Differentiation

Single Correct Answer Type

1. The second derivative of a single valued function parametrically represented by $x=\phi(t)$ and $y=\psi(t)$, (where $\phi(t)$ and $\psi(t)$ are different functions and $\phi'(t)\neq 0$) is given by

A) $\frac{d^2y}{dx^2}=\frac{\left(\frac{dx}{dt}\right)\left(\frac{d^2y}{dt^2}\right)-\left(\frac{d^2x}{dt^2}\right)\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)^3}$

B) $\frac{d^2y}{dx^2}=\frac{\left(\frac{dx}{dt}\right)\left(\frac{d^2y}{dt^2}\right)-\left(\frac{d^2x}{dt^2}\right)\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)^2}$

C) $\frac{d^2y}{dx^2}=\frac{\left(\frac{d^2x}{dt^2}\right)\frac{dy}{dt}-\frac{dx}{dt}\left(\frac{d^2y}{dt^2}\right)}{\left(\frac{dx}{dt}\right)^3}$

D) $\frac{d^2y}{dx^2}=\frac{\left(\frac{d^2x}{dt^2}\right)\left(\frac{dy}{dt}\right)-\left(\frac{d^2y}{dt^2}\right)\frac{dx}{dt}}{\left(\frac{dy}{dt}\right)^3}$

Key. A

Sol. $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy/dt}{dx/dt}\right)$$

2. $y=f(x)$ be a real valued twice differentiable function defined on R, then $\frac{d^2y}{dx^2}\left(\frac{dx}{dy}\right)^3+\frac{d^2x}{dy^2}=$

(A) $\frac{dy}{dx}$ (B) $\frac{dx}{dy}$ (C) $\frac{d^2y}{dx^2}$ (D)0

Key. D

Sol. $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{1}{\frac{dx}{dy}}\right)$

$$=\frac{d}{dy}\left(\frac{1}{\frac{dx}{dy}}\right)\frac{dy}{dx}=\frac{-\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^2}.\frac{dy}{dx}=-\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3}$$

$$\Rightarrow\frac{d^2y}{dx^2}.\left(\frac{dx}{dy}\right)^3+\frac{d^2x}{dy^2}=0$$

3. A function $f:R\to[1,\infty)$ satisfies the equation

$f(xy)=f(x)f(y)-f(x)-f(y)+2$. If f is differentiable on $R-\{0\}$ and $f(2)=5$,

$f'(x)=\frac{f(x)-1}{x}.\lambda$ then $\lambda=$_____

(A) $2f'(1)$ (B) $3f'(1)$ (C) $\frac{1}{2}f'(1)$ (D) $f'(1)$

Key. D

Sol. $f'(x)=\lim_{x\to 0}\frac{f(x+h)-f(x)}{h}$

$=\lim_{x\to 0}\frac{f\left\{x\left(1+\frac{h}{x}\right)\right\}-f(x)}{h}(x\neq 0\ given)$

$=\lim_{x\to 0}\frac{f\left(1+\frac{h}{x}\right)-2}{\frac{h}{x}}.\frac{f(x)-1}{x}$

Putting $x=1,y=2$ in the given functional equation, f(1)=2

$\therefore f'(x)=\frac{f(x)-1}{x}.\lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)-f(1)}{h/x}$

$=\frac{f(x)-1}{x}.f'(1)$

4. If $F(x)=f(x).g(x)$ and $f'(x)g'(x)=c$, where 'c' is a constant then $\frac{f''}{f}+\frac{g''}{g}+\frac{2C}{fg}=$

(A) $\frac{f''}{F}$ (B) $\frac{f''}{F'}$ (C) $\frac{F''}{F}$ (D) none of these

Key. C

Sol. Given $F=fg\Rightarrow F'=f'g+fg'$

$F''=f''g+2f'g'+fg''$

$\Rightarrow F''=f''g+2C+fg''$

$\Rightarrow \frac{f''}{F}=\frac{f''}{f}+\frac{g''}{g}+\frac{2C}{fg}$

5. Let $g(x)=\log f(x)$ where $f(x)$ is a twice differentiable positive function on $(0,\infty)$ such that $f(x+1)=xf(x)$. Then, for $N=1,2,3.....,g''\left(N+\frac{1}{2}\right)-g''\left(\frac{1}{2}\right)=$

A) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+...+\frac{1}{(2N-1)^2}\right\}$ B) $4\left\{1+\frac{1}{9}+\frac{1}{25}+...+\frac{1}{(2N-1)^2}\right\}$

C) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+...+\frac{1}{(2N+1)^2}\right\}$ D) $4\left\{1+\frac{1}{9}+\frac{1}{25}+...+\frac{1}{(2N+1)^2}\right\}$

Key. 1

Sol. $g''(x+1)-g''(x)=-\frac{1}{x^2}$

$x\to x-\frac{1}{2}$, we get $g''\left(x+\frac{1}{2}\right)-g''\left(x-\frac{1}{2}\right)=-\frac{4}{(2x-1)^2}$

Put x=1,2,.....N adding we get result

6. $\frac{d^2x}{dy^2}$ equals :

A) $\left(\frac{d^2y}{dx^2}\right)^{-1}$ B) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$

C) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$ D) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$

Key. 4

Sol. $\frac{dx}{dy}=\left(\frac{dy}{dx}\right)^{-1}\Rightarrow\frac{d^2x}{dy^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)^{-1}.\frac{dx}{dy}=-\left(\frac{dy}{dx}\right)^{-2}\frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^{-1}=-\left(\frac{dy}{dx}\right)^{-3}\left(\frac{d^2y}{dx^2}\right)$

7. If $x^2+y^2=1$, then

A) $yy''-2(y')^2+1=0$ B) $yy''+(y')^2+1=0$

C) $yy''-(y')^2+1=0$ D) $yy''+2(y')^2+1=0$

Key. 2

Sol. Differentiate successively

8. If $y=x\log\left(\frac{x}{a+bx}\right)$, then $x^n\frac{d^2y}{dx^2}=\left(x\frac{dy}{dx}-y\right)^m$, where :

A) n=3,m=2 B) $n=2,m=3$

C) $m=n=2$ D) $m=n=3$

Key. 1

Sol. Conceptual

9. If $y=\left(\sin^{-1}x\right)^2+\left(\cos^{-1}x\right)^2$, then $\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=$

A) -4 B) 4 C) 2 D) -2

Key. 2

Sol. $y_1=\frac{2\sin^{-1}x}{\sqrt{1-x^2}}-\frac{2\cos^{-1}x}{\sqrt{1-x^2}}\Rightarrow\sqrt{1-x^2}\,y_1=2\left(\sin^{-1}x-\cos^{-1}x\right)$

$\Rightarrow\sqrt{1-x^2}\,y_2-\frac{2x}{2\sqrt{1-x^2}}y_1=2\left(\frac{1}{\sqrt{1-x^2}}+\frac{1}{\sqrt{1-x^2}}\right)$

10. If $y = x\sin\left(\log x\right) + x\log x,$ then $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + 2y =$

A) $\ln(x)$ B) $x\ln(x)$ C) $-x\ln(x)$ D) $\dfrac{\ln x}{x}$

Key. 2

Sol. $y_1 = \sin\left(\log x\right) + x\cos\left(\log x\right) + \log x + 1$

$y_2 = \dfrac{\cos\left(\log x\right)}{x} - \dfrac{\sin\left(\log x\right)}{x} + \dfrac{1}{x}$

11. By introducing a new variable t, putting $x = \cos t$, the expression $\left(1-x^2\right)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y$ is transformed into :

A) $\dfrac{d^2y}{dt^2} + y$ B) $\dfrac{d^2y}{dt^2} - t\dfrac{dy}{dt} + y$ C) $\dfrac{d^2y}{dt^2} - y$ D) $\dfrac{d^2y}{dt^2}$

Key. 1

Sol. $t = \cos^{-1}x \Rightarrow \dfrac{dt}{dx} = \dfrac{-1}{\sqrt{1-x^2}}$

$\dfrac{dy}{dx} = \dfrac{dy}{dx}.\dfrac{dt}{dx} = -\dfrac{1}{\sqrt{1-x^2}}.\dfrac{dy}{dt} \Rightarrow \sqrt{1-x^2}\dfrac{dy}{dx} = -\dfrac{dy}{dt}$

$\Rightarrow \left(1-x^2\right)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = \dfrac{d^2y}{dt^2}$

12. Let $f(x)$ and $g(x)$ be two functions having finite non-zero third order derivatives $f'''(x)$ and $g'''(x)$ for all $x \in R$. If $f(x)g(x) = 1$ for all $x \in R,$ then $\dfrac{f'''}{f'} - \dfrac{g'''}{g'}$ is equal to :

A) $3\left(\dfrac{f''}{g} - \dfrac{g''}{f}\right)$ B) $3\left(\dfrac{f''}{f} - \dfrac{g''}{g}\right)$

C) $3\left(\dfrac{g''}{g} - \dfrac{f''}{g}\right)$ D) $3\left(\dfrac{f''}{g} - \dfrac{g''}{f}\right)$

Key. 2

Sol. $fg = 1 \Rightarrow fg_1 + f_1g = 0$

$\Rightarrow fg_2 + gf_2 + 2f_1g_1 = 0$

$\Rightarrow fg_3 + gf_3 = -3\left[f_1g_2 + f_2g_1\right]$

$\Rightarrow \dfrac{f_3}{f_1} - \dfrac{g_3}{g_1} = 3\left[\dfrac{f_2}{f} - \dfrac{g_2}{g}\right]$

(using $fg_1 = -f_1g$)

13. If $y^{1/m} = \left[x + \sqrt{1+x^2}\right]$, then $\left(1+x^2\right)y_2 + xy_1$ is equal to :

A) m^2y B) my^2 C) m^2y^2 D) my

Key. 1

Sol. $y = \left(x + \sqrt{1+x^2}\right)^m \Rightarrow y_1 = m\left(x + \sqrt{1+x^2}\right)^{m-1}.\left\{1 + \dfrac{2x}{2\sqrt{1+x^2}}\right\}$

$$\Rightarrow \sqrt{1+x^2}\,y_1 = m\left(x+\sqrt{1+x^2}\right)^m \Rightarrow \sqrt{1+x^2}\,y_1 = my$$

$$\Rightarrow \left(1+x^2\right)y_2 + xy_1 = m^2 y$$

14. If $x+\cos\theta=\sec\theta, y+\cos^8\theta=\sec^8\theta$ then $\left(\frac{x^2+4}{y^2+4}\right)\left(\frac{dy}{dx}\right)^2 =$

a) 8 b) 16 c) 64 d) 49

Key. C

Sol. $$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{8\sec^8\theta\tan\theta + 8\cos^7\theta\sin\theta}{\sec\theta\tan\theta+\sin\theta} = \frac{8\tan\theta\left(\sec^8\theta+\cos^8\theta\right)}{\tan\theta\left(\sec\theta+\cos\theta\right)}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{64\left(\sec^8\theta+\cos^8\theta\right)^2}{\left(\sec\theta+\cos\theta\right)^2} = \frac{64\left[\left(\sec^8\theta-\cos^8\theta\right)^2+4\right]}{\left[\left(\sec\theta-\cos\theta\right)^2+4\right]} = \frac{64\left(y^2+4\right)}{\left(x^2+4\right)}$$

$$\left(\frac{x^2+4}{y^2+4}\right)\left(\frac{dy}{dx}\right)^2 = 64$$

15. Let $y=f(x)$ and $f:R\to R$ be an odd function which is differentiable such that $f'''(x)>0$ and $f(\alpha,\beta)=\sin^8\alpha+\cos^8\beta+2-4\sin^2\alpha\cos^2\beta$.

If $f''\left(f(\alpha,\beta)\right)=0$ then $\sin^2\alpha+\sin^2\beta=$

a) 0 b) 1 c) 2 d) 3

Key. B

Sol. $f''(x)$ is an odd function

$$\therefore f(\alpha,\beta)=0 \Rightarrow \left(\sin^4\alpha-1\right)^2+\left(\cos^4\beta-1\right)^2+2\left(\sin^2\alpha-\cos^2\beta\right)^2=0$$

$$\Rightarrow \sin^2\alpha+\sin^2\beta=1$$

16. If $y=\sin(\sin x)$ and $\frac{d^2y}{dx^2}+\frac{dy}{dx}\tan x+f(x)=0$, then $f(x)$ equals.

A) $\sin^2 x\sin(\cos x)$ B) $\sin^2 x\cos(\sin x)$ C) $\cos^2 x\sin(\cos x)$ D) $\cos^2 x\sin(\sin x)$

Key. D

Sol. $$\frac{dy}{dx}=\cos(\sin x)\cos x$$

$$\frac{d^2y}{dx^2}=-\cos(\sin x)\sin x+\cos x\left[-\sin(\sin x)\right]\cos x$$

$$\therefore \frac{d^2y}{dx^2}+\frac{dy}{dx}\tan x=-\cos^2 x\sin(\sin x)$$

$$\therefore \frac{d^2y}{dx^2}+\frac{dy}{dx}\tan x+\cos^2 x\sin(\sin x)=0$$

$\therefore f(x) = \cos^2 x \sin(\sin x)$

17. $\dfrac{d^2x}{dy^2}$ equals

A) $\left(\dfrac{d^2y}{dx^2}\right)^{-1}$ B) $-\left(\dfrac{d^2y}{dx^2}\right)^{-1}\left(\dfrac{dy}{dx}\right)^{-3}$ C) $\dfrac{d^2y}{dx^2}\left(\dfrac{dy}{dx}\right)^{-2}$ D) $-\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)^{-3}$

Key. D

Sol. $\dfrac{d^2x}{dy^2} = \dfrac{d}{dy}\left(\dfrac{dx}{dy}\right) = \dfrac{d}{dx}\left(\dfrac{dx}{dy}\right) \times \dfrac{dx}{dy}$

$$= \frac{d}{dx}\left[\frac{1}{\frac{dy}{dx}}\right] \times \frac{1}{\left(\frac{dy}{dx}\right)}$$

$$= \frac{-1}{\left(\frac{dy}{dx}\right)^2} \times \frac{d^2y}{dx^2} \times \frac{1}{\frac{dy}{dx}}$$

$$= -\left(\frac{dy}{dx}\right)^{-3} \cdot \frac{d^2y}{dx^2}$$

18. If $(a+bx)e^{\frac{y}{x}} = x$ then $x^3 \dfrac{d^2y}{dx^2}$ equals

A) $\left(\dfrac{dy}{dx} + x\right)^2$ B) $\left(x\dfrac{dy}{dx} - y\right)^2$ C) $\left(\dfrac{dy}{dx} - y\right)^2$ D) $\left(x\dfrac{dy}{dx} + y\right)^2$

Key. B

Sol. $\dfrac{y}{x} = \log x - \log(a+bx)$

Differentiable

$$\frac{x\frac{dy}{dx} - y}{x^2} = \frac{1}{x} - \frac{1}{a+bx}b = \frac{a}{x(a+bx)}$$

$$\therefore x\frac{dy}{dx} - y = \frac{ax}{a+bx} \rightarrow ....(1)$$

Differentiable again w.r.t x

$$x\frac{d^2y}{dx^2} = \frac{a^2}{(a+bx)^2}$$

$$\therefore x^3 \frac{d^2y}{dx^2} = \frac{a^2x^2}{(a+bx)^2} = \left(\frac{xdy}{dx} - y\right)^2$$

19. If $y = Tan^{-1}\left(\frac{3+2\log_e x}{1-6\log_e x}\right) + Tan^{-1}\left(\frac{\log_e\left(\frac{e}{x^2}\right)}{\log_e\left(ex^2\right)}\right)$, then $\frac{d^2y}{dx^2}$ is

A) 0 B) 1 C) -1 D) 2

Key. A

Sol. $y = Tan^{-1}\left(\frac{1-2\log_e^{x}}{1+2\log_e x}\right) + Tan^{-1}\left(\frac{3+2\log_e x}{1-6\log_e x}\right)$

$= Tan^{-1}(1) - Tan^{-1}(2\log_e x) + Tan^{-1}(3) + Tan^{-1}(2\log_e x)$

$= Tan^{-1}(1) + Tan^{-1}(3)$

$\therefore \frac{dy}{dx} = 0$

20. Let f be a function such that $f(x+y) = f(x) + f(y)$ for all x and y and $f(x) = (2x^2+3x)g(x)$ for all x where $g(x)$ is continuous and $g(0) = 9$ then $f'(x)$ is equals to

A) 9 B) 3 C) 27 D) 6

Key. C

Sol. $f'(x) = \underset{h\to 0}{Lt} \frac{f(x+h) - f(x)}{h}$

$= \underset{h\to 0}{Lt} \frac{f(x) + f(h) - f(x)}{h}$

$= \underset{h\to 0}{Lt} \frac{(2h^2+3h)g(h)}{h}$

$= \underset{h\to 0}{Lt} (2h+3)g(h)$

$= 3g(0) = 3.9 = 27$

21. If $y = (Tax)^{(Tanx)^{Tanx}}$ then $\frac{dy}{dx}$ $at\ x = \frac{\pi}{4}$ is

A) 1 B) 2 C) 3 D) 4

Key. B

Sol. $\log y = (\tan x)^{\tan x} \log(\tan x) \quad \to (1)$

Taking $\log$ again, we get from (1)

$\log(\log y) = \tan x \log(\tan x) + \log(\log(\tan x))$

Differentiate with respect to x

$$\frac{1}{\log y}\cdot\frac{1}{y}\frac{dy}{dx}=\sec^2 x\log(Tanx)+Tanx\frac{1}{Tanx}Sec^2x+\frac{1}{\log(Tanx)}\frac{1}{Tanx}Sec^2x$$

$$\therefore\frac{dy}{dx}=y\log y\ \sec^2 x\left(\log(Tanx)+1+\frac{1}{Tanx\log(Tanx)}\right)$$

$$=y(Tanx)^{Tanx}\log Tan.Sec^2x\left[(\log(Tanx)+1)+\frac{1}{Tanx\log(Tanx)}\right]$$

$$=y(Tanx)^{Tanx}Sec^2x\left[\log(Tanx)(\log Tanx+1)+Cotx\right]$$

When at $x=\frac{\pi}{4},\ y=1$

$$\therefore\frac{dy}{dx}=1.1.2(0+1)=2$$

22. The triangle formed by the tangent to the curve $f(x)=x^2+bx-b$ at the point (1, 1) and the co-ordinate axes, lies in the first quadrant. If its area is 2 then the value of b is

A) -1 B) 3 C) -3 D) 1

Key. C

Sol. $\frac{dy}{dx}=2x+b=2+b\ at\ (1,1)$

Equation of tangent is $y-1=(2+b)(x-1)$ is intercepts A and B on the axis are obtained by putting y = 0 and then x = 0

$$\therefore A=\frac{b+1}{b+2},\ B=-(b+1)$$

$$\Delta=\frac{1}{2}AB=2$$

$$\therefore AB=4\Rightarrow-(b+1)(b+1)=4(b+2)$$

$$\therefore b=-3$$

23. If $t(1+x^2)=x$ and $x^2+t^2=y$ then at x = 2, the value of $\frac{dy}{dx}$ is

a) $\frac{24}{5}$ b) $\frac{101}{125}$ c) $\frac{488}{125}$ d) $\frac{358}{125}$

Key. 3

Sol. $\frac{dy}{dx}=2x+2t.\frac{dt}{dx}$

$t=\frac{x}{1+x^2}\quad\frac{dt}{dx}=\frac{1-x^2}{(1+x^2)^2}$ put x =2

24. Suppose f and g are functions having second derivatives f'' and g'' every where, if $f(x).g(x)=1$ for all x and f' and g' are never zero, then $\frac{f''(x)}{f'(x)}-\frac{g''(x)}{g'(x)}$ equals

a) $2\frac{f'(x)}{f(x)}$ b) $\frac{2g'(x)}{g(x)}$ c) $-\frac{f'(x)}{f(x)}$ d) $\frac{g'(x)}{g(x)}$

Key. 1

Sol. We have $f'(x)g(x)+g'(x)f(x)=0 \Rightarrow \frac{g(x)}{g'(x)}+\frac{f(x)}{f'(x)}=0$ --- (1)

Further $f''(x)g(x)+2f'(x)g'(x)+g''(x)f(x)=0$

Divide throughout by $f'(x)g'(x)$ and use (1)

25. If $x=\varphi(t)$ and $y=\psi(t)$, then $\frac{d^2y}{dx^2}$ is equal to (dashes denote the derivative w.r.t 't')

a) $\frac{\varphi'\psi''-\psi'\varphi''}{\varphi'}$ b) $\frac{\varphi'\psi''-\psi'\varphi''}{\varphi'^2}$ c) $\frac{\varphi''}{\psi''}$ d) $\frac{\psi''}{\varphi'^2}-\frac{\psi'.\varphi''}{\varphi'^3}$

Key. 4

Sol. $\frac{dy}{dx}=\frac{\psi'}{\varphi'}$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}=\frac{\frac{\varphi'\psi''-\psi'\varphi''}{(\varphi')^2}}{\varphi'}$$

26. If $x=a\sin 2\theta(1+\cos 2\theta)$ and $y=b\cos 2\theta(1-\cos 2\theta)$, then $\left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{3}}=$

a) 0 b) $\frac{ab}{\sqrt{3}}$ c) $\frac{b}{a}\sqrt{3}$ d) 1

Key. 3

Sol. $x=a\sin 2\theta\left(2\cos^2\theta\right)=2a\sin 2\theta\cos^2\theta$

$y=b\cos 2\theta\left(2\sin^2\theta\right)=2b\cos 2\theta\sin^2\theta$

$$\frac{dy}{dx}=\frac{2b\left(-2\sin 2\theta\sin^2\theta+\cos 2\theta\sin 2\theta\right)}{2a\left(2\cos 2\theta\cos^2\theta+\sin 2\theta(-\sin 2\theta)\right)}$$

$$\Rightarrow\left(\frac{dy}{dx}\right)_{\theta=\pi/3}=\frac{b}{a}.\sqrt{3}$$

27. If $x=\frac{1+t}{t^3}, y=\frac{3}{2t^2}+\frac{2}{t}$ satisfies $f(x)\left(\frac{dy}{dx}\right)^3=1+\frac{dy}{dx}$, then $f(x)$

a) x b) $\frac{x^2}{1+x^2}$ c) $x+\frac{1}{x}$ d) $x-\frac{1}{x}$

Key. 1

Sol. $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{-\frac{(3+2t)}{t^3}}{-\frac{(3+2t)}{t^4}}=t$

Since

$$f(x)\left(\frac{dy}{dx}\right)^3=1+\frac{dy}{dx}\Rightarrow f(x)t^3=1+t \qquad \Rightarrow f(x)=\frac{1+t}{t^3}=x$$

28. If $y=(1+x)(1+x^2)(1+x^4)$, then $\frac{dy}{dx}$ $at\ x=0$ is

a) 1 b) – 1 c) 0 d) 2

Key. 1

Sol. Multiplying numerator & denominator by (1 - x)

$$y=\frac{(1-x)(1+x)(1+x^2)(1+x^4)}{1-x} \qquad \Rightarrow y=\frac{1-x^8}{1-x} \qquad \left(\frac{dy}{dx}\right)_{x=0}=1$$

29. $y=f(x)$ be a real valued twice differentiable function defined on R, then

$\frac{d^2y}{dx^2}\left(\frac{dx}{dy}\right)^3+\frac{d^2x}{dy^2}=$

(A) $\frac{dy}{dx}$ (B) $\frac{dx}{dy}$

(C) $\frac{d^2y}{dx^2}$ (D) 0

Key. D

Sol. $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{1}{\frac{dx}{dy}}\right)$

$$=\frac{d}{dy}\left(\frac{1}{\frac{dx}{dy}}\right)\frac{dy}{dx}=\frac{-\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^2}\cdot\frac{dy}{dx}=-\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3}$$

$$\Rightarrow \frac{d^2y}{dx^2}\cdot\left(\frac{dx}{dy}\right)^3+\frac{d^2x}{dy^2}=0$$

30. If cos y = x cos(a+y) and $\frac{dy}{dx}=\frac{k}{1+x^2-2x\cos a}$ then the value of k is

a) sin a b) cos a c) 1 d) – sin a

Key. A

Sol. $x=\frac{\cos y}{\cos(a+y)}\Rightarrow\frac{dx}{dy}=\frac{\sin a}{\cos^2(a+y)}\Rightarrow\frac{dy}{dx}=\frac{\cos^2 y}{x^2\sin a}=\frac{\sin a}{1+x^2-2x\cos a}\Rightarrow K=\sin a$

31. Given that f(x) = $\begin{cases}\frac{x.g(x)}{|x|}, & x\neq 0\\ 0, & x=0\end{cases}$ g(0) = 0 = g′(0), then f′(0) equals

(A) 1
(B) –1
(C) 2
(D) 0

Key. D

Sol. f′(0) = $\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$

= $\lim_{h\to 0}\frac{\frac{hg(h)}{|h|}-0}{h}=\lim_{h\to 0}\frac{g(h)}{|h|}=0$ = $\lim_{h\to 0}\frac{g'(h)}{1}=0$

32. If $f(x)=x+\tan x,$ and f is inverse of g, then $g'(x)$ equal to

a) $\frac{1}{1+[g(x)-x]^2}$ b) $\frac{1}{2-[g(x)-x]^2}$ c) $\frac{1}{2+[g(x)-x]^2}$ d) $\frac{1}{1-[g(x)-x]^2}$

Key. C

Sol. $f(x)=x+\tan x$

$f\left(f^{-1}(y)\right)=f^{-1}(y)+\tan\left(f^{-1}(y)\right)$

$y=g(y)+\tan(g(y))$

$x=g(x)+\tan(g(x))$

diff

$1=g'(x)+\sec^2(g(x)).g'(x)$

$g'(x)=\frac{1}{2+[g(x)-x]^2}$

33. If 'f' is an increasing function from R $\to$ R such that f''(x) > 0 and f^{-1} exists then $\frac{d^2\left(f^{-1}(x)\right)}{dx^2}$ is

a) < 0
b) > 0
c) = 0
d) cannot be determined

Key. A

Sol. f'(x) > 0 and f'' (x) > 0

Let g(x) = $f^{-1}(x)$

f (g(x)) = x

f' (g(x)) . g'(x) = 1

$g'(x) = \dfrac{1}{f'(g(x))}$

$g''(x) = -\dfrac{1}{\left(f'(g(x))\right)^2} f''(g(x)).\ g'(x)$

$\therefore g''(x). < 0.$

$(\because f''(x)\ \&\ f'(x) > 0)$

$\dfrac{d^2}{dx^2}\left(f^{-1}(x) = gw''(x)\right). < 0$

34. If a function $f:[-2a, 2a] \to R$ is an odd function such that $f(2a-x) = f(x), \forall x \in [a, 2a]$ and left hand derivative at x = a is 0 then find left hand derivative at x = -a

(a) 0 (b) -1 (c) $\dfrac{1}{2}$ (d) a

Key. A

Sol. $f'(-a^-) = \underset{h\to 0}{Lt} \dfrac{f(-a) - f(-a-h)}{h}$

$\underset{h\to 0}{Lt} \dfrac{-f(a) + f(a+h)}{h} = \underset{h\to 0}{Lt} \dfrac{-f(a) + f(a+h)}{h} = 0$

35. $\dfrac{d^2x}{dy^2} =$

(a) $\left(\dfrac{d^2y}{dx^2}\right)^{-1}$ (b) $-\left(\dfrac{d^2y}{dx^2}\right)^{-1}\left(\dfrac{dy}{dx}\right)^{-3}$

(c) $\dfrac{d^2y}{dx^2}\left(\dfrac{dy}{dx}\right)^{-2}$ (d) $-\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)^{-3}$

Key. D

Sol. $\dfrac{dx}{dy} = \left(\dfrac{dy}{dx}\right)^{-1}$

Again differentiating with respective y

36. If $\sqrt{x^2+y^2} = a e^{Tan^{-1}\left(\frac{y}{x}\right)} (a > 0), (y(0) > 0)$ then $y^{11}(0) =$

(a) $\dfrac{a}{2}e^{\pi/2}$ (b) $ae^{-\pi/2}$ (c) $\dfrac{-2}{a}e^{-\pi/2}$ (d) $\dfrac{a}{2}e^{-\pi/2}$

Key. C

Sol. Differentiating with respect to x two times

$\dfrac{d^2y}{dx^2} = \dfrac{2(x^2+y^2)}{(x-y)^3}$

$Put\ x = 0 \Rightarrow \dfrac{2y(0)^2}{(-y(0))^3}$

37. If $y = e^{-x}\cos x$ and $y_4 + ky = 0$, where $y_4 = \frac{d^4y}{dx^4}$, then k =

A) 4 B) – 4 C) 2 D) – 2

Key. A

Sol. $y = e^{-x}\cos x$

$y_1 = -e^{-x}\sin x - e^{-x}\cos x = -e^{-x}\sin x - y$

$y_2 = -e^{-x}\cos x + e^{-x}\sin x - y_1$

$\Rightarrow \quad y_2 = -y - y_1 + e^{-x}\sin x$

$\Rightarrow \quad y_2 = -2(y + y_1)$

$\Rightarrow \quad y_3 = -2(y_1 + y_2)$

$\Rightarrow \quad y_3 = -2(e^{-x}\sin x - y)$

$\Rightarrow \quad y_4 = 2y_1 - 2y - 2y - 2y_1$

or $\quad y_4 + 4y = 0 \quad \Rightarrow \quad k = 4$

38. Let $y = e^{2x}$. Then $\left(\frac{d^2y}{dx^2}\right)\left(\frac{d^2x}{dy^2}\right)$ is:

A) 1 B) e^{-2x} C) $2e^{-2x}$ D) $-2e^{-2x}$

Key. D

Sol. $y = e^{2x}$

$\therefore \quad \frac{dy}{dx} = 2e^{2x}$ and $\frac{d^2y}{dx^2} = 4e^{2x}$

$\frac{dx}{dy} = \frac{1}{2e^{2x}} = \frac{1}{2y}$

$\therefore \quad \frac{d^2x}{dy^2} = -\frac{1}{2y^2} = -\frac{1}{2}e^{-4x}$

$\therefore \quad \frac{d^2y}{dx^2}.\frac{dx^2}{dy^2} = 4.\,e^{2x}.\left(\frac{-e^{-2x}}{2e^{2x}}\right) = -2e^{-2x}$

39. If $y^2 = P(x)$, is a polynomial of degree 3, then $\left(\frac{d}{dx}\right)\left(y^3.\frac{d^2y}{dx^2}\right)$ equals:

A) $P'''(x) + P'(x)$ B) $P''(x).P'''(x)$ C) $P(x).P'''(x)$ D) a constant

Key. C

Sol. $y^2 = P(x) \Rightarrow 2y\frac{dy}{dx} = P'(x)$

Or $\quad 2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = P''(x)$

Or $\quad 2y\frac{d^2y}{dx^2} = P'' - 2\left(\frac{dy}{dx}\right)^2 = P'' - \frac{P'^2}{2y^2}$

$\because \quad 2y^3\frac{d^2y}{dx^2} = y^2P'' - \frac{1}{2}P'^2 = PP'' - \frac{1}{2}P'^2$

$\therefore \quad 2\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right) = P'P'' + PP''' - P'P'' = PP'''$

40. If $f'(x) = -f(x)$ and g(x) = f(x) and $F(x) = \left(f\left(\frac{x}{2}\right)\right)^2 + \left(g\left(\frac{x}{2}\right)\right)^2$ and given that F(5) = 5, then F(10) is equal to

A) 5 B) 10 C) 0 D) 15

Key. A

Sol. $F'(x)\left[f\left(\frac{x}{2}\right).f'\left(\frac{x}{2}\right)+g\left(\frac{x}{2}\right)g'\left(\frac{x}{2}\right)\right]$

Here $g(x) = f'(x)$ & $g'(x) = f''(x) = -f(x)$

So $F'(x) = f\left(\frac{x}{2}\right)g\left(\frac{x}{2}\right) - f\left(\frac{x}{2}\right)g\left(\frac{x}{2}\right) = 0$

$\Rightarrow$ F(x) is constant function

So F(10) = 5

41. If $f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$ (where p is constant), then at x = 0, $\frac{d^3 f(x)}{dx^3} = 0$

A) p B) p^2 C) – p D) 0

Key. D

Sol. $f'(x) = \begin{vmatrix} 3x^2 & \cos x & -\sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$

$f''(x) = \begin{vmatrix} 6x & -\sin x & -\cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$

$f'''(x) = \begin{vmatrix} 6 & -1 & 0 \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix} = 0$

42. If $x^y = e^{x-y}$, then $\frac{dy}{dx} =$

A) $(1+\log x)^{-1}$ B) $(1+\log x)^{-2}$

C) $\log x(1+\log x)^{-2}$ D) $\log x(1+\log x)^{-1}$

Key. C

Sol. $x^y = e^{x-y}$ i.e. $y\,ln\,x = x - y$ i.e. $y = \frac{x}{1+ln\,x}$

$\therefore$ $\frac{dy}{dx} = \frac{ln\,x}{(1+ln\,x)^2}$

43. If $y = \tan^{-1}\left(\frac{2^x}{1+2^{2x+1}}\right)$, then $\frac{dy}{dx}$ at x = 0 is

A) 1 B) 2 C) ln 2 D) none of these

Key. D

Sol. $y = \tan^{-1}\left(\frac{2^{x+1}-2^x}{1+2^x.2^{x+1}}\right) = \tan^{-1}2^{(x+1)} - \tan^{-1}2^x \Rightarrow y' = \frac{2^{x+1}\ln 2}{1+\left(2^{x+1}\right)^2} - \frac{2^x \ln 2}{1+\left(2^x\right)^2} \Rightarrow y'(0) = -\frac{1}{10}\ln 2$

44. A function g defined for all real x > 0 satisfies g(1) = 1, $g'\left(x^2\right)=x^3$ for all x > 0, then g(4) equals

A) $\frac{13}{3}$ B) 3 C) $\frac{67}{5}$ D) none of these

Key. C

Sol. $g'\left(x^2\right)=x^3$

$\Rightarrow \quad g'(t)=t^2, \quad$ where $\quad x^2=t$

$\Rightarrow \quad g(t)=\frac{t^{5/2}}{5/2}+c$

$\therefore \quad g(1)=\frac{(1)^{5/2}}{5/2}+c$

$\therefore \quad c=\frac{3}{5}$

$\therefore \quad g(t)=\frac{t^{5/2}}{5/2}+\frac{3}{5}$

$g(4)=\frac{(4)^{5/2}}{5/2}+\frac{3}{5}=\frac{67}{5}$

45. If f(x) = log x, then $f'(\log x)$ equals

A) $\frac{x}{\log x}$ B) $\frac{\log x}{x}$ C) $\frac{1}{x\log x}$ D) $\frac{1}{\log x}$

Key. D

Sol. $f(x)=\log x$

$f'(x)=\frac{1}{x}$

$f'(\log x)=\frac{1}{\log x}$

23. f is a strictly monotonic differentiable function with $f^1(x)=\frac{1}{\sqrt{1+x^3}}$. If g is the inverse of f then $g^{11}(x)=$

A) $\frac{3x^2}{2\sqrt{1+x^3}}$ B) $\frac{3g^2(x)}{2\sqrt{1+g^2(x)}}$ C) $\frac{3}{2}g^2(x)$ D) $\frac{x^2}{\sqrt{1+x^3}}$

Key. C

Sol. $y=f(x)\Leftrightarrow x=f^{-1}(y)\Leftrightarrow x=g(y)$ since $g=f^{-1}$

$\frac{dx}{dy}=g^1(y)=\frac{1}{f^1(x)}=\sqrt{1+x^3}$

$\frac{d^2x}{dy^2}=g^{11}(y)=\frac{d}{dy}^{\left(\frac{dx}{dy}\right)}=\frac{d}{dy}^{\sqrt{1+x^3}}$

$=\frac{d}{dx}^{\left(\sqrt{1+x^3}\right)}\times\frac{dx}{dy}=\frac{3x^2}{2\sqrt{1+x^3}}\times g^1(y)=\frac{3x^2}{2}$

$g^{11}(y)=\frac{3}{2}g^{2}(y)$

24. Let $f(x)=\frac{g(x)}{x}$ when $x\neq 0$ and $f(0)=0$. If $g(0)=g^{1}(0)=0$ and $g^{11}(0)=17$ then $f^{1}(0)=$

A) 3/4 B) –1/2 C) 17/3 D) 17/2

Key. D

Sol. $f^{1}(0)=\underset{x\to 0}{Lt}\frac{f(x)-f(0)}{x-0}=\underset{x\to 0}{Lt}\frac{f(x)}{x}$

$=\underset{x\to 0}{Lt}\frac{g(x)}{x^{2}}=\underset{x\to 0}{Lt}\frac{g^{1}(x)}{2x}=\underset{x\to 0}{Lt}\frac{g^{11}(x)}{2}=\frac{g^{11}(0)}{2}=\frac{17}{2}$

Differentiation

Multiple Correct Answer Type

1. If $\frac{d}{dx}\{f(x)\} = g(x)$ where $g(x) = e^{-x}.f(x) + e^{x}.f(-x)$ then $(given\ f(0)=0)$.

(A) $g(x)$ is an even function

(B) $f(x)$ is an odd function

(C) $g^1(0) = f^1(0) = g(0)$

(D) $g^1(0) \neq f(0)$

Key. A,B

Sol. Clearly $g(x)$ in even hence $f(x)$ is odd as $f(0)=0$

$g^1(0)=0; f^1(0)=g(0)$ (obvious)

2. If $y=\sec(\tan^{-1}x)$, then $\frac{dy}{dx}$ at x = 1 is equal to

A) $\frac{1}{\sqrt{2}}$

B) $\sin^{-1}\left(\sin\frac{1}{\sqrt{2}}\right)$

C) 1

D) none of these

Key. A

Sol. $y=\sec(\tan^{-1}x)$

$\therefore \quad \frac{dy}{dx} = \frac{\sec(\tan^{-1}x).\tan(\tan^{-1}x)}{1+x^2}$

$\therefore \quad \left.\frac{dy}{dx}\right|_{x=1} = \frac{1}{\sqrt{2}} = \sin^{-1}\left(\sin\frac{1}{\sqrt{2}}\right)$

3. If $y=10^{10^x}$ then $\frac{1}{y}\frac{dy}{dx}=10^x.\lambda$ then value of λ is

A) $ln\ 10$

B) $(ln\ 10)^2$

C) $e^{ln(ln\ 10)^2}$

D) $(\log_{10}e)^2$

Key. B,C

Sol. $y=10^{10^x}$

$\therefore \quad \frac{dy}{dx}=10^{10^x}\,ln10.\ 10^x\ ln\ 10 = y\,10^x\,(ln\ 10)^2$

$\therefore \quad \frac{1}{y}\frac{dy}{dx}=10^x\,(ln\ 10)^2$

$\therefore \quad \lambda=(ln\ 10)^2=e^{ln(ln\ 10)^2}$

4. Given $f(x)=-\frac{x^3}{3}+x^2\sin\ 1.5a - x\sin a.\sin 2a - 5\ \sin^{-1}(a^2-8a+17)$ then:

A) $f'(x)=-x^2+2x\sin 6-\sin 4\sin 8$

B) $f'(\sin 8)>0$

C) $f'(x)$ is not defined at x = sin 8

D) $f'(\sin 8)<0$

Key. A,D

Sol. $f(x)=-\frac{x^3}{3}+x^2\sin 1.5a - x\sin a.\sin 2a - 5\sin^{-1}(a^2-8a+17)$

f(x) is defined when $-1\le a^2-8a+17\le 1$

$$-1\le(a-4)^2+1\le 1 \quad \Rightarrow \quad a=4$$

$\therefore \quad f(x) = -\frac{x^3}{3} + x^2 \sin 6 - x \sin 4 \sin 8 - \frac{5\pi}{2}$

$\therefore \quad f'(x) = -x^2 + 2x \sin 6 - \sin 4 \sin 8$

$f'(\sin 8) = -\sin^2 8 + 2\sin 8 \sin 6 - \sin 4 \sin 8 = \sin 8\left[2\sin 6 - (\sin 8 + \sin 4)\right]$

= sin 8 [2 sin 6 – 2 sin 6 cos 2] = 2 sin 6 sin 8 (1 – cos 2)

Sin 6 < 0, sin 8 > 0, 1 – cos 2 > 0

$\therefore \quad f'(\sin 8) < 0$

Differentiation

Assertion Reasoning Type

1. Statement – 1: Let $g(x)=x[x]$ and $[.]$ denotes greatest integral function, when x is not an integral, then rule for $g'(x)$ is given by $[x]$

Statement – 2: $g'(x)$ does not exist for any $x\in$ integer.

Key. A

Sol. $g(x)=x[x]\ \begin{cases}-x, & -1\le x<0\\ 0, & 0\le x<1\\ x, & 1\le x<2\\ 2x, & 2\le x<3\\ .. & ....\end{cases}$

$$\Rightarrow g'(x)=\begin{cases}-1, & -1<x<0\\ 0, & 0<x<1\\ 1, & 1<x<2\\ 2, & 2<x<3\\ .. & ....\end{cases}\qquad \Rightarrow g'(x)=[x]$$

2. Statement – 1: Let $f:[0,\alpha)\to[0,\alpha)$, be a function defined by $y=f(x)=x^2$, then

$$x^3\left(\frac{d^2y}{dx^2}\right)\left(\frac{d^2x}{dy^2}\right)=\frac{-1}{2}$$

Statement – 2: $\frac{d^2x}{dy^2}=\frac{d}{dy}\left(\frac{dx}{dy}\right)$

Key. A

Sol. $\because y=x^2$

$\Rightarrow \frac{dy}{dx}=2x$

$\Rightarrow \frac{d^2y}{dx^2}=2$

Now $y=x^2$

$\Rightarrow 1=2x\frac{dx}{dy}\Rightarrow \frac{dx}{dy}=\frac{1}{2x}$

$\Rightarrow \frac{d^2x}{dy^2}=\frac{-1}{2x^2}.\frac{dx}{dy}=\frac{-1}{4x^3}$

$$\therefore x^3\left(\frac{d^2y}{dx^2}\right)\left(\frac{d^2x}{dy^2}\right) = -\frac{1}{2}$$

3. statement – 1 : If f : $R \to (0,\infty)$ be a function satisfying the relation

$$f(x+y) - f(x-y) = f(x)\left(f(y) - f(-y)\right),\ \forall\ x, y \in R \text{ and}$$

$f'(o) = \dfrac{\pi}{2};$ when f(x) is not an even function then $f(x) = e^{\frac{\pi}{2}x}$

Because

Statement – 2 : $2f'(x) = \lim\limits_{h\to 0}\left(\dfrac{f(x+h)-f(x)}{h} + \dfrac{f(x-h)-f(x)}{-h}\right)$

Key. A

Sol. If $f(x) = \dfrac{e^x}{1+e^x}$

$\Rightarrow f(a) + f(-a) = 1$ and also using the property $\int_p^q f(x)dx = \int_p^q f(p+q-x)dx$

We get $\dfrac{I_2}{I_1} = 2$

4. Let f and g be real valued functions defined on interval (–1, 1) such that g″(x) is continuous g(0) ≠ 0, g′(0) = 0 g″(0) ≠ 0, and f(x) = g(x) sinx.

STATEMENT–1

$\lim\limits_{x\to 0}$ [g(x) cotx – g(x) cosec x] = f″ (0).

because

STATEMENT–2

f′(0) = g(0)

Key. A

Sol. f(x) = g(x) sinx

f″(x) = g″(x) sinx + 2g′(x) cosx g(x) sinx

f″(0) = 2g′(0) = 0

f′(x) = g′(x) sinx + g(x) cosx

f′(0) = g(0)

For statement 1: $\lim\limits_{x\to 0}\dfrac{g(x)\cos x - g(x)}{\sin x} = \lim\limits_{x\to 0}\dfrac{g'(x)\cos x - g(x)\sin x - g'(x)}{\cos x}$ = g′(0) = 0 = f″(0)

For statement 2

f′(0) = g(0)

5. Let $p(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$.

Statement – 1 : If $|p(x)| \le |e^{x-1} - 1|$ for all $x \ge 0$, then $|a_1 + 2a_2 + ... + na_n| \le 1$.

Statement – 2 : $|p(x)| \le |e^{x-1} - 1| \Rightarrow p(1) = 0$ and $p'(1) = \lim\limits_{h\to 0}\dfrac{p(1+h) - p(1)}{h}$.

Key. A

Sol. Given $|p(x)| \le |e^{x-1} - 1|$ for all $x \ge 0$.

Put $x = 1$.

$|p(1)| \le 0 \Rightarrow p(1) = 0$.

Put $x = 1+h$

$|p(1+h)| \le |e^h - 1|$

Divide by $|h|$

$$\left|\frac{p(1+h)}{h}\right| \le \left|\frac{e^h-1}{h}\right|$$

$$\Rightarrow \left|\frac{p(1+h)-p(1)}{h}\right| \le \left|\frac{e^h-1}{h}\right| \quad \text{since } p(1)=0.$$

Now take limit both sides as $h \to 0$.

$$|p'(1)| \le 1$$

$$\Rightarrow |a_1 + 2a_2 + ... + na_n| \le 1.$$

$\therefore$ $Ans: a$.

6. Statement – 1 For x < 0, $\frac{d}{dx}(l\,n|x|) = \frac{1}{x}$

Statement – 2 For x < 0, $|x| = -x \quad \Rightarrow \quad \frac{d}{dx}|x| = -1$

Key. A

Sol. $\frac{d}{dx} l\,n|x| = \frac{d}{dx} l\,n(-x) = \frac{1}{(-x)}(-1) = \frac{1}{x}$

7. Statement – 1 $\frac{d}{dx}\left\{\tan^{-1}(\sec x + \tan x)\right\} = \frac{d}{dx}\left\{\cot^{-1}(\operatorname{cosec} x + \cot x)\right\}$, $x \in \left(0, \frac{\pi}{4}\right)$.

Statement – 2 $\sec^2 x - \tan^2 x = 1 = \operatorname{cosec}^2 x - \cot^2 x$.

Key. B

Sol. $\frac{d}{dx}\left\{\tan^{-1}(\sec x + \tan x)\right\} = \frac{d}{dx}\left\{\tan^{-1}\left(\frac{1+\sin x}{\cos x}\right)\right\}$

$$= \frac{d}{dx}\left\{\tan^{-1}\left(\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right)\right\} = \frac{d}{dx}\left(\frac{\pi}{4}+\frac{x}{2}\right) = \frac{1}{2}$$

and $\frac{d}{dx}\left\{\cot^{-1}(\operatorname{cosec} x + \cot x)\right\} = \frac{d}{dx}\left\{\cot^{-1}\left(\frac{1+\cos x}{\sin x}\right)\right\}$

$$= \frac{d}{dx}\left\{\cot^{-1}\left(\cot\frac{x}{2}\right)\right\} = \frac{d}{dx}\left(\frac{x}{2}\right) = \frac{1}{2}$$

Differentiation

Comprehension Type

Passage - 1

Let f be a differentiable function Define $D^* f(x) = \lim_{h\to 0} \frac{f^2(x+h) - f^2(x)}{h}$ where $f^2(x) = (f(x))^2$

Answer the following

1. If $f(x) = \tan x, g(x) = \ln x$ then $D^*(fg)(1)$ =

   A. $\tan 1$ B. 0 C. $(\tan 1)^2$ D. $2(\tan 1)^2$

Key. B

Sol. $D^*(fg)(x) = f^2(x).2g(x)g^1(x) + g^2(x).2f(x)f^1(x)$

$= \tan^2 x.2\frac{\ln x}{x} + (\ln x)^2.2\tan x \sec^2 x$

$\therefore D^*(fg)(1) = 0$

2. If $f(x) = x^5$ and $g(x) = e^x$ then $D^*(fg)(1)$ =

   A. e^2 B. $4e^2$ C. $8e^2$ D. $12e^2$

Key. D

Sol. $D^*(fg)(x) = x^{10}.2e^{2x} + e^{2x}.10x^9$

$\Rightarrow D^*(fg)(1) = 2e^2 + 10e^2 = 12e^2$

3. If $f(x) = \sin x, g(x) = \cos x,$ then $D^*(fog)(x) =$

   A. $-\sin(2\cos x)\sin x$ B. $-\sin(\cos x)\sin x$ C. $-\sin^2(\cos x)\sin x$ D. $-\sin(\cos x)\sin^2 x$

Key. A

Sol. $(fog)(x) = \sin(\cos x) \Rightarrow D^*((fog)(x) = 2(fog)(x).(fog)^1(x)$

$= 2\sin(\cos x).\cos(\cos x).(-\sin x) = -\sin x \sin(2\cos x)$

Passage - 2

$f(x)$ is a polynomial function $f : R \to R$ such that $f(2x) = f'(x)f''(x)$

4. The value of $f(2)$ is

   A) 4 B)12 C) 15 D) $\frac{32}{9}$

Key. D

5. $f(x)$ is

A) one – one and onto B) one – one and into
C) many – one and onto D) many – one and into

Key. A

6. The equation $f(x) = x$ has

A) three real and positive roots
B) three real and negative roots
C) one real root
D) three real roots such that sum of roots is zero.

Key. D

Sol. 4. Suppose degree of $f(x) = n$, then degree of $f' = n-1$ and degree of $f'' = n-2$

So = $n = n-1+n-2 \Rightarrow n = 3$

Let $f(x) = ax^3 + bx^2 + cx + d$ (where $a \neq 0$)

$f(2x) = f'(x) f''(x)$

$\Rightarrow 8ax^3 + 4bx^2 + 2cx + d = (3ax^2 + 2bx + c)(6ax + 2b)$

Comparing coefficients of terms, we have

$a = \frac{4}{9}, b = 0, c = 0, d = 0$

$\therefore f(x) = \frac{4x^3}{9}$, which is clearly one one and onto

$\Rightarrow f(2) = \frac{32}{9}$

5. See the above it is a clearly one one and onto

6. $\frac{4x^3}{9} = x \Rightarrow x = 0, \pm\frac{3}{2}$

Sum of roots of equation is zero.

Passage - 3

If two function f(x) and F(x) are differentiable in [a, b] and $F'(x) \neq 0$ for any $x \in [a, b]$, then there exists at least one $c \in (a, b)$ such that $\frac{f(b)-f(a)}{F(b)-F(a)} = \frac{f'(c)}{F'(c)}$.

7. If $f(x) = e^x$ and $F(x) = e^{-x}$, then $c \in (a, b)$ is such that a, c, b are in

(A) AP
(B) GP
(C) HP
(D) none of these

Key. A

8. If f(x) = sin x, F(x) = cos x and $c \in (a, b)$, where $\frac{f(b)-f(a)}{F(b)-F(a)} = \frac{f'(c)}{F'(c)}$, then c is

(A) AM of a, b
(B) GM of a, b

(C) HM of a, b (D) none of these

Key. A

9. If f(x) is differentiable in [a, b] and for $c \in (a, b)$, $(a^2 - b^2)\, f'(C) = k\{f(a) - f(b)\}$, then k is equal to

(A) 2 (B) 2c

(C) c (D) c^2

Key. B

Sol. 7. (A)

We have $\dfrac{f(b)-f(a)}{F(b)-F(a)} = \dfrac{e^b - e^a}{e^{-b} - e^{-a}} = \dfrac{f'(c)}{F'(c)} = \dfrac{e^c}{-e^c}$

$\Rightarrow \dfrac{(e^b - e^a)e^a e^b}{e^a - e^b} = -e^{2c}$

$\Rightarrow e^{2c} = e^{a+b}$

$\Rightarrow 2c = a + b$

8. (A)

We have, $\dfrac{\sin b - \sin a}{\cos b - \cos a} = \dfrac{\cos c}{-\sin c}$

$\Rightarrow \dfrac{\sin\frac{b-a}{2}\cos\frac{a+b}{2}}{\sin\frac{a-b}{2}\sin\frac{a+b}{2}} = -\cot c$

$\Rightarrow \cot\dfrac{a+b}{2} = \cot c$

$\Rightarrow c = \dfrac{a+b}{2}$

9. (B)

Let $f(x) = x^2$

Hence $\dfrac{f(b)-f(a)}{b^2 - a^2} = \dfrac{f'(c)}{2c}$

$\Rightarrow k = 2c.$

Passage - 4

Consider following two infinite series in real r and θ

$$C = 1 + r\cos\theta + \frac{r^2\cos 2\theta}{2!} + \frac{r^3\cos 3\theta}{3!} + ...$$

$$S = r\sin\theta + \frac{r^2\sin 2\theta}{2!} + \frac{r^3\sin 3\theta}{3!} + ...$$

If θ remains constant and r varies, then

10. The expression $C\dfrac{dC}{dr} + S\dfrac{dS}{dr}$ is equal to

(A) $C^2 + S^2$ (B) $(C^2 + S^2)\cos\theta$

(C) $(C^2 + S^2)\sin\theta$ (D) CS

Key. B

11. The expression $C\dfrac{dS}{dr} - S\dfrac{dC}{dr}$ is equal to

(A) $C^2 + S^2$ (B) $(C^2 + S^2)\cos\theta$

(C) $(C^2 + S^2)\sin\theta$ (D) CS

Key. C

12. $\left(\frac{dC}{dr}\right)^2 + \left(\frac{dS}{dr}\right)^2$ is equal to

(A) $C^2 + S^2$ (B) $(C^2 + S^2)\cos\theta$

(C) $(C^2 + S^2)\sin^2\theta$ (D) 1

Key. A

Sol. 10-12. We have $C + iS = 1 + re^{i\theta} + \frac{r^2e^{2i\theta}}{2!} + \frac{r^3e^{3i\theta}}{3!} + ... = e^{re^{i\theta}}$ (1)

and, $C - iS = 1 + re^{-i\theta} + \frac{r^2e^{-2i\theta}}{2!} + \frac{r^3e^{-3i\theta}}{3!} + ... = e^{re^{-i\theta}}$ (2)

Clearly, $C^2 + S^2 = |e^{re^{i\theta}}|^2 = |e^{r\cos\theta}e^{ir\sin\theta}|^2 = e^{2r\cos\theta}$ (3)

On differentiating equation (1) with respect to r, we get

$\frac{dC}{dr} + i\frac{dS}{dr} = e^{i\theta} + re^{2i\theta} + \frac{r^2e^{3i\theta}}{2!} + ... = e^{i\theta}e^{re^{i\theta}}$ (4)

$\therefore \left(\frac{dC}{dr}\right)^2 + \left(\frac{dS}{dr}\right)^2 = |e^{i\theta} + e^{re^{i\theta}}| = |e^{re^{i\theta}}|^2 = C^2 + S^2$ from equation (3)

On multiplying equations (2) and (3), we get

$\left(\frac{dC}{dr} + i\frac{dS}{dr}\right)(C - iS) = e^{i\theta}e^{r\left(e^{i\theta}+e^{-i\theta}\right)} = = e^{i\theta}e^{2r\cos\theta} = (\cos\theta + i\sin\theta)e^{2r\cos\theta}$

Equating real and imaginary parts and from equation (3),

$C\frac{dC}{dr} + S\frac{dS}{dr} = \left(C^2 + S^2\right)\cos\theta$ and $C\frac{dS}{dr} - S\frac{dC}{dr} = \left(C^2 + S^2\right)\sin\theta$

Passage – 5

Let $f(x)$ be a real valued function not identically zero, such that

$f\left(x + y^n\right) = f(x) + \left(f(y)\right)^n \ \forall\ x, y \in R$ where $n \in N(n \neq 1)$ and $f'(0) \geq 0$

13. The value of $f'(0)$ is

a) 1 b) 1 + n c) n d) 2

Key. A

14. The value of $f(5)$ is___

a) 2 b) 3 c) 5n d) 5

Key. D

15. $\int_0^1 f(x)\,dx =$ _____

a) $\frac{1}{2n}$ b) 2n c) $\frac{1}{2}$ d) 2

Key. C

Sol. 13-15. Putting $x = y = 0$ in the given equation

$$f(0) = f(0) + \left(f(0)\right)^n$$

$$\Rightarrow f(0) = 0$$

$$f'(0)=\lim_{h\to 0}\frac{f(0)+\left(f\left(h^{1/n}\right)\right)^n-f(0)}{\left(h^{1/n}\right)^n}$$

$$=\lim_{h\to 0}\left(\frac{\left(f\left(h^{1/n}\right)\right)}{\left(h^{1/n}\right)}\right)^n=l^n$$

$\therefore l=l^n \Rightarrow l=0,1,-1$

But $l\neq -1\left(\because f'(0)\geq 0\right)$

If $l=0$ then

$$f'(x)=\lim_{h\to 0}\frac{f(n+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(x)+\left(f\left(h^{1/n}\right)\right)^n-f(x)}{h}=l^n=0$$

$\Rightarrow f'(x)=0$

But $f(0)=0\Rightarrow$ 'f' is identically zero which can't be possible

$\therefore l-1 \therefore f(x)=x+c$ but $f(0)=0\Rightarrow C=0\Rightarrow f(x)=x\ f'(0)=1$ and $f(5)=5$

Passage – 6

Let $f:R\to R$ be a differentiable function satisfying $f\left(\frac{x+y}{3}\right)=\frac{2+f(x)+f(y)}{3}$ for all real *x* and *y* and $f'(2)=2$.

16. If $g(x)=|f(|x|)-3|$ for all $x\in R$, then $g(x)$ has

a) one non-differentiable point
b) two non-differentiable points
c) three non-differentiable points
d) four non-differentiable points

Key. C

17. If $x\in[-2,3]$, then range of $f(x)$ is

a) $[-2,3]$ b) $[-3,4]$
c) $[-2,8]$ d) $[-3,10]$

Key. C

18. The number of solutions of the equation $x^2+(f|x|)^2=9$ is

a) 0 b) 2 c) 3 d) 5

Key. B

Sol. $\because\ f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to 0}\frac{f\left(\frac{3x+3h}{3}\right)-f\left(\frac{3x+0}{3}\right)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{2+f(3x)+f(3h)}{3}-\frac{2+f(3x)+f(0)}{3}}{h}$$

$$=\lim_{h\to 0}\frac{f(3h)-f(0)}{3h}=f'(0)$$

$\Rightarrow f'(2)=f'(0)=2\ \left(\because\ f'(2)=2\right)$

$\Rightarrow f'(x)=2$

$\Rightarrow f(x)=2x+c$. ...... (i)

Put $x=y=0$ in $f\left(\frac{x+y}{3}\right)=\frac{2+f(x)+f(y)}{3}$.

$\Rightarrow f(0)=2$.

Now, from Eq. (i), $f(0)=0+c=2$

$\therefore\quad c=2$.

From Eq.(i),

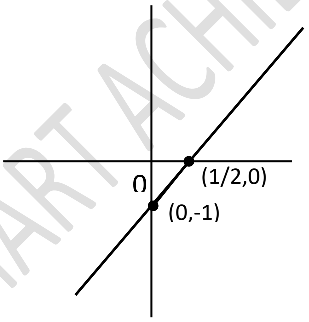
$$f(x)=2x+2.$$

16. $\because f(|x|)=2|x|+2$;

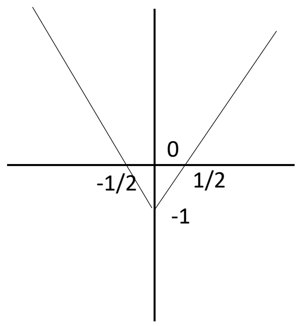
$\therefore f(|x|)-3=2|x|-1$

and $g(x)=|f(|x|)-3|=|2|x|-1|$

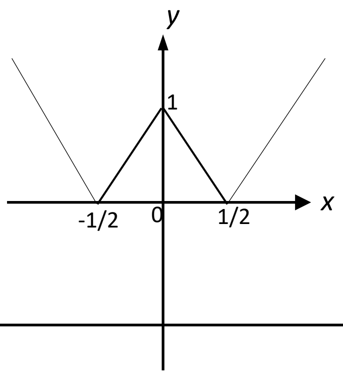
now, graph of $y=2x-1$



Now, graph of $y=2|x|-1$



Graph of $g(x)=|2|x|-1|$



Number of non-differentiable points = 3.

17. $\because f(x)=2x+2$

Here, given $x\in[-2,3]$

or $-2\le x\le 3$

$\Rightarrow \quad -4\le 2x\le 6$

$\Rightarrow \quad -4+2\le 2x+2\le 6+2$

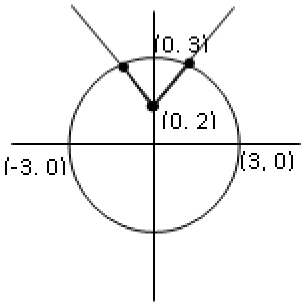
$\therefore -2\le f(x)\le 8$

Hence, range of $f(x)$ is $[-2,8]$.

18. Let $y=f(|x|)=(2|x|+2)$

then $x^2+y^2=3^2$

$\therefore$ Number of solutions = 2.



Passage – 7

The graph of y = f(x) is given with six labeled points Out of these points answer the following questions.

19. The point which has the greatest value of $f'(x)$ is

A) B B) C C) D D) E

Key. A

20. The point where f' and f'' are non-zero and of the same sign are

A) B & D B) D & E C) A & F D) None of these

Key. C

21. The points where atleast two of f, f' and f'' are zero

A) C & D B) A and D C) A & F D) None of these

Key. B

Sol. clear from the graph

Passage – 8

In certain problem the differentiation of {f(x). g(x)} appears. One student commits mistake and differentiates as $\frac{df}{dx}.\frac{dg}{dx}$ but the gets correct result if $f(x)=x^3$ and $g(0)=\frac{1}{3}$.

22. The function g(x) is

A) $\frac{3}{|x-3|^3}$ B) $\frac{4}{|x-3|^3}$ C) $\frac{9}{|x-3|^3}$ D) $\frac{27}{|x-3|^3}$

Key. C

Sol. $f(x)g(x) = x^3 g(x)$

$3x^2.\, g'(x) = 3x^2 g(x) + x^3 g'(x)$

$3g'(x) = 3g(x) + xg'(x)$

$(3-x)g'(x) = 3g(x)$

$\int \frac{g'(x)}{g(x)}\,dx = \int \frac{3}{3-x}\,dx + \ln c$

$\ln g(x) = -3\ln|3-x| + \ln c$

$\therefore \quad g(x) = \frac{c}{|3-x|^3}$

$g(0) = \frac{c}{27} = \frac{1}{3} \quad \therefore \quad c = 9$

$\therefore \quad g(x) = \frac{9}{|3-x|^3}$

23. Derivative of {f(x – 3). g(x)} with respect to x at x = 100 is

A) 0 B) – 1 C) 1 D) 2

Key. A

Sol. $f(x-3).g(x) = (x-3)^3.\, g(x) = 9$

$\therefore$ derivative of f(x – 3). g(x) is 0

24. $\lim_{x\to 0} \frac{f(x).g(x)}{x(1+g(x))}$ will be

A) 0 B) – 1 C) 1 D) 2

Key. A

Sol. $\lim_{x\to 0} \frac{f(x).g(x)}{x(1+g(x))} = \lim_{x\to 0} \frac{x^3 \frac{9}{|x-3|^3}}{x\left(1+\frac{9}{|x-3|^3}\right)} = 0$

Passage – 9

Let $f(x) = \frac{1}{1+x^2}$. Let m be the slope, a be the x-intercept and be the y-intercept of a tangent to y = f(x), then

25. Abscissa of the point of contact of the tangent for which m is greatest

A) $\frac{1}{\sqrt{3}}$ B) 1 C) – 1 D) $-\frac{1}{\sqrt{3}}$

Key. D

Sol. $f'(x) = -\frac{2x}{(1+x^2)^2}$

$f''(x) = \frac{-2+6x^2}{(1+x^2)^3} \qquad f''(x) = 0 \text{ if } x = \pm\frac{1}{\sqrt{3}}$

$\therefore$ $f'(x)$ is greatest $x = -\frac{1}{\sqrt{3}}$

26. Find the greatest value of b

A) $\frac{9}{8}$ B) $\frac{3}{8}$ C) $\frac{1}{8}$ D) $\frac{5}{8}$

Key. A

Sol. Equation of tangent at $x = \alpha$ is

$$y - \frac{1}{1+\alpha^2} = \frac{-2\alpha}{(1+\alpha^2)^2}(x-\alpha)$$

$$\therefore \quad b = \frac{1}{1+\alpha^2} + \frac{2\alpha^2}{(1+\alpha^2)^2} = \frac{1+3\alpha^2}{(1+\alpha^2)^2}$$

$$\therefore \quad \frac{db}{d\alpha} = \frac{(1+\alpha^2)^2 .\, 6\alpha - 2(1+3\alpha^2)(1+\alpha^2)2\alpha}{(1+\alpha^2)^4} = \frac{2\alpha(1-3\alpha^2)}{(1+\alpha^2)^3}$$

$$\frac{db}{d\alpha} = 0 \text{ if } \quad \alpha = 0, \pm\frac{1}{\sqrt{3}}$$

At $\alpha = \pm\frac{1}{\sqrt{3}}$, $b = \frac{9}{8}$

27. Find the abscissa of the point of contact of tangent for which $\frac{1}{a}$ is greatest

A) $\frac{1}{\sqrt{3}}$ B) 1 C) – 1 D) $-\frac{1}{\sqrt{3}}$

Key. A

Sol. $a = \frac{1+3\alpha^2}{2\alpha}$

$\therefore$ $\frac{1}{a} = \frac{2\alpha}{1+3\alpha^2}$ its greatest value is $\frac{1}{\sqrt{3}}$

Differentiation

Integer Answer Type

1. If the function $f\left(x\right)=x^{3}+e^{\frac{x}{2}}$ and $g\left(x\right)=f^{-1}\left(x\right)$, then the value of $g^{1}\left(1\right)$ is

Key. 2

Sol. Given that $f(x)= x^{3}+ e^{x/2}$

Let $g(x)= f^{-1}(x)$ then we should have $gof(x)= x$

$\Rightarrow \quad g(f(x))= x$

$\Rightarrow \quad g(x^{3}+ e^{x/2})= x$

Differentiating both sides with respect to we get

$$g'(x^{3}+ e^{x/2}).\left(3x^{2}+ e^{x/2}.\frac{1}{2}\right)= 1$$

$$\Rightarrow \quad g'(x^{3}+ e^{x/3})= \frac{1}{3x^{2}+ e^{x/2}.\frac{1}{2}}$$

For $x= 0$, we get $g'(1)= \frac{1}{1/2}= 2$

2. Find $\frac{dy}{dx}$ at x = -1 , when $\left(\sin y\right)^{\sin\left(\frac{\pi}{2}x\right)}+\frac{\sqrt{3}}{2}\sec^{-1}\left(2x\right)+2^{x}\tan\left(\ln\left(x+2\right)\right)=0$

Key. 0

Sol. We are given the function

$$\left(\sin y\right)^{\sin\left(\frac{\pi x}{2}\right)}+\frac{\sqrt{3}}{2}\sec^{-1}\left(2x\right)+2^{x}\tan\left[\log\left(x+2\right)\right]=0$$

Differentiating w.r.to x and put x = - 1 we get $\frac{dy}{dx}=0$

3. $y=\frac{2x^{2}}{(x-2)(x-3)(x-4)}+\frac{3x}{(x-3)(X-4)}+\frac{x}{x-4}$. If $\frac{x}{y}\frac{dy}{dx}=\frac{a}{2-x}+\frac{b}{3-x}+\frac{c}{4-x}$ then a+b+c is

Key. 9

Sol. $y=\frac{2x^{2}}{(x-2)(x-3)(x-4)}+\frac{3x+x^{2}-3x}{(x-3)(x-4)}=\frac{x^{3}}{(x-2)(x-3)(x-4)}$

$$\log y=3\log x-\log(x-2)-\log(x-3)-\log(x-4)$$

$$\frac{\frac{dy}{dx}}{y}=\frac{1}{x}-\frac{1}{x-2}+\frac{1}{x}-\frac{1}{x-3}+\frac{1}{x}-\frac{1}{x-4}$$

$$=\frac{-2}{x(x-2)}-\frac{3}{x(x-3)}-\frac{4}{x(x-4)}$$

$\frac{x}{y}\frac{dy}{dx}=\frac{2}{2-x}+\frac{3}{3-x}+\frac{4}{4-x}\Rightarrow a+b+c=9$

4. If the function $f(x)=4x^5+3x^3+2x^2+e^{x/7}$ and $g(x)=f^{-1}(x)$ then the value of $g'(1)$ is.

Key. 7

Sol. Clearly $g'[f(0)]=g'[1]=\frac{1}{f'(0)}$ (Here $f(0)=1$)

$\because gof=I\Rightarrow g[f(x)]=x$ as 'g' is inverse of f.

$\Rightarrow g'[f(x)].f'(x)=1$

$\therefore g'(1)=7$

5. If $f(x)=x^3+x^2.f'(1)+x.f''(2)-f'''(3),\ \forall x\in R$ then $|f(0)+f(3)+16|=$

Key. 8

Sol. $f(x)=x^3+Ax^2+Bx-C$ where $A=f'(1)$

$f'(x)=3x^2+2Ax+B$ $B=f''(2)$

$f''(x)=6x+2A;\ f'''(x)=6$ $c=-f'''(3)$

$B=12+2A$ - (1) $C=-6$

$A=3+2A+B$ - (2)

$A+B=-3$

$2A-B=-12$

$3A=-15$, A = -5, B = 2

$\therefore f(x)=x^3-5x^2+2x^2-6$

$f(0)=-6$

$f(3)=27-5\times9+2\times9-6$

$=27-45+18-6$

$=-6$

$\therefore f(0)+f(3)=0$ Hence $f(0)+f(3)+16=4$

6. Let $f:R\to R$ and $g:R\to R$ be twice differentiable functions satisfying $f''(x)=g''(x), 2f'(1)=g'(1)=4$ and $3f(2)=g(2)=9$ then the value of $15+f(4)-g(4)$ is equal to

Key. 5

Sol. $f''(x)=g''(x)\Rightarrow f'(x)=g'(x)+c_1$

$\Rightarrow f(x)=g(x)+c_1x+c_2$

$f'(1)=g'(1)+c_1$ and $f(2)=g(2)+2c_1+c_2$

$\therefore f(4)-g(4)=4c_1+c_2$

$C_1=f'(1)-g'(1)=2-4=-2$

$$C_2 = f'(2) - g(2) = 2c_1 = -2$$
$$\therefore f(4) - g(4) = -10$$

7. Find $\frac{dy}{dx}$ ln each of the following case: $y = \tan^{-1}\frac{4x}{1+5x^2} + \tan^{-1}\frac{2+3x}{3-2x}$

Sol. $\frac{dy}{dx} = \frac{5}{(1+25x^2)}$

$$y = \tan^{-1}\frac{4x}{1+5x^2} + \tan^{-1}\frac{2+3x}{3-2x} = \tan^{-1}\frac{5x-x}{1+5x.x} + \tan^{-1}\frac{\frac{2}{3}+x}{1-\frac{2}{3}.x}$$

$$= \tan^{-1}5x - \tan^{-1}x + \tan^{-1}\frac{2}{3} + \tan^{-1}x = \tan^{-1}5x + \tan^{-1}\frac{2}{3}$$

$\therefore \quad \frac{dy}{dx} = \frac{5}{1+25x^2}$

8. If α be a repeated root of a quadratic equation f(x) = 0 & A(x), B(x) be the polynomials of degree 3, 4 & 5 respectively, then show that $\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$ is divisible by f(x), where dash denotes the derivative.

Sol. let $g(x) = \begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$

$\therefore \quad g(\alpha) = 0$

$$g'(x) = \begin{vmatrix} A'(x) & B'(x) & C'(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$$

$\therefore \quad g'(\alpha) = 0$

$\therefore \quad \alpha$ is a repeated root of g(x)

$\therefore \quad$ g(x) is divisible by f(x)

(f(x) is a quadratic equation)

9. Show that the function y = f(x) defined by the parametric equations $x = e'\sin t, y = e'\cos t$ satisfies the relation $y''(x+y)^2 = 2(xy'-y)$.

Sol. $x = e^t \sin t$ and $y = e^t \cos t$

$\Rightarrow \quad x^2 + y^2 = e^{2t} \Rightarrow e^t = \sqrt{x^2+y^2}$ …(1)

and $\tan t = \frac{x}{y} \qquad \Rightarrow t = \tan^{-1}\left(\frac{x}{y}\right)$ put in (1)

$\therefore \quad e^{\tan^{-1}\left(\frac{x}{y}\right)} = \sqrt{x^2+y^2}$ …(2)

taking log of both sides.

$$\tan^{-1}\left(\frac{x}{y}\right) = \frac{1}{2} ln(x^2+y^2)$$

Differentiate both sides w.r.t. 'x'

$$\Rightarrow \left(\frac{1}{1+\frac{x^2}{y^2}}\right)\left(\frac{y.1-x.y'}{y^2}\right)=\frac{1}{2}.\frac{(2x+2yy')}{(x^2+y^2)}$$

$$\Rightarrow \quad y'=\frac{y-x}{x+y} \qquad \ldots(3)$$

Again differentiate equation (3) w.r.t. 'x'

$$\Rightarrow \quad y''=\frac{(x+y)(y'-1)-(y-x)(1+y')}{(x+y)^2}$$

$$\Rightarrow \quad y''(x+y)^2=y'(2x)-2y$$

$$\Rightarrow \quad y''(x+y)^2=2(xy'-y)$$. Hence proved.

Differentiation

Matrix-Match Type

1. Match the following:

Column – I		Column – II
A)	If $f(0)=0,\ f'(0)=2$ then the derivative of $y=f\left(f\left(f\left(f(x)\right)\right)\right)$ at x=0 assuming it exists is	p) 1
B)	If $y=(1+x)(1+x^2)(1+x^4).......(1+x^{2^n})$ then $\frac{dy}{dx}$ $at\ x=0$ is	q) 16
C)	If $y=\cos^{-1}\left(\frac{5\cos x-12\sin x}{13}\right),\ x\in\left(0,\frac{\pi}{2}\right)$ then $\frac{dy}{dx}$ is equal to	r) 2
D)	If $xe^{xy}-y=\sin^2 x$ then $\frac{dy}{dx}$ at $x=0$ is	s) -2

Key. a-q; b-p; c-p ; d-p

Sol. A) $y_1(x)=f'\left(f\left(f\left(f(x)\right)\right)\right)f'\left(f\left(f(x)\right)\right).\ f'\left(f(x)\right).f'(x)$

$\Rightarrow y_1(0)=f'\left(f\left(f\left(f(0)\right)\right)\right)\ f'\left(f\left(f(0)\right)\right).\ f'\left(f(0)\right).f'(0)$

$=f'\left(f\left(f(0)\right)\right)\ f'\left(f(0)\right).\ f'(0).\ f'(0)$

$=f'\left(f(0)\right)\ f'(0).\ f'(0).\ f'(0)$

$=f'(0)\ f'(0)\ f'(0)\ f'(0)=16$

B) $y=\frac{(1-x)(1+x)(1+x^2).....(1+x^{2^n})}{1-x}$

$=\frac{(1-x^2)(1+x^2).......(1+x^{2^n})}{1-x}$

$=\frac{(1-x^4)(1+x^4)......(1+x^{2^n})}{1-x}$

$=\frac{(1-x^{2^n})(1+x^{2^n})}{1-x}=\frac{1-x^{2^{n+1}}}{1-x}$

$\frac{dy}{dx}=\frac{-2^{n+1}.x^{2^{n+1}-1}.(1-x).(1-x^{2^{n+1}})(-1)}{(1-x)^2}$

$\left(\frac{dy}{dx}\right)_{x=0} = \frac{0+1}{1^2} = 1$

C) Let $Cos\alpha = \frac{5}{13}$ and $\sin\alpha = \frac{12}{13}$

So, $y = \cos^{-1}(\cos\alpha.\cos x - \sin\alpha.\sin x)$

$\Rightarrow y = \cos^{-1}(\cos(x+\alpha)) = x+\alpha$

$\because$ $x+\alpha$ is in the first quadrant.

$\Rightarrow \frac{dy}{dx} = 1$

D) $xe^{xy} - y = \sin^2 x$

Differentiating w.r.t x

$e^{xy} + x\,e^{xy}\left(x\frac{dy}{dx} + y\right) - \frac{dy}{dx} = 2\sin x\cos x$

$\therefore \frac{dy}{dx} = \frac{2\sin x\cos x - e^{xy} - xye^{xy}}{x^2e^{xy} - 1}$

When $x = 0, y = 0$

$\therefore \frac{dy}{dx} = \frac{0-1-0}{0-1} = 1$

2. Match the following:

	Column – I		Column – II
A)	Let f and g be differentiable functions satisfying $g'(a) = 2, g(a) = b$ and $fog = I$ (Identity function) then $f'(b)$ is equal to	p)	2
B)	If $f(x) = \cos x.\cos 2x.\cos 4x.\cos 8x.\cos 16x$ Then $f'\left(\frac{\pi}{4}\right)$ is equal to	q)	$\frac{1}{2}$
C)	If $f'(x) = \sqrt{2x^2 - 1}$ and $y = f(x^2)$ then $\frac{dy}{dx}$ at $x = 1$ is	r)	$\sqrt{2}$
D)	If $y = \sin^2 \cot^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right)$ then $\frac{dy}{dx} =$	s)	$-\sqrt{2}$

Key. a-q; b-r; c-p; d-q

Sol. A) $fog = I$

$\Rightarrow fog(x) = x, \forall x$

$\Rightarrow f'(g(x))g'(x) = 1, \forall x$

$\Rightarrow f'(g(x)) = \frac{1}{g'(x)}$

$\Rightarrow f'(g(a)) = \frac{1}{2} \Rightarrow f'(b) = \frac{1}{2}$

B) We know that

$Cosx.\cos 2x.\cos 2^2 x........\cos 2^{n-1} x = \frac{Sin 2^n x}{2^n \sin x}$

$\therefore f(x) = \frac{\sin 32x}{32 \sin x}$

$\Rightarrow f'(x) = \frac{1}{32}\left[\frac{\sin x \ \cos(32x).32 - (\sin 32x)\cos x}{\sin^2 x}\right]$

$\therefore f'\left(\frac{\pi}{4}\right) = \sqrt{2}$

C) $y = f(x^2) \Rightarrow \frac{dy}{dx} = f'(x^2)2x$

$= 2x\sqrt{2(x^2)^2 - 1}$

At $x = 1 \Rightarrow \frac{dy}{dx} = 2$

D) $Put \ \ x = \cos\theta \Rightarrow \theta = \cos^{-1} x$

$\Rightarrow y = \sin^2 \cot^{-1}\left(\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right)$

$= \sin^2\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$

$= \cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos\theta}{2}$

$= \frac{1+x}{2}$

$\therefore \frac{dy}{dx} = \frac{1}{2}$

3. Match the following : -

	Column – I		Column – II
(A)	$\lim\limits_{x\to\frac{1}{\sqrt{2}}}\dfrac{x-\cos\left(\sin^{-1}x\right)}{1-\tan\left(\sin^{-1}x\right)}$ is equal to	(p)	Does not exist
(B)	If $f(x)=\log_{x^2}(\log x)$, then $f'\left(\frac{1}{2}\right)$ is equal to	(q)	4
(C)	For the function $f(x)=ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$ If $\frac{dy}{dx}=\sec x+p$, then p is equal to	(r)	28
(D)	$\lim\limits_{x\to0}\frac{1}{x}\sqrt{\frac{1-\cos2x}{1+\cos2x}}$ is equal to	(s)	1
		(t)	0

Key. A $\to$ t; B $\to$ t; C $\to$ r; D $\to$ p

Sol. (A) $\lim\limits_{x\to\frac{1}{\sqrt{2}}}\dfrac{x-\cos\left(\sin^{-1}x\right)}{1-\tan\left(\sin^{-1}x\right)}=\lim\limits_{x\to\frac{1}{\sqrt{2}}}\dfrac{x-\sqrt{1-x^2}}{1-\dfrac{x}{\sqrt{1-x^2}}}=\lim\limits_{x\to\frac{1}{\sqrt{2}}}\left(-\sqrt{1-x^2}\right)=-\dfrac{1}{\sqrt{2}}$

(B) $f(x)=ln_{x^2}\ x=\frac{1}{2}$ is not in the domain

$\therefore$ $f'\left(\frac{1}{2}\right)$ does not exists

(C) $y=f(x)=ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$

$\therefore$ $\dfrac{dy}{dx}=\dfrac{\sec^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)}.\dfrac{1}{2}=\dfrac{1}{2}\dfrac{1}{\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}=\dfrac{1}{\sin\left(\frac{\pi}{2}+x\right)}=\sec x$

(D) $\lim\limits_{x\to0}\frac{1}{x}\sqrt{\frac{1-\cos2x}{1+\cos2x}}=\lim\limits_{x\to0}\frac{|\tan x|}{x}=$ does not exists

4. Match the following : -

	Column – I		Column – II
(A)	If $f'(x)\sqrt{3x^2+6}\ \&\ y=f\left(x^3\right)$ then at x = 1, $\frac{dy}{dx}=$	(p)	- 2
(B)	If f be a diffr. Fun. Such that f(xy) = f(x) + f(y) + x, y $\in$ R then f(e) + f(1/e) =	(q)	4
(C)	If f be a twice diffr. Fun. Such that $f''(x)=-f(x)\ \&\ f'(x)=g(x)$, If $h(x)=(f(x))^2+(g(x))^2\ \&\ h(5)=9$ then h(10) = ?	(r)	0
(D)	$y=\tan^{-1}(\cot x)+\cot^{-1}(\tan x),\frac{\pi}{2}<x<\pi$ then $\frac{dy}{dx}$	(s)	- 1

		(t)	9

Key. A $\to$ t; B $\to$ r; C $\to$ t; D $\to$ p

Sol. (A) $y = f\left(x^3\right)$

$\therefore \quad \frac{dy}{dx} = f'\left(x^3\right).\,3x^2$

$\therefore \quad \left.\frac{dy}{dx}\right|_{x-1} = f'(1).\,3 = 9$

(B) $f(xy) = f(x) + f(y)$

f(1) = f(1) + f(1)

$\therefore \quad f(1) = 0$

Also $\quad f(1) = f(e) + f\left(\frac{1}{e}\right)$

$\therefore \quad f(e) + f\left(\frac{1}{e}\right) = 0$

(C) $\quad f''(x) = -f(x),\ f'(x) = g(x)$

$\therefore \quad g'(x) = f''(x) = -f(x)$

$h(x) = \left(f(x)\right)^2 + \left(g(x)\right)^2$

$\therefore \quad h'(x) = 2f(x).\,f'(x) + 2g(x).g'(x)$

= 2 f(x). g(x) + 2 gx (- f(x)) = 0

$\therefore \quad h(x) = c, x \in R$

$\therefore \quad$ h(10) = h (5) = 9.

(D) $\quad y = \tan^{-1}(\cot x) + \cot^{-1}(\tan x), \frac{\pi}{2} < x < \pi$

$\frac{dy}{dx} = \frac{-\operatorname{cosec}^2 x}{1+\cot^2 x} + \frac{-1}{1+\tan^2 x}.\sec^2 x$

= 1 – 1 = - 2

5. Match the following : -

Column – I		Column – II	
(A)	If $y = \cos^{-1}(\cos x)$, then y' at x = 5 is equal to	(p)	Does not exist
(B)	For the function f(x) = *l*n |tan x| $f'\left(-\frac{\pi}{4}\right)$ is equal to	(q)	0
(C)	The derivative of $\tan^{-1}\left(\frac{1+x}{1-x}\right)$ at x = - 1 is	(r)	$\frac{1}{2}$
(D)	The derivative of $\frac{\log\|x\|}{x}$ at x = - 1 is	(s)	1
		(t)	- 1

Key. A $\to$ t; B $\to$ s; C $\to$ r; D $\to$ s

Sol. (A) $y = \cos^{-1}(\cos x)$

$y' = \frac{-1}{\sqrt{1-\cos^2 x}}.(-\sin x) = \frac{\sin x}{|\sin x|} \quad \therefore \quad y'$ at x = 5 is – 1

(B) $\quad$ y = f(x) = *l*n |tan x|

$\therefore \quad f'(x) = (1/\tan x)(\sec^2 x).\left(\frac{|\tan x|}{\tan x}\right)$

$f'\left(-\frac{\pi}{4}\right) = 1$

(C) $\frac{d}{dx}\tan^{-1}\left(\frac{1+x}{1-x}\right) = \frac{1}{1+\left(\frac{1+x}{1-x}\right)^2}.\frac{d}{dx}\left(\frac{1+x}{1-x}\right) = \frac{(1-x)^2}{2(1+x^2)}.\frac{2}{(1-x)^2} = \frac{1}{1+x^2}$ at x = - 1

$\frac{d}{dx}\tan^{-1}\left(\frac{1+x}{1-x}\right) = \frac{1}{2}$

(D) $\frac{d}{dx}\frac{l n|x|}{x} = \frac{x.\frac{1}{x} - l n|x|}{x^2} = \frac{1 - l n|x|}{x^2}$ at x = - 1

$\frac{d}{dx}\frac{l n|x|}{x} = 1$