Tangent & Normals

Single Correct Answer Type

1. The points of contact of the tangents drawn from the origin to the curve $y = x^2 + 3x + 4$ are

1. (2, 14), (-2, 12) 2. (2, 12), (-2, 2) 3. (2, 14), (-2, 2) 4. (2, 12), (-2, 14)

Key. 3

Sol. Let $P(x_1, y_1)$ be a point on the curve $y = x^2 + 3x + 4$

$\Rightarrow y_1 = x_1^2 + 3x_1 + 4$ ….(1)

$\left(\frac{dy}{dx}\right)_{at(x_1,y_1)} = 2x_1 + 3$

Equation of tangent is : $y - y_1 = m(x - x_1)$

It is passes through (0, 0)

Then $y_1 = 2x_1^2 + 3x_1$ …..(2)

From (1) & (2) $x_1 = \pm 2$

$\therefore$ the points are $(2,14) \& (-2, 2)$

2. If $3x + 2y = 1$ acts as a tangent to $y = f(x)$ at $x = 1/2$ and if

$p = \lim\limits_{x \to 0} \frac{x(x-1)}{f\left(\frac{e^{2x}}{2}\right) - f\left(\frac{e^{-2x}}{2}\right)}$, then, $\sum_{r=1}^{\infty} p^r =$ ______

a) 1/2 b) 1/3 c) 1/6 d) 1/7

Key. A

Sol. slope of $3x + 2y = 1$ is $\frac{-3}{2}$

$\Rightarrow f^1\left(\frac{1}{2}\right) = \frac{-3}{2}$

$p = \lim\limits_{x \to 0} \frac{x(x-1)}{f\left(\frac{e^{2x}}{2}\right) - f\left(\frac{e^{-2x}}{2}\right)}\left(\frac{0}{0}\right) = \frac{-1}{f^1\left(\frac{1}{2}\right) + f^1\left(\frac{1}{2}\right)} = \frac{1}{3}$

$\therefore \sum_{r=1}^{\infty} p^r = \frac{1}{3} + \frac{1}{3^2} + \ldots..\infty = \frac{1/3}{1 - 1/3} = \frac{1/3}{2/3} = \frac{1}{2}$

3. If the tangent drawn at $P\left(t = \frac{\pi}{4}\right)$ to the curve $x = \sec^2 t, y = \cot t$ meets the curve again at R, then, PR=________

a) $\frac{3\sqrt{5}}{2}$ b) $\frac{2\sqrt{5}}{3}$ c) $\frac{5\sqrt{5}}{4}$ d) $\frac{4\sqrt{5}}{5}$

Key. A

Sol. At $t=\frac{\pi}{4}, x=2, y=1 \Rightarrow P$ is $(2,1)$

$$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}}=\frac{-\operatorname{cosec}^2 t}{2\sec t.\sec t.\tan t}=-1/2$$

$\therefore$ tangent at $P(2,1)$ is, $y=\frac{4-x}{2}$

Elimating 't' curve equation is, $x=2,5 \Rightarrow R(5,-1/2) \Rightarrow PR=\frac{3}{2}\sqrt{5}$

4. If the points of contact of tangents to $y=\sin x$, drawn from origin always lie on $\frac{a}{y^2}-\frac{b}{x^2}=c$, then,

a) a,b,c are in AP, but not in GP and HP
b) a,b,c are in GP, but not in HP and AP
c) a,b,c are in HP, but not in AP and GP
d) a,b,c are in AP,GP and HP

Key. D

Sol. Let $P(h,k)$ be any point on $y=\sin x$

$\Rightarrow k=\sin h$. tangent P is $y-k=\cos h(x-h)$

$(0,0) \Rightarrow -k=\cos h.(0-n) \Rightarrow \cos h=\frac{k}{h}$

$\Rightarrow \frac{1}{y^2}-\frac{1}{x^2}=1 \Rightarrow a=1, b=1, c=1$

5. $A(1,0), B(e,1)$ are two points on the curve $y=\log_e x$. If P is a point on the curve at which the tangent to the curve is parallel to the chord AB, then, abscissa of P, is

a) $\frac{e-1}{2}$ b) $\frac{e+1}{2}$ c) $e-1$ d) $e+1$

Key. C

Sol. By LMVT, applied to $f(x)=\log_e x$ on $[1,e], \exists$ an $x_0 \in (1,e) \ni f^1(x_0)=\frac{f(e)-f(1)}{e-1}$

$\Rightarrow x_0=e-1$

6. The abscissa of the points. Where the tangent to the curve $y=x^3-3x^2-9x+5$ is parallel to x-axis is

1) 0 and 0 2) x=1 and -1 3) x=1 and -3 4) x=-1 and 3

Key. 4

Sol. Tangent is parallel to x-axis $\Rightarrow \frac{dy}{dx}=0 \Rightarrow x=-1,3$

7. Co-ordinates of a point on the curve $y=x\log x$ at which the normal is parallel to the line $2x-2y=3$, are

1) (0,0) 2) (e,e) 3) $\left(e^{-2}, 2e^{-2}\right)$ 4) $\left(e^{-2}, -2e^{-2}\right)$

Key. 4

Sol. Slope of the normal $=\frac{-1}{1+\log x} \Rightarrow \frac{-1}{1+\log x}=1 \Rightarrow x=e^{-2}$

8. If the point on $y=x\tan\alpha-\frac{ax^2}{2u^2\cos^2\alpha}\left(0<\alpha<\frac{\pi}{2}\right)$ where the tangent is parallel to y=x has an ordinate $\frac{u^2}{4a}$ then the value of α is

1) $\frac{\pi}{2}$ 2) $\frac{\pi}{6}$ 3) $\frac{\pi}{3}$ 4) $\frac{\pi}{4}$

Key. 3

Sol. Given m=1 $\Rightarrow \tan\alpha-\frac{ax}{u^2\cos^2\alpha}=1 \Rightarrow x=\frac{(\tan\alpha-1)}{a}u^2\cos^2\alpha$ substitute x and y values in given equation $\frac{u^2}{4a}=\frac{u^2}{a}\left[\sin^2\alpha-\frac{1}{2}\right] \Rightarrow \alpha=\frac{\pi}{3}$

9. If at each point of the curve $y=x^3-ax^2+x+1$ the tangent is inclined at an acute angle with the positive direction of the x-axis, then a lies in the interval

1) $[-3,3]$ 2) $[-2,2]$ 3) $\left[-\sqrt{3},\sqrt{3}\right]$ 4) R

Key. 3

Sol. $\frac{dy}{dx}=3x^2-2ax+1, \frac{dy}{dx}>0$ $3x^2-2ax+1>0$

10. The number of tangents to the curve $x^{3/2}+y^{3/2}=a^{3/2}$, where the tangents are equally inclined to the axes, is

1) 2 2) 1 3) 0 4) 4

Key. 2

Sol. $\Rightarrow \frac{dy}{dx} = -\frac{x^{1/2}}{y^{1/2}}$

$\therefore \left(\frac{dy}{dx}\right)_{\alpha,\beta} = 1 \Rightarrow \alpha^{1/2} + \beta^{1/2} = 0$

$\alpha^{3/2} + \beta^{3/2} = a^{3/2} \quad \{\because (\alpha, \beta) \text{ is on the curve}\}$

$\left(\frac{dy}{dx}\right)_{\alpha,\beta} = -1 \quad \Rightarrow \alpha^{1/2} = \beta^{1/2}$

$\therefore \alpha = \beta = \frac{a}{2^{2/3}}$

there is only one point

11. The tangent at any point on the curve $x = a\cos^3\theta, y = a\sin^3\theta$ meets the axes in P and Q. The locus of the mid point of PQ is

1) $x^2 + y^2 = a^2$ 2) $2(x^2 + y^2) = a^2$ 3) $4(x^2 + y^2) = a^2$ 4) $x^2 + y^2 = 4a^2$

Key. 3

Sol. Equation of tangent at θ is $\Rightarrow P = (a\cos\theta, 0), Q = (0, a\sin\theta)$ .Locus of midpoint of PQ is $4(x^2 + y^2) = a^2$

12. If the curves $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{l^2} - \frac{y^2}{m^2} = 1$ cut each other orthogonally then .

1) $a^2 + b^2 = l^2 + m^2$ 2) $a^2 - b^2 = l^2 - m^2$ 3) $a^2 - b^2 = l^2 + m^2$ 4) $a^2 + b^2 = l^2 - m^2$

Key. 3

Sol. If the curves $a_1x^2 + b_1y^2 = 1$, $a_2x^2 + b_2y^2 = 1$ cut each other orthogonally then apply

$\frac{1}{a} - \frac{1}{b} = \frac{1}{a_1} - \frac{1}{b_1}$

13. If the relation between the sub-normal and sub-tangent at any point on the curve

$y^2 = (x+a)^3$ is $p(S.N) = q(S.T)^2 \text{ then } \frac{p}{q} =$

1) $\frac{8}{27}$ 2) $\frac{27}{8}$ 3) $\frac{4}{9}$ 4) $\frac{9}{4}$

Key. 1

Sol. Length of sub normal = $|y_1 m|$

Length of sub tangent = $\left|\frac{y_1}{m}\right|$

14. The sum of the lengths of subtangent and tangent to the curve

$$x = c\left[2\cos\theta - \log\left(\operatorname{cosec}\theta + \cot\theta\right)\right], y = c\sin 2\theta \text{ at } \theta = \frac{\pi}{3}$$ is

1) $\frac{c}{2}$ 2) $2c$ 3) $\frac{3c}{2}$ 4) $\frac{5c}{2}$

Key. 3

Sol. Length of tangent $= \left|\frac{y_1\sqrt{1+m^2}}{m}\right|$

Length of sub-tangent $= \left|\frac{y_1}{m}\right|$

15. The curves $C_1 : y = x^2 - 3; C_2 : y = kx^2, k < 1$ intersect each other at two different points. The tangent drawn to C_2, at one of the points of intersection $A = (a, y_1)(a > 0)$ meets C_1 again at $B(1, y_2)$. $(y_1 \neq y_2)$. Then value of $a = \_\_\_?$

a) 4 b) 3 c) 2 d) 1

Sol : ans: b

solving

$$C_1 \& C_2 \Rightarrow A\left(\sqrt{\frac{3}{1-k}}, \frac{3k}{1-k}\right) = (a, ka^2) \equiv (a, a^2 - 3).$$

$\tan\text{gent } l \text{ to } C_2 \text{ at A is } y + a^2 - 3 = 2kx - - - - - (1)$

$\Rightarrow B = (1, -2) \ (A \neq 1).$

$\text{from expression (1) } -2 + a^2 - 3 = 2a\left(1 - \frac{3}{a^2}\right).$

$\Rightarrow a = 3, a = -2, a = 1$

$\therefore a = 3$

16. Let $f\left(\frac{x+y}{2}\right)=\frac{1}{2}(f(x)+f(y))$ for real x and y. If $f'(0)$ exists and equals to -1 and f(0)=1 then the value of f(2) is

a) 1 b) -1 c) $\frac{1}{2}$ d) 2

KEY : B

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{f(2x)+f(2h)}{2}-f(x)}{h}$$

$$f'(x)=-1 \qquad ;\ f(2x)=2f(x)\text{-}1$$

$$\Rightarrow f(x)=1-x$$

17. If the length of subnormal is equal to length of sub-tangent at point $(3,4)$ on the curve $y=f(x)$ and the tangent at $(3,4)$ to $y=f(x)$ meets the coordinate axes at A and B, then maximum area of the ΔOAB where O is origin, is

(A) $\frac{45}{2}$ sq.units (B) $\frac{49}{2}$ sq.units

(C) $\frac{51}{2}$ sq.units (D) $\frac{81}{2}$ sq.units

KEY : B

Sol : Length of subnormal = length of subtangent

$$\Rightarrow \left|y_1\left(\frac{dy}{dx}\right)_{(x_1y_1)}\right|=\left|\frac{y_1}{\left(\frac{dy}{dx}\right)_{(x_1y_1)}}\right|$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\pm 1$$

If $\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=1$

Then the equation of tangent is $y-x=1$ and area of $\Delta OAB=\frac{1}{2}\times 1\times 1=\frac{1}{2}$

If $\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=-1$

Then the equation of tangent is $x+y=7$ and area of $\Delta OAB=\frac{1}{2}\times 7\times 7=\frac{49}{2}$

18. The equation of normal to the curve $x + y = x^y$, where it cuts the x-axis is

(A) y = x - 1 (B) x + y = 1

(C) 12x + y + 2 = 0 (D) 3x + y = 3

Key: A

Sol: At x-axis, y = 0 $\Rightarrow$ x = 1

$x + y = x^y \Rightarrow \ln(x + y) = y \ln x$

$$\frac{1}{x+y}\left(1+\frac{dy}{dx}\right)=\frac{y}{x}+\frac{dy}{dx}\ln x$$

$$\left(\frac{dy}{dx}(1,0\right) = -1$$

So equation of normal y - 0 = x - 1.

19. Maximum no. of parallel tangents of curves $y = x^3 - x^2 - 2x + 5$ and $y = x^2 - x + 3$ is

(A) 2 (B) 3

(C) 4 (D) none of these

Key: D

Sol: Let m be slope is common tangent

Then m = 2x - 1 and m = $3x^2 - 2x - 2$,

So, infinite common tangents

20. The equation of the straight lines which are both tangent and normal to the curve $27x^2 = 4y^3$ are

a) $x = \pm\sqrt{2}(y-2)$ b) $x = \pm\sqrt{3}(y-2)$

c) $x = \pm\sqrt{2}(y-3)$ d) $x = \pm\sqrt{3}(y-3)$

Key. A

Sol. $x = 2t^3, y = 3t^2 \Rightarrow$ tangent at t is $x - yt = -t^3$ Normal at t_1 is, $xt_1 + y = 2t_1^4 + 3t_1^2$

$$\Rightarrow \frac{1}{t_1} = -t = \frac{-t^3}{2t_1^4 + 3t_1^2} \Rightarrow t^6 - 3t^2 - 2 = 0 \Rightarrow t^2 = 2 \Rightarrow t = \pm\sqrt{2}$$

$\therefore$ lines are $x = \pm\sqrt{2}(y-2)$

21. If $f(x) + f(y) = f(x)f(y) + f(xy), f(1) = 0, f^1(1) = -2$ then, equation to the tangent, drawn to the curve $y = f(x)$ at $x = \sqrt{2}$ is,

a) $2\sqrt{2}x - y - 3 = 0$ b) $2\sqrt{2}x + y - 3 = 0$

c) $2\sqrt{2}x + y + \sqrt{3} = 0$ d) $2\sqrt{2}x + 2y - 3 = 0$

Key. B

Sol. Clearly $f(x) = 1 - x^2$ at $x = \sqrt{2}, y = -1 \Rightarrow$ tangent at $(\sqrt{2}, -1)$ is,

$$y + 1 = -2\sqrt{2}(x - \sqrt{2})$$

22. Let $f(x)$ be a polynomial of degree 5. When f(x) is divided by $(x-1)^3$, the remainder 33, and when f(x) is divided by $(x+1)^3$, the remainder is –3. Then, equation to the tangent drawn to $y = f(x)$ at x = 0 is

a) $135x + 4y + 60 = 0$ b) $135x - 4y - 60 = 0$

c) $135x - 4y + 60 = 0$ d) $135x - 4y + 75 = 0$

Key. C

Sol. $f(x) = \frac{27x^5}{4} - \frac{45x^3}{2} + \frac{135x}{4} + 15$ at $x = 0, y = 15 \Rightarrow f^1(0) = \frac{135}{4}$

$\Rightarrow$ tangent equation is $y-15=\frac{135}{4}(x)\Rightarrow 135x-4y+60=0$

23. If the equation $x^{5/3}-5x^{2/3}=K$ has exactly one positive root, then, the complete solution set of K is,
a) $(-\infty,\infty)$ b) $(-\infty,0)$ c) $(3,\infty)$ d) $(0,\infty)$

Key. D

Sol. Sketch $y=x^{5/3}-5x^{2/3}$ and $y=K$

24. The equation of the straight lines which are both tangent and normal to the curve $27x^2=4y^3$ are
a) $x=\pm\sqrt{2}(y-2)$ b) $x=\pm\sqrt{3}(y-2)$
c) $x=\pm\sqrt{2}(y-3)$ d) $x=\pm\sqrt{3}(y-3)$

Key. A

Sol. $x=2t^3, y=3t^2\Rightarrow$ tangent at t is $x-yt=-t^3$ Normal at t_1 is, $xt_1+y=2t_1^4+3t_1^2$
$\Rightarrow\frac{1}{t_1}=-t=\frac{-t^3}{2t_1^4+3t_1^2}\Rightarrow t^6-3t^2-2=0\Rightarrow t^2=2\Rightarrow t=\pm\sqrt{2}$
$\therefore$ lines are $x=\pm\sqrt{2}(y-2)$

25. If $f(x)+f(y)=f(x)f(y)+f(xy), f(1)=0, f^1(1)=-2$ then, equation to the tangent, drawn to the curve $y=f(x)$ at $x=\sqrt{2}$ is,
a) $2\sqrt{2}x-y-3=0$ b) $2\sqrt{2}x+y-3=0$
c) $2\sqrt{2}x+y+\sqrt{3}=0$ d) $2\sqrt{2}x+2y-3=0$

Key. B

Sol. Clearly $f(x)=1-x^2$ at $x=\sqrt{2}, y=-1\Rightarrow$ tangent at $(\sqrt{2},-1)$ is, $y+1=-2\sqrt{2}(x-\sqrt{2})$

26. Let $f(x)$ be a polynomial of degree 5. When f(x) is divided by $(x-1)^3$, the remainder 33, and when f(x) is divided by $(x+1)^3$, the remainder is –3. Then, equation to the tangent drawn to $y=f(x)$ at x = 0 is
a) $135x+4y+60=0$ b) $135x-4y-60=0$
c) $135x-4y+60=0$ d) $135x-4y+75=0$

Key. C

Sol. $f(x)=\frac{27x^5}{4}-\frac{45x^3}{2}+\frac{135x}{4}+15$ at $x=0, y=15\Rightarrow f^1(0)=\frac{135}{4}$

$\Rightarrow$ tangent equation is $y-15=\frac{135}{4}(x)\Rightarrow 135x-4y+60=0$

27. Two runners A and B start at the origin and run along positive x-axis, with B running three times as fast as A. An observer, standing one unit above the origin, keeps A and B in view. Then the maximum angle of sight 'θ' between the observes view of A and B is
a) $\pi/8$ b) $\pi/6$ c) $\pi/3$ d) $\pi/4$

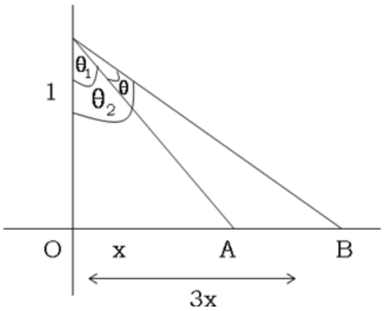
Key. B

Sol. $\tan\theta = \tan(\theta_2 - \theta_1) \Rightarrow \tan\theta = \dfrac{3x - x}{1 + 3x.x} = \dfrac{2x}{1+3x^2}$

$$\text{let } y = \frac{2x}{1+3x^2}\ \frac{dy}{dx} = \frac{2(1-3x^2)}{(1+3x^2)^2}$$

$$\frac{dy}{dx} = 0 \Rightarrow x = \frac{1}{\sqrt{3}} \text{ and } \frac{d^2y}{dx^2} = \frac{-24x}{(1+3x^2)^3} < 0 \text{ for } x = 1/\sqrt{3}$$

$$\Rightarrow \theta = \pi \backslash 6$$



28. If the line joining the points $(0,3)$ and $(5,-2)$ is a tangent to the curve $y = \dfrac{c}{x+1}$, then value of c is

A) 1 B) -2 C) 4 D) -4.

Key. 3

Sol. Eqn. of the line joining given points is $(y+2) = \dfrac{-2-3}{5-0}(x-5)$.

$\Rightarrow y + x = 3.$

29. The number of points on the curve $y^3 - 3xy + 2 = 0$ where the tangent is either horizontal or vertical is

A) 0 B) 1 C) 2 D) > 2.

Key. 2

Sol. $3yy^1 - 3y - 3xy^1 = 0 \Rightarrow y^1 = \dfrac{y}{y^2 - x}$.

$y^1 = 0 \Rightarrow y = 0,$ no real x

$y^1 = \infty \Rightarrow y^2 = x \Rightarrow y^3 = 1 \Rightarrow y = 1.$

The point is $(1,1)$.

30. The tangent to the curve $y = \dfrac{1+3x^2}{3+x^2}$ drawn at the points for which $y = 1$, intersect at

A) $(0,0)$ B) $(0,1)$ C) $(1,0)$ D) $(1,1)$

Key. 1

Sol. $y = 1 \Rightarrow x = \pm 1$ points are $(1,1),(-1,1) \Rightarrow \dfrac{dy}{dx} = \dfrac{16x}{(3+x^2)^2}, \left(\dfrac{dy}{dx}\right)_{(1,1)} = 1, \left(\dfrac{dy}{dx}\right)_{(-1,1)} = -1$

Eq. of tangent at (1,1) is y – 1 = (x – 1) => x – y = 0

Eq. of tangent at (-1, 1) y – 1 = -1 (x + 1) => x + y = 0

Both tangents pass through origin.

31. The equation of the normal to the curve $x + y = x^y$, where it cuts x-axis is

A) $y = x$ B) $y = x + 1$

C) $y = x - 1$ D) $x + y = 1$.

Key. 3

Sol. Point is $(1, 0)$

After doing logarithmic differentiation, we get $\left(\frac{dy}{dx}\right)_{(1,0)} = -1$.

normal equation is $y = x - 1$.

32. The distance of the origin from the normal to the curve $y = (1+x)^y + \sin^{-1}(\sin^2 x)$ at x= 0is

A) 1 B) $\frac{1}{2}$ C) $\sqrt{2}$ D) $\frac{1}{\sqrt{2}}$

Key. 4

Sol. $x = 0 \rightarrow y = 1$ .Differentiating the given relation

$$y' = (1+x)^y \left(\frac{y}{1+x} + y' \ln(1+x)\right) + \frac{2\sin x \cos x}{\sqrt{1-\sin^4 x}}$$

$$y'(0) = 1$$

Normal is 1. $(y-1) + (x-0) = 0 \rightarrow x + y - 1 = 0$

The distance of the origin from it is $\frac{1}{\sqrt{2}}$

33. The number of tangents to the curve $y = \cos(x+y), |x| \le 2\pi$, that are parallel to the line $x + 2y = 0$ is

A) 0 B) 1 C) 2 D) > 2

Key. 3

Sol. $y' = -\sin(x+y)(1+y')$

Slope of tangent is $-\frac{1}{2} = y'$

$\frac{1}{2} = \sin(x+y)\frac{1}{2} \Rightarrow \sin(x+y) = 1$, $\cos(x+y) = 0$

$\rightarrow y = 0 \rightarrow 0 = \cos x \rightarrow x = \frac{\pi}{2}, \frac{-3\pi}{2}$

which satisfies the above equation.

34. The slope of the straight line which is both tangent and normal to the curve $4x^3 = 27y^2$ is

A) ± 1 B) $\pm \frac{1}{2}$ C) $\pm \frac{1}{\sqrt{2}}$ D) $\pm\sqrt{2}$.

Key. 4

Sol. $x = 3t^2, y = 2t^3, \frac{dy}{dx} = t$.

The tangent at t, $y - 2t^3 = t(x - 3t^2)$

The normal at t_1, $t_1 y + x = 2t_1^4 + 3t_1^2$.

(1) , (2) are identical ,

Comparing we get, $-t^3 = 2t_1^3 + 3t_1, t_1 = \frac{1}{t}$. Eliminating t_1, we get $t^6 = 2 + 3t^2$.

$\Rightarrow t^2 = 2, t = \pm\sqrt{2}$

35. The tangent at any point P on the curve $x^{2/3} + y^{2/3} = 4$ meets the coordinate axes at A and B Then $AB =$

A) 2 B) 4 C) 8 D) 16

Key. 3

Sol. $x = 8\cos^3\theta$, $y = 8\sin^3\theta$, $\frac{dy}{dx} = -\frac{\sin\theta}{\cos\theta}$.

Tangent at θ, $y - 8\sin^2\theta = -\frac{\sin\theta}{\cos\theta}\left(x - 8\cos^3\theta\right)$

$x\sin\theta + y\cos\theta = 8\sin\theta\cos\theta$

$OA = 8\cos\theta$, $OB = 8\sin\theta$

$AB = \sqrt{OA^2 + OB^2} = 8$.

36. If the tangent to the curve $x = 1 - 3t^2$, $y = t - 3t^3$ at the point $P(-2,2)$ meets the curve again at Q, the angle between the tangents at P and Q is

A) $\frac{\pi}{6}$ B) $\frac{\pi}{4}$ C) $\frac{\pi}{3}$ D) $\frac{\pi}{2}$.

Key. 4

Sol. $\frac{dy}{dx} = \frac{9t^2 - 1}{6t}$

$x = -2, y = 2 \Rightarrow t = -1, \frac{dy}{dx} = -\frac{4}{3}$

The tangent at P, $y - 2 = -\frac{4}{3}(x + 2) \Rightarrow 4x + 3y = -2$.

$4\left(1 - 3t^2\right) + 3\left(t - 3t^3\right) = -2$

$\Rightarrow (t+1)^2(3t - 2) = 0$

$t = \frac{2}{3}$, Slope of tangent at Q is $\frac{dy}{dx} = \frac{9\left(\frac{4}{9}\right) - 1}{6\left(\frac{2}{3}\right)} = \frac{3}{4}$.

(1), (2) $\Rightarrow$ The tangents are perpendicular.

37. The curves $x^3 - 3xy^2 = a$ and $3x^2y - y^3 = b$ intersect at an angle of

A) $\frac{\pi}{4}$ B) $\frac{\pi}{3}$ C) $\frac{\pi}{2}$ D) $\frac{\pi}{6}$.

Key. 3

Sol. Clearly $m_1m_2 = -1$.

38. The cosine of the angle of intersection of curves $f(x) = 2^x \log_e x$ and $g(x) = x^{2x} - 1$ is

A) 1 B) 0 C) $\frac{1}{2}$ D) $\frac{\sqrt{3}}{2}$.

Key. 1

Sol. Clearly, $(1,0)$ is the point of intersection of the given curves.

Now, $f'(x) = \dfrac{2^x}{x} + 2^x(\log_e 2)(\log_e x)$

$\therefore$ Slope of tangent to the curve $f(x)$ at $(1,0)$, $m_1 = 2$.

$$g'(x) = \frac{d}{dx}\left(e^{2x\log x} - 1\right) = x^{2x}\left(2x \times \frac{1}{x} + 2\log_e x\right)$$

$\therefore$ Slope of tangent to the curve $g(x)$ at $(1,0)$, $m_2 = 2$.

Since $m_1 = m_2 = 2$.

$\therefore$ Two curves touch each other, so the angle between them is 0.

Hence, $\cos\theta = \cos 0 = 1$.

39. The curves $\dfrac{x^2}{a} + \dfrac{y^2}{b} = 1$ and $\dfrac{x^2}{a_1} + \dfrac{y^2}{b_1} = 1$ will cut orthogonally if

A) $ab = a_1 b_1$ B) $\dfrac{a}{b} = \dfrac{a_1}{b_1}$ C) $a + b = a_1 + b_1$ D) $a - b = a_1 - b_1$

Key. 4

Sol. $\dfrac{x^2}{a} + \dfrac{y^2}{b} = 1$ . . . (1)

$\dfrac{x^2}{a_1} + \dfrac{y^2}{b_1} = 1$ . . . (2)

(1)-(2) $\rightarrow$ $x^2\left(\dfrac{1}{a} - \dfrac{1}{a_1}\right) + y^2\left(\dfrac{1}{b} - \dfrac{1}{b_1}\right) = 0$.

$\rightarrow$ $\dfrac{x^2(a_1 - a)}{a_1 a} = -\dfrac{y^2(b_1 - b)}{b_1 b}$ . . .(3)

Differentiating (1), $\dfrac{x}{a} + \dfrac{y m_1}{b} = 0$

$\Rightarrow m_1 = \dfrac{-bx}{ay},\ m_2 = \dfrac{-b_1 x}{a_1 y}$

$$m_1 m_2 = \frac{b_1 b_2 x^2}{a a_1 y^2} = -\left(\frac{b_1 - b}{a_1 - a}\right) = -1.$$

40. The value of n in the equation of curve $y = a^{1-n}x^n$, so that the sub-normal may be of constant length is

A) 2 B) $\dfrac{3}{2}$ C) $\dfrac{1}{2}$ D) 1

Key. 3

Sol. Taking log and differentiating both sides, we get $\dfrac{dy}{dx} = \dfrac{ny}{x}$ . . . (1)

Length of sub-normal $= na^{2-2n}.x^{2n-1}$

$$n=\frac{1}{2}.$$

41. Let $f(x)=x^2+xg'(1)+g''(2)$ and $g(x)=f(1)x^2+xf'(x)+f''(x)$, then $f(3)+g(3)=$

A) 7 B) -7 C) 0 D) 6

Key. 2

Sol. Let $g'(1)=a, g^{11}(a)=b$ then $f(x)=x^2+ax+b$ then $f(1)=1+a+b$

$$g(x)=(1+a+b)x^2+x(2x+a)+b$$

$$g'(x)=2x(3+a+b)+a$$

$$g'(1)=a \Rightarrow a+b+3=0\,,\quad g''(2)=b \Rightarrow 2a+b=-6$$

42. Tangents are drawn from origin to the curve $y=\sin x+\cos x$ .Then their points of contact lie on the curve

a) $\frac{1}{x^2}+\frac{2}{y^2}=1$ b) $\frac{2}{x^2}-\frac{1}{y^2}=1$ c) $\frac{2}{x^2}+\frac{1}{y^2}=1$ d) $\frac{2}{y^2}-\frac{1}{x^2}=1$

Key. D

Sol. $y_1=\sqrt{2}\sin\left(x_1+\frac{\pi}{4}\right),\frac{dy}{dx}=\frac{y_1}{x_1}$ where (x_1,y_1) is point on the curve

$$\therefore \frac{y_1^2}{x_1^2}=2\cos^2\left(x_1+\frac{\pi}{4}\right)=2\left(\frac{y_1^2}{2}+1\right)$$

$\Rightarrow$ Locus of (x_1,y_1) is $\frac{2}{y^2}-\frac{1}{x^2}=1$

43. The abscissa of two points on $y=(2010)x^2+(2011)x-2011$ are 2010 and 2012. if the chord joining those two points is parallel to tangent at P on the curve then the ordinate of P is equal to

a) $(2009)(2010)(2011)$ b) $(2010)(2011)(2012)$

c) $(2011)(2012)(2013)$ d) none

Key. B

Sol. Apply LMVT with a = 2010, b = 2012

$$f(x)=2010x^2+2011x-2011.$$

$$\frac{f(b)-f(a)}{b-a}=f'(c)\Rightarrow c=2011, f(c)=(2010)(2011)(2012)$$

44. Tangent at P_1 other than origin on the curve $y = x^3$ meets the curve again at P_2. The tangent at P_2 meets the curve again at P_3 and so on .... then $\frac{\text{area of }\Delta P_1P_2P_3}{\text{area of }\Delta P_2P_3P_4}$ equals

a) 1 : 20 b) 1 : 16 c) 1 : 8 d) 1 : 2

Key. B

Sol. Let $P_1=\left(t_1,t_1^3\right)P_2=\left(t_2,t_2^3\right),P_3\left(t_3,t_3^3\right)$.......

Solving tangent equation at P_1 with the curve again we get $t_2=-2t_1$. Repeating the process we have $t_3=4t_1$ $t_4=-8t_1$.............

$$\therefore \frac{\Delta P_1P_2P_3}{\Delta P_2P_3P_4}=\begin{vmatrix} t_1 & t_1^3 & 1 \\ t_2 & t_2^3 & 1 \\ t_3 & t_3^3 & 1 \end{vmatrix} \div \begin{vmatrix} t_2 & t_2^3 & 1 \\ t_3 & t_3^3 & 1 \\ t_4 & t_4^3 & 1 \end{vmatrix}=\frac{1}{16}$$

45. The value of parameter t so that the line $(4-t)x+ty+\left(a^3-1\right)=0$ is normal to the curve $xy=1$ may lie in the interval

A) (1, 4) B) $(-\alpha,0)\cup(4,\alpha)$ C) $(-4,4)$ D) $[3,4]$

Key. B

Sol. Slope of line $(4-t)x+ty+\left(a^3-1\right)=0$

is $\frac{-(4-t)}{t}$ (or) $\frac{t-4}{t}$

$\because xy=1$

$\therefore \frac{dy}{dx}=\frac{-y}{x}=\frac{-1}{x^2}$

$\therefore$ slope of normal = $x^2=\frac{t-4}{t}$

$\therefore x^2>0$

$\frac{t-4}{t}>0$

$t\in(-\infty,0)\cup(4,\infty)$

46. The angle of intersection of curves $y=\left[\,|\sin x|+|\cos x|\,\right]$ and $x^2+y^2=5$, where $[\bullet]$ denotes greatest integral function is

A) $Tan^{-1}(2)$ B) $Tan^{-1}\left(\sqrt{2}\right)$ C) $Tan^{-1}\left(\sqrt{3}\right)$ D) $Tan^{-1}(3)$

Key. A

Sol. We know that $1\le|\sin x|+|\cos x|\le\sqrt{2}$

$\therefore y=\left[|\sin x|+\cos x\right]=1$

Let P and Q be the points of intersection of given curves clearly the given curves meet at points where $y=1$, so we get

$x^2+1=5$

$\Rightarrow x=\pm 2$

$\therefore P(2,1)\ and\ Q(-2,1)$

Now $x^2+y^2=5$

$\Rightarrow x=\pm 2$

$\therefore P(2,1)\ and\ Q(-2,1)$

Now $x^2+y^2=5$

$$\Rightarrow \frac{dy}{dx}=\frac{-x}{y},\left(\frac{dy}{dx}\right)_{(2,1)}=-2,\left(\frac{dy}{dx}\right)_{(-2,1)}=2$$

Clearly the slope of a line $y=1,$ is 0 and the slope of tangent at P and Q are -2 and 2 respectively.

$\therefore$ The angle of intersection is $\tan^{-1}(2)$

47. If the tangent at (1, 1) on $y^2=x(2-x)^2$ meets the curve again at P, then P is

a) (4, 4) b) (-1, 2) c) (9/4, 3/8) d) (9/5, 3/8)

Key. C

Sol. $2y\frac{dy}{dx}=(2-x)^2-2x(2-x),$ so $\left.\frac{dy}{dx}\right|_{(1,1)}=-\frac{1}{2}$ Therefore, the equation of tangent at (1, 1) is

$$y-1=-\frac{1}{2}(x-1)$$

$$\Rightarrow y=\frac{-x+3}{2}$$

The intersection of the tangent and the curve is given by $(1/4)(-x+3)^2=x(4+x^2-4x)$

$$\Rightarrow x^2-6x+9=16x+4x^3-16x^2$$

$$\Rightarrow 4x^3-17x^2+22x-9=0$$

$$\Rightarrow (x-1)(4x^2-13x+9)=0 \quad \Rightarrow (x-1)^2(4x-9)=0$$

Since x = 1 is already the point of tangency, x = 9/4 and $y^2=\frac{9}{4}\left(2-\frac{9}{4}\right)^2=\frac{9}{24}$. Thus the required point is $(9/4,3/8)$.

48. The equation of the normal to the curve parametrically represented by $x=t^2+3t-8$ and $y=2t^2-2t-5$ at the point P(2, -1) is

a) $2x+3y-1=0$ b) $6x-7y-11=0$

c) $7x+6y-8=0$ d) $3x+y-1=0$

Key. C

Sol. $\left.\begin{array}{l} t^2+3t-8=2 \Rightarrow t=2,-5 \\ 2t^2-2t-5=-1 \Rightarrow t=2,-1 \end{array}\right\} \Rightarrow t=2,\ \frac{dy}{dx}=\frac{4t-2}{2t+3} \Rightarrow \left(\frac{dy}{dx}\right)_{t=2}=\frac{6}{7}$

$equaiton\ of\ normal\ \ y+1=\frac{-7}{6}(x-2)$

49. Tangents are drawn from origin to the curve y = cos x, their points of contact lie on the curve

a) $x^2+y^2=x^2y^2$ b) $y^2-x^2=x^2y^2$

c) $x^2+y^2=1$ d) $x^2-y^2=x^2y^2$

Key. D

Sol. Let point of contact is (h, k)

$\Rightarrow k=\cos h--(1)\quad eq. of\ \tan gent\ at\ (h,k)\quad y-k=-\sin h(x-h), it\ passes\ through$
$origin \Rightarrow -k=h.\sin h---(2)$

$\cos^2 h+\sin^2 h=k^2+\frac{k^2}{h^2} \Rightarrow 1=y^2+\frac{y^2}{x^2}\ is\ the\ locus\ of\ po\int of\ contact$

50. The angle between tangents at the point of intersection of two curves $x^3-3xy^2+2=0, 3x^2y-y^3=2$ is

a) $\frac{\pi}{6}$ b) $\frac{\pi}{4}$ c) $\frac{\pi}{3}$ d) $\frac{\pi}{2}$

Key. D

Sol. Let the point of intersection is (x, y)

$x^3-3xy^2+2=0 \Rightarrow \frac{dy}{dx}=\frac{x^2-y^2}{2xy},\quad 3x^2y-y^3=2 \Rightarrow \frac{dy}{dx}=\frac{2xy}{y^2-x^2},\quad m_1.m_2=-1 \Rightarrow \theta=90^0$

51. Let the equation of a curve in parametric form be $x=a(\theta+\sin\theta),\ y=a(1-\cos\theta)$. The angle between the tangent drawn at the point $\theta=\frac{\pi}{3}$ and normal drawn at the point $\theta=\frac{2\pi}{3}$ is

a) $\frac{\pi}{6}$ b) $\frac{\pi}{4}$ c) $\frac{\pi}{3}$ d) $\frac{\pi}{2}$

Key. C

Sol. $\frac{dy}{dx}=\frac{a\sin\theta}{a(1+\cos\theta)}=\tan\frac{\theta}{2}$

$$m_1 = \tan\frac{\frac{\pi}{3}}{2} = \frac{1}{\sqrt{3}},\ m_2 = -\frac{1}{\tan\frac{\theta}{2}} = -\frac{1}{\tan\frac{\pi}{3}} = \frac{-1}{\sqrt{3}},\ \tan\theta = \left|\frac{m_1 - m_2}{1+m_1m_2}\right| = \sqrt{3} \Rightarrow \theta = 60^0$$

52. Let the equation of a curve be $\frac{x^2}{4}+\frac{y^2}{3}=1$ where $\left(2\cos\theta, \sqrt{3}\sin\theta\right)$ is a general point on the curve. If the tangent to the given curve intersects the co-ordinate axes at points A, B, then the locus of midpoint of AB is

a) $2x^2+\sqrt{3}y^2=4$ b) $3x^2+4y^2=4x^2y^2$

c) $3x^2+4y^2=x^2y^2$ d) $4x^2+3y^2=4x^2y^2$

Key. B

Sol. Equation of tangent is

$$y-\sqrt{3}\sin\theta = \frac{-\sqrt{3}}{2}.\cot\theta\left(x-2\cos\theta\right) \Rightarrow x\ intercept\left(x_0\right) = \frac{2}{\cos\theta} \Rightarrow \cos\theta = \frac{2}{x_0},$$

$$y\ intercept\left(y_0\right) = \frac{\sqrt{3}}{\sin\theta} \Rightarrow \sin\theta = \frac{\sqrt{3}}{y_0},\ if\ mid\ point\ is\left(h,k\right)$$

$$h=\frac{x_0}{2},\ k=\frac{y_0}{2},\ \cos\theta=\frac{1}{h}, \sin\theta=\frac{\sqrt{3}}{2k} \Rightarrow \frac{1}{h^2}+\frac{3}{4k^2}=1$$

53. If the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $\frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2}=1$ cut each other orthogonally, then

a) $a^2+b^2=\alpha^2+\beta^2$ b) $a^2-b^2=\alpha^2-\beta^2$

c) $a^2-b^2=\alpha^2+\beta^2$ d) $a^2+b^2=\alpha^2-\beta^2$

Key. C

Sol.

$$Slope\ of\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ at\ P\left(x_0,y_0\right) is\ -\frac{b^2x_0}{a^2y_0},\ Slope\ of\ \frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2}=1\ at\ P\left(x_0,y_0\right)\ is\ \frac{\beta^2x_0}{\alpha^2y_0}$$

$$\because M_1M_2=-1 \Rightarrow b^2\beta^2x_0^2 = a^2\alpha^2y_0^2\ ---(1)$$

now solving the curves

$$x_0^2\left(\frac{1}{a^2}-\frac{1}{\alpha^2}\right) = -y_0^2\left(\frac{1}{b^2}+\frac{1}{\beta^2}\right)\ ----(2)$$

from (1) & (2)

$$\frac{\frac{1}{a^2}-\frac{1}{\alpha^2}}{\frac{1}{b^2}+\frac{1}{\beta^2}} = \frac{b^2\beta^2}{a^2\alpha^2} \Rightarrow a^2-b^2=\alpha^2+\beta^2$$

54. The rate of change of $\sqrt{x^2+16}$ with respect to $\frac{x}{x-1}$ at x = 3 is

a) 1 b) $\frac{11}{5}$ c) $-\frac{12}{5}$ d) –3

Key. C

Sol. $u=\sqrt{x^2+16}\ \frac{du}{dx}=\frac{2x}{2\sqrt{x^2+16}}=\frac{x}{\sqrt{x^2+16}},\ V=\frac{x}{x-1}\Rightarrow\frac{dv}{dx}=\frac{-1}{(x-1)^2}$

$\frac{du}{dv}=\frac{du/dx}{dv/dx}=\frac{-12}{5}$

55. A curve represented parametrically by the equation $x=t^3-4t^2-3t$ and $y=2t^2+3t-5$ where $t\in R.$ If H denotes the number of point(s) on the curve where the tangent is horizontal and V is the number of point(s) where the tangent is vertical then

a) H = 2, V = 1 b) H = 1, V = 2 c) H = 2, V = 2 d) H = 1, V = 1

Key. B

Sol. $\frac{dy}{dt}=4t+3,\ \frac{dx}{dt}=3t^2-8t-3$ Tangents are horizontal if $\frac{dy}{dx}=0$

$\Rightarrow\frac{dy}{dt}=0,\ \frac{dx}{dt}\neq 0,\ 4t+3=0\Rightarrow t=\frac{-3}{4}$

Tangents are vertical if $\frac{dx}{dy}=0$ $\frac{dx}{dt}=0,\frac{dy}{dt}\neq 0$ $3t^2-8t-3=0\Rightarrow t=3,\frac{-1}{3}$

56. The tangents to the curve $y=\frac{1+3x^2}{3+x^2}$ drawn at the points for which y = 1, intersect at

a) (0, 0) b) (0, 1) c) (1, 0) d) (1, 1)

Key. A

Sol. $y=1\Rightarrow x=\pm 1$ $point\,s\,are(1,1),(-1,1)\Rightarrow\frac{dy}{dx}=\frac{16x}{\left(3+x^2\right)^2},\left(\frac{dy}{dx}\right)_{(1,1)}=1,\left(\frac{dy}{dx}\right)_{(-1,1)}=-1$

Eq. of tangent at (1,1) is y – 1 = (x – 1) => x – y = 0

Eq. of tangent at (-1, 1) y – 1 = -1 (x + 1) => x + y = 0

Both tangents pass through origin.

57. A cyclist moving on a level road at 4 m/s stops , pedalling and free wheels to rest. The retardation of the cycle has two components, a constant 0.08 m/s^2 due to friction in the working parts, and resistance of 0.02 v^2/s^2 where v is speed in meter per second. The distance traversed by the cycle before it comes to rest (approximately) is

a) $40\frac{1}{4}mts$ b) $40\frac{1}{2}mts$ c) $20\frac{1}{2}mts$ d) $20\frac{1}{4}mts$

Key. A

Sol. Let x be the displace ment of the particle and let its acceleration of particle at P is a

$$\mathrm{v}=\frac{\mathrm{dx}}{\mathrm{dt}}\text{ and } \mathrm{a}=\frac{\mathrm{dv}}{\mathrm{dt}}=\frac{\mathrm{d^2x}}{\mathrm{dt^2}}=\mathrm{v}\frac{\mathrm{dv}}{\mathrm{dx}}$$

By dater retardation = 0.08+ 0.02 v^2 = 0.02 (4+x^2)

$$v\frac{dx}{dt}=-0.02\left(4+v^2\right)\Rightarrow\int_0^{x^1}dx=\frac{-1}{0.04}\int_4^0\frac{2v}{4+v^2}dv\ \Rightarrow x^1=\frac{\log 5}{0.04}\approx 40\frac{1}{4}\text{mts}$$

58. For $x=t^2-1, y=t^2-t$, the tangent line is perpendicular to x-axis when

A) $t=0$ B) $t=\infty$ C) $t=1/\sqrt{3}$ D) $t=-1/\sqrt{3}$

Key. A

Sol. $\frac{dy}{dx}=\tan\theta=\infty\Rightarrow\frac{dx}{dy}=0\quad\frac{dy}{dt}\neq 0$

$2t=0$ and $2t-1\neq 0\Rightarrow t=0$ and $t\neq 1/2$

59. The acute angle between the curves $y=|x^2-1|$ and $y=|x^2-3|$ at their points of intersection is

a) $\pi/4$ b) $\tan^{-1}(4\sqrt{2}/7)$

c) $\tan^{-1}(4\sqrt{7})$ d) $\tan^{-1}(2\sqrt{2}/7)$

Key. B

Sol. The point of intersection is $x^2=2, y=1$. The given equations represent four parabolas.

$y=\pm(x^2-1)$ and $y=\pm(x^2-3)$

The curves intersect when $1<x^2<3$ or $1<x<\sqrt{3}$ or $-\sqrt{3}<x<-1$

$\therefore\quad y=x^2-1$ and $y=-(x^2-3)$

The points of intersection are $\left(\pm\sqrt{2},1\right)$

At $\left(\sqrt{2},1\right)$, $m_1=2x=2\sqrt{2}, m_2=-2x=-2\sqrt{2}$

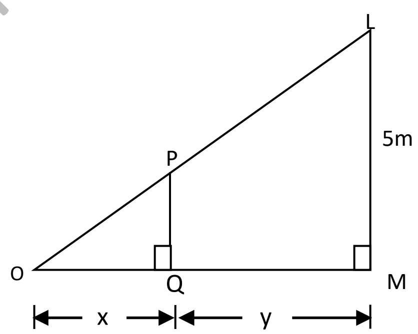
$$\therefore\ \tan\theta=\left|\frac{4\sqrt{2}}{1-8}\right|=\frac{4\sqrt{2}}{7}\Rightarrow\theta=\tan^{-1}\left(\frac{4\sqrt{2}}{7}\right).$$

60. A man of height 2m walks directly away from a lamp at a height of 5m, on a level road at 3m/s. The rate at which the length of his shadow is increasing is

a) 1m/s b) 2m/s c) 3m/s d) 4m/s

Key. B

Sol. Let *L* be the lamp and *PQ* be the man and *OQ* = *x* metre be his shadow and let *MQ* = *y metre.*



$\therefore\quad\frac{dy}{dt}=$ speed of the man

$=3\ m/s$ (given)

$\because \Delta OPQ$ and ΔOLM are similar

$\therefore \quad \frac{OM}{OQ} = \frac{LM}{PQ}$

$\Rightarrow \quad \frac{x+y}{x} = \frac{5}{2}$

$\Rightarrow \quad y = \frac{3}{2}x$

$\therefore \quad \frac{dy}{dt} = \frac{3}{2}\frac{dx}{dt}$

$\Rightarrow \quad 3 = \frac{3}{2}\frac{dx}{dt}$

$\Rightarrow \quad \frac{dx}{dt} = 2\ m/s.$

61. The parametric equations of a curve are $x = t^2$ and $y = t^3$. $A(t_1), B(t_2)$ are points on the curve. If $t_1 = 1, t_2 = 3$ then the abscissa of the point P on the curve the tangent at which is parallel to chord AB is

A) 13 / 6 B) 169 / 36 C) 17 / 36 D) 27 / 4

Key. B

Sol. (t^2, t^3) is a/pt on the curve.

$\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 3t^2$

$\frac{dy}{dx} = \frac{3t^2}{2t} = \frac{3}{2}t$

A=(1,1) and B =(9,27)

Slope of AB = $\frac{27-1}{9-1} = \frac{26}{8} = \frac{13}{4}$

$\frac{3}{2}t = \frac{13}{14} \Rightarrow t = \frac{13}{6}$

$\therefore Tgt$ at $\left(\left(\frac{13}{6}\right)^2, \left(\frac{13}{6}\right)^3\right)$ is parallel to AB

62. The sum of the coordinates of the point on the graph of $f(x) = x^3 + 4x$ the tangent at which is parallel to the chord joining the points (-2, -16) and (1, 5) is

A) –6 B) 4 C) –8 D) 5/2

Key. A

Sol. Slope of chord $= \frac{5-(16)}{1-(-2)} = \frac{21}{3} = 7$

$f^1(x) = 3x^2 + 4$

By L.M.V.T $\exists C \in (-2,1)$ such that $f^1(C) = 7$

$3c^2 + 4 = 7 \Rightarrow C = \pm 1$

$\therefore C = -1$

Point $=(C, f(C))=(-1,-5)$

63. The maximum value of the sum of the intercepts made by any tangent to the curve $\left(a\sin^2\theta, 2a\sin\theta\right)$ with the axes is

(a) 2a (b) a/4 (c) a/2 (d) a

Key. A

Sol. Equation of tangent $\dfrac{y-2a\sin\theta}{x-a\sin^2\theta}=\dfrac{1}{\sin\theta}$

$\Rightarrow \quad \dfrac{x}{-a\sin^2\theta}+\dfrac{y}{a\sin\theta}=1$

Sum of intercepts = $a\left(\sin^2\theta+\sin\theta\right)$

which is maximum when sin θ = 1

(sum of intercepts)$_{max}$ = 2a

64. The tangent to the curve y = e^x drawn at the point $\left(c, e^c\right)$ intersects the line joining the points $\left(c-1, e^{c-1}\right)$ and $\left(c+1, e^{c+1}\right)$

(a) on the left of x = c (b) on the right of x = c

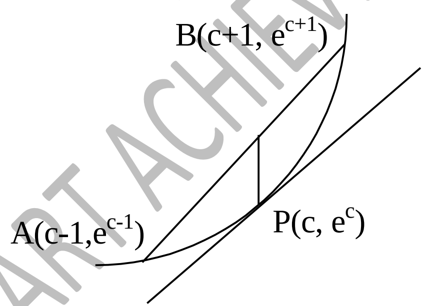
(c) at no point (d) at all points

Key. A

Sol. Slope of AB = $\dfrac{e^{c+1}-e^{c-1}}{2}$

Slope of tangent is e^c

$\dfrac{e^{c+1}-e^{c-1}}{2}>e^c \quad \left(\because\ e-\dfrac{1}{e}>2\right)$



y-coordinate of straight line AB at x = c will be more than y-coordinate of the tangent at x = c for this graph.

Also rate of increasing of AB is more than tangent. So already these two lines had interested before x = c.

Tangent & Normals

Multiple Correct Answer Type

1. If the tangent to the curve $2y^3 = ax^2 + x^3$ at the point (a,a) cuts off intercepts α and β on the coordinate axes, (where $\alpha^2 + \beta^2 = 61$) then the value of a is

A) -30 B) 10 C) 20 D) 30

Key. A,D

Sol. $\because \quad 2y^3 = ax^2 + x^3 \quad \therefore \quad 6y^2 \frac{dy}{dx} = 2ax + 3x^2$

$\Rightarrow \quad \frac{dy}{dx} = \left(\frac{2ax+3x^2}{6y^2}\right) \qquad \therefore \quad \left.\frac{dy}{dx}\right|_{(a,a)} = \frac{5a^2}{6a^2} = \frac{5}{6}$

$\Rightarrow$ Equation of tangent is $\quad y - a = \frac{5}{6}(x-a)$

or $\quad 6y - 6a = 5x - 5a$

or $\quad -5x + 6y = a$ or $\frac{x}{(-a/5)} + \frac{y}{(a/6)} = 1$

$\therefore \quad \alpha = \frac{a}{5}, \beta = \frac{a}{6}$

$\left(\frac{-a}{5}\right)^2 + \left(\frac{a}{6}\right)^2 = 61$ (given) $\Rightarrow \frac{61a^2}{(30)^2} = 61 \therefore \quad a = \pm 30$

2. The coordinates of the point on the curve $(x^2+1)(y-3) = x$ where a tangent to the curve has the greatest slope are given by

A) $(\sqrt{3}, 3+\sqrt{3}/4)$ B) $(\sqrt{3}, 3-\sqrt{3}/4)$ C) (0, 3) D) (3, 0)

Key. A,B

Sol. $(x^2+1)(y-3) = x \quad \Rightarrow y = 3 + \frac{x}{x^2+1}$

$m = \frac{dy}{dx} = \frac{1-x^2}{(1+x^2)^2} \quad \Rightarrow \frac{dm}{dx} = \frac{-2x(3-x^2)}{(1+x^2)^2}$

For extremum $f'(x) = 0$

x=0, $x = \pm\sqrt{3}$

x=0, y=3 (0,3)

$x = \sqrt{3}, y = 3 + \frac{\sqrt{3}}{4} \quad \left(\sqrt{3}, 3 + \frac{\sqrt{3}}{4}\right)$

$x = -\sqrt{3}, y = 3 - \frac{\sqrt{3}}{4} \quad \left(\sqrt{3}, 3 - \frac{\sqrt{3}}{4}\right)$

3. A normal is drawn at a point $P(x, y)$ of a curve. It meet the x-axis at Q. If PQ is of constant length k. Then

a) The differential equation describing such curves is $y\frac{dy}{dx} = \pm\sqrt{k^2 - y^2}$

b) The curve is passing through (0, k)

c) The curve is passing through (k, 0)

d) The equation of the curve represents circle with centre as origin

Key. A,B,C,D

Sol. Equation of the normal at a point $P(x, y)$ is given by

$$Y - y = -\frac{1}{dy/dx}(X - x) \quad \text{---(1)}$$

Let the point Q at the x-axis be $(x_1, 0)$. From (1) we get

$$y\frac{dy}{dx} = x_1 - x \quad \text{--(2)}$$

Now given that $PQ^2 = k^2$

We have $(x - x_1)^2 + y^2 = k^2$

or $x - x_1 = \pm\sqrt{k^2 - y^2}$,

Hence using (2) we obtain $y\frac{dy}{dx} = \pm\sqrt{k^2 - y^2}$ ---(3)

(3) is the required differential equation for such curves

Now solving (3) we get $\int \frac{-ydy}{\sqrt{k^2 - y^2}} = \int -dx$

or $x^2 + y^2 = k^2$ which passes through (0,k)

4. Which of the following pairs of curves intersect orthogonally?

(A) $\frac{x^2}{31} - \frac{y^2}{41} = 1$ and $\frac{x^2}{91} + \frac{y^2}{19} = 1$

(B) $\frac{x^2}{71} + \frac{y^2}{17} = 1$ and $\frac{x^2}{31} - \frac{y^2}{23} = 1$

(C) $\frac{x^2}{37} - \frac{y^2}{41} = 1$ and $\frac{x^2}{47} - \frac{y^2}{31} = 1$

(D) $\frac{x^2}{13} + \frac{y^2}{17} = 1$ and $\frac{x^2}{19} + \frac{y^2}{23} = 1$

KEY - A,B

HINT. (C) and (D) don't intersect. The system of equations has no real solution so only (A) and (B) are correct.

5. The equations of the tangents to the curve $y = x^4$ drawn from the point (2, 0), are given by

(A) y = 0

(B) y = 4x + 8

(C) $y-\frac{4096}{81}=\frac{2048}{27}\left(x-\frac{8}{3}\right)$ (D) $y-\frac{320}{243}=\frac{80}{81}\left(x-\frac{2}{3}\right)$

Key. A,C

Sol. Let $\left(x_0, x_0^4\right)$ be the point of tangency. Then the equation of the tangent will be $y-x_0^4=y'\left(x_0\right)\left(x-x_0\right)$. Since this tangent passes through the point (2, 0), we have $0-x_0^4=4x_0^3\left(2-x_0\right)$, or $3x_0^4-8x_0^3=0$. That is, $x_0=0$ or $x_0=\frac{8}{3}$, so that the points of tangency are (0, 0) and (8/3, 4096/81). Therefore, the equations of the tangents are

$y=0$ and $y-\frac{4096}{81}=\frac{2048}{27}\left(x-\frac{8}{3}\right)$

6. The coordinates of the point on the curve $(x^2+1)(y-3)=x$ where a tangent to the curve has the greatest slope are given by

a) $(\sqrt{3},3+\sqrt{3}/4)$ b) $(\sqrt{3},3-\sqrt{3}/4)$ c) (0, 3) d) (3, 0)

Key. A,B

Sol. $\left(x^2+1\right)\left(y-3\right)=x$

$y=3+\frac{x}{x^2+1}$

$m=\frac{dy}{dx}=\frac{1-x^2}{\left(1+x^2\right)^2}$

$\frac{dm}{dx}=\frac{-2x\left(3-x^2\right)}{\left(1+x^2\right)^2}$

For extremum $f'(x)=0$

x=0, $x=\pm\sqrt{3}$

x=0, y=3 (0,3)

$x=\sqrt{3}, y=3+\frac{\sqrt{3}}{4}$ $\left(\sqrt{3},3+\frac{\sqrt{3}}{4}\right)$

$x=-\sqrt{3}, y=3-\frac{\sqrt{3}}{4}$ $\left(\sqrt{3},3-\frac{\sqrt{3}}{4}\right)$

7. The values of the parameter ‘a’ so that the line $(3-a)x+ay+a^2-1=0$ is a normal to the curve $xy=1$ may belong to the interval

a) $(3,\infty)$ b) $(-\infty,0)$ c) $(0,1)$ d) (1, 2)

Key. A,B

Sol. Slope of given line = $\frac{a-3}{a}>0\Rightarrow a>3$ or $a<0$.

26. Tangent at a point P_1 [other than (0, 0)] or the curve $y = x^3$ meets the curve again at P_2. The tangent at P_2 meets the curve at P_3 and so on. Then

(a) abscissae of $P_1, P_2, P_3, \ldots\ldots P_n$ are in A. P.

(b) abscissae of $P_1, P_2, P_3, \ldots. P_n$ form a G. P.

(c) area $\left(\Delta P_2P_3P_4\right) = 16$ area $\left(\Delta P_1P_2P_3\right)$

(d) area $\left(\Delta P_1P_2P_3\right) = 16$ area $\left(\Delta P_2P_3P_4\right)$

Key. B,C

Sol. Let $P_i\left(x_i, y_i\right)$ i = 1, 2, 3, ….n

$y = x^3$ (1) $\Rightarrow \frac{dy}{dx} = 3x^2$

Equation of tangent at $P_1(x_1, y_1)$

$y - x_1^3 = 3x_1^2\left(x - x_1\right)$ (2)

Solving (1) and (2), $x = -2x_1$

$\therefore$ $x_2 = -2x_1$ and $y_2 = -8x_1^3$

$P_2 \equiv \left(-2x_1, -8x_1^3\right)$

and like wise $P_3 \equiv \left(-2\left(-2x_1\right), +64x_1^3\right)$

Abscissa of $P_1, P_2, \ldots, P_n$ are given by

$x_1, -2x_1, 4x_1, -8x_1, \ldots..$ which is G. P. with common ratio –2.

$$\text{Area of } \left(\Delta P_1P_2P_3\right) = \frac{1}{2}\begin{vmatrix} x_1 & x_1^3 & 1 \\ -2x_1 & -8x_1^3 & 1 \\ 4x_1 & 64x_1^3 & 1 \end{vmatrix} = \frac{x_1^4}{2}\begin{vmatrix} 1 & 1 & 1 \\ -2 & -8 & 1 \\ 4 & 64 & 1 \end{vmatrix}$$

$$\text{Area of } \left(\Delta P_2P_3P_4\right) = \frac{1}{2}\begin{vmatrix} -2x_1 & -8x_1^3 & 1 \\ 4x_1 & 64x_1^3 & 1 \\ -8x_1 & -512x_1^3 & 1 \end{vmatrix} = 8x_1^4\begin{vmatrix} 1 & 1 & 1 \\ -2 & -8 & 1 \\ 4 & 64 & 1 \end{vmatrix}$$

So area of $\left(\Delta P_1P_2P_3\right) = \frac{1}{16}$ area of $\left(\Delta P_2P_3P_4\right)$

Tangent & Normals

Assertion Reasoning Type

1. **Statement – 1:** The tangent at $x=1$, to the curve $y=x^3-x^2-x+2$ again meets the curve at $x=0$

Statement – 2: For third degree equation when the equation of a tangent solved, with the given curve repeated roots are obtained at point of tangency.

Key. D

Sol. When x = 1, y = 1

$$\frac{dy}{dx}=3x^2-2x-1 \Rightarrow \left(\frac{dy}{dx}\right)_{x=1}=0$$

$\Rightarrow$ equation of the tangent is y = 1

Solving with the curve, $x^3-x^2-x+2=1$

$\Rightarrow x^3-x^2-x+1=0 \Rightarrow x=-1,1(1\ \textit{is repeated root})$

$\therefore$ The tangent meets the curve again at $x=-1$

2. **Statement – 1:** Equation of tangents to the curve $y=f(x)=x^2$ at the point where slope of tangent is equal to functional value of the curve is $4x-y-4=0,\ y=0$

Statement – 2: $f'(x)=f(x)$

Key. A

Sol. Given $f'(x)=f(x)$

$\Rightarrow 2x=x^2 \Rightarrow x=0,2$

At $x=0, y=0$ *and* $x=2, y=4$

So, we have to find equation of tangents at (0,0) and (2,4)

at $(0,0)$, $f'(0)=0$ at $(2,4)$, $f'(2)=4$

$\therefore$ tangents are $y=0$ and $4x-y-4=0$

3. Statement – 1 : The points on the curve $y^2=x+\sin x$ at which the tangent is parallel to x-axis lie on a straight line.

Statement – 2 : If tangent is parallel to x-axis, then $\frac{dy}{dx}=0$ or $\frac{dx}{dy}$ is undefined.

Key. D

Sol. $\because y^2 = x + \sin x$ ....(i)

$\therefore 2y\dfrac{dy}{dx} = 1 + \cos x = 0\left(\because \dfrac{dy}{dx} = 0\right)$

$\therefore \cos x = -1,$

Then, $\sin x = 0$

From Eq. (i), $y^2 = x$ (parabola).

4. A point P describes the circle $x^2 + y^2 = 25$.

Statement – 1 : If the ordinate of *P* decreases at the rate of 1.5 cm/s, then the rate of change of abscissa of the point when ordinate equals 4 cm is 2 cm/s.

Statement – 2 : $x\,dx + y\,dy = 0$

Key. A

Sol. $\because$ $x^2 + y^2 = 25$

$\Rightarrow$ $2x\,dx + 2y\,dy = 0$

or $\dfrac{dy}{dx} = -\dfrac{x}{y}$

$\Rightarrow$ $\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\dfrac{x}{y}$

$\Rightarrow$ $\dfrac{-1.5}{\frac{dx}{dt}} = -\dfrac{3}{4}$ $\left(\because x = \sqrt{25 - y^2}\right)$

or $\dfrac{dx}{dt} = \dfrac{1.5 \times 4}{3} = 2\ cm/s$

Tangent & Normals

Comprehension Type

Paragraph – 1

A curve is represented parametrically by the equations x= e^t cost and y= e^t sint where "t" is a parameter. Then

1. The relation between the parameter "t" and the angle α between the tangent to the given curve and the y-axis is given by, "t" equals

a) $\frac{\pi}{2}-\alpha$ b) $\frac{\pi}{4}+\alpha$ c) $\frac{\pi}{4}-\alpha$ d) $\alpha-\frac{\pi}{4}$

2. The value of $\frac{d^2y}{dx^2}$ at the point where $t=\frac{\pi}{2}$ is

a) 1 b) $-2e^{-\pi/2}$ c) $2e^{\pi/2}$ d) 0

3. $F(t)=\int(x+y)dt$ then the value of $F(\pi)-F\left(\frac{\pi}{2}\right)$ is

a) 0 b) $e^{\pi/2}$ c) 1 d) $-e^{\pi/2}$

Sol. 1. (C) $\frac{dy}{dx}=\tan\left(\frac{\pi}{4}+t\right)=\tan\alpha$

2. (B) Find y_2 at $t=\frac{\pi}{2}$

3. (D) $F(t)=e^t\sin t+c$

Paragraph – 2

Let the Tangent to the cubic curve $x^3+y^3=a^3$ at $P(x_1,y_1)$ meet the curve again at $Q(h,k)$. Put $A=\frac{h}{x_1}$ and $B=\frac{k}{y_1}$, $A\neq B$.

Then answer the following questions.

4. x_1^3 is equal to

A) $\frac{a^3(1-B)}{A-B}$ B) $a^3(1-A)$ C) $\frac{a^3(1-B)}{A+B}$ D) $\frac{a^3(1+B)}{A-B}$

Key. A

5. Which of the following is true

A) $(A+B)^2-AB(A+B)+AB+1=0$ B) $(A+B)^2+AB(A+B)+AB+1=0$

C) $(A+B)^2-AB(A+B)-AB-1=0$ D) $(A+B)^2-AB(A+B)+AB-1=0$

Key. C

6. The value of $\frac{h}{x_1}+\frac{k}{y_1}$ must be

A) 1 B) –1 C) 0 D) 3

Key. B

Sol. 4. The equation of the tangent at (x_1, y_1) of $x^3 + y^3 = a^3$ is $xx_1^2 + yy_1^2 = a^3$. If this tangent passes through (h,k) we have $hx_1^2 + ky_1^2 = a^3$ --- (1)

Also $x_1^3 + y_1^3 = a^3$ --- (2)

$h^3 + k^3 = a^3$ --- (3)

Now, let $A = \frac{h}{x_1}, B = \frac{k}{y_1}$, then the three relations become

$Ax_1^3 + By_1^3 = a^3, x_1^3 + y_1^3 = a^3, A^3x_1^3 + B^3y_1^3 = a^3$

Solving first two for x_1^3 and y_1^3, we get $x_1^3 = \frac{a^3(1-B)}{A-B}, y_1^3 = -\frac{a^3(1-A)}{A-B}$

5. On putting in the last relation and canceling a^3 we get $A^3(1-B) - B^3(1-A) = A - B$

$\Rightarrow (A^3 - B^3) - AB(A^2 - B^2) = A - B \Rightarrow (A-B)\left[(A+B)^2 - AB(A+B) - AB - 1\right] = 0$

$\Rightarrow (A+B)^2 - AB(A+B) - AB - 1 = 0 \ (\because A \neq B)$

6. Solving the quadratic for $A+B$ we get $A+B = \frac{AB \pm (AB+2)}{2}$

On taking –sign, we get $A+B = -1$

$\Rightarrow \frac{h}{x_1} + \frac{k}{y_1} = -1 \Rightarrow$ (B) is true.

We have to rule out $A = B$ and $A + B = AB + 1$

If $A = B$, i.e., $\frac{h}{x_1} = \frac{k}{y_1} = \alpha$ (say), then the point (h,k) and (x_1, y_1) coincide

If $A + B = AB + 1$, then $(1-A)(1-B) = 0 \Rightarrow A = 1, B = 1$

$\Rightarrow (x_1, y_1)$ and (h,k) coincide.

Paragraph – 3

The rate at which a body undergoes a change in temperature is proportional to the difference between its temperature and temperature of the surrounding medium. If $y = f(t)$ is the temperature of the body at time *t* and if $M(t)$ denotes the temperature of the surrounding medium, Newton's law leads to the differential equation $y^1 = -k[y - M(t)]$ or $y^1 + ky = k[M(t)]$, where *k* is a positive constant. This first-order linear equation is the mathematical model we use for cooling problems. The unique solution of the equation satisfying the initial condition $f(a) = b$ is given by the formula

$$f(t) = be^{-kt} + e^{-kt}\int_a^t kM(z)e^{kz}dz.$$

7. A body cools from 200^0 to 100^0 in 40 minutes while immersed in a medium whose temperature is kept constant. Let $M(t) = 10^0$. If we measure *t* in minutes and $f(t)$ in degree then $f(t)$ must be equal to

A) $10 + 180e^{-kt}$ B) $10 + 140e^{-kt}$ C) $10 + 100e^{-kt}$ D) $10 + 190e^{-kt}$

Key. D

8. The value of *k* must be

A) $(\log 19 - \log 9)/100$
B) $(\log 19 - \log 9)/50$
C) $(\log 19 - \log 9)/40$
D) $(\log 19 - \log 9)/20$

Key. C

9. Suppose in the same system a body cools from 200^0 to 400^0 with $M(t) = 5^0$, then time taken for cooling must be equal to

A) $40 \log 19$
B) $40 \log 9$
C) $40 \dfrac{\log 19 - \log 9}{\log 39 - \log 19}$
D) $40 \dfrac{\log 39 - \log 19}{\log 19 - \log 9}$

Key. D

Sol. From the equation given in comprehension

$$f(t) = 200e^{-kt} + 10ke^{-kt}\int_0^t e^{kz}dz = 200e^{-kt} = 10(1 - e^{-kt}) = 10 + 190e^{-kt}$$

On putting $t = 40$, we get $100 = 10 + 190e^{-kt} \Rightarrow e^{-kt} = \dfrac{90}{190}$

$$\Rightarrow k = \frac{1}{40}(\log 19 - \log 9)$$

As earlier, we will get $f(t) = 5 + 195e^{-kt}$ --- (1) (initial condition changes)

Since *k* is determined, the time taken to cool from 200^0 to 100^0 can be determined by putting $f(t) = 100$ in equation (1)

We get, $100 = 5 + 195e^{-kt} \Rightarrow t = \dfrac{1}{k}(\log 39 - \log 19) \cong 38.5\text{minutes}$

(On putting *k* we get the answer as given in choice D)

Paragraph – 4

Let $f(x) = \dfrac{1}{1 + x^2}$. Let m be the slope, 'a' be the x-intercept and 'b' be the y-intercept of tangent to y=f(x),

Then answer the following questions:

10. Abscissa of point of contact of the tangent for which 'm' is greatest

a) $\dfrac{1}{\sqrt{3}}$
b) 1
c) -1
d) $\dfrac{-1}{\sqrt{3}}$

11. Greatest value of b = _____

a) $\dfrac{9}{8}$
b) $\dfrac{3}{8}$
c) $\dfrac{1}{8}$
d) $\dfrac{5}{8}$

12. Abscissa of the point of contact of tangent for which $\dfrac{1}{a}$ is greatest

a) $\dfrac{1}{\sqrt{3}}$
b)1
c) -1
d) $\dfrac{-1}{3}$

KEY : D-A-A

HINT

(10-12 Passage)

10. $f(x)=\frac{-2x}{(1+x^2)^2}$; $f^{||}(x)=\frac{-2x+6x^2}{(1+x^2)^3}$;

$f^{||}(x)=0\Rightarrow x=\pm\frac{1}{\sqrt{3}}$

$f^{|}(x)$ is greatest $x=\frac{-1}{\sqrt{3}}$

11. Equation of tangents

$y-\frac{1}{1+\alpha^2}=\frac{-2\alpha}{(1+\alpha^2)^2}(x-\alpha)$

$b=\frac{1+3\alpha^2}{(1+\alpha^2)^2}$; $b^{|}=\frac{2\alpha(1-3\alpha^2)}{(1+\alpha^2)^3}$

$b^{|}=0 \Rightarrow \alpha=0, \pm\frac{1}{\sqrt{3}}$

at $\alpha=\pm\frac{1}{\sqrt{3}}$, $b=\frac{9}{8}$

12.

$a=\frac{1+3\alpha^2}{2\alpha}$; $\frac{1}{a}=\frac{2\alpha}{1+3\alpha^2}$

its value is greatest if $\alpha=\frac{1}{\sqrt{3}}$

Paragraph – 5

A right circular cone with radius R and height H contains a liquid which evaporates at a rate proportional to its surface area as the liquid is in contact with air. Let 'K' be the proportionality constant, with, $K>0$. Let $r(t)$ be the radius of liquid in the cone, at time 't' Then, answer the following:

13. The time after which the cone is empty is,

a) $\frac{H}{K}$ b) $\frac{H}{2K}$ c) $\frac{H}{3K}$ d) $\frac{2H}{K}$

Key. A

14. The radius of water in the cone at t = 1 is,

a) $R\left(1+\frac{K}{H}\right)$ b) $R\left(1+\frac{H}{K}\right)$ c) $R\left(1-\frac{H}{K}\right)$ d) $R\left(1-\frac{K}{H}\right)$

Key. D

15. The value of $\sum_{i=1}^{10} r(i)=$

a) $10R\left(2-\frac{K}{H}\right)$ b) $5R\left(2+\frac{11K}{H}\right)$

c) $5R\left(2-\frac{11K}{H}\right)$ d) $4R\left(2-\frac{11K}{H}\right)$

Key. C

Sol. Let θ be the semi – vertical angle of the cone, so that $\tan\theta = R/H$. Let the radius height of the metal in the cone at time t be r and h respectively

$\Rightarrow \tan\theta = \frac{r}{h}$. If V is the volume of the water and S is the surface area of the water in the cone, in contact with air, at time t, then, $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^3 \cot\theta$

and $S = \pi r^2$ by hypothesis $\frac{dv}{dt}\alpha S \Rightarrow \frac{dv}{dt} = -ks(\therefore V \text{ is decreasing})$

$\Rightarrow \frac{1}{3}\pi(3r^2)\cot\frac{\theta dr}{dt} = -k(\pi r^2) \Rightarrow \frac{dr}{dt} = -K\tan\theta$ integrating, $r = (-K\tan\theta)t + c$

When $t = 0, r = R \Rightarrow C = R \Rightarrow r = (-K\tan\theta)t + R$ when cone is empty, r = 0. If T is the time taken for the cone to be empty, then $0 = (-K\tan\theta)T + R$

$$\Rightarrow T = \frac{R}{K\tan\theta} = \frac{R}{K(R/H)} = \frac{H}{K}$$

hence, cone will be empty in time $\frac{H}{K}$ also,

$$r(1) = -K\tan\theta + R = -K\frac{R}{H} + R = R\left(1-\frac{K}{H}\right)$$

$\sum_{r=1}^{10} r(i) = 10R - \frac{R.K}{H}\sum_{i=1}^{10} i = 10R - \frac{RK}{H}55$

Paragraph – 6

One of the roots of the equation $f(x) = 0$ is an even prime integer where

$f(x) = x^2 - ax + b$ ('a' is an odd positive integer). Also known that $\sum_{i=1}^{5} f(i) = 20$ and two curves are defined as

$C_1 : y^2 = f(x)$ and $C_2 : y^2 = -f(x)$

16. Number of distinct normals to the curve C_1 which passes through the centre of the curve C_2 is/are

(A) 1 (B) 2

(C) 3 (D) 4

Key: A

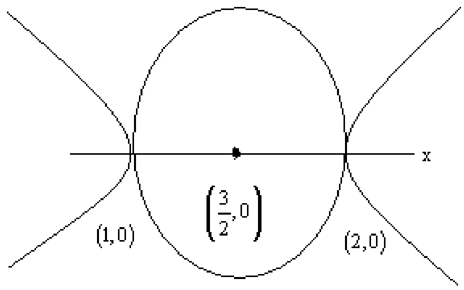
Hint: $f(x) = x^2 - ax + b$

One root must be 2

$f(2)=0$

$2a-b=4$ ....(1)

$\sum_{i=1}^{5} f(x)=20 \Rightarrow 13a-4b=31$ ....(2)



From (i) and (ii) we get

$a=3, b=2$

$\therefore f(x)=x^2-3x+2$

Curve $C_1:\ x^2-y^2-3x+2=0$

Rectangular hyperbola with centre $\left(\frac{3}{2},0\right)$ & $a^2=b^2=\frac{1}{4}$

Curve $C_2: x^2+y^2-3x+2=0$

Circle with centre $\left(\frac{3}{2},0\right)$ & radius is $\frac{1}{2}$

Exactly one normal of C_1 passes through centre of C_2

17. Equation of normals to the curve y = f(x) which are parallel to the tangent from origin to the curve C_2 are $\pm\frac{x}{\sqrt{t}}=ty-v$, (where 't' & 'v' are real constants) then 't' equals.

(A) 1 (B) 2
(C) 4 (D) 9

Key: B

Hint: Slope of tangents drawn from origin to the circle is $\pm\frac{1}{2\sqrt{2}}$

Given that $\frac{1}{t\sqrt{t}}=\frac{1}{2\sqrt{2}} \Rightarrow t=2$

18. The smaller area bounded by $y=f(x)$ and curve C_2 is $\frac{3\pi-\lambda}{6\lambda}$ units then 'λ' equals.

(A) $\frac{4}{3}$ (B) 1
(C) 2 (D) 4

Key: D

Hint: y= f(x)

$y=x^2-3x+2$

Required Area $=\frac{\pi}{8}-\frac{1}{6}$

$=\frac{3\pi-4}{24}$

$\therefore \lambda=4$

Paragraph – 7

A right circular cone with radius R and height H contains a liquid which evaporates at a rate proportional to its surface area in contact with air (proportionality constant = k > 0). Suppose that r(t) is the radius of liquid cone at time t.

19. The time after which the cone is empty is

(a) H/2k (b) H/k (c) H/3k (d) 2H/k

Key: B

Hint:

20. The radius of water cone at t = 1 is

(a) R[1 – k/H] (b) R[1 – H/k] (c) R[1 + H/k] (d) R[1 + k/H]

Key: A

21. The value of $\sum_{i=1}^{10} r(i)$ is equal to

(a) $10R\left[2-\frac{k}{H}\right]$ (b) $5R\left[2+\frac{11k}{H}\right]$ (c) $5R\left[2-\frac{11k}{H}\right]$ (d) $4R\left[2-\frac{11k}{H}\right]$

Key: C

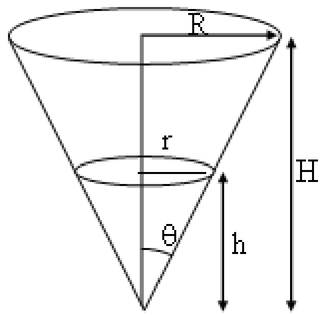
Hint:

19-21. Let θ be the semi vertical angle of the cone so that tan θ = R/H.

Let the radius and height of water cone at time t be r and h respectively. So

$$\tan\theta = \frac{r}{h}$$

If V is the volume of water and S is the surface of the cone in contact with air at time t, then



$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^3 \cot\theta$ and S = πr^2

We are given that $\frac{dV}{dt} \propto S$

$\Rightarrow \frac{dV}{dt} = -kS$ (V is decreasing) $\Rightarrow \frac{1}{3}\pi(3r^2)\cot\theta\frac{dr}{dt} = -k(\pi r^2)$ $\Rightarrow \frac{dr}{dt} = -k\tan\theta$

Integrating, we get r = $-(k\tan\theta)t + C$

When t = 0, r = R, ∴ C = R

Thus, r = (–k tan θ)t + R

When cone is empty, r = 0. If T is the time taken for the cone to be empty, then

0 + (–k tan θ)T + R $\Rightarrow T = \frac{R}{k\tan\theta} = \frac{R}{k(R/H)} = \frac{H}{k}$

Hence, cone will be empty in time H/k.

r(1) = –k tan θ + R = $-k\frac{R}{H} + R = R\left(1-\frac{k}{H}\right)$

$$\sum_{i=1}^{10} r(i) = 10R - \frac{R}{H}k\sum_{i=1}^{10} i = 10R - \frac{Rk}{H}55$$

Paragraph – 8

Let $y = a\sqrt{x} + bx$ be a curve and $(2x - y) + \lambda(2x + y - 4) = 0$ be family of lines.

22. If curve has slope $-1/2$ at $(9,0)$, then a tangent belonging to family of lines is

a) $x + 2y - 5 = 0$ b) $x - 2y + 3 = 0$

c) $3x - y - 1 = 0$ d) $3x + y - 5 = 0$

Key. B

23. A line of the family is cutting positive intercepts on the axes so that it forms a triangle with coordinate axes. Then minimum length of the line segment between axes is

a) $(2^{2/3} - 1)^{3/2}$ b) $(2^{2/3} + 1)^{3/2}$ c) $7^{3/2}$ d) 27

Key. B

24. Two perpendicular chords of curve $y^2 - 4x - 4y + 4 = 0$ belonging to family of lines form diagonals of a quadrilateral. Minimum area of quadrilateral is

a) 16 b) 32 c) 64 d) 50

Key. B

Sol. Conceptual

Paragraph – 9

The curve $y = f(x) = ax^3 + bx^2 + cx + 5$ touches the x-axis at $P(-2,0)$ and cuts the y-axis at a point Q where its gradient is 3.

25. The equation of the curve is given by

A) $-\frac{1}{2}x^3 + 3x - 5$ B) $-\frac{1}{2}x^3 - \frac{3}{4}x^2 + 3x + 5$ C) $-\frac{1}{2}x^3 + \frac{3}{4}x^2 - 3x + 5$ D) $x^3 - x^2 + 6x - 12$

Key. B

26. The equation of tangent at Q is

A) $3x - y + 5 = 0$ B) $2x - 3y - 5 = 0$

C) $x + y + 5 = 0$ D) $3x + y - 5 = 0$

Key. A

27. The value of $f(2)$ is

A) 1 B) 2 C) -3 D) 4

Key. D

Sol. 25. As the x-axis touches the curve at $P(-2,0)$, we have $\left(\frac{dy}{dx}\right)_{(-2,0)} = Tan\,0^0 = 0$.........(1)

And $(y)_{x=-2} = 0$..........(2)

As the curve cuts y – axis at 'Q' where the gradient of the curve is 3, $\left(\frac{dy}{dx}\right)_{x=0} = 3$.....(3)

Differentiating $y = ax^3 + bx^2 + cx + 5$ w.r.t x

$\Rightarrow \dfrac{dy}{dx} = 3ax^2 + 2bx + c$

$\therefore (1) \Rightarrow \ 0 = 3a(-2)^2 + 2b(-2) + c$

$\Rightarrow 12a - 4b + c = 0$ ....(4)

(2) $\Rightarrow 0 = a(-2)^3 + b(-2)^2 + c(-2) + 5$

$\Rightarrow -8a + 4b - 2c + 5 = 0$........(5)

(3) $\Rightarrow 3 = 3a.0^2 + 2b(0) + c \Rightarrow c = 3$

Solve (4) and (5), we get

$a = -\dfrac{1}{2}, b = \dfrac{-3}{4}$

$\therefore$ the curve is $y = -\dfrac{1}{2}x^3 - \dfrac{3}{4}x^2 + 3x + 5$

26. Equation of tangent at Q is $3x - y + 5 = 0$

27. $f(2) = 4$

Paragraph – 10

Let the tangent to the cubic curve $x^3 + y^3 = a^3$ at $P(x_1, y_1)$ meets the curve again at Q(h, k),

Put $\quad A = \dfrac{h}{x_1}$ and $B = \dfrac{k}{y_1}$

$A \neq B$ then

28. x_1^3 must be equal to

(a) $\dfrac{a^3(1-B)}{A-B}$ (b) $\dfrac{a^3(1-A)}{A-B}$

(c) $\dfrac{a^3(1-B)}{A+B}$ (d) $\dfrac{a^3(1-A)}{A+B}$

Key. A

29. Which of the following is true

(a) $(A + B)^2 - AB(A + B) + AB + 1 = 0$ (b) $(A + B)^2 + AB(A + B) + AB + 1 = 0$

(c) $(A + B)^2 - AB(A + B) - AB - 1 = 0$ (d) $(A + B)^2 + AB(A + B) - AB + 1 = 0$

Key. C

30. The value of $\dfrac{h}{x_1} + \dfrac{k}{y_1}$ must be equal to

(a) 1 (b) – 1

(c) 0 (d) 3

Key. B

Sol. **18, 29, 30**

Slope of tangent at $P(x_1, y_1)$

$= -\frac{x_1^2}{y_1^2}$

Slope of PQ $= \frac{k - y_1}{h - x_1}$

$\Rightarrow \quad = \frac{k - y_1}{h - x_1} = -\frac{x_1^2}{y_1^2}$

$\Rightarrow \quad = \frac{h - x_1}{y_1^2} = \frac{k - y_1}{x_1^2} = t(\text{say})$

$h = x_1 - ty_1^2$ .....(i)

$k = y_1 + tx_1^2$ .....(ii)

as Q(h, k) lie on $x^3 + y^3 = a^3$

$\Rightarrow \quad (x_1 - ty_1^2)^3 + (y_1 + tx_1)^2 = a^3$

$\Rightarrow \quad t = \frac{3x_1y_1}{y_1^3 - x_1^3}$

from (i) & (ii)

$A = 1 - \frac{3y_1^3}{y_1^3 - x_1^3}$ ....(iii)

$B = 1 + \frac{3x_1^3}{y_1^3 - x_1^3}$ .....(iv)

$B - A = \frac{3a^3}{y_1^3 - x_1^3}$

$y_1^3 - x_1^3 = \frac{3a^3}{B - A}$ .....(v)

from (iv) & (iii) $x_1^3 = \frac{a^3(1 - B)}{A - B}$

from (iii) & (iv)

A + B = –1

***Paragraph** – 11*

To find the point of contact $P \equiv (x_1, y_1)$ of a tangent to the graph of $y = f(x)$ passing through origin 0, we equate the slope of tangent to y = f(x) at P to the slope of OP. Hence we solve the equation $f'(x_1) = \frac{f(x_1)}{x_1}$ to get x_1 and y_1.

31. The equation $|\log mx| = px$ where m is a positive constant has a single root for

a) $0 < p < \frac{m}{e}$ b) $p < \frac{e}{m}$ c) $0 < p < \frac{e}{m}$ d) $p > \frac{m}{e}$

Key. D

32. The equation $|\log mx| = px$ where m is a positive constant has exactly two roots for

a) $p = \frac{m}{e}$ b) $p = \frac{e}{m}$ c) $0 < p \le \frac{e}{m}$ d) $0 < p \le \frac{m}{e}$

Key. A

33. The equation $|\log mx| = px, m$ is a + ve const has exactly three roots for

a) $p<\frac{m}{e}$ b) $0<p<\frac{m}{e}$ c) $0<p<\frac{e}{m}$ d) $p<\frac{e}{m}$

Key. B

Sol. 31. 32. 33. Slope of tangent at P = slope of OP

$\Rightarrow \frac{1}{t}=\frac{\log mt}{t} \Rightarrow t=\frac{e}{m}$

$\Rightarrow P=\left(\frac{e}{m},1\right) \Rightarrow \tan\alpha = p=\frac{m}{e}$

***Paragraph** – 12*

In second degree curves, a line which once touches the curve cannot meet the curve again but in cubic and other non-algebraic curves, the tangent can meet the curve again. If we solve the equation of tangent and cubic curve, we will, in general, get three roots, two of which will be equal, since they will correspond to the point where the tangent was initially drawn.

34. *P* is a point (β,β^3) different from $(0,0)$ on the curve $y=x^3$. If the tangent at *P* meets the curve again at *Q* and tangent at *Q* meets the curve again at *R*, then abscissa of the point *R* must be

a) 8β b) -4β c) 4β d) -2β

Key. C

35. The tangent at (t, t^2-t^3) on the curve $y=x^2-x^3$ meets the curve again at *Q*, then abscissa of *Q* must be

a) 1 + 2*t* b) 1 – 2*t* c) –1 – 2*t* d) 2*t* –1

Key. B

36. If the tangent at t of the curve $y=8t^3-1, x=4t^2+3$ meets the curve at $t^{|}$ and is normal to the curve at that point, then value of *t* must be

a) $\pm\frac{1}{\sqrt{3}}$ b) $\pm\frac{1}{\sqrt{2}}$

c) $\pm\frac{\sqrt{2}}{3}$ d) $\pm\frac{\sqrt{3}}{2}$

Key. C

Sol. 34. Let Q be (x_1, y_1) and R be (x_2, y_2).

Eq. of tangent at P is $y-\beta^3=3\beta^2(x-\beta) \longrightarrow (1)$

Put $y=x^3$ in (1)

$x^3-3\beta^2x+2\beta^3=0$ (Roots are β,β,x_1)

Sum of roots $=2\beta+x_1=0 \Rightarrow x_1=-2\beta$.

Similarly $x_2=-2(-2\beta)=4\beta$.

35. Let Q be (x_1, y_1).

Eq. of tangent at (t, t^2-t^3) is $y-(t^2-t^3)=(2t-3t^2)(x-t) \longrightarrow (1)$.

Put $y = x^2 - x^3$ in (1)

$x^3 - x^2 + \left(2t - 3t^2\right)x + \left(2t^3 - t^2\right) = 0$ (Roots are t, t, x_1)

Sum of roots $= 2t + x_1 = 1 \Rightarrow x_1 = 1 - 2t$.

36. Let $t' = p$

Slope of tangent at $t =$ Slope of normal at p

$=$ Slope of the chord joining t and p.

$$\Rightarrow 3t = -\frac{1}{3p} = \frac{2\left(t^2 + p^2 + tp\right)}{t + p}.$$

Eliminating p from above equations, we get

$$81t^4 - 9t^2 - 2 = 0 \Rightarrow t^2 = \frac{2}{9} \; or \; -\frac{1}{9}$$

$$\Rightarrow t = \pm\frac{\sqrt{2}}{3}.$$

Passage – II

A curve C which is not a straight line lies in the first quadrant. The tangent at any point on C meets the positive directions of the coordinate axes at the points A, B. Let 'd' be the minimum distance of the curve C from the origin O.

17. If $OA + OB = 1$ then d =

A) $\frac{1}{2\sqrt{2}}$ B) $\frac{1}{2}$ C) $\frac{1}{\sqrt{2}}$ D) $\sqrt{2}$

Key. A

18. If $OA.OB = 4$ then d =

A) $\frac{1}{2\sqrt{2}}$ B) $\frac{1}{2}$ C) $\frac{1}{\sqrt{2}}$ D) $\sqrt{2}$

Key. D

19. If $AB = 1$ then d =

A) $\frac{1}{2\sqrt{2}}$ B) $\frac{1}{2}$ C) $\frac{1}{\sqrt{2}}$ D) $\sqrt{2}$

Key. B

Sol. **(17 – 19)**

Equation of tangent at (x, y) is $Y - y = p(X - x)$

Where $p = \frac{dy}{dx}$. Then $OA = x - \frac{y}{p}$ and $OB = y - px$

$$OA + OB = 1 \Rightarrow y = px + \frac{p}{p-1}$$

$$OA.OB = 4 \Rightarrow y = px + 2\sqrt{-p}$$

$$AB = 1 \Rightarrow y = px - \frac{p}{\sqrt{1+p^2}}$$

Tangent & Normals

Integer Answer Type

1. The parametric equations of a curve are $x = \sec^2 t \quad y = \cot t$. If the tangent, drawn to the curve at $P\left(t = \frac{\pi}{4}\right)$ meets the curve again at Q, and if $PQ = \frac{\alpha}{\beta}\sqrt{5}, (\alpha, \beta) = 1$ then, the numerical value of $\alpha + \beta$ is

Key. 5

Sol. $P = (2,1), Q = (5, -1/2) \Rightarrow PQ = \frac{3\sqrt{5}}{2}$

2. The number of points lying in $\{(x,y) / |x| \le 10, |y| \le 3\}$, on the curve $y^2 = x - \sin x$, at which the tangents to the curve are parallel to x – axis, is\are

Key. 2

Sol. $\frac{dy}{dx} = \frac{1 - \cos x}{2\sqrt{x + \sin x}} = 0 \Rightarrow \cos x = 1 \Rightarrow x = 0, \pm 2\pi, \pm 4\pi \ldots$

$\therefore x \in [-10, 10] \Rightarrow x = 0, \pm 2\pi$

For x = 0, y = 0

$\left.\begin{array}{l} x = 0, y = 0 \\ x = 2\pi, y = \pm\sqrt{2\pi} \end{array}\right\}$ they satisfy $-3 \le y \le 3$

X $x = -2\pi, y^2 = -2\pi$

Also, slope at $(0,0)$ is undefined hence points are $(2\pi, 2\pi)$ and $\left(2\pi, -\sqrt{2\pi}\right)$

3. The sum of all the integral values of K such that the variable point $\left(K, \frac{1}{K}\right)$ remains on or inside the triangle formed by the x-axis and the tangents drawn at $(2,1)$ on the curve $y = e^{-|x-2|}$ is,

Key. 2

Sol. $\frac{1 + \sqrt{5}}{2} \le K \le \frac{3 + \sqrt{5}}{2} \Rightarrow$ hence K = 2

4. Let $f(x)$ be a continuous function which satisfies,

$f^3(x) - 5f^2(x) + 10f(x) - 12 \le 0$,

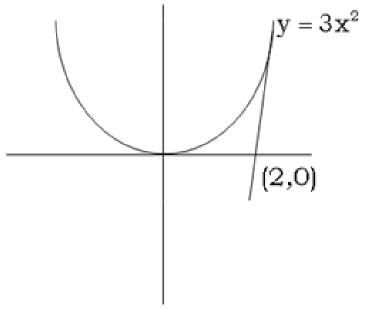
$f^2(x) - 4f(x) + 3 \ge 0$

$f^2(x) - 6f(x) + 8 \le 0$

If the equation to the tangent drawn from $(2,0)$ to the curve, $y = x^2 f(\sin x)$ is of the form, $y = \alpha x - 2\alpha,\ \alpha \in R^+$, then the numerical value of $\frac{\alpha}{6}$ is,

Key. 4

Sol.



$$y = 24x - 48$$

$$\alpha = 24 \Rightarrow \frac{\alpha}{6} = 4$$

5. Let y = f(x) be drawn with f(0) = 2 and for each real number 'a', the tangent to y = f(x) at (a, f(a)), has x intercept (a – 2). If f(x) is of the form $k\,e^{px}$, then $\left(\frac{k}{p}\right)$ has the value equal to

Key. 4

Sol. We have f(0) = 2

Now $y - f(a) = f'(a)[x - a]$

For x intercept y = 0, so

$$x = a - \frac{f(a)}{f'(a)} = a - 2 \Rightarrow \frac{f(a)}{f'(a)} = 2$$

$$\Rightarrow \frac{f'(a)}{f(a)} = \frac{1}{2}$$

$\therefore$ On integrating both sides w.r.t. a, we get

$$\ln f(a) = \frac{a}{2} + C$$

$$f(a) = Ce^{a/2}$$

$$f(x) = Ce^{x/2}$$

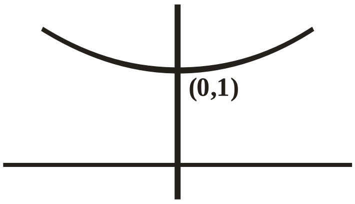
$$f(0) = C \qquad \Rightarrow C = 2$$

$$\therefore f(x) = 2e^{x/2}$$

Hence $k = 2,\ p = \frac{1}{2} \Rightarrow \frac{k}{p} = 4$

6. The shortest distance of the point (0,0) from the curve $y = \frac{1}{2}\left(e^{x} + e^{-x}\right)$ is

Key. 1

Sol. 

7. The segment of the tangent to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 16$, contained between '$x$' and '$y$' axes, has length equal to λ^2 then the value of λ is

Key. 8

Sol. Let $x = 64\cos^2 t$

$y = 64\sin^{t}$

$\frac{dx}{dt} = -192\cos^2 t\left(-\sin t\right)$

Equation of tangent at 't'

$\Rightarrow y - 64\sin^2 t = \frac{-\sin t}{\cos t}\left(x - 64\cos^2 t\right)$

$\Rightarrow \frac{y}{\sin t} - 64\sin^2 t = \frac{-\sin t}{\cos t}\left(x - 64\cos^2 t\right)$

$\Rightarrow \frac{x}{64\cos t} + \frac{y}{64\sin t} = 1$

$\Rightarrow$ Segment of tangent between α and β

$= \sqrt{(64)^2\cos^2 t + 64^2\sin^2 t} = 64$

Length $= 64 = x^2 \therefore \lambda = 8$

8. Find the value of a for which the area of the triangle included between the axes and any tangent to the curve $x^a y = k^a$ is constant

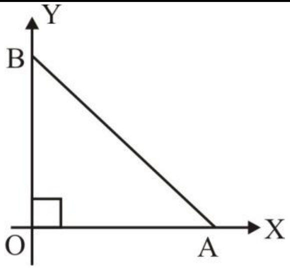
Key. 1

Sol. $\because$ $x^a y = k^a$

$\Rightarrow$ a ln x + ln y = a ln k

$\therefore$ $\frac{a}{x} + \frac{1}{y}\frac{dy}{dx} = 0$

or $\frac{dy}{dx} = -\frac{ay}{x}$



Equation of tangent at (x, y)

$$Y-y=-\frac{ay}{x}(X-x)$$

or $$\frac{Y}{\left(\frac{ay}{x}\right)}-\frac{x}{a}=-X+x$$

or $$\frac{X}{1}+\frac{Y}{\left(\frac{ay}{x}\right)}=x\left(1+\frac{1}{a}\right)$$

or $$\frac{X}{x\left(1+\frac{1}{a}\right)}+\frac{Y}{ay\left(1+\frac{1}{a}\right)}=1$$

$\therefore$ Area $$=\frac{1}{2}.x\left(1+\frac{1}{a}\right)ay\left(1+\frac{1}{a}\right)$$

$$=\frac{ax}{2}.\frac{k^a}{x^a}.\left(1+\frac{1}{a}\right)^2$$

$$=\frac{ak^a}{2x^{a-1}}\left(1+\frac{1}{a}\right)^2$$

$\because$ Area is constant

$\therefore$ a – 1 = 0

$\therefore$ a = 1

9. The sum of all the integral values of K such that the variable point $\left(K,\frac{1}{K}\right)$ remains on or inside the triangle formed by the x-axis and the tangents drawn at $(2,1)$ on the curve $y=e^{-|x-2|}$ is,

Key. 2

Sol. $\frac{1+\sqrt{5}}{2}\le K\le\frac{3+\sqrt{5}}{2}\Rightarrow$ hence K = 2

10. The number of points lying in $\{(x,y)/|x|\le 10,|y|\le 3\}$, on the curve $y^2=x-\sin x$, at which the tangents to the curve are parallel to x – axis, is\are

Key. 2

Sol. $\frac{dy}{dx}=\frac{1-\cos x}{2\sqrt{x+\sin x}}=0\Rightarrow\cos x=1\Rightarrow x=0,\pm 2\pi,\pm 4\pi....$

$\therefore x\in[-10,10]\Rightarrow x=0,\pm 2\pi$

For x = 0, y = 0

$\left.\begin{array}{l} x = 0, y = 0 \\ x = 2\pi, y = \pm\sqrt{2\pi} \end{array}\right\}$ they satisfy $-3 \le y \le 3$

X $x = -2\pi, y^2 = -2\pi$

Also, slope at $(0,0)$ is undefined hence points are $(2\pi, 2\pi)$ and $\left(2\pi, -\sqrt{2\pi}\right)$

11. Let $f(x)$ be a continuous function which satisfies,

$f^3(x) - 5f^2(x) + 10f(x) - 12 \le 0$,

$f^2(x) - 4f(x) + 3 \ge 0$

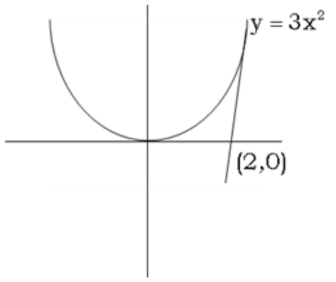
$f^2(x) - 6f(x) + 8 \le 0$

If the equation to the tangent drawn from $(2,0)$ to the curve, $y = x^2 f(\sin x)$ is of the form, $y = \alpha x - 2\alpha,\ \alpha \in R^+$, then the numerical value of $\frac{\alpha}{6}$ is,

Key. 4

Sol. $y = 24x - 48$

$\alpha = 24 \Rightarrow \frac{\alpha}{6} = 4$



12. The parametric equations of a curve are $x = \sec^2 t$ $y = \cot t$. If the tangent, drawn to the curve at $P\left(t = \frac{\pi}{4}\right)$ meets the curve again at Q, and if $PQ = \frac{\alpha}{\beta}\sqrt{5}, (\alpha, \beta) = 1$ then, the numerical value of $\alpha + \beta$ is

Key. 5

Sol. $P = (2,1), Q = (5, -1/2) \Rightarrow PQ = \frac{3\sqrt{5}}{2}$

13. The curve $y = ax^3 + bx^2 + cx + 5$, touches the x-axis at P(–2, 0) and cuts the y-axis at a point Q, where its gradient is 3, then find the value of 4b – 2a + c.

Key. 1

Sol. Let y = f(x), $f'(-2) = 0$, $f(-2) = 0$

$f'(0) = 3$, $f'(x) = 3ax^2 + 2bx + c$

Solving $a = -\frac{1}{2}, b = -\frac{3}{4}$ and $c = 3$

$\Rightarrow$ 4b – 2a + c = 1

14. Three normals are drawn from the point (c, 0) to the curve $y^2 = x$, $c > \frac{1}{2}$. One normal is always the x-axis. Then the value of 4c for which other two normals are perpendicular to each other is

Key. 3

Sol. Equation of normal is $y = mx - 2am - am^3$ to the curve $y^2 = 4ax$.

$a = \frac{1}{4}$

$\Rightarrow\ y = mx - \frac{m}{2} - \frac{m^3}{4}$ passes through (c, 0) then $-m\left(\frac{m^2}{4} + \frac{1}{2} - c\right) = 0$

$\Rightarrow$ Remaining normals are perpendicular.

$\Rightarrow$ Product of the roots of equation $\frac{m^2}{4} + \frac{1}{2} - c = 0$ will be -1.

$$\Rightarrow\ \frac{\frac{1}{2} - c}{\frac{1}{4}} = -1 \Rightarrow\ c = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

so 4c = 3

Tangent & Normals

Matrix-Match Type

1. Match the following

Column I — Column II

a) The tangent at any point on the curve $x = at^3,\ y = at^4$ divides the abscissa of the point of contact in the ration m:n then $|n+m|$ is equal to (m and n are co-prime) — P) 0

b) the area of the triangle formed by normal at the point (1,0) on the curve $x = e^{\sin y}$ with co-ordinate axes is — Q) 1/2

c) If the angle between the curves $x^2 y = 1$ and $y = e^{2(1-x)}$ at the point (1,1) is θ, then $\tan\theta$ is equal to — R) 4

d) The length of the subtangent at any point on the curve $y = be^{x/3}$ is equal to — S) 3

Key. A-r ; B-q ; C-p ; D-s

Sol. a) $x = at^3, y = at^4$

$\therefore \frac{dy}{dx} = \frac{4}{3}t$

Equation of tangent is $y - at^4 = \frac{4}{3}t\left(x - at^3\right)$

Abscissa of pt.of intersection of tangent with axis is $\frac{at^3}{4}$ & 0

Point of contact divides this abscissa in -3:4 ratio

b) $x = e^{\sin y} \Rightarrow \sin y = \ln x$

$\cos y \frac{dy}{dx} = \frac{1}{x} \Rightarrow \frac{dx}{dy} = x\cos y$

Slope of normal at (1,0) = -1

Equation of normal y = -1(x-1)

Area of triangle $\frac{1}{2}$

c) For 1st curve $\frac{dy}{dx} = \frac{-2}{x} \Rightarrow \frac{dy}{dx} at \left(1,1\right) = -2$

For 2nd curve $\frac{dy}{dx} = -2e^{2(1-x)} \Rightarrow \frac{dy}{dx} at \left(1,1\right) = -2$

So angle between two curves is = 0

d)Length of subtangent = $y_1 . \frac{dx}{dy}$

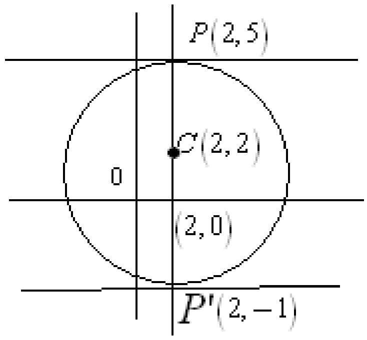
$$= be^{x_1/3}.\frac{1}{\frac{b}{3}e^{x_1/3}} = 3$$

2. For the $x^2 + y^2 - 4x - 4y - 1 = 0$ curve at the point P whose ordinate is 5. Then match the following.

Column - I	Column - II
a) y - intercept of tangent	p) 2
b) x - intercept of normal	q) 10
c) Area of quadrilateral formed by tangent & normal and co-ordinate axes	r) 1
d) A tangent to the above curve is drawn such that it is parallel to tangent at P. Then its distance from origin is	s) 5

Key. a – s ; b – p ; c – q; d – r

Sol. $P = (2,5)$; $m = y^1 = \frac{2-x}{y-2}$ is the slope of tangent. y–intercept of tangent $= y - mx_1 = 5$



x–intercept of normal = $x_1 = my_1 = 2$

Area of required quadrilateral $= 2 \times 5 = 10$ squ

Distance of origin from tangent at $P^1 = 1$

3. Column - I Column - II

a) $f:[0,4]\to R$ is differentiable and $a, b \in (0,4)$ are some constants then $\frac{(f(4))^2 - (f(0))^2}{f'(a)f(b)} =$ p) 2

b) Let $F(x) = f(x)g(x)h(x)$ for all real x where f,g,h are differentiable functions at some point x_0 then $F'(x_0) = 21F(x_0), f'(x_0) = 4f(x_0), g'(x_0) = -7g(x_0)$ and $h'(x_0) = kh(x_0)$ then $\frac{k}{6} =$ — q) 4

c) $f(x)$ is function such that $\left|f(x_1) - f(x_2)\right| \le \left|x_1 - x_2\right|^2 \ \forall$ $x_1, x_2 \in R$ and normal at some point on $y = f(x)$ passes through (1,2). Equation of normal is $lx + my - n = 0$ Then $lm - mn + nl =$ — r) 8

d) The normal at $\left(at_1^2, 2at_1\right)$ on the curve $y^2 = 4ax$ meets the curve again at $\left(at_2^2, 2at_2\right)$ then t_2^2 takes the least value — s) 1

Key. a – r ; b – q ; c – p,q,r,s; d – r

Sol. a) By LMVT there exists $a \in (0,4)$ such that $f(4) - f(0) = 4f^1(a)$

By Intermediate value theorem $\Im\, b \in (0,4)$ such that $\frac{f(4) + f(0)}{2} = f(b)$

b) $F^1(x) = fgh^1 + fg^1h + f^1gh$ by product rule at x = x_0

$$21F = kfgh - 7fgh + 4fgh$$

$$\therefore 21fgh = (k-3)fgh \Rightarrow k = 24$$

c) $\left|\lim_{x_1 \to x_2} \frac{f(x) - f(x_2)}{x_1 - x_2}\right| \le 0 \Rightarrow f(x) = c$ for all $x \in R$ Normal is x = 1 . Since it passes through (1,2)

$$\therefore l = 1, m = 0, n = 1$$

d) Normal at "t_1" is $y + xt_1 = 2at_1 + at_1^3$

It passes through "t_2" $\Rightarrow 2a(t_2 - t_1) = at_1(t_1^2 - t_2^2) \quad \Rightarrow t_2 + t_1 = \frac{-2}{t_1}$

$$\Rightarrow t_2 = -t_1 - \frac{2}{t_1} \qquad \Rightarrow \qquad |t_2| \ge 2\sqrt{2} \qquad \Rightarrow t_2^2 \ge 8$$

4. Match the following:

	Column – I		Column – II
A)	The curve $y = 2e^{2x}$ intersect the y-axis at an angle $\cot^{-1}\left(\left|\frac{8n-4}{3}\right|\right)$ then the value of n is	p)	2
B)	The area of triangle formed by normal at the point (1,0) on the curve $x = e^{\sin y}$ with axes is $\left|\frac{2t+1}{6}\right|$ sq.units. then the value of t is	q)	-1

C)	length of sub-tangent to the curve $x^2y^2=16$ at the point (-2,2) is $\lvert K\rvert$ then the value of K is	r)	-2
D)	The slope of the tangent to the curve $y=\sqrt{4-x^2}$ at the point where the ordinate and the abscissa are equal is	s)	4

Key. a-pq; b-r; c-pr; d-q

Sol. A) $y=2e^{2x}$ intersect y-axis at (0,2)

$$\left(\frac{dy}{dx}\right)_{at\ x=0}=4$$

$\therefore$ angle of intersection with y-axis = $\frac{\pi}{4}-Tan^{-1}4=Cot^{-1}4$

$\Rightarrow n=2\ or\ -1$

B) $x=e^{\sin y}$

$$1=e^{\sin y}\cos y\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{e^{\sin y}\cos y}$$

$$\left(\frac{dy}{dx}\right)_{(1,0)}=\frac{1}{1}=1$$

Slope of the normal at (1,0) = -1 equation of normal is $x+y=1$

Area = $\frac{1}{2}$

$\Rightarrow t=1,-2$

C) $x^2y^2=16 \Rightarrow xy=\pm 4$

$$L_{S.T}=\left|\frac{y}{\frac{dy}{dx}}\right|=\lvert x\rvert$$

$\Rightarrow L_{S.T}=2\Rightarrow K=\pm 2$

D) Here $y>0$, putting $y=x$

in $y=\sqrt{4-x^2}$, we get $x=\sqrt{2},-\sqrt{2}$

So the point $\left(\sqrt{2},\sqrt{2}\right)$

Differentiating $y^2+x^2=4$

w.r.t x, we get $\frac{dy}{dx} = \frac{-x}{y}$

at $\left(\sqrt{2}, \sqrt{2}\right), \frac{dy}{dx} = -1$