Date: 22/10/2021

Q1. Imagine removing one electron from He^4 and He^3. Their energy levels, as worked out on the basis of Bohr model will be very close. Explain why.

Q2. When an electron falls from a higher energy to a lower energy level, the difference in the energies appears in the form of electromagnetic radiation. Why cannot it be emitted as other forms of energy?

Q3. Consider two different hydrogen atoms. The electron in each atom is in an excited state. Is it possible for the electrons to have different energies but the same orbital angular momentum according to the Bohr model?

Q4. Would the Bohr formula for the H-atom remain unchanged if proton had a charge $(+4/3)\,e$ and electron a charge $(-3/4)\,e$, where $e = 1.6 \times 10^{-19}$ C. Give reasons for your answer.

Q5. The mass of a H-atom is less than the sum of the masses of a proton and electron. Why is this?

Q6. Assume that there is no repulsive force between the electrons in an atom but the force between positive and negative charges is given by Coulomb's law as usual. Under such circumstances, calculate the ground state energy of a He-atom.

Q7. Positronium is just like a H-atom with the proton replaced by the positively charged anti-particle of the electron (called the positron which is as massive as the electron). What would be the ground state energy of positronium?

Q8. Using Bohr model, calculate the electric current created by the electron when the H-atom is in the ground state.

Q9. Show that the first few frequencies of light that is emitted when electrons fall to the n^{th} level from levels higher than n, are approximate harmonics (*i.e.*, in the ratio 1 : 2 : 3...) when $n \gg 1$.

Q10. What is the minimum energy that must be given to a H atom in ground state so that it can emit an H_γ line in Balmer series. If the angular momentum of the system is conserved, what would be the angular momentum of such H_γ photon?

Q11. Deuterium was discovered in 1932 by Harold Urey by measuring the small change in wavelength for a particular transition in ^1H and ^2H. This is because, the wavelength of transition depend to a certain extent on the nuclear mass. If nuclear motion is taken into account then the electrons and nucleus revolve around their common centre of mass. Such a system is equivalent to a single particle with a reduced mass μ, revolving around the nucleus at a distance equal to the electron-nucleus separation. Here $\mu = m_e M/(m_e + M)$ where M is the nuclear mass and me is the electronic mass. Estimate the percentage difference in wavelength for the 1st line of the Lyman series in ^1H and ^2H. (Mass of ^1H nucleus is 1.6725×10^{-27} kg, Mass of ^2H nucleus is 3.3374×10^{-27} kg, Mass of electron = 9.109×10^{-31} kg.)

Q12. In the Auger process an atom makes a transition to a lower state without emitting a photon. The excess energy is transferred to an outer electron which may be ejected by the atom. (This is called an Auger electron). Assuming the nucleus to be massive, calculate the kinetic energy of an $n = 4$ Auger electron emitted by Chromium by absorbing the energy from a $n = 2$ to $n = 1$ transition.

Q13. The Bohr model for the H-atom relies on the Coulomb's law of electrostatics. Coulomb's law has not directly been verified for very short distances of the order of angstroms. Supposing Coulomb's law between two opposite charge $+q_1, -q_2$ is modified to:

$$F = \frac{q_1 q_2}{(4\pi\varepsilon_0)} \frac{1}{r}, \quad r \geq R_0$$

$$= \frac{q_1 q_2}{4\pi\varepsilon_0} \frac{1}{R_0^2} \left(\frac{R_0}{r}\right)^{\varepsilon}, \quad r \leq R_0$$

Calculate in such a case, the ground state energy of a H-atom, if $\varepsilon = 0.1$, $R_0 = 1$Å.

Q14. If a proton had a radius R and the charge was uniformly distributed, calculate using Bohr theory, the ground state energy of a H-atom when (i) $R = 0.1$Å, and (ii) $R = 10$ Å.

Q15. The first four spectral lines in the Lyman serics of a H-atom are $\lambda = 1218$ Å, 1028Å, 974.3 Å and 951.4Å. If instead of Hydrogen, we consider Deuterium, calculate the shift in the wavelength of these lines.

Q16. The inverse square law in electrostatics is $|F| = \frac{e^2}{(4\pi\varepsilon_0).r^2}$ for the force between an electron and a proton. The $\left(\frac{1}{r}\right)$ dependence of $|F|$ can be understood in quantum theory as being due to the fact that the 'particle' of light (photon) is massless. If photons had a mass m_p, force would be modified to $|F| = \frac{e^2}{(4\pi\varepsilon_0)r^2}\left[\frac{1}{r^2} + \frac{\lambda}{r}\right]$, $\exp(-\lambda r)$ where $\lambda = m_p c/h$ and $h = \frac{h}{2\pi}$.

Estimate the change in the ground state energy of a H-atom if m_p were 10^{-6} times the mass of an electron.

Date: 22/10/2021

S1. On removing one electron from He^4 and He^3, the energy levels, as worked out on the basis of Bhor model will be very close as both the nuclei are very heavy as compared to electron mass. Also after removing one electron from He^4 and He^3 atoms contain one electron and are hydrogen like atoms.

S2. The transition of an electron from a higher energy to a lower energy level can appear in the form of electromagnetic radiation because electrons interact only electromagnetically.

S3. According to Bohr model electrons having different energies belong to different levels having different values of *n*. So, their angular momenta will be different, as

$$L = \frac{nh}{2\pi} \quad \text{or} \quad L \propto n.$$

S4. If proton had a charge (+4/3) *e* and electron a charge (– 3/4) *e*, then the Bohr formula for the H-atom remain same, since the Bohr formula involves only the product of the charges which remain constant for given values of charges.

S5. Since, the difference in mass of a nucleus and its constituents, ΔM, is called the mass defect and is given by

$$\Delta M = [Zm_p + (A - Z)\,m_n] - M \qquad [m_p = \text{mass of proton and } m_n = \text{mass of neutrons}]$$

Also, the binding energy is given by B = mass defect $(\Delta M) \times c^2$.

Thus, the mass of a H-atom is $m_p + m_e - \frac{B}{c^2}$ where $B \approx 13.6$ eV is the binding energy.

S6. For a He-nucleus with charge 2e and electrons of charge – e, the energy level in ground state is

$$-E_n = Z^2\,\frac{-13.6\text{ eV}}{n^2}$$

$$= 2^2\,\frac{-13.6\text{ eV}}{1^2} = -54.4\text{ eV}$$

Thus, the ground state will have two electrons each of energy *E* and the total ground state energy would be $-(4 \times 13.6)$ e = -54.4 eV.

S7. The total energy of the electron in the stationary states of the hydrogen atom is given by

$$E_n = -\frac{me^4}{8n^2\varepsilon_0^2h^2}$$

where signs are as usual and the *m* that occurs in the Bohr formula is the reduced mass of electron and proton. Also, the total energy of the electron in the ground state of the hydrogen atom is – 13.6 eV. For H-atom reduced mass m_e. Whereas for positronium, the reduced mass is

$$m = \frac{m_e}{2}$$

Hence, the total energy of the electron in the ground state of the positronium atom is

$$\frac{-13.6 \text{ eV}}{2} = -6.8 \text{ eV}.$$

S8. v = velocity of electron

a_0 = Bohr radius.

$\therefore$ Number of revolutions per unit time = $\frac{2\pi a_0}{v}$

$\therefore$ Current = $\frac{2\pi a_0}{v} e$.

S9. The frequency of any line in a series in the spectrum of hydrogen like atoms corresponding to the transition of electrons from $(n + p)$ level to n^{th} level can be expressed as a difference of two terms.

$$\nu_{mn} = cRZ^2 \left[\frac{1}{(n+p)^2} - \frac{1}{n^2} \right],$$

where $m = n + p$, ($p = 1, 2, 3, ...$) and R is Rydberg constant.

For $p << n$.

$$\nu_{mn} = cRZ^2 \left[\frac{1}{n^2} \left(1 + \frac{P}{n}\right)^{-2} - \frac{1}{n^2} \right],$$

$$\nu_{mn} = cRZ^2 \left[\frac{1}{n^2} - \frac{2p}{n^3} - \frac{1}{n^2} \right],$$

$$\nu_{mn} = cRZ^2 \frac{2p}{n^3}; \quad \left(\frac{2cRZ^2}{n^3} \right) p$$

Thus, ν_{mn} are approximately in the order 1, 2, 3..........

S10. H_γ in Balmer series corresponds to transition $n = 5$ to $n = 2$.

So the electron in ground state $n = 1$ must first be put in state $n = 5$.

Energy required = $E_1 - E_5 = 13.6 - 0.54 = 13.06$ eV.

If angular momentum is conserved,

angular momentum of photon = change in angular momentum of electron

$= L_5 - L_2 = 5h - 2h = 3h$

$$= 3 \times 1.06 \times 10^{-34}$$

$$= 3.18 \times 10^{-34} \text{ kg m}^2/\text{s}.$$

S11. Taking into account the nuclear motion, the stationary state energies shall be,

$$E_n = -\frac{\mu Z^2 e^4}{8\varepsilon_0^2 h^2}\left(\frac{1}{n^2}\right).$$

Let μ_H be the reduced mass of Hydrogen and μ_D that of Deutrium. Then the frequency of the 1st Lyman line in Hydrogen is

$$h\nu_H = \frac{\mu_H e^4}{8\varepsilon_0^2 h^2}\left(1-\frac{1}{4}\right) = \frac{3}{4}\frac{\mu_H e^4}{8\varepsilon_0^2 h^2}.$$

Thus the wavelength of the transition is

$$\lambda_H = \frac{3}{4}\frac{\mu_H e^4}{8\varepsilon_0^2 h^2}.$$

The wavelength of the transition for the same line in Deutrium is

$$\lambda_D = \frac{3}{4}\frac{\mu_D e^4}{8\varepsilon_0^2 h^3 c}.$$

$\therefore$ $$\Delta\lambda = \lambda_D - \lambda_H$$

Hence the percentage difference is

$$100 \times \frac{\Delta\lambda}{\lambda_H} = \frac{\lambda_D - \lambda_H}{\lambda_H} \times 100 = \frac{\mu_D - \mu_H}{\mu_H} \times 100$$

$$= \frac{\dfrac{m_e M_D}{(m_e + M_D)} - \dfrac{m_e M_H}{(m_e + M_H)}}{(m_e M_H)/(m_e + M_H)} \times 100$$

$$= \left[\left(\frac{m_e + M_H}{m_e + M_D}\right)\frac{M_D}{M_H} - 1\right] \times 100$$

Since, $m_e << M_H < M_D$.

$$\frac{\Delta\lambda}{\lambda_H} \times 100 = \left[\frac{M_H}{M_D} \times \frac{M_D}{M_H}\left(\frac{1 + m_e/M_H}{1 + m_e/M_D}\right) - 1\right] \times 100$$

$$= (1 + m_e/M_H)(1 + m_e/M_D)^{-1} - 1 \times 100$$

$$\approx \left[1 + \frac{m_e}{M_H} - \frac{m_e}{M_D} - 1\right] \times 100$$

$$\approx m_e\left[\frac{1}{M_H} - \frac{1}{M_D}\right] \times 100$$

$$= 9.1 \times 10^{-31}\left[\frac{1}{1.6725 \times 10^{-27}} - \frac{1}{3.3374 \times 10^{-27}}\right] \times 100$$

$$= 9.1 \times 10^{-4}[0.5979 - 0.2996] \times 100$$
$$= 2.714 \times 10^{-2}\,\%.$$

S12. As the nucleus is massive, recoil momentum of the atom may be neglected and the entire energy of the transition may be considered transferred to the Auger electron. As there is a single valence electron in Cr, the energy states may be thought of as given by the Bohr model.

The energy of the n^{th} state $E_n = -Z^2R\frac{1}{n^2}$ where R is the Rydberg constant and $Z = 24$.

The energy released in a transition from 2 to 1 is $\Delta E = Z^2R\left(1 - \frac{1}{4}\right) = \frac{3}{4}Z^2R$. The energy required to eject a $n = 4$ electron is $E_4 = Z^2R\frac{1}{16}$.

Thus the kinetic energy of the Auger electron is

$$\text{K.E.} = Z^2R\left(\frac{3}{4} - \frac{1}{16}\right) = \frac{11}{16}Z^2R$$

$$= \frac{11}{16} \times 24 \times 24 \times 13.6 \text{ eV} = 5385.6 \text{ eV}.$$

S13. Let $\varepsilon = 2 + \delta$

$$F = \frac{q_1q_2}{4\pi\varepsilon_0} \cdot \frac{R_0^\delta}{r^{2+\delta}} = \wedge\frac{R_0^\delta}{r^{2+\delta}},$$

where $\frac{q_1q_2}{4\pi_0\varepsilon} = \wedge$

$$\wedge = (1.6 \times 10^{-19})^2 \times 9 \times 10^9$$
$$= 23.04 \times 10^{-29}$$

$$= \frac{mv^2}{r}$$

$$v^2 = \frac{\wedge R_0^\delta}{mr^{1+\delta}}$$

$$mvr = nh, \quad \Rightarrow \quad r = \frac{nh}{mv} = \frac{nh}{m}\left[\frac{m}{\wedge R_0^\delta}\right]^{1/2} r^{1/2+\delta/2}$$

Solving this for r, we get $r_n = \left[\frac{n^2\hbar^2}{m\wedge R_0^\delta}\right]^{\frac{1}{1-\delta}}$

For $n = 1$ and substituting the values of constant, we get

$$r_1 = \left[\frac{\hbar^2}{m \wedge R_0^{\delta}}\right]^{\frac{1}{1-\delta}}$$

$$r_1 = \left[\frac{1.05^2 \times 10^{68}}{9.1\times 10^{-31} \times 2.3 \times 10^{-28} \times 10^{+19}}\right]^{\frac{1}{2.9}}$$

$$= 8 \times 10^{-11} = 0.08 \text{ nm} \quad (< 0.1 \text{ nm})$$

This is the radius of orbit of electron in ground state of hydrogen atom.

$$v_n = \frac{n\hbar}{mr_n} = n\hbar\left(\frac{m \wedge R_0^{\delta}}{n^2\hbar^2}\right)^{\frac{1}{1-\delta}}.$$

For $$n = 1, \quad v_1 = \frac{\hbar}{mr_1} = 1.44 \times 10^6 \text{ m/s}$$

$$\text{K.E.} = \frac{1}{2}mv_1^2 = 9.43 \times 10^{-19} \text{ J} = 5.9 \text{ eV}$$

$$\text{P.E. till } R_0 = -\frac{\wedge}{R_0}$$

$$\text{P.E. from } R_0 \text{ to } r = +\wedge R_0^{\delta}\int_{R_0}^{r}\frac{dr}{r^{2+\delta}} = +\frac{\wedge R_0^{\delta}}{-1-\delta}\left[\frac{1}{r^{1+\delta}}\right]_{R_0}^{r}$$

$$= -\frac{\wedge R_0^{\delta}}{1+\delta}\left[\frac{1}{r^{1+\delta}} - \frac{1}{R_0^{1+\delta}}\right]$$

$$= -\frac{\wedge}{1+\delta}\left[\frac{R_0^{\delta}}{r^{1+\delta}} - \frac{1}{R_0}\right]$$

$$\text{P.E.} = -\frac{\wedge}{1+\delta}\left[\frac{R_0^{\delta}}{r^{1+\delta}} - \frac{1}{R_0} + \frac{1+\delta}{R_0}\right]$$

$$\text{P.E.} = -\frac{\wedge}{-0.9}\left[\frac{R_0^{-1.9}}{r^{-0.9}} - \frac{1.9}{R_0}\right]$$

$$= \frac{2.3}{0.9} \times 10^{-18}[(0.8)^{0.9} - 1.9] \text{ J} = -17.3 \text{ eV}$$

Total energy is $(-17.3 + 5.9) = -11.4$ eV.

S14. For a point nucleus in H-atom:

Ground state: $$\frac{mv^2}{r_B} = -\frac{e^2}{r_B^2}\cdot\frac{1}{4\pi\varepsilon_0}$$

$\therefore$ $$m\frac{h^2}{m^2 r_B^2}\cdot\frac{1}{r_B} = \left(\frac{e^2}{4\pi\varepsilon_0}\right)\frac{1}{r_B^2}$$

$\therefore$ $$\frac{\hbar^2}{m}\cdot\frac{4\pi\varepsilon_0}{e^2} = r_B = 0.51\,\text{Å}$$

Potential energy

$$-\left(\frac{e^2}{4\pi r_0}\right)\frac{1}{r_B^2} = -27.2\text{ eV};\quad \text{K.E.} = \frac{mv^2}{2} = \frac{1}{2}m\,,\frac{\hbar^2}{m^2 r_B^2} = \frac{\hbar}{2mr_B^2} = +13.6\text{ eV}$$

For an spherical nucleus of radius R,

If $R < r_B$, same result.

If $R >> r_B$: the electron moves inside the sphere with radius r'_B (r'_B = new Bohr radius).

Charge inside

$$r_B'^4 = e\left(\frac{r_B'^3}{R^3}\right)$$

$\therefore$ $$r'_B = \frac{h^2}{m}\left(\frac{4\pi\varepsilon_0}{e^2}\right)\frac{R^3}{r_B'^3}$$

$$r_B'^4 = (0.51\,\text{Å})\,.\,R^3,\quad R = 10\,\text{Å}$$
$$= 510\,(\text{Å})^4$$
$$r'_B \approx (510)^{1/4}\,\text{Å} < R.$$

$$\text{K.E.} = \frac{1}{2}mv^2 = \frac{m}{2}\cdot\frac{h}{m^2 r_B'^2} = \frac{h}{2m}\cdot\frac{1}{r_B'^2}$$

$$\left(\frac{h^2}{2mr_B^2}\right)\cdot\left(\frac{r_B^2}{r_B'^2}\right) = (13.6\text{ eV})\frac{(0.51)^2}{(510)^{1/2}} = \frac{3.54}{22.6} - 0.16\text{ eV}$$

$$\text{P.E.} = +\left(\frac{e^2}{4\pi\varepsilon_0}\right)\cdot\left(\frac{(r_B'^2 - 3R^2)}{2R^3}\right)$$

$$= +\left(\frac{e^2}{4\pi\varepsilon_0}\cdot\frac{1}{r_B}\right)\cdot\left(\frac{r_B(r_B'^2 - 3R^2)}{R^3}\right)$$

$$= +(27.2\text{ eV})\left[\frac{0.51(\sqrt{510} - 300)}{1000}\right]$$

$$= +(27.2\text{ eV})\,.\,\frac{-141}{1000} = -3.83\text{ eV}\,.$$

S15. The total energy of the electron in the stationary states of the hydrogen atom is given by

$$E_n = \frac{me^4}{8n^2\varepsilon_0^2h^2}$$

where, signs are as usual and the m that occurs in the Bohr formula is the reduced mass of electron and proton in hydrogen atoms,

By Bhor model, $h\nu_{if} = E_{n_i} - E_{n_f}$

On simplifying, $\nu_{if} = \frac{me^4}{8\varepsilon_0^2h^3}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$

Reduced mass for $H = \mu_H = \frac{m_e}{1+\frac{m_e}{M}};\quad m_e\left(1-\frac{m_e}{M}\right)$

Reduced mass for $D = m_e\left(1-\frac{m_e}{2M}\right) = m_e\left(1-\frac{m_e}{2M}\right)\left(1+\frac{m_e}{2M}\right)$

$$h\nu_{ij} = (E_i - E_j)\,\alpha\mu.\ \text{Thus,}\ \lambda_{ij}\,\alpha\,\frac{1}{\mu}$$

If for Hydrogen/Deuterium the wavelength is λ_H/λ_D

$$\frac{\lambda_D}{\lambda_H} = \frac{\mu_H}{\mu_D};\quad \left(1+\frac{m_e}{2M}\right)^{-1};\quad \left(1-\frac{1}{2\times 1840}\right)$$

$$\lambda_D = \lambda_H \times (0.99973)$$

Thus lines are 1217.7 Å, 1027.7 Å, 974.04 Å, 951.143 Å.

S16. For $m_p = 10^{-6}$ times, the mass of an electron, the energy associated with it is given by

$$\begin{aligned} m_pc^2 &= 10^{-6} \times \text{electron mass} \times c^2 \\ &\approx 10^{-6} \times 0.5\,\text{MeV} \\ &\approx 10^{-6} \times 0.5 \times 1.6 \times 10^{-13} \\ &\approx 0.8 \times 10^{-19}\,\text{J} \end{aligned}$$

$$\frac{h}{m_pc} = \frac{hc}{m_pc^2} = \frac{10^{-34}\times 3\times 10^8}{0.8\times 10^{-19}} \approx 4\times 10^{-7}\,\text{m} >> \text{Bohr radius.}$$

$$|\boldsymbol{F}| = \frac{e^2}{4\pi\varepsilon_0}\left[\frac{1}{r^2} + \frac{\lambda}{r}\right]\exp(-\lambda r)$$

where $\lambda^{-1} = \frac{\hbar}{m_pc} \approx 4\times 10^{-7}\,\text{m} >> r_B$

$\therefore$ $$\lambda << \frac{1}{r_B} \quad i.e., \quad \lambda r_B << 1$$

$$U(r) = -\frac{e^2}{4\pi\varepsilon_0} \cdot \frac{\exp(-\lambda r)}{r}$$

$$mvr = h \qquad \therefore \quad v = \frac{h}{mr}$$

Also : $$\frac{mv^2}{r} \approx \left(\frac{e^2}{4\pi\varepsilon_0}\right)\left[\frac{1}{r^2} + \frac{\lambda}{r}\right]$$

$\therefore$ $$\frac{h^2}{mr^3} = \left(\frac{e^2}{4\pi\varepsilon_0}\right)\left[\frac{1}{r^2} + \frac{\lambda}{r}\right]$$

$\therefore$ $$\frac{h^2}{m} = \left(\frac{e^2}{4\pi\varepsilon_0}\right)[r + \lambda r^2]$$

If $$\lambda = 0; \quad r = r_B = \frac{h}{m} \cdot \frac{4\pi\varepsilon_0}{e^2}$$

$$\frac{h^2}{m} = \frac{e^2}{4\pi\varepsilon_0} \cdot r_B$$

Since, $\lambda^{-1} >> r_B$, put $r = r_B + \delta$

$\therefore$ $$r_B = r_B + \delta + \lambda(r_B^2 + \delta^2 + 2\delta r_B); \quad \text{neglect } \delta^2$$

or $$0 = \lambda r_B^2 + \delta(1 + 2\lambda r_B)\ B\ B\ r\ r$$

$$\delta = \frac{-\lambda r_B^2}{1 + 2\lambda r_B} \approx r_B^2(1 - 2\lambda r_B) = -\lambda r_B^2 \text{ since } \lambda r_B << 1$$

$\therefore$ $$V(r) = -\frac{e^2}{4\pi\varepsilon_0} \cdot \frac{\exp(-\lambda\delta - \lambda r_B)}{r_B + \delta}$$

$\therefore$ $$V(r) = -\frac{e^2}{4\pi\varepsilon_0}\frac{1}{r_B}\left[\left(1 - \frac{\delta}{r_B}\right) \cdot (1 - \lambda r_B)\right]$$

$$\approx (-27.2\text{ eV}) \text{ remains unchanged.}$$

$$\text{K.E.} = \frac{1}{2}mv^2 = \frac{1}{2}m \cdot \frac{h^2}{mr^2} = \frac{h^2}{2(r_B + \delta)^2} = \frac{h^2}{2r_B^2}\left(1 - \frac{2\delta}{r_B}\right)$$

$$= (13.6\text{ eV})[1 + 2\lambda r_B]$$

$$\text{Total energy} = -\frac{e^2}{4\pi\varepsilon_0 r_B} + \frac{h^2}{2r_B^2}[1 + 2\lambda r_B]$$

$$= -27.2 + 13.6[1 + 2\lambda r_B] \text{ eV}$$

Change in energy = $13.6 \times 2\lambda r_B$ eV = $27.2\ \lambda r_B$ eV.