x)\cos^2 x$

$= 0.$ **Hence proved.**

S16. $y = (\sin^{-1} x)^2$

$\Rightarrow \quad \dfrac{dy}{dx} = 2\sin^{-1} x\cdot\dfrac{1}{\sqrt{1-x^2}}$

$\Rightarrow \quad \dfrac{d^2y}{dx^2} = 2\left[\dfrac{\sqrt{1-x^2}\cdot\dfrac{1}{\sqrt{1-x^2}} - \sin^{-1}x\cdot\dfrac{1}{2\sqrt{1-x^2}}(-2x)}{(1-x^2)}\right]$

$= 2\left[\dfrac{\sqrt{1-x^2} + x\sin^{-1}x}{(1-x^2)^{3/2}}\right]$

Now, $\quad (1-x^2)\dfrac{d^2y}{dx^2} = 2\left[\dfrac{\sqrt{1-x^2} + x\sin^{-1}x}{\sqrt{1-x^2}}\right]$

and $\quad x\dfrac{dy}{dx} = \dfrac{2x\sin^{-1}x}{\sqrt{1-x^2}}$

Again now, $\quad (1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - 2$

$= 2 + \dfrac{2x\sin^{-1}x}{\sqrt{1-x^2}} - \dfrac{2x\sin^{-1}x}{\sqrt{1-x^2}} - 2 = 0.$

S17. $y = e^{\tan^{-1} x}$

$$\Rightarrow \quad \frac{dy}{dx} = e^{\tan^{-1} x} \cdot \frac{1}{1+x^2} \quad \dots (i)$$

$$(1 + x^2)\frac{dy}{dx} = e^{\tan^{-1} x}$$

Diff. w.r.t. x both side

$$\Rightarrow \quad (1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = e^{\tan^{-1} x} \cdot \frac{1}{1+x^2} \quad \text{from eq. (i)}$$

$$(1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = \frac{dy}{dx}$$

$$(1+x^2)\frac{d^2y}{dx^2} + (2x-1)\frac{dy}{dx} = 0$$

S18. $y = 500\, e^{7x} + 600\, e^{-7x}$

$$\Rightarrow \quad \frac{dy}{dx} = 500.7\, e^{7x} - 600.7\, e^{-7x}$$

$$\Rightarrow \quad \frac{dy}{dx} = 700\,(5\, e^{7x} - 6\, e^{-7x})$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 700\,(35e^{7x} + 42e^{-7x})$$

$$= 4900\,(5e^{7x} + 6e^{-7x})$$

$$= 49\,(500e^{7x} + 600e^{-7x})$$

$$= 49\,y$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 49\,y.$$

S19. $y = \log(1 + \cos x)$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{1+\cos x} \cdot (-\sin x)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2\sin\frac{x}{2}\cdot\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\tan\frac{x}{2}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\frac{1}{2}\sec^2\frac{x}{2}$$

$$\Rightarrow \quad \frac{d^3y}{dx^3} = -\frac{1}{2}\cdot 2\sec\frac{x}{2}\left(\sec\frac{x}{2}\tan\frac{x}{2}\right)\frac{1}{2}$$

$$= -\frac{1}{2}\sec^2\frac{x}{2}\tan\frac{x}{2}$$

Now, $$\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2}\cdot\frac{dy}{dx} = -\frac{1}{2}\sec^2\frac{x}{2}\tan\frac{x}{2} - \frac{1}{2}\sec^2\frac{x}{2}\left(-\tan\frac{x}{2}\right)$$

$$= -\frac{1}{2}\sec^2\frac{x}{2}\tan\frac{x}{2} + \frac{1}{2}\sec^2\frac{x}{2}\tan\frac{x}{2}$$

$$= 0.$$

S20. $x = 2\cos t - \cos 2t$ and $y = 2\sin t - \sin 2t$

$$\frac{dx}{dt} = -2\sin t + 2\sin 2t \quad \text{and} \quad \frac{dy}{dt} = 2\cos t - 2\cos 2t$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos t - 2\cos 2t}{2\sin 2t - 2\sin t}$$

$$\Rightarrow \quad \frac{\cos t - \cos 2t}{\sin 2t - \sin t} = \frac{2\sin\frac{3t}{2}\cdot\sin\frac{t}{2}}{2\cos\frac{3t}{2}\cdot\sin\frac{t}{2}} = \tan\frac{3t}{2}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{3}{2}\sec^2\frac{3t}{2}\cdot\frac{dt}{dx} = \frac{3}{2}\sec^2\frac{3t}{2}\cdot\frac{1}{2(\sin 2t - \sin t)}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{2}} = \frac{3}{2}\sec^2\frac{3\pi}{4}\cdot\frac{1}{2\left(\sin\pi - \sin\frac{\pi}{2}\right)}$$

$$= \frac{3}{2}(\sqrt{2})^2\cdot\frac{1}{2(0-1)} = \frac{-3}{2}$$

S21. Given that, $y = a\left(\cos t + \log\tan\frac{t}{2}\right)$; $x = a\sin t$

Differentiating both equations w.r.t. t, we get

$$\frac{dy}{dt} = a\left[\frac{d}{dt}(\cos t) + \frac{d}{dt}\left(\log\tan\frac{t}{2}\right)\right]$$

$\Rightarrow \qquad \frac{dy}{dt} = a\left[-\sin t + \frac{1}{\tan\left(\frac{t}{2}\right)} \times \frac{d}{dt}\left(\tan\frac{t}{2}\right)\right]$

$$\frac{dy}{dt} = a\left[-\sin t + \frac{1}{\tan\left(\frac{t}{2}\right)} \times \sec^2\left(\frac{t}{2}\right) \times \frac{1}{2}\right]$$

$$= a\left[-\sin t + \frac{1}{2} \times \frac{\cos\left(\frac{t}{2}\right)}{\sin\left(\frac{t}{2}\right)} \times \frac{1}{\cos^2\left(\frac{t}{2}\right)}\right]$$

$$= a\left[-\sin t + \frac{1}{2\sin\frac{t}{2} \cdot \cos\frac{t}{2}}\right]$$

$$= a\left[-\sin t + \frac{1}{\sin t}\right] \qquad \left[\because \sin t = 2\sin\frac{t}{2} \cdot \cos\frac{t}{2}\right]$$

$$= a\left[\frac{1 - \sin^2 t}{\sin t}\right]$$

$$= a\frac{\cos^2 t}{\sin t} \qquad [\because \sin^2 t + \cos^2 t = 1]$$

and $\qquad \frac{dx}{dt} = \frac{d}{dt}(a\sin t) = a\cos t$

$\therefore \qquad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\left[\frac{a\cos^2 t}{\sin t}\right]}{a\cos t} = \cot t$

$\therefore \qquad \frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}(\cot t)$

$$= \frac{d}{dt}(\cot t) \times \left(\frac{dt}{dx}\right)$$

$$= (-\text{cosec}^2 t)\left(\frac{dt}{dx}\right)$$

$$= -(\text{cosec}^2 t) \cdot \frac{1}{a\cos t}$$

$$= -\frac{\text{cosec}^2 t}{a\cos t}.$$

S22. Consider

$$x = a(\cos t + t \sin t)$$

Differentiating *w.r.t.* t, we have

$$\frac{dx}{dt} = a(-\sin t + 1 \cdot \sin t + t \cos t)$$

$$= at \cos t$$

Again, considering

$$y = a(\sin t - t \cos t)$$

Differentiating w.r.t. t, we get

$$\frac{dy}{dt} = a(\cos t - \cos t \cdot 1 + t \sin t)$$

$$= at \sin t \qquad \ldots \text{(i)}$$

Now, $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{at \sin t}{at \cos t} = \tan t$$

Again, differentiating w.r.t., x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dt}\right) \cdot \frac{dt}{dx} = \sec^2 t \left(\frac{1}{\frac{dx}{dt}}\right)$$

$$= \frac{\sec^2 t}{at \cos t} = \frac{1}{at} \sec^3 t$$

We need to find

$$\frac{d^2y}{dt^2} = \frac{d}{dt}.\left(\frac{dy}{dt}\right) = \frac{d}{dt}(at \sin t) \qquad \left[\text{from (i)}, \frac{dy}{dt} = at \sin t\right]$$

$$= a(\sin t + t \cos t).$$

S23. Here, we use chain rule, *i.e.*, if

$$y = f_1(\theta) \quad \text{and} \quad x = f_2(\theta).$$

then $$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

Given that $x = a(\theta - \sin \theta)$ and $y = a(1 + \cos \theta)$

First, we find $\frac{dy}{dx}$.

Now, $x = a(\theta - \sin\theta)$ and $y = a(1 + \cos\theta)$

Differentiating both equation w.r.t. θ, we have

$$\frac{dx}{d\theta} = a(1 - \cos\theta), \quad \frac{dy}{d\theta} = -a\sin\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{-a\sin\theta}{a(1-\cos\theta)} = \frac{-\sin\theta}{1-\cos\theta}$$

or $$\frac{dy}{dx} = \frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = -\cot\frac{\theta}{2}$$

Now, $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-\cot\frac{\theta}{2}\right) = \frac{d}{d\theta}\left(-\cot\frac{\theta}{2}\right)\times\frac{d\theta}{dx} \qquad \left[\because \frac{d}{dx}(f\theta) = \frac{d}{d\theta}(f\theta)\times\frac{d\theta}{dx}\right]$$

$$= \frac{1}{2}\operatorname{cosec}^2\frac{\theta}{2}\times\frac{1}{a(1-\cos\theta)} \qquad \left[\because \frac{d}{d\theta}\cot\theta = -\operatorname{cosec}^2\theta\right]$$

$$= \frac{1}{2a}\operatorname{cosec}^2\frac{\theta}{2}\times\frac{1}{2\sin^2\frac{\theta}{2}} \qquad \left[\because 1-\cos\theta = 2\sin^2\frac{\theta}{2}; \frac{1}{\sin\theta} = \operatorname{cosec}\theta\right]$$

$$= \frac{1}{4a}\operatorname{cosec}^4\frac{\theta}{2}.$$

S24. Given that $x = a(\cos\theta + \theta\sin\theta)$ ... (i)

and $y = a(\sin\theta - \theta\cos\theta)$ ... (ii)

First, we find $\frac{dy}{dx}$ using the formula

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} \quad \text{... (iii)}$$

Differentiating Eq. (i) and Eq. (ii) w.r.t. θ, we get

$$\frac{dx}{d\theta} = a(\cancel{-\sin\theta} + \theta\cos\theta + \cancel{\sin\theta})$$

$$\therefore \quad \frac{dx}{d\theta} = a\theta\cos\theta$$

and $$\frac{dy}{d\theta} = a(\cancel{\cos\theta} + \theta\sin\theta - \cancel{\cos\theta})$$

$\therefore$ $$\frac{dy}{d\theta} = a\theta\sin\theta$$

$\therefore$ By using Eq. (iii), we get

$$\frac{dy}{dx} = \frac{\cancel{a\theta}\sin\theta}{\cancel{a\theta}\cos\theta} = \tan\theta$$

Now, $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\tan\theta)$$

$$= \frac{d}{d\theta}(\tan\theta).\frac{d\theta}{dx} \qquad \left[\because \frac{d}{dx}[f(\theta)] = \frac{d}{d\theta}f(\theta)\times\frac{d\theta}{dx}\right]$$

$$= \sec^2\theta\times\frac{1}{a\theta\cos\theta} \qquad \left[\because \frac{1}{\cos\theta} = \sec\theta\right]$$

$\therefore$ $$\frac{d^2y}{dx^2} = \frac{\sec^3\theta}{a\theta}.$$

S25. Given that $x = a(\theta + \sin\theta)$ ... (i)

and $y = a(1 - \cos\theta)$ ... (ii)

First, we find $\frac{dy}{dx}$ using the formula

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} \qquad \text{... (iii)}$$

Differentiating Eq. (i) and Eq. (ii) w.r.t. x, we get

$$\frac{dx}{d\theta} = a(1 + \cos\theta),\ \frac{dy}{d\theta} = a\sin\theta \qquad \left[\because \frac{d}{d\theta}\sin\theta = \cos\theta, \frac{d}{d\theta}\cos\theta = -\sin\theta\right]$$

$\therefore$ $$\frac{dy}{dx} = \frac{a\sin\theta}{a(1+\cos\theta)} \qquad \text{[By Eq. (iii)]}$$

$$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} \qquad \left[\begin{array}{l}\because \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} \\ \text{and } 1+\cos x = 2\cos^2\frac{x}{2}\end{array}\right]$$

$\therefore \qquad \dfrac{dy}{dx} = \tan\dfrac{\theta}{2}$

Now, $\qquad \dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$

$$= \frac{d}{dx}\left(\tan\frac{\theta}{2}\right) \qquad \left[\because \frac{d}{dx}f(\theta) = \frac{d}{dx}f(\theta)\cdot\frac{d\theta}{dx}\right]$$

$$= \frac{d}{d\theta}\left(\tan\frac{\theta}{2}\right)\times\frac{d\theta}{dx}$$

$$= \sec^2\frac{\theta}{2}\cdot\frac{d}{d\theta}\left(\frac{\theta}{2}\right)\cdot\frac{d\theta}{dx} \qquad \left[\begin{array}{l}\because \dfrac{dx}{d\theta} = a(1+\cos\theta) \\ \therefore \dfrac{d\theta}{dx} = \dfrac{1}{a(1+\cos\theta)}\end{array}\right]$$

$$= \frac{1}{2}\sec^2\frac{\theta}{2}\times\frac{1}{a\times 2\cos^2\frac{\theta}{2}} \qquad \left[\because \frac{1}{\cos\theta} = \sec\theta \text{ and } 1+\cos x = 2\cos^2\frac{x}{2}\right]$$

$$= \frac{1}{4a}\sec^4\frac{\theta}{2}.$$

S26. Consider

$$x = \cos t + \log\tan\left(\frac{t}{2}\right)$$

Differentiating w.r.t. t, we have

$$\Rightarrow \qquad \frac{dx}{dt} = -\sin t + \frac{1}{\tan\left(\frac{t}{2}\right)}\cdot\sec^2\frac{t}{2}\cdot\frac{1}{2}$$

$$= -\sin t + \frac{\cos\left(\frac{t}{2}\right)}{\sin\left(\frac{t}{2}\right)}\cdot\frac{1}{\cos^2\left(\frac{t}{2}\right)}\cdot\frac{1}{2}$$

$$= -\sin t + \frac{1}{2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)}$$

$$= -\sin t + \frac{1}{\sin t} = \frac{-\sin^2 t + 1}{\sin t} = \frac{\cos^2 t}{\sin t}$$

Again, consider

$$y = \sin t$$

Differentiating w.r.t. t, we get

$$\frac{dy}{dt} = \cos t$$

Again differentiating w.r.t. t, we get

$$\frac{d^2y}{dt^2} = \frac{d}{dt}.\left(\frac{dy}{dt}\right) = \frac{d}{dt}(\cos t) = -\sin t$$

We need to find

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos t}{\frac{\cos^2 t}{\sin t}} = \frac{\sin t}{\cos t} = \tan t$$

Again, differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \sec^2 t \cdot \frac{dt}{dx} = \sec^2 t \cdot \frac{\sin t}{\cos^2 t} = \sec^4 t \sin t$$

at $t = \frac{\pi}{4}, \frac{d^2y}{dt^2} = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$

at $t = \frac{\pi}{4}, \frac{d^2y}{dx^2} = \sec^4\frac{\pi}{4} \cdot \sin\frac{\pi}{4} = (\sqrt{2})^4 \cdot \frac{1}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$.

S27. Given that $x = a\cos^3\theta$ and $y = a\sin^3\theta$

Differentiating w.r.t. θ, we get

$\Rightarrow$ $$\frac{dx}{d\theta} = 3a\cos^2\theta\frac{d}{d\theta}(\cos\theta)$$

$$\frac{dx}{d\theta} = 3a\cos^2\theta \cdot (-\sin\theta) = -3a\cos^2\theta \cdot \sin\theta$$

and $$\frac{dy}{d\theta} = 3a\sin^2\theta\frac{d}{d\theta}(\sin\theta)$$

$\Rightarrow$ $$\frac{dy}{d\theta} = 3a\sin^2\theta \cdot (\cos\theta) = 3a\sin^2\theta \cdot \cos\theta$$

$\therefore$ $$\frac{dy}{dx} = \left(\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\right) = \frac{3a\sin^2\theta \cdot \cos\theta}{-3a\cos^2\theta \cdot \sin\theta} = -\tan\theta$$

Again differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -\frac{d}{d\theta}(\tan\theta)\frac{d\theta}{dx} = -\sec^2\theta\cdot\frac{d\theta}{dx}$$

$$= -\sec^2\theta\cdot\left(\frac{-1}{3a\cos^2\theta\cdot\sin\theta}\right)$$

$$\frac{d^2y}{dx^2} = \frac{1}{3a\cos^4\theta\cdot\sin\theta}$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)_{\left(at\ \theta=\frac{\pi}{6}\right)} = \frac{1}{3a\left(\cos\frac{\pi}{6}\right)^4\left(\sin\frac{\pi}{6}\right)} = \frac{1}{3a\left(\frac{\sqrt{3}}{2}\right)^4\left(\frac{1}{2}\right)} = \frac{1}{3a\left(\frac{9}{16}\right)\left(\frac{1}{2}\right)} = \frac{32}{27a}.$$

S28. Given that $y = a\sin x + b\cos x$ ... (i)

To prove $y^2 + \left(\frac{dy}{dx}\right)^2 = a^2 + b^2$ ... (ii)

Differentiating Eq. (i) w.r.t. x on both sides, we get

$$\frac{dy}{dx} = a\cos x - b\sin x$$

Now, we take L.H.S. of Eq. (ii), we get

$$\text{L.H.S.} = y^2 + \left(\frac{dy}{dx}\right)^2$$

Putting value of y and $\frac{dy}{dx}$, we get

$$\text{L.H.S.} = (a\sin x + b\cos x)^2 + (a\cos x - b\sin x)^2$$

$[\because (a \pm b)^2 = a^2 + b^2 \pm 2ab]$

$$= a^2\sin^2 x + b^2\cos^2 x + 2ab\sin x\cos x + a^2\cos^2 x + b^2\sin^2 x - 2ab\sin x\cos x$$

$$= a^2\sin^2 x + b^2\cos^2 x + a^2\cos^2 x + b^2\sin^2 x$$

Taking a^2 and b^2 common, we get

$$\text{L.H.S.} = a^2(\sin^2 x + \cos^2 x) + b^2(\sin^2 x + \cos^2 x)$$

$$= a^2 + b^2 \qquad [\because \sin^2 x + \cos^2 x = 1]$$

$= \text{R.H.S.}$ **Hence proved.**

S29. Given, $\log(\sqrt{1+x^2} - x) = y\sqrt{1+x^2}$ ... (i)

To prove $(1+x^2)\frac{dy}{dx} + xy + 1 = 0$

Differentiating Eq. (i) w.r.t. x on both sides, we get

$$\frac{1}{\sqrt{1+x^2}-x}\frac{d}{dx}\left[\sqrt{1+x^2}-x\right] = y\frac{d}{dx}\sqrt{1+x^2}+\sqrt{1+x^2}\frac{dy}{dx}$$

$$\frac{1}{\sqrt{1+x^2}-x}\left[\frac{1}{2\sqrt{1+x^2}}\frac{d}{dx}(x)^2-1\right] = \frac{y}{2\sqrt{1+x^2}}\frac{d}{dx}(1+x^2)+\sqrt{1+x^2}\frac{dy}{dx}$$

$$\left[\because \frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}\right]$$

$$\Rightarrow \quad \frac{1}{\sqrt{1+x^2}-x}\left[\frac{2x}{2\sqrt{1+x^2}}-1\right] = y\times\frac{2x}{2\sqrt{1+x^2}}+\sqrt{1+x^2}\cdot\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{1}{\sqrt{1+x^2}-x}\left[\frac{x-\sqrt{1+x^2}}{\sqrt{1+x^2}}\right] = \frac{xy}{\sqrt{1+x^2}}+\sqrt{1+x^2}\cdot\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{-1}{\sqrt{1+x^2}} = \frac{xy+(1+x^2)\frac{dy}{dx}}{\sqrt{1+x^2}}$$

$$\Rightarrow \quad -1 = xy+(1+x^2)\frac{dy}{dx}$$

$$\Rightarrow \quad (1+x^2)\frac{dy}{dx}+xy+1 = 0.$$ **Hence proved.**

S30. Given that $y = e^x(\sin x + \cos x)$ ... (i)

To show $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y = 0$ ... (ii)

Differentiating Eq. (i) w.r.t. x on both sides by using the product rule

$$\Rightarrow \quad \frac{dy}{dx} = e^x\cdot\frac{d}{dx}(\sin x+\cos x)+(\sin x+\cos x)\cdot\frac{d}{dx}(e^x)$$

$$\Rightarrow \quad \frac{dy}{dx} = e^x(\cos x-\sin x)+(\sin x+\cos x)\cdot e^x$$

$$\Rightarrow \quad \frac{dy}{dx} = e^x(2\cos x) = 2e^x\cos x$$

$$\therefore \quad \frac{dy}{dx} = 2e^x\cos x \quad \text{... (iii)}$$

Differentiating Eq. (iii) w.r.t. x on both sides, we get

$$\frac{d^2y}{dx^2} = (2e^x)\cdot\frac{d}{dx}(\cos x)+\cos x\cdot\frac{d}{dx}(2e^x)$$

$$= 2e^x(-\sin x)+\cos x\,.\,2e^x$$

$\Rightarrow \qquad = 2e^x \cos x - 2e^x \sin x \qquad \text{... (iv)}$

Now, we put value of $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ from Eqs. (iv) and (iii) along with value of y in L.H.S. of the Eq. (ii).

$$\therefore \qquad \text{L.H.S.} = \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y$$

$$= (2e^x \cos x - 2e^x \sin x) - 2(2e^x \cos x) + 2e^x(\sin x + \cos x)$$

$$= 2e^x \cos x - 2e^x \sin x - 4e^x \cos x + 2e^x \sin x + 2e^x \cos x$$

$$= 4e^x \cos x - 4e^x \cos x$$

$$= 0 = \text{R.H.S.}$$

$\therefore \qquad$ L.H.S. = R.H.S. **Hence proved.**

S31. Given that $\qquad y = e^x \sin x \qquad$ ... (i)

First, we find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Differentiating Eq. (i) w.r.t. x on both sides, we get

$$\Rightarrow \qquad \frac{dy}{dx} = e^x \cdot \frac{d}{dx}(\sin x) + \sin x \cdot \frac{d}{dx}(e^x)$$

$$\left[\because \text{Using product rule, } \frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}\right]$$

$$\Rightarrow \qquad \frac{dy}{dx} = e^x \cos x + \sin x \cdot e^x \qquad \left[\because \frac{d}{dx}(\sin x) = \cos x; \frac{d}{dx}(e^x) = e^x\right]$$

Now, $\qquad \frac{dy}{dx} = e^x(\cos x + \sin x) \qquad$ ... (ii)

Again, differentiating Eq. (ii) on both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = e^x \cdot \frac{d}{dx}(\cos x + \sin x) + (\cos x + \sin x) \cdot \frac{d}{dx}(e^x)$$

$$\Rightarrow \qquad \frac{d^2y}{dx^2} = e^x(-\sin x + \cos x) + (\cos x + \sin x) \cdot e^x$$

$$= e^x[-\sin x + \cos x + \cos x + \sin x]$$

$$\therefore \qquad \frac{d^2y}{dx^2} = 2\cos x\, e^x \qquad \text{... (iii)}$$

Now, we have to show that

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

Putting the values of $\frac{d^2y}{dx^2}$ from Eq. (iii), $\frac{dy}{dx}$ from Eq. (ii) and that of y from Eq. (i) on L.H.S., we get

$$\text{L.H.S.} = \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y$$

$$= 2e^x \cos x - 2e^x (\cos x + \sin x) + 2e^x \sin x$$

$$= 2e^x \cos x - 2e^x \cos x - 2e^x \sin x + 2e^x \sin x$$

$= 0 = $ R.H.S. **Hence proved.**

S32. We have,

$$y = \left\{x + \sqrt{x^2+1}\right\}^m$$

Differentiating w.r.t. x both side

$$\therefore \quad \frac{dy}{dx} = m\cdot\left\{x + \sqrt{x^2+1}\right\}^{m-1} \times \frac{d}{dx}\left\{x + \sqrt{x^2+1}\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = m\left\{x + \sqrt{x^2+1}\right\}^{m-1} \times \left\{1 + \frac{2x}{2\sqrt{x^2+1}}\right\}$$

$$= m\frac{\left\{x + \sqrt{x^2+1}\right\}^m}{x + \sqrt{x^2+1}} \times \left\{\frac{\sqrt{1+x^2}+x}{\sqrt{x^2+1}}\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = m\cdot\frac{\left\{\sqrt{x^2+1}+x\right\}^m}{\sqrt{x^2+1}} = \frac{my}{\sqrt{x^2+1}}$$

$$\Rightarrow \quad y_1 = \frac{my}{\sqrt{x^2+1}}$$

$$\Rightarrow \quad y_1\sqrt{x^2+1} = my$$

Differentiating again w.r.t. x both side

$$\Rightarrow \quad y_1\frac{2x}{2\sqrt{x^2+1}} + \sqrt{x^2+1}\cdot y_2 = my_1$$

$$\Rightarrow \quad xy_1 + (x^2+1)\,y_2 = m\sqrt{1+x^2}\,y_1$$

$$\Rightarrow \quad y_2(1+x^2) + xy_1 = m\sqrt{1+x^2}\,\frac{my}{\sqrt{1+x^2}}$$

$$\Rightarrow \quad y_2(x^2+1) + xy_1 - m^2y = 0.$$

S33. To show $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0$ and

Given that $y = 3\cos(\log x) + 4\sin(\log x)$

Differentiating on both sides w.r.t. x, we get

$$\frac{dy}{dx} = -3\sin(\log x)\frac{d}{dx}(\log x) + 4\cos(\log x)\frac{d}{dx}(\log x)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-3\sin(\log x)}{x} + \frac{4\cos(\log x)}{x}$$

$$\Rightarrow \quad x\frac{dy}{dx} = -3\sin(\log x) + 4\cos(\log x)$$

Differentiating again on both sides w.r.t. x by using product rule, we get

$$x\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\cdot\frac{d}{dx}(x) = \frac{d}{dx}[-3\sin(\log x) + 4\cos(\log x)]$$

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot 1 = -3\cos(\log x)\frac{d}{dx}(\log x) - 4\sin(\log x)\frac{d}{dx}(\log x)$$

$$\Rightarrow \quad x\cdot\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot 1 = \frac{-3\cos(\log x)}{x} - \frac{4\sin(\log x)}{x}$$

$$\Rightarrow \quad x\cdot\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{-[3\cos(\log x) + 4\sin(\log x)]}{x}$$

$$\Rightarrow \quad x^2\cdot\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -[3\cos(\log x) + 4\sin(\log x)]$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -y \qquad [\because 3\cos(\log x) + 4\sin(\log x) = y\text{(given)}]$$

Hence, $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0$. **Hence proved.**

S34. We have, $y = x\log\left(\dfrac{x}{a+bx}\right)$

$$\Rightarrow \quad y = x\{\log x - \log(a+bx)\}$$

$$\Rightarrow \quad \frac{y}{x} = \log x - \log(a+bx)$$

Differentiating w.r.t. x, we get

$$\Rightarrow \quad \frac{x\frac{dy}{dx} - y}{x^2} = \frac{1}{x} - \frac{1}{a+bx}\frac{d}{dx}(a+bx)$$

$$\Rightarrow \qquad x\frac{dy}{dx} - y = x^2\left\{\frac{1}{x} - \frac{b}{a+bx}\right\}$$

$$\Rightarrow \qquad x\frac{dy}{dx} - y = x^2\left\{\frac{a + bx - bx}{x(a+bx)}\right\}$$

$$\Rightarrow \qquad x\frac{dy}{dx} - y = \frac{ax}{a+bx} \qquad \text{... (i)}$$

Differentiating both sides of (i) w.r.t. x, we get

$$\Rightarrow \qquad x\frac{d^2y}{dx^2} + \frac{dy}{dx} - \frac{dy}{dx} = \frac{(a+bx)\times a - ax(0+b)}{(a+bx)^2}$$

$$\Rightarrow \qquad x\frac{d^2y}{dx^2} = \frac{a^2}{(a+bx)^2}$$

$$\Rightarrow \qquad x^3\frac{d^2y}{dx^2} = \frac{a^2x^2}{(a+bx)^2} \qquad \text{[Multiplying both sides by } x^2\text{]}$$

$$\Rightarrow \qquad x^3\frac{d^2y}{dx^2} = \left(\frac{ax}{a+bx}\right)^2 \qquad \text{... (ii)}$$

From (i) and (ii), we have

$$x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2.$$

S35.

$$y = \log[x + \sqrt{x^2 + a^2}]$$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{x + \sqrt{x^2+a^2}}\frac{d}{dx}\left(x + \sqrt{x^2+a^2}\right)$$

$$\left[\because \frac{d}{dx}(\log x) = \frac{1}{x}\frac{d}{dx}(x)\right]$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+a^2}}\left(1 + \frac{2x}{2\sqrt{x^2+a^2}}\right)$$

$$= \frac{1}{x+\sqrt{x^2+a^2}}\cdot\left(\frac{x+\sqrt{x^2+a^2}}{\sqrt{x^2+a^2}}\right)$$

$$\therefore \qquad y_1 = \frac{1}{x+\sqrt{x^2+a^2}}\left(\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}\right)$$

$\therefore \qquad y_1(\sqrt{x^2+a^2}) = 1$

Differentiating again w.r.t. x, we get

$$\sqrt{x^2+a^2}\,\frac{d}{dx}(y_1) + y_1\frac{d}{dx}(\sqrt{x^2+a^2}) = \frac{d}{dx}(1)$$

$$\therefore \qquad y_2(\sqrt{x^2+a^2}) + \frac{1.2x\cdot y_1}{2\sqrt{x^2+a^2}} = 0 \qquad \left[\because \frac{d}{dx}(u\cdot v) = u\cdot\frac{dv}{dx} + v\cdot\frac{du}{dx}\right]$$

$$\therefore \qquad y_2(x^2 + a^2) + xy_1 = 0 \qquad \left[\text{where, } y_1 = \frac{dy}{dx} \text{ and } y_2 = \frac{d^2y}{dx^2}\right]$$

$$(x^2+a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$$

S36. To show $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - a^2y = 0$

Given that $\qquad y = e^{a\cos^{-1}x}, -1 \le x \le 1,$

Differentiating the given equation w.r.t. x on both sides, we get

$$\frac{dy}{dx} = e^{a\cos^{-1}x}\cdot\frac{d}{dx}(a\cos^{-1}x)\backslash$$

$$\frac{dy}{dx} = e^{a\cos^{-1}x}\times\frac{-a}{\sqrt{1-x^2}}$$

$$\left[\because \frac{d}{dx}e^x = e^x, \frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \qquad \sqrt{1-x^2}\,\frac{dy}{dx} = -ae^{a\cos^{-1}x}$$

$$\Rightarrow \qquad \sqrt{1-x^2}\,\frac{dy}{dx} = -ay \qquad \dots \text{(i)}$$

Again, differentiating above equation w.r.t. x on both sides, we get

$$\sqrt{1-x^2}\times\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\cdot\frac{d}{dx}\sqrt{1-x^2} = -a\frac{dy}{dx}$$

$$\sqrt{1-x^2}\cdot\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot\frac{1}{2\sqrt{1-x^2}}\cdot\frac{d}{dx}(1-x^2) = -a\cdot\frac{dy}{dx}$$

$$\Rightarrow \qquad \sqrt{1-x^2}\cdot\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot\frac{-2x}{2\sqrt{1-x^2}} = -a\cdot\frac{dy}{dx}$$

$$\Rightarrow \quad \sqrt{1-x^2}\,\frac{d^2y}{dx^2} - \frac{x}{\sqrt{1-x^2}}\cdot\frac{dy}{dx} = -a\frac{dy}{dx}$$

Multiplying on both sides by $\sqrt{1-x^2}$, we get

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = -a\sqrt{1-x^2}\cdot\frac{dy}{dx} \quad \text{... (ii)}$$

But from Eq. (i), we have

$$\Rightarrow \quad (\sqrt{1-x^2})\frac{dy}{dx} = -ay$$

or

$$\frac{dy}{dx} = \frac{-ay}{\sqrt{1-x^2}} \quad \text{... (iii)}$$

Putting value of $\frac{dy}{dx}$ from Eq. (iii) in Eq. (ii), we get

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = -a\sqrt{1-x^2}\times\frac{-ay}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = a^2y$$

Hence, $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$. **Hence proved**.

S37. To show

$$x(x^2-1)\frac{d^2y}{dx^2} + (2x^2-1)\frac{dy}{dx} = 0$$

Given that $y = \operatorname{cosec}^{-1} x$; $x > 1$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{-1}{x\sqrt{x^2-1}}$$

$$x\sqrt{x^2-1}\,\frac{dy}{dx} = -1$$

Differentiating again w.r.t. x both side

$$\therefore \quad (x\sqrt{x^2-1})\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\cdot\frac{d}{dx}(x\sqrt{x^2-1}) = \frac{d}{dx}(-1)$$

$$\left[\because \frac{d}{dx}(u\cdot v) = u\cdot\frac{dv}{dx} + v\cdot\frac{du}{dx}\right]$$

$$\Rightarrow \quad x\sqrt{x^2-1}\cdot\frac{d^2y}{dx^2}+\frac{dy}{dx}\left\{x\times\frac{d}{dx}\sqrt{x^2-1}+\sqrt{x^2-1}\times\frac{d}{dx}(x)\right\} = 0$$

$$\Rightarrow \quad x\sqrt{x^2-1}\frac{d^2y}{dx^2}+\frac{dy}{dx}\left\{\frac{x}{2\sqrt{x^2-1}}\frac{d}{dx}(x^2-1)+\sqrt{x^2-1}\times 1\right\} = 0$$

$$\Rightarrow \quad x\sqrt{x^2-1}\frac{d^2y}{dx^2}+\frac{dy}{dx}\left\{\frac{x\cdot 2x}{2\sqrt{x^2-1}}+\sqrt{x^2-1}\right\} = 0$$

$$\Rightarrow \quad x\sqrt{x^2-1}\frac{d^2y}{dx^2}+\frac{dy}{dx}\left\{\frac{x^2}{\sqrt{x^2-1}}+\sqrt{x^2-1}\right\} = 0$$

$$\Rightarrow \quad x\sqrt{x^2-1}\frac{d^2y}{dx^2}+\frac{x^2}{\sqrt{x^2-1}}\frac{dy}{dx}+\sqrt{x^2-1}\frac{dy}{dx} = 0$$

Multiplying on both sides by $\sqrt{x^2-1}$, we get

$$x(x^2-1)\frac{d^2y}{dx^2}+x^2\frac{dy}{dx}+(x^2-1)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x(x^2-1)\frac{d^2y}{dx^2}+(x^2+x^2-1)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x(x^2-1)\frac{d^2y}{dx^2}+(2x^2-1)\frac{dy}{dx} = 0.$$

Hence proved.

S38. Given, $y = (\cot^{-1} x)^2$

To show $(x^2+1)^2\frac{d^2y}{dx^2}+2x(x^2+1)\frac{dy}{dx}=2$

Now, $y = (\cot^{-1} x)^2$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 2\cdot\cot^{-1}x\cdot\frac{d}{dx}(\cot^{-1}x)$$

$$\frac{dy}{dx} = 2(\cot^{-1}x)\cdot\frac{-1}{1+x^2} \qquad \left[\because \frac{d}{dx}\cot^{-1}x=\frac{-1}{1+x^2}\right]$$

$$\Rightarrow \quad (1+x^2)\frac{dy}{dx} = -2\cot^{-1}x$$

Differentiating w.r.t. x again on both sides, we get

$$\Rightarrow \quad (1+x)^2\times\frac{d}{dx}\left(\frac{dy}{dx}\right)+\frac{dy}{dx}\times\frac{d}{dx}(1+x^2) \qquad \left[\because \frac{d}{dx}(u\cdot v)=u\frac{dv}{dx}+v\frac{du}{dx}\right]$$

$\Rightarrow \qquad = \frac{d}{dx}(-2\cot^{-1} x)$

$\Rightarrow \qquad (1+x)^2 \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot 2x = \frac{(-2)\times(-1)}{1+x^2}$ $\left[\because \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} \quad \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}\right]$

Multiplying on both sides by $(1 + x^2)$, we get

$(1+x^2)^2 \frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} = 2$. **Hence proved**.

S39. Given that

$$y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$$

Differentiating both sides w.r.t. x by using quotient rule in differentiation, we get

$$\frac{dy}{dx} = \frac{\sqrt{1-x^2} \times \frac{d}{dx}(\sin^{-1} x) - (\sin^{-1} x) \times \frac{d}{dx}\sqrt{1-x^2}}{(\sqrt{1-x^2})^2}$$

$$\frac{dy}{dx} = \frac{\sqrt{1-x^2} \times \frac{1}{\sqrt{1-x^2}} - \sin^{-1} x \cdot \frac{1}{2\sqrt{1-x^2}} \cdot \frac{d}{dx}(1-x^2)}{(\sqrt{1-x^2})}$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{\left[\sqrt{1-x^2} \cdot \frac{1}{\sqrt{1-x^2}} - (\sin^{-1} x) \cdot \frac{-2x}{2\sqrt{1-x^2}}\right]}{1-x^2}$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{1 + \frac{x \sin^{-1} x}{\sqrt{1-x^2}}}{(1-x^2)}$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{1+xy}{1-x^2} \qquad \left[\because \frac{\sin^{-1} x}{\sqrt{1-x^2}} = y \text{ (Given)}\right]$$

$$\Rightarrow \qquad (1-x^2)\frac{dy}{dx} = 1 + xy$$

Differentiating above equation w.r.t. x on both sides by using product rule, we get

$$(1-x^2) \cdot \frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx} \cdot \frac{d}{dx}(1-x^2) = \frac{d}{dx}(1+xy)$$

$$(1-x^2)\frac{d^2y}{dx^2} + \frac{dy}{dx}(-2x) = x\frac{dy}{dx} + y \cdot 1$$

$$\Rightarrow \qquad (1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - x\frac{dy}{dx} - y = 0$$

$$\Rightarrow \qquad (1-x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - y = 0.$$ **Hence proved.**

S40. We have,

$$x^2 + y^2 = (a\cos\theta + b\sin\theta)^2 + (a\sin\theta - b\cos\theta)^2$$

$$x^2 + y^2 = a^2\cos^2\theta + b^2\sin^2\theta + 2ab\sin\theta\cos\theta$$
$$+ a^2\sin^2\theta + b^2\cos^2\theta - 2ab\sin\theta\cos\theta$$

$$\Rightarrow \qquad x^2 + y^2 = a^2(\cos^2\theta + \sin^2\theta) + b^2(\sin^2\theta + \cos^2\theta)$$

$$\Rightarrow \qquad x^2 + y^2 = a^2 + b^2$$

Differentiating w.r.t. x, we get

$$2x + 2y \times \frac{dy}{dx} = 0$$

$$\Rightarrow \qquad \frac{dy}{dx} = -\frac{x}{y} \qquad \text{... (i)}$$

Differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -\left\{\frac{y\times 1 - x\frac{dy}{dx}}{y^2}\right\} = -\left\{\frac{y - x\left(-\frac{x}{y}\right)}{y^2}\right\} \qquad \text{[Using (i)]}$$

$$\Rightarrow \qquad \frac{d^2y}{dx^2} = -\frac{(x^2+y^2)}{y^3} \qquad \text{... (ii)}$$

Now,

$$y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = -y^2\left(\frac{x^2+y^2}{y^3}\right) - x\left(-\frac{x}{y}\right) + y \qquad \text{[Using (i) and (ii)]}$$

$$y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \frac{-x^2 - y^2 + x^2 + y^2}{y} = 0$$

$$y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$$ **Hence proved.**

S41. $x = \sin t$ and $y = \sin pt$

$$\frac{dx}{dt} = \cos t \quad \text{and} \quad \frac{dy}{dt} = p\cos pt$$

$$\therefore \qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{p\cos pt}{\cos t}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{\cos t \cdot -p \sin pt \cdot p\frac{dt}{dx} - p\cos pt \cdot (-\sin t)\frac{dt}{dx}}{\cos^2 t}$$

$$= \frac{-p^2 \sin pt \cos t + p\cos pt \sin t}{\cos^2 t} \cdot \frac{dt}{dx}$$

$$= \frac{-p^2 \sin pt \cos t + p\cos pt \sin t}{\cos^2 t} \cdot \frac{1}{\cos t}$$

$$= \frac{-p^2 \sin pt \cos t + p\cos pt \sin t}{\cos^3 t}$$

Now, $(1-x^2)\frac{d^2y}{dx^2} = (1-\sin^2 t)\left[\frac{-p^2 \sin pt \cos t + p\cos pt \sin t}{\cos^3 t}\right]$

$= -p^2 \sin pt + p \cos pt \tan t$

and $x\frac{dy}{dx} = \sin t \cdot \frac{p\cos pt}{\cos t} = p \cos pt \cdot \tan t$

and $p^2y = p^2 \sin pt$

Again now, $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y$

$= -p^2 \sin pt + p\cos pt \tan t - p\cos pt \tan t + p^2 \sin pt$

$= 0.$ **Hence Proved.**

S42. $y = A \sin 3x + B\cos 3x$

$\Rightarrow \quad \frac{dy}{dx} = 3A\cos 3x - 3B\sin 3x$

$\Rightarrow \quad \frac{d^2y}{dx^2} = -9A\sin 3x - 9B\cos 3x$

Now, $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = (-9A\sin 3x - 9B\cos 3x) + 4(3A\cos 3x - 3B\sin 3x)$
$+ 3(A\sin 3x + B\cos 3x)$

$= -6A\sin 3x - 12B\sin 3x - 6B\cos 3x + 12A\cos 3x$

$= (-6A - 12B)\sin 3x + (12A - 6B)\cos 3x$

But $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 10\cos 3x$

$\Rightarrow \quad (-6A - 12B)\sin 3x + (12A - 6B)\cos 3x = 10\cos 3x$

Comparing the coff. of $\sin 3x$ and $\cos 3x$ both side

$-6A - 12B = 0$ and $12A - 6B = 10$

$$A = -2B \qquad \text{and} \qquad 6A - 3B = 5$$

$$-12B - 3B = 5$$

$$B = \frac{-5}{15} = \frac{-1}{3} \quad \text{and} \quad A = \frac{2}{3}$$

Hence $B = \frac{-1}{3}$ and $A = \frac{2}{3}$

Date: 18/10/2021

Q1. Examine if Rolle's theorem is applicable on the function $f(x) = [x]$ for $x \in [5, 9]$. Can you say something about the converse of Rolle's theorem from the function?

Q2. Discuss the applicability of *Rolle's* theorem for the function $f(x) = 3 + (x-2)^{\frac{2}{3}}$ on the interval [1, 3].

Q3. Verify Rolle's Theorem for $f(x) = x^2 - x - 6$ on [– 2, 3].

Q4. Verify Rolle's Theorem for $f(x) = 2(x + 1)(x - 2)$ on [– 1, 2].

Q5. Verify Rolle's Theorem for $f(x) = x^2 - 6x + 5$ on [1, 5].

Q6. Verify Rolle's Theorem for $f(x) = x^2 - 8x + 12$ on [2, 6]:

Q7. Verify Rolle's Theorem for the function $f(x) = x^2 + 2x - 8$, $x \in [-4, 2]$.

Q8. Verify Rolle's Theorem for the function $f(x) = x^2 - x - 12$ in [– 3, 4].

Q9. Verify Rolle's Theorem for $f(x) = x^2 - 2x - 3$ on [–1, 3].

Q10. Verify Rolle's Theorem for $f(x) = x^2 + 5x + 6$ on [– 3, – 2].

Q11. Verify Rolle's theorem for the function $f(x) = x^2 - 5x + 6$ on the interval [2, 3].

Q12. Verify Roll's Theorem for the function $f(x) = x^3 - 6x^2 + 11x - 6$ in the interval [1, 3].

Q13. Verify Rolle's Theorem for the following function: $f(x) = (x - 1)(x - 2)^2$, [1, 2].

Q14. Verify Rolle's theorem for the function $f(x) = x^2 - 4x + 3$ on the interval [1, 3].

Q15. Verify Rolle's theorem for the function $f(x) = x(x - 3)^2$, $0 \le x \le 3$.

Q16. Verify Rolle's Theorem for $f(x) = (x^2 - 1)(x - 2)$ on [– 1, 2].

Q17. Verify Rolle's Theorem for the function $f(x) = x^3 + 3x^2 - 24x - 80$ in the interval [– 4, 5].

Q18. Discuss the applicability of *Rolle's* theorem for the function $f(x) = |x|$ on the interval [– 1, 1] .

Q19. Verify Rolle's theorem for the function $f(x) = \sqrt{4 - x^2}$ on [– 2, 2].

Q20. Discuss the applicability of Rolle's theorem on the function

$$f(x) = \begin{cases} x^2 + 1, & \text{when } 0 \le x \le 1 \\ 3 - x, & \text{when } 1 < x \le 2 \end{cases}.$$

Q21. Verify Rolle's theorem for the function $f(x) = (x - a)^m (x - b)^n$ on the interval $[a, b]$, where m, n are positive integers.

Q22. Verify Rolle's theorem for the function $f(x) = \sin^2 x$ on the interval $0 \le x \le \pi$.

Q23. Verify Rolle's theorem for the function $f(x) = \sin x + \cos x - 1$ on the interval $\left[0, \frac{\pi}{2}\right]$.

Q24. Verify Rolle's Theorem for the function $f(x) = \cos 2x$ in the interval $[0, \pi]$.

Q25. Verify Rolle's Theorem for $f(x) = e^x \sin x$ on $[0, \pi]$.

Q26. Verify Rolle's Theorem for $f(x) = \sin x + \cos x$ on $\left[0, \frac{\pi}{2}\right]$.

Q27. Verify Rolle's Theorem for $f(x) = \sin^4 x + \cos^4 x$ on $\left[0, \frac{\pi}{2}\right]$.

Q28. Verify Rolle's Theorem for $f(x) = e^{1-x^2}$ on $[-1, 1]$.

Q29. Verify Rolle's Theorem for $f(x) = \dfrac{\sin x}{e^x}$ on $[0, \pi]$.

Q30. Verify Rolle's theorem for the function $f(x) = e^x(\sin x - \cos x)$ on the interval $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$.

Q31. Verify Rolle's theorem for the function $f(x) = x(x+3)e^{\frac{-x}{2}}$ on the interval $[-3, 0]$.

Q32. Verify Rolle's theorem for the function $f(x) = \log\left\{\dfrac{x^2 + ab}{x(a+b)}\right\}$ on $[a, b]$, where $0 < a < b$.

Q33. Verify Rolle's theorem for the function $f(x) = \sin x - \sin 2x$ on the interval $[0, \pi]$.

Q34. Verify Rolle's Theorem for $f(x) = \log(x^2 + 2) - \log(3)$ on $[-1, 1]$.

Q35. At what point is the tangent to the curve $y = \log x$ parallel to the chord joining the points $A(1, 0)$ and $B(e, 1)$?

Q36. Using Rolle's theorem, find points on the curve $y = 16 - x^2$, $x \in [-1, 1]$, where tangent is parallel to x-axis.

Q37. It is given that for the function f given by $f(x) = x^3 + bx^2 + ax$, $x \in [1, 3]$ Rolle's theorem holds with $c = 2 + \dfrac{1}{\sqrt{3}}$. Find the values of a and b.

Q38. It is given that for the function $f(x) = x^3 - 6x^2 + ax + b$ on $[1, 3]$, Rolle's theorem holds with $c = 2 + \dfrac{1}{\sqrt{3}}$. Find the values of a and b, if $f(1) = f(3) = 0$.

Date: 18/10/2021

S1. $f(x) = [x]$ is discontinuous at $x = 5, 6, 7, 8, 9$ in $[5, 9]$. So, Rolle's theorem is not applicable.

The converse of Rolle's theorem does not hold good, because $f'(x) = 0$ for all $x \in (5, 6) \cup (6, 7) \cup (7, 8) \cup (8, 9)$. But $f(x)$ is neither continuous nor differentiable on $[5, 9]$.

S2. We have,

$$f(x) = 3 + (x-2)^{\frac{2}{3}},\ x \in [1, 3]$$

$$\Rightarrow \quad f'(x) = \left(\frac{2}{3}\right)(x-2)^{\frac{-1}{3}}$$

$$\Rightarrow \quad f'(x) = \frac{2}{3}\frac{1}{(x-2)^{1/3}}$$

Clearly, $\lim_{x \to 2} f'(x) = \infty$

So, $f(x)$ is not differentiable at $x = 2 \in (1, 3)$.

Hence, Rolle's theorem is not applicable to $f(x) = 3 + (x-2)^{\frac{2}{3}}$ on the interval $[1, 3]$.

S3. We have

Here observe that

(i) $f(x)$ being polynomial function, is continuous on $[-2, 3]$

(ii) $f'(x) = 2x - 1$, which clearly exists for all values of $x \in (-2, 3)$

(iii) Also, $f(-2) = 4 + 2 - 6 = 0$

$f(3) = 9 - 3 - 6 = 0$

$\Rightarrow \quad f(-2) = f(3)$

Thus, all the conditions of Rolle's theorem are satisfied.

Now, we have to show that there exists some $c \in (-2, 3)$, such that $f'(c) = 0$

we have $\quad f'(x) = 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$

Thus $\quad c = \frac{1}{2} \in (-2, 3)$ such that $f'(c) = 0$

Hence, Rolle's Theorem is verified.

S4. We have $f(x) = 2(x + 1)(x - 2)$

$$= (2x + 2)(x - 2) = 2x^2 - 4x + 2x - 4 = 2x^2 - 2x - 4 \text{ on } [-1, 2]$$

Here we observe that

(i) $f(x)$ being polynomial function is continuous on $[-1, 2]$

(ii) $f'(x) = 4x - 2$, which clearly exists for all values of $x \in (-1, 2)$

(iii) Also, $f(-1) = 2(-1)^2 - 2(-1) - 4 = 2 + 2 - 4 = 0$

And $f(2) = 2(2)^2 - 2(2) - 4 = 8 - 4 - 4 = 0$

$\Rightarrow$ $f(-1) = f(2)$

Thus all the conditions of Rolle's theorem are satisfied.

Now, we have to show there exists, some $c \in (-1, 2)$, such that $f'(c) = 0$

We have $f'(x) = 4x - 2 = 0 \quad \Rightarrow \quad x = \frac{1}{2}$

Thus $c = \frac{1}{2} \in (-1, 2)$ such that $f'(c) = 0$

Hence, Rolle's Theorem is verified.

S5. We have

Here we observe that

(i) $f(x)$ being polynomial function, is continuous on $[1, 5]$

(ii) $f'(x) = 2x - 6$, which clearly exists for all values of $x \in (1, 5)$, so $f(x)$ is differentiable on interval $(1, 5)$.

(iii) Also, $f(1) = 1 - 6 + 5 = 0,$

$f(5) = 25 - 30 + 5 = 0$

$\Rightarrow$ $f(1) = f(5)$

Thus all the conditions of Rolle's theorem are satisfied.

Now, we have to show that there exists some $c \in (1, 5)$, such that $f'(c) = 0$

We have, $f(x) = x^2 - 6x + 5 \quad \Rightarrow \quad f'(x) = 2x - 6$

$\therefore$ $f'(x) = 2x - 6 = 0 \quad \Rightarrow \quad x = 3$

Thus, $c = 3 \in (1, 5)$ such that $f'(c) = 0$

Here, Rolle's theorem is verified.

S6. We have,

Here we observe that

(i) $f(x)$ being polynomial function, is continuous on $[2, 6]$

(ii) $f'(x) = 2x - 8$, which clearly exists for all values of $x \in (2, 6)$.

So, $f(x)$ is differentiable on interval (2, 6).

(iii) Also, $f(2) = 4 - 16 + 12 = 0$,

$f(6) = 36 - 48 + 12 = 0$

$\Rightarrow$ $f(2) = f(6)$

Thus, all the conditions of Rolle's theorem are satisfied

Now, we have to show that there exists some $c \in (2, 6)$, such that $f'(c) = 0$

We have, $f(x) = x^2 - 8x + 12 \Rightarrow f'(x) = 2x - 8$

$\therefore$ $f'(x) = 2x - 8 = 0 \Rightarrow x = 4$

Thus, $c = 4 \in (2, 6)$ such that $f'(c) = 0$

Here, Rolle's theorem is verified.

S7. We have, $f(x) = x^2 + 2x - 8$

Here, we observe that

(i) $f(x)$ being polynomial function, is continuous on $[-4, 2]$

(ii) $f'(x) = 2x + 2$, which clearly exists for all value of $x \in (-4, 2)$

So, $f(x)$ is differentiable on the interval $(-4, 2)$

(iii) Also, $f(-4) = (-4)^2 + 2(-4) - 8 = 16 - 8 - 8 = 0$

and $f(2) = (2)^2 + (2)(2) - 8 = 4 + 4 - 8 = 0$

$\Rightarrow$ $f(-4) = f(2)$

Thus, all the conditions of Rolle's theorem are satisfied.

Now, we have to show that there exists some $c \in (-4, 2)$, such that $f'(c) = 0$

We have, $f(x) = x^2 + 2x - 8 \Rightarrow f'(x) = 2x + 2$

$\therefore$ $f'(x) = 2x + 2 = 0 \Rightarrow x = -1$

Thus, $c = -1 \in (-4, 2)$ such that $f'(c) = 0$.

Hence, Rolle's theorem is verified.

S8. Here, $f(x) = x^2 - x - 12$ in $[-3, 4]$.

(i) Being a polynomial, $f(x)$ is continuous in $[-3, 4]$

(ii) $f(x)$ is differentiable in $(-3, 4)$.

(iii) Also $f(-3) = (-3)^2 - (-3) - 12 = 9 + 3 - 12 = 0$

and $f(4) = 4^2 - 4 - 12 = 16 - 4 - 12 = 0$

$\Rightarrow$ $f(-3) = f(4) = 0$

$\therefore$ All the conditions of Rolle's theorem are satisfied.

$\Rightarrow$ $c \in (-3, 4)$ such that $f'(c) = 0$

Now, $f'(x) = 0 \;\Rightarrow\; 2x - 1 = 0 \;\Rightarrow\; x = \frac{1}{2}$

Clearly, $c = \frac{1}{2} \in (-3, 4)$

$\therefore$ Rolle's theorem is verified.

S9. We have,

Here we observe that

(i) $f(x)$ being polynomial function, is continuous on $[-1, 3]$

(ii) $f'(x) = 2x - 2$, which clearly exists for all values of $x \in (-1, 3)$, so $f(x)$ is differentiable on interval $(-1, 3)$.

(iii) Also, $f(-1) = 1 + 2 - 3 = 0$,

$f(3) = 9 - 6 - 3 = 0$

$\Rightarrow$ $f(-1) = f(3)$

Thus all the conditions of Rolle's theorem are satisfied.

Now, we have to show that there exists some $c \in (-1, 3)$, such that $f'(c) = 0$

We have $f(x) = x^2 - 2x - 3 \;\Rightarrow\; f'(x) = 2x - 2$

$f'(x) = 2x - 2 = 0 \;\Rightarrow\; x = 1$

Thus, $c = 1 \in (-1, 3)$ such that $f'(c) = 0$

Hence, Rolle's theorem is verified.

S10. We have

Here we observe that

(i) $f(x)$ being polynomial function, is continuous on $[-3, -2]$

(ii) $f'(x) = 2x + 5$, which clearly exists for all values of $x \in (-3, -2)$

(iii) Also, $f(-3) = 9 - 15 + 6 = 0$,

$f(-2) = 4 - 10 + 6 = 0$

$\Rightarrow$ $f(-3) = f(-2)$

Thus all the conditions of Rolle's Theorem are satisfied.

Now, we have to show that there exists some $c \in (-3, -2)$, such that $f'(c) = 0$

We have $f'(x) = 2x + 5 = 0 \Rightarrow x = -\frac{5}{2}$

Thus $c = -\frac{5}{2} \in (-3, -2)$ such that $f'(c) = 0$

Hence, Rolle's Theorem is verified.

S11. Since a polynomial function is every where differentiable and so continuous also. Therefore,

(i) $f(x)$ is continuous on [2, 3] and (ii) $f(x)$ is differentiable on (2, 3)

Also, $f(2) = 2^2 - 5 \times 2 + 6 = 0$ and $f(3) = 3^2 - 5 \times 3 + 6 = 0$

$\therefore$ $f(2) = f(3)$

Thus all the conditions of Rolle's theorem are satisfied. Now we have to show that there exists some $c \in (2, 3)$ such that $f'(c) = 0$.

We have, $f(x) = x^2 - 5x + 6 \Rightarrow f'(x) = 2x - 5$

$\therefore$ $f'(x) = 2x - 5 = 0 \Rightarrow x = 2.5$

Thus, $c = 2.5 \in (2, 3)$ such that $f'(c) = 0$.

Hence, Rolle's theorem is verified.

S12. We have,

$$f(x) = x^3 - 6x^2 + 11x - 6 \quad \text{(polynomial)}$$

We know that a polynomial function is everywhere continuous and differentiable. Therefore

(i) It is continuous on [1, 3]

(ii) It is differentiable on (1, 3)

(iii) Also, $f(1) = 1^3 - 6 \times 1^2 + 11 \times 1 - 6 = 1 - 6 + 11 - 6 = 12 - 12 = 0$

and $f(3) = 3^3 - 6 \times 3^2 + 11 \times 3 - 6 = 27 - 54 + 33 - 6 = 60 - 60 = 0$

$\Rightarrow$ $f(1) = f(3)$

Thus, all the conditions of Rolle's Theorem are satisfied So, there must exist some $c \in (1, 3)$, such that $f'(c) = 0$

Now, $f'(c) = 3c^2 - 12c + 11 = 0$

$$\Rightarrow \quad c = \frac{12 \pm \sqrt{144 - 132}}{6}$$

$$\Rightarrow \quad c = \left(2 \pm \frac{1}{\sqrt{3}}\right)$$

Clearly, both the value of c lie. in the interval (1, 3).

Hence, Rolle's Theorem is verified.

S13. We have,

$$f(x) = (x-1)(x-2)^2,\ x \in [1, 2] \qquad \ldots (1)$$

$\Rightarrow \quad f(x) = (x-1)(x^2-4x+4),\ x \in [1, 2]$

$\Rightarrow \quad f(x) = x^3 - 4x^2 + 4x - x^2 + 4x - 4,\ x \in [1, 2]$

$\Rightarrow \quad f(x) = x^3 - 5x^2 + 8x - 4,\ x \in [1, 2]$

(i) $f(x)$ being a polynomial of x, is continuous in [1, 2]

(ii) $f'(x) = 3x^2 - 10x + 8$ which exists on (1, 2)

$\Rightarrow$ $f(x)$ is differentiable on (1, 2)

(iii) $f(1) = (1-1)(1-2)^2 = 0$

and $f(2) = (2-1)(2-2)^2 = 0$

$\Rightarrow \quad f(1) = f(2)$

Thus, all the conditions of Rolle's theorem are satisfied. So there must exist at least one value of c in (1, 2) such that $f'(c) = 0$

$\Rightarrow \quad 3c^2 - 10c + 8 = 0$

$\Rightarrow \quad c = \dfrac{10 \pm \sqrt{100-96}}{6} \quad \Rightarrow \quad c = \dfrac{10 \pm \sqrt{4}}{6} = \dfrac{10 \pm 2}{6} = 2 \text{ or } \dfrac{4}{3}$

Neglecting $c = 2$ as $2 \notin (1, 2)$

so, $c = \dfrac{4}{3} \in (1, 2)$

Hence, Rolle's theorem is verified.

S14. We have, $f(x) = x^2 - 4x + 3,\ x \in [1, 3]$

Clearly, $f(x)$ being a polynomial function, is continuous on [1, 3] and differentiable on (1, 3).

Also $f(1) = (1)^2 - 4 \times 1 + 3 = 1 - 4 + 3 = 0$

and $f(3) = 3^2 - 4 \times 3 + 3 = 9 - 12 + 3 = 0$

$\therefore \quad f(1) = f(3)$

Thus, $f(x)$ satisfies all the conditions of Rolle's theorem.

Now, $f(x) = x^2 - 4x + 3$

$\Rightarrow \quad f'(x) = 2x - 4$

$\therefore \quad f'(x) = 0 \Rightarrow x = 2$

Clearly, $c = 2 \in (1, 3)$ such that $f'(c) = 0$.

Hence, Rolle's theorem is verified.

S15. We have, $f(x) = x^3 - 6x^2 + 9x$

We know that a polynomial function is everywhere differentiable and so continuous also. So, $f(x)$ being a polynomial function is continuous on [0, 3] and differentiable on (0, 3). Also

$$f(0) = 0(0-3)^2 = 0 \text{ and } f(3) = 3(3-3)^2 = 3(0)^2 = 0$$

$\therefore$ $f(0) = f(3) = 0$

Thus, all the conditions of Rolle's theorem are satisfied. Now we have to show that there exists $c \in (0, 3)$ such that $f'(c) = 0$.

We have, $f(x) = x^3 - 6x^2 + 9x$

$\Rightarrow$ $f'(x) = 3x^2 - 12x + 9$

$\therefore$ $f'(x) = 0 \Rightarrow 3x^2 - 12x + 9 = 0$

$\Rightarrow$ $x^2 - 4x + 3 = 0 \Rightarrow x = 1, 3$

Thus, $c = 1 \in (0, 3)$ such that $f'(c) = 0$

Hence, Rolle's theorem is verified.

S16. $f(x) = (x^2 - 1)(x - 2)$ on $[-1, 2]$

As $f(x)$ is a polynomial, $f(x)$ is continuous on [–1, 2] and differentiable on (–1, 2). Further

$$f(-1) = ((-1)^2 - 1)(-1-2) = 0 \times (-3) = 0$$

and $f(2) = (2^2 - 1)(2 - 2) = 3 \times 0 = 0$

$\therefore$ $f(-1) = f(2)$

So that all hypothesis of Rolle's theorem are fulfilled.

There must be some $c \in (-1, 2)$. Such that $f'(c) = 0$

We have $f'(x) = (x^2 - 1) + (x - 2)2x = 3x^2 - 4x - 1$

$f'(c) = 0$

$\Rightarrow$ $3c^2 - 4c - 1 = 0 \Rightarrow c = \dfrac{4 \pm \sqrt{16+12}}{6} = \dfrac{2 \pm \sqrt{7}}{3}$

The value of c in Rolle's theorem is

$$c = \frac{2 \pm \sqrt{7}}{3} \in (-1, 2)$$

s.t $f'(c) = 0.$

S17. Here, $f(x) = x^3 + 3x^2 - 24x - 80$ in $[-4, 5]$.

We know that a polynomial function is everywhere continuous and differentiable. Therefore:

(i) $f(x)$ being a polynomial, is continuous in $[-4, 5]$.

(ii) $f(x)$ being a polynomial is derivable in $(-4, 5)$.

(iii) Also, $f(-4) = (-4)^3 + 3(-4)^2 - 24(-4) - 80 \Rightarrow f(-4) = -64 + 48 + 96 - 80 = 0$.

and $f(5) = (5)^3 + 3(5)^2 - 24(5) - 80 \Rightarrow f(5) = 125 + 75 - 120 - 80 = 0$

$\Rightarrow \quad f(-4) = f(5)$

$\therefore$ All the conditions of the Rolle's Theorem are satisfied.

$\Rightarrow \quad c \in [-4, 5]$ such that $f'(c) = 0$.

Now, $\quad f'(x) = 3x^2 + 6x - 24$.

$\Rightarrow \quad f'(x) = 0 \Rightarrow 3(x^2 + 2x - 8) = 0$

$\Rightarrow \quad (x + 4)(x - 2) = 0$

$\Rightarrow \quad x = -4, 2$.

Clearly, $\quad c = 2 \in (-4, 5)$ such that $f'(c) = 0$

Hence, Rolle's Theorem is verified.

S18. We have,

$$f(x) = |x| = \begin{cases} -x, \text{ when } -1 \le x < 0 \\ x, \text{ when } 0 \le x < 1 \end{cases}$$

Since a polynomial function is everywhere continuous and differentiable. Therefore $f(x)$ is continuous and differentiable for all $x < 0$ and for all $x > 0$ except possibly at $x = 0$.

So, consider the point $x = 0$.

We have, $\quad \lim_{x \to 0^-} f(x) = \lim_{x \to 0} -x = 0$ and $\lim_{x \to 0^+} f(x) = \lim_{x \to 0} x = 0$

$\therefore \quad \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$

Thus $f(x)$ is continuous at $x = 0$

Hence, $f(x)$ is continuous on $[-1, 1]$.

Now, $\quad (\text{LHD at } x = 0) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0}$

$\Rightarrow \quad (\text{LHD at } x = 0) = \lim_{x \to 0} \frac{-x - 0}{x}$ $\quad [\because f(x) = -x \text{ for } x < 0 \text{ and } f(0) = 0]$

$\Rightarrow \quad (\text{LHD at } x = 0) = \lim_{x \to 0} \frac{-x}{x} = \lim_{x \to 0} -1 = -1$

and $\quad (\text{RHD at } x = 0) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x - 0}{x}$ $\quad [\because f(x) = x \text{ for } x \ge 0]$

$\Rightarrow \quad (\text{RHD at } x = 0) = \lim_{x \to 0} \frac{x}{x} = \lim_{x \to 0} 1 = 1$

$\therefore \quad (\text{LHD at } x = 0) \ne (\text{RHD at } x = 0)$

This shows that $f(x)$ is not differentiable at $x = 0 \in (-1, 1)$. Thus, the condition of derivability at each point of $(-1, 1)$ is not satisfied.

Hence, Rolle's theorem is not applicable to $f(x) = |x|$ on $[-1, 1]$.

S19. Clearly, $f(x)$ is defined for all $x \in [-2, 2]$ and has a unique value for each $x \in [-2, 2]$. So, at each point of $[-2, 2]$, the limit of $f(x)$ is equal to the value of the function. Therefore, $f(x)$ is continuous on $[-2, 2]$.

Also, $f'(x) = \dfrac{-x}{\sqrt{4-x^2}}$ exists for all $x \in (-2, 2)$

So, $f(x)$ s differentiable on $(-2, 2)$.

Also, $f(-2) = f(2) = 0$

Thus, all the three conditions of Rolle's theorem are satisfied.

Now, we have to show that three exists $c \in (-2, 2)$ such that $f'(c) = 0$.

We have, $f(x) = \sqrt{4-x^2} \Rightarrow f'(x) = \dfrac{-x}{\sqrt{4-x^2}}$

$\therefore$ $f'(x) = 0 \Rightarrow \dfrac{-x}{\sqrt{4-x^2}} = 0 \Rightarrow x = 0$

Since $c = 0 \in (-2, 2)$ such that $f'(c) = 0$.

Hence, Rolle's theorem is verified.

S20. Since a polynomial function is everywhere continuous and differentiable. Therefore, $f(x)$ is continuous and differentiable at all points except possibly at $x = 1$.

Now, we consider the differentiability of $f(x)$ at $x = 1$.

We have, $(\text{LHD at } x = 1) = \lim_{x\to1^-} \dfrac{f(x)-f(1)}{x-1}$

$\Rightarrow$ $(\text{LHD at } x = 1) = \lim_{x\to1} \dfrac{(x^2+1)-(1+1)}{x-1}$ $[\because f(x) = x^2 + 1 \text{ for } 0 \le x \le 1]$

$\Rightarrow$ $(\text{LHD at } x = 1) = \lim_{x\to1} \dfrac{x^2-1}{x-1} = \lim_{x\to1} (x+1) = 2$

and, $(\text{RHD at } x = 1) = \lim_{x\to1^+} \dfrac{f(x)-f(1)}{x-1} = \lim_{x\to1} \dfrac{(3-x)-(1+1)}{x-1}$

$\Rightarrow$ $(\text{RHD at } x = 1) = \lim_{x\to1} \dfrac{-(x-1)}{x-1} = -1$

$\therefore$ $(\text{LHD at } x = 1) \neq (\text{RHD at } x = 1)$

So, $f(x)$ is not differentiable at $x = 1$.

Since the condition of differentiability at each point of the given interval is not satisfied, hence Roll's theorem is not applicable to the given function on [0,2].

S21. We have, $f(x) = (x - a)^m (x - b)^n$ where $m, n \in N$

On expanding $(x - a)^m$ and $(x - b)^n$ by binomial theorem and then taking the product, we find that $f(x)$ is a polynomial of degree $(m + n)$. Since a polynomial function is everywhere differentiable and so continuous also. Therefore,

(i) $f(x)$ is continuous on $[a, b]$ and (ii) $f(x)$ is derivable on (a, b)

Also, $f(a) = (a - a)^m (a - b)^n = 0$

$f(b) = (b - a)^m (b - b)^n = 0$

$\therefore$ $f(a) = f(b) = 0$

Thus, all the three conditions of Rolle's theorem are satisfied.

Now, we have to show that there exists $c \in (a, b)$ such that $f'(c) = 0$.

We have, $f(x) = (x - a)^m (x - b)^n$

$\Rightarrow$ $f'(x) = m(x - a)^{m-1}(x - b)^n + (x - a)^m n(x - b)^{n-1}$

$\Rightarrow$ $f'(x) = (x - a)^{m-1}(x - b)^{n-1}\{m(x - b) + n(x - a)\}$

$\Rightarrow$ $f'(x) = (x - a)^{m-1}(x - b)^{n-1}\{x(m + n) - (mb + na)\}$

$\therefore$ $f'(x) = 0$

$\Rightarrow$ $(x - a)^{m-1}(x - b)^{n-1}\{x(m + n) - (mb + na)\} = 0$

$\Rightarrow$ $(x - a) = 0$ or $(x - b) = 0$ or $x(m + n) - (mb + na) = 0$

$\Rightarrow$ $x = a$ or $x = b$ or $x = \dfrac{mb + na}{m + n}$

Since $x = \dfrac{mb + na}{m + n}$ divides (a, b) into the ratio $m : n$. Therefore, $\dfrac{mb + na}{m + n} \in (a, b)$

Thus, $c = \dfrac{mb + na}{m + n} \in (a, b)$ such that $f'(c) = 0$

Hence, Rolle's theorem is verified.

S22. Since $\sin x$ is everywhere continuous and differentiable and the product of two continuous (differentiable) functions is continuous (differentiable). Therefore, $f(x) = \sin^2 x = \sin x \cdot \sin x$ is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$.

Also, $f(0) = \sin^2 0 = 0$ and $f(\pi) = \sin^2 \pi = 0$

$\therefore$ $f(0) = f(\pi)$

Thus, $f(x)$ satisfies all the three conditions of Rolle's theorem.

Now, we have to show that there exists $c \in (0, \pi)$ such that $f'(c) = 0$.

We have, $f(x) = \sin^2 x \Rightarrow f'(x) = 2 \sin x \cos x = \sin 2x$

$\therefore$ $f'(x) = 0 \Rightarrow \sin 2x = 0 \Rightarrow 2x = \pi \Rightarrow x = \dfrac{\pi}{2}$

Thus $\qquad c = \dfrac{\pi}{2} \in (0, \pi)$ such that $f'(c) = 0$

Hence, Rolle's theorem is verified.

S23. Since sin x and cos x are everywhere continuous and differentiable.

Therefore, $f(x) = \sin x + \cos x - 1$ is continuous on $\left[0, \dfrac{\pi}{2}\right]$ and differentiable on $\left(0, \dfrac{\pi}{2}\right)$

Also, $\qquad f(0) = \sin 0 + \cos 0 - 1 = 0 + 1 - 1 = 0$

$$f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} + \cos\frac{\pi}{2} - 1 = 1 + 0 - 1 = 0$$

$\therefore \qquad f(0) = f\left(\dfrac{\pi}{2}\right)$

Thus, $f(x)$ satisfies conditions of Rolle's theorem on $\left[0, \dfrac{\pi}{2}\right]$. Therefore, there exists $c \in \left(0, \dfrac{\pi}{2}\right)$ such that $f'(c) = 0$.

Now, $\qquad f(x) = \sin x + \cos x - 1 \;\Rightarrow\; f'(x) = \cos x - \sin x$

$\therefore \qquad f'(x) = 0 \;\Rightarrow\; \cos x - \sin x = 0 \Rightarrow \sin x = \cos x \Rightarrow \tan x = 1 \Rightarrow x = \dfrac{\pi}{4}$

Thus $c = \dfrac{\pi}{4} \in \left(0, \dfrac{\pi}{2}\right)$ such that $f'(c) = 0$

Hence, Rolle's theorem is verified.

S24. We have,

$$f(x) = \cos 2x, \quad x \in [0, \pi]$$

Since, cos x is continuous and differentiable everywhere, therefore.

(i) $f(x)$ is continuous in $[0, \pi]$

(ii) $f(x)$ is differentiable on $(0, \pi)$

(iii) Also, $f(0) = \cos 0 = 1$ and $f(x) = \cos 2\pi = 1$

$\Rightarrow \qquad f(0) = f(\pi)$

Thus, all the conditions of Rolle's Theorem are satisfied. So, there must exist at least one real value of c in $(0, \pi)$, such that $f'(c) = 0$

$$f'(x) = -2 \sin 2x = 0 \;\Rightarrow\; 2x = \pi \;\Rightarrow\; x = \frac{\pi}{2}.$$

$\Rightarrow \qquad c = \dfrac{\pi}{2} \in (0, \pi)$, such that $f'(c) = 0$

Hence Rolle's Theorem is verified.

S25. $f(x) = e^x \sin x$ on $[0, \pi]$

$f(x)$ is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$. Further.

$f(0) = e^0 \sin 0 = 1 \times 0 = 0$

$f(\pi) = e^\pi \sin \pi = e^\pi \times 0 = 0$

$\therefore$ $f(0) = f(\pi)$

Thus, $f(x)$ satisfies all the conditions of Rolle's theorem. Therefore there must exist $c \in (0, \pi)$

Such that $f'(c) = 0$

$f'(x) = e^x \cos x + e^x \sin x = e^x(\cos x + \sin x)$.

$\Rightarrow$ $f'(c) = e^c (\cos c + \sin c)$

$\because$ $f'(c) = 0 \Rightarrow e^c (\cos c + \sin c) = 0$

$\Rightarrow$ $\cos c + \sin c = 0$

$\Rightarrow$ $\tan c = -1 \Rightarrow c = 3(\pi/4)$

Thus, there exists $c = 3(\pi/4) \in (0, \pi)$ *s.t.* $f'(c) = 0$.

Hence Rolle's Theorem is verified.

S26. $f(x) = \sin x + \cos x$

Clearly, $f(x)$ is continuous on $[0, \pi/2]$ and differentiable on $(0, \pi/2)$. Further

$f(0) = \sin(0) + \cos(0) = 1$ and $f(\pi/2) = \sin(\pi/2) + \cos(\pi/2) = 1$

$\therefore$ $f(0) = f(\pi/2)$

Thus, all the hypothesis of Rolle's theorem are satisfied.

Thus there must exist $c \in (0, \pi/2)$

such that $f'(\text{c}) = 0$

$f'(\text{c}) = \cos \text{c} - \sin \text{c}$

$f'(\text{c}) = 0$

$\Rightarrow$ $\cos c - \sin c = 0$

$\Rightarrow$ $\cos c = \sin c \Rightarrow \tan c = 1$

$\Rightarrow$ $c = \pi/4 \in (0, \pi/2)$

Thus value of c in Rolle's theorem is $c = \pi/4$ such that $f'(\text{c}) = 0$.

Hence Rolle's Theorem is verified.

S27. $f(x) = \sin^4 x + \cos^4 x$

$= (\sin^2 x + \cos^2 x)^2 - 2 \sin^2 x.\cos^2 x$

$= 1 - \frac{1}{2}(\sin 2x)^2$ on $[0, \pi/2]$

Function is continuous on $[0, \pi/2]$ and differentiable on $(0, \pi/2)$. Further

$$f(0) = 1 - \frac{1}{2}(\sin 0) = 1$$

$$f(\pi/2) = 1 - \frac{1}{2}(\sin \pi)^2 = 1$$

$\therefore \qquad f(0) = f(\pi/2)$

All the conditions of Rolle's theorem are satisfied thus there must exist a real no. $c \in \left(0, \frac{\pi}{2}\right)$ such that $\quad f'(c) = 0$

$$f'(x) = 4 \sin^3 x . \cos x - 4 \cos^3 x \sin x$$

$$f'(c) = 4 \sin^3 c . \cos c - 4 \cos^3 c . \sin c$$

$$f'(c) = 0 \quad \Rightarrow \quad 4 \sin^3 c . \cos c - 4 \cos^3 c . \sin c = 0$$

$\Rightarrow \qquad 4 \sin^3 c . \cos c = 4 \cos^3 c . \sin c \quad$ or $\quad \tan^3 c = \tan c \quad$ or $\quad \tan c(\tan^2 c - 1) = 0$

$\Rightarrow \qquad \tan c = 0 \quad$ or $\quad \tan^2 c = 1 \quad \Rightarrow \quad c = 0 \quad$ or $\quad c = \pi/4$

$0 \notin (0, \pi/2)$ but $\pi/4 \in (0, \pi/2)$.

Thus value of c in $(0, \pi/2)$ is $c = \pi/4$ such that $f'(c) = 0$.

Hence Rolle's Theorem is verified.

S28. We have,

$$f(x) = e^{1-x^2}$$

Since, the exponential function is continuous for all real x, therefore,

(i) $f(x)$ is continuous in $[-1, 1]$

(ii) $f'(x) = -2xe^{1-x^2}$ which exists for all $x \in (-1, 1)$

(iii) Also, $f(-1) = e^{1-1} = e^0 = 1$ and $f(1) = e^{1-1} = e^0 = 1$

$\therefore \qquad f(-1) = f(1)$

Thus, all the conditions of Rolle's Theorem are satisfied. So there must exist a real value in $(-1, 1)$ such that $f'(c) = 0$

$\Rightarrow \qquad -2ce^{1-c^2} = 0 \quad \Rightarrow \quad c = 0 \in (-1, 1) \qquad \left[\because \ e^{1-c^2} \neq 0, \forall c \in R\right]$

Hence, Rolle's Theorem is verified.

S29. We have,

$$f(x) = \frac{\sin x}{e^x}$$

Since, $\sin x$ and e^x are continuous everywhere, and $e^x \neq 0$ for any real value of x.

$\therefore$ (i) $f(x)$ is continuous in the closed interval $[0, \pi]$

(ii) $f'(x) = \dfrac{e^x \cos x - \sin x\, e^x}{(e^x)^2} = \dfrac{e^x(\cos x - \sin x)}{(e^x)^2} = \dfrac{\cos x - \sin x}{e^x}$ which exists for all x.

$f(x)$ is derivable in the open interval $(0, \pi)$

$\therefore$ (iii) $f(0) = \dfrac{\sin 0}{e^0} = \dfrac{0}{1} = 0$ and $f(\pi) = \dfrac{\sin \pi}{e^\pi} = \dfrac{0}{e^\pi} = 0$

Thus, all the conditions of Rolle's theorem are satisfied. So, there must exist a real number c such that $f'(c) = 0$, $c \in (0, \pi)$

Now, $f'(c) = 0 \Rightarrow \dfrac{\cos c - \sin c}{e^c} = 0 \Rightarrow \cos c - \sin c = 0$ [From (ii)]

$\Rightarrow \cos c = \sin c \Rightarrow \tan c = 1 \Rightarrow c = \dfrac{\pi}{4} \in (0, \pi)$

Hence, Rolle's Theorem is verified.

S30. Since an exponential function and sine and cosine functions are everywhere continuous and differentiable. Therefore, $f(x)$ is continuous on $\left[\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right]$ and differentiable on $\left(\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right)$.

Also, $f\left(\dfrac{\pi}{4}\right) = e^{\frac{\pi}{4}}\left(\sin\dfrac{\pi}{4} - \cos\dfrac{\pi}{4}\right) = e^{\frac{\pi}{4}}\left(\dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{2}}\right) = 0$

$f\left(\dfrac{5\pi}{4}\right) = e^{\frac{5\pi}{4}}\left(\sin\dfrac{5\pi}{4} - \cos\dfrac{5\pi}{4}\right) = e^{\frac{5\pi}{4}}\left(-\sin\dfrac{\pi}{4} + \cos\dfrac{\pi}{4}\right)$

$\Rightarrow$ $f\left(\dfrac{5\pi}{4}\right) = e^{\frac{5\pi}{4}}\left(-\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}\right) = 0$

$\therefore$ $f\left(\dfrac{\pi}{4}\right) = f\left(\dfrac{5\pi}{4}\right)$

Thus, $f(x)$ satisfies all the three conditions of Rolle's theorem on $\left[\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right]$.

Consequently, there exists $c \in \left(\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right)$ such that $f'(c) = 0$.

Now, $f(x) = e^x(\sin x - \cos x)$

$\Rightarrow$ $f'(x) = e^x(\sin x - \cos x) + e^x(\cos x + \sin x) = 2e^x \sin x$

$\therefore$ $f'(x) = 0$

$\Rightarrow$ $2e^x \sin x = 0$

$\Rightarrow \qquad \sin x = 0 \qquad [\because e^x \neq 0]$

$\Rightarrow \qquad x = \pi$

Thus, $c = \pi \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ such that $f'(c) = 0$. Hence, Rolle's theorem is verified.

S31. Since a polynomial function and an exponential function are everywhere continuous and differentiable. Therefore, $f(x)$ being product of these two, is continuous on $[-3, 0]$ and differentiable on $(-3, 0)$.

Also, $\qquad f(-3) = -3\,(-3 + 3)\,e^{\frac{3}{2}} = 0$

$f(0) = 0\,(0 + 3)\,e^{o} = 0$

$\therefore \qquad f(-3) = f(0)$

Thus, $f(x)$ satisfies all the three conditions of Rolle's theorem on $[-3, 0]$.

Consequently, there exists $c \in (-3, 0)$ such that $f'(c) = 0$.

Now, $\qquad f(x) = x\,(x+3)e^{\frac{-x}{2}}$

$\Rightarrow \qquad f'(x) = (2x+3)e^{\frac{-x}{2}} + (x^2+3x)\left(-\frac{1}{2}\right)e^{\frac{-x}{2}}$

$= e^{\frac{-x}{2}}\left\{\frac{4x+6-x^2-3x}{2}\right\} = e^{\frac{-x}{2}}\left\{\frac{-x^2+x+6}{2}\right\}$

$\therefore \qquad f'(x) = 0$

$\Rightarrow \qquad e^{\frac{-x}{2}}\left\{\frac{-x^2+x+6}{2}\right\} = 0$

$\Rightarrow \qquad -x^2 + x + 6 = 0$

$\Rightarrow \qquad x^2 - x - 6 = 0$

$\Rightarrow \qquad (x-3)\,(x+2) = 0$

$\Rightarrow \qquad x = -2, 3$

Thus, $c = -2 \in (-3, 0)$ such that $f'(c) = 0$.

Hence, Rolle's theorem is verified.

S32. We have

$$f(x) = \log\left\{\frac{x^2+ab}{x(a+b)}\right\} = \log(x^2 + ab) - \log x - \log(a + b)$$

Since logarithmic function is differentiable and so continuous on its domain. Therefore, $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b).

Also, $f(a) = \log\left(\frac{a^2 + ab}{a(a+b)}\right) = \log\left(\frac{a(a+b)}{a(a+b)}\right) = \log 1 = 0$

and $f(b) = \log\left(\frac{b^2 + ab}{b(a+b)}\right) = \log\left(\frac{b(a+b)}{b(a+b)}\right) = \log 1 = 0$

$\therefore$ $f(a) = f(b)$

Thus, all the three conditions of Rolle's theorem are satisfied.

Now, we have to show that there exists $c \in (a, b)$ such that $f'(c) = 0$.

We have, $f(x) = \log(x^2 + ab) - \log x - \log(a + b)$

$$\Rightarrow \quad f'(x) = \frac{2x}{x^2 + ab} - \frac{1}{x}$$

$$= \frac{2x^2 - x^2 - ab}{x(x^2 + ab)}$$

$$= \frac{x^2 - ab}{x\left(x^2 + ab\right)}$$

$$\therefore \quad f'(x) = 0 \Rightarrow \frac{x^2 - ab}{x\left(x^2 + ab\right)} = 0$$

$$\Rightarrow \quad x^2 = ab \;\Rightarrow\; x = \sqrt{ab}$$

Since $a < \sqrt{ab} < b$. Therefore, $c = \sqrt{ab} \in (a, b)$ is such that $f'(c) = 0$.

Hence, Rolle's theorem is verified.

S33. Since sine function is everywhere continuous and differentiable, Consequently, $f(x) = \sin x - \sin 2x$ is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$.

Also, $f(0) = \sin 0 - \sin 0 = 0$ and $f(\pi) = \sin \pi - \sin 2\pi = 0$

$\therefore$ $f(0) = f(\pi)$

Thus, $f(x)$ satisfies all the three conditions of Rolle's theorem on $[0, \pi]$. Consequently there exists $c \in (0, \pi)$ such that $f'(c) = 0$.

Now, $f(x) = \sin x - \sin 2x \;\Rightarrow\; f'(x) = \cos x - 2\cos 2x$

$\therefore$ $f'(x) = 0 \;\Rightarrow\; \cos x - 2\cos 2x = 0 \Rightarrow \cos x - 2(2\cos^2 x - 1) = 0$

$\Rightarrow$ $4\cos^2 x - \cos x - 2 = 0$

$$\Rightarrow \quad \cos x = \frac{1 \pm \sqrt{33}}{8} = 0.8431 \quad \text{or} \quad -0.5931$$

$\Rightarrow \qquad x = \cos^{-1}(0.8431) \quad \text{or} \quad \cos^{-1}(-0.5931)$

$\Rightarrow \qquad x = \cos^{-1}(0.8431) \quad \text{or} \quad 180° - \cos^{-1}(0.5931)$

$$[\because \cos^{-1}(-x) = \pi - \cos^{-1} x]$$

$\Rightarrow \qquad x = 32°32' \quad \text{or} \quad x = 126°23'$

Thus, $\qquad c = 32°32', 126°23' \in (0, \pi)$ such that $f'(c) = 0$

Hence, Rolle's theorem is verified.

S34. (i) Since, logarithmic function is continuous for $x > 0$;

$\therefore$ $f(x)$ is continuous in closed interval $[-1, 1]$

(ii) $f'(x) = \dfrac{2x}{x^2 + 2}$ which exists in $(-1, 1)$.

$\therefore$ $f(x)$ is differentiable on $(-1, 1)$.

(iii) $f(-1) = \log(1 + 2) - \log 3 = 0$ and $f(1) = \log(1 + 2) - \log 3 = 0$

$\therefore \qquad f(-1) = f(1)$

Thus, all the conditions of Rolle's Theorem are satisfied.

Now, we have to show that there exists some $c \in (-1, 1)$ such that $f'(c) = 0$.

We have, $\qquad f(x) = \log(x^2 + 2) - \log 3$

$\Rightarrow \qquad f'(x) = \dfrac{2x}{x^2 + 2} = 0 \quad \Rightarrow \quad 2x = 0 \quad \Rightarrow \quad x = 0$

Thus, $\qquad c = 0 \in (-1, 1)$. such that $f'(c) = 0$

Hence, Rolle's Theorem is verified.

S35. From mean value theorem. We have

$$\frac{f(b) - f(a)}{b - a} = f'(c) \text{ for } c \in [a, b]$$

$$y = \log x$$

$\therefore \qquad f'(x) = \dfrac{dy}{dx} = \dfrac{1}{x} \quad \text{or} \quad f'(c) = \dfrac{1}{c}$

Hence $\qquad \dfrac{\log e - \log 1}{e - 1} = \dfrac{1}{c}$

or $\qquad \dfrac{1}{c} = \dfrac{1}{e - 1} \quad \text{or} \quad c = e - 1$

Corresponding values of $\qquad y = \log(e - 1)$

Thus at $(e - 1, \log(e - 1))$ tangent to the curve $y = \log x$ is parallel to chord joining the points $A(1, 0)$ and $B(e, 1)$.

S36. The equation of the curve is $y = 16 - x^2$. Let $P(x_1, y_1)$ be a point on it where tangent is parallel to x-axis. Then,

$$\left(\frac{dy}{dx}\right)_P = 0$$

Now, $$y = 16 - x^2 \Rightarrow \frac{dy}{dx} = -2x \Rightarrow \left(\frac{dy}{dx}\right)_P = -2x_1$$

$\therefore$ $$\left(\frac{dy}{dx}\right)_P = 0 \Rightarrow -2x_1 = 0 \Rightarrow x_1 = 0$$

Since $P(x_1, y_1)$ lies on $y = 16 - x^2$. Therefore, $y_1 = 16 - x_1^2$

When $x_1 = 0$, $y_1 = 16 - 0 \Rightarrow y_1 = 16$

Hence (0, 16) is the required point.

S37. It is given that the Rolle's theorem holds for $f(x)$ defined on [1, 3] with $c = 2 + \frac{1}{\sqrt{3}}$.

$\therefore$ $f(1) = f(3)$ and $f'(c) = 0$

$\Rightarrow$ $1 + b + a = 27 + 9b + 3a$ and $3c^2 + 2bc + a = 0$

$\Rightarrow$ $2a + 8b + 26 = 0$ and $3\left(2 + \frac{1}{\sqrt{3}}\right)^2 + 2b\left(2 + \frac{1}{\sqrt{3}}\right) + a = 0$

$\Rightarrow$ $a + 4b + 13 = 0$ ...(i)

$\Rightarrow$ $a + 4b + 13 = 0$ and $3\left(4 + \frac{1}{3} + \frac{4}{\sqrt{3}}\right) + 4b + \frac{2b}{\sqrt{3}} + a = 0$ ...(ii)

$\Rightarrow$ $a + 4b + 13 = 0$

$\Rightarrow$ Solving eq. (i) and (ii), we get

$\Rightarrow$ $a = 11$ and $b = -6$

S38. We are given that $f(1) = f(3) = 0$.

$\therefore$ $1^3 - 6 \times 1 + a + b = 3^3 - 6 \times 3^2 + 3a + b = 0$

$\Rightarrow$ $a + b = 5$ and $3a + b = 27$

Solving these two equations for a and b, we get $a = 11$ and $b = -6$.

We now verify whether for these values of a and b, $f'(c)$ is zero or not.

We have, $f(x) = x^3 - 6x^2 + ax + b$

$\Rightarrow$ $f(x) = x^3 - 6x^2 + 11x - 6$ $[\because a = 11, b = -6]$

$\Rightarrow$ $f'(x) = 3x^2 - 12x + 11$

Now, $f'(c) = 3c^2 - 12c + 11$

$$\Rightarrow \qquad f'(c) = 3\left(2+\frac{1}{\sqrt{3}}\right)^2 - 12\left(2+\frac{1}{\sqrt{3}}\right)+11$$

$$\Rightarrow \qquad f'(c) = 12+\frac{12}{\sqrt{3}}+1-24-\frac{12}{\sqrt{3}}+11 = 0$$

Hence, $a = 11$ and $b = -6$.

Date: 18/10/2021

Q1. Verify Lagrange's mean value theorem for the function $f(x) = x(x - 2)$ on the interval [1, 3]. Also, find a point c in the indicated interval.

Q2. Verify Lagrange's Mean value Theorem for the function:

$$f(x) = x^2 - 2x + 4 \text{ on } [1, 5].$$

Q3. Verify Lagrange's Mean Value Theorem for the function $f(x) = x^2 + x - 1$ in the interval [0, 4].

Q4. Verify mean value Theorem, if $f(x) = x^2 - 4x - 3$ in the interval $[a, b]$, where $a = 1$ and $b = 4$.

Q5. State the Mean value theorem in the equation $f(b) - f(a) = (b - a) f'(c)$, determine c lying between a and b, if

$$f(x) = (x - 1)(x - 2)(x - 3) \text{ and } a = 0, b = 4.$$

Q6. Verify Lagrange's Mean value Theorem for $f(x) = \sqrt{x^2 - 4}$ in the interval [2, 4].

Q7. Verify Lagrange's Mean value Theorem for $f(x) = x + \dfrac{1}{x}$ on [1, 3].

Q8. Verify Mean Value Theorem for the following function:

$$f(x) = x^3 - 6x \text{ in the range } 1 \le x \le 3.$$

Q9. Test, whether the Lagrange's Mean Value Theorem of differential calculus holds for the function: $f(x) = x - x^3$ on the interval [– 2, 1] and find appropriate value of c.

Q10. Using Lagrange's mean value theorem, find a point on the curve $y = \sqrt{x - 2}$ defined on the interval [2, 3], where the tangent is parallel to the chord joining the end points of the curve.

Q11. Verify Lagrange's mean value theorem for the function $f(x) = (x - 3)(x - 6)(x - 9)$ on the interval [3, 5].

Q12. Verify Lagrange's mean value theorem for the function $f(x) = x^3 - 5x^2 - 3x$ on the interval [1, 3]. Find a point 'c' in the indicated interval as stated by the Lagrange's mean value theorem.

Q13. Verify Lagrange's mean value theorem for the function $f(x) = x(x - 1)(x - 2)$ on the interval $\left[0, \dfrac{1}{2}\right]$. Also, find a point c in the indicated interval.

Q14. Verify Lagrange's mean value theorem for the function $f(x) = x - 2 \sin x$ on the interval $[-\pi, \pi]$.

Q15. Verify Lagrange's mean value theorem for the function

$$f(x) = \begin{cases} 2 + x^3 & \text{if } x \le 1 \\ 3x & \text{if } x > 1 \end{cases} \text{ on the interval } [-1, 2].$$

Q16. Verify Lagrange's mean value theorem for the function $f(x) = 2\sin x + \sin 2x$ on the interval $[0, \pi]$.

Q17. Verify Lagrange's mean value theorem for the function $f(x) = \log_e x$ on the interval [1, 2].

Q18. Verify Mean Value Theorem for the following function:

$f(x) = e^x$ on [0, 1]

Q19. If $f(x + h) = f(x) + h f'(x + \theta h)$; find θ for $f(x) = ax^2 + bx + c$ in the interval (0, 1).

Q20. Apply Mean-Value Theorem on $f(x) = \sqrt{x}$ to prove $4 < \sqrt{16.4} < 4.05$.

Q21. Using mean value theorem, prove that $\tan x > x$ for all $x \in \left(0, \frac{\pi}{2}\right)$.

Q22. Using Lagrange's Mean Value Theorem, prove that:

$$\frac{b-a}{b} < \log\left(\frac{b}{a}\right) < \frac{b-a}{a}, \quad \text{where } 0 < a < b.$$

Q23. Applying Mean-value Theorem, show that

$$x > \log_e(1 + x) > x - \frac{1}{2}x^2 \quad \text{for} \quad x > 0.$$

Q24. Using Lagrange's Mean Value Theorem, prove that:

$$\frac{x}{1+x} < \log(1+x) < x \quad \text{for} \quad \text{all } x > 0.$$

Q25. Using Lagrange's mean value theorem, show that $\sin x < x$ for $x > 0$.

Q26. Find a point on the parabola $y = (x + 3)^2$; where the tangent is parallel to the chord joining (– 3, 0) and (– 4, 1).

Q27. Find the coordinates of the point, at which the tangent to the curve given by $f(x) = x^2 - 6x + 1$ is parallel to the chord joining the points (1, – 4), and (3, – 8).

Q28. Let f and g be differentiable on [0, 1] such that $f(0) = 2$, $g(0) = 0$, $f(1) = 6$ and $g(1) = 2$, show that there exists a point $c \in (0, 1)$, such that $f'(c) = 2g'(c)$.

Date: 18/10/2021

S1. We have, $f(x) = x(x-2) = x^2 - 2x$

We know that polynomial function is everywhere continuous and differentiable. So, $f(x)$ being a polynomial, is continuous on [1, 3] and differentiable on (1, 3).

Thus, $f(x)$ satisfies both the conditions of Lagrange's mean value theorem on [1, 3].

So, there must exist at least one real number $c \in (1, 3)$ such that

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}$$

Now, $f(x) = x^2 - 2x$

$\Rightarrow$ $f'(x) = 2x - 2,$

$f(3) = 9 - 6 = 3$ and $f(1) = 1 - 2 = -1$

$\therefore$ $$f'(x) = \frac{f(3) - f(1)}{3 - 1}$$

$\Rightarrow$ $$2x - 2 = \frac{3 - (-1)}{3 - 1}$$

$\Rightarrow$ $2x - 2 = 2$

$\Rightarrow$ $x = 2$

Thus, $c = 2 \in (1, 3)$ such that $f'(c) = \dfrac{f(3) - f(1)}{3 - 1}$

Hence, Lagrange's mean value theorem is verified for $f(x)$ on [1, 3].

S2. We have, $f(x) = x^2 - 2x + 4 \Rightarrow f'(x) = 2x - 2$

Since $f(x)$ is polynomial, therefore continuous and differentiable everywhere. Hence :

(a) $f(x)$ is continuous on closed interval [1, 5]

(b) $f(x)$ is differentiable on open interval (1, 5).

Thus, both the conditions of Lagrange's Mean Value Theorem are satisfied, now there must exist.

$c \in (1, 5)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$, $f'(c) = 2c - 2$ ... (i)

$f(1) = 1^2 - 2(1) + 4 = 1 - 2 + 4 = 3$

$f(5) = 5^2 - 2(5) + 4 = 25 - 10 + 4 = 19$

$\Rightarrow$ $f'(c) = \dfrac{f(5) - f(1)}{5 - 1} \Rightarrow \dfrac{19 - 3}{5 - 1} = 2c - 2$ [from (i)]

$\Rightarrow \qquad 2c - 2 = \dfrac{16}{4} \quad \Rightarrow \quad 2c - 2 = 4$

$\Rightarrow \qquad 2c = 4 + 2 \quad \Rightarrow \quad 2c = 6 \quad \Rightarrow \quad c = 3 \in (1, 5)$

Thus, $c \in (1, 5)$ such that $f'(c) = \dfrac{f(5) - f(1)}{5 - 1}$

Hence, Lagrange's Mean Value Theorem is verified.

S3. We have, $f(x) = x^2 + x - 1 \quad \Rightarrow \quad f'(x) = 2x + 1$

Since $f(x)$ is polynomial, therefore continuous and differentiable everywhere. Hence,

(a) $f(x)$ is continuous in [0, 4].

(b) $f(x)$ is differentiable in (0, 4)

All the conditions of Lagrange's Mean value Theorem are satisfied.

$\Rightarrow \quad \exists c \in (0, 4)$ such that

$\Rightarrow \qquad f'c = \dfrac{f(b) - f(a)}{b - a} = \dfrac{f(4) - f(0)}{4 - 0}$

$\Rightarrow \qquad 2c + 1 = \dfrac{(4^2 + 4 - 1) - (-1)}{4} = \dfrac{19 + 1}{4} = 5$

$\Rightarrow \qquad 2c = 5 - 1 = 4 \quad \Rightarrow \quad c = 2$ and $2 \in (0, 4)$.

Hence, Lagrange's Mean Value Theorem is verified.

S4. We have, $f(x) = x^2 - 4x - 3 \quad \Rightarrow \quad f'(x) = 2x - 4$

Since $f(x)$ is polynomial, therefore continuous and differentiable everywhere. Hence,

(a) $f(x)$ is continuous on closed interval [1, 4].

(b) $f(x)$ is differentiable on open interval (1, 4).

Thus, both the conditions of Lagrange's Mean value Theorem are satisfied. Now there must exist.

$c \in (1, 4)$ such that $\qquad f'(c) = \dfrac{f(b) - f(a)}{b - a},\ f'(c) = 2c - 4$

$f(1) = 1^2 - 4(1) - 3 = 1 - 4 - 3 = -6$

$f(4) = 4^2 - 4(4) - 3 = 16 - 16 - 3 = -3$

$\Rightarrow \qquad f'(c) = \dfrac{f(4) - f(1)}{4 - 1} \quad \Rightarrow \quad \dfrac{-3 - (-6)}{3} = 2c - 4$

$\Rightarrow \quad 2c - 4 = \frac{3}{3} \quad \Rightarrow \quad 2c - 4 = 1 \quad \Rightarrow \quad c = \frac{5}{2} \in (1, 4)$

Thus, $c \in (1, 4)$ such that $f'(c) = \frac{f(4) - f(1)}{4 - 1}$,

Hence, Lagrange's Mean value Theorem is verified.

S5.

$$f(x) = (x - 1)(x - 2)(x - 3)$$
$$f(0) = (0 - 1)(0 - 2)(0 - 3) = -6$$
$$f(4) = (4 - 1)(4 - 2)(4 - 3) = 6$$

$$f'(x) = (x - 2)(x - 3) + (x - 1)(x - 3) + (x - 1)(x - 2)$$
$$= (x^2 - 5x + 6) + (x^2 - 4x + 3) + (x^2 - 3x + 2)$$
$$= 3x^2 - 12x + 11$$

$$f'(c) = 3c^2 - 12c + 11 \quad \ldots \text{(i)}$$

By Mean value Theorem, we have

$$f(b) - f(a) = (b - a) f'(c)$$
$$f(4) - f(0) = (4 - 0) f'(c)$$

$$6 - (-6) = 4f'(c)$$
$$3 = f'(c)$$

Putting the value of $f'(c)$ in (i), we get

$$3 = 3c^2 - 12c + 11$$
$$3c^2 - 12c + 8 = 0$$
$$c = \frac{12 \pm \sqrt{144 - 96}}{6}$$

$$c = \frac{12 \pm \sqrt{48}}{6} = \frac{12 \pm 4\sqrt{3}}{6} = 2 \pm \frac{2}{3}\sqrt{3}$$

$= 2 \pm \frac{2}{3}\sqrt{3}$ lies between 0 and 4.

S6. We have,

$$f(x) = \sqrt{x^2 - 4}$$

(a) Since, $f(x) = \sqrt{x^2 - 4}$ has a definite and unique value in [2, 4].

$\therefore$ $f(x)$ is continuous on the closed interval [2, 4]

(b) $f'(x) = \frac{2x}{2\sqrt{x^2 - 4}} = \frac{x}{\sqrt{x^2 - 4}}$ which exists for all $x \in (2, 4)$.

$\therefore$ $f(x)$ is differentiable on the open interval (2, 4).

Thus, both the conditions of Lagrange's Mean value Theorem are satisfied.

So, there must exist some $c \in (2, 4)$, such that

$$f'(c) = \frac{f(4)-f(2)}{4-2} = \frac{\sqrt{12}-0}{4-2} = \frac{\sqrt{12}-0}{2} = \sqrt{3}$$

$$f'(c) = \sqrt{3} \quad \Rightarrow \quad \frac{c}{\sqrt{c^2-4}} = \sqrt{3}$$

$$\Rightarrow \qquad c^2 = 3(c^2-4) \quad \Rightarrow \quad c = \pm\sqrt{6} \in (2, 4)$$

Thus, $\quad c \in (2, 4)$ such that $f'(c) = \dfrac{f(4)-f(2)}{4-2}$

Hence, Lagrange's Mean value Theorem is verified.

S7. We have,

$$f(x) = x + \frac{1}{x} \text{ on } [1, 3].$$

Since, $f(x)$ is polynomial, therefore it is continuous and differentiable every where. Thus,

(a) $f(x)$ is continuous on closed interval [1, 3].

(b) $f(x)$ is differentiable on open interval (1, 3).

Thus, both the conditions of Lagrange's Mean value Theorem are satisfied. So, there must exist $c \in [1, 3]$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{f(3)-f(1)}{3-1}$$

$$\Rightarrow \qquad 1-\frac{1}{c^2} = \frac{\left(3+\frac{1}{3}\right)-\left(1+\frac{1}{1}\right)}{2} = \frac{3+\frac{1}{3}-2}{2} \qquad \left[\because \; f'(x) = 1-\frac{1}{x^2}\right]$$

$$= \frac{1}{2}\left(1+\frac{1}{3}\right) = \frac{2}{3}$$

$$\Rightarrow \qquad \frac{1}{c^2} = \frac{1}{3} \quad \Rightarrow \quad c^2 = 3 \quad \Rightarrow \quad c = \sqrt{3} = 1.732 \in (1, 3)$$

Thus, Lagrange's Mean Value Theorem is applicable.

S8. We have $f(x) = x^3 - 6x \quad \Rightarrow \quad f'(x) = 3x^2 - 6$

Since $f(x)$ is polynomial, therefore continuous and differentiable every where, Hence

(a) $f(x)$ is continuous on closed interval [1, 3]

(b) $f(x)$ is differentiable on open interval (1, 3)

Thus both the conditions of Lagrange's mean value.

Theorem are satisfied, now there must exist

$c \in (1, 3)$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}, f'(c) = 3c^2 - 6$ ... (i)

$$f(1) = 1^3 - 6(1) = 1 - 6 = -5$$

$$f(3) = 3^3 - 6(3) = 27 - 18 = 9$$

$$\Rightarrow \quad f'(c) = \frac{f(3)-f(1)}{3-1} \Rightarrow 3c^2 - 6 = \frac{9-(-5)}{3-1}$$

$$\Rightarrow \quad 3c^2 - 6 = \frac{14}{2} \Rightarrow 3c^2 - 6 = 7 \Rightarrow 3c^2 = 13 \Rightarrow c = \pm\sqrt{\frac{13}{3}}$$

$$\Rightarrow \quad c = \sqrt{\frac{13}{3}} = 2.08 \in (1, 3)$$

Thus $c \in (1, 3)$ such that $f'(c) = \dfrac{f(3)-f(1)}{3-1}$

Hence Lagrange's Mean value Theorem is verified.

S9. $f(x) = x - x^3,\ x \in [-2, 1]$

$$f'(x) = 1 - 3x^2$$

$$f'(c) = 1 - 3c^2$$

By Mean value Theorem,

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow \quad 1 - 3c^2 = \frac{0-6}{1-(-2)} = \frac{-6}{3} = -2$$

$$\Rightarrow \quad 1 - 3c^2 = -2 \Rightarrow 3c^2 = 3 \Rightarrow c^2 = 1 \Rightarrow c = \pm 1$$

Thus, $c = -1 \in (-2, 1)$.

S10. Let $f(x) = \sqrt{x-2}$. Since for each $x \in [2, 3]$, the function $f(x)$ attains a unique definite value. So $f(x)$ is continuous on $[2, 3]$.

Also $f'(x) = \dfrac{1}{2\sqrt{x-2}}$ exists for all $x \in (2, 3)$. So, $f(x)$ is differentiable on $(2, 3)$.

Thus, both the conditions of Lagrange's mean value theorem are satisfied. Consequently, there must exist some $c \in (2, 3)$ such that

$$f'(c) = \frac{f(3)-f(2)}{3-2}$$

Now, $f(x) = \sqrt{x-2}$

$\Rightarrow$ $f'(x) = \dfrac{1}{2\sqrt{x-2}}$,

$f(3) = \sqrt{3-2} = 1$ and $f(2) = \sqrt{2-2} = 0$

$\therefore$ $f'(x) = \dfrac{f(3)-f(2)}{3-2}$

$\Rightarrow$ $\dfrac{1}{2\sqrt{x-2}} = \dfrac{1-0}{3-2}$

$\Rightarrow$ $\dfrac{1}{2\sqrt{x-2}} = 1$

$\Rightarrow$ $4(x-2) = 1$

$\Rightarrow$ $x - 2 = \dfrac{1}{4}$ $\Rightarrow$ $x = \dfrac{9}{4}$

Thus, $c = \dfrac{9}{4} \in (2, 3)$ such that $f'(c) = \dfrac{f(3)-f(2)}{3-2}$

Now, $f(c) = \sqrt{\dfrac{9}{4} - 2} = \dfrac{1}{2}$

Thus, $(c, f(c))$ *i.e.*, $\left(\dfrac{9}{4}, \dfrac{1}{2}\right)$ is a point on the curve $y = \sqrt{x-2}$ such that the tangent at it is parallel to the chord joining the end points of the curve.

S11. We have, $f(x) = (x-3)(x-6)(x-9) = x^3 - 18x^2 + 99x - 162$

Since a polynomial function is everywhere continuous and differentiable.

Therefore $f(x)$ is continuous on [3, 5] and differentiable on (3, 5).

Thus, both the conditions of Lagrange's mean value theorem are satisfied.

So, there must exists at least one real number $c \in (3, 5)$ such that

$$f'(c) = \frac{f(5)-f(3)}{5-3}$$

Now, $f(x) = x^3 - 18x^2 + 99x - 162$

$\Rightarrow$ $f'(x) = 3x^2 - 36x + 99$,

$f(5) = (5-3)(5-6)(5-9) = 8$

and $f(3) = (3-3)(3-6)(3-9) = 0$

$$\therefore \qquad f'(x) = \frac{f(5)-f(3)}{5-3}$$

$$\Rightarrow \qquad 3x^2 - 36x + 99 = \frac{8-0}{5-3}$$

$$\Rightarrow \qquad 3x^2 - 36x + 99 = 4$$

$$\Rightarrow \qquad 3x^2 - 36x + 95 = 0 \;\Rightarrow\; x = \frac{36 \pm \sqrt{1296-1140}}{6} = \frac{36 \pm 12.49}{6} = 8.8,\ 4.8$$

Thus, $c = 4.8 \in (3, 5)$ such that $f'(c) = \dfrac{f(5)-f(3)}{5-3}$.

S12. Clearly, $f(x)$ being a polynomial function, is continuous on [1, 3] and differentiable on (1, 3).

Now, $\qquad f(x) = x^3 - 5x^2 - 3x$

$\Rightarrow \qquad f'(x) = 3x^2 - 10x - 3,$

$\qquad f(1) = (1)^3 - 5(1)^2 - 3 = -7$

and $\qquad f(3) = 27 - 45 - 9 = -27$

$$\therefore \qquad f'(x) = \frac{f(3)-f(1)}{3-1}$$

$$\Rightarrow \qquad 3x^2 - 10x - 3 = \frac{-27+7}{2}$$

$$\Rightarrow \qquad 3x^2 - 10x + 7 = 0$$

$$\Rightarrow \qquad (3x-7)(x-1) = 0 \;\Rightarrow\; x = 1,\ \frac{7}{3}$$

Clearly, $c = \dfrac{7}{3} \in (1, 3)$ such that $f'(c) = \dfrac{f(3)-f(1)}{3-1}$.

Hence, Lagrange's mean value theorem is verified.

S13. We have, $\qquad f(x) = x(x-1)(x-2) = x^3 - 3x^2 + 2x$

Since $f(x)$ is a polynomial function and a polynomial function is everywhere continuous and differentiable. Therefore, $f(x)$ is continuous on $\left[0, \frac{1}{2}\right]$ and differentiable on $\left(0, \frac{1}{2}\right)$.

Thus, both the conditions of Lagrange's mean value theorem are satisfied.

So, there must exist at least one real number $c \in \left(0, \frac{1}{2}\right)$ such that

$$f'(c) = \frac{f\left(\frac{1}{2}\right) - f(0)}{\frac{1}{2} - 0}$$

Now, $f(x) = x^3 - 3x^2 + 2x$

$\Rightarrow$ $f'(x) = 3x^2 - 6x + 2,$

$$f(0) = 0 \quad \text{and} \quad f\left(\frac{1}{2}\right) = \frac{1}{8} - \frac{3}{4} + 1 = \frac{3}{8}$$

$$\Rightarrow \quad f'(x) = \frac{f\left(\frac{1}{2}\right) - f(0)}{\left(\frac{1}{2}\right) - 0}$$

$$\Rightarrow \quad 3x^2 - 6x + 2 = \frac{\left(\frac{3}{8}\right) - 0}{\left(\frac{1}{2}\right) - 0}$$

$$\Rightarrow \quad 3x^2 - 6x + 2 = \frac{3}{4}$$

$$\Rightarrow \quad 12x^2 - 24x + 5 = 0$$

$$\Rightarrow \quad x = \frac{24 \pm \sqrt{336}}{24} = 1 \pm \frac{\sqrt{21}}{6}$$

Since $c = 1 - \frac{\sqrt{21}}{6} \in \left(0, \frac{1}{2}\right)$ such that $f'(c) = \frac{f\left(\frac{1}{2}\right) - f(0)}{\left(\frac{1}{2}\right) - 0}$.

Hence, Lagrange's mean value theorem is verified.

S14. Since x and $\sin x$ are everywhere continuous and differentiable, therefore $f(x)$ is continuous on $[-\pi, \pi]$ and differentiable on $(-\pi, \pi)$. Thus, both the conditions of Lagrange's mean value theorem are satisfied. So, there must exist at least one $c \in (-\pi, \pi)$ such that

$$f'(c) = \frac{f(\pi) - f(-\pi)}{\pi - (-\pi)}$$

Now, $f(x) = x - 2\sin x$

$\Rightarrow$ $f'(x) = 1 - 2\cos x,$

$$f(\pi) = \pi - 2\sin\pi = \pi - 0 = \pi$$

and $f(-\pi) = -\pi - 2\sin(-\pi) = -\pi - 0 = -\pi$

$$\therefore \quad f'(x) = \frac{f(\pi) - f(-\pi)}{\pi - (-\pi)}$$

$$\Rightarrow \quad 1 - 2\cos x = \frac{\pi - (-\pi)}{\pi - (-\pi)}$$

$$\Rightarrow \quad 1 - 2\cos x = 1$$

$\Rightarrow \qquad \cos x = 0 \;\Rightarrow\; x = \pm\frac{\pi}{2}$

Thus, $\qquad c = \pm\left(\frac{\pi}{2}\right) \in (-\pi, \pi)$ such that $f'(c) = \frac{f(\pi) - f(-\pi)}{\pi - (-\pi)}$

Hence, Lagrange's mean value theorem is verified.

S15. Since $2 + x^3$ and $3x$ are polynomial functions. Therefore $f(x)$ is continuous and differentiable for all values of x except possibly at $x = 1$.

Continuity at $x = 1$:

We have,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1} 2 + x^3 = 2 + 1^3 = 3$$

and $$\lim_{x \to 1^-} f(x) = \lim_{x \to 1} 3x = 3 \times 1 = 3$$

Also, $$f(1) = 2 + 1^3 = 3$$

$\therefore$ $$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$$

So, $f(x)$ is continuous at $x = 1$

Differentiability at $x = 1$:

We have

$$(\text{LHD at } x = 1) = \lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{2 + x^3 - 3}{x - 1} = \lim_{x \to 1} \frac{x^3 - 1}{x - 1}$$

$\Rightarrow$ $$(\text{LHD at } x = 1) = \lim_{x \to 1} \frac{(x-1)\left(x^2 + x + 1\right)}{x - 1} = \lim_{x \to 1} x^2 + x + 1 = 1^2 + 1 + 1 = 3$$

$\Rightarrow$ $$(\text{RHD at } x = 1) = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{3x - 3}{x - 1} = \lim_{x \to 1} \frac{3(x-1)}{x - 1} = 3$$

$\therefore \qquad (\text{LHD at } x = 1) = (\text{RHD at } x = 1)$

So, $f(x)$ is differentiable at $x = 1$

Thus $f(x)$ is continuous and differentiable on $[-1, 2]$. So, both the conditions of Lagrange's mean value theorem are satisfied. Consequently, there must exist some $c \in (-1, 2)$ such that

$$f'(c) = \frac{f(2) - f(-1)}{2 - (-1)}$$

Now, $$f(x) = \begin{cases} 2 + x^3 &, x \le 1 \\ 3x &, x > 1 \end{cases} \Rightarrow f'(x) = \begin{cases} 3x^2 &, x \le 1 \\ 3 &, x > 1 \end{cases}$$

$$f(-1) = 2 + (-1)^3 = 1 \quad \text{and} \quad f(2) = 3(2) = 6$$

$\therefore \quad f'(x) = \dfrac{f(2) - f(-1)}{2 - (-1)} = \dfrac{6 - 1}{2 - (-1)} = \dfrac{5}{3}$

Since $f'(x) = 3$ for $x > 1$, the value of x must be less than 1

$\therefore \quad f'(x) = \dfrac{5}{3} \Rightarrow 3x^2 = \dfrac{5}{3}$ $\quad [\because x < 1 \text{ and for } x < 1, f'(x) = 3x^2]$

$\Rightarrow \quad x^2 = \dfrac{5}{9} \Rightarrow x = \pm\dfrac{\sqrt{5}}{3}$

Since $\quad c = \pm\left(\dfrac{\sqrt{5}}{3}\right) \in (-1, 2)$ such that $f'(c) = \dfrac{f(2) - f(-1)}{2 - (-1)}$.

Hence, Lagrange's mean value theorem is verified.

S16. Since sin x and sin $2x$ are everywhere continuous and differentiable, therefore $f(x)$ is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$. Thus, $f(x)$ satisfies both the conditions of Lagrange's mean value theorem. Consequently, there exist at least one $c \in (0, \pi)$ such that

$$f'(c) = \frac{f(\pi) - f(0)}{\pi - 0}$$

Now, $\quad f(x) = 2\sin x + \sin 2x$

$\Rightarrow \quad f'(x) = 2\cos x + 2\cos 2x,$

$\quad f(0) = 2 \times \sin 0° + \sin 0° = 0$

and $\quad f(\pi) = 2\sin\pi + \sin 2\pi = 0$

$\therefore \quad f'(x) = \dfrac{f(\pi) - f(0)}{\pi - 0}$

$\Rightarrow \quad 2\cos x + 2\cos 2x = \dfrac{0 - 0}{\pi - 0}$

$\Rightarrow \quad 2\cos x + 2\cos 2x = 0$

$\Rightarrow \quad \cos x + \cos 2x = 0$

$\Rightarrow \quad \cos 2x = -\cos x$

$\Rightarrow \quad \cos 2x = \cos(\pi - x)$

$\Rightarrow \quad 2x = \pi - x \Rightarrow 3x = \pi \Rightarrow x = \dfrac{\pi}{3}$

Thus, $\quad c = \dfrac{\pi}{3} \in (0, \pi)$ such that $f'(c) = \dfrac{f(\pi) - f(0)}{\pi - 0}$

Hence, Lagrange's mean value theorem is verified.

S17. Since $f(x) = \log_e x$ is differentiable and so continuous for all $x > 0$. So, $f(x)$ is continuous on [1, 2] and differentiable on (1, 2). Thus, both the conditions of Lagrange's mean value theorem are satisfied. Consequently, there must exist some $c \in (1, 2)$ such that

$$f'(c) = \frac{f(2) - f(1)}{2 - 1}$$

Now, $f(x) = \log_e x$

$\Rightarrow$ $f'(x) = \frac{1}{x}$,

$f(2) = \log_e 2$ and $f(1) = \log_e 1 = 0$

$\therefore$ $f'(x) = \frac{f(2) - f(1)}{2 - 1}$

$\Rightarrow$ $\frac{1}{x} = \frac{\log_e 2 - 0}{2 - 1}$

$\Rightarrow$ $\frac{1}{x} = \log_e 2$

$\Rightarrow$ $x = \frac{1}{\log_e 2} = \log_2 e$ $\left[\because \log_b a = \frac{1}{\log_a b}\right]$

Now, $2 < e < 4 \Rightarrow \log_2 2 < \log_2 e < \log_2 4 \Rightarrow 1 < \log_2 e < 2$

Thus, $c = \log_2 e \in (1, 2)$ such that $f'(c) = \frac{f(2) - f(1)}{2 - 1}$

Hence, Lagrange's mean value theorem is verified.

S18. We have, $f(x) = e^x \Rightarrow f'(x) = e^x$

f is continuous and differentiable for all x.

Therefore, f is continuous in [0, 1] and derivable in (0, 1).

$\therefore$ $\frac{f(b) - f(a)}{b - a} = f'(c)$

$\Rightarrow$ $\frac{f(1) - f(0)}{1 - 0} = e^c$

$\Rightarrow$ $e - 1 = e^c$

$\Rightarrow$ $\log (e - 1) = \log e^c$

$\Rightarrow$ $\log(e - 1) = c$ $[\because \log e = 1]$

Since $2 < e < 3$ $\therefore$ $1 < e - 1 < 2$ so that $\log (e - 1)$ lies in (0, 1)

S19. We have, $f(x) = ax^2 + bx + c$

$\Rightarrow$ $f'(x) = 2ax + b$ ... (i)

$h = 1 - 0 = 1$

$$f(1) = a(1)^2 + b(1) + c = a + b + c$$

$$f(0) = c$$

Given, $\quad f(x + h) = f(x) + h f'(x + \theta h)$

$$\frac{f(x+h) - f(x)}{h} = f'(x + \theta h)$$

$$\frac{f(1) - f(0)}{1 - 0} = f'(0 + \theta.1)$$

$$\frac{(a+b+c) - c}{1} = f'(\theta)$$

$$a + b = f'(\theta) \quad \ldots \text{(ii)}$$

From (i), we have $\quad f'(\theta) = 2a\theta + b \quad \ldots$ (iii)

From (ii) and (iii) we get

$$2a\theta + b = a + b$$

$$2a\theta = a \text{ or } \theta = \frac{1}{2}.$$

S20. We have $\quad f(x) = x^{\frac{1}{2}}$

$\Rightarrow \quad f'(x) = \frac{1}{2}\frac{1}{\sqrt{x}}$

Now, $\quad 4 < \sqrt{16.4} < 4.05$

$\Rightarrow \quad 16 < 16.4 < 16.4025$

By Mean value theorem

Let $\quad f(b) - f(a) = (b - a) f'(c), \quad a < c < b$

$$f(16.4) - f(16) = (16.4 - 16)\frac{1}{2\sqrt{c}}$$

$$\sqrt{16.4} - \sqrt{16} = (0.4)\frac{1}{2\sqrt{c}}$$

$$\sqrt{16.4} - \sqrt{16} = \left(\frac{0.2}{\sqrt{c}}\right)$$

$$\sqrt{16.4} - \sqrt{16} = +\text{ve quantity} ; 16 < c < 16.4$$

$\therefore \quad \sqrt{16.4} > 4 \quad \ldots$ (i)

Again $\quad f(b) - f(a) = (b - a) f'(c)$

Let $\quad a = 16.4,\ b = 16.4025, \quad 16.4 < c < 16.4025$

$$f(16.4025) - f(16.4) = (16.4025 - 16.4)\ f'(c)$$

$$\sqrt{(16.4025)} - \sqrt{(16.4)} = (0.0025)\,\frac{1}{2\sqrt{c}}$$

$$\sqrt{(16.4025)} - \sqrt{(16.4)} = \frac{0.00125}{\sqrt{c}}$$

$$\sqrt{(16.4025)} - \sqrt{(16.4)} = +\text{ ve quantity}$$

$$\sqrt{4.05} - \sqrt{(16.4)} = +\text{ ve}$$

$$4.05 > \sqrt{(16.4)} \qquad \ldots \text{(ii)}$$

From (i) and (ii), we have

$$4 < \sqrt{(16.4)} < 4.05.$$

S21. Let x be any point in the interval $\left(0, \frac{\pi}{2}\right)$. Consider the function f given by

$$f(x) = \tan x - x, \quad \text{where} \quad x \in [0, x] \subset \left(0, \frac{\pi}{2}\right)$$

Clearly, $f(x)$ is continuous on $[0, x]$ and differentiable on $(0, x)$. So, there exists $c \in (0, x)$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0}$$

Now $\qquad f(x) = \tan x - x$

$$f'(c) = \sec^2 c - 1.$$

$$f(0) = \tan 0° - 0 = 0$$

$$\Rightarrow \qquad \sec^2 c - 1 = \frac{(\tan x - x) - 0}{x - 0}$$

$$\Rightarrow \qquad \frac{\tan x - x}{x} > 0 \qquad \left[\because \sec^2 c > 1, c \in (0, x) \subset \left(0, \frac{\pi}{2}\right)\right]$$

$$\Rightarrow \qquad \tan x - x > 0 \qquad [\because x > 0]$$

$$\Rightarrow \qquad \tan x > x \quad \text{for all } x \in \left(0, \frac{\pi}{2}\right).$$

S22. Let the function f is given by,

$$f(x) = \log_e x, \quad x \in [a, b], \quad 0 < a < b.$$

Clearly, it is continuous on $[a, b]$ and differentiable on (a, b). So, by Lagrange's Mean Value Theorem there exist $c \in (a, b)$ such that :

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$\Rightarrow$ $$\frac{1}{c} = \frac{\log b - \log a}{b - a} \qquad \left[\because\ f(x) = \log x \Rightarrow f'(x) = \frac{1}{x} \Rightarrow f'(c) = \frac{1}{c}\right]$$

Now, $c \in (a, b) \quad \Rightarrow \quad a < c < b.$

$\Rightarrow$ $$\frac{1}{b} < \frac{1}{c} < \frac{1}{a} \qquad [\because \quad 0 < a < b]$$

$\Rightarrow$ $$\frac{1}{b} < \frac{\log b - \log a}{b - a} < \frac{1}{a} \qquad \Rightarrow \quad \frac{b - a}{b} < \log b - \log a < \frac{b - a}{a}$$

$\Rightarrow$ $$\frac{b - a}{b} < \log\left(\frac{b}{a}\right) < \frac{b - a}{a}.$$

S23. Let, $f(x) = x - \log_e (1 + x)$

$\Rightarrow$ $$f'(x) = 1 - \frac{1}{1 + x}$$

$$f'(x) > 0 \quad \text{for } x > 0$$

$f(x)$ is increasing for $x > 0$.

$\therefore$ $f(x) > f(0)$

$\Rightarrow$ $x - \log_e (1 + x) > 0$

$$x > \log_e(1 + x) \qquad \ldots \text{(i)}$$

Let $$F(x) = \log_e(1 + x) - x + \frac{x^2}{2}$$

$$F'(x) = \frac{1}{1 + x} - 1 + x$$

$$F'(x) > 0 \quad \text{for} \quad x > 0$$

$F'(x)$ is increasing for $x > 0$.

$\therefore$ $F(x) > F(0)$

$\Rightarrow$ $$\log_e (1 + x) - x + \frac{x^2}{2} > 0$$

i.e., $$\log_e (1 + x) > x - \frac{x^2}{2} \qquad \ldots \text{(ii)}$$

From (i) and (ii), we have

$$x > \log_e (1 + x) > x - \frac{x^2}{2}$$

S24. Let $f(x) = x - \log(1 + x)$

$$f'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x}$$

$f(x)$ is increasing for $x > 0$

$\therefore$ $f(x) > f(0)$

$x - \log(1 + x) > 0$

$\therefore$ $x > \log(1 + x)$ ... (i)

Again $F(x) = \log(1 + x) - \frac{1}{1+x}$

$$F'(x) = \frac{1}{1+x} - \frac{1}{(1+x)^2}$$

$F'(x) > 0$ for $x > 0$

$F'(x)$ is increasing for $x > 0$

$\therefore$ $F(x) > F(0)$

$\log(1 + x) - \frac{x}{1+x} > 0$ ... (ii)

$\therefore$ $\log(1 + x) > \frac{x}{1+x}$

From (i) and (ii), we have

$$\frac{x}{1+x} < \log(1 + x) < x$$

S25. Consider the function $f(x) = x - \sin x$ defined on the interval $[0, x]$, where $x > 0$.

Clearly, $f(x)$ is everywhere continuous and differentiable. So, it is continuous on $[0, x]$ and differentiable on $(0, x)$. Consequently, there exists $c \in (0, x)$ such that

$f'(c) = \frac{f(x) - f(0)}{x - 0}$ [By Lagrange's mean value theorem]

$\Rightarrow$ $1 - \cos c = \frac{x - \sin x - 0}{x - 0}$ $[\because f(x) = x - \sin x, f'(x) = 1 - \cos x]$

$\Rightarrow$ $\frac{x - \sin x}{x} > 0$ $[\because 1 - \cos c > 0]$

$\Rightarrow$ $x - \sin x > 0$ $[\because x > 0]$

$\Rightarrow$ $x > \sin x$

$\Rightarrow$ $\sin x < x$ for all x.

S26. Let $y = f(x) = (x + 3)^2$

The function being polynomial,

(a) $f(x)$ is continuous on closed interval $[-4, -3]$.

(b) Since $f'(x) = 2(x + 3)$, which exists for all $x \in (-4, -3)$. So $f(x)$ is derivable in $(-4, -3)$.

Thus, both the conditions of Lagrange's Mean value Theorem are satisfied.

Hence, there must exist at least one $c \in (-4, -3)$, such that

$$\frac{f(b) - f(a)}{b - a} = f'(c), \text{ where } a = -4,\ b = -3 \qquad \text{... (i)}$$

$\therefore$ $f'(x) = 2(x + 3)$

$f'(c) = 2(c + 3)$

$f(b) = f(-3) = (-3 + 3)^2 = 0$

and $f(a) = f(-4) = (-4 + 3)^2 = (-1)^2 = 1$

From (i), we have

$$\frac{0 - 1}{-3 - (-4)} = 2(c + 3) \quad \Rightarrow \quad \frac{-1}{1} = 2(c + 3)$$

$\Rightarrow$ $-1 = 2c + 6 \quad \Rightarrow \quad 2c = -7$

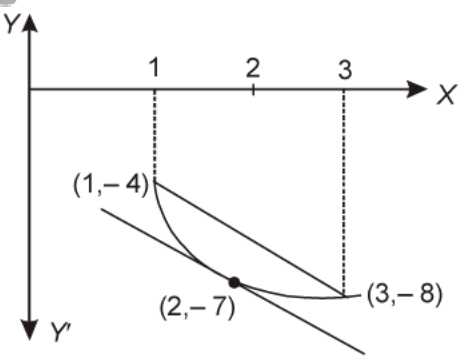
$\Rightarrow$ $c = -\frac{7}{2} \in (-4, -3)$

When $x = -\frac{7}{2}$ then $y = \left(-\frac{7}{2} + 3\right)^2 = \left(-\frac{1}{2}\right)^2 = \frac{1}{4}$

Thus, the tangent at the point $\left(-\frac{7}{2}, \frac{1}{4}\right)$ on the given curve is parallel. to chord joining the point $(-3, 0)$ and $(-4, 1)$.

S27. We have,

$f(x) = x^2 - 6x + 1$



Being polynomial,

(a) $f(x)$ is continuous on closed interval $[1, 3]$.

(b) Since $f'(x) = 2x - 6$, which exists for all $x \in (1, 3)$.

$\Rightarrow$ $f(x)$ is derivable in the interval $(1, 3)$.

Thus both the conditions of Lagrange's Mean Value Theorem are satisfied.

Hence, there exists at least one $c \in (1,3)$.

Such that $$\frac{f(b) - f(a)}{b - a} = f'(c), \text{ where } a = 1,\ b = 3 \qquad \text{... (i)}$$

Now, $f'(x) = 2x - 6$

$\Rightarrow$ $f'(c) = 2c - 6$

$f(b) = f(3) = (3)^2 - 6(3) + 1 = 9 - 18 + 1 = -8$

$f(a) = f(1) = (1)^2 - 6(1) + 1 = 1 - 6 + 1 = -4$

From (i), we get

$$\frac{-8-(-4)}{3-1} = 2c - 6$$

$$\Rightarrow \quad \frac{-4}{2} = 2c - 6$$

$$\Rightarrow \quad -2 = 2c - 6 \quad \Rightarrow \quad -1 = c - 3$$

$$\Rightarrow \quad c = -1 + 3 = 2 \in (1, 3)$$

When $x = 2$, then $y = (2)^2 - 6(2) + 1 = 4 - 12 + 1 = -7$

Thus, the tangent at $(2, -7)$, on the curve $y = x^2 - 6x + 1$ is parallel to the chord joining the points $(1, -4)$ and $(3, -8)$.

S28. If $f(x)$ is a function defined on $[a, b]$ such that :

(a) $f(x)$ is continuous on closed interval $[a, b]$,

(b) $f(x)$ is derivable on open interval (a, b).

Then, there exists a real number $c \in (a, b)$, such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \qquad \text{[Lagrange's Mean Value Theorem]}$$

$$\Rightarrow \quad f'(c) = \frac{f(1) - f(0)}{1 - 0}, \text{ where } c \in (0, 1)$$

$$\Rightarrow \quad f'(c) = \frac{6-2}{1-0} = 4 \qquad \text{... (i)}$$

and $$g'(c) = \frac{g(1) - g(0)}{1 - 0} \text{ where } c \in (0, 1)$$

$$\Rightarrow \quad g'(c) = \frac{2-0}{1-0} = 2 \qquad \text{... (ii)}$$

From (i) and (ii), we get

$$f'(c) = 2.g'(c).$$