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Q1. Using properties of determinants, show that

$$\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix} = 0.$$

Q2. Without expanding, prove that

$$\begin{vmatrix} a+bx & c+dx & p+qx \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix} = (1-x^2)\begin{vmatrix} a & c & p \\ b & d & q \\ u & v & w \end{vmatrix}.$$

Q3. If *a*, *b*, *c* are in *A.P.*, find the value of

$$\begin{vmatrix} 2y+4 & 5y+7 & 8y+a \\ 3y+5 & 6y+8 & 9y+b \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix}.$$

Q4. Find the values of *k* if area of triangle is 4 sq. units and vertices are: (–2, 0), (0, 4), (0, *k*)

Q5. Find the values of *k* if area of triangle is 4 sq. units and vertices are: (*k*, 0), (4, 0), (0, 2)

Q6. Show that the points *A*(*a*, *b* + *c*), *B* (*b, c* + *a*), *C* (*c*, *a* + *b*) are collinear.

Q7. Find the equation of the line joining (*a, b*) and (*b, a*), using determinants.

Q8. Find the value of *k* so that the points *A* (5, 5) *B* (*k*, 1) and *C* (11, 7) are collinear.

Q9. Prove, using properties of determinants

$$\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix} = k^2(3y+k).$$

Q10. Show that

$$\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}.$$

Q11. Using properties of determinants, prove that

$$\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix} = x^3.$$

Q12. Prove that

$$\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 1+3p+2q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix} = 1.$$

Q13. Evaluate:

$$\Delta = \begin{vmatrix} 10! & 11! & 12! \\ 11! & 12! & 13! \\ 12! & 13! & 14! \end{vmatrix}$$

Q14. Solve for x

$$\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0.$$

Q15. Solve for x

$$\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0.$$

Q16. Without expanding evaluate the determinant

$$\begin{vmatrix} (a^x + a^{-x})^2 & (a^x - a^{-x})^2 & 1 \\ (a^y + a^{-y})^2 & (a^y - a^{-y})^2 & 1 \\ (a^z + a^{-z})^2 & (a^z - a^{-z})^2 & 1 \end{vmatrix}, \quad \text{where } a > 0 \text{ and } (x, y, z) \in R$$

Q17. Without expanding show that

$$\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix} = 0$$

Q18. Using properties of determinants, prove the following:

$$\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2.$$

Q19. Using properties of determinants, prove the following

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz(x-y)(y-z)(z-x).$$

Q20. Prove that

$$\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc.$$

Q21. Prove that

$$\begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ ax+by & bx+cy & 0 \end{vmatrix} = (b^2-ac)(ax^2+2bxy+cy^2).$$

Q22. Show that

$$\begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & b+a & c \end{vmatrix} = (a+b+c)(a^2+b^2+c^2).$$

Q23. Prove that

$$\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2.$$

Q24. Show that

$$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2.$$

Q25. Prove that

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix} = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha+\beta+\gamma).$$

Q26. Using properties of determinants, prove that

$$\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3.$$

Q27. Using properties of determinants, prove that

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3.$$

Q28. Using properties of determinants, prove the following:

$$\begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix} = 3(x+y+z)(xy+yz+zx).$$

Q29. Using properties of determinants, prove the following:

$$\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix} = 9y^2(x+y).$$

Q30. Prove that

$$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc\,(a+b+c)^3.$$

Q31. Using properties of determinants, prove that

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1 + a^2 + b^2 + c^2.$$

Q32. Prove that

$$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca).$$

Q33. For any scalar *p* prove that

$$\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = (1+pxyz)(x-y)(y-z)(z-x).$$

Q34. Using properties of determinants, prove that

$$\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = xyz + xy + yz + zx.$$

Q35. Show that

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3.$$

Q36. Show that

$$\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a^3.$$

Q37. Show that

$$\begin{vmatrix} b^2+c^2 & ab & ac \\ ba & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2.$$

Q38. Prove that

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = abc + bc + ca + ab.$$

Q39. Using properties of determinants, prove the following

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c).$$

Q40. Using properties of determinants, prove the following:

$$\begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2).$$

Q41. Show that if $x \neq y \neq z$ and

$$\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0, \text{ then } 1+xyz = 0.$$

Q42. Prove that

$$\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3+b^3+c^3-3abc.$$

Q43. Show that, $abc(ab+bc+ca) = a+b+c$. Where a, b, c are all different such that

$$\begin{vmatrix} a & a^3 & a^4-1 \\ b & b^3 & b^4-1 \\ c & c^3 & c^4-1 \end{vmatrix} = 0.$$

Q44. If a, b, c are real numbers such that:

$$\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0,$$ then show that either $a+b+c=0$ or, $a=b=c$.

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S1.

Let L.H.S. $\Delta = \begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ac \\ 1 & c & c^2-ab \end{vmatrix}$.

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + \begin{vmatrix} 1 & a & -bc \\ 1 & b & -ca \\ 1 & c & -ab \end{vmatrix}$$

$\Rightarrow$ $\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$ [Taking (– 1) common from C_3 of second det.]

$\Rightarrow$ $\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \dfrac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix}$

[Multiplying R_1, R_2 and R_3 of second det. by *a, b* and *c* respectively]

$\Rightarrow$ $\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \dfrac{abc}{abc}\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$ [Taking *abc* common from C_3 of second det].

$\Rightarrow$ $\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + \begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix}$ [Applying $C_2 \leftrightarrow C_3$ in second det.]

$\Rightarrow$ $\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ [Applying $C_1 \leftrightarrow C_2$ in second det.]

$\Rightarrow$ $\Delta = 0 =$ R.H.S.

S2. Let Δ be the given determinant.

$$\Delta = \begin{vmatrix} a+bx & c+dx & p+qx \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix}$$

$$\Delta = \begin{vmatrix} a(1-x^2) & c(1-x^2) & p(1-x^2) \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix} \quad \text{Applying } R_1 \to R_1 - xR_2 \text{ we get}$$

$$\Rightarrow \quad \Delta = (1-x^2)\begin{vmatrix} a & c & p \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix} \quad [\text{taking common } (1-x^2) \text{ From } R_1]$$

$$\Rightarrow \quad \Delta = (1-x^2)\begin{vmatrix} a & c & p \\ b & d & q \\ u & v & w \end{vmatrix} \quad [\text{Applying } R_2 \to R_2 - xR_1]$$

$= \text{R.H.S.}$

S3. Let Δ be the given determinant.

$$\Delta = \begin{vmatrix} 2y+4 & 5y+7 & 8y+a \\ 3y+5 & 6y+8 & 9y+b \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix}$$

$$\Rightarrow \quad \Delta = \frac{1}{2}\begin{vmatrix} 2y+4 & 5y+7 & 8y+a \\ 6y+10 & 12y+16 & 18y+2b \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix} \quad \text{Applying } R_2 \to 2R_2\text{, we get}$$

$$\Rightarrow \quad \Delta = \frac{1}{2}\begin{vmatrix} 2y+4 & 5y+7 & 8y+a \\ 0 & 0 & 2b-(a+c) \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix} \quad \text{Applying } R_2 \to R_2 - (R_1 + R_3)$$

$$\Rightarrow \quad \Delta = \frac{1}{2}\begin{vmatrix} 2y+4 & 5y+7 & 8y+a \\ 0 & 0 & 0 \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix} \quad [\because a, b, c \text{ are in } A.P \therefore 2b = a + c]$$

$$\Rightarrow \quad \Delta = 0.$$

S4. Let $A(-2, 0)$, $B(0, 4)$, $C(0, k)$ be three vertices of triangle ABC. Then

$$\text{area of } \Delta ABC = \frac{1}{2}\begin{vmatrix} -2 & 0 & 1 \\ 0 & 4 & 1 \\ 0 & k & 1 \end{vmatrix}$$

$$= \frac{1}{2}[-2(4-k) - 0(0-0) + 1(0-0)]$$

$$= \frac{1}{2}[-8+2k] = (k-4) \text{ sq. units}$$

But area of ΔABC is 4 sq. units

$\therefore \quad k - 4 = \pm 4$

$\therefore \quad k - 4 = 4 \quad$ or $\quad k - 4 = -4$

$\Rightarrow \quad k = 4 + 4 \quad$ or $\quad k = -4 + 4$

$\Rightarrow \quad k = 8 \quad$ or $\quad k = 0$

S5. Let $A\,(k, 0)$, $B(4, 0)$ and $C\,(0, 2)$ be three vertices of triangle ABC. Then

$$\text{area of } \Delta ABC = \frac{1}{2}\begin{vmatrix} k & 0 & 1 \\ 4 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix}$$

$$= \frac{1}{2}[k(0-2) - 0(4-0) + 1(8-0)]$$

$$= \frac{1}{2}[-2k+8] = (-k+4) \text{ sq. units}$$

But area of ΔABC is 4 sq. units

$\therefore \quad -k + 4 = \pm 4$

$\therefore \quad -k + 4 = 4 \quad$ or $\quad -k + 4 = -4$

$\Rightarrow \quad k = 0 \quad$ or $\quad k = 8$

Note : Since area is always a positive quantity, therefore we always take the absolute value of the determinant for the area.

S6. Let $A\,(a, b + c)$, $B(b, c + a)$ and $C(c, a + b)$ be three vertices of triangle ABC. Then

$$\text{area of } \Delta ABC = \frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$$

$$= \frac{1}{2}[a(c+a-a-b) - (b+c)(b-c) + 1(ab+b^2-c^2-ac)]$$

$$= \frac{1}{2}[\cancel{ac} - \cancel{ab} - \cancel{b^2} + \cancel{c^2} + \cancel{ab} + \cancel{b^2} - \cancel{c^2} - \cancel{ac}]$$

$$= \frac{1}{2} \times 0 = 0$$

$\therefore \quad$ area of $\Delta ABC = 0$

Thus, points $A(a, b + c)$, $B\,(b, c + a)$, $C\,(c, a + b)$ are collinear points.

S7. Let $A(a, b)$ and $B(b, a)$ be any two points. Let $P(x, y)$ be any point on line AB.

$\therefore \quad$ area of $\Delta ABP = 0$

$$\frac{1}{2}\begin{vmatrix} a & b & 1 \\ b & a & 1 \\ x & y & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad \frac{1}{2}[a(a-y) - b(b-x) + 1(by - ax)] = 0$

$\Rightarrow \quad a^2 - ay - b^2 + bx + by - ax = 0$

$\Rightarrow \quad (b - a)x + (b - a)\, y = b^2 - a^2$

$\Rightarrow \quad x + y = a + b$

which is the equation of required line.

S8. Let A (5,5), B (k, 1) and C(11, 7) be three vertices of a ΔABC.

Then area of $\Delta ABC = \frac{1}{2}\begin{vmatrix} 5 & 5 & 1 \\ k & 1 & 1 \\ 11 & 7 & 1 \end{vmatrix}$

$$= \frac{1}{2}[5(1-7) - 5(k-11) + 1(7k-11)]$$

$$= \frac{1}{2}[-30 - 5k + 55 + 7k - 11]$$

$$= \frac{1}{2}[2k + 14] = (k + 7) \text{ sq. units}$$

But it is given that A(5, 5), B(k, 1) and C(11, 7) are collinear.

$\therefore \quad$ area of $\Delta ABC = 0$

$\therefore \quad k + 7 = 0 \quad \Rightarrow \quad k = -7$

S9. Let,

$$\text{L.H.S.} = \begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we get

$$= \begin{vmatrix} 3y+k & 3y+k & 3y+k \\ y & y+k & y \\ y & y & y+k \end{vmatrix}$$

Taking $(3y + k)$ common from R_1, we get

$$= (3y+k)\begin{vmatrix} 1 & 1 & 1 \\ y & y+k & y \\ y & y & y+k \end{vmatrix}$$

Now, applying $C_1 \to C_1 - C_2,\ C_2 \to C_2 - C_3$, we get

$$= (3y+k)\begin{vmatrix} 0 & 0 & 1 \\ -k & k & y \\ 0 & -k & y+k \end{vmatrix}$$

Expanding along R_1, we get

$$= (3y + k)\ [1(k^2)] = k^2(3y + k)$$

= R.H.S. **Hence proved.**

S10. Let,

$$\text{L.H.S.} = \begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix}$$

$$\Rightarrow \quad = \begin{vmatrix} 2(a+b+c) & c+a & a+b \\ 2(p+q+r) & r+p & p+q \\ 2(x+y+z) & z+x & x+y \end{vmatrix} \quad \{\text{Applying } C_1 \to C_1 + C_2 + C_3\}$$

$$\Rightarrow \quad = 2\begin{vmatrix} a+b+c & c+a & a+b \\ p+q+r & r+p & p+q \\ x+y+z & z+x & x+y \end{vmatrix} \quad \{\text{Taking 2 common from } C_1\}$$

$$\Rightarrow \quad = 2\begin{vmatrix} a+b+c & -b & -c \\ p+q+r & -q & -r \\ x+y+z & -y & -z \end{vmatrix} \quad \left\{\begin{matrix}\text{Applying } C_2 \to C_2 - C_1 \\ \text{and } C_3 \to C_3 - C_1\end{matrix}\right\}$$

$$\Rightarrow \quad = 2\begin{vmatrix} a & -b & -c \\ p & -q & -r \\ x & -y & -z \end{vmatrix} \quad \{\text{Applying } C_1 \to C_1 + C_2 + C_3\}$$

$$\Rightarrow \quad = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} \quad \{\text{Taking } (-1) \text{ common from both } C_2 \text{ and } C_3\}$$

$\Rightarrow$ = R.H.S.

S11.

$$\text{L.H.S.} = \begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix}$$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$, we get

$$= \begin{vmatrix} x+y & x & x \\ 3x+2y & 2x & 0 \\ 7x+5y & 5x & 0 \end{vmatrix}$$

Expanding along C_3, we get

$$= x\begin{vmatrix} 3x+2y & 2x \\ 7x+5y & 5x \end{vmatrix}$$

$$= x[5x(3x + 2y) - 2x(7x + 5y)]$$

$$= x[15x^2 + 10xy - (14x^2 + 10xy)] = x^3 = \text{R.H.S.}$$

Hence proved.

S12. Let,

$$\text{L.H.S.} = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 1+3p+2q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix}$$

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, we get

$$= \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & p-1 \\ 0 & 3 & 3p-2 \end{vmatrix}$$

Now, expanding along C_1, we get

$$= 1\times\begin{vmatrix} 1 & p-1 \\ 3 & 3p-2 \end{vmatrix}$$

$$= 1[(3p-2)-(3p-3)]$$

$$= 3p - 2 - 3p + 3 = 1 = \text{R.H.S.}$$ **Hence proved.**

S13. We have,

$$\Rightarrow \qquad \Delta = \begin{vmatrix} 10! & 11! & 12! \\ 11! & 12! & 13! \\ 12! & 13! & 14! \end{vmatrix}$$

$$\Rightarrow \qquad \Delta = \begin{vmatrix} 10! & 11\times 10! & 12\times 11\times 10! \\ 11! & 12\times 11! & 13\times 12\times 11! \\ 12! & 13\times 12! & 14\times 13\times 12! \end{vmatrix}$$

$$\Rightarrow \quad \Delta = 10! \times 11! \times 12! \begin{vmatrix} 1 & 11 & 132 \\ 1 & 12 & 156 \\ 1 & 13 & 182 \end{vmatrix}$$

[Taking 10!, 11! and 12! common from R_1, R_2 and R_3 respectively]

$$\Rightarrow \quad \Delta = 10! \times 11! \times 12! \begin{vmatrix} 1 & 11 & 132 \\ 0 & 1 & 24 \\ 0 & 2 & 50 \end{vmatrix} \quad \text{[Applying } R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1\text{]}$$

$$\Rightarrow \quad \Delta = 10! \times 11! \times 12! \times \begin{vmatrix} 1 & 24 \\ 2 & 50 \end{vmatrix} \quad \text{[Expanding along } C_1\text{]}$$

$$\Rightarrow \quad \Delta = (10! \times 11! \times 12!) \times 2$$

S14.

Let $$\Delta = \begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix}$$

$$\Rightarrow \quad \begin{vmatrix} 3a-x & a-x & a-x \\ 3a-x & a+x & a-x \\ 3a-x & a-x & a+x \end{vmatrix} \quad \{\text{Applying } C_1 \to C_1 + C_2 + C_3\}$$

$$\Rightarrow \quad (3a-x)\begin{vmatrix} 1 & a-x & a-x \\ 1 & a+x & a-x \\ 1 & a-x & a+x \end{vmatrix} \quad \{\text{Taking } (3a-x) \text{ common from } C_1\}$$

$$\Rightarrow \quad (3a-x)\begin{vmatrix} 1 & a-x & a-x \\ 0 & 2x & 0 \\ 0 & 0 & 2x \end{vmatrix} \quad \left\{\begin{matrix} \text{Applying } R_2 \to R_2 - R_1 \\ \text{and } R_3 \to R_3 - R_1 \end{matrix}\right\}$$

$$\Rightarrow \quad (3a-x) \times 1 \begin{vmatrix} 2x & 0 \\ 0 & 2x \end{vmatrix} \quad \{\text{Expanding along } C_1\}$$

$$\Rightarrow \quad (3a-x)\, 4x^2$$

Now $\Delta = 0$

$\Rightarrow \quad (3a-x)4x^2 = 0$

$\Rightarrow \quad x = 0, 3a$

S15. Let,

$$\Delta = \begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix}$$

$$\Rightarrow \quad \begin{vmatrix} x-2 & 1 & 2 \\ x-4 & -1 & -4 \\ x-8 & -11 & -40 \end{vmatrix} \quad \{\text{Applying } C_2 \to C_2 - 2C_1 \text{ and } C_3 \to C_3 - 3C_1\}$$

$$\Rightarrow \quad \begin{vmatrix} x-2 & 1 & 2 \\ -2 & -2 & -6 \\ -6 & -12 & -42 \end{vmatrix} \quad \{\text{Applying } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1\}$$

$$\Rightarrow \quad (-2)\,(-6)\begin{vmatrix} x-2 & 1 & 2 \\ 1 & 1 & 3 \\ 1 & 2 & 7 \end{vmatrix}$$

{Taking (–2) and (–6) common from R_2 and R_3}

$$\Rightarrow \quad 12\begin{vmatrix} x-2 & 1 & 2 \\ 1 & 1 & 3 \\ 0 & 1 & 4 \end{vmatrix} \quad \{\text{Applying } R_3 \to R_3 - R_2\}$$

$$\Rightarrow \quad 12\left\{(x-2)\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix}\right\} \quad \text{Expanding along } C_1.$$

$$\Rightarrow \quad 12\,\{(x-2)\,(4-3) - 1\,(4-2)\} = 12\,(x-4)$$

Now $\quad \Delta = 0$

$\Rightarrow \quad 12\,(x-4) = 0$

$\Rightarrow \quad x = 4.$

S16. Let Δ be the given determinant.

$$\Delta = \begin{vmatrix} (a^x + a^{-x})^2 & (a^x - a^{-x})^2 & 1 \\ (a^y + a^{-y})^2 & (a^y - a^{-y})^2 & 1 \\ (a^z + a^{-z})^2 & (a^z - a^{-z})^2 & 1 \end{vmatrix} \quad \text{Applying } C_1 \to C_1 - C_2, \text{ we get}$$

$$= \begin{vmatrix} (a^x + a^{-x})^2 - (a^x - a^{-x})^2 & (a^x - a^{-x})^2 & 1 \\ (a^y + a^{-y})^2 - (a^y - a^{-y})^2 & (a^y - a^{-y})^2 & 1 \\ (a^z + a^{-z})^2 - (a^z - a^{-z})^2 & (a^z - a^{-z})^2 & 1 \end{vmatrix} \quad [\text{Using: } (a+b)^2 - (a-b)^2 = 4ab]$$

$$= \begin{vmatrix} 4a^x \cdot a^{-x} & (a^x - a^{-x})^2 & 1 \\ 4a^y \cdot a^{-y} & (a^y - a^{-y})^2 & 1 \\ 4a^z \cdot a^{-z} & (a^z - a^{-z}) & 1 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} 4 & (a^x - a^{-x})^2 & 1 \\ 4 & (a^y - a^{-y})^2 & 1 \\ 4 & (a^z - a^{-z})^2 & 1 \end{vmatrix} \quad [\because \quad a^m \cdot a^{-m} = a^0 = 1]$$

$$\Delta = 4\begin{vmatrix} 1 & (a^x - a^{-x})^2 & 1 \\ 1 & (a^y - a^{-y})^2 & 1 \\ 1 & (a^z - a^{-z})^2 & 1 \end{vmatrix}$$

[Taking out 4 common from C_1]
[C_1 and C_3 are identical]

$$\Delta = 4 \times 0 = 0.$$

S17. Let,

$$\Delta = \begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix}$$

Applying $R_1 \to R_1(a)$, $R_2 \to R_2(b)$ and $R_3 \to R_3(c)$, we get

$$\Delta = \frac{1}{abc}\begin{vmatrix} ab^2c^2 & abc & ab+ac \\ bc^2a^2 & abc & bc+ba \\ ca^2b^2 & abc & ac+bc \end{vmatrix}$$

[$\because$ R_1, R_2 R_3 are multiplied by *a, b* and *c* respectively, therefore we divide by *abc*]

$$\Rightarrow \quad \Delta = \frac{1}{abc}(abc)^2\begin{vmatrix} bc & 1 & ab+ac \\ ca & 1 & bc+ca \\ ab & 1 & ac+bc \end{vmatrix}$$ [Taking out *abc* common from C_1 and C_2]

$$\Rightarrow \quad \Delta = abc\begin{vmatrix} bc & 1 & ab+bc+ca \\ ca & 1 & ab+bc+ca \\ ab & 1 & ab+bc+ca \end{vmatrix}$$ [Applying $C_3 \to C_3 + C_1$]

$$\Rightarrow \quad \Delta = abc(ab+bc+ca)\begin{vmatrix} bc & 1 & 1 \\ ca & 1 & 1 \\ ab & 1 & 1 \end{vmatrix}$$ [Taking out $ab + bc + ca$ common from C_3]

$$\Rightarrow \quad \Delta = abc(ab+bc+ca)\cdot 0 = 0$$ [$\because$ C_2 and C_3 are identical]

S18. Applying $R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix} x+4+2x+2x & 2x+x+4+2x & 2x+2x+x+4 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

$$= \begin{vmatrix} 5x+4 & 5x+4 & 5x+4 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

$$= (5x+4)\begin{vmatrix} 1 & 1 & 1 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} \quad [\because \text{ Taking } 5x+4 \text{ common from } R_1]$$

Now, applying $C_2 \to C_2 - C_1$

$$= (5x+4)\begin{vmatrix} 1 & 0 & 1 \\ 2x & 4-x & 2x \\ 2x & 0 & 4+x \end{vmatrix}$$

Expanding along C_2, we get

$$= (5x+4)(4-x)\begin{vmatrix} 1 & 1 \\ 2x & 4+x \end{vmatrix}$$

$$= (5x+4)(4-x)(4+x-2x)$$

$= (5x+4)(4-x)^2 =$ R.H.S. **Hence proved.**

S19. Let,

$$\text{L.H.S.} = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} \quad [\because \text{ Taking } x, y \text{ and } z \text{ common from } C_1, C_2 \text{ and } C_3]$$

Applying $C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$, we get

$$= xyz\begin{vmatrix} 0 & 0 & 1 \\ x-y & y-z & z \\ x^2-y^2 & y^2-z^2 & z^2 \end{vmatrix}$$

Expanding the above determinant along R_1, we get

$$= xyz\begin{vmatrix} x-y & y-z \\ x^2-y^2 & y^2-z^2 \end{vmatrix}$$

Taking common $(x-y)$ from C_1 and $(y-z)$ from C_2.

$$= xyz(x-y)(y-z)\begin{vmatrix} 1 & 1 \\ x+y & y+z \end{vmatrix}$$

$$= xyz(x-y)(y-z)(z-x)$$

= R.H.S. **Hence proved.**

S20. Let,

$$\text{L.H.S.} = \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we get

$$= \begin{vmatrix} 2b+2c & 2a+2c & 2a+2b \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

$$= 2\begin{vmatrix} b+c & a+c & a+b \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$, we get

$$= 2\begin{vmatrix} c & 0 & a \\ b-c & a & -a \\ c & c & a+b \end{vmatrix}$$

Expanding along R_1, we get

$$= 2[c\,(a^2 + ab + ac) + a(cb - c^2 - ac)]$$

$$= 2\,[\cancel{ca^2} + abc + \cancel{ac^2} + acb - \cancel{ac^2} - \cancel{a^2c}]$$

$$= 2[2\,abc]$$

$$= 4abc = \text{R.H.S.}$$

$\therefore$ L.H.S. = R.H.S. **Hence proved**

S21. Let,

$$\text{L.H.S.} = \Delta = \begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ ax+by & bx+cy & 0 \end{vmatrix}$$ Applying $C_3 \to C_3 - xC_1 - yC_2$, we get

$$\Delta = \begin{vmatrix} a & b & 0 \\ b & c & 0 \\ ax+by & bx+cy & -x(ax+by)-y(bx+cy) \end{vmatrix}$$

$\Rightarrow$ $$\Delta = \begin{vmatrix} a & b & 0 \\ b & c & 0 \\ ax+by & bx+cy & -(ax^2+2bxy+cy^2) \end{vmatrix}$$

$\Rightarrow$ $$\Delta = -(ax^2+2bxy+cy^2)\begin{vmatrix} a & b \\ b & c \end{vmatrix}$$ [Expanding along C_3]

$\Rightarrow$ $\Delta = -\,(ax^2 + 2bxy + cy^2)\,(ac - b^2)$

$\Rightarrow$ $\Delta = (b^2 - ac)\,(ax^2 + 2bxy + cy^2)$ = R.H.S.

S22. Let,

$$\text{L.H.S.} = \Delta = \begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & b+a & c \end{vmatrix}.$$

Multiplying first column by *a*, we get

$$\Delta = \frac{1}{a}\begin{vmatrix} a^2 & b-c & c+b \\ a^2+ac & b & c-a \\ a^2-ab & b+a & c \end{vmatrix}$$

$\Rightarrow$ $$\Delta = \frac{1}{a}\begin{vmatrix} a^2+b^2+c^2 & b-c & c+b \\ a^2+b^2+c^2 & b & c-a \\ a^2+b^2+c^2 & b+a & c \end{vmatrix}$$ [Applying $C_1 \to C_1 + bC_2 + cC_3$]

$\Rightarrow$ $$\Delta = \frac{1}{a}(a^2+b^2+c^2)\begin{vmatrix} 1 & b-c & c+b \\ 1 & b & c-a \\ 1 & b+a & c \end{vmatrix}$$ [Taking $a^2 + b^2 + c^2$ common from C_1]

$\Rightarrow$ $$\Delta = \frac{1}{a}(a^2+b^2+c^2)\begin{vmatrix} 1 & b-c & c+b \\ 0 & c & -a-b \\ 0 & a+c & -b \end{vmatrix}$$ [Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$]

$\Rightarrow$ $$\Delta = \frac{1}{a}(a^2+b^2+c^2)\times 1\times\begin{vmatrix} c & -a-b \\ a+c & -b \end{vmatrix}$$ [Expanding along C_1]

$\Rightarrow$ $$\Delta = \frac{1}{a}(a^2+b^2+c^2)(-bc+a^2+ac+ba+bc)$$

$\Rightarrow$ $$\Delta = (a^2+b^2+c^2)(a+b+c).$$

S23. Let,

$$\text{L.H.S.} = \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we get

$$= \begin{vmatrix} 1+x+x^2 & 1+x+x^2 & 1+x+x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

Taking $1 + x + x^2$ common from row R_1, we get

$$= (1+x+x^2)\begin{vmatrix} 1 & 1 & 1 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$, we get

$$= (1+x+x^2)\begin{vmatrix} 0 & 0 & 1 \\ x^2-1 & 1-x & x \\ x-x^2 & x^2-1 & 1 \end{vmatrix}$$

$$= (1+x+x^2)\begin{vmatrix} 0 & 0 & 1 \\ (x-1)(x+1) & 1-x & x \\ x(1-x) & (x-1)(x+1) & 1 \end{vmatrix}$$

Taking $(1 - x)$ common from columns C_1 and C_2, we get

$$= (1+x+x^2)(1-x)^2\begin{vmatrix} 0 & 0 & 1 \\ -(x+1) & 1 & x \\ x & -(x+1) & 1 \end{vmatrix}$$

Expanding the above determinant along row R_1, we get

$$= (1+x+x^2)(1-x)^2\left(1\times\begin{vmatrix} -(x+1) & 1 \\ x & -(x+1) \end{vmatrix}\right)$$

$= (1 + x + x^2)(1 - x)^2 [(x + 1)^2 - x]$

$= (1 + x + x^2)(1 - x)^2 (1 + x^2 + 2x - x)$

$= (1 + x + x^2)(1 - x)^2 (1 + x + x^2)$

$= [(1 + x + x^2)(1 - x)]^2 = (1 - x^3)^2$ $\quad [\because (1 + x + x^2)(1 - x) = 1 - x^3]$

= R.H.S. **Hence proved.**

S24. Let,

$$\text{L.H.S.} = \Delta = \begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$$

$$\Delta = abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

[Taking a, b, c common from C_1, C_2 and C_3 respectively]

$$\Rightarrow \qquad \Delta = abc\begin{vmatrix} 2(a+c) & c & a+c \\ 2(a+b) & b & a \\ 2(b+c) & b+c & c \end{vmatrix}$$

[Applying $C_1 \to C_1 + C_2 + C_3$]

$$\Rightarrow \qquad \Delta = 2abc\begin{vmatrix} a+c & c & a+c \\ a+b & b & a \\ b+c & b+c & c \end{vmatrix}$$

$$\Rightarrow \qquad \Delta = 2abc\begin{vmatrix} a+c & -a & 0 \\ a+b & -a & -b \\ b+c & 0 & -b \end{vmatrix}$$ [Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$]

$$\Rightarrow \qquad \Delta = 2abc\begin{vmatrix} c & -a & 0 \\ 0 & -a & -b \\ c & 0 & -b \end{vmatrix}$$ [Applying $C_1 \to C_1 + C_2 + C_3$]

$$\Rightarrow \qquad \Delta = 2abc \times abc\begin{vmatrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & -1 \end{vmatrix}$$

` [Taking *c, a* and *b* common from C_1, C_2 and C_3 respectively]

$$\Rightarrow \qquad \Delta = 2a^2b^2c^2\begin{vmatrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & -1 \end{vmatrix}$$ [Applying $R_3 \to R_3 - R_1$]

$$\Rightarrow \qquad \Delta = 4a^2\, b^2\, c^2 = \text{R.H.S}$$ [Expanding along C_1]

S25. Let,

$$\text{L.H.S.} = \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix}$$

$$= \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \alpha+\beta+\gamma & \alpha+\beta+\gamma & \alpha+\beta+\gamma \end{vmatrix}$$ {Applying $R_3 \to R_1 + R_3$}

$$= (\alpha+\beta+\gamma)\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ 1 & 1 & 1 \end{vmatrix}$$ {Taking out ($\alpha + \beta + \gamma$) common from R_3}

$$= (\alpha+\beta+\gamma)\begin{vmatrix} \alpha & \beta-\alpha & \gamma-\alpha \\ \alpha^2 & \beta^2-\alpha^2 & \gamma^2-\alpha^2 \\ 1 & 0 & 0 \end{vmatrix} \quad \{\text{Applying } C_2 \to C_2 - C_1,\ C_3 \to C_3 - C_1\}$$

$$= (\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha)\begin{vmatrix} \alpha & 1 & 1 \\ \alpha^2 & \beta+\alpha & \gamma+\alpha \\ 1 & 0 & 0 \end{vmatrix} \quad \begin{Bmatrix} \text{Taking common } (\beta-\alpha) \text{ from } C_2 \\ \text{and } (\gamma-\alpha) \text{ from } C_3 \end{Bmatrix}$$

$$= (\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha)\times 1\begin{vmatrix} 1 & 1 \\ \beta+\alpha & \gamma+\alpha \end{vmatrix} \quad \{\text{expanding along } R_3\}$$

$$= (\alpha + \beta + \gamma)(\beta - \alpha)(\gamma - \alpha)(\gamma + \alpha - \beta - \alpha)$$

$$= (\alpha + \beta + \gamma)(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha) = \text{R.H.S.}$$

Hence proved.

S26. Let,

$$\text{L.H.S.} = \begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$= \begin{vmatrix} 2x+2y+2z & x & y \\ 2x+2y+2z & y+z+2x & y \\ 2x+2y+2z & x & z+x+2y \end{vmatrix}$$

Taking $2(x + y + z)$ common from C_1

$$= 2(x+y+z)\begin{vmatrix} 1 & x & y \\ 1 & y+z+2x & y \\ 1 & x & z+x+2y \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$= 2(x+y+z)\begin{vmatrix} 1 & x & y \\ 0 & y+z+x & 0 \\ 0 & 0 & z+x+y \end{vmatrix}$$

Taking common $(x + y + z)$ from R_2 and R_3

$$= 2(x+y+z)^3\begin{vmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along C_1, we get

$$= 2(x + y + z)^3 [1(1 - 0) - 0 + 0]$$

$$= 2(x + y + z)^3$$

= R.H.S. **Hence proved.**

S27. Let,

$$\text{L.H.S.} = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we get

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Taking $(a + b + c)$ common from R_1, we get

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Now, applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$

$$= (a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ b+c+a & -(a+b+c) & 2b \\ 0 & c+a+b & c-a-b \end{vmatrix}$$

Taking $a + b + c$ common from C_1 and C_2, we get

$$= (a+b+c)^3\begin{vmatrix} 0 & 0 & 1 \\ 1 & -1 & 2b \\ 0 & 1 & c-a-b \end{vmatrix}$$

Expanding above determinant along R_1, we get

$$= (a + b + c)^3 [1 - 0]$$

$$= (a + b + c)^3 = \text{R.H.S.}$$

Hence proved.

S28.

$$\text{L.H.S.} = \begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$= \begin{vmatrix} x+y+z & -x+y & -x+z \\ x+y+z & 3y & z-y \\ x+y+z & y-z & 3z \end{vmatrix}$$

Taking common $(x + y + z)$ in C_1, we get

$$= (x+y+z)\begin{vmatrix} 1 & -x+y & -x+z \\ 1 & 3y & z-y \\ 1 & y-z & 3z \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$= (x+y+z)\begin{vmatrix} 1 & -x+y & -x+z \\ 0 & 2y+x & x-y \\ 0 & x-z & 2z+x \end{vmatrix}$$

Now, expanding along C_1, we get

$$= (x + y + z) \cdot 1 \cdot \{(2y + x)(2z + x) - (x - y)(x - z)\}$$

$$= (x + y + z)\{4yz + 2xz + 2yx + x^2 - x^2 + xy + zx - yz\}$$

$$= 3(x + y + z) \cdot (xy + yz + zx) = \text{R.H.S.}$$ **Hence proved.**

S29. Let,

$$\text{L.H.S.} = \begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we get

$$= \begin{vmatrix} 3x+3y & 3x+3y & 3x+3y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$$

Taking $(3x + 3y)$ common from R_1, we get

$$= (3x+3y)\begin{vmatrix} 1 & 1 & 1 \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get

$$= 3(x+y)\begin{vmatrix} 1 & 0 & 0 \\ x+2y & -2y & -y \\ x+y & y & -y \end{vmatrix}$$

Now, expanding along R_1, we get

$$= 3(x+y)\cdot 1\cdot[(-2y)\cdot(-y)-(y)\cdot(-y)]$$
$$= 3(x+y)[2y^2+y^2]$$
$$= 3(x+y)(3y^2) = 9y^2(x+y) = \text{R.H.S.} \quad \textbf{Hence proved.}$$

S30. Let,

$$\text{L.H.S.} = \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we get

$$= \begin{vmatrix} (b+c)^2 & a^2-(b+c)^2 & a^2-(b+c)^2 \\ b^2 & (a+c)^2-b^2 & 0 \\ c^2 & 0 & (a+b)^2-c^2 \end{vmatrix}$$

$$= \begin{vmatrix} (b+c)^2 & (a+b+c)(a-b-c) & (a+b+c)(a-b-c) \\ b^2 & (a+c+b)(a+c-b) & 0 \\ c^2 & 0 & (a+b+c)(a+b-c) \end{vmatrix}$$

Taking $a+b+c$ common from C_2 and C_3, we get

$$= (a+b+c)^2 \begin{vmatrix} (b+c)^2 & a-b-c & a-b-c \\ b^2 & a+c-b & 0 \\ c^2 & 0 & a+b-c \end{vmatrix}$$

Applying $R_1 \to R_1 - (R_2 + R_3)$, we get

$$= (a+b+c)^2 \begin{vmatrix} 2bc & -2c & -2b \\ b^2 & a+c-b & 0 \\ c^2 & 0 & a+b-c \end{vmatrix}$$

Applying $C_2 \to C_2 + \frac{1}{b}C_1, C_3 \to C_3 + \frac{1}{c}C_1$, we get

$$= (a+b+c)^2 \begin{vmatrix} 2bc & 0 & 0 \\ b^2 & a+c & b^2/c \\ c^2 & c^2/b & a+b \end{vmatrix}$$

Expanding along R_1, we get

$$= (a+b+c)^2\,[2bc(a^2+ab+ac+bc-bc)]$$
$$= (a+b+c)^2\,[2bc(a^2+ab+ac)]$$

$$= (a + b + c)^2 \cdot 2abc(a + b + c)$$
$$= 2abc(a + b + c)^3 = \text{R.H.S.}$$ **Hence proved.**

S31.

$$\text{L.H.S.} = \begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$$

Applying $R_1 \to \frac{1}{a}R_1, R_2 \to \frac{1}{b}R_2$ and $R_3 \to \frac{1}{c}R_3$, we get

$$= abc\begin{vmatrix} a+\frac{1}{a} & b & c \\ a & b+\frac{1}{b} & c \\ a & b & c+\frac{1}{c} \end{vmatrix}$$

Applying $C_1 \to aC_1,\ C_2 \to bC_2$ and $C_3 \to cC_3$

$$= \frac{abc}{abc}\begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} 1+a^2+b^2+c^2 & b^2 & c^2 \\ 1+a^2+b^2+c^2 & b^2+1 & c^2 \\ 1+a^2+b^2+c^2 & b^2 & c^2+1 \end{vmatrix}$$

$$= (1+a^2+b^2+c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 1 & b^2+1 & c^2 \\ 1 & b^2 & c^2+1 \end{vmatrix}$$

[By taking $1 + a^2 + b^2 + c^2$ common from C_1]

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$= (1+a^2+b^2+c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along C_1, we get

$$= (1 + a^2 + b^2 + c^2)\,[1(1 - 0)]$$
$$= 1 + a^2 + b^2 + c^2$$
$$= \text{R.H.S.}$$ **Hence proved.**

S32. Let,

$$\text{L.H.S.} = \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$$

Multiply columns C_1, C_2, and C_3 by a, b and c respectively and dividing the determinant by abc, we get

$$= \frac{1}{abc}\begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ abc & abc & abc \end{vmatrix}$$

Taking abc common from R_3, we get

$$= \frac{1}{abc}\cdot abc\begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ 1 & 1 & 1 \end{vmatrix}$$

Now, applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$

$$= \begin{vmatrix} a^2-b^2 & b^2-c^2 & c^2 \\ a^3-b^3 & b^3-c^3 & c^3 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} (a-b)(a+b) & (b-c)(b+c) & c^2 \\ (a-b)\,(a^2+ab+b^2) & (b-c)(b^2+bc+c^2) & c^3 \\ 0 & 0 & 1 \end{vmatrix}$$

Take $a - b$ and $b - c$ common from C_1 and C_2, respectively and expand along R_3, we get

$$= (a-b)(b-c)\cdot 1\begin{vmatrix} a+b & b+c \\ a^2+ab+b^2 & b^2+bc+c^2 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, we get

$$= (a-b)(b-c)\begin{vmatrix} a+b & c-a \\ a^2+ab+b^2 & (c^2-a^2)+b(c-a) \end{vmatrix}$$

Take $(c - a)$ common from C_2,

$$= (a-b)(b-c)(c-a)\begin{vmatrix} a+b & 1 \\ a^2+ab+b^2 & c+a+b \end{vmatrix}$$

$= (a-b)(b-c)(c-a)[a^2+ab+ac+ab+b^2+bc)-(a^2+ab+b^2)]$

$= (a-b)(b-c)(c-a)(ab+bc+ca)$

= R.H.S. **Hence proved.**

S33. Let,

$$\text{L.H.S.} = \begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & px^3 \\ y & y^2 & py^3 \\ z & z^2 & pz^3 \end{vmatrix} \{\because \text{ each element in III column is sum of two element}\}$$

$$\Rightarrow -\begin{vmatrix} 1 & x^2 & x \\ 1 & y^2 & y \\ 1 & z^2 & z \end{vmatrix} + pxyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

{Interchanging C_1 and C_3 in I[st] determinant taking x, y, z common from R_1, R_2, R_3 respectively and p from C_3 in 2[nd] determinant.}

$$\Rightarrow \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + pxyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

{Interchanging C_2 and C_3 in I[st] determinant}

$$\Rightarrow (1+pxyz)\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$\Rightarrow (1+pxyz)\begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix}$$

{Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$}

$$\Rightarrow (1+pxyz)(y-x)(z-x)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 1 & z+x \end{vmatrix}$$

{Taking $(y-x)$ and $(z-x)$ common from R_2 and R_3 respectively}

$$\Rightarrow (1+pxyz)(y-x)(z-x)\begin{vmatrix} 1 & y+x \\ 1 & z+x \end{vmatrix}$$

expanding along C_1

$\Rightarrow (1+pxyz)(y-x)(z-x)(z+x-y-x)$

$\Rightarrow (1+pxyz)(x-y)(y-z)(z-x) =$ **R.H.S.**

S34. Let,

$$\text{L.H.S.} = \begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix}$$

Dividing row R_1 by x, R_2 by y and R_3 by z and multiplying the determinant by xyz, we get

$$= xyz\begin{vmatrix} \frac{1}{x}+1 & \frac{1}{x} & \frac{1}{x} \\ \frac{1}{y} & \frac{1}{y}+1 & \frac{1}{y} \\ \frac{1}{z} & \frac{1}{z} & \frac{1}{z}+1 \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we get

$$= xyz\begin{vmatrix} 1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z} & 1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z} & 1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \\ \frac{1}{y} & \frac{1}{y}+1 & \frac{1}{y} \\ \frac{1}{z} & \frac{1}{z} & \frac{1}{z}+1 \end{vmatrix}$$

Taking $1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ common from row R_1

$$= xyz\left(1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{y} & \frac{1}{y}+1 & \frac{1}{y} \\ \frac{1}{z} & \frac{1}{z} & \frac{1}{z}+1 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we get

$$= (xyz + yz + zx + xy)\begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{y} & 1 & 0 \\ \frac{1}{z} & 0 & 1 \end{vmatrix}$$

$= (xyz + xy + yz + zx)\,[1(1-0)]$ [$\therefore$ Expanding along R_1]

$= xyz + xy + yz + zx =$ **R.H.S. Hence proved**

S35. Let,

$$\text{L.H.S.} = \begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1+a^2+b^2 & 0 & -2b \\ 0 & 1+a^2+b^2 & 2a \\ b(1+a^2+b^2) & -a(1+a^2+b^2) & 1-a^2-b^2 \end{vmatrix} \quad \left\{\begin{matrix}\text{Applying } C_1 \to C_1 - bC_3 \\ \text{and } C_2 \to C_2 + aC_3\end{matrix}\right\}$$

$$= (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix}$$ {Taking $(1 + a^2 + b^2)$ common from both C_1 and C_2}

$$= (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ 0 & 0 & 1+a^2+b^2 \end{vmatrix}$$ {Applying $R_3 \to R_3 - bR_1 + aR_2$}

$$= (1+a^2+b^2)^2 \times \begin{vmatrix} 1 & 2a \\ 0 & 1+a^2+b^2 \end{vmatrix}$$ {Expanding along C_1}

$= (1 + a^2 + b^2)^3$ = R.H.S.

S36. Let,

$$\text{L.H.S.} = \begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} a & a & a+b+c \\ 2a & 3a & 4a+3b+2c \\ 3a & 6a & 10a+6b+3c \end{vmatrix} + \begin{vmatrix} a & b & a+b+c \\ 2a & 2b & 4a+3b+2c \\ 3a & 3b & 10a+6b+3c \end{vmatrix}$$

{Since each element of 2nd column is sum of two elements}

$$\Rightarrow \begin{vmatrix} a & a & a+b+c \\ 2a & 3a & 4a+3b+2c \\ 3a & 6a & 10a+6b+3c \end{vmatrix} + ab \begin{vmatrix} 1 & 1 & a+b+c \\ 2 & 2 & 4a+3b+2c \\ 3 & 3 & 10a+6b+3c \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} a & a & a+b+c \\ 2a & 3a & 4a+3b+2c \\ 3a & 6a & 10a+6b+3c \end{vmatrix} + ab \times 0$$ {In 2nd determinant Ist and 2nd column is identical}

$$\Rightarrow \begin{vmatrix} a & a & a \\ 2a & 3a & 4a \\ 3a & 6a & 10a \end{vmatrix} + \begin{vmatrix} a & a & b \\ 2a & 3a & 3b \\ 3a & 6a & 6b \end{vmatrix} + \begin{vmatrix} a & a & c \\ 2a & 3a & 2c \\ 3a & 6a & 3c \end{vmatrix}$$

{Each element of 3rd column is sum of three element}

$$\Rightarrow a^3 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix} + a^2b \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 3 & 6 & 6 \end{vmatrix} + a^2c \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 6 & 3 \end{vmatrix}$$

$\Rightarrow \quad a^3\begin{vmatrix}1 & 1 & 1\\2 & 3 & 4\\3 & 6 & 10\end{vmatrix} + a^2b\times 0 + a^2c\times 0$ $\left\{\begin{matrix}\because \text{ In II}^{\text{nd}} \text{ determinant } C_2, C_3 \text{ are identical}\\ \text{and in III}^{\text{rd}} \text{ determinant } C_1, C_3 \text{ are identical}\end{matrix}\right\}$

$\Rightarrow \quad a^3\begin{vmatrix}1 & 0 & 0\\2 & 1 & 2\\3 & 3 & 7\end{vmatrix}$ {Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$}

$\Rightarrow \quad a^3\times 1\begin{vmatrix}1 & 2\\3 & 7\end{vmatrix}$ expanding along R_1

$\Rightarrow \quad a^3(7-6) = a^3 = \text{R.H.S.}$

S37. Let,

$$\text{L.H.S.} = \begin{vmatrix}b^2+c^2 & ab & ac\\ ba & c^2+a^2 & bc\\ ca & cb & a^2+b^2\end{vmatrix}$$

$$\Rightarrow \quad \frac{1}{abc}\begin{vmatrix}a(b^2+c^2) & a^2b & a^2c\\ b^2a & b(c^2+a^2) & b^2c\\ c^2a & c^2b & c(a^2+b^2)\end{vmatrix}$$

$\Rightarrow \quad \frac{abc}{abc}\begin{vmatrix}b^2+c^2 & a^2 & a^2\\ b^2 & c^2+a^2 & b^2\\ c^2 & c^2 & a^2+b^2\end{vmatrix}$ {take common a, b, c from C_1, C_2 and C_3}

$\Rightarrow \quad \begin{vmatrix}2(b^2+c^2) & 2(a^2+c^2) & 2(a^2+b^2)\\ b^2 & c^2+a^2 & b^2\\ c^2 & c^2 & a^2+b^2\end{vmatrix}$ {Applying $R_1 \to R_1 + R_2 + R_3$}

$\Rightarrow \quad 2\begin{vmatrix}b^2+c^2 & a^2+c^2 & a^2+b^2\\ b^2 & c^2+a^2 & b^2\\ c^2 & c^2 & a^2+b^2\end{vmatrix}$ {Take 2 common from R_1}

$\Rightarrow \quad 2\begin{vmatrix}b^2+c^2 & c^2+a^2 & a^2+b^2\\ -c^2 & 0 & -a^2\\ -b^2 & -a^2 & 0\end{vmatrix}$ $\left\{\begin{matrix}\text{Applying } R_2 \to R_2 - R_1\\ \text{and } R_3 \to R_3 - R_1\end{matrix}\right\}$

$\Rightarrow \quad 2\begin{vmatrix}0 & c^2 & b^2\\ -c^2 & 0 & -a^2\\ -b^2 & -a^2 & 0\end{vmatrix}$ {Applying $R_1 \to R_1 + R_2 + R_3$}

$$\Rightarrow \quad 2\left\{-c^2\begin{vmatrix}-c^2 & -a^2\\ -b^2 & 0\end{vmatrix}+b^2\begin{vmatrix}-c^2 & 0\\ -b^2 & -a^2\end{vmatrix}\right\} \qquad \{\text{Expanding along } R_1\}$$

$\Rightarrow \quad 2\{a^2b^2c^2 + a^2b^2c^2\} = 4a^2b^2c^2 =$ **R.H.S.**

S38. Let,

$$\text{L.H.S.} = \begin{vmatrix}1+a & 1 & 1\\ 1 & 1+b & 1\\ 1 & 1 & 1+c\end{vmatrix}$$

$$\Rightarrow \quad abc\begin{vmatrix}\frac{1}{a}+1 & \frac{1}{b} & \frac{1}{c}\\ \frac{1}{a} & 1+\frac{1}{b} & \frac{1}{c}\\ \frac{1}{a} & \frac{1}{b} & 1+\frac{1}{c}\end{vmatrix} \quad \{\text{take } a,\ b \text{ and } c \text{ common from } C_1,\ C_2,\ C_3 \text{ respectively}\}$$

$$\Rightarrow \quad abc\begin{vmatrix}1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & \frac{1}{b} & \frac{1}{c}\\ 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{b} & \frac{1}{c}\\ 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & \frac{1}{b} & 1+\frac{1}{c}\end{vmatrix} \qquad \{\text{Applying } C_1 \to C_1 + C_2 + C_3\}$$

$$\Rightarrow \quad abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix}1 & \frac{1}{b} & \frac{1}{c}\\ 1 & 1+\frac{1}{b} & \frac{1}{c}\\ 1 & \frac{1}{b} & 1+\frac{1}{c}\end{vmatrix} \quad \left\{\text{taking common }\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\text{ from } C_1\right\}$$

$$\Rightarrow \quad abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix}1 & \frac{1}{b} & \frac{1}{c}\\ 0 & 1 & 0\\ 0 & 0 & 1\end{vmatrix} \qquad \left\{\begin{matrix}\text{Applying } R_2 \to R_2 - R_1\\ \text{and } R_3 \to R_3 - R_1\end{matrix}\right\}$$

$$\Rightarrow \quad abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\times 1\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix} \qquad \{\text{Expanding along } C_1\}$$

$$\Rightarrow \quad abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = abc + bc + ca + ab = \textbf{R.H.S.}$$

S39. Let,

$$\text{L.H.S.} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$, we get

$$= \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^3-b^3 & b^3-c^3 & c^3 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ (a-b)(a^2+ab+b^2) & (b-c)(b^2+bc+c^2) & c^3 \end{vmatrix}$$

$[\because\ a^3 - b^3 = (a-b)(a^2+ab+b^2)]$

Taking $(a - b)$ common from C_1 and $(b - c)$ from C_2, we get

$$= (a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a^2+ab+b^2 & b^2+bc+c^2 & c^3 \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$, we get

$$= (a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & c \\ (a^2-c^2)+(ab-bc) & b^2+bc+c^2 & c^3 \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & c \\ (a-c)(a+b+c) & b^2+bc+c^2 & c^3 \end{vmatrix} \quad \begin{bmatrix} \because (a^2-c^2)+ab-bc \\ =(a-c)(a+c)+b(a-c) \\ =(a-c)(a+b+c) \end{bmatrix}$$

Taking $(c - a)(a + b + c)$ common from C_1

$$= (a-b)(b-c)(c-a)(a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & c \\ -1 & b^2+bc+c^2 & c^3 \end{vmatrix}$$ Expanding along C_1, we get

$$= (a-b)(b-c)(c-a)(a+b+c)\left(-1\times\begin{vmatrix} 0 & 1 \\ 1 & c \end{vmatrix}\right)$$

$$= (a-b)(b-c)(c-a)(a+b+c)[-1(0-1)]$$

$$= (a-b)(b-c)(c-a)(a+b+c)$$

= **R.H.S. Hence proved.**

S40. Let,

$$\text{L.H.S.} = \begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, we get

$$= \begin{vmatrix} a^2 & -(b-c)^2 & bc \\ b^2 & -(c-a)^2 & ca \\ c^2 & -(a-b)^2 & ab \end{vmatrix} = -\begin{vmatrix} a^2 & b^2+c^2-2bc & bc \\ b^2 & c^2+a^2-2ac & ca \\ c^2 & a^2+b^2-2ab & ab \end{vmatrix}$$

Applying $C_2 \to C_2 + C_1 + 2C_3$, we get

$$= -\begin{vmatrix} a^2 & a^2+b^2+c^2 & bc \\ b^2 & a^2+b^2+c^2 & ca \\ c^2 & a^2+b^2+c^2 & ab \end{vmatrix} = -(a^2+b^2+c^2)\begin{vmatrix} a^2 & 1 & bc \\ b^2 & 1 & ca \\ c^2 & 1 & ab \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$, we get

$$= -(a^2+b^2+c^2)\begin{vmatrix} a^2-b^2 & 0 & c(b-a) \\ b^2-c^2 & 0 & a(c-b) \\ c^2 & 1 & ab \end{vmatrix}$$

$$= -(a^2+b^2+c^2)(a-b)(b-c)\begin{vmatrix} a+b & 0 & -c \\ b+c & 0 & -a \\ c^2 & 1 & ab \end{vmatrix}$$

Applying $C_1 \to C_1 - C_3$, we get

$$= -(a^2+b^2+c^2)(a-b)(b-c)\begin{vmatrix} a+b+c & 0 & -c \\ a+b+c & 0 & -a \\ c^2-ab & 1 & ab \end{vmatrix}$$

Expanding along C_2, we get

$$= -(a^2+b^2+c^2)(a-b)(b-c)(-1)^{3+2}[(a+b+c)(-a+c)]$$

$$= (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2)$$

$$= \text{R.H.S.}$$

S41. First apply properties of determinants in L.H.S. and reduce it into its lowest term. Then, equate the lowest term to zero and use the given fact that $x \neq y \neq z$ and get the desired result.

Given that, $$\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$$

and $x \neq y \neq z$

We apply various properties in L.H.S. and equate the lowest term to zero. Then, we use the fact $x \neq y \neq z$.

The given determinant equation can be written as

$$\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0$$

Taking x, y, z common from R_1, R_2 and R_3, respectively from second determinant.

Now, we know that when any two columns of a determinant gets interchanged, its value changes by (– 1) sign.

$\therefore$ Applying $C_2 \leftrightarrow C_3$ in first determinant, we get

$$\Rightarrow \quad -\begin{vmatrix} x & 1 & x^2 \\ y & 1 & y^2 \\ z & 1 & z^2 \end{vmatrix} + xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0$$

Using the same property again in the first determinant, *i.e.*, applying $C_1 \leftrightarrow C_2$, we get

$$(-)(-)\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}(1 + xyz) = 0$$

Now, apply $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$, we get

$$\begin{vmatrix} 0 & x-y & x^2-y^2 \\ 0 & y-z & y^2-z^2 \\ 1 & z & z^2 \end{vmatrix}(1+xyz) = 0$$

$$\Rightarrow \quad (1+xyz)\begin{vmatrix} 0 & x-y & (x-y)(x+y) \\ 0 & y-z & (y-z)(y+z) \\ 1 & z & z^2 \end{vmatrix} = 0$$

Taking $(x-y)$ common from R_1 and $(y-z)$ from R_2, we get

$$(1+xyz)(x-y)(y-z)\begin{vmatrix} 0 & 1 & x+y \\ 0 & 1 & y+z \\ 1 & z & z^2 \end{vmatrix} = 0$$

Expanding along C_1, we get

$$(1+xyz)(x-y)(y-z)[(y+z)-(x+y)] = 0$$

$$\Rightarrow \quad (1+xyz)(x-y)(y-z)(z-x) = 0$$

$\Rightarrow$ Either $1 + xyz = 0$

or $x - y = y - z = z - x = 0$

$\Rightarrow$ $x = y = z$

But this is contradictions as given that

$$x \neq y \neq z$$

$\therefore$ $1 + xyz = 0$. **Hence proved.**

S42. Let,

$$\text{L.H.S.} = \begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}$$ Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$= \begin{vmatrix} a+b+c & b & c \\ 0 & b-c & c-a \\ 2(a+b+c) & c+a & a+b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 2 & c+a & a+b \end{vmatrix}$$

[$\because$ Taking $a + b + c$ common from column C_1]

Applying $R_3 \to R_3 - 2R_1$, we get

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 0 & c+a-2b & a+b-2c \end{vmatrix}$$

Expanding along C_1, we get

$$= (a+b+c)\cdot 1\begin{vmatrix} b-c & c-a \\ c+a-2b & a+b-2c \end{vmatrix}$$

Applying $R_2 \to R_2 + 2R_1$, we get

$$= (a+b+c)\begin{vmatrix} b-c & c-a \\ a-c & b-a \end{vmatrix}$$

$$= (a + b + c) [(b - c) (b - a) - (a - c) (c - a)]$$
$$= (a + b + c) [(b^2 - ab - bc + ac) + (a^2 + c^2 - 2ca)]$$
$$= (a + b + c) (a^2 + b^2 + c^2 - ab - bc - ca)$$
$$= a^3 + b^3 + c^3 - 3abc = \textbf{R.H.S.}$$

Hence proved.

S43. Let,

$$\Delta = \begin{vmatrix} a & a^3 & a^4-1 \\ b & b^3 & b^4-1 \\ c & c^3 & c^4-1 \end{vmatrix}. \quad \text{Then,}$$

$$\Rightarrow \qquad \Delta = \begin{vmatrix} a & a^3 & a^4 \\ b & b^3 & b^4 \\ c & c^3 & c^4 \end{vmatrix} + \begin{vmatrix} a & a^3 & -1 \\ b & b^3 & -1 \\ c & c^3 & -1 \end{vmatrix}$$

$$\Rightarrow \qquad \Delta = abc\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} - \begin{vmatrix} a & a^3 & 1 \\ b & b^3 & 1 \\ c & c^3 & 1 \end{vmatrix}$$

$$\Rightarrow \qquad \Delta = abc\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b^2-a^2 & b^3-a^3 \\ 0 & c^2-a^2 & c^3-a^3 \end{vmatrix} - \begin{vmatrix} a & a^3 & 1 \\ b-a & b^3-a^3 & 0 \\ c-a & c^3-a^3 & 0 \end{vmatrix}$$

[Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$]

$$\Rightarrow \qquad \Delta = abc(b-a)(c-a)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b+a & b^2+a^2+ab \\ 0 & c+a & c^2+a^2+ac \end{vmatrix} - (b-a)(c-a)\begin{vmatrix} a & a^3 & 1 \\ 1 & b^2+a^2+ab & 0 \\ 1 & c^2+a^2+ac & 0 \end{vmatrix}$$

$$\Rightarrow \quad \Delta = abc(b-a)(c-a)\begin{vmatrix} b+a & b^2+a^2+ab \\ c+a & c^2+a^2+ac \end{vmatrix} - (b-a)(c-a)\begin{vmatrix} 1 & b^2+a^2+ab \\ 1 & c^2+a^2+ac \end{vmatrix}$$

$$\Rightarrow \quad \Delta = abc(b-a)(c-a)\begin{vmatrix} b+a & b^2+a^2+ab \\ c-b & c^2-b^2+a(c-b) \end{vmatrix} - (b-a)(c-a)\begin{vmatrix} 1 & b^2+a^2+ab \\ 0 & c^2-b^2+a(c-b) \end{vmatrix}$$

[Applying $R_2 \to R_2 - R_1$]

$$\Rightarrow \quad \Delta = abc\,(b-a)(c-a)(c-b)\begin{vmatrix} b+a & b^2+a^2+ab \\ 1 & a+b+c \end{vmatrix}$$
$$-(b-a)(c-a)(c-b)\begin{vmatrix} 1 & b^2+a^2+ab \\ 0 & a+b+c \end{vmatrix}$$

$$\Rightarrow \quad \Delta = abc(b-a)(c-a)(c-b)\{(b+a)(a+b+c)-(b^2+a^2+ab)\}$$
$$-(b-a)(c-a)(c-b)(a+b+c-0)$$

$$\Rightarrow \quad \Delta = abc(a-b)(b-c)(c-a)(bc+ca+ab)-(a-b)(b-c)(c-a)(a+b+c)$$

$$\Rightarrow \quad \Delta = (a-b)(b-c)(c-a)\,\{abc(ab+bc+ca)-(a+b+c)\}$$

Now, $\quad \Delta = 0$

$$\Rightarrow \quad (a-b)(b-c)(c-a)\{abc(ab+bc+ca)-(a+b+c)\} = 0$$

$$\Rightarrow \quad abc(ab+bc+ca)-(a+b+c) = 0 \qquad [\because a \neq b \neq c \therefore a-b \neq 0, b-c \neq 0, c-a \neq 0]$$

$$\Rightarrow \quad abc(ab+bc+ca) = a+b+c.$$

S44. We have,

$$\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} 2(a+b+c) & c+a & a+b \\ 2(a+b+c) & a+b & b+c \\ 2(a+b+c) & b+c & c+a \end{vmatrix} = 0 \qquad \text{[Applying } C_1 \to C_1 + C_2 + C_3\text{]}$$

$$\Rightarrow \quad 2(a+b+c)\begin{vmatrix} 1 & c+a & a+b \\ 1 & a+b & b+c \\ 1 & b+c & c+a \end{vmatrix} = 0$$

$$\Rightarrow \quad 2(a+b+c)\begin{vmatrix} 1 & c+a & a+b \\ 0 & b-c & c-a \\ 0 & b-a & c-b \end{vmatrix} = 0$$

[Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$]

$\Rightarrow \quad 2(a + b + c)\{(b - c)(c - b) - (c - a)(b - a)\} = 0$ [Expanding along C_1]

$\Rightarrow \quad 2(a + b + c)\{-(b^2 - 2bc + c^2) - (bc - ca - ab + a^2)\} = 0$

$\Rightarrow \quad 2(a + b + c)(-a^2 - b^2 - c^2 + bc + ca + ab) = 0$

$\Rightarrow \quad 2(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = 0$

$\Rightarrow \quad (a + b + c)(2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca) = 0$

$\Rightarrow \quad (a + b + c)\{(a - b)^2 + (b - c)^2 + (c - a)^2\} = 0$

$\Rightarrow \quad a + b + c = 0$ or $(a - b)^2 + (b - c)^2 + (c - a)^2 = 0$

$\Rightarrow \quad a + b + c = 0$

or $\quad a = b = c$

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Q1. show that

$$A^{-1} = \frac{1}{19}A. \text{ If } A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}.$$

Q2. Verify A (adj A) = (adj A) $A = |A|\, I_3$. If

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}.$$

Q3. Find adj A and verify that A (adj A) = (adj A) $A = |A|\, I_3$ If

$$A = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Q4. Let A = diag [a b c], where a, b, c are non zero, find A^{-1}

Q5. Find A^{-1}. If

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix},$$

Q6. Find a matrix B such that $AB = I$. If

$$A = \begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix}.$$

Q7. Verify that $(A^{-1})^{-1} = A$. If

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

Q8. For the matrix

$A = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$, Find x and y so that $A^2 + xI = yA$. Hence find A^{-1}.

Q9. Show that $A^2 - 6A + 17I = 0$. Hence find A^{-1}. If

$$A = \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}.$$

Q10. Show that $A^2 - 5A + 7I = 0$. Hence, find A^{-1}. If

$$A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix},$$

Q11. Verify that $A^2 - 4A + I = 0$, Hence, find A^{-1}. If

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix},$$

Q12. Show that the matrix

$A = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix}$ satisfies the equation $A^2 - 4A - 5I_3 = 0$ and hence find A^{-1}.

Q13. Show that $A^{-1} = A^2$. If

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Q14. Find the inverse of matrix

$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ and verify that $A^{-1}A = I_3$.

Q15. If

$A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, find the values of a and b such that $A^2 + aA + bI = 0$. Hence find A^{-1}.

Q16. Show that $A^2 - 12A + I = 0$. Hence Find A^{-1}. If

$$A = \begin{bmatrix} 6 & 5 \\ 7 & 6 \end{bmatrix}.$$

Q17. Verify that $(AB)^{-1} = B^{-1}A^{-1}$. If

$$A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix},$$

Q18. Verify that $(AB)^{-1} = B^{-1}A^{-1}$. If

$$A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}.$$

Q19. Verify that $(AB)^{-1} = B^{-1}A^{-1}$. If

$$A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 6 \\ 3 & 2 \end{bmatrix}.$$

Q20. Compute $(AB)^{-1}$. Given

$$A = \begin{bmatrix} 5 & 0 & 4 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}.$$

Q21. Find $(AB)^{-1}$. If

$$A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}.$$

Q22. Find $(AB)^{-1}$, where

$$A=\begin{bmatrix}1&0&2\\2&3&2\\1&2&1\end{bmatrix}, B^{-1}=\begin{bmatrix}1&2&1\\1&4&3\\1&3&4\end{bmatrix}$$

Q23. Find the matrix A satisfying the matrix equation

$$\begin{bmatrix}2&1\\3&2\end{bmatrix}\cdot A\cdot\begin{bmatrix}-3&2\\5&-3\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

Q24. If

$$A=\begin{bmatrix}1&\tan x\\-\tan x&1\end{bmatrix}, \text{ show that } A^T A^{-1}=\begin{bmatrix}\cos 2x&-\sin 2x\\\sin 2x&\cos 2x\end{bmatrix}$$

Q25. Using the concept of inverse of a matrix, solve the matrix equation.

$$\begin{bmatrix}1&-4\\3&-2\end{bmatrix}X=\begin{bmatrix}-16&-6\\7&2\end{bmatrix}$$

Q26. Find a 2×2 matrix B such that

$$B\begin{bmatrix}1&-2\\1&4\end{bmatrix}=\begin{bmatrix}6&0\\0&6\end{bmatrix}$$

Q27. Verify that $(A^{-1})^{-1} = A$. If

$$A=\begin{bmatrix}1&-2&1\\-2&3&1\\1&1&5\end{bmatrix}.$$

Q28. Verify that $[\text{adj } A]^{-1} = \text{adj }(A^{-1})$. If

$$A=\begin{bmatrix}1&-2&1\\-2&3&1\\1&1&5\end{bmatrix}.$$

Q29. Verify that $A^3-6A^2+9A-4I=0$ and hence find A^{-1}. If

$$A=\begin{bmatrix}2&-1&1\\-1&2&-1\\1&-1&2\end{bmatrix}.$$

Q30. Show that $A^3-6A^2+5A+11I=0$. Hence find A^{-1}. Where the matrix

$$A=\begin{bmatrix}1&1&1\\1&2&-3\\2&-1&3\end{bmatrix}.$$

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S1. We have,

$$|A| = \begin{vmatrix} 2 & 3 \\ 5 & -2 \end{vmatrix} = -4 - 15 = -19 \neq 0.$$

Therefore, A is invertible. Let C_{ij} be the cofactor of a_{ij} in $A = [a_{ij}]$. Then,

$$C_{11} = -2,\ C_{12} = -5,\ C_{21} = -3 \text{ and } C_{22} = 2.$$

$$\therefore \qquad \text{adj } A = \begin{bmatrix} -2 & -5 \\ -3 & 2 \end{bmatrix}^T = \begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix}$$

$$\Rightarrow \qquad A^{-1} = \frac{1}{|A|} adj\ A = \frac{1}{-19}\begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix} = \frac{1}{19}\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix} = \frac{1}{19} A.$$

S2. Let,

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{vmatrix}$$

$$= 1(0-0) - (-1)(9+2) + 2(0-0)$$

$$= 0 + 11 + 0 = 11$$

$$A_{11} = (0-0) = 0,$$

$$A_{12} = -(9+2) = -11$$

$$A_{13} = (0-0) = 0$$

$$A_{21} = -(-3-0) = 3,$$

$$A_{22} = (3-2) = 1,$$

$$A_{23} = -(0+1) = -1$$

$$A_{31} = (2-0) = 2,$$

$$A_{32} = -(-2-6) = 8,$$

$$A_{33} = (0+3) = 3$$

$$\text{adj}\,A = \begin{bmatrix} 0 & -11 & 0 \\ 3 & 1 & -1 \\ 2 & 8 & 3 \end{bmatrix}^T = \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}$$

$$A(\text{adj}\,A) = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0+11+0 & 3-1-2 & 2-8+6 \\ 0+0+0 & 9+0+2 & 6+0-6 \\ 0+0+0 & 3+0-3 & 2+0+9 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|\, I_3$$

$$(\text{adj}\,A)A = \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0+9+2 & 0+0+0 & 0-6+6 \\ -11+3+8 & 11+0+0 & -22-2+24 \\ 0-3+3 & 0+0+0 & 0+2+9 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|\, I_3$$

Thus $\quad A\,(\text{adj}\,A) = (\text{adj}\,A)\,A = |A|\, I_3$

S3. Let C_{ij} be the cofactor of a_{ij} in $A = [a_{ij}]$.

Then, $\quad C_{11} = (-1)^{1+1}\begin{vmatrix} \cos\alpha & 0 \\ 0 & 1 \end{vmatrix} = \cos\alpha, \quad C_{12} = (-1)^{1+2}\begin{vmatrix} \sin\alpha & 0 \\ 0 & 1 \end{vmatrix} = -\sin\alpha,$

$$C_{13} = (-1)^{1+3}\begin{vmatrix}\sin\alpha & \cos\alpha\\ 0 & 0\end{vmatrix} = 0,\ C_{21} = (-1)^{1+2}\begin{vmatrix}-\sin\alpha & 0\\ 0 & 1\end{vmatrix} = \sin\alpha,$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix}\cos\alpha & 0\\ 0 & 1\end{vmatrix} = \cos\alpha,\ C_{23} = (-1)^{2+3}\begin{vmatrix}\cos\alpha & -\sin\alpha\\ 0 & 0\end{vmatrix} = 0,$$

and, $$C_{31} = (-1)^{3+1}\begin{vmatrix}-\sin\alpha & 0\\ \cos\alpha & 0\end{vmatrix} = 0,\ C_{32} = (-1)^{3+2}\begin{vmatrix}\cos\alpha & 0\\ \sin\alpha & 0\end{vmatrix} = 0,$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{vmatrix} = \cos^2\alpha + \sin^2\alpha = 1,$$

$\therefore$ $$\text{adj } A = \begin{bmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{bmatrix}^T = \begin{bmatrix}\cos\alpha & \sin\alpha & 0\\ -\sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{bmatrix}$$

and, $|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

$\Rightarrow$ $|A| = \cos\alpha \cdot \cos\alpha + (-\sin\alpha)(-\sin\alpha) + 0 \times 0 = \cos^2\alpha + \sin^2\alpha = 1.$

Now, $$A.(\text{adj } A) = \begin{bmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{bmatrix} \cdot \begin{bmatrix}\cos\alpha & \sin\alpha & 0\\ -\sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{bmatrix}$$

$\Rightarrow$ $$A.(\text{adj } A) = \begin{bmatrix}\cos^2\alpha + \sin^2\alpha & \sin\alpha\cos\alpha - \cos\alpha\sin\alpha & 0\\ \sin\alpha\cos\alpha - \sin\alpha\cos\alpha & \sin^2\alpha + \cos^2\alpha & 0\\ 0 & 0 & 1\end{bmatrix}$$

$\Rightarrow$ $$A.(\text{adj } A) = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = |A|\, I \qquad [\because\ |A| = 1]$$

and, $$(\text{adj } A)A = \begin{bmatrix}\cos\alpha & \sin\alpha & 0\\ -\sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{bmatrix}$$

$\Rightarrow$ $$(\text{adj } A)A = \begin{bmatrix}\cos^2\alpha + \sin^2\alpha & -\sin\alpha\cos\alpha + \cos\alpha\sin\alpha & 0\\ -\sin\alpha\cos\alpha + \cos\alpha\sin\alpha & \sin^2\alpha + \cos^2\alpha & 0\\ 0 & 0 & 1\end{bmatrix}$$

$\Rightarrow$ $$(\text{adj } A)A = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = |A|\, I_3 \qquad [\because\ |A| = 1]$$

Hence, $A(\text{adj } A) = |A|\, I_3 = (\text{adj } A)\, A$ is verified.

S4. Given $A = \text{diag}\ [a\ b\ c]$

$$\therefore \quad A = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$$

Now $$|A| = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc \neq 0 \qquad [\because\ a \neq 0,\ b \neq 0,\ c \neq 0]$$

Hence A^{-1} exists.

Now $$A_{11} = bc,\ A_{12} = 0,\ A_{13} = 0,$$
$$A_{21} = 0,\ A_{22} = ac,\ A_{23} = 0,$$
$$A_{31} = 0,\ A_{32} = 0,\ A_{33} = ab$$

$$\therefore \quad \text{adj}\, A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} bc & 0 & 0 \\ 0 & ac & 0 \\ 0 & 0 & ab \end{bmatrix}^T$$

$$\therefore \quad \text{adj}\, A = \begin{bmatrix} bc & 0 & 0 \\ 0 & ac & 0 \\ 0 & 0 & ab \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{adj\ A}{|A|}$$

$$= \frac{1}{abc}\begin{bmatrix} bc & 0 & 0 \\ 0 & ac & 0 \\ 0 & 0 & ab \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}$$

$$\therefore \quad A^{-1} = \text{diag}\left[\frac{1}{a}\frac{1}{b}\frac{1}{c}\right] = \text{diag}\ [a^{-1}\ b^{-1}\ c^{-1}].$$

S5. Let,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{vmatrix}$$

$$= 1(-\cos^2\alpha - \sin^2\alpha)$$

$$= -(\cos^2\alpha + \sin^2\alpha) = -1$$

$|A| \neq 0$

$\Rightarrow$ A^{-1} exist

$$A_{11} = (-\cos^2\alpha - \sin^2\alpha)$$

$$= -(\cos^2\alpha + \sin^2\alpha) = -1$$

$$A_{12} = -(0 - 0) = 0,$$

$$A_{13} = (0 - 0) = 0$$

$$A_{21} = -(0 - 0) = 0,$$

$$A_{22} = (-\cos\alpha - 0) = -\cos\alpha,$$

$$A_{23} = -(\sin\alpha - 0) = -\sin\alpha$$

$$A_{31} = (0 - 0) = 0,$$

$$A_{32} = -(\sin\alpha - 0) = -\sin\alpha,$$

$$A_{33} = (\cos\alpha - 0) = \cos\alpha$$

$$\text{adj } A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}^T$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$$

$\therefore$

$$A^{-1} = \frac{1}{|A|} adj\ A$$

$$= \frac{1}{-1}\begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$$

S6. Given, $AB = I$

Pre-multiplying by A^{-1}

$$A^{-1}AB = A^{-1}I \qquad [\because \quad A^{-1}A = I,\ A^{-1}I = A^{-1} \text{ and } IA = A]$$

$$IB = A^{-1}$$

$$B = A^{-1}$$

Given, $A = \begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix}$

Now $|A| = \begin{vmatrix} 3 & -4 \\ -1 & 2 \end{vmatrix} = 6 - 4 = 2 \neq 0$

$\Rightarrow$ A^{-1} exist.

$$A_{11} = 2,\ A_{12} = 1,\ A_{21} = 4,\ A_{22} = 3$$

$\therefore$ $\text{adj } A = \begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix}^T$

$\therefore$ $\text{adj } A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$

$\therefore$ $A^{-1} = \dfrac{1}{|A|} \text{ adj } A = \dfrac{1}{2}\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$

Second method :

Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Given $AB = I$

$\Rightarrow$ $\begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow$ $\begin{bmatrix} 3a - 4c & 3b - 4d \\ -a + 2c & -b + 2d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore$ $3a - 4c = 1$ ... (i)

$3b - 4d = 0$ ... (ii)

$-a + 2c = 0$ ... (iii)

$-b + 2d = 1$ ... (iv)

Solving (i) and (iii), we get

$$a = 1,$$

$$c = \frac{1}{2}$$

Solving (ii) and (iv), we get

$$b = 2,$$

$$d = \frac{3}{2}$$

Hence $$B = \begin{bmatrix} 1 & 2 \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}.$$

S7. Given,

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Now $$|A| = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$$

$$= \cos^2\theta + \sin^2\theta = 1 \neq 0$$

Now $$A_{11} = (-1)^{1+1} \cos\theta = \cos\theta$$
$$A_{12} = (-1)^{1+2} (-\sin\theta) = \sin\theta$$
$$A_{21} = (-1)^{2+1} \sin\theta = -\sin\theta$$
$$A_{22} = (-1)^{2+2} \cos\theta = \cos\theta$$

Now $$\text{adj } A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$\therefore$ $$A^{-1} = \frac{adj\ A}{|A|} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Also $$|A^{-1}| = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$$

$\Rightarrow$ $(A^{-1})^{-1}$ exist

$$A_{11}^{-1} = \cos\theta, \quad A_{12}^{-1} = -\sin\theta,$$
$$A_{21}^{-1} = \sin\theta, \quad A_{22}^{-1} = \cos\theta$$

$\therefore$ $$\text{adj}\ (A^{-1}) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^T$$

$$\text{adj}\ (A^{-1}) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$\therefore$ $$(A^{-1})^{-1} = \frac{adj\ (A^{-1})}{|A^{-1}|} = \frac{adj\ (A^{-1})}{1} = \text{adj}\ (A^{-1})$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = A$$

Hence, $(A^{-1})^{-1} = A$.

S8. We have,

$$A^2 = AA = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} 9+7 & 3+5 \\ 21+35 & 7+25 \end{bmatrix} = \begin{bmatrix} 16 & 8 \\ 56 & 32 \end{bmatrix}$$

Now, $A^2 + xI = yA$

$\Rightarrow$ $$\begin{bmatrix} 16 & 8 \\ 56 & 32 \end{bmatrix} + x\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = y\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$$

$\Rightarrow$ $$\begin{bmatrix} 16+x & 8+0 \\ 56+0 & 32+x \end{bmatrix} = \begin{bmatrix} 3y & y \\ 7y & 5y \end{bmatrix}$$

$\Rightarrow$ $16 + x = 3y,\ y = 8,\ 5y = 32 + x.$

Putting $y = 8$ in $16 + x = 3y$, we get $x = 24 - 16 = 8$.

Clearly $x = 8$ and $y = 8$ also satisfy $7y = 56$ and $5y = 32 + x$.

Hence, $x = 8$ and $y = 8$.

$\because$ $$|A| = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix} = 8 \neq 0$$

So, A is invertible.

Putting $x = 8$, $y = 8$ in $A^2 + xI = yA$, we get

$$A^2 + 8I = 8A$$

$\Rightarrow$ $A^{-1}(A^2 + 8I) = 8A^{-1}A$ [Pre-multiplying throughout by A^{-1}]

$\Rightarrow$ $A^{-1}A^2 + 8A^{-1}I = 8A^{-1}A$

$\Rightarrow$ $A + 8A^{-1} = 8I$

$\Rightarrow$ $8A^{-1} = 8I - A$

$\Rightarrow$ $$A^{-1} = \frac{1}{8}(8I - A) = \frac{1}{8}\left\{\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}\right\}$$

$$\Rightarrow \qquad A^{-1} = \frac{1}{8}\begin{bmatrix} 8-3 & 0-1 \\ 0-7 & 8-5 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 5 & -1 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} 5/8 & -1/8 \\ -7/8 & 3/8 \end{bmatrix}.$$

S9. Here $\quad A = \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}$

$$\therefore \qquad A^2 = A.A = \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4-9 & -6-12 \\ 6+12 & -9+16 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & -18 \\ 18 & 7 \end{bmatrix}$$

$$\therefore \qquad A^2 - 6A + 17I = \begin{bmatrix} -5 & -18 \\ 18 & 7 \end{bmatrix} - 6\begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix} + 17\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & -18 \\ 18 & 7 \end{bmatrix} - \begin{bmatrix} 12 & -18 \\ 18 & 24 \end{bmatrix} + \begin{bmatrix} 17 & 0 \\ 0 & 17 \end{bmatrix}$$

$$= \begin{bmatrix} -5-12+17 & -18+18+0 \\ 18-18+0 & 7-24+17 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

But $\quad |A| = \begin{vmatrix} 2 & -3 \\ 3 & 4 \end{vmatrix} = 8 + 9 = 17 \neq 0$

$\Rightarrow \quad A^{-1}$ exist

Now $\quad A^2 - 6A + 17I = 0$

$\Rightarrow \qquad A^2 - 6A = -17I$

Premultiplying by A^{-1} on both sides, we get

$\Rightarrow \qquad A^{-1}A^2 - 6A^{-1}A = -17A^{-1}I \qquad [\because \; A^{-1}A = I,\; A^{-1}I = A^{-1}$ and $IA = A]$

$\Rightarrow \qquad IA - 6I = -17A^{-1}$

$\Rightarrow \qquad 17A^{-1} = 6I - A$

$$\therefore \qquad A^{-1} = \frac{1}{17}(6I - A)$$

$$\Rightarrow \qquad A^{-1} = \frac{1}{17}\left(6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}\right)$$

$$= \frac{1}{17}\left(\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}\right)$$

$$= \frac{1}{17}\begin{bmatrix} 4 & 3 \\ -3 & 2 \end{bmatrix}$$

Thus $$A^{-1} = \frac{1}{17}\begin{bmatrix} 4 & 3 \\ -3 & 2 \end{bmatrix}.$$

S10. $$A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$\therefore$ $$A^2 = A \cdot A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$\therefore$ $$A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

But $$|A| = \begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix} = 6+1 = 7 \neq 0$$

$\Rightarrow$ A^{-1} exist

$\Rightarrow$

Now $A^2 - 5A + 7I = 0$

pre-multiplying by A^{-1} both sides, we get

$\Rightarrow$ $A^{-1}A^2 - 5A^{-1}A = -7A^{-1}I$ $\quad [\because A^{-1}A = I,\ A^{-1}I = A^{-1}$ and $IA = A]$

$\Rightarrow$ $IA - 5I = -7A^{-1}$

$\Rightarrow$ $7A^{-1} = (5I - A)$

$\Rightarrow$ $$A^{-1} = \frac{1}{7}(5I - A)$$

$$= \frac{1}{7}\left(5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\right)$$

$$= \frac{1}{7}\left(\begin{bmatrix}5 & 0\\0 & 5\end{bmatrix} - \begin{bmatrix}3 & 1\\-1 & 2\end{bmatrix}\right)$$

$$= \frac{1}{7}\begin{bmatrix}2 & -1\\1 & 3\end{bmatrix}$$

Thus $$A^{-1} = \frac{1}{7}\begin{bmatrix}2 & -1\\1 & 3\end{bmatrix}.$$

S11. We have,

$$A = \begin{bmatrix}2 & 3\\1 & 2\end{bmatrix}$$

$\therefore$ $$A^2 = AA = \begin{bmatrix}2 & 3\\1 & 2\end{bmatrix}\begin{bmatrix}2 & 3\\1 & 2\end{bmatrix}$$

$\Rightarrow$ $$A^2 = \begin{bmatrix}7 & 12\\4 & 7\end{bmatrix}$$

$\therefore$ $$A^2 - 4A + I = \begin{bmatrix}7 & 12\\4 & 7\end{bmatrix} - 4\begin{bmatrix}2 & 3\\1 & 2\end{bmatrix} + \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = 0$$

but $|A| = \begin{bmatrix}2 & 3\\1 & 2\end{bmatrix} = 1 \neq 0 \quad \Rightarrow \quad A^{-1}$ exists.

Now $A^2 - 4A + I = 0$

$\Rightarrow \quad A^{-1}(A^2 - 4A + I) = A^{-1}0$ [Multiplying both sides by A^{-1}]

$\Rightarrow \quad A^{-1}A^2 - 4A^{-1}A + A^{-1}I = 0$

$\Rightarrow \quad A - 4I + A^{-1} = 0$

$\Rightarrow$ $$A^{-1} = 4I - A = \begin{bmatrix}4 & 0\\0 & 4\end{bmatrix} - \begin{bmatrix}2 & 3\\1 & 2\end{bmatrix} = \begin{bmatrix}2 & -3\\-1 & 2\end{bmatrix}.$$

S12.

$$A^2 = \begin{bmatrix}1 & 2 & 2\\2 & 1 & 2\\2 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 2\\2 & 1 & 2\\2 & 2 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1.1+2.2+2.2 & 1.2+2.1+2.2 & 1.2+2.2+2.1\\2.1+1.2+2.2 & 2.2+1.1+2.2 & 2.2+1.2+2.1\\2.1+2.2+1.2 & 2.2+2.1+1.2 & 2.2+2.2+1.1\end{bmatrix}$$

$$= \begin{bmatrix}9 & 8 & 8\\8 & 9 & 8\\8 & 8 & 9\end{bmatrix}$$

$$A^2 - 4A - 5I_3 = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now $$A^2 - 4A - 5I_3 = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9-4-5 & 8-8-0 & 8-8-0 \\ 8-8-0 & 9-4-5 & 8-8-0 \\ 8-8-0 & 8-8-0 & 9-4-5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

but $$|A| = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix}$$

$$= 1(1-4) - 2(2-4) + 2(4-2)$$
$$= -3 + 4 + 4 = 5 \neq 0$$

$\Rightarrow$ A^{-1} exists

Now $A^2 - 4A - 5I_3 = 0$

$\Rightarrow$ $A^2 - 4A = 5I_3$

$\Rightarrow$ $A^{-1} . A^2 - 4A^{-1}. A = 5A^{-1}. I_3$ [pre-multiplying throughout by A^{-1}]

$\Rightarrow$ $(A^{-1}.A)A - 4I_3 = 5A^{-1}$ [$\because$ $A^{-1}A = I$, $A^{-1}I = A^{-1}$ and $IA = A$]

$\Rightarrow$ $A - 4I_3 = 5A^{-1}$

$\Rightarrow$ $$A^{-1} = \frac{1}{5}(A - 4I_3)$$

$$= \frac{1}{5} \cdot \left\{ \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \right\}$$

$$= \frac{1}{5}\begin{bmatrix} 1-4 & 2-0 & 2-0 \\ 2-0 & 1-4 & 2-0 \\ 2-0 & 2-0 & 1-4 \end{bmatrix}$$

$$= \frac{1}{5}\cdot\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}.$$

S13. We know that a matrix B is the inverse of a matrix A if $AB = I = BA$. Here we have to show that A^2 is the inverse of A. Therefore, it is sufficient to prove that $A^2 \cdot A = I$ or, $A^3 = I$.

Now, $$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1-2+1 & -1+1+0 & 1+0+0 \\ 2-2+0 & -2+1+0 & 2+0+0 \\ 1+0+0 & -1+0+0 & 1+0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$$

$\Rightarrow$ $$A^3 = A^2A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0+0+1 & 0+0+0 & 0+0+0 \\ 0-2+2 & 0+1+0 & 0+0+0 \\ 1-2+1 & -1+1+0 & 1+0+0 \end{bmatrix}$$

$\Rightarrow$ $$A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence, $A^2 = A^{-1}$.

S14. We have,

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$\Rightarrow$ $$|A| = (-1)^{1+1}\begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} + 3(-1)^{1+2}\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} + 3(-1)^{1+3}\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix}$$

$\Rightarrow$ $$|A| = \begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} - 3\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} + 3\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix}$$

$\Rightarrow$ $$|A| = (16-9) - 3(4-3) + 3(3-4) = 7 - 3 - 3 = 1 \neq 0.$$

So, A is invertible.

Let C_{ij} be the cofactor of a_{ij} in $A = [a_{ij}]$. Then,

$$C_{11} = (-1)^{1+1}\begin{vmatrix}4 & 3\\3 & 4\end{vmatrix} = 7, C_{12} = (-1)^{1+2}\begin{vmatrix}1 & 3\\1 & 4\end{vmatrix} = -1,$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix}1 & 4\\1 & 3\end{vmatrix} = -1, C_{21} = (-1)^{2+1}\begin{vmatrix}3 & 3\\3 & 4\end{vmatrix} = -3,$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix}1 & 3\\1 & 4\end{vmatrix} = 1, C_{23} = (-1)^{2+3}\begin{vmatrix}1 & 3\\1 & 3\end{vmatrix} = 0,$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix}3 & 3\\4 & 3\end{vmatrix} = -3, C_{32} = (-1)^{3+2}\begin{vmatrix}1 & 3\\1 & 3\end{vmatrix} = 0,$$

and $$C_{33} = (-1)^{3+3}\begin{vmatrix}1 & 3\\1 & 4\end{vmatrix} = 1$$

$\therefore$ $$\text{adj } A = \begin{bmatrix}7 & -1 & -1\\-3 & 1 & 0\\-3 & 0 & 1\end{bmatrix}^T = \begin{bmatrix}7 & -3 & -3\\-1 & 1 & 0\\-1 & 0 & 1\end{bmatrix}$$

Hence, $$A^{-1} = \frac{1}{|A|}\text{adj}\, A$$

$\Rightarrow$ $$A^{-1} = \frac{1}{1}\begin{bmatrix}7 & -3 & -3\\-1 & 1 & 0\\-1 & 0 & 1\end{bmatrix} = \begin{bmatrix}7 & -3 & -3\\-1 & 1 & 0\\-1 & 0 & 1\end{bmatrix}$$

Now, $$A^{-1}A = \begin{bmatrix}7 & -3 & -3\\-1 & 1 & 0\\-1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 3 & 3\\1 & 4 & 3\\1 & 3 & 4\end{bmatrix}$$

$\Rightarrow$ $$A^{-1}A = \begin{bmatrix}7-3-3 & 21-12-9 & 21-9-12\\-1+1+0 & -3+4+0 & -3+3+0\\-1+0+1 & -3+0+3 & -3+0+4\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

S15. Here,

$$A = \begin{bmatrix}3 & 2\\1 & 1\end{bmatrix}$$

$\Rightarrow$ $$|A| = \begin{vmatrix}3 & 2\\1 & 1\end{vmatrix} = 3 \times 1 - 1 \times 2 = 1 \neq 0$$

$\Rightarrow$ A^{-1} exists

Also $$A^2 = A.A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} . \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9+2 & 6+2 \\ 3+1 & 2+1 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix}$$

Now, $$A^2 + aA + bI = 0 \qquad \ldots (i)$$

$$\Rightarrow \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix} + a\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 11+3a+b & 8+2a \\ 4+a & 3+a+b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore \quad 11 + 3a + b = 0; 8 + 2a = 0$$

$$\Rightarrow \quad 4 + a = 0; 3 + a + b = 0$$

$$\Rightarrow \quad a = -4; \quad b = 1$$

From (i), $$A^2 - 4A + I = 0$$

$$\Rightarrow \quad I = 4A - A^2$$

Post multiplying by A^{-1}

$$\Rightarrow \quad IA^{-1} = (4A - A^2)A^{-1}$$

$$IA^{-1} = 4AA^{-1} - AAA^{-1} \quad [\because \quad A^{-1}A = I, A^{-1}I = A^{-1} \text{ and } IA = A]$$

$$\Rightarrow \quad A^{-1} = 4I - A$$

$$= 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3 & 0-2 \\ 0-1 & 4-1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}.$$

S16. Here, $$A = \begin{bmatrix} 6 & 5 \\ 7 & 6 \end{bmatrix}$$

$$\therefore \quad A^2 = \begin{bmatrix} 6 & 5 \\ 7 & 6 \end{bmatrix}\begin{bmatrix} 6 & 5 \\ 7 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 71 & 60 \\ 84 & 71 \end{bmatrix}$$

Now $\quad A^2 - 12A + I = \begin{bmatrix} 71 & 60 \\ 84 & 71 \end{bmatrix} - 12\begin{bmatrix} 6 & 5 \\ 7 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 71 & 60 \\ 84 & 71 \end{bmatrix} - \begin{bmatrix} 72 & 60 \\ 84 & 72 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 71 - 72 + 1 & 60 - 60 + 0 \\ 84 - 84 + 0 & 71 - 72 + 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Now, $\quad |A| = \begin{vmatrix} 6 & 5 \\ 7 & 6 \end{vmatrix} = 6 \times 6 - 7 \times 5 = 1 \neq 0$

$\Rightarrow \quad A^{-1}$ exist

Now $\quad A^2 - 12A + I = 0$

$\Rightarrow \quad I = 12A - A^2$

[Post multiplying both sides by A^{-1}]

$\Rightarrow \quad IA^{-1} = (12A - A^2)A^{-1}$

$$= 12AA^{-1} - A(AA^{-1}) \qquad [\because \ A^{-1}A = I,\ A^{-1}I = A^{-1} \text{ and } IA = A]$$

$$= 12I - AI$$

$$= 12\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 6 & 5 \\ 7 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 0 \\ 0 & 12 \end{bmatrix} - \begin{bmatrix} 6 & 5 \\ 7 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 12 - 6 & 0 - 5 \\ 0 - 7 & 12 - 6 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 6 & -5 \\ -7 & 6 \end{bmatrix}$$

S17. We have,

$$|A| = \begin{vmatrix} 3 & 2 \\ 7 & 5 \end{vmatrix} = 15 - 14 = 1 \neq 0.$$

So, A is invertible.

Let A_{ij} be the cofactors of elements a_{ij} in $A = [a_{ij}]$. Then,

$$A_{11} = (-1)^{1+1}\,5 = 5,\ A_{12} = (-1)^{1+2}\,7 = -7,$$

$$A_{21} = (-1)^{2+1}\,2 = -2 \quad \text{and} \quad A_{22} = (-1)^{2+2}\,3 = 3.$$

$$\therefore \qquad \text{adj}\,A = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}^T = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$

Hence, $$A^{-1} = \frac{1}{|A|} adj\ A = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$

We have, $$|B| = \begin{vmatrix} 6 & 7 \\ 8 & 9 \end{vmatrix} = 54 - 56 = -2 \neq 0.$$

So, B is invertible.

Let B_{ij} be the cofactors of b_{ij} in $B = [b_{ij}]$. Then,

$$B_{11} = (-1)^{1+1}\,9 = 9,\ B_{12} = (-1)^{1+2}\,8 = -8,$$
$$B_{21} = (-1)^{2+1}\,7 = -7 \quad \text{and} \quad B_{22} = (-1)^{2+2}\,6 = 6.$$

$$\therefore \qquad adj\ B = \begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}^T = \begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix}$$

Hence, $$B^{-1} = \frac{1}{|B|} adj\ B = -\frac{1}{2}\begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix}$$

We know that adj AB = adj B. adj A.

$$\therefore \qquad \text{adj}\,AB = \begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} 94 & -39 \\ -82 & 34 \end{bmatrix}$$

We also know that $|AB| = |A|\,|B|$.

$$\therefore \qquad |AB| = 1 \times (-2) = -2 \neq 0.$$

So, AB is invertible

Hence, $$(AB)^{-1} = \frac{1}{|AB|} adj(AB) = -\frac{1}{2}\begin{bmatrix} 94 & -39 \\ -82 & 34 \end{bmatrix} \qquad \text{... (i)}$$

Now, $$B^{-1}A^{-1} = -\frac{1}{2}\begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 94 & -39 \\ -82 & 34 \end{bmatrix} \qquad \text{... (ii)}$$

From (i) and (ii), we get

$$(AB)^{-1} = B^{-1}A^{-1}$$

S18. Here,

$$A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$$

$\therefore \quad |A| = \begin{vmatrix} 3 & 7 \\ 2 & 5 \end{vmatrix} = 15 - 14 = 1$

$A_{11} = 5,\ A_{12} = -2,\ A_{21} = -7,\ A_{22} = 3$

$\text{adj } A = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}^T = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} adj\ A = \frac{1}{1}\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$

$B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$

$\therefore \quad |B| = \begin{vmatrix} 6 & 8 \\ 7 & 9 \end{vmatrix} = 54 - 56 = -2 \neq 0$

$\Rightarrow \quad B^{-1}$ exist $\quad B_{11} = 9,\ B_{12} = -7,\ B_{21} = -8,\ B_{22} = 6$

$\text{adj } B = \begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix}^T = \begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}$

$\therefore \quad B^{-1} = \frac{1}{|B|} adjB = \frac{1}{-2}\begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -9 & 8 \\ 7 & -6 \end{bmatrix}$

Now $\quad AB = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$

$= \begin{bmatrix} 18+49 & 24+63 \\ 12+35 & 16+45 \end{bmatrix}$

$= \begin{bmatrix} 67 & 87 \\ 47 & 61 \end{bmatrix}$

$\therefore \quad |AB| = \begin{vmatrix} 67 & 87 \\ 47 & 61 \end{vmatrix} = 4087 - 4089 = -2 \neq 0$

$\Rightarrow \quad (AB)^{-1}$ exist

$AB_{11} = 61,\ AB_{12} = -47,\ AB_{21} = -87,\ AB_{22} = 67$

$\therefore \quad \text{adj } |AB| = \begin{bmatrix} 61 & -47 \\ -87 & 67 \end{bmatrix}^T$

$= \begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix}$

$\therefore \quad (AB)^{-1} = \frac{1}{|AB|} adj\ |AB|$

$$= \frac{1}{-2}\begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -61 & 87 \\ 47 & -67 \end{bmatrix}$$

Now $$B^{-1}A^{-1} = \frac{1}{2}\begin{bmatrix} -9 & 8 \\ 7 & -6 \end{bmatrix}\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -45 - 16 & 63 + 24 \\ 35 + 12 & -49 - 18 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -61 & 87 \\ 47 & -67 \end{bmatrix}$$

Thus $(AB)^{-1} = B^{-1}A^{-1}$.

S19. Here,

$$A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & 2 \\ 7 & 5 \end{vmatrix} = 15 - 14 = 1$$

$$A_{11} = 5,\ A_{12} = -7,\ A_{21} = -2,\ A_{22} = 3 \qquad \because\ |A| \neq 0$$

$\Rightarrow$ A^{-1} exist

$$\text{adj } A = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}^T = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$

$\therefore$ $$A^{-1} = \frac{1}{|A|} adj\ A$$

$$= \frac{1}{1}\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$

Here $$B = \begin{bmatrix} 4 & 6 \\ 3 & 2 \end{bmatrix}$$

$$|B| = \begin{vmatrix} 4 & 6 \\ 3 & 2 \end{vmatrix} = 8 - 18 = -10 \neq 0$$

$\Rightarrow$ B^{-1} exist

$\therefore$ $$B_{11} = 2,\ B_{12} = -3,\ B_{21} = -6,\ B_{22} = 4$$

$$\text{adj } B = \begin{bmatrix} 2 & -3 \\ -6 & 4 \end{bmatrix}^T = \begin{bmatrix} 2 & -6 \\ -3 & 4 \end{bmatrix}$$

$$\therefore \qquad B^{-1} = \frac{1}{|B|} adj\ B$$

$$= \frac{1}{-10}\begin{bmatrix} 2 & -6 \\ -3 & 4 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} -2 & 6 \\ 3 & -4 \end{bmatrix}$$

$$AB = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} 4 & 6 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 12+6 & 18+4 \\ 28+15 & 42+10 \end{bmatrix} = \begin{bmatrix} 18 & 22 \\ 43 & 52 \end{bmatrix}$$

$$|AB| = \begin{vmatrix} 18 & 22 \\ 43 & 52 \end{vmatrix} = 936 - 946 = -10 \neq 0$$

$\Rightarrow \quad (AB)^{-1}$ exist

$$AB_{11} = 52,\ AB_{12} = -43,\ AB_{21} = -22,\ AB_{22} = 18$$

$$\text{adj } AB = \begin{bmatrix} 52 & -43 \\ -22 & 18 \end{bmatrix}^T = \begin{bmatrix} 52 & -22 \\ -43 & 18 \end{bmatrix}$$

$$\therefore \qquad (AB)^{-1} = \frac{1}{|AB|} adj\ AB$$

$$= \frac{1}{-10}\begin{bmatrix} 52 & -22 \\ -43 & 18 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} -52 & 22 \\ 43 & -18 \end{bmatrix}$$

Also $\quad B^{-1}A^{-1} = \frac{1}{10}\begin{bmatrix} -2 & 6 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

$$= \frac{1}{10}\begin{bmatrix} -10-42 & 4+18 \\ 15+28 & -6-12 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} -52 & 22 \\ 43 & -18 \end{bmatrix}$$

Thus $\quad (AB)^{-1} = B^{-1}A^{-1}$.

S20. We have to find (AB^{-1}) and we are given the values of A and B^{-1}. But, $(AB)^{-1} = B^{-1}A^{-1}$. So, we need to find A^{-1}.

$$A = \begin{bmatrix} 5 & 0 & 4 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$\Rightarrow$ $$|A| = 5(3-4) - 0(2-2) + 4(4-3) = -5 + 4 = -1 \neq 0$$

So, A^{-1} exists.

Let C_{ij} be cofactor of a_{ij} in $A = [a_{ij}]$. Then,

$$C_{11} = 3 - 4 = -1,\ C_{12} = -(2-2) = 0,\ C_{13} = 4 - 3 = 1$$
$$C_{21} = -(0-8) = 8,\ C_{22} = 5 - 4 = 1,\ C_{23} = -(10-0) = -10$$
$$C_{31} = (0-12) = -12,\ C_{32} = -(10-8) = -2,\ C_{33} = 15$$

$\therefore$ $$\text{adj } A = \begin{bmatrix} -1 & 0 & 1 \\ 8 & 1 & -10 \\ -12 & -2 & 15 \end{bmatrix}^T$$

Hence, $$A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{-1}\begin{bmatrix} -1 & 8 & -12 \\ 0 & 1 & -2 \\ 1 & -10 & 15 \end{bmatrix} = \begin{bmatrix} 1 & -8 & 12 \\ 0 & -1 & 2 \\ -1 & 10 & -15 \end{bmatrix}$$

Hence, $$(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & -8 & 12 \\ 0 & -1 & 2 \\ -1 & 10 & -15 \end{bmatrix} = \begin{bmatrix} -2 & 19 & -27 \\ -2 & 18 & -25 \\ -3 & 29 & -42 \end{bmatrix}.$$

S21. Here,

$$A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

and $$B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$\therefore$ $$|B| = \begin{vmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{vmatrix}$$

$$= 1(3-0) - 2(-1-0) - 2(2-0)$$

$$= 3 + 2 - 4 = 1 \neq 0$$

$\Rightarrow$ B^{-1} exist

$$B_{11} = (3-0) = 3,$$
$$B_{12} = -(-1-0) = 1,$$

$$B_{13} = -(2-0) = -2$$
$$B_{21} = -(2-4) = 2,$$
$$B_{22} = (1-0) = 1,$$
$$B_{23} = -(-2-0) = 2$$
$$B_{31} = (0+6) = 6,$$
$$B_{32} = -(0-2) = 2,$$
$$B_{33} = (3+2) = 5$$

$\therefore$ $$\text{adj } B = \begin{bmatrix} 3 & 1 & -2 \\ 2 & 1 & 2 \\ 6 & 2 & 5 \end{bmatrix}^T = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ -2 & 2 & 5 \end{bmatrix}$$

$\therefore$ $$B^{-1} = \frac{1}{|B|} adjB$$

$$= \frac{1}{1}\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ -2 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ -2 & 2 & 5 \end{bmatrix}$$

Now $$(AB)^{-1} = B^{-1}A^{-1}$$

$$= \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9-30+30 & -3+12-12 & 3-10+12 \\ 3-15+10 & -1+6-4 & 1-5+4 \\ 6-30+25 & -2+12-10 & 2-10+10 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -3 & 5 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

S22. We know that $(AB^{-1}) = B^{-1} A^{-1}$ ...(i)

Now $$|A| = \begin{vmatrix} 1 & 0 & 2 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{vmatrix} = 1(3-4) - 0(2-2) + 2(4-3) = 1 \neq 0$$

Hence A^{-1} exist.

Let A_{ij} be the cofactor of the elements in the i^{th} row and j^{th} column of A, then

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} = -1,$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} = 0$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 1$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} = 4$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = 2$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} 0 & 2 \\ 3 & 2 \end{vmatrix} = -6$$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} = 2$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} = 3$$

Now, $$\text{adj } A = \begin{bmatrix} -1 & 0 & 1 \\ 4 & -1 & 2 \\ -6 & 2 & 3 \end{bmatrix}^T$$

$$= \begin{bmatrix} -1 & 4 & -6 \\ 0 & -1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$\therefore$ $$A^{-1} = \frac{1}{|A|} adj\ A = \frac{1}{(1)} adj\ A$$

$$= \text{adj } A = \begin{bmatrix} 1 & 4 & -6 \\ 0 & -1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

From (i), $(AB)^{-1} = B^{-1} A^{-1}$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 4 & -6 \\ 0 & -1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+1 & 4-2+2 & -6+4-3 \\ 1+0+3 & 4-4+6 & -6+8+9 \\ 1+0+4 & 4-3+8 & -6+6+12 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 & 1 \\ 4 & 6 & 11 \\ 5 & 9 & 12 \end{bmatrix}.$$

S23. Let,

$$B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

and $$C = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$$

Now $$|B| = \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = (4 - 3) = 1 \neq 0$$

$$|C| = \begin{vmatrix} -3 & 2 \\ 5 & -3 \end{vmatrix} = (9 - 10) = -1 \neq 0$$

Hence B^{-1} and C^{-1} exist.

$\therefore$ The given matrix equation becomes $BAC = I_2$

Now. $BAC = I_2$

Post-multiplying by C^{-1} and Pre - multiplying by B^{-1} on both sides, we get

$\Rightarrow \quad B^{-1}(BAC)\, C^{-1} = B^{-1} I_2\, C^{-1}$ $\quad [\because \quad A^{-1}A = I,\ A^{-1}I = A^{-1} \text{ and } IA = A]$

$\Rightarrow \quad (B^{-1}\, B)\, A(CC^{-1}) = B^{-1}\, (I_2 C^{-1})$

$\Rightarrow \quad I_2\, AI_2 = B^{-1}\, C^{-1}$

$\Rightarrow \quad A = B^{-1}\, C^{-1}$ ... (i)

Now, the co-factors of the elements of B are

$$B_{11} = 2;\ B_{12} = -3;\ B_{21} = -1;\ B_{22} = 2.$$

$$\therefore \quad \text{adj } B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}^T = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

$\therefore \qquad B^{-1} = \frac{1}{|B|} \cdot adj\ B = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \qquad [\because\ |B| = 1]$

Again, the co-factors of the elements of C are:

$$C_{11} = -3;\ C_{12} = -5;\ C_{21} = -2;\ C_{22} = -3$$

$\therefore \qquad \text{adj } C = \begin{bmatrix} -3 & -5 \\ -2 & -3 \end{bmatrix}^T$

$$= \begin{bmatrix} -3 & -2 \\ -5 & -3 \end{bmatrix}$$

$\therefore \qquad C^{-1} = \frac{1}{|C|} \cdot adj\ C$

$$= \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} \qquad [\because\ |C| = -1]$$

Now from (i), $\qquad A = B^{-1}\ C^{-1}$

$$= \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

S24. We have, $\qquad |A| = \begin{vmatrix} 1 & \tan x \\ -\tan x & 1 \end{vmatrix} = 1 + \tan^2 x \neq 0$

So, A is invertible. Let C_{ij} be the cofactor of a_{ij} in $A = [a_{ij}]$. Then,

$$C_{11} = (-1)^{1+1}1 = 1,\ C_{12} = (-1)^{1+2}(-\tan x) = \tan x$$

$$C_{21} = (-1)^{2+1}\tan x = -\tan x \text{ and } C_{22} = (-1)^{2+2}1 = 1$$

$\therefore \qquad \text{adj } A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$

Now, $\qquad A^{-1} = \frac{1}{|A|}\text{adj } A$

$\Rightarrow \qquad A^{-1} = \frac{1}{1+\tan^2 x}\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$

$\Rightarrow \qquad A^{-1} = \begin{bmatrix} \frac{1}{1+\tan^2 x} & \frac{-\tan x}{1+\tan^2 x} \\ \frac{\tan x}{1+\tan^2 x} & \frac{1}{1+\tan^2 x} \end{bmatrix}$

$\therefore \quad A^T A^{-1} = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{1+\tan^2 x} & \frac{-\tan x}{1+\tan^2 x} \\ \frac{\tan x}{1+\tan^2 x} & \frac{1}{1+\tan^2 x} \end{bmatrix}$

$\Rightarrow \quad A^T A^{-1} = \begin{bmatrix} \frac{1}{1+\tan^2 x} - \frac{\tan^2 x}{1+\tan^2 x} & -\frac{\tan x}{1+\tan^2 x} - \frac{\tan x}{1+\tan^2 x} \\ \frac{\tan x}{1+\tan^2 x} + \frac{\tan x}{1+\tan^2 x} & -\frac{\tan^2 x}{1+\tan^2 x} + \frac{1}{1+\tan^2 x} \end{bmatrix}$

$\Rightarrow \quad A^T A^{-1} = \begin{bmatrix} \frac{1-\tan^2 x}{1+\tan^2 x} & \frac{-2\tan x}{1+\tan^2 x} \\ \frac{2\tan x}{1+\tan^2 x} & \frac{1-\tan^2 x}{1+\tan^2 x} \end{bmatrix}$

$\Rightarrow \quad A^T A^{-1} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$.

S25. Let $A = \begin{bmatrix} 1 & -4 \\ 3 & -2 \end{bmatrix}$

and $B = \begin{bmatrix} -16 & -6 \\ 7 & 2 \end{bmatrix}$

Now $|A| = \begin{vmatrix} 1 & -4 \\ 3 & -2 \end{vmatrix} = -2 + 12 = 10 \neq 0$

Hence A^{-1} exist.

$\therefore$ Given equation become $AX = B$

Now $AX = B$

$\Rightarrow \quad X = A^{-1} B \quad \ldots (1)$

Now $A_{11} = (-1)^{1+1}(-2) = -2$

$A_{12} = (-1)^{1+2}(3) = -3$

$A_{21} = (-1)^{2+1}(-4) = 4$

$A_{22} = (-1)^{2+2}(1) = 1$

$\therefore \quad \text{Adj } A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^T$

$= \begin{bmatrix} -2 & -3 \\ 4 & 1 \end{bmatrix}^T$

$\therefore$ $\quad$ $\text{adj } A = \begin{bmatrix} -2 & 4 \\ -3 & 1 \end{bmatrix}$

$\therefore$ $\quad$ $A^{-1} = \dfrac{adj\ A}{|A|}$

$$= \frac{1}{10}\begin{bmatrix} -2 & 4 \\ -3 & 1 \end{bmatrix}$$

$\therefore$ From Eq. (1), $\quad X = A^{-1}B = \dfrac{1}{10}\begin{bmatrix} -2 & 4 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} -16 & -6 \\ 7 & 2 \end{bmatrix}$

$$= \frac{1}{10}\begin{bmatrix} 60 & 20 \\ 55 & 20 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ \frac{11}{2} & 2 \end{bmatrix}.$$

S26. Let,

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

Then, $\quad |A| = \begin{vmatrix} 1 & -2 \\ 1 & 4 \end{vmatrix} = 6 \neq 0$

So, A is invertible.

The given matrix equation is $B\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$

$\Rightarrow$ $\quad BA = C$

$\Rightarrow$ $\quad (BA)A^{-1} = CA^{-1}$ $\quad$ [Post-multiplying throughout by A^{-1}]

$\Rightarrow$ $\quad BAA^{-1} = CA^{-1}$

$\Rightarrow$ $\quad BI = CA^{-1} \Rightarrow B = CA^{-1}$.

Let C_{ij} be the cofactor of a_{ij} in $A = [a_{ij}]$. Then,

$$C_{11} = (-1)^{1+1}4 = 4,\ C_{12} = (-1)^{1+2}1 = -1,\ C_{21} = (-1)^{2+1}(-2) = 2$$

and, $\quad C_{22} = (-1)^{2+2}1 = 1.$

$\therefore$ $\quad \text{adj } A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}^T = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$

$\Rightarrow$ $\quad A^{-1} = \dfrac{1}{|A|}\text{adj } A = \dfrac{1}{6}\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$

Now, $\quad B = CA^{-1}$

$\Rightarrow$ $$B = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} . \frac{1}{6}\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$$

$\Rightarrow$ $$B = \frac{1}{6}\begin{bmatrix} 24+0 & 12+0 \\ 0-6 & 0+6 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}.$$

S27.

Here $$A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

$\therefore$ $$|A| = \begin{vmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{vmatrix}$$

$$= 1(15-1) - (-2)(-10-1) + 1(-2-3)$$

$$= 14 - 22 - 5 = -13$$

$$A_{11} = (15-1) = 14,$$

$$A_{12} = -(-10-1) = 11$$

$$A_{13} = (-2-3) = -5$$

$$A_{21} = -(-10-1) = 11,$$

$$A_{22} = (5-1) = 4,$$

$$A_{23} = -(1+2) = -3$$

$$A_{31} = (-2-3) = -5,$$

$$A_{32} = -(1+2) = -3,$$

$$A_{33} = (3-4) = -1$$

$\therefore$ $$\text{adj } A = \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}^T$$

$$= \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$$

$\therefore$ $$A^{-1} = \frac{1}{|A|}\text{adj } A$$

Let $$A^{-1} = C = \frac{1}{-13}\begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$$

$$= \frac{1}{13}\begin{bmatrix} -14 & -11 & 5 \\ -11 & -4 & 3 \\ 5 & 3 & 1 \end{bmatrix}$$

$\therefore$ $$|C| = \begin{vmatrix} -\frac{14}{13} & -\frac{11}{13} & \frac{5}{13} \\ -\frac{11}{13} & -\frac{4}{13} & \frac{3}{13} \\ \frac{5}{13} & \frac{3}{13} & \frac{1}{13} \end{vmatrix}$$

$$= -\frac{14}{13}\left[\left(-\frac{4}{169}-\frac{9}{169}\right)\right]+\frac{11}{13}\left(-\frac{11}{169}-\frac{15}{169}\right)+\frac{5}{13}\left(-\frac{33}{169}+\frac{20}{169}\right)$$

$$= -\frac{14}{13}\times\left(-\frac{13}{169}\right)+\frac{11}{13}\times\left(-\frac{26}{169}\right)+\frac{5}{13}\times\left(-\frac{13}{169}\right)$$

$$= \frac{14}{169}-\frac{22}{169}-\frac{5}{169}$$

$$= -\frac{13}{169} = -\frac{1}{13}$$

$$C_{11} = -\frac{4}{169}-\frac{9}{169} = -\frac{13}{169}$$

$$C_{12} = -\left(-\frac{11}{169}-\frac{15}{169}\right) = \frac{26}{169}$$

$$C_{13} = -\frac{33}{169}+\frac{20}{169} = -\frac{13}{169}$$

$$C_{21} = -\left(-\frac{11}{169}-\frac{15}{169}\right) = \frac{26}{169}$$

$$C_{22} = \left(-\frac{14}{169}-\frac{25}{169}\right) = -\frac{39}{169}$$

$$C_{23} = -\left(-\frac{42}{169}+\frac{55}{169}\right) = -\frac{13}{169}$$

$$C_{31} = \left(-\frac{33}{169}+\frac{20}{169}\right) = -\frac{13}{169}$$

$$C_{32} = -\left(-\frac{42}{169}+\frac{55}{169}\right) = -\frac{13}{169}$$

$$C_{33} = \left(\frac{56}{169}-\frac{121}{169}\right) = -\frac{65}{169}$$

$$\therefore \qquad \text{adj } C = \begin{bmatrix} -\frac{1}{13} & \frac{2}{13} & -\frac{1}{13} \\ \frac{2}{13} & -\frac{3}{13} & -\frac{1}{13} \\ -\frac{1}{13} & -\frac{1}{13} & -\frac{5}{13} \end{bmatrix}^T$$

$$= \begin{bmatrix} -\frac{1}{13} & \frac{2}{13} & -\frac{1}{13} \\ \frac{2}{13} & -\frac{3}{13} & -\frac{1}{13} \\ -\frac{1}{13} & -\frac{1}{13} & -\frac{5}{13} \end{bmatrix}$$

and $$\text{adj } (A^{-1}) = \frac{1}{13}\begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & -5 \end{bmatrix}$$

$$\therefore \qquad (A^{-1})^{-1} = \frac{1}{|A^{-1}|} \text{adj}\,(A^{-1})$$

$$= \frac{1}{-\frac{1}{13}} \times \frac{1}{13}\begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} = A$$

Thus $(A^{-1})^{-1} = A.$

S28.

Here $$A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

$$\therefore \qquad |A| = \begin{vmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{vmatrix}$$

$$= 1(15 - 1) - (-2)(-10 - 1) + 1(-2 - 3)$$

$$= 14 - 22 - 5 = -13$$

$$A_{11} = (15 - 1) = 14,$$

$$A_{12} = -(-10 - 1) = 11$$

$$A_{13} = (-2 - 3) = -5$$

$$A_{21} = -(-10-1) = 11,$$
$$A_{22} = (5-1) = 4,$$
$$A_{23} = -(1+2) = -3$$
$$A_{31} = (-2-3) = -5,$$
$$A_{32} = -(1+2) = -3,$$
$$A_{33} = (3-4) = -1$$

$\therefore$ $$\text{adj } A = \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}^T$$

$$= \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$$

$\therefore$ $$A^{-1} = \frac{1}{|A|} adj\ A$$

$$= \frac{1}{-13}\begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$$

$$= \frac{1}{13}\begin{bmatrix} -14 & -11 & 5 \\ -11 & -4 & 3 \\ 5 & 3 & 1 \end{bmatrix}$$

Let $$\text{adj } A = B = \begin{bmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{bmatrix}$$

$\therefore$ $$|B| = \begin{vmatrix} 14 & 11 & -5 \\ 11 & 4 & -3 \\ -5 & -3 & -1 \end{vmatrix}$$

$$= 14(-4-9) - 11(-11-15) - 5(-33+20)$$
$$= -182 + 286 + 65 = 169$$
$$B_{11} = (-4-9) = -13$$
$$B_{12} = -(-11-15) = 26$$
$$B_{13} = (-33+20) = -13$$
$$B_{21} = -(-11-15) = 26,$$

$$B_{22} = (-14 - 25) = -39,$$
$$B_{23} = -(-42 + 55) = -13$$
$$B_{31} = (-33 + 20) = -13,$$
$$B_{32} = -(-42 + 55) = -13,$$
$$B_{33} = 56 - 121 = -65$$

$\therefore$ $$\text{adj } B = \begin{bmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & -65 \end{bmatrix}^T$$

$$= \begin{bmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & -65 \end{bmatrix}$$

$\therefore$ $$B^{-1} = \frac{1}{|B|}\text{adj } B$$

$$= \frac{1}{169}\begin{bmatrix} -13 & 26 & -13 \\ 26 & -39 & -13 \\ -13 & -13 & -65 \end{bmatrix}$$

$$= \frac{1}{13}\begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & -5 \end{bmatrix}$$

$\therefore$ $$(\text{adj } A)^{-1} = \frac{1}{13}\begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & -5 \end{bmatrix}$$

Let $$A^{-1} = C = \frac{1}{13}\begin{bmatrix} -14 & -11 & 5 \\ -11 & -4 & 3 \\ 5 & 3 & 1 \end{bmatrix}$$

$\therefore$ $$|C| = \begin{vmatrix} -\frac{14}{13} & -\frac{11}{13} & \frac{5}{13} \\ -\frac{11}{13} & -\frac{4}{13} & \frac{3}{13} \\ \frac{5}{13} & \frac{3}{13} & \frac{1}{13} \end{vmatrix}$$

$$= -\frac{14}{13}\left[\left(-\frac{4}{169} - \frac{9}{169}\right)\right] + \frac{11}{13}\left(-\frac{11}{169} - \frac{15}{169}\right) + \frac{5}{13}\left(-\frac{33}{169} + \frac{20}{169}\right)$$

$$= -\frac{14}{13} \cdot \left(-\frac{13}{169}\right) + \frac{11}{13} \cdot \left(-\frac{26}{169}\right) + \frac{5}{13} \cdot \left(-\frac{13}{169}\right)$$

$$= \frac{14}{169} - \frac{22}{169} - \frac{5}{169}$$

$$= -\frac{13}{169} = -\frac{1}{13}$$

$$C_{11} = -\frac{4}{169} - \frac{9}{169} = -\frac{13}{169}$$

$$C_{12} = -\left(-\frac{11}{169} - \frac{15}{169}\right) = \frac{26}{169}$$

$$C_{13} = -\frac{33}{169} + \frac{20}{169} = -\frac{13}{169}$$

$$C_{21} = -\left(-\frac{11}{169} - \frac{15}{169}\right) = \frac{26}{169}$$

$$C_{22} = \left(-\frac{14}{169} - \frac{25}{169}\right) = -\frac{39}{169}$$

$$C_{23} = -\left(-\frac{42}{169} + \frac{55}{169}\right) = -\frac{13}{169}$$

$$C_{31} = \left(-\frac{33}{169} + \frac{20}{169}\right) = -\frac{13}{169}$$

$$C_{32} = -\left(-\frac{42}{169} + \frac{55}{169}\right) = -\frac{13}{169}$$

$$C_{33} = \left(\frac{56}{169} - \frac{121}{169}\right) = -\frac{65}{169}$$

$$\therefore \quad \text{adj } C = \begin{bmatrix} -\frac{1}{13} & \frac{2}{13} & -\frac{1}{13} \\ \frac{2}{13} & -\frac{3}{13} & -\frac{1}{13} \\ -\frac{1}{13} & -\frac{1}{13} & -\frac{5}{13} \end{bmatrix}^T$$

$$= \begin{bmatrix} -\frac{1}{13} & \frac{2}{13} & -\frac{1}{13} \\ \frac{2}{13} & -\frac{3}{13} & -\frac{1}{13} \\ -\frac{1}{13} & -\frac{1}{13} & -\frac{5}{13} \end{bmatrix}$$

$$= \frac{1}{13}\begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & -5 \end{bmatrix} = \text{adj } (A^{-1})$$

Thus (adj $A)^{-1}$ = adj (A^{-1}).

S29.

Here $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

$\therefore$ $A^2 = A.A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 4+1+1 & -2-2-1 & 2+1+2 \\ -2-2-1 & 1+4+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 12+5+5 & -6-10-5 & 6+5+10 \\ -10-6-5 & 5+12+5 & -5-6-10 \\ 10+5+6 & -5-10-6 & 5+5+12 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

Now $A^3 - 6A^2 + 9A - 4I$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix} + \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 22-36+18-4 & -21+30-9-0 & 21-30+9-0 \\ -21+30-9-0 & 22-36+18-4 & -21+30-9-0 \\ 21-30+9-0 & -21+30-9-0 & 22-36+18-4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$$|A| = \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix} = 2(4-1) + 1(-2+1) + 1(-1-2) = 4 \neq 0$$

Hence A^{-1} exists.

Thus $A^3 - 6A^2 + 9A - 4I = 0$

Pre-multiplying by A^{-1}, on both sides, we get

$A^{-1}(A^3 - 6A^2 + 9A - 4I) = A^{-1}\,0.$ $[\because\ A^{-1}A = I,\ A^{-1}I = A^{-1}$ and $IA = A]$

$\Rightarrow\ A^{-1}A^3 - 6A^{-1}A^2 + 9A^{-1}A - 4A^{-1}I = 0$

$\Rightarrow\ A^2 - 6A + 9I - 4A^{-1} = 0$

$$\Rightarrow \quad A^{-1} = \frac{1}{4}(A^2 - 6A + 9I)$$

$$= \frac{1}{4}\left(\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$= \frac{1}{4}\left(\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - \begin{bmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}\right)$$

$$= \frac{1}{4}\begin{bmatrix} 6-12+9 & -5+6+0 & 5-6+0 \\ -5+6+0 & 6-12+9 & -5+6+0 \\ 5-6+0 & -5+6+0 & 6-12+9 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Thus $\quad A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$

S30.

Here $\quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

$\therefore \quad A^2 = A \,.\, A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 1+1+2 & 1+2-1 & 1-3+3 \\ 1+2-6 & 1+4+3 & 1-6-9 \\ 2-1+6 & 2-2-3 & 2+3+9 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}$$

$$A^3 = A^2 \,.\, A = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+2+2 & 4+4-1 & 4-6+3 \\ -3+8-28 & -3+16+14 & -3-24-42 \\ 7-3+28 & 7-6-14 & 7+9+42 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix}$$

Now $\quad A^3 - 6A^2 + 5A + 11I$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - 6\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} + 5\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix} + \begin{bmatrix} 5 & 5 & 5 \\ 5 & 10 & -15 \\ 10 & -5 & 15 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 8-24+5+11 & 7-12+5+0 & 1-6+5+0 \\ -23+18+5+0 & 27-48+10+11 & -69+84-15+0 \\ 32-42+10+0 & -13+18-5+0 & 58-84+15+11 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{vmatrix} = 1(6+3) - 1(3+6) + 1(-1-4) = -5 \neq 0$$

Hence A^{-1} exists

Thus $A^3 - 6A^2 + 5A + 11I = 0$

Pre-multiplying both sides by A^{-1}, we get

$\Rightarrow \quad A^{-1}(A^3 - 6A^2 + 5A + 11I) = A^{-1}.\,0$

$\Rightarrow \quad A^{-1}.\,A^3 - 6A^{-1}.A^2 + 5A^{-1}.A + 11A^{-1} = 0$ $\quad [\because A^{-1}A = I,\ A^{-1}I = A^{-1} \text{ and } IA = A]$

$\Rightarrow \quad A^2 - 6A + 5I + 11A^{-1} = 0$

$$\Rightarrow \quad A^{-1} = \frac{1}{11}(-A^2 + 6A - 5I)$$

$$= \frac{1}{11}\left(-\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} + 6\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$= \frac{1}{11}\left(\begin{bmatrix} -4 & -2 & -1 \\ 3 & -8 & 14 \\ -7 & 3 & -14 \end{bmatrix} + \begin{bmatrix} 6 & 6 & 6 \\ 6 & 12 & -18 \\ 12 & -6 & 18 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \right)$$

$$= \frac{1}{11}\begin{bmatrix} -4+6-5 & -2+6-0 & -1+6-0 \\ 3+6-0 & -8+12-5 & 14-18-0 \\ -7+12-0 & 3-6-0 & -14+18-5 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$$

Thus $A^{-1} = \frac{1}{11}\begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$.

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Q1. Using matrices, solve

$$x - y + 2z = 1; \quad 2y - 3z = 1; \quad 3x - 2y + 4z = 2.$$

Q2. Using matrices, solve

$$2x + y + z = 7; \quad x - y - z = -4; \quad 3x + 2y + z = 10.$$

Q3. Using matrices, solve

$$2x + 3y + 3z = 5; \quad x - 2y + z = -4; \quad 3x - y - 2z = 3.$$

Q4. Using matrices, solve the following system of equations:

$$x + y + z = 3; \quad x - 2y + 3z = 2 \quad \text{and} \quad 2x - y + z = 2$$

Q5. Solve the following system of equations by matrix method

$$5x + 3y + z = 16; \quad 2x + y + 3z = 19 \quad \text{and} \quad x + 2y + 4z = 25.$$

Q6. Solve the following system of equations by matrix method

$$x - y + z = 4; \quad 2x + y - 3z = 0 \quad \text{and} \quad x + y + z = 2$$

Q7. Solve the following system of equations by matrix method

$$2x + y + z = 1; \quad x - 2y - z = \frac{3}{2} \quad \text{and} \quad 3y - 5z = 9$$

Q8. Using matrices, solve the following system of equations

$$3x - 2y + 3z = 8; \quad 2x + y - z = 1; \quad 4x - 3y + 2z = 4.$$

Q9. Using matrices, solve the following system of equations

$$8x + 4y + 3z = 18; \quad 2x + y + z = 5; \quad x + 2y + z = 5.$$

Q10. Using matrices, solve

$$2x + 8y + 5z = 5; \quad x + y + z = -2; \quad x + 2y - z = 2.$$

Q11. Using matrices, solve the following system of equations

$$x + y + z = 1; \quad x - 2y + 3z = 2; \quad x - 3y + 5z = 3.$$

Q12. Using matrices, solve system of linear equations

$$x + y + z = 6; \quad x + 2z = 7; \quad 3x + y + z = 12.$$

Q13. Using matrices, solve the following system of equations.

$$x + 2y + z = 7; \quad x + 3z = 11; \quad 2x - 3y = 1.$$

Q14. Using matrices, solve the following system of equations.

$$4x + 3y + 2z = 60; \quad x + 2y + 3z = 45; \quad 6x + 2y + 3z = 70.$$

Q15. Using matrices, solve the following system of equations.

$$x - y + 2z = 7; \quad 3x + 4y - 5z = -5 \quad \text{and} \quad 2x - y + 3z = 12$$

Q16. Using matrices, solve the following system of linear equation

$$x + y - z = 3; \quad 2x + 3y + z = 10; \quad 3x - y - 7z = 1$$

Q17. Using matrix method, solve the following system of equations.

$$\frac{2}{x}+\frac{3}{y}+\frac{10}{z}=4,\quad \frac{4}{x}-\frac{6}{y}+\frac{5}{z}=1,\quad \frac{6}{x}+\frac{9}{y}-\frac{20}{z}=2,\quad x,y,z\neq 0.$$

Q18. Find A^{-1}, using A^{-1} solve the system of equations $2x - 3y + 5z = 11$; $3x + 2y - 4z = -5$; $x + y - 2z = -3$. Where

Q19. Find A^{-1} and hence solve the following system of equations $8x - 4y + z = 5$, $10x + 6z = 4$,

$8x + y + 6z = \frac{5}{2}$. If $A = \begin{bmatrix} 8 & -4 & 1 \\ 10 & 0 & 6 \\ 8 & 1 & 6 \end{bmatrix}$.

Q20. Solve the following system of equations $3x + 2y + z = 6$; $4x - y + 2z = 5$; $7x + 3y - 3z = 7$.

If $A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & -1 & 2 \\ 7 & 3 & -3 \end{bmatrix}$ find A^{-1}.

Q21. Find A^{-1} and hence solve the following system of equations $3x - 4y + 2z = -1$, $2x + 3y + 5z = 7$

and $x + z = 2$. Where $A = \begin{bmatrix} 3 & -4 & 2 \\ 2 & 3 & 5 \\ 1 & 0 & 1 \end{bmatrix}$.

Q22. Find AB, hence solve system of equations $x - 2y = 10$; $2x + y + 3z = 8$ and $-2y + z = 7$.

Where $A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 3 \\ 0 & -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 2 & -6 \\ -2 & 1 & -3 \\ -4 & 2 & 5 \end{bmatrix}$.

Q23. Find AB, use this to solve the system of equations $x - y = 3$, $2x + 3y + 4z = 17$, $y + 2z = 7$.

Where $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$.

Q24. Find A^{-1} and hence solve the system of equations $x + 2y + z = 4$, $-x + y + z = 0$ and

$x - 3y + z = 4$. Where $A = \begin{vmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 1 & -3 & 1 \end{vmatrix}$.

Q25. Find A^{-1} and hence solve the following of equations $2x - y + z = -3$, $3x - z = 0$ and

$2x + 6y - 2 = 0$. Where $A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 0 & -1 \\ 2 & 6 & 0 \end{bmatrix}$.

Q26. Find A^{-1} and hence solve the following of equations $x + 2y - 3z = -4$, $2x + 3y + 2z = 2$ and $3x - 3y - 4z = 11$.

where $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix}$.

Q27. Find A^{-1} and hence solve the following system of equations $x - 2y + z = 0$, $-y + z = -2$, $2x - 3z = 10$. Where $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & -1 & 1 \\ 2 & 0 & -3 \end{bmatrix}$.

Q28. Using matrices, solve the following system of equations:

$$x + y + z = 6; \quad x + 2y + 3z = 14; \quad x + 4y + 7z = 30.$$

Q29. Show that the following system of equation is consistent: $x - y + z = 3$; $2x + y - z = 2$; $-x - 2y + 2z = 1$. Also, find the solution.

Q30. Examine the consistency of the following system of equations:

$$3x - y + 7z = 3; \quad 2x + y + 3z = 5; \quad x + 4y - 2z = 1.$$

Q31. Use matrix method to solve the system of equations $x + y = 2$ and $2x + 2y = 4$.

Q32. Use the following product and then use to solve the system of equations $x - y + z = 4$, $x - 2y - 2z = 9$ and $2x + y + 3z = 1$. $\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$

Q33. Use the following product to solve the system of equations $x - y + 2z = 1$, $2y - 3z = 1$ and $3x - 2y + 4z = 2$.

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$$

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S1. Given system of equation is $AX = B$.

where $$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

We know that by matrix method, we have

$$X = A^{-1}B \qquad \dots \text{(i)}$$

where $$A^{-1} = \frac{\text{adj}(A)}{|A|}$$

Now, $$|A| = 1(8-6) + 1(0+9) + 2(0-6)$$
$$= 2 + 9 - 12 = -1$$

$|A| \neq 0$,

Hence A is non-singular. Therefore the given system of equations will have the unique solution given by $X = A^{-1} B$.

and $$\text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

Now, $$A_{11} = (-1)^2 (2) = 1 \times 2 = 2$$
$$A_{12} = (-1)^3 (9) = -1 \times 9 = -9$$
$$A_{13} = (-1)^4 (-6) = 1 \times -6 = -6$$

$$A_{21} = (-1)^3 (0) = -1 \times 0 = 0$$
$$A_{22} = (-1)^4 (-2) = 1 \times -2 = -2$$
$$A_{23} = (-1)^5 (1) = -1 \times 1 = -1$$

$$A_{31} = (-1)^4 (-1) = 1 \times -1 = -1$$
$$A_{32} = (-1)^5 (-3) = -1 \times 3 = 3$$
$$A_{33} = (-1)^6 (2) = 1 \times 2 = 2$$

$\therefore$ $$\text{adj}(A) = \begin{bmatrix} 2 & -9 & -6 \\ 0 & -2 & -1 \\ -1 & 3 & 2 \end{bmatrix}^T = \begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix}$$

$\therefore$ $$A^{-1} = \frac{-1}{1}\begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$$

$\therefore$ By using Eq. (i), we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2+0+2 \\ 9+2-6 \\ 6+1-4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}$$

Hence, $x = 0$, $y = 5$ and $z = 3$.

S2. The given system of equations is $AX = B$.

where $$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 7 \\ -4 \\ 10 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

We know that by matrix method, we have

$$X = A^{-1}B \qquad \text{... (i)}$$

where $$A^{-1} = \frac{\text{adj}(A)}{|A|}$$

Now, $$|A| = 2(-1 + 2) - 1(1 + 3) + 1(2 + 3)$$
$$= 2 - 4 + 5 = 3$$

$|A| \neq 0$,

Hence A is non-singular. Therefore the given system of equations will have the unique solution given by $X = A^{-1} B$.

and $$\text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

Now, $$A_{11} = (-1)^2 (1) = 1 \times 1 = 1$$
$$A_{12} = (-1)^3 (4) = (-1) \times 4 = -4$$
$$A_{13} = (-1)^4 (5) = 1 \times 5 = 5$$

$$A_{21} = (-1)^3 (-1) = (-1)\times(-1) = 1$$
$$A_{22} = (-1)^4 (-1) = 1 \times (-1) = -1$$
$$A_{23} = (-1)^5 (1) = -1 \times 1 = -1$$

$$A_{31} = (-1)^4 (0) = 1 \times 0 = 0$$
$$A_{32} = (-1)^5 (-3) = (-1)\times(-3) = 3$$
$$A_{33} = (-1)^6 (-3) = 1 \times (-3) = -3$$

$\therefore$ $$\text{adj}(A) = \begin{bmatrix} 1 & -4 & 5 \\ 1 & -1 & -1 \\ 0 & 3 & -3 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ -4 & -1 & 3 \\ 5 & -1 & -3 \end{bmatrix}$$

$\therefore$ $$A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 0 \\ -4 & -1 & 3 \\ 5 & -1 & -3 \end{bmatrix}$$

$\therefore$ By using Eq. (i), we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 0 \\ -4 & -1 & 3 \\ 5 & -1 & -3 \end{bmatrix}\begin{bmatrix} 7 \\ -4 \\ 10 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 7-4+0 \\ -28+4+30 \\ 35+4-30 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

$\therefore$ $$x = \frac{3}{3}, \quad y = \frac{6}{3}, \quad z = \frac{9}{3}$$

Hence, $x = 1$, $y = 2$ and $z = 3$.

S3. The given system of equations is $AX = B$ where

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

We know that by matrix method, we have

$$X = A^{-1}B \qquad \ldots \text{(i)}$$

where $$A^{-1} = \frac{1}{|A|} \cdot \text{adj}\,(A)$$

Now, $$|A| = 2 \times \begin{vmatrix} -2 & 1 \\ -1 & -2 \end{vmatrix} - 3 \times \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} + 3 \times \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix}$$

$$= 2(5) - 3(-5) + 3(5) = 10 + 15 + 15 = 40$$

$|A| \neq 0$,

Hence A is non-singular. Therefore the given system of equations will have the unique solution given by $X = A^{-1} B$.

and $$\text{adj}\,(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t$$

Now,

$$A_{11} = (-1)^2 \begin{vmatrix} -2 & 1 \\ -1 & -2 \end{vmatrix} = 5$$

$$A_{12} = (-1)^3 \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} = 5$$

$$A_{13} = (-1)^4 \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} = 5$$

$$A_{21} = (-1)^3 \begin{vmatrix} 3 & 3 \\ -1 & -2 \end{vmatrix} = 3$$

$$A_{22} = (-1)^4 \begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} = -13$$

$$A_{23} = (-1)^5 \begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} = 11$$

$$A_{31} = (-1)^4 \begin{vmatrix} 3 & 3 \\ -2 & 1 \end{vmatrix} = 9$$

$$A_{32} = (-1)^5 \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 1$$

$$A_{33} = (-1)^6 \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -7$$

$\therefore$

$$\text{adj}(A) = \begin{bmatrix} 5 & 5 & 5 \\ 3 & -13 & 11 \\ 9 & 1 & -7 \end{bmatrix}^T$$

$$= \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

$\therefore$

$$A^{-1} = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

Now, by using Eq. (i), we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

$$= \frac{1}{40}\begin{bmatrix} 25-12+27 \\ 25+52+3 \\ 25-44-21 \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 40 \\ 80 \\ -40 \end{bmatrix}$$

$\therefore$ $$x = \frac{40}{40}, \quad y = \frac{80}{40},$$

$$z = -\frac{40}{40}$$

Hence, $x = 1$, $y = 2$ and $z = -1$.

S4. Given system of equations in matrix form is:

$$AX = B, \qquad \text{... (i)}$$

where $$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 3 \\ 2 & -1 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$$

Then the matrix equation of the given system of equations become $AX = B$.

Now $$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -2 & 3 \\ 2 & -1 & 1 \end{vmatrix}$$

$$= 1(-2+3) - 1(1-6) + 1(-1+4) = 9 \neq 0$$

Hence A is non-singular. Therefore the given system of equations will have the unique solution given by $X = A^{-1} B$.

$$A_{11} = (-2+3) = 1$$
$$A_{12} = -(1-6) = 5$$
$$A_{13} = (-1+4) = 3$$

$$A_{21} = -(1+1) = -2$$
$$A_{22} = (1-2) = -1$$
$$A_{23} = -(-1-2) = 3$$

$$A_{31} = (3+2) = 5$$
$$A_{32} = -(3-1) = -2$$
$$A_{33} = (-2-1) = -3$$

$$\text{adj } A = \begin{bmatrix} 1 & 5 & 3 \\ -2 & -1 & 3 \\ 5 & -2 & -3 \end{bmatrix}^T$$

Now, $\text{adj } A = \begin{bmatrix} 1 & -2 & 5 \\ 5 & -1 & -2 \\ 3 & 3 & -3 \end{bmatrix}$

$\therefore$ $A^{-1} = \frac{1}{|A|} \text{adj } A$

$$= \frac{1}{9}\begin{bmatrix} 1 & -2 & 5 \\ 5 & -1 & -2 \\ 3 & 3 & -3 \end{bmatrix}$$

From (i), $X = A^{-1} B$

$\therefore$ $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 1 & -2 & 5 \\ 5 & -1 & -2 \\ 3 & 3 & -3 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 1.3 + (-2)2 + 5.2 \\ 5.3 + (-1)2 + (-2).2 \\ 3.3 + 3.2 + (-3).2 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 \\ 9 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ $x = 1, \quad y = 1, \quad z = 1$

S5.

Let $A = \begin{bmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 16 \\ 19 \\ 25 \end{bmatrix}$

Then the matrix equation of the given system of equations become $AX = B$.

Now $|A| = \begin{vmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{vmatrix}$

$$= 5(4-6) - 3(8-3) + 1(4-1) = -22 \neq 0$$

Hence A is non-singular. Therefore the given system of equations will have the unique solution given by $X = A^{-1} B$.

Let C be the matrix whose elements are the cofactors of the corresponding elements of A, then

$$A_{11} = (4-6) = -2$$
$$A_{12} = -(8-3) = -5$$
$$A_{13} = (4-1) = 3$$

$A_{21} = -(12-2) = -10$
$A_{22} = (20-1) = 19$
$A_{23} = -(10-3) = -7$

$A_{31} = (9-1) = 8$
$A_{32} = -(15-2) = -13$
$A_{33} = (5-6) = -1$

$$C = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$C = \begin{bmatrix} -2 & -5 & 3 \\ -10 & 19 & -7 \\ 8 & -13 & -1 \end{bmatrix}$$

$\therefore$ $$\text{adj } A = C^T = \begin{bmatrix} -2 & -10 & 8 \\ -5 & 19 & -13 \\ 3 & -7 & -1 \end{bmatrix}$$

$\therefore$ $$A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{-22}\begin{bmatrix} -2 & -10 & 8 \\ -5 & 19 & -13 \\ 3 & -7 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2/22 & 10/22 & -8/22 \\ 5/22 & -19/22 & 13/22 \\ -3/22 & 7/22 & 1/22 \end{bmatrix}$$

$\therefore$ $$X = A^{-1} B$$

$$= \begin{bmatrix} 2/22 & 10/22 & -8/22 \\ 5/22 & -19/22 & 13/22 \\ -3/22 & 7/22 & 1/22 \end{bmatrix}\begin{bmatrix} 16 \\ 19 \\ 25 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{32}{22} + \frac{190}{22} - \frac{200}{22} \\ \frac{80}{22} - \frac{361}{22} + \frac{325}{22} \\ -\frac{48}{22} + \frac{133}{22} + \frac{25}{22} \end{bmatrix} = \begin{bmatrix} \frac{22}{22} \\ \frac{44}{22} \\ \frac{110}{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

$\therefore$ $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

$\therefore$ $x = 1, \quad y = 2, \quad z = 5.$

S6. The system of equations can be written in the form $AX = B$,

where $$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

Then the matrix equation of the given system of equations become $AX = B$.

Now $$|A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 1(1 + 3) - (-1)(2 + 3) + 1(2 - 1)$$

$$= 4 + 5 + 1 = 10 \neq 0$$

Hence A is non-singular. Therefore the given system of equations will have the unique solution given by $X = A^{-1} B$.

$$A_{11} = (1 + 3) = 4,$$
$$A_{12} = -(2 + 3) = -5,$$
$$A_{13} = (2 - 1) = 1$$

$$A_{21} = -(-1 - 1) = 2,$$
$$A_{22} = (1 - 1) = 0,$$
$$A_{23} = -(1 + 1) = -2$$

$$A_{31} = (3 - 1) = 2,$$
$$A_{32} = -(-3 - 2) = 5,$$
$$A_{33} = (1 + 2) = 3$$

$$\text{adj } A = \begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}^T = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

$\therefore$ $$A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

$\therefore$ $$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 16+0+4 \\ -20+0+10 \\ 4-0+6 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 20 \\ -10 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$\therefore$ $x = 2, \quad y = -1 \quad \text{and} \quad z = 1.$

S7. The system of equations can be written in the form $AX = B$,

where $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & -4 & -2 \\ 0 & 3 & -5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$

Then the matrix equation of the given system of equations become $AX = B$.

Now $$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 2 & -4 & -2 \\ 0 & 3 & -5 \end{vmatrix}$$

$$= 2(20 + 6) - 1(-10 - 0) + 1(6 - 0)$$
$$= 52 + 10 + 6 = 68 \neq 0$$

Hence A is non-singular. Therefore the given system of equations will have the unique solution given by $X = A^{-1} B$.

$$A_{11} = (20 + 6) = 26,$$
$$A_{12} = -(-10 - 0) = 10,$$
$$A_{13} = (6 - 0) = 6$$
$$A_{21} = -(-5 - 3) = 8,$$
$$A_{22} = (-10 - 0) = -10,$$
$$A_{23} = -(6 - 0) = -6$$
$$A_{31} = (-2 + 4) = 2,$$
$$A_{32} = -(-4 - 2) = 6,$$
$$A_{33} = (-8 - 2) = -10$$

$$\text{adj } A = \begin{bmatrix} 26 & 10 & 6 \\ 8 & -10 & -6 \\ 2 & 6 & -10 \end{bmatrix}^T$$

$$= \begin{bmatrix} 26 & 8 & 2 \\ 10 & -10 & 6 \\ 6 & -6 & -10 \end{bmatrix}$$

$\therefore$ $$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$= \frac{1}{68} \begin{bmatrix} 26 & 8 & 2 \\ 10 & -10 & 6 \\ 6 & -6 & -10 \end{bmatrix}$$

$\therefore$ $$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{68}\begin{bmatrix} 26 & 8 & 2 \\ 10 & -10 & 6 \\ 6 & -6 & -10 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$$

$$= \frac{1}{68}\begin{bmatrix} 26+24+18 \\ 10-30+54 \\ 6-18-90 \end{bmatrix}$$

$$= \frac{1}{68}\begin{bmatrix} 68 \\ 34 \\ -102 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ -3/2 \end{bmatrix}$$

$\therefore$ $x = 1, \quad y = \frac{1}{2} \quad$ and $\quad z = -\frac{3}{2}$.

S8. The given system of equation is $AX = B$

where, $A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Its solution is given by

$$X = A^{-1}B \qquad \text{...(i)}$$

where, $A^{-1} = \frac{1}{|A|}\cdot \text{adj}(A)$

Now, $|A| = 3(-1) + 2(8) + 3(-10)$

$= -3 + 16 - 30 = -17$

$|A| \neq 0$,

Hence A is non-singular. Therefore the given system of equations will have the unique solution given by $X = A^{-1} B$.

and $\text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$

Now, $A_{11} = (-1)^2 (-1) = -1$

$A_{12} = (-1)^3 (8) = -8$

$A_{13} = (-1)^4 (-10) = -10$

$A_{21} = (-1)^3 (5) = -5$

$A_{22} = (-1)^4 (-6) = -6$

$A_{23} = (-1)^5 (-1) = 1$

$$A_{31} = (-1)^4 (-1) = -1$$
$$A_{32} = (-1)^5 (-9) = 9$$
$$A_{33} = (-1)^6 (7) = 7$$

Now, $$\text{adj}(A) = \begin{bmatrix} -1 & -8 & -10 \\ -5 & -6 & 1 \\ -1 & 9 & 7 \end{bmatrix}^T$$

$$= \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$$

Hence, $$A^{-1} = -\frac{1}{17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$$

$\therefore$ From Eq. (i), we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}\begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

$$= -\frac{1}{17}\begin{bmatrix} -8-5-4 \\ -64-6+36 \\ -80+1+28 \end{bmatrix} = -\frac{1}{17}\begin{bmatrix} -17 \\ -34 \\ -51 \end{bmatrix}$$

$\therefore$ $$x = \frac{-17}{-17}, \quad y = \frac{-34}{-17}, \quad z = \frac{-51}{-17}$$

$\therefore$ $x = 1$, $y = 2$ and $z = 3$.

S9. The given system of equation can be written as $AX = B$

where, $$A = \begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 18 \\ 5 \\ 5 \end{bmatrix} \text{ and } X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Solution of above system of equations is given by

$$X = A^{-1}B \qquad \dots \text{(i)}$$

where $$A^{-1} = \frac{\text{adj}(A)}{|A|}$$

Now, $|A| = 8(1-2) - 4(2-1) + 3(4-1)$

$= -8 - 4 + 9 = -3$

$|A| \neq 0,$

Hence A is non-singular. Therefore the given system of equations will have the unique solution given by $X = A^{-1}B$.

and $$\text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

Now,
$$A_{11} = (-1)^2(1-2) = 1(-1) = -1$$
$$A_{12} = (-1)^3(2-1) = -1(1) = -1$$
$$A_{13} = (-1)^4(4-1) = 1(3) = 3$$
$$A_{21} = (-1)^3(4-6) = -1(-2) = 2$$
$$A_{22} = (-1)^4(8-3) = 1(5) = 5$$
$$A_{23} = (-1)^5(16-4) = -1(12) = -12$$
$$A_{31} = (-1)^4(4-3) = 1(1) = 1$$
$$A_{32} = (-1)^5(8-6) = -1(2) = -2$$
$$A_{33} = (-1)^6(8-8) = 1(0) = 0$$

Now, $$\text{adj}(A) = \begin{bmatrix} -1 & -1 & 3 \\ 2 & 5 & -12 \\ 1 & -2 & 0 \end{bmatrix}^T$$

$$= \begin{bmatrix} -1 & 2 & 1 \\ -1 & 5 & -2 \\ 3 & -12 & 0 \end{bmatrix}$$

$\therefore$ $$A^{-1} = -\frac{1}{3}\begin{bmatrix} -1 & 2 & 1 \\ -1 & 5 & -2 \\ 3 & -12 & 0 \end{bmatrix}$$

$\therefore$ From Eq. (i), we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} -1 & 2 & 1 \\ -1 & 5 & -2 \\ 3 & -12 & 0 \end{bmatrix}\begin{bmatrix} 18 \\ 5 \\ 5 \end{bmatrix}$$

$$= -\frac{1}{3}\begin{bmatrix} -18+10+5 \\ -18+25-10 \\ 54-60+0 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} -3 \\ -3 \\ -6 \end{bmatrix}$$

$\therefore$ $$x = \frac{-3}{-3}, \quad y = \frac{-3}{-3}, \quad z = \frac{-6}{-3}$$

Hence, $x = 1$, $y = 1$ and $z = 2$.

S10. Given system of equation is $AX = B$.

where $$A = \begin{bmatrix} 2 & 8 & 5 \\ 1 & 1 & 1 \\ 1 & 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 5 \\ -2 \\ 2 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Solution of above system of eq. is given by

$$X = A^{-1}B \qquad \text{... (i)}$$

where $$A^{-1} = \frac{\text{adj}(A)}{|A|}$$

Now, $$|A| = 2(-1-2) - 8(-1-1) + 5(2-1)$$
$$= -6 + 16 + 5 = 15$$

$|A| \neq 0$,

Hence A is non-singular. Therefore the given system of equations will have the unique solution given by $X = A^{-1} B$.

and $$\text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

Now, $$A_{11} = (-1)^2(-1-2) = 1(-3) = -3$$
$$A_{12} = (-1)^3(-1-1) = -1(-2) = 2$$
$$A_{13} = (-1)^4(2-1) = 1(1) = 1$$

$$A_{21} = (-1)^3(-8-10) = -1(-18) = 18$$
$$A_{22} = (-1)^4(-2-5) = 1(-7) = -7$$
$$A_{23} = (-1)^5(4-8) = -1(-4) = 4$$

$$A_{31} = (-1)^4(8-5) = 1(3) = 3$$
$$A_{32} = (-1)^5(2-5) = -1(-3) = 3$$
$$A_{33} = (-1)^6(2-8) = 1(-6) = -6$$

Now, $\text{adj}(A) = \begin{bmatrix} -3 & 2 & 1 \\ 18 & -7 & 4 \\ 3 & 3 & -6 \end{bmatrix}^T$

$$= \begin{bmatrix} -3 & 18 & 3 \\ 2 & -7 & 3 \\ 1 & 4 & -6 \end{bmatrix}$$

$\therefore$ $A^{-1} = \frac{1}{15}\begin{bmatrix} -3 & 18 & 3 \\ 2 & -7 & 3 \\ 1 & 4 & -6 \end{bmatrix}$

Hence, by Eq. (i), we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{15}\begin{bmatrix} -3 & 18 & 3 \\ 2 & -7 & 3 \\ 1 & 4 & -6 \end{bmatrix}\begin{bmatrix} 5 \\ -2 \\ 2 \end{bmatrix}$$

$$= \frac{1}{15}\begin{bmatrix} -15-36+6 \\ 10+14+6 \\ 5-8-12 \end{bmatrix}$$

$$= \frac{1}{15}\begin{bmatrix} -45 \\ 30 \\ -15 \end{bmatrix}$$

$\therefore$ $x = -\frac{45}{15}, \quad y = \frac{30}{15}, \quad z = -\frac{15}{15}$

Hence, $x = -3$, $y = 2$ and $z = -1$.

S11. The given system of equation can be written as $AX = B$

where, $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 3 \\ 1 & -3 & 5 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Solution of above system of equations is given by

$X = A^{-1}B$ ...(i)

where $A^{-1} = \frac{\text{adj}(A)}{|A|}$

Now, $|A| = 1(-10 + 9) - 1(5 - 3) + 1(-3 + 2)$

$= -1 - 2 - 1$

$\Rightarrow \quad |A| = -4$

$|A| \neq 0,$

Hence A is non-singular. Therefore the given system of equations will have the unique solution given by $X = A^{-1} B$.

and
$$\text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

Now,
$$A_{11} = (-1)^2 \begin{vmatrix} -2 & 3 \\ -3 & 5 \end{vmatrix} = 1(-10 + 9) = -1$$

$$A_{12} = (-1)^3 \begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} = -1(5 - 3) = -2$$

$$A_{13} = (-1)^4 \begin{vmatrix} 1 & -2 \\ 1 & -3 \end{vmatrix} = 1(-3 + 2) = -1$$

$$A_{21} = (-1)^3 \begin{vmatrix} 1 & 1 \\ -3 & 5 \end{vmatrix} = -1(5 + 3) = -8$$

$$A_{22} = (-1)^4 \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} = 1(5 - 1) = 4$$

$$A_{23} = (-1)^5 \begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} = -1(-3 - 1) = 4$$

$$A_{31} = (-1)^4 \begin{vmatrix} 1 & 1 \\ -2 & 3 \end{vmatrix} = 1(3 + 2) = 5$$

$$A_{32} = (-1)^5 \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -1(3 - 1) = -2$$

$$A_{33} = (-1)^6 \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = 1(-2 - 1) = -3$$

Now,
$$\text{adj}(A) = \begin{bmatrix} -1 & -2 & -1 \\ -8 & 4 & 4 \\ 5 & -2 & -3 \end{bmatrix}^T$$

$$= \begin{bmatrix} -1 & -8 & 5 \\ -2 & 4 & -2 \\ -1 & 4 & -3 \end{bmatrix}$$

$\therefore \quad A^{-1} = -\frac{1}{4}\begin{bmatrix} -1 & -8 & 5 \\ -2 & 4 & -2 \\ -1 & 4 & -3 \end{bmatrix} \qquad \left[\because \; A^{-1} = \frac{\text{adj}(A)}{|A|}\right]$

$\therefore$ From Eq. (i), we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} -1 & -8 & 5 \\ -2 & 4 & -2 \\ -1 & 4 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= -\frac{1}{4}\begin{bmatrix} -1 -16 + 15 \\ -2 + 8 - 6 \\ -1 + 8 - 9 \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix}$$

$\therefore \quad x = \frac{-2}{-4}, \quad y = -\frac{0}{4}, \quad z = \frac{-2}{-4}$

Hence, $\quad x = \frac{1}{2}, \quad y = 0 \quad \text{and} \quad z = \frac{1}{2}$

S12. The given system of equation can be written as $AX = B$

where, $\quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Its solution is given by

$$X = A^{-1}B \qquad \ldots \text{(i)}$$

where $\quad A^{-1} = \frac{\text{adj}(A)}{|A|}$

$$|A| = 1(-2) - 1(1 - 6) + 1(1)$$

$$= -2 + 5 + 1 = 4$$

$\therefore \quad |A| \neq 0,$

Hence A is non-singular. Therefore the given system of equations will have the unique solution given by $X = A^{-1} B$.

and $\quad \text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$

Now, $\quad A_{11} = (-1)^2(-2) = 1(-2) = -2$

$$A_{12} = (-1)^3(-5) = -1(-5) = 5$$

$$A_{13} = (-1)^4(1) = 1(1) = 1$$

$$A_{21} = (-1)^3(0) = 0$$

$$A_{22} = (-1)^4 (-2) = 1(-2) = -2$$

$$A_{23} = (-1)^5 (-2) = -1(-2) = 2$$

$$A_{31} = (-1)^4 (2) = 1(2) = 2$$

$$A_{32} = (-1)^5 (1) = -1(1) = -1$$

$$A_{33} = (-1)^6 (-1) = 1(-1) = -1$$

$\therefore$ $$\text{adj}(A) = \begin{bmatrix} -2 & 5 & 1 \\ 0 & -2 & 2 \\ 2 & -1 & -1 \end{bmatrix}^T = \begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$$

$\therefore$ $$A^{-1} = \frac{1}{4}\begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$$

$\therefore$ From Eq. (i), we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -12+0+24 \\ 30-14-12 \\ 6+14-12 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 12 \\ 4 \\ 8 \end{bmatrix}$$

$\therefore$ $$x = \frac{12}{4}, \quad y = \frac{4}{4} \quad \text{and} \quad z = \frac{8}{4}$$

Hence, $x = 3$, $y = 1$ and $z = 2$.

S13. The given system of equation can be written as $AX = B$,

where $$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Its solution is given by

$$X = A^{-1}B, \qquad \text{...(i)}$$

where $$A^{-1} = \frac{\text{adj } A}{|A|}$$

$$|A| = 1\times\begin{vmatrix} 0 & 3 \\ -3 & 0 \end{vmatrix} - 2\times\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} + 1\times\begin{vmatrix} 1 & 0 \\ 2 & -3 \end{vmatrix}$$

$$|A| = 1(0+9) - 2(0-6) + 1(-3-0)$$

$$= 9 + 12 - 3 = 18$$

$\therefore$ $$|A| = 18$$

$\therefore \quad |A| \neq 0,$

Hence A is non-singular. Therefore the given system of equations will have the unique solution given by $X = A^{-1} B$.

and $$\text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t$$

Now, $$A_{11} = (-1)^2 \begin{vmatrix} 0 & 3 \\ -3 & 0 \end{vmatrix} = 1(0+9) = 9$$

$$A_{12} = (-1)^3 \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = -1(0-6) = -1(-6) = 6$$

$$A_{13} = (-1)^4 \begin{vmatrix} 1 & 0 \\ 2 & -3 \end{vmatrix} = 1(-3-0) = -3$$

$$A_{21} = (-1)^3 \begin{vmatrix} 2 & 1 \\ -3 & 0 \end{vmatrix} = -1(0+3) = -1(3) = -3$$

$$A_{22} = (-1)^4 \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = 1(0-2) = 1(-2) = -2$$

$$A_{23} = (-1)^5 \begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix} = -1(-3-4) = -1(-7) = 7$$

$$A_{31} = (-1)^4 \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} = 1(6-0) = 1(6) = 6$$

$$A_{32} = (-1)^5 \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -1(3-1) = -1(2) = -2$$

$$A_{33} = (-1)^6 \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 1(0-2) = 1(-2) = -2$$

$\therefore$ $$\text{adj}(A) = \begin{bmatrix} 9 & 6 & -3 \\ -3 & -2 & 7 \\ 6 & -2 & -2 \end{bmatrix}^T = \begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}$$

$\therefore \quad A^{-1} = \frac{1}{18}\begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}$

$\therefore$ Using Eq. (i), we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}\begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 63-33+6 \\ 42-22-2 \\ -21+77-2 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 36 \\ 18 \\ 54 \end{bmatrix}$$

$\therefore \quad x = \frac{36}{18}, y = \frac{18}{18} \text{ and } z = \frac{54}{18}$

Hence, $x = 2$, $y = 1$ and $z = 3$.

S14. The given system of equation can be written as $AX = B$,

where $A = \begin{bmatrix} 4 & 3 & 2 \\ 1 & 2 & 3 \\ 6 & 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 60 \\ 45 \\ 70 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Its solution is given by

$$X = A^{-1}B \quad \text{... (i)}$$

where $A^{-1} = \frac{\text{adj } A}{|A|}$

Now, $|A| = 4 \times \begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} - 3 \times \begin{vmatrix} 1 & 3 \\ 6 & 3 \end{vmatrix} + 2 \times \begin{vmatrix} 1 & 2 \\ 6 & 2 \end{vmatrix}$

$= 4(6 – 6) – 3(3 – 18) + 2(2 – 12)$

$\therefore \quad |A| = 4(0) – 3(–15) + 2(–10) = 0 + 45 – 20 = 25$

$\therefore \quad |A| = 25$

$\therefore \quad |A| \neq 0$,

Hence A is non-singular. Therefore the given system of equations will have the unique solution given by $X = A^{-1} B$.

and $\text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$

Now, $A_{11} = (-1)^2 \begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} = 1(6-6) = 0$

$$A_{12} = (-1)^3 \begin{vmatrix} 1 & 3 \\ 6 & 3 \end{vmatrix} = -1(3-18) = -1(-15) = 15$$

$$A_{13} = (-1)^4 \begin{vmatrix} 1 & 2 \\ 6 & 2 \end{vmatrix} = 1(2-12) = 1(-10) = -10$$

$$A_{21} = (-1)^3 \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = -1(9-4) = -1(5) = -5$$

$$A_{22} = (-1)^4 \begin{vmatrix} 4 & 2 \\ 6 & 3 \end{vmatrix} = 1(12-12) = 0$$

$$A_{23} = (-1)^5 \begin{vmatrix} 4 & 3 \\ 6 & 2 \end{vmatrix} = -1(8-18) = -1(-10) = 10$$

$$A_{31} = (-1)^4 \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 1(9-4) = 1(5) = 5$$

$$A_{32} = (-1)^5 \begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} = -1(12-2) = -1(10) = -10$$

$$A_{33} = (-1)^6 \begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix} = 1(8-3) = 5$$

$\therefore$ $$\text{adj}(A) = \begin{bmatrix} 0 & 15 & -10 \\ -5 & 0 & 10 \\ 5 & -10 & 5 \end{bmatrix}^T = \begin{bmatrix} 0 & -5 & 5 \\ 15 & 0 & -10 \\ -10 & 10 & 5 \end{bmatrix}$$

$\therefore$ $$A^{-1} = \frac{1}{25}\begin{bmatrix} 0 & -5 & 5 \\ 15 & 0 & -10 \\ -10 & 10 & 5 \end{bmatrix}$$

$\therefore$ Using Eq. (i), we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 0 & -5 & 5 \\ 15 & 0 & -10 \\ -10 & 10 & 5 \end{bmatrix}\begin{bmatrix} 60 \\ 45 \\ 70 \end{bmatrix}$$

$$= \frac{1}{25}\begin{bmatrix} 0-225+350 \\ 900+0-700 \\ -600+450+350 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 125 \\ 200 \\ 200 \end{bmatrix}$$

$\therefore \quad x = \frac{125}{25}, y = \frac{200}{25}, z = \frac{200}{25}$

Hence, $x = 5$, $y = 8$ and $z = 8$.

S15. Given system of equations can be written in matrix form $AX = B$. Firstly determine the cofactors of A and then determine A^{-1} and then using the relation $X = A^{-1}B$ to get the values of x, y and z. Given equations are

$$x - y + 2z = 7; \quad 3x + 4y - 5z = -5 \quad \text{and} \quad 2x - y + 3z = 12$$

Let
$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$$

According to matrix, we have

$$X = A^{-1}B \qquad \text{...(i)}$$

where
$$A^{-1} = \frac{\text{adj A}}{|A|}$$

Now,
$$|A| = 1(12 - 5) + 1(9 + 10) + 2(-3 - 8)$$
$$|A| = 1(7) + 1(19) + 2(-11)$$
$$= 7 + 19 - 22 = 4$$

$\therefore \quad |A| \neq 0,$

Hence A is non-singular. Therefore the given system of equations will have the unique solution given by $X = A^{-1} B$.

and
$$\text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

Now,
$$A_{11} = (-1)^2 \begin{vmatrix} 4 & -5 \\ -1 & 3 \end{vmatrix} = 1(12 - 5) = 7$$

$$A_{12} = (-1)^3 \begin{vmatrix} 3 & -5 \\ 2 & 3 \end{vmatrix} = -1(9 + 10) = -19$$

$$A_{13} = (-1)^4 \begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix} = 1(-3 - 8) = -11$$

$$A_{21} = (-1)^3 \begin{vmatrix} -1 & 2 \\ -1 & 3 \end{vmatrix} = -1(-3 + 2) = 1$$

$$A_{22} = (-1)^4 \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 1(3-4) = -1$$

$$A_{23} = (-1)^5 \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = -1(-1+2) = -1$$

$$A_{31} = (-1)^4 \begin{vmatrix} -1 & 2 \\ 4 & -5 \end{vmatrix} = 1(5-8) = -3$$

$$A_{32} = (-1)^5 \begin{vmatrix} 1 & 2 \\ 3 & -5 \end{vmatrix} = -1(-5-6) = 11$$

$$A_{33} = (-1)^6 \begin{vmatrix} 1 & -1 \\ 3 & 4 \end{vmatrix} = 1(4+3) = 7$$

$$\therefore \qquad \text{adj}\,(A) = \begin{bmatrix} 7 & -19 & -11 \\ 1 & -1 & -1 \\ -3 & 11 & 7 \end{bmatrix}^T = \begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}$$

$$\Rightarrow \qquad A^{-1} = \frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}$$

Putting A^{-1} in Eq. (ii), we get

$$X = \frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}\begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 49-5-36 \\ -133+5+132 \\ -77+5+84 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 8 \\ 4 \\ 12 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$\therefore \qquad x = 2, \quad y = 1 \quad \text{and} \quad z = 3.$

S16. Given system of equations are

$$x + y - z = 3; \quad 2x + 3y + z = 10; \quad 3x - y - 7z = 1$$

Above system of eq. can be written as

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 3 & -1 & -7 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 \\ 10 \\ 1 \end{bmatrix}$$

By matrix method, we have

$$X = A^{-1}B \quad ..(i)$$

where, $A^{-1} = \dfrac{\text{adj } A}{|A|}$

Now, $|A| = 1(-21 + 1) - 1(-14 - 3) - 1(-2 - 9)$

$= 1(-20) - 1(-17) - 1(-11)$

$= -20 + 17 + 11 = 8$

$\therefore$ $|A| \neq 0$, hence unique solution.

and adj $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$

$\therefore$ $A_{11} = (-1)^2 \begin{vmatrix} 3 & 1 \\ -1 & -7 \end{vmatrix} = 1(-21+1) = -20$

$A_{12} = (-1)^3 \begin{vmatrix} 2 & 1 \\ 3 & -7 \end{vmatrix} = -1(-14-3) = 17$

$A_{13} = (-1)^4 \begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} = 1(-2-9) = -11$

$A_{21} = (-1)^3 \begin{vmatrix} 1 & -1 \\ -1 & -7 \end{vmatrix} = -1(-7-1) = 8$

$A_{22} = (-1)^4 \begin{vmatrix} 1 & -1 \\ 3 & -7 \end{vmatrix} = 1(-7+3) = -4$

$A_{23} = (-1)^5 \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} = -1(-1-3) = 4$

$A_{31} = (-1)^4 \begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} = 1(1+3) = 4$

$A_{32} = (-1)^5 \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = -1(1+2) = -3$

$A_{33} = (-1)^6 \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1(3-2) = 1$

$\therefore$ adj $(A) = \begin{bmatrix} -20 & 17 & -11 \\ 8 & -4 & 4 \\ 4 & -3 & 1 \end{bmatrix}^T = \begin{bmatrix} -20 & 8 & 4 \\ 17 & -4 & -3 \\ -11 & 4 & 1 \end{bmatrix}$

$$\therefore \qquad A^{-1} = \frac{1}{8}\begin{bmatrix} -20 & 8 & 4 \\ 17 & -4 & -3 \\ -11 & 4 & 1 \end{bmatrix}$$

Putting A^{-1} in Eq. (i), we get

$$X = \frac{1}{8}\begin{bmatrix} -20 & 8 & 4 \\ 17 & -4 & -3 \\ -11 & 4 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 10 \\ 1 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} -60+80+4 \\ 51-40-3 \\ -33+40+1 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 24 \\ 8 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore \qquad x = 3, \quad y = 1 \quad \text{and} \quad z = 1.$$

S17. First let $\frac{1}{x} = u$, $\frac{1}{y} = v$, and $\frac{1}{z} = w$ and then apply concept of matrix method to reduced system of equations in terms of u, v and w. Get the values of u, v and w and then find x, y and z from above mentioned substitutions.

The given system of equation are

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4, \quad \frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1, \quad \frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2 \quad \text{and} \quad x, y, z \neq 0$$

Let $\frac{1}{x} = u$, $\frac{1}{y} = v$ and $\frac{1}{z} = w$, we get the reduced system of equations as

$$2u + 3v + 10w = 4; \quad 4u - 6v + 5w = 1; \quad 6u + 9v - 20w = 2$$

Now, we find values of u, v and w by using matrix method.

The given system Eq. (i) can be written as $AX = B$,

where $\qquad AX = B,$

where $\qquad A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$

Its solution is given by

$$X = A^{-1}B \qquad \ldots \text{(i)}$$

where $\qquad A^{-1} = \frac{\text{adj}\,A}{|A|}$

Now, $|A| = 2 \times \begin{vmatrix} -6 & 5 \\ 9 & -20 \end{vmatrix} - 3 \times \begin{vmatrix} 4 & 5 \\ 6 & -20 \end{vmatrix} + 10 \times \begin{vmatrix} 4 & -6 \\ 6 & 9 \end{vmatrix}$

$= 2(120 - 45) - 3(-80 - 30) + 10(36 + 36)$

$= 2(75) - 3(-110) + 10(72)$

$= 150 + 330 + 720 = 1200$

$\therefore \quad |A| = 1200$

$\therefore \quad |A| \neq 0,$ hence unique solution.

and $\text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$

Now, $A_{11} = (-1)^2 \begin{vmatrix} -6 & 5 \\ 9 & -20 \end{vmatrix} = 1(120 - 45) = 75$

$A_{12} = (-1)^3 \begin{vmatrix} 4 & 5 \\ 6 & -20 \end{vmatrix} = -1(-80 - 30) = 110$

$A_{13} = (-1)^4 \begin{vmatrix} 4 & -6 \\ 6 & 9 \end{vmatrix} = (36 + 36) = 72$

$A_{21} = (-1)^3 \begin{vmatrix} 3 & 10 \\ 9 & -20 \end{vmatrix} = -1(-60 - 90) = 150$

$A_{22} = (-1)^4 \begin{vmatrix} 2 & 10 \\ 6 & -20 \end{vmatrix} = 1(-40 - 60) = -100$

$A_{23} = (-1)^5 \begin{vmatrix} 2 & 3 \\ 6 & 9 \end{vmatrix} = -1(18 - 18) = 0$

$A_{31} = (-1)^4 \begin{vmatrix} 3 & 10 \\ -6 & 5 \end{vmatrix} = 1(15 + 60) = 75$

$A_{32} = (-1)^5 \begin{vmatrix} 2 & 10 \\ 4 & 5 \end{vmatrix} = -1(10 - 40) = 30$

$A_{33} = (-1)^6 \begin{vmatrix} 2 & 3 \\ 4 & -6 \end{vmatrix} = 1(-12 - 12) = -24$

$\therefore \quad \text{adj}(A) = \begin{bmatrix} 75 & 110 & 72 \\ 150 & -100 & 0 \\ 75 & 30 & -24 \end{bmatrix}^T = \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$

$\therefore \qquad A^{-1} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$

Putting value of A^{-1} Eq. (i), we get

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{1200}\begin{bmatrix} 300+150+150 \\ 440-100+60 \\ 288+0-48 \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix}$$

$\therefore \qquad u = \frac{600}{1200}, \quad v = \frac{400}{1200}, \quad w = \frac{240}{1200}$

$\therefore \qquad u = \frac{1}{2}, \quad v = \frac{1}{3} \quad \text{and} \quad w = \frac{1}{5}$

But $\qquad \frac{1}{x} = u, \ \frac{1}{y} = v \quad \text{and} \quad \frac{1}{z} = w$

$\Rightarrow \qquad \frac{1}{x} = \frac{1}{2}, \frac{1}{y} = \frac{1}{3} \quad \text{and} \quad \frac{1}{z} = \frac{1}{5}$

$\Rightarrow \qquad x = 2, \quad y = 3 \quad \text{and} \quad z = 5$

S18.

$$\boldsymbol{A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}}.$$

Here $\qquad A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{vmatrix}$$

$$= 2(-4+4) - (-3)(-6+4) + 5(3-2)$$

$$= 0 - 6 + 5 = -1 \neq 0$$

$\Rightarrow \quad A^{-1}$ exists

$$A_{11} = (-4+4) = 0,$$
$$A_{12} = -(-6+4) = 2,$$

$$A_{13} = (3-2) = 1$$
$$A_{21} = -(6-5) = -1,$$
$$A_{22} = (-4-5) = -9,$$
$$A_{23} = -(2+3) = -5$$
$$A_{31} = (12-10) = 2,$$
$$A_{32} = -(-8-15) = 23,$$
$$A_{33} = 4+9 = 13$$

$$\text{adj } A = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}^T$$

$$= \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$\therefore$ $$A^{-1} = \frac{1}{|A|} adj\ A$$

$$= \frac{1}{-1}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$\therefore$ $$A^{-1} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$$

Now $$X = A^{-1}B$$

where $$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0-5+6 \\ -22-45+69 \\ -11-25+39 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\therefore$ $x = 1, \quad y = 2 \quad$ and $\quad z = 3.$

S19. The given system of equation is $AX = B$

where, $$A = \begin{bmatrix} 8 & -4 & 1 \\ 10 & 0 & 6 \\ 8 & 1 & 6 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 \\ 4 \\ \frac{5}{2} \end{bmatrix}$$

By matrix method, we know that

$$X = A^{-1}B \qquad \ldots\text{(i)}$$

Note that the matrix A is given also. So, we find A^{-1} and use Eq. (i) to find x, y, z.

We know that $$A^{-1} = \frac{\text{adj}(A)}{|A|}$$

Now, $$|A| = 8(0-6) + 4(60-48) + 1(10-0)$$
$$= -48 + 48 + 10 = 10$$

$|A| \neq 0$, hence unique solution.

$$\text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

Now, $$A_{11} = (-1)^2(-6) = 1 \times -6 = -6$$
$$A_{12} = (-1)^3(12) = -1 \times 12 = -12$$
$$A_{13} = (-1)^4(10) = 1 \times 10 = 10$$

$$A_{21} = (-1)^3(-25) = -1 \times -25 = 25$$
$$A_{22} = (-1)^4(40) = 1 \times 40 = 40$$
$$A_{23} = (-1)^5(40) = -1 \times 40 = -40$$

$$A_{31} = (-1)^4(-24) = 1 \times -24 = -24$$
$$A_{32} = (-1)^5(38) = -1 \times 38 = -38$$
$$A_{33} = (-1)^6(40) = 1 \times 40 = 40$$

$\therefore$ $$\text{adj}(A) = \begin{bmatrix} -6 & -12 & 10 \\ 25 & 40 & -40 \\ -24 & -38 & 40 \end{bmatrix}^T = \begin{bmatrix} -6 & 25 & -24 \\ -12 & 40 & -38 \\ 10 & -40 & 40 \end{bmatrix}$$

$\therefore$ $$A^{-1} = \frac{1}{10}\begin{bmatrix} -6 & 25 & -24 \\ -12 & 40 & -38 \\ 10 & -40 & 40 \end{bmatrix}$$

$\therefore$ Using Eq. (i), we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{10}\begin{bmatrix} -6 & 25 & -24 \\ -12 & 40 & -38 \\ 10 & -40 & 40 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \\ 5/2 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} -30+100-60 \\ -60+160-95 \\ 50-160+100 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 10 \\ 5 \\ -10 \end{bmatrix}$$

$\therefore$ $x = \frac{10}{10}$, $y = \frac{5}{10}$ and $z = -\frac{10}{10}$

Hence, $x = 1$, $y = \frac{1}{2}$ and $z = -1$.

S20. Given that

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & -1 & 2 \\ 7 & 3 & -3 \end{bmatrix}$$

The given system of equations can be written as

$$AX = B$$

where, $A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & -1 & 2 \\ 7 & 3 & -3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 6 \\ 5 \\ 7 \end{bmatrix}$

Solution of above system of equations is given by

$$X = A^{-1}B \qquad \text{... (i)}$$

So, now we find A^{-1},

where $A^{-1} = \frac{\text{adj(A)}}{|A|}$

Now, $|A| = 3(3-6) - 2(-12-14) + 1(12+7)$

$= 3(-3) - 2(-26) + 1(19)$

$= -9 + 52 + 19 = 62$

$|A| \neq 0$, hence unique solution.

and $\text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$

$\therefore$ $$A_{11} = (-1)^2 \begin{vmatrix} -1 & 2 \\ 3 & -3 \end{vmatrix} = (-1)^2 \times (3-6) = -3$$

$$A_{12} = (-1)^3 \begin{vmatrix} 4 & 2 \\ 7 & -3 \end{vmatrix} = -1(-12-14) = 26$$

$$A_{13} = (-1)^4 \begin{vmatrix} 4 & -1 \\ 7 & 3 \end{vmatrix} = 1(12+7) = 19$$

$$A_{21} = (-1)^3 \begin{vmatrix} 2 & 1 \\ 3 & -3 \end{vmatrix} = -1(-6-3) = 9$$

$$A_{22} = (-1)^4 \begin{vmatrix} 3 & 1 \\ 7 & -3 \end{vmatrix} = 1(-9-7) = -16$$

$$A_{23} = (-1)^5 \begin{vmatrix} 3 & 2 \\ 7 & 3 \end{vmatrix} = -1(9-14) = 5$$

$$A_{31} = (-1)^4 \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 1(4+1) = 5$$

$$A_{32} = (-1)^5 \begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} = -1(6-4) = -2$$

$$A_{33} = (-1)^6 \begin{vmatrix} 3 & 2 \\ 4 & -1 \end{vmatrix} = 1(-3-8) = -11$$

$\therefore$ $$\text{adj}(A) = \begin{bmatrix} -3 & 26 & 19 \\ 9 & -16 & 5 \\ 5 & -2 & -11 \end{bmatrix}^T$$

$$= \begin{bmatrix} -3 & 9 & 5 \\ 26 & -16 & -2 \\ 19 & 5 & -11 \end{bmatrix}$$

$\Rightarrow$ $$A^{-1} = \frac{1}{62}\begin{bmatrix} -3 & 9 & 5 \\ 26 & -16 & -2 \\ 19 & 5 & -11 \end{bmatrix}\begin{bmatrix} 6 \\ 5 \\ 7 \end{bmatrix}$$ $\left[\because\ A^{-1} = \dfrac{\text{adj}(A)}{|A|}\right]$

Now, by using Eq. (i), we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{62}\begin{bmatrix} -3 & 9 & 5 \\ 26 & -16 & -2 \\ 19 & 5 & -11 \end{bmatrix}\begin{bmatrix} 6 \\ 5 \\ 7 \end{bmatrix}$$

$$= \frac{1}{62}\begin{bmatrix} -18+45+35 \\ 156-80-14 \\ 114+25-77 \end{bmatrix} = \frac{1}{62}\begin{bmatrix} 62 \\ 62 \\ 62 \end{bmatrix}$$

$\therefore$ $$x = \frac{62}{62}; \quad y = \frac{62}{62}; \quad z = \frac{62}{62}$$

Hence, $x = 1$, $y = 1$ and $z = 1$.

S21. First we find A^{-1} and then use it to find x, y and z in the given system of linear equations.

Now, $$A = \begin{bmatrix} 3 & -4 & 2 \\ 2 & 3 & 5 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 \\ 7 \\ 2 \end{bmatrix}$$ [Given]

We know that

$$A^{-1} = \frac{\text{adj}(A)}{|A|}$$

Now, $$|A| = 3(3-0) + 4(2-5) + 2(0-3)$$

$$|A| = (3 \times 3) - (4 \times 3) - (2 \times 3) = 9 - 12 - 6 = 9 - 18 = -9$$

$|A| \neq 0$, hence unique solution.

and $$\text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

Now,

$$A_{11} = (-1)^2\,(3) = 1 \times 3 = 3$$
$$A_{12} = (-1)^3\,(-3) = -1 \times -3 = 3$$
$$A_{13} = (-1)^4\,(-3) = 1 \times -3 = -3$$

$$A_{21} = (-1)^3(-4) = -1 \times -4 = 4$$
$$A_{22} = (-1)^4\,(1) = 1 \times 1 = 1$$
$$A_{23} = (-1)^5\,(4) = -1 \times 4 = -4$$

$$A_{31} = (-1)^4\,(-26) = 1 \times -26 = -26$$
$$A_{32} = (-1)^5\,(11) = -1 \times 11 = -11$$
$$A_{33} = (-1)^6\,(17) = 1 \times 17 = 17$$

$\therefore$ $$\text{adj}(A) = \begin{bmatrix} 3 & 3 & -3 \\ 4 & 1 & -4 \\ -26 & -11 & 17 \end{bmatrix}^T = \begin{bmatrix} 3 & 4 & -26 \\ 3 & 1 & -11 \\ -3 & -4 & 17 \end{bmatrix}$$

$$\therefore \qquad A^{-1} = -\frac{1}{9}\begin{bmatrix} 3 & 4 & -26 \\ 3 & 1 & -11 \\ -3 & -4 & 17 \end{bmatrix}$$

Hence, by matrix method, we know that

$$X = A^{-1}B$$

$$\therefore \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{9}\begin{bmatrix} 3 & 4 & -26 \\ 3 & 1 & -11 \\ -3 & -4 & 17 \end{bmatrix}\begin{bmatrix} -1 \\ 7 \\ 2 \end{bmatrix}$$

$$= -\frac{1}{9}\begin{bmatrix} -3+28-52 \\ -3+7-22 \\ 3-28+34 \end{bmatrix} = -\frac{1}{9}\begin{bmatrix} -27 \\ -18 \\ 9 \end{bmatrix}$$

$$\therefore \qquad x = \frac{-27}{-9}, y = \frac{-18}{-9}, \ z = \frac{-9}{9}$$

Hence, $x = 3$, $y = 2$ and $z = -1$.

S22. Given that,

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 3 \\ 0 & -2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 7 & 2 & -6 \\ -2 & 1 & -3 \\ -4 & 2 & 5 \end{bmatrix}$$

First we find product AB and then use it to find A^{-1}.

Now, $$AB = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 3 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 7 & 2 & -6 \\ -2 & 1 & -3 \\ -4 & 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 7+4-0 & 2-2+0 & -6+6+0 \\ 14-2-12 & 4+1+6 & -12-3+15 \\ 0+4-4 & 0-2+2 & 0+6+5 \end{bmatrix}$$

$$\therefore \qquad AB = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix} = 11I$$

$\therefore$ We get $\qquad AB = 11I \qquad$ ...(i)

Premultiplying both sides of Eq. (i) by A^{-1}, we get

$$A^{-1}AB = 11A^{-1}I$$

$$\Rightarrow \quad IB = 11A^{-1} \qquad [\because A^{-1}A = I \text{ and } A^{-1}I = A^{-1}]$$

$$\Rightarrow \quad B = 11A^{-1} \qquad [IB = B]$$

$$\Rightarrow \quad A^{-1} = \frac{1}{11}\cdot B$$

$$\Rightarrow \quad A^{-1} = \frac{1}{11}\begin{bmatrix} 7 & 2 & -6 \\ -2 & 1 & -3 \\ -4 & 2 & 5 \end{bmatrix}$$

Now, we know that by matrix method the solution of system of equations,

$$X = A^{-1}D,$$

where $D = \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\Rightarrow \quad X = \frac{1}{11}\begin{bmatrix} 7 & 2 & -6 \\ -2 & 1 & -3 \\ -4 & 2 & 5 \end{bmatrix}\begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} 70+16-42 \\ -20+8-21 \\ -40+16+35 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 44 \\ -33 \\ 11 \end{bmatrix}$$

$$\therefore \quad x = \frac{44}{11}, \quad y = -\frac{33}{11} \quad \text{and} \quad z = \frac{11}{11}$$

or $\quad x = 4, \quad y = -3, \quad z = 1.$

S23. First find the product AB and then premultiply both sides of product AB by A^{-1} and obtain A^{-1}. Then, using the relation $X = A^{-1}C$ and simplify it to get the result.

First we find the product AB.

$$AB = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2+4+0 & 2-2-0 & -4+4+0 \\ 4-12+8 & 4+6-4 & -8-12+20 \\ 0-4+4 & 0+2-2 & 0-4+10 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 6I$$

$\therefore \quad AB = 6I \quad$ ...(i)

Now, given system of equations can be written as

$$AX = C \Rightarrow X = A^{-1}C \quad \text{... (ii)}$$

where $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad C = \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$

Now, again from Eq. (i)

$$AB = 6I$$

$\Rightarrow \quad A^{-1}AB = 6A^{-1}I \quad$ [Premultiplying by A^{-1} on both sides]

$\Rightarrow \quad B = 6A^{-1} \quad$ [$\because A^{-1}A = I$ and $IB = B$]

$$\therefore \quad A^{-1} = \frac{1}{6}B = \frac{1}{6}\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$$

Now from Eq. (ii), we get

$$X = A^{-1}C \quad \text{where} \quad C = \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix} \text{ and } X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 6+34-28 \\ -12+34-28 \\ 6-17+35 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 12 \\ -6 \\ 24 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

$\therefore \quad x = 2, \quad y = -1 \quad$ and $\quad z = 4$.

S24. Given,

$$A = \begin{vmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 1 & -3 & 1 \end{vmatrix}$$

$$|A| = 1(1 + 3) - 2(-1 - 1) + 1(3 - 1)$$

$$= 4 + 4 + 2 = 10$$

$|A| \neq 0,$ hence unique solution.

$$A^{-1} = \frac{1}{|A|}.\text{adj A}$$

Now, to find adj A.

$$\text{adj }(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

$\therefore$

$$A_{11} = (-1)^2 \begin{vmatrix} 1 & 1 \\ -3 & 1 \end{vmatrix} = (1+3) = 4$$

$$A_{12} = (-1)^3 \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -(-1-1) = 2$$

$$A_{13} = (-1)^4 \begin{vmatrix} -1 & 1 \\ 1 & -3 \end{vmatrix} = 3 - 1 = 2$$

$$A_{21} = (-1)^3 \begin{vmatrix} 2 & 1 \\ -3 & 1 \end{vmatrix} = -(2+3) = -5$$

$$A_{22} = (-1)^4 \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 1 - 1 = 0$$

$$A_{23} = (-1)^5 \begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix} = -1(-3-2) = 5$$

$$A_{31} = (-1)^4 \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 2 - 1 \quad 1$$

$$A_{32} = (-1)^5 \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = -(1+1) = -2$$

$$A_{33} = (-1)^6 \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = (1+2) = 3$$

$\therefore$

$$\text{adj}(A) = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}^T = \begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}$$

$$\Rightarrow \qquad A^{-1} = \frac{1}{10}\begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}$$

From the given system of equations, we have

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 1 & -3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\therefore \qquad X = A^{-1} B$$

$$X = \frac{1}{10}\begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 16+0+4 \\ 8+0-8 \\ 8+0+12 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 20 \\ 0 \\ 20 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

$$\therefore \qquad x = 2, \quad y = 0, \quad \text{and} \quad z = 2.$$

S25. Since the given matrix A is same as the coefficient matrix of the given system of linear equations, so find A^{-1} and put it in the equation $X = A^{-1}B$ to get x, y and z.

Given that $$A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 0 & -1 \\ 2 & 6 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$$

This matrix is same as the coefficient matrix of given system of equations. So, we find A^{-1} and using this get the values of x, y and z.

We know that $$A^{-1} = \frac{\text{adj}(A)}{|A|}$$

Now,
$$|A| = 2(0 + 6) + 1(0 + 2) + 1(18 - 0)$$
$$|A| = 2(6) + 1(2) + 1(18)$$
$$= 12 + 2 + 18 = 32$$

$|A| \neq 0$, hence unique solution.

and $$\text{adj }(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

$\therefore \quad A_{11} = (-1)^2 \begin{vmatrix} 0 & -1 \\ 6 & 0 \end{vmatrix} = 1(0+6) = 6$

$A_{12} = (-1)^3 \begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} = -1(0+2) = -1(2) = -2$

$A_{13} = (-1)^4 \begin{vmatrix} 3 & 0 \\ 2 & 6 \end{vmatrix} = 1(18-0) = 1(18) = 18$

$A_{21} = (-1)^3 \begin{vmatrix} -1 & 1 \\ 6 & 0 \end{vmatrix} = -1(0-6) = -1(-6) = 6$

$A_{22} = (-1)^4 \begin{vmatrix} 2 & 1 \\ 2 & 0 \end{vmatrix} = 1(0-2) = -2$

$A_{23} = (-1)^5 \begin{vmatrix} 2 & -1 \\ 2 & 6 \end{vmatrix} = -1(12+2) = -14$

$A_{31} = (-1)^4 \begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix} = 1(1-0) = 1$

$A_{32} = (-1)^5 \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} = -1(-2-3) = 5$

$A_{33} = (-1)^6 \begin{vmatrix} 2 & -1 \\ 3 & 0 \end{vmatrix} = 1(0+3) = 3$

$$\therefore \quad \text{adj}(A) = \begin{bmatrix} 6 & -2 & 18 \\ 6 & -2 & -14 \\ 1 & 5 & 3 \end{bmatrix}^T$$

$$= \begin{bmatrix} 6 & 6 & 1 \\ -2 & -2 & 5 \\ 18 & -14 & 3 \end{bmatrix}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{32}\begin{bmatrix} 6 & 6 & 1 \\ -2 & -2 & 5 \\ 18 & -14 & 3 \end{bmatrix} \qquad \left[\because \; A^{-1} = \frac{\text{adj}(A)}{|A|}\right]$$

Now, by matrix method, we have

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{32}\begin{bmatrix} 6 & 6 & 1 \\ -2 & -2 & 5 \\ 18 & -14 & 3 \end{bmatrix}\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$$

$$= \frac{1}{32}\begin{bmatrix} -18+0+2 \\ 6-0+10 \\ -54-0+6 \end{bmatrix} = \frac{1}{32}\begin{bmatrix} -16 \\ 16 \\ -48 \end{bmatrix}$$

$\therefore$ $$x = -\frac{16}{32}, \quad y = \frac{16}{32}, \quad z = -\frac{48}{32}$$

$\therefore$ $$x = -\frac{1}{2}, \quad y = \frac{1}{2}, \quad z = -\frac{3}{2}.$$

S26.

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix}$$

$$|A| = 1(-12 + 6) -2(-8 -6) - 3(-6 - 9)$$

$$= -6 + 28 + 45 = 67$$

$|A| \neq 0$, hence unique solution.

$$A^{-1} = \frac{1}{|A|}.\text{adj } A$$

Now, to find adj A.

$$\text{adj } (A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

$\therefore$ $$A_{11} = (-1)^2\begin{vmatrix} 3 & 2 \\ -3 & -4 \end{vmatrix} = -12 + 6 = -6$$

$$A_{12} = (-1)^3\begin{vmatrix} 2 & 2 \\ 3 & -4 \end{vmatrix} = -(1)(-8-6) = 14$$

$$A_{13} = (-1)^4\begin{vmatrix} 2 & 3 \\ 3 & -3 \end{vmatrix} = -6 - 9 = -15$$

$$A_{21} = (-1)^3\begin{vmatrix} 2 & -3 \\ -3 & -4 \end{vmatrix} = -(-8-9) = 17$$

$$A_{22} = (-1)^4 \begin{vmatrix} 1 & -3 \\ 3 & -4 \end{vmatrix} = (-4+9) = 5$$

$$A_{23} = (-1)^5 \begin{vmatrix} 1 & 2 \\ 3 & -3 \end{vmatrix} = -(-3-6) = 9$$

$$A_{31} = (-1)^4 \begin{vmatrix} 2 & -3 \\ 3 & 2 \end{vmatrix} = (4+9) = 13$$

$$A_{32} = (-1)^5 \begin{vmatrix} 1 & -3 \\ 2 & 2 \end{vmatrix} = -(2+6) = -8$$

$$A_{33} = (-1)^6 \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = (3-4) = -1$$

$\therefore$ $$\text{adj}(A) = \begin{bmatrix} -6 & 14 & -15 \\ 17 & 5 & 9 \\ 13 & -8 & -1 \end{bmatrix}^T = \begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}$$

$\Rightarrow$ $$A^{-1} = \frac{1}{67}\begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}$$

From the given system of equations, we have

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} -4 \\ 2 \\ 11 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\therefore$ $$X = A^{-1}B$$

$$X = \frac{1}{67}\begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}\begin{bmatrix} -4 \\ 2 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{67}\begin{bmatrix} 24+34+143 \\ -56+10-88 \\ +60+18-11 \end{bmatrix} = \frac{1}{67}\begin{bmatrix} 201 \\ -134 \\ 67 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

$\therefore$ $x = 3, \quad y = -2, \quad z = 1.$

S27. Given that,

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & -1 & 1 \\ 2 & 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -2 \\ 10 \end{bmatrix}$$

First we find A^{-1}, where

$$A^{-1} = \frac{\text{adj}(A)}{|A|}$$

Now,

$$|A| = 1(3-0) + 2(0-2) + 1(0+2)$$

$$|A| = 1(3) + 2(-2) + 1(2) = 3 - 4 + 2 = 1$$

$|A| \neq 0$, hence unique solution.

and $$\text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

Now,

$$A_{11} = (-1)^2 (3-0) = 1 \times 3 = 3$$

$$A_{12} = (-1)^3 (-2) = -1 \times -2 = 2$$

$$A_{13} = (-1)^4 (0+2) = 1 \times 2 = 2$$

$$A_{21} = (-1)^3 (6-0) = -1 \times 6 = -6$$

$$A_{22} = (-1)^4 (-3-2) = 1 \times -5 = -5$$

$$A_{23} = (-1)^5 (0+4) = -1 \times 4 = -4$$

$$A_{31} = (-1)^4 (-2+1) = 1 \times -1 = -1$$

$$A_{32} = (-1)^5 (1-0) = -1 \times 1 = -1$$

$$A_{33} = (-1)^6 (-1+0) = 1 \times -1 = -1$$

$\therefore$ $$\text{adj}(A) = \begin{bmatrix} 3 & 2 & 2 \\ -6 & -5 & -4 \\ -1 & -1 & -1 \end{bmatrix}^T = \begin{bmatrix} 3 & -6 & -1 \\ 2 & -5 & -1 \\ 2 & -4 & -1 \end{bmatrix}$$

$\therefore$ $$A^{-1} = \begin{bmatrix} 3 & -6 & -1 \\ 2 & -5 & -1 \\ 2 & -4 & -1 \end{bmatrix} \qquad \left[\because A^{-1} = \frac{\text{adj}(A)}{|A|} \text{ and } |A| = 1\right]$$

We know that by matrix method, we have

$$X = A^{-1} B$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -6 & -1 \\ 2 & -5 & -1 \\ 2 & -4 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \\ 10 \end{bmatrix} = \begin{bmatrix} 0+12-10 \\ 0+10-10 \\ 0+8-10 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$$

Hence, $x = 2$, $y = 0$ and $z = -2$.

S28. The given equations are

$$x + y + z = 6 \qquad \text{... (i)}$$

$$x + 2y + 3z = 14 \qquad \text{... (ii)}$$

$$x + 4y + 7z = 30 \qquad \text{... (iii)}$$

The system of equations in matrix form is $AX = B$,

where $$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$$

Now $$|A| = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix}$$

$$= 1(14 - 12) - 1(7 - 3) + 1(4 - 2)$$

$$= 2 - 4 + 2 = 0$$

Hence given system of equations has either no solution or infinitely many solutions.

Given system of equations will be consistent or inconsistent according as (adj A). $B = 0$ or (adj A) $B \neq 0$.

$$A_{11} = (14 - 12) = 2$$

$$A_{12} = -(7 - 3) = -4$$

$$A_{13} = (4 - 2) = 2$$

$$A_{21} = -(7 - 4) = -3$$

$$A_{22} = (7 - 1) = 6$$

$$A_{23} = -(4 - 1) = -3$$

$$A_{31} = (3 - 2) = 1$$

$$A_{32} = -(3 - 1) = -2$$

$$A_{33} = (2 - 1) = 1$$

$$\text{adj } A = \begin{bmatrix} 2 & -4 & 2 \\ -3 & 6 & -3 \\ 1 & -2 & 1 \end{bmatrix}^T$$

$$\text{adj } A = \begin{bmatrix} 2 & -3 & 1 \\ -4 & 6 & -2 \\ 2 & -3 & 1 \end{bmatrix}$$

$\therefore$ $$(\text{adj } A) \, . \, B = \begin{bmatrix} 2 & -3 & 1 \\ -4 & 6 & -2 \\ 2 & -3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$$

$$= \begin{bmatrix} 12 - 42 + 30 \\ -24 + 84 - 60 \\ 12 - 42 + 30 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \text{ (zero matrix)}$$

$\Rightarrow$ The given system of equations is consistent with infinitely many solutions.

Let $z = k$

Equation (i) $\Rightarrow$ $x + y = 6 - k$

Equation (ii) $\Rightarrow$ $x + 2y = 14 - 3k$

Equation (iii) $\Rightarrow$ $x + 4y = 30 - 7k$

Solving the first two equations, we get

$$x = -2 + k;$$

$$y = 8 - 2k$$

Which clearly satisfies the third equation.

$\therefore$ $x = -2 + k,$

$$y = 8 - 2k$$

and $z = k \; \forall \, k \in R$ is the required solutions.

S29. The given system of equations in matrix form is, $AX = B$.

Where $$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & -2 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Now, $$|A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & -2 & 2 \end{vmatrix}$$

$$= 1(2 - 2) + 1(4 - 1) + 1(-4 + 1) = 0$$

Hence given system of equations has either no solution or infinitely many solutions.

Given system of equations will be consistent or inconsistent according as (adj A). $B = 0$ or (adj A).$B \neq 0$.

Let C be the matrix whose elements are cofactors of the corresponding elements of A, then

$$A_{11} = (2-2) = 0$$
$$A_{12} = -(4-1) = -3$$
$$A_{13} = (-4+1) = -3$$
$$A_{21} = (-2+2) = 0$$
$$A_{22} = (2+1) = 3$$
$$A_{23} = -(-2-1) = 3$$
$$A_{31} = (1-1) = 0$$
$$A_{32} = -(-1-2) = 3$$
$$A_{33} = (1+2) = 3$$

$$C = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & -3 & -3 \\ 0 & 3 & 3 \\ 0 & 3 & 3 \end{bmatrix}$$

$\therefore$ $$\text{adj } A = C^T = \begin{bmatrix} 0 & 0 & 0 \\ -3 & 3 & 3 \\ -3 & 3 & 3 \end{bmatrix}$$

Now $$(\text{adj } A).\, B = \begin{bmatrix} 0 & 0 & 0 \\ -3 & 3 & 3 \\ -3 & 3 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0+0+0 \\ -9+6+3 \\ -9+6+3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus, (adj A) . $B = 0$ (zero matrix)

Hence given system of equations is consistent and will have infinite number of solutions.

To find the solution : Given equation are

$$x - y + z = 3 \quad \ldots \text{(i)}$$
$$2x + y - z = 2 \quad \ldots \text{(ii)}$$
$$-x - 2y + 2z = 1 \quad \ldots \text{(iii)}$$

(i) + (ii) ⇒ $3x = 5$ ⇒ $x = \frac{5}{3}$

Let $y = t$

From (i), $z = 3 + y - x = 3 + y - \frac{5}{3} = \frac{4}{3} + y = \frac{4}{3} + t$

Thus solution is given by

$$x = \frac{5}{3}, \quad y = t, \quad z = \frac{4}{3} + t$$

where t is an arbitrary number.

S30. The given system of equations can be written in matrix form as $AX = B$, where

$$A = \begin{bmatrix} 3 & -1 & 7 \\ 2 & 1 & 3 \\ 1 & 4 & -2 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$$

Now $|A| = \begin{vmatrix} 3 & -1 & 7 \\ 2 & 1 & 3 \\ 1 & 4 & -2 \end{vmatrix}$

$$= 3(-2 - 12) + 1(-4 - 3) + 7(8 - 1)$$
$$= -42 - 7 + 49 = 0$$

Therefore given system of equations will have either no solution or infinite number of solutions. Given system of equations will be consistent or inconsistent according as (adj A). $B = 0$ or (adj A).$B \neq 0$

Let C be the matrix whose elements are cofactors of the corresponding elements of A.

$$A_{11} = (-2 - 12) = -14$$
$$A_{12} = -(-4 - 3) = 7$$
$$A_{13} = (8 - 1) = 7$$
$$A_{21} = -(2 - 28) = 26$$
$$A_{22} = (-6 - 7) = -13$$
$$A_{23} = -(12 + 1) = -13$$
$$A_{31} = (-3 - 7) = -10$$
$$A_{32} = -(9 - 14) = 5$$
$$A_{33} = (3 + 2) = 5$$

$$C = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Then, $$C = \begin{bmatrix} -14 & 7 & 7 \\ 26 & -13 & -13 \\ -10 & 5 & 5 \end{bmatrix}$$

$\therefore$ $$\text{adj } A = C^T$$

$$= \begin{bmatrix} -14 & 26 & -10 \\ 7 & -13 & 5 \\ 7 & -13 & 5 \end{bmatrix}$$

Now $$(\text{adj } A)\, B = \begin{bmatrix} -14 & 26 & -10 \\ 7 & -13 & 5 \\ 7 & -13 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -42 + 130 - 10 \\ 21 - 65 + 5 \\ 21 - 65 + 5 \end{bmatrix} = \begin{bmatrix} 78 \\ -39 \\ -39 \end{bmatrix} \neq 0 \text{ (zero matrix)}$$

Hence given system of equations is inconsistent *i.e.*, it has no solution.

S31. Given system of equations is

$$x + y = 2 \quad \text{... (i)}$$

$$2x + 2y = 4 \quad \text{... (ii)}$$

Let $$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix},$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$

and $$B = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Now $$|A| = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 2 - 2 = 0$$

Therefore, given system of equations will have either no solution or infinite number of solutions.

Given system of equations will be consistent or inconsistent according as (adj A). $B = 0$ or (adj A).$B \neq 0$.

Let C be the matrix whose elements are cofactors of corresponding elements of A, then

$$A_{11} = 2$$
$$A_{12} = -2$$
$$A_{21} = -1$$
$$A_{22} = 1$$

$\therefore$ $$C = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}$$

$\therefore$ $$\text{adj } A = C^T = \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}$$

Now $$(\text{adj } A) \cdot B = \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4-4 \\ -4+4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \text{ (zero matrix)}$$

Hence given system of equations has infinite number of solutions.

Let $x = t$, then from equation (i), $y = 2 - t$.

Thus solutions of given system of equations is given by

$x = t$ and $y = 2 - t$, where t is an arbitrary number.

S32.

Let $$C = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$$

$\therefore$ $$AC = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -4+7+5 & 4-1-3 & 4-3-1 \\ -4+14-10 & 4-2+6 & 4-6+2 \\ -8-7+15 & 8+1-9 & 8+3-3 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 8I$$

$$A^{-1}(AC) = 8IA^{-1}$$

$$IC = 8A^{-1} \qquad [AA^{-1} = I \text{ and } IA^{-1} = A^{-1}]$$

$$A^{-1} = \frac{1}{8}C$$

$$\therefore \qquad A^{-1} = \frac{1}{8}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$$

Now, for X, we know that

$$AX = B$$

$$\Rightarrow \qquad X = A^{-1}B,$$

where, $A^{-1} = \frac{1}{8}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\therefore \qquad X = \frac{1}{8}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} -16+36+4 \\ -28+9+3 \\ 20-27-1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 24 \\ -16 \\ -8 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

$$\therefore \qquad x = 3, \quad y = -2 \quad \text{and} \quad z = -1.$$

S33. First find the product of given matrices and then premultiply both sides of the product by A^{-1} and obtain A^{-1}. Then, by using A^{-1} and concept of matrix method find x, y and z.

Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$

and $C = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$

We have to first find the product AC and then use it to solve the given system of linear equations.

So, $AC = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$

$$= \begin{bmatrix} -2-9+12 & 0-2+2 & 1+3-4 \\ 0+18-18 & 0+4-3 & 0-6+6 \\ -6-18+24 & 0-4+4 & 3+6-8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ We get $AC = I$

Now, premultiply both sides by A^{-1}, we get

$$A^{-1}AC = A^{-1} I$$

$\therefore \quad IC = A^{-1}$ $\quad [\because \; A^{-1}A = I \text{ and } A^{-1}I = A^{-1}]$

$\Rightarrow \quad C = A^{-1}$

$\therefore$ We observed that

$$A^{-1} = C = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$$

Now, we know that by matrix method, we have

$$X = A^{-1}B$$

$\Rightarrow \quad X = CB$ $\quad [\because \; A^{-1} = C]$

where $C = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2+0+2 \\ 9+2-6 \\ 6+1-4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}$$

$\therefore \quad x = 0, \; y = 5$ and $z = 3$.