Date: 9/10/2021

Q1. Give an example of a statement $P(n)$ which is true for all $n \geq 4$ but $P(1)$, $P(2)$ and $P(3)$ are not true. Justify your answer.

Q2. Give an example of a statement $P(n)$ which is true for all n. Justify your answer.

Q3. Prove the statement by the Principle of Mathematical Induction:

$$\sum_{t=1}^{n-1} t(t+1) = \frac{n(n-1)(n+1)}{3}, \qquad \text{for all natural numbers } n \geq 2.$$

Q4. Prove the statement by the Principle of Mathematical Induction:

$$\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right) = \frac{n+1}{2n}, \qquad \text{for all natural numbers, } n \geq 2.$$

Q5. Define the sequence $a_1, a_2, a_3, \ldots$ as follows:

$a_1 = 2$, $a_n = 5 \,.\, a_{n-1}$, for all natural numbers $n \geq 2$.

(i) Write the first four terms of the sequece.

(ii) Use the Principle of Mathematical Induction to show that the terms of the sequence satisfy the formula $a_n = 2.5^{n-1}$ for all natural numbers.

Q6. Prove that for all $n \in N$:

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos(\alpha+(n-1)\beta) = \frac{\cos\left(\alpha+\left(\frac{n-1}{2}\right)\beta\right)\sin\left(\frac{n\beta}{2}\right)}{\sin\frac{\beta}{2}}.$$

Q7. A sequence $d_1, d_2, d_3, \ldots$ is defined by letting $d_1 = 2$ and $d_k = \frac{d_{k-1}}{k}$ for all natural numbers $k \geq 2$. Show that $d_n = \frac{2}{n!}$ for all $n \in N$.

Q8. A sequence $b_0, b_1, b_2, \ldots$ is defined by letting $b_0 = 5$ and $b_k = 4 + b_{k-1}$ for all natural numbers k. Show that $b_n = 5 + 4n$ for all natural numbers n using Mathematical Induction.

Q9. Use the principle of Mathematical Induction in the following question:

A sequence $a_1, a_2, a_3, \ldots$ is defined by $a_1 = 3$ and $a_k = 7a_{k-1}$ for all natural numbers $k \geq 2$. Show that $a_n = 3.7^{n-1}$ for all natural numbers.

Q10. Prove that, $\cos\theta \cos 2\theta \cos 2^2\theta \ldots \cos 2^{n-1}\theta = \frac{\sin 2^n\theta}{2^n \sin\theta}$, for all $n \in N$.

Q11. Prove the statement by the Principle of Mathematical Induction: $n(n^2 + 5)$ is divisible by 6, for each natural number n.

Q12. Prove the statement by the Principle of Mathematical Induction: $n^3 - n$ is divisible by 6, for each natural number $n \geq 2$.

Q13. Prove the statement by the Principle of Mathematical Induction: For any natural number n, $7^n - 2^n$ is divisible by 5.

Q14. Prove the statement by the Principle of Mathematical Induction: $3^{2n} - 1$ is divisible by 8, for all natural number n.

Q15. Prove the statement by the Principle of Mathematical Induction: $n^3 - 7n + 3$ is divisible by 3, for all natural number n.

Q16. Prove the statement by the Principle of Mathematical Induction: $4^n - 1$ is divisible by 3, for each natural number n.

Q17. Prove the statement by the Principle of Mathematical Induction: $2^{2n} - 1$ is divisible by 3.

Q18. Prove that number of subsets of a set containing n distinct elements is 2^n, for all $n \in N$.

Q19. Show that $\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}$ is a natural number for all $n \in N$.

Q20. Prove that:

$$\sin\theta + \sin 2\theta + \sin 3\theta + ... + \sin n\theta = \frac{\frac{\sin n\theta}{2} \sin\frac{(n+1)}{2}\theta}{\sin\frac{\theta}{2}}, \text{ for all } n \in N.$$

Q21. Prove that: $\frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n} > \frac{13}{24}$ for all natural number $n > 1$.

Q22. Prove the statement by the Principle of Mathematical Induction:

$$\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{n}} \text{ for all natural number } n \geq 2.$$

Q23. Prove the statement by the Principle of Mathematical Induction: $2n < (n + 2)!$ for all natural number n.

Q24. $2n + 1 < 2^n$, for all natural numbers $n \geq 3$.

Q25. Prove the statement by the Principle of Mathematical Induction: $n^2 < 2^n$ for all natural number $n \geq 5$.

Q26. Prove by induction that for all natural number n:

$$\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + ... + \sin(\alpha + (n-1)\beta) = \frac{\sin\left(\alpha + \frac{n-1}{2}\beta\right)\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}.$$

Q27. Show by the Principle of Mathematical Induction that the sum S_n of the n term of the series $1^2 + 2 \times 2^2 + 3^2 + 2 \times 4^2 + 5^2 + 2 \times 6^2$ ... is given by

$$S_n = \begin{cases} \frac{n(n+1)^2}{2}, & \text{if } n \text{ is even} \\ \frac{n^2(n+1)}{2}, & \text{if } n \text{ is odd} \end{cases}.$$

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S1. Consider the statement

$$P(n) : 2n < n!$$

For $n = 1$, $2 \times 1 < 1! \Rightarrow 2 < 1$ (False)

For $n = 2$, $2 \times 2 < 2! \Rightarrow 4 < 2$ (False)

For $n = 3$, $2 \times 3 < 3! \Rightarrow 6 < 6$ (False)

For $n = 4$, $2 \times 4 < 4! \Rightarrow 8 < 24$ (True)

For $n = 5$, $2 \times 5 < 5! \Rightarrow 10 < 120$ (True)

S2. $P(n)$: $1 + 2 + 3 + ... + n = \dfrac{n(n+1)}{2}$

For $n = 1$, $1 = \dfrac{1(1+1)}{2} = 1$ (True).

$n = 2$, $1 + 2 = \dfrac{2(2+1)}{2} = 3$ (True).

$n = 3$, $1 + 2 + 3 = \dfrac{3(3+1)}{2} = 6$ (True).

S3. Let the given statement be $P(n)$, be given as

$$P(n) : \sum_{t=1}^{n-1} t(t+1) = \frac{n(n-1)(n+1)}{3}, \text{ for all natural numbers } n \geq 2.$$

We observe that

$$P(2) : \sum_{t=1}^{2-1} t(t+1) = \sum_{t=1}^{1} t(t+1) = 1.2 = \frac{1.2.3}{3}$$

$$= \frac{2.(2-1)(2+1)}{3}$$

Thus, $P(n)$ in true for $n = 2$.

Assume that $P(n)$ is true for $n = k \in \boldsymbol{N}$.

i.e., $P(k) : \sum_{t=1}^{k-1} t(t+1) = \dfrac{k(k-1)(k+1)}{3}$

To prove that $P(k+1)$ is true, we have

$$\sum_{t=1}^{(k+1-1)} t(t+1) = \sum_{t=1}^{k} t(t+1)$$

$$= \sum_{t=1}^{k-1} t(t+1) + k(k+1) = \frac{k(k-1)(k+1)}{3} + k(k+1)$$

$$= k(k+1)\left[\frac{k-1+3}{3}\right] = \frac{k(k+1)(k+2)}{3}$$

$$= \frac{(k+1)((k+1)-1)((k+1)+1)}{3}$$

Thus, $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the Principle of Mathematical Induction, $P(n)$ is true for all natural numbers, $n \geq 2$.

S4. Let the given statement be $P(n)$, *i.e.*,

$$P(n): \quad \left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right) = \frac{n+1}{2n}, \text{ for all natural numbers } n \geq 2.$$

We observe that $P(2)$ is true, since

$$\left(1-\frac{1}{2^2}\right) = 1-\frac{1}{4} = \frac{4-1}{4} = \frac{3}{4} = \frac{2+1}{2\times 2}$$

Assume that $P(n)$ is true for $k \in \mathbf{N}$, *i.e.*,

$$P(k): \quad \left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{k^2}\right) = \frac{k+1}{2k}$$

Now, to prove that $P(k+1)$ is true, we have

$$\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{k^2}\right)\cdot\left(1-\frac{1}{(k+1)^2}\right) = \frac{k+1}{2k}\left(1-\frac{1}{(k+1)^2}\right) = \frac{k^2+2k}{2k(k+1)} = \frac{(k+1)+1}{2(k+1)}$$

Thus, $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, by the Principle of Mathematical Induction, $P(n)$ is true for all natural numbers, $n \geq 2$.

S5. (i) We have

$a_1 = 2$

$a_2 = 5a_{2-1} = 5a_1 = 5.2 = 10$

$a_3 = 5a_{3-1} = 5a_2 = 5.10 = 50$

$a_4 = 5a_{4-1} = 5a_3 = 5.50 = 250$

(ii) Let $P(n)$ be the statement, *i.e.*,

$P(n): a_n = 2.5^{n-1}$ for all natural numbers. We observe that $P(1)$ is true.

Assume that $P(n)$ is true for some natural number k, *i.e.*, $P(k): a_k = 2.5^{k-1}$.

Now to prove that $P(k + 1)$ is true, we have

$$P(k+1) : a_{k+1} = 5 \,.\, a_k = 5 \,.\, (2.5^{k-1})$$
$$= 2.5^k = 2.5^{(k+1)-1}$$

Thus, $P(k + 1)$ is true, whenever $P(k)$ is true.

Hence, by the Principle of Mathematical Induction, $P(n)$ is true for all natural numbers, $n \geq 2$.

S6. Let $P(n)$: $\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \ldots + \cos(\alpha + (n-1)\beta)$

$$= \frac{\cos\left(\alpha + \left(\frac{n-1}{2}\right)\beta\right)\sin\left(\frac{n\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

Basic step:

To prove: $P(1)$ is true

Proof: For $n = 1$,

$$\text{R.H.S.} = \frac{\cos\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right]\sin\left(\frac{n\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

$$= \frac{\cos(\alpha)\sin\frac{\beta}{2}}{\sin\frac{\beta}{2}} = \cos\alpha = T_1$$

Thus, $P(1)$ is true.

Induction Step: Given, $P(k)$ is true

i.e., $\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \ldots + \cos[(\alpha + (k-1)\beta)]$

$$= \frac{\cos\left[\alpha + \left(\frac{k-1}{2}\right)\beta\right]\sin\left(\frac{k\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

To prove: $P(k + 1)$ is true

i.e., $\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \ldots + \cos(\alpha + k\beta)$

$$= \frac{\cos\left(\alpha + \frac{k\beta}{2}\right)\sin\left[(k+1)\left(\frac{\beta}{2}\right)\right]}{\sin\frac{\beta}{2}}$$

Proof: LHS $= \cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \ldots + \cos(\alpha + k\beta)$

$$= \frac{\cos\left[\alpha + \left(\frac{k-1}{2}\right)\beta\right]\sin\left(\frac{k\beta}{2}\right)}{\sin\frac{\beta}{2}} + \cos(\alpha + k\beta)$$

$$= \frac{\cos\left(\alpha + \frac{k-1}{2}\beta\right)\sin\frac{k\beta}{2} + \cos(\alpha + k\beta)\sin\frac{\beta}{2}}{\sin\frac{\beta}{2}}$$

$$= \frac{\sin\left(\alpha + k\beta - \frac{\beta}{2}\right) - \sin\left(\alpha - \frac{\beta}{2}\right) + \sin\left[\alpha + k\beta + \frac{\beta}{2}\right] - \sin\left(\alpha + k\beta - \frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}}$$

$$= \frac{\sin\left(\alpha + k\beta + \frac{\beta}{2}\right) - \sin\left(\alpha - \frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}}$$

$$= \frac{2\cos\left(\frac{2\alpha + k\beta}{2}\right)\sin\left(\frac{k\beta + \beta}{2}\right)}{2\sin\frac{\beta}{2}}$$

$$= \frac{\cos\left(\alpha + \frac{k\beta}{2}\right)\sin\left[(k+1)\frac{\beta}{2}\right]}{\sin\frac{\beta}{2}} = \text{R.H.S.}$$

Thus, $P(k + 1)$ is true.

Hence, $P(n)$ is true.

S7. Let $P(n): \quad d_n = \frac{2}{n!}, \quad \forall\, n \in N$

Basic step:

To prove: $P(2)$ is true

i.e., $d_2 = \frac{d_1}{2} = \frac{2}{2} = 1$

Proof: For $n = 2$, $d_k = \frac{2}{n!}$

$\Rightarrow$ $d_2 = \frac{2}{2!} = 1$

which is true.

Thus, $P(2)$ is true.

Induction Step: Given, $P(k)$ is true

i.e., $d_k = \frac{2}{k!}$

To prove: $P(k+1)$ is true

i.e., $$d_{k+1} = \frac{2}{(k+1)!}$$

Proof: $$\text{L.H.S.} = d_{k+1} = \frac{d_k}{k+1}$$
$$= \frac{2}{(k)!} \times \frac{1}{(k+1)}$$
$$= \frac{2}{(k+1)!} = \text{R.H.S.}$$

Thus, $P(k+1)$ is true.

Hence, $P(n)$ is true.

S8. Let $P(n)$: $b_n = 5 + 4n \in N$

Now, $b_0 = 5$ and $b_k = 4 + b_{k-1}\ \forall k \in N$

Basic step:

To prove: $P(1)$ is true

Proof: For $n = 1$, $b_k = 4 + b_{k-1}$

$\Rightarrow$ $b_1 = 4 + b_0 = 4 + 5 = 9$

which is true.

Thus, $P(1)$ is true.

Induction Step: Given, $P(k)$ is true

i.e., $b_k = 5 + 4k$

To prove: $P(k+1)$ is true

i.e., $b_{k+1} = 5 + 4(k+1)$

or $b_{k+1} = 9 + 4k$

Proof: $$\text{L.H.S.} = b_{k+1} = 4 + b_k$$
$$= 4 + 5 + 4k$$
$$= 9 + 4k = \text{R.H.S.}$$

Thus, $P(k+1)$ is true.

Hence, $P(n)$ is true.

S9. A sequence $a_1, a_2, a_3, \ldots$ is defined by letting $a_1 = 3$ and $a_k = 7a_{k-1}$ for all natural numbers $k \geq 2$

Let $P(n)$: $a_n = 3.7^{n-1}$ for all natural numbers $n \in N$

Basic step:

To prove: $P(2)$ is true

Proof: For $n = 2$, $a_n = 3 \cdot 7^{n-1}$

$\Rightarrow$ $a_2 = 3 \cdot 7^{2-1} = 3 \times 7 = 21$

As $a_1 = 3$ and $a_k = 7a_{k-1}$

$\Rightarrow$ $a_2 = 7 \times a_1 = 7 \times 3 = 21$

$\therefore$ $P(1)$ is true.

Induction Step: Given, $P(k)$ is true

i.e., $a_k = 3 \cdot 7^{k-1}$

To prove: $P(k+1)$ is true

i.e., $a_{k+1} = 3 \cdot 7^k$

Proof:
$$\text{L.H.S.} = a_{k+1} = 3 \cdot 7^k$$
$$= 3 \cdot 7^{k-1} \times 7 = 7a_k$$

which is true.

Thus, $P(k+1)$ is true.

Hence, $P(n)$ is true.

S10. Let $P(n) : \cos\theta \cos 2\theta \cos 2^2\theta \ldots \cos 2^{n-1}\theta = \dfrac{\sin 2^n \theta}{2^n \sin\theta}$, for all $n \in N$

Basic step:

To prove: $P(1)$ is true

Proof: For $n = 1$,

$$\text{R.H.S.} = \frac{\sin 2^n \theta}{2^n \sin\theta} = \frac{\sin 2\theta}{2\sin\theta}$$
$$= \frac{2\sin\theta\cos\theta}{2\sin\theta} = \cos\theta = T_1$$

Thus, $P(1)$ is true.

Induction Step: Given, $P(k)$ is true

i.e., $\cos\theta \cos 2\theta \cos 2^2\theta \ldots \cos 2^{k-1}\theta = \dfrac{\sin 2^k \theta}{2^k \sin\theta}$

To prove: $\cos\theta \cos 2\theta \cos 2^2\theta \ldots \cos 2^k\theta = \dfrac{\sin (2^{k+1}\theta)}{2^{k+1}\sin\theta}$

$\therefore$
$$\text{L.H.S.} = \cos\theta \cos 2\theta \cos 2^2\theta \ldots \cos 2^k\theta = \frac{\sin 2^k\theta}{2^k \sin\theta} \times \cos 2^k\theta$$
$$= \frac{2\sin 2^k\theta \cos 2^k\theta}{2^{k+1}\sin\theta} = \frac{\sin(2^{k+1}\theta)}{2^{k+1}\sin\theta} = \text{R.H.S.}$$

Thus, $P(k+1)$ is true.

Hence, $P(n)$ is true.

S11. Let $P(n) : n(n^2 + 5)$ is divisible by 6, for $n \in N$.

Basic step:

To prove: $P(1)$ is true

Proof: For $n = 1$, $n(n^2 + 5) = 1(1^2 + 5) = 6$

which is divisible by 6.

$\therefore$ $P(1)$ is true.

Induction step:

Given: $P(k)$ is true

i.e., $k(k^2 + 5)$ is divisible by 6.

To prove: $P(k + 1)$ is true

i.e., $(k + 1)[(k + 1)^2 + 5]$ is divisible by 6.

Proof: For
$$\begin{aligned}(k + 1)[(k + 1)^2 + 5] &= (k + 1)(k^2 + 2k + 1 + 5)\\ &= k(k^2 + 5) + k(2k + 1) + (k^2 + 2k + 6)\\ &= k(k^2 + 5) + 3k^2 + 3k + 6\\ &= k(k^2 + 5) + 3(k^2 + k + 2) = A + B\end{aligned}$$

Now, A is divisible by 6, and $k^2 + k + 2$ is divisible by 2 whether k is odd or even..

Thus, $A + B$ is divisible by 6.

Thus, $P(k + 1)$ is true.

Hence, $P(n)$ is true.

S12. Let $P(n) : n^3 - n$ is divisible by 6, for $n \in N$ and $n \geq 2$.

Basic step:

To prove: $P(2)$ is true

Proof: For $n = 2$, $n^3 - n = 2^3 - 2 = 8 - 2 = 6$

which is divisible by 6.

$\therefore$ $P(2)$ is true.

Induction step:

Given: $P(k)$ is true

i.e., $k^3 - k$ is divisible by 6.

To prove: $P(k + 1)$ is true

i.e., $(k + 1)^3 - (k + 1)$ is divisible by 6.

Proof: For P $(k + 1)$:
$$\begin{aligned}(k + 1)^3 - (k + 1) &= k^3 + 3k^2 + 3k + 1 - k - 1\\ &= (k^3 - k) + 3k^2 + 3k\\ &= (k^3 - k) + 3k(k + 1) = A + B\end{aligned}$$

Now, A is given to be divisible by 6 and out of k and $k + 1$ one is even *i.e.*, divisible by 2.

Therefore, $3k(k + 1)$ is divisible by 6.

$\therefore$ $A + B$ is divisible by 6.

Thus, $P(k + 1)$ is true.

Hence, $P(n)$ is true.

S13. Let $P(n) : 7^n - 2^n$ is divisible by 5, for $n \in N$.

Basic step:

To prove: $P(1)$ is true

Proof: For $n = 1$, $7^n - 2^n = 7^1 - 2^1 = 5$

which is divisible by 5.

$\therefore$ $P(1)$ is true.

Induction step:

Given: $P(k)$ is true

Or

$7^k - 2^k$ is divisible by 5.

To prove: $P(k + 1)$ is true

Or

$7^{k+1} - 2^{k+1}$ is divisible by 5.

Proof: For P $(k + 1)$: $7^{k+1} - 2^{k+1} = 7 \cdot 7^k - 2^k \cdot 2$

$$= 7(7^k - 2^k) + 7 \cdot 2^k - 2^k \cdot 2$$

$$= 7(7^k - 2^k) + 2^k(7 - 2)$$

$$= \underbrace{7(7^k - 2^k)}_{A} + \underbrace{2^k \cdot 5}_{B}$$

Now, A is divisible by 5 and B is also divisible by 5.

Thus, $A + B$ is divisible by 5.

Thus, $P(k + 1)$ is true.

Hence, $P(n)$ is true.

S14. Let $P(n) : 3^{2n} - 1$ is divisible by 8 for all $n \in N$.

Basic step:

To prove: $P(1)$ is true

Proof: For $n = 1$, $3^{2n} - 1 = 9 - 1 = 8$

which is clearly divisible by 8.

$\therefore$ $P(1)$ is true.

Induction step:

Given: $P(k)$ is true

Or

$3^{2k} - 1$ is divisible by 8.

To prove: $P(k + 1)$ is true

Or

$3^{2(k+1)} - 1$.

Proof: For P $(k + 1)$: $3^{2(k+1)} - 1 = 3^{2k} \cdot 3^2 - 1,$

$$= 9(3^{2k} - 1) + 9 - 1$$

$$= \underbrace{9(3^{2k} - 1)}_{A} + \underbrace{8}_{B}$$

Now, A is given to be divisible by 8 and B is also divisible by 8.

Thus, $A + B$ is divisible by 8.

Thus, $P(k + 1)$ is true.

Hence, $P(n)$ is true.

S15. Let $P(n): n^3 - 7n + 3$ is divisible by 3 for all natural numbers n.

Basic step:

To prove: $P(1)$ is true

Proof: For $n = 1$, $n^3 - 7n + 3 = (1)^3 - 7 \times 1 + 3 = 1 - 7 + 3 = -3$

which is clearly divisible by 3.

Induction step:

Given: $P(k)$ is true

Or

$k^3 - 7k + 3$ is divisible by 3.

To prove: $P(k + 1)$ is true

Or

$(k + 1)^3 - 7(k + 1) + 3$ is divisible by 3.

Proof: For P $(k + 1)$: $(k + 1)^3 - 7(k + 1) + 3 = k^3 + 3k^2 + 3k + 1 - 7k - 7 + 3$

$$= (k^3 - 7k + 3) + 3k^2 + 3k - 6$$

$$= \underbrace{(k^3 - 7k + 3)}_{A} + \underbrace{3(k^2 + k - 2)}_{B}$$

A is given to be divisible by 3 and B contains a factor 3.

Thus, $A + B$ is divisible by 3.

Thus, $P(k + 1)$ is true.

Hence, $P(n)$ is true.

S16. Let $P(n): 4^n - 1$ is divisible by 3

Basic step:

To prove: $P(n)$ is true

Proof: For $n = 1$,

$$4^n - 1 = 4^1 - 1 = 3$$

which is clearly divisible by 3.

Induction step:

Given: $P(k)$ is true

Or

$4^k - 1$ is divisible by 3.

To prove: $P(k + 1)$ is true

Or

$4^{k+1} - 1$ is divisible by 3.

Proof: For $4^{k+1} - 1 = 4 \cdot 4^k - 1,$

$$= 4 \cdot 4^k - 4 + 4 - 1$$

$$= \underbrace{4(4^k - 1)}_{A} + \underbrace{3}_{B}$$

Clearly A is given to be divisible and B is also divisible by 3.

$\therefore \quad A + B$ is divisible by 3.

Thus, $P(k + 1)$ is true.

Hence, $P(n)$ is true.

S17. Let the statement $P(n)$ given as

$P(n) : 2^{2n} - 1$ is divisible by 3, for every natural number n.

We observe that $P(1)$ is true, since

$$2^2 - 1 = 4 - 1 = 3 . 1 \text{ is divisible by 3.}$$

Assume that $P(n)$ is true for some natural number k, *i.e.*,

$P(k) : 2^{2k} - 1$ is divisible by 3, *i.e.*, $2^{2k} - 1 = q$, where $q \in N$

Now, to prove that $P(k + 1)$ is true, we have

$$P(k + 1) : 2^{2(k+1)} - 1 = 2^{2k+2} - 1 = 2^{2k} . 2^2 - 1$$
$$= 2^{2k} . 4 - 1 = 3 . 2^{2k} + (2^{2k} - 1)$$
$$= 3 . 2^{2k} + q$$
$$= 3(2^{2k} + q) = 3m, \text{ where } m \in N$$

Thus, $P(k + 1)$ is true whenever $P(k)$ is true.

Hence, by the Principle of Mathematical Induction $P(k)$ is true for all natural numbers n.

S18. $P(n)$: Number of subsets of a set containing n distinct elements is 2^n for all $n \in N$.

Basic step:

To prove: $P(1)$ is true

Proof: For $n = 1$, subsets of a set containing 1 distinct elements is 2^1 which is true.

Induction Step: Given : Number of subsets of a set containing k distinct elements is 2^k.

To prove: Number of subsets of a set containing $k + 1$ distinct elements is 2^{k+1}.

Proof: We know that with the addition of one element in the set, the number of subsets become double.

Now number of subsets of a set containing $k + 1$ distinct elements $= 2 \times 2^k = 2^{k+1}$.

Thus, $P(k + 1)$ is true.

Hence, $P(n)$ is true.

S19. $P(n) : \dfrac{n^5}{5} + \dfrac{n^3}{3} + \dfrac{7n}{15}$ is a natural number for all $n \in N$

Basic step:

To prove: $P(1)$ is true

Proof: For $n = 1$, $\frac{n^5}{5}+\frac{n^3}{3}+\frac{7n}{15} = \frac{(1)^5}{5}+\frac{(1)^3}{3}+\frac{7(1)}{15}$

$$= \frac{1}{5}+\frac{1}{3}+\frac{7}{15}=\frac{3+5+7}{15}=\frac{15}{15}=1\in N$$

Thus, $P(1)$ is true.

Induction Step: Given, $P(k)$ is true

Or $\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}$ is a natural number.

To prove: $P(k + 1)$ is true

Or $\frac{(k+1)^5}{5}+\frac{(k+1)^3}{3}+\frac{7(k+1)}{15}$ is a natural number.

Proof: $\frac{(k+1)^5}{5}+\frac{(k+1)^3}{3}+\frac{7(k+1)}{15}$

$$= \frac{(k^5+5k^4+10k^3+10k^2+5k+1)}{5}+\frac{k^3+3k^2+3k+1}{3}+\frac{7k+7}{15}$$

$$= \left(\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}\right)+k^4+2k^3+(2k^2+k^2)+(k+k)+\left(\frac{1}{5}+\frac{1}{3}+\frac{7}{15}\right)$$

$$= \left(\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}\right)+(k^4+2k^3+3k^2+2k+1)=A+B$$

Now, A is given to be a natural number and B is also a natural as k is a rational.

$\therefore$ $A + B$ is a natural number.

Thus, $P(k + 1)$ is true.

Hence, $P(n)$ is true.

S20.

Let $P(n)$: $\sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta = \dfrac{\frac{\sin n\theta}{2}\sin\frac{(n+1)}{2}\theta}{\sin\frac{\theta}{2}}$

Basic step:

To prove: $P(1)$ is true

Proof: For $n = 1$,

$$\text{R.H.S.} = \frac{\frac{\sin n\theta}{2}\sin\frac{(n+1)}{2}\theta}{\sin\frac{\theta}{2}}$$

$$= \frac{\frac{\sin\theta}{2}\sin\theta}{\sin\frac{\theta}{2}} = \sin\theta = T_1$$

Thus, $P(1)$ is true.

Induction Step: Given, $P(k)$ is true

i.e., $\sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin k\theta = \dfrac{\frac{\sin k\theta}{2}\sin\frac{(k+1)}{2}\theta}{\sin\frac{\theta}{2}}$

To prove: $P(k+1)$ is true

i.e., $\sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin(k+1)\theta = \dfrac{\frac{\sin(k+1)\theta}{2}\sin\left(\frac{k+2}{2}\right)\theta}{\sin\frac{\theta}{2}}$

$\therefore$ L.H.S. $= \sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin(k+1)\theta$

$$= \frac{\frac{\sin k\theta}{2}\sin\frac{k+1}{2}\theta}{\sin\frac{\theta}{2}} + \sin(k+1)\theta$$

$$= \frac{\sin\frac{k\theta}{2}\sin\left(\frac{k+1}{2}\right)\theta \sin(k+1)\theta\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

$$= \frac{\cos\left(\frac{\theta}{2}\right) - \cos\frac{2k+1}{2}\theta + \cos\left(k+\frac{1}{2}\right)\theta - \cos\left(k+\frac{3}{2}\right)\theta}{2\sin\frac{\theta}{2}}$$

$$= \frac{\cos\frac{\theta}{2} - \cos\left(k+\frac{3}{2}\right)\theta}{2\sin\frac{\theta}{2}}$$

$$= \frac{2\sin\left(\frac{k+\frac{3}{2}-\frac{1}{2}}{2}\right)\theta\sin\left(\frac{\frac{1}{2}+k+\frac{3}{2}}{2}\right)\theta}{2\sin\frac{\theta}{2}}$$

$$= \frac{\sin\left(\frac{k+1}{2}\right)\theta \sin\left(\frac{k+2}{2}\right)\theta}{\sin\frac{\theta}{2}} = \text{R.H.S.}$$

Thus, $P(k+1)$ is true.

Hence, $P(n)$ is true.

S21. Let $P(n): \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} > \frac{13}{24}$ for $n \in N$ and $n > 1$.

Basic step:

To prove: $P(2)$ is true

Proof: For $n = 2$, $\frac{1}{n+1} + \frac{1}{n+2} > \frac{13}{24} \Rightarrow \frac{1}{2+1} + \frac{1}{2+2} > \frac{13}{24}$

$$\Rightarrow \frac{1}{3} + \frac{1}{4} > \frac{13}{24}$$

$$\Rightarrow \frac{7}{12} > \frac{13}{24} \text{ which is true.}$$

$\therefore$ $P(1)$ is true.

Induction step:

Given: $P(k)$ is true

Or, $$\frac{1}{k+1} + \frac{1}{k+2} + \ldots + \frac{1}{2k} > \frac{13}{24}$$

To prove: $P(k+1)$ is true

Or, $$\frac{1}{k+1} + \frac{1}{k+2} + \ldots + \frac{1}{2(k+1)} > \frac{13}{24}$$

Proof: Given $$\frac{1}{k+1} + \frac{1}{k+2} + \ldots + \frac{1}{2k} > \frac{13}{24}$$

$$= \frac{1}{k+1} + \frac{1}{k+2} + \ldots + \frac{1}{2k} + \frac{1}{2(k+1)} > \frac{13}{24} + \frac{1}{2(k+1)}$$

Now, $$\frac{13}{24} + \frac{1}{2(k+1)} > \frac{13}{24}$$

Hence, $$\frac{1}{k+1} + \frac{1}{k+2} + \ldots + \frac{1}{2k} + \frac{1}{2(k+1)} > \frac{13}{24}$$

Thus, $P(k+1)$ is true.

Hence, $P(n)$ is true.

S22. Let $P(n): \sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{n}}$ for $n \in N$ and $n \geq 2$

Basic step:

To prove: $P(2)$ is true

Proof: For $n = 2$, $$\sqrt{2} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}}$$

which is true.

$\therefore$ $P(1)$ is true.

Induction step:

Given: $P(k)$ is true

i.e., $$\sqrt{k} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{k}}$$

To prove: $P(k + 1)$ is true

i.e., $$\sqrt{k+1} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{k+1}}$$

Proof: Given $$\sqrt{k} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{k}}$$

$$\Rightarrow \sqrt{k} + \frac{1}{\sqrt{k+1}} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}$$

$$\Rightarrow \frac{\sqrt{k}\sqrt{k+1}+1}{\sqrt{k+1}} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \quad ... \text{(i)}$$

If, $$\sqrt{k+1} < \frac{\sqrt{k}\sqrt{k+1}+1}{\sqrt{k+1}} \quad ... \text{(ii)}$$

$$\Rightarrow k + 1 < \sqrt{k}\sqrt{k+1}+1$$

$$\Rightarrow k < \sqrt{k}\,(k+1)$$

$$\Rightarrow \sqrt{k} < \sqrt{k+1}$$

which is true.

From Eq. (i) and (ii), we get

$$\sqrt{k+1} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{k+1}}$$

Thus, $P(k + 1)$ is true.

Hence, $P(n)$ is true.

S23. Let $P(n) : 2n < (n + 2)!$ for $n \in N$.

Basic step:

To prove: $P(1)$ is true

Proof: For $n = 1$, $2n < (n + 2)!$

$\Rightarrow \quad 2 \times 1 < (1 + 2)!$

$\Rightarrow \quad 2 < 3!$

which is true.

$\therefore \quad P(1)$ is true.

Induction step:

Given: $P(k)$ is true

i.e., $2k < (k + 2)!$

To prove: $P(k + 1)$ is true

i.e., $2(k + 1) < (k + 3)!$

Proof: For $2k < (k + 2)!$

$\Rightarrow \quad 2k + 2 < (k + 2)! + 2$

$\Rightarrow \quad 2(k + 1) < (k + 2)! + 2 \quad \ldots \text{(i)}$

Now, $(k + 2)! + 2 < (k + 3)! \quad \ldots \text{(ii)}$

From Eq. (i) and (ii), we get

$$2(k + 1) < (k + 3)!$$

Thus, $P(k + 1)$ is true.

Hence, $P(n)$ is true.

S24. Let $P(n)$ be the given statement, *i.e.*, $P(n) : 2n + 1 < 2^n$ for all natural numbers, . We observe that $P(3)$ is true, since

$$2 . 3 + 1 = 7 < 8 = 2^3.$$

Assume that $P(n)$ is true for some natural number k, *i.e.*, $2k + 1 < 2^k$

To prove $P(k + 1)$ is true, we have to show that $2(k + 1) + 1 < 2^{k+1}$. Now, we have

$$2(k + 1) + 1 = 2k + 3$$
$$= 2k + 1 + 2 < 2^k + 2 < 2^k . 2 = 2^{k+1}.$$

Thus, $P(k + 1)$ is true, whenever $P(k)$ is true.

Hence, by the Principle of Mathematical Induction $P(n)$ is true for all natural numbers, $n \geq 3$.

S25. Let $P(n) : n^2 < 2^n$ for $n \in N$ and $n \geq 5$.

Basic step:

To prove: $P(5)$ is true

Proof: For $n = 5$, $n^2 < 2^n$

$\Rightarrow$ $5^2 < 2^5$

$\Rightarrow$ $25 < 32$

$\therefore$ $P(1)$ is true.

Induction step:

Given: $P(k)$ is true

i.e., $k^2 < 2^k$

To prove: $P(k + 1)$ is true

i.e., $(k + 1)^2 < 2^{k+1}$

Proof: For $k^2 < 2^k$

$\Rightarrow$ $k^2 + 2k + 1 < 2^k + 2k + 1$

$\Rightarrow$ $(k + 1)^2 < 2^k + 2k + 1$ ... (i)

Now, $2k + 1 < 2^k$

$\Rightarrow$ $2^k + 2k + 1 < 2^k + 2^k$

$\Rightarrow$ $2^k + 2k + 1 < 2 \cdot 2^k$

$\Rightarrow$ $2^k + 2k + 1 < 2^{k+1}$ ... (ii)

From Eq. (i) and (ii), we get

$$(k + 1)^2 < 2^{k+1}$$

Thus, $P(k)$ is true.

Hence, $P(n)$ is true.

S26. Consider $P(n)$: $\sin \alpha + \sin (\alpha + \beta) + \sin (\alpha + 2\beta) + ... + \sin (\alpha + (n - 1)\beta)$

$$= \frac{\sin\left(\alpha + \frac{n-1}{2}\beta\right)\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}, \text{ for all natural number } n.$$

We observe that $P(1)$ is true, since

$$P(1) : \quad \sin \alpha = \frac{\sin(\alpha + 0)\sin\frac{\beta}{2}}{\sin\frac{\beta}{2}}$$

Assume that $P(n)$ is true for some natural numbers k, *i.e.*,

$P(k)$: $\sin \alpha + \sin (\alpha + \beta) + \sin (\alpha + 2\beta) + ... + \sin (\alpha + (k - 1)\beta)$

$$= \frac{\sin\left(\alpha + \frac{k-1}{2}\beta\right)\sin\left(\frac{k\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

Now, to prove that $P(k + 1)$ is true, we have

$P(k + 1)$: $\sin \alpha + \sin (\alpha + \beta) + \sin (\alpha + 2\beta) + \ldots + \sin (\alpha + (k - 1) \beta) + \sin (\alpha + k\beta)$

$$= \frac{\sin\left(\alpha + \frac{k-1}{2}\beta\right)\sin\left(\frac{k\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)} + \sin(\alpha + k\beta)$$

$$= \frac{\sin\left(\alpha + \frac{k-1}{2}\beta\right)\sin\frac{k\beta}{2} + \sin(\alpha + k\beta)\sin\frac{\beta}{2}}{\sin\frac{\beta}{2}}$$

$$= \frac{\cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left(\alpha + k\beta - \frac{\beta}{2}\right) + \cos\left(\alpha + k\beta - \frac{\beta}{2}\right) - \cos\left(\alpha + k\beta + \frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}}$$

$$= \frac{\cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left(\alpha + k\beta + \frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}}$$

$$= \frac{\sin\left(\alpha + \frac{k\beta}{2}\right)\sin\left(\frac{k\beta + \beta}{2}\right)}{\sin\frac{\beta}{2}}$$

$$= \frac{\sin\left(\alpha + \frac{k\beta}{2}\right)\sin(k+1)\left(\frac{\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

Thus, $P(k + 1)$ is true, whenever $P(k)$ is true.

Hence, by the Principle of Mathematical Induction, $P(n)$ is true for all natural numbers, n.

S27.

Here, $$S_n = \begin{cases} \frac{n(n+1)^2}{2}, & \text{if } n \text{ is even} \\ \frac{n^2(n+1)}{2}, & \text{if } n \text{ is odd} \end{cases}$$

Also, note that any term T_n of the series is given by

$$T_n = \begin{cases} n^2, & \text{if } n \text{ is odd} \\ 2n^2, & \text{if } n \text{ is even} \end{cases}$$

We observe that $P(1)$ is true, since

$$P(1): \quad S_1 = 1^2 = 1 = \frac{1 \cdot 2}{2} = \frac{1^2 \cdot (1+1)}{2}$$

Assume that $P(k)$ is true for some natural number k, *i.e.*,

Case 1 : When k is odd, then $k + 1$ is even. We have

$$P(k+1) : S_{k+1} = 1^2 + 2 \times 2^2 + \ldots + k^2 + 2 \times (k+1)^2$$

$$= \frac{k^2(k+1)}{2} + 2 \times (k+1)^2$$

$$= \frac{(k+1)}{2}[k^2 + 4(k+1)] \quad \left(\text{as } k \text{ is odd, } 1^2 + 2 \times 2^2 + \ldots + k^2 = k^2 \frac{(k+1)}{2}\right)$$

$$= \frac{k+1}{2}[k^2 + 4k + 4]$$

$$= \frac{k+1}{2}(k+2)^2 = (k+1)\frac{[(k+1)+1]^2}{2}$$

So, $P(k + 1)$ is true, whenever $P(k)$ is true in the case when k is odd.

Case 2 : When k is even, then $k + 1$ is odd. Now,

$$P(k+1) : 1^2 + 2 \times 2^2 + \ldots + 2 \,.\, k^2 + (k+1)^2$$

$$= \frac{k(k+1)^2}{2} + (k+1)^2 \quad \left(\text{as } k \text{ is even, } 1^2 + 2 \times 2^2 + \ldots + 2k^2 = k\frac{(k+1)^2}{2}\right)$$

$$= \frac{(k+1)^2(k+2)}{2} = \frac{(k+1)^2((k+1)+1)}{2}$$

Therefore, $P(k + 1)$ is true, whenever $P(k)$ is true for the case when k is even.

Thus, $P(k + 1)$ is true, whenever $P(k)$ is true for any natural numbers k. Hence $P(n)$ is true for all natural numbers.