Date: 9/10/2021

Q1. Evaluate : $(1 + i)^6 + (1 - i)^3$.

Q2. Find the value of a such that the sum of the squares of the roots of the equation $x^2 - (a - 2)x - (a + 1) = 0$ is least.

Q3. Evaluate:

$$\sum_{i=1}^{13} (i^n + i^{n+1}), \text{ where } n \in N.$$

Q4. Find the value of P such that the difference of the roots of the equation $x^2 - Px + 8 = 0$ is 2.

Q5. Find the value of $2x^4 + 5x^3 + 7x^2 - x + 41$. when $x = -2 - \sqrt{3}\,i$.

Q6. If $|z^2 - 1| = |z|^2 + 1$, then show that z lies on imaginary axis.

Q7. Locate the points for which $3 < |z| < 4$.

Q8. If $z = x + iy$, then show that $z\bar{z} + 2(z + \bar{z}) + b = 0$, where $b \in R$, represents a circle.

Q9. If z_1 and z_2 are two complex numbers such that $|z_1| = |z_2|$ and arg. (z_1) + arg. $(z_2) = \pi$, then show that $z_1 = -\bar{z}_2$.

Q10. If $\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = x + iy$, find (x, y).

Q11. If $\frac{(1+i)^2}{2-i} = x + iy$, then find the value of $x + y$.

Q12. Let z_1 and z_2 be two complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$. Then show that arg. (z_1) – arg. $(z_2) = 0$.

Q13. Let z_1 and z_2 be two complex numbers such that $\bar{z}_1 + i\bar{z}_2 = 0$ and arg. $(z_1 z_2) = \pi$. Then find arg (z_1).

Q14. If z and w are the two complex numbers such that $|zw| = 1$ and arg. (z) – arg. $(w) = \frac{\pi}{2}$, then show that $\bar{z}w = -i$.

Q15. If for complex numbers z_1 and z_2, arg. (z_1) – arg. $(z_2) = 0$, then show that $|z_1 - z_2| = |z_1| - |z_2|$.

Q16. If z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers, then find:

$$\text{arg.}\left(\frac{z_1}{z_4}\right) + \text{arg.}\left(\frac{z_2}{z_3}\right).$$

Q17. If arg. $(z - 1)$ = arg. $(z + 3i)$, then find $(x - 1) : y$, where $z = x + iy$.

Q18. Show that the complex number z satisfying the condition arg. $\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$ lies on a circle.

Q19. If $|z_1| = 1$, $(z_1 \neq -1)$ and $z_2 = \frac{z_1 - 1}{z_1 + 1}$, then show that the real part of z_2 is zero.

Q20. If $\frac{z-1}{z+1}$ is a purely imaginary number ($z \neq -1$), then find the value of $|z|$.

Q21. If $\left(\frac{1-i}{1+i}\right)^{100} = a + ib$, then find (a, b).

Q22. If the imaginary part of $\frac{2z+1}{iz+1}$ is -2, then show that the locus of the point representing z in the argand plane is straight line.

Q23. Solve the eqation $z^2 = \bar{z}$, where $z = x + iy$.

Q24. If $(x+iy)^{\frac{1}{3}} = a + ib$, where $x, y, a, b \in R$, show that $\frac{x}{a} - \frac{y}{b} = -2(a^2 + b^2)$.

Q25. Find the complex number satisfying the equation: $z + \sqrt{2} \cdot |(z+1)| + i = 0$.

Q26. Solve the system of equations: $\text{Re}(z^2) = 0$, $|z| = 2$.

Q27. Show that $\left|\frac{z-2}{z-3}\right| = 2$ represents a circle. Find its centre and radius.

Q28. If $a = \cos\theta + i\sin\theta$, find the value of $\frac{1+a}{1-a}$.

Q29. If $|z+1| = z + 2(1+i)$, then find z.

Q30. Solve the equation : $|z| = z + 1 + 2i$.

Q31. If $(1+i)z = (1-i)\bar{z}$, then show that $z = -i\bar{z}$.

Q32. If the real part of $\frac{\bar{z}+2}{\bar{z}-1}$ is 4, then show that the locus of the point representing z in the complex plane is a circle.

Q33. If a complex number z lies in the interior or on the boundary of a circle of radius 3 units and centre $(-4, 0)$, find the greatest and least values of $|z+1|$.

Q34. Find the value of k if for the complex numbers z_1 and z_2,

$$|1-\bar{z}_1 z_2|^2 - |z_1 - z_2|^2 = k(1-|z_1|^2)(1-|z_2|^2).$$

Q35. If z_1 and z_2 both satisfy $z + \bar{z} = 2|z-1|$ and $\arg.(z_1 - z_2) = \frac{\pi}{2}$, then find $\text{Im}(z_1 + z_2)$.

Q36. Write the complex number $z = \frac{1-i}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$ in polar form.

Q37. For a positive integer n, find the value of: $(1-i)^n\left(1-\frac{1}{i}\right)^n$.

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S1. Let $(1 + i)^6 = [(1 + i)^2]^3 = (1 + i^2 + 2i)^3 = (1 - 1 + 2i)^3 = 8i^3 = -8i$

and $(1 - i)^3 = 1 - i^3 - 3i + 3i^2 = 1 + i - 3i - 3 = -2 - 2i$

Therefore, $(1 + i)^6 + (1 - i)^3 = -8i - 2 - 2i = -2 - 10i.$

S2. Let α, β be the roots of the equation

Therefore, $\alpha + \beta = a - 2$ and $\alpha\beta = -(a + 1)$

$$(\alpha^2 + \beta^2) = (\alpha + \beta)^2 - 2\alpha\beta$$

$$= (a - 2)^2 + 2(a + 1)$$

$$= (a - 1)^2 + 5$$

Therefore, $\alpha^2 + \beta^2$ will be minimum if $(a - 1)^2 = 0$, *i.e.*, $a = 1$.

S3. Given $\sum_{i=1}^{13} (i^n + i^{n+1}) = (i + i^2 + i^3 + i^4 + \ldots + i^{13}) + (i^2 + i^3 + i^4 + \ldots + i^{14})$

$$= [(i - 1 - i + 1 + i - 1 - i + 1 + i - 1 - i + 1 - i)]$$

$$= + [(i - 1) + (-i + 1) + (i - 1) + (-i + 1) + (i - 1) + (-i + 1) - 1]$$

$$= i - 1.$$

S4. Let α, β be the roots of the equation $x^2 - Px + 8 = 0$.

Therefore $\alpha + \beta = P$ and $\alpha \cdot \beta = 8$

Now, $\alpha - \beta = \pm\sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$

Therefore, $2 = \pm\sqrt{P^2 - 32}$

$\Rightarrow$ $P^2 - 32 = 4$, *i.e.*, $P = \pm 6$.

S5. Let $x + 2 = -\sqrt{3}\, i \Rightarrow x^2 + 4x + 7 = 0$

Therefore, $2x^4 + 5x^3 + 7x^2 - x + 41 = (x^2 + 4x + 7)(2x^2 - 3x + 5) + 6$

$$= 0 \cdot (2x^2 - 3x + 5) + 6 = 6.$$

S6. Let $z = x + iy$

Then, $|z^2 - 1| = |z|^2 + 1$

$\Rightarrow$ $|x^2 - y^2 - 1 + i2xy| = |x + iy|^2 + 1$

$\Rightarrow$ $(x^2 - y^2 - 1)^2 + 4x^2y^2 = (x^2 + y^2 + 1)$

$\Rightarrow$ $4x^2 = 0$ *i.e.*, $x = 0$

Hence, z lies on y-axis.

S7. $|z| < 4 \Rightarrow x^2 + y^2 < 16$, which is the interior of circle with centre at origin and radius 4 units, and $|z| > 3 \Rightarrow x^2 + y^2 > 9$, which is exterior of circle with centre at origin and radius 3 units. Hence, $3 < |z| < 4$ is the portion between two circles $x^2 + y^2 = 9$ and $x^2 + y^2 = 16$.

S8. Let $z = x + iy$

Now, $z\bar{z} + 2(z + \bar{z}) + b = 0$

$\Rightarrow (x + iy)(x - iy) + 2(x + iy + x - iy) + b = 0$

$\Rightarrow x^2 + y^2 + 2x + b = 0$

which is the equation of a circle.

S9. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Now, $|z_1| = |z_2| \Rightarrow r_1 = r_2$

Also, $\arg.(z_1) + \arg.(z_2) = \pi$

$\Rightarrow \theta_1 + \theta_2 = \pi$

$\Rightarrow \theta_1 = \pi - \theta_2$

$\therefore z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$\Rightarrow z_1 = r_2[\cos(\pi - \theta_2) + i\sin(\pi - \theta_2)]$

$= r_2[-\cos\theta_2 + i\sin\theta_2)$

$= -r_2[\cos\theta_2 - i\sin\theta_2)$

$= -\bar{z}_2$

Hence, $z_1 = -\bar{z}_2$.

S10. Given, $\left(\frac{1+i}{1-i}\right)^3 = \frac{1+i^3+3i(1+i)}{1-i^3-3i(1-i)}$

$$= \frac{1-i+3i-3}{1+i-3i+3i^2}$$

$$= \frac{-2+2i}{-2-2i}$$

$$= \frac{2-2i}{2+2i}$$

$$= \frac{1-i}{1+i} \times \frac{1-i}{1-i}$$

$$= \frac{(1-i)^2}{1-i^2}$$

$$= \frac{1+i^2-2i}{1+1}$$

$$= \frac{1-1-2i}{2} = -i$$

Similarly, $\left(\frac{1-i}{1+i}\right)^3 = -\frac{1}{i} = \frac{i^2}{i} = i$

$\therefore$ $-i - i = x + iy$

$\Rightarrow$ $-2i = x + iy$

$x = 0, \quad y = -2$

Hence, (x, y) is $(0, -2)$.

S11. Given, $\frac{(1+i)^2}{2-i} = x + iy$

$\Rightarrow$ $\frac{1+i^2+2i}{2-i} = x + iy$

$\Rightarrow$ $\frac{1-1+2i}{2-i} = x + iy$

$\Rightarrow$ $\frac{2i}{2-i} \times \frac{2+i}{2+i} = x + iy$

$\Rightarrow$ $\frac{4i+2i^2}{4+1} = x + iy$

$\Rightarrow$ $\frac{4i-2}{5} = x + iy$

$\Rightarrow$ $-\frac{2}{5} + \frac{4i}{5} = x + iy$

Comparing real and imaginary parts on both sides, we get

$$x = -\frac{2}{5}, \quad y = \frac{4}{5}$$

Now, $x + y = -\frac{2}{5} + \frac{4}{5} = \frac{2}{5}$.

S12. Let $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$

and $z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$

where, $r_1 = |z_1|, \quad \text{arg. } (z_1) = \theta_1$

$r_2 = |z_2|, \quad \text{arg. } (z_2) = \theta_2$

We have, $|z_1 + z_2| = |z_1| + |z_2|$

$= |r_1(\cos\theta_1 + i\sin\theta_1) + r_2(\cos\theta_2 + i\sin\theta_2)| = r_1 + r_2$

$= r_1^2 + r_2^2 + 2r_1 r_2\cos(\theta_1 - \theta_2) = (r_1 + r_2)^2 \Rightarrow \cos(\theta_1 - \theta_2) = 1$

$\Rightarrow \theta_1 - \theta_2 = 0$, *i.e.*, arg. z_1 = arg. z_2.

S13. Given, $\bar{z}_1 + i\bar{z}_2 = 0$

$\Rightarrow$ $z_1 = iz_2$, *i.e.*, $z_2 = -iz_1$

Thus, arg. $(z_1 z_2)$ = arg. z_1 + arg. $(-iz_1) = \pi$

$\Rightarrow$ arg. $(-iz_1^2) = \pi$

$\Rightarrow$ arg. $(-i)$ + arg. $(z_1^2) = \pi$

$\Rightarrow$ arg. $(-i)$ + 2 arg. $(z_1) = \pi$

$\Rightarrow$ $\frac{-\pi}{2}$ + 2 arg. $(z_1) = \pi$

$\Rightarrow$ arg. $(z_1) = \frac{3\pi}{4}$.

S14. Let $z = r_1(\cos\theta_1 + i\sin\theta_1)$

and $w = r_2(\cos\theta_2 + i\sin\theta_2)$

Now, $|zw| = 1 \Rightarrow r_1 r_2 = 1$

Also, arg. (z) – arg. $(w) = \frac{\pi}{2} \Rightarrow \theta_1 - \theta_2 = \frac{\pi}{2}$

$\arg.\left(\frac{z}{w}\right) = \frac{\pi}{2}$

Now, L.H.S. $= \bar{z}w$

$= r_1(\cos\theta_1 - i\sin\theta_1)\, r_2(\cos\theta_2 + i\sin\theta_2)$

$= r_1 r_2[\cos(\theta_2 - \theta_1) + i\sin(\theta_2 - \theta_1)]$

$= 1\left[\cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)\right]$

$= 0 - i = -i$.

S15. Let $z_1 = x_1 + iy_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

and $z_2 = x_2 + iy_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\therefore$ arg. (z_1) = arg. (z_2)

$\Rightarrow$ $\theta_1 = \theta_2$

$\therefore$ $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

and $z_2 = r_2(\cos\theta_1 + i\sin\theta_1)$

$$\text{L.H.S.}\quad |z_1 - z_2| = \sqrt{(r_1\cos\theta_1 - r_2\cos\theta_1)^2 + (r_1\sin\theta_1 - r_1\sin\theta_1)^2}$$

$$= \sqrt{r_1^2 + r_2^2 - 2r_1r_2}$$

$$= r_1 - r_2$$

$$= |z_1| - |z_2| = \text{R.H.S.}$$

S16. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$z_2 = r_1(\cos\theta_1 - i\sin\theta_1)$

$= r_1[\cos(-\theta_1) + i\sin(-\theta_1)]$

and $z_3 = r_2(\cos\theta_2 + i\sin\theta_2)$

$z_4 = r_2(\cos\theta_2 - i\sin\theta_2)$

$= r_2[\cos(-\theta_2) + i\sin(-\theta_2)]$

Now, $\arg.\left(\frac{z_1}{z_4}\right) + \arg.\left(\frac{z_2}{z_3}\right) = \arg.(z_1) - \arg.(z_4) + \arg.(z_2) - \arg.(z_3)$

$= \theta_1 - (-\theta_2) + (-\theta_1) - (\theta_2)$

$= 0.$ **[Hence Proved]**

S17. Let, $\arg.(z - 1) = \arg.(z + 3i)$

$z = x + iy$ (Given)

$\therefore$ $\arg.(x + iy - 1) = \arg.(x + iy + 3i)$

$\Rightarrow$ $\arg.(x - 1 + iy) = \arg.[x + i(y + 3)]$

$\Rightarrow$ $\tan\theta_1 = \frac{y}{x-1}$ and $\tan\theta_2 = \frac{y+3}{x}$ $(\because \tan\theta_1 = \tan\theta_2)$

$\therefore$ $\frac{y}{x-1} = \frac{y+3}{x}$

$\Rightarrow$ $xy = xy - 3 + 3x - y$

$\Rightarrow$ $0 = 3x - y - 3$

$\Rightarrow$ $y = 3x - 3$

$\therefore$ $\frac{x-1}{y} = \frac{x-1}{3x-3} = \frac{x-1}{3(x-1)} = \frac{1}{3}.$

S18. Let $z = x + iy$

$$\arg.\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$$

$\Rightarrow$ $\arg.(z - 1) - \arg.(z + 1) = \frac{\pi}{4}$

$\Rightarrow$ arg. $(x + iy - 1)$ – arg. $(x + iy + 1) = \frac{\pi}{4}$

$$\Rightarrow \quad \tan^{-1}\left(\frac{y}{x-1}\right) - \tan^{-1}\left(\frac{y}{x+1}\right) = \frac{\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{\frac{y}{x-1} - \frac{y}{x+1}}{1+\left(\frac{y}{x-1}\right)\left(\frac{y}{x+1}\right)}\right] = \frac{\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{yx + y - yx + y}{x^2 - 1 + y^2}\right] = \frac{\pi}{4}$$

$$\Rightarrow \quad \frac{2y}{x^2 + y^2 - 1} = \tan\frac{\pi}{4}$$

$$\Rightarrow \quad 2y = x^2 + y^2 - 1$$

$$\Rightarrow \quad x^2 + y^2 - 2y - 1 = 0$$

which is a circle.

S19. Let $z_1 = x + iy$

$$|z_1| = \sqrt{x^2 + y^2} = 1$$

$$\Rightarrow \quad x^2 + y^2 = 1$$

Again, $z_2 = \frac{z_1 - 1}{z_1 + 1}$

$$= \frac{x + iy - 1}{x + iy + 1} = \frac{(x-1) + iy}{(x+1) + iy}$$

$$= \frac{[(x-1) + iy][(x+1) - iy]}{[x + 1 + iy][(x+1) - iy]}$$

$$= \frac{[x^2 - 1 + y^2] + i\,[y\,(x+1) - y\,(x-1)]}{(x+1)^2 - i^2 y^2}$$

$$= \frac{(x^2 - 1 + y^2) + i\,(y + y)}{(x+1)^2 + y^2}$$

Now, Real part of $z_2 = \frac{x^2 + y^2 - 1}{(x+1)^2 + y^2}$

$$= \frac{1-1}{(x+1)^2 + y^2} = 0$$

[Hence Proved.]

S20. Let $z = x + iy$

$$\therefore \quad \frac{z-1}{z+1} = \frac{x+iy-1}{x+iy+1}$$

$$= \left[\frac{(x-1)+iy}{(x+1)+iy}\right]\left[\frac{(x+1)-iy}{(x+1)-iy}\right]$$

$$= \frac{[(x^2-1)-i^2y^2]+i\,[y\,(x+1)-y\,(x-1)]}{(x+1)^2-i^2y^2}$$

$$= \frac{(x^2-1+y^2)+i\,(2y)}{(x+1)^2+y^2}$$

It is imaginary if $x^2 + y^2 - 1 = 0$

$\Rightarrow \quad x^2 + y^2 = 1$

Now, $|z| = \sqrt{x^2+y^2} = \sqrt{1} = 1.$

S21. Given, $\left(\frac{1-i}{1+i}\right)^{100} = a + ib$

$\Rightarrow \quad \left(\frac{1-i}{1+i}\times\frac{1-i}{1-i}\right)^{100} = a + ib$

$\Rightarrow \quad \left[\frac{(1-i)^2}{1-i^2}\right]^{100} = a + ib$

$\Rightarrow \quad \left[\frac{1+i^2-2i}{1+1}\right]^{100} = a + ib$

$\Rightarrow \quad \left[\frac{1-1-2i}{2}\right]^{100} = a + ib$

$\Rightarrow \quad (-i)^{100} = a + ib$

$\Rightarrow \quad i^{100} = a + ib$

$\Rightarrow \quad (i^2)^{50} = a + ib$

$\Rightarrow \quad (-1)^{50} = a + ib$

$\Rightarrow \quad 1 = a + ib$

$\therefore \quad a = 1, \quad b = 0$

or $(a, b) = (1, 0).$

S22. Let $z = x + iy$

Then $\frac{2z+1}{iz+1} = \frac{2(x+iy)+1}{i(x+iy)+1} = \frac{(2x+1)+i2y}{(1-y)+ix}$

$$= \frac{\{(2x+1)+i\,2y\}}{\{(1-y)+ix\}} \times \frac{\{(1-y)-ix\}}{\{(1-y)-ix\}}$$

$$= \frac{(2x+1-y)+i\,(2y-2y^2-2x^2-x)}{1+y^2-2y+x^2}$$

Thus, $\operatorname{Im}\left(\frac{2z+1}{iz+1}\right) = \frac{2y-2y^2-2x^2-x}{1+y^2-2y+x^2}$

But $\operatorname{Im}\left(\frac{2z+1}{iz+1}\right) = -2$ (Given)

So, $\frac{2y-2y^2-2x^2-x}{1+y^2-2y+x^2} = -2$

$\Rightarrow$ $2y - 2y^2 - 2x^2 - x = -2 - 2y^2 + 4y - 2x^2$

i.e., $x + 2y - 2 = 0$

which is the equation of a line.

S23. Given, $z^2 = \bar{z} \;\Rightarrow\; x^2 - y^2 + i\,2xy = x - iy$

Therefore, $x^2 - y^2 = x$ ... (i)

and $2xy = -y$ ... (ii)

From Eq. (ii), we have

$$y = 0 \quad \text{or} \quad x = -\frac{1}{2}$$

When $y = 0$, From Eq. (i), we get

$$x^2 - x = 0 \qquad i.e., \quad x = 0 \quad \text{or} \quad x = 1.$$

When $x = -\frac{1}{2}$, from Eq. (i), we get

$$y^2 = \frac{1}{4} + \frac{1}{2} \quad \text{or} \quad y^2 = \frac{3}{4}, \quad i.e., \quad y = \pm\frac{\sqrt{3}}{2}$$

Hence, the solutions of the given equation are

$$0 + i0, \quad 1 + i0, \quad -\frac{1}{2} + i\,\frac{\sqrt{3}}{2}, \quad -\frac{1}{2} - i\,\frac{\sqrt{3}}{2}$$

S24. Given, $(x+iy)^{\frac{1}{3}} = a + ib$

$\Rightarrow$ $x + iy = (a + ib)^3$

i.e., $x + iy = a^3 + i^3b^3 + 3iab\,(a + ib)$

$$= a^3 - ib^3 + i\,3a^2b - 3ab^2$$

$$= a^3 - 3ab^2 + i(3a^2b - b^3)$$

$\Rightarrow$ $$x = a^3 - 3ab^2 \quad \text{and} \quad y = 3a^2b - b^3$$

Thus, $$\frac{x}{a} = a^2 - 3b^2 \quad \text{and} \quad \frac{y}{b} = 3a^2 - b^2$$

So $$\frac{x}{a} - \frac{y}{b} = a^2 - 3b^2 - 3a^2 + b^2 = -2a^2 - 2b^2 = -2(a^2 + b^2).$$

S25. Let $$z = x + iy$$

$$z + 1 = (x + 1) + iy$$

$$|(z + 1)| = \sqrt{(x+1)^2 + y^2}$$

$\therefore$ Equation is

$$x + iy + \sqrt{2}\sqrt{(x+1)^2 + y^2} + i = 0$$

$$x + i(y + 1) = \sqrt{2}\sqrt{(x+1)^2 + y^2}$$

$$x = \sqrt{2}\sqrt{(x+1)^2 + y^2}, \quad y + 1 = 0$$

$$x^2 = 2[(x + 1)^2 + y^2], \quad y = -1$$

$$x^2 = 2[(x + 1)^2 + 1]$$

$\Rightarrow$ $$x^2 = 2[x^2 + 1 + 2x + 1]$$

$\Rightarrow$ $$x^2 = 2x^2 + 4x + 4$$

$\Rightarrow$ $$x^2 + 4x + 4 = 0$$

$\Rightarrow$ $$(x + 2)^2 = 0$$

$$x = -2$$

Hence, $$x + iy = -2 - i.$$

S26. Let $$z = x + iy$$

$$z^2 = (x + iy)^2$$

$$z^2 = x^2 + i^2y^2 + 2xyi$$

$$z^2 = x^2 - y^2 + 2xyi$$

Now, $$\text{Re}(z^2) = 0$$

$\Rightarrow$ $$x^2 - y^2 = 0$$

$\Rightarrow$ $$x^2 = y^2$$

$\Rightarrow$ $$y = \pm x$$

$$|z| = 2 \Rightarrow \sqrt{(x^2+y^2)} = 2$$
$$\Rightarrow x^2 + y^2 = 4$$
$$\Rightarrow 2x^2 = 4, \quad x^2 = 2 \Rightarrow x = \pm\sqrt{2}$$

When $x = \sqrt{2}$, then $y = \pm\sqrt{2}$

$\therefore$ $x + iy = \sqrt{2} \pm \sqrt{2}\, i$

When $x = -\sqrt{2}$, then $y = \pm\sqrt{2}$

$\therefore$ $x + iy = -\sqrt{2} \pm \sqrt{2}\, i.$

S27. Let $z = x + iy$

$$\left|\frac{z-2}{z-3}\right| = 2$$

$$\left|\frac{x+iy-2}{x+iy-3}\right| = 2$$

$$\Rightarrow \sqrt{(x-2)^2+y^2} = 2\sqrt{(x-3)^2+y^2}$$

$$\Rightarrow x^2 + 4 - 4x + y^2 = 4\,[x^2 + 9 - 6x + y^2]$$

$$\Rightarrow 3x^2 + 3y^2 - 20x + 32 = 0$$

$$\Rightarrow x^2 + y^2 - \frac{20}{3}x + \frac{32}{3} = 0$$

$$\therefore \quad 2g = -\frac{20}{3}, \quad 2f = 0, \quad c = \frac{32}{3}$$

Centre is $(-g, -f) = \left(\frac{10}{3}, 0\right)$

and radius $= \sqrt{\left(\frac{-10}{3}\right)^2 + (0)^2 - \frac{32}{3}}$

$$= \sqrt{\frac{100-96}{9}} = \sqrt{\frac{4}{9}} = \frac{2}{3}.$$

S28.
$$\frac{1+a}{1-a} = \frac{1+\cos\theta + i\sin\theta}{1-\cos\theta - i\sin\theta}$$

$$= \frac{2\cos^2\frac{\theta}{2} + i\cdot 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2} - i\cdot 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= \frac{2\cos\frac{\theta}{2}\left[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right]}{2\sin\frac{\theta}{2}\left[\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right]}$$

$$= -i\cot\frac{\theta}{2}\frac{\left[\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right]}{\left[\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right]} = -i\cot\frac{\theta}{2}.$$

S29. Let $z = x + iy$

$\therefore$ $z + 1 = (x + 1) + iy$

$|z + 1| = \sqrt{(x+1)^2 + y^2}$

$\therefore$ $\sqrt{(x+1)^2+y^2} = x + iy + 2(1 + i)$

$\Rightarrow$ $(x + 1)^2 + y^2 = (x + 2)^2 + [i(y + 2)]^2 + 2i(x + 2)(y + 2)$

$\Rightarrow$ $x^2 + 1 + 2x + y^2 = x^2 + 4 + 4x + i^2(y^2 + 4 + 4y) + 2i(xy + 2y + 2x + 4)$

$\Rightarrow$ $x^2 + y^2 + 2x + 1 = x^2 + 4x + 4 - y^2 - 4 - 4y) + 2i(xy + 2y + 2x + 4)$

$\Rightarrow$ $2y^2 + 2x + 1 = 4x - 4y + 2i(xy + 2y + 2x + 4)$

Comparing real and imaginary parts, we get

$2y^2 + 2x + 1 = 4x - 4y$

$2y^2 - 2x + 4y + 1 = 0$

and $2(x + 2)(y + 2) = 0$

$\Rightarrow$ $x = -2, \quad y = -2$

When, $y = -2, \quad 8 + 2x + 1 = 4x + 8$

$\Rightarrow$ $2x + 1 = 4x$

$\Rightarrow$ $2x = 1 \quad \Rightarrow \quad x = \frac{1}{2}$

$\therefore$ $z = x + iy = \frac{1}{2} - 2i$

When $x = -2$

$2y^2 - 2x + 4y + 1 = 0$

$2y^2 + 4 + 4y + 1 = 0$

$2y^2 + 4y + 5 = 0$. Not possible.

Hence, $z = \frac{1}{2} - 2i.$

S30. Let $z = x + iy \quad (x \in R, y \in R)$

$|z| = \sqrt{x^2 + y^2}$

$\therefore$ $|z| = z + 1 + 2i$

$\Rightarrow$ $\sqrt{x^2+y^2} = x + iy + 1 + 2i$

$\Rightarrow$ $x^2 + y^2 = (x + 1)^2 + [(i(y + 2)]^2 + 2i(x + 1)(y + 2)$

$\Rightarrow$ $x^2 + y^2 = x^2 + 1 + 2x + i^2(y^2 + 4 + 4y) + 2i(x + 1)(y + 2)$

$\Rightarrow$ $x^2 + y^2 = x^2 + 2x + 1 - y^2 - 4 - 4y + 2i(x + 1)(y + 2)$

$2y^2 = 2x - 4y - 3 + 2i(x + 1)(y + 2)$

Comparing real and imaginary parts, we get

$$2y^2 = 2x - 4y - 3$$

and $\quad 2(x + 1)(y + 2) = 0$

$\Rightarrow \quad x + 1 = 0, \quad y + 2 = 0$

$\Rightarrow \quad x = -1, \quad y = -2$

When, $\quad x = -1, \quad 2y^2 = 2x - 4y - 3$

$$2y^2 = -2 - 4y - 3$$

$$2y^2 + 4y + 5 = 0$$

$$y = \frac{-4 \pm \sqrt{16 - 40}}{4} \notin R$$

Now, let $\quad y = -2$

$$2y^2 = 2x - 4y - 3$$

$\Rightarrow \quad 8 = 2x + 8 - 3$

$\Rightarrow \quad 0 = 2x - 3 \quad = \quad x = \dfrac{3}{2}$

Now, $\quad z = x + iy$

$$= \frac{3}{2} - 2i \text{ is the solution.}$$

S31. Let $\quad (1 + i)z = (1 - i)\overline{z}$

$\Rightarrow \quad z = \dfrac{1-i}{1+i}\overline{z}$

$\Rightarrow \quad z = \dfrac{1-i}{1+i} \times \dfrac{1-i}{1-i}\overline{z}$

$$= \frac{(1-i)^2}{1-i^2}\overline{z}$$

$$= \frac{1+i^2-2i}{2}\overline{z}$$

$$= \frac{1-1-2i}{2}\overline{z}$$

$$= \frac{-2i}{2}\overline{z}$$

$$= -i\overline{z} = \text{R.H.S.}$$

S32. Let $z = x + iy$

Now,
$$\frac{\bar{z}+2}{\bar{z}-1} = \frac{x-iy+2}{x-iy-1}$$
$$= \frac{(x+2)-iy}{(x-1)-iy}$$
$$= \frac{[(x+2)-iy][(x-1)+iy]}{[(x-1)-iy][(x-1)+iy]}$$
$$= \frac{(x+2)(x-1)-i^2y^2+iy(x+2)-iy(x-1)}{(x-1)^2+y^2}$$

Now real part $\frac{(x+2)(x-1)+y^2}{(x-1)^2+y^2} = 4$

$\Rightarrow \quad \frac{x^2+2x-x-2+y^2}{(x-1)^2+y^2} = 4$

$\Rightarrow \quad \frac{x^2+y^2+x-2}{x^2+y^2+1-2x} = 4$

$\Rightarrow \quad x^2 + y^2 + x - 2 = 4x^2 + 4y^2 + 4 - 8x$

$\Rightarrow \quad 3x^2 + 3y^2 - 9x + 6 = 0$

which represents, a circle.

Hence, $z = x + iy$ lies on a circle.

S33. Distance of the point representing z from the centre of the circle is
$$|z-(-4+i0)| = |z+4|$$
According to given condition $|z+4| \le 3$.

Now, $|z+1| = |z+4-3|$
$$\le |z+4| + |-3|$$
$$\le 3 + 3 = 6$$
Therefore, greatest value of $|z+1|$ is 6.

Since, least value of the modulus of a complex number is zero, hence, the least value of $|z+1| = 0$.

S34.
$$\text{L.H.S.} = |1-\bar{z}_1z_2|^2 - |z_1-z_2|^2$$
$$= (1-\bar{z}_1z_2)(\overline{1-\bar{z}_1z_2}) - (z_1-z_2)(\overline{z_1-z_2})$$
$$= (1-\bar{z}_1z_2)(1-z_1\bar{z}_2) - (z_1-z_2)(\bar{z}_1-\bar{z}_2)$$
$$= 1+\bar{z}_1z_1z_2\bar{z}_2 - z_1\bar{z}_1 - z_2\bar{z}_2$$
$$= 1+|z_1|^2\cdot|z_2|^2 - |z_1|^2 - |z_2|^2$$
$$= (1-|z_1|^2)\cdot(1-|z_2|^2)$$

$$\text{R.H.S.} = k(1-|z_1|^2)(1-|z_2|^2)$$

$$\Rightarrow \quad k = 1$$

Hence, equating L.H.S. and R.H.S., we get $k = 1$.

S35. Let $z = x + iy, \quad z_1 = x_1 + iy_1 \quad \text{and} \quad z_2 = x_2 + iy_2$

Since, $z + \overline{z} = 2|z - 1|$

$$\Rightarrow \quad (x + iy) + (x - iy) = 2|x - 1 + iy|$$

$$\Rightarrow \quad 2x = 1 + y^2 \qquad \ldots \text{(i)}$$

Since, z_1 and z_2 both satisfy Eq. (i), we have

$$\Rightarrow \quad 2(x_1 - x_2) = (y_1 + y_2)(y_1 - y_2)$$

$$\Rightarrow \quad 2 = (y_1 + y_2)\left(\frac{y_1 - y_2}{x_1 - x_2}\right) \qquad \ldots \text{(ii)}$$

Again, $z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$

Therefore $\tan\theta = \dfrac{y_1 - y_2}{x_1 - x_2}$, where $\theta = \arg.(z_1 - z_2)$

$$\Rightarrow \quad \tan\frac{\pi}{4} = \frac{y_1 - y_2}{x_1 - x_2} \qquad \left(\text{since } \theta = \frac{\pi}{4}\right)$$

i.e., $1 = \dfrac{y_1 - y_2}{x_1 - x_2}$

From Eq. (ii), we get

$$2 = y_1 + y_2, \qquad i.e., \quad \text{Im}(z_1 + z_2) = 2.$$

S36. Let $z = \dfrac{1-i}{\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}}$

$$= \frac{-(-1+i)}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$$

$$= \frac{-\sqrt{2}\left(\frac{-1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$$

$$= \frac{-\sqrt{2}\left[\cos\left(\pi - \frac{\pi}{4}\right) + i\sin\left(\pi - \frac{\pi}{4}\right)\right]}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$$

$$= \frac{-\sqrt{2}\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right]}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$$

$$= -\sqrt{2}\left[\cos\left(\frac{3\pi}{4} - \frac{\pi}{3}\right) + i\sin\left(\frac{3\pi}{4} - \frac{\pi}{3}\right)\right]$$

$$= -\sqrt{2}\left[\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right].$$

S37. Given expression $(1-i)^n\left(1-\frac{1}{i}\right)^n = (1-i)^n\left(\frac{i-1}{i}\right)^n$

$$= \frac{1}{i^n}(-1)^n (1-i)^n (1-i)^n$$

$$= \frac{1}{i^n}(-1)^n [(1-i)^2]^n$$

$$= \frac{(-1)^n}{i^n}[1+i^2-2i]^n$$

$$= \frac{(-1)^n}{i^n}[1-1-2i]^n$$

$$= \left(\frac{i^2}{i}\right)^n \times (-2i)^n$$

$$= (i)^n\ (-1)^n\ (2^n)\ (i^n)$$

$$= (i^n)\ (i^2)^n\ 2^n \cdot i^n$$

$$= (i^2)^n\ (i^2)^n\ 2^n$$

$$= (i^2)^{2n} 2^n$$

$$= (-1)^{2n} \cdot 2^n = 2^n.$$