Definite, Indefinite Integration & Areas

Single Correct Answer Type

1. If a circle of radius r is touching the lines $x^2 - 4xy + y^2 = 0$ in the first quadrant at points A and B, then area of triangle OAB (O being the origin) is

(A) $\frac{3\sqrt{3}r^2}{4}$ (B) $\frac{\sqrt{3}r^2}{4}$ (C) $\frac{3r^2}{4}$ (D) r^2

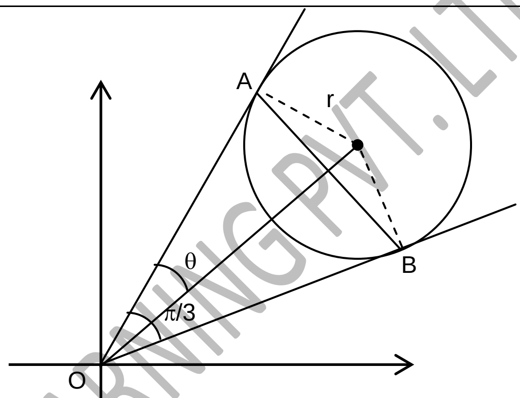
Key. A

Sol.

Here $\tan 2\theta = \frac{2\sqrt{4-1}}{2} = \sqrt{3} \Rightarrow \theta = \frac{\pi}{6}$

Area of $\Delta OAB = \frac{1}{2}(r\cot\theta)^2 (\sin 2\theta)$

$= \frac{1}{2}(r\sqrt{3})^2 \frac{\sqrt{3}}{2}$.



2. ABCD is a quadrilateral with side lengths AB = 4, BC = 10, CD = 6 and AD = 6, and diagonal BD = 8 units. If the incircles of triangles ABD and BCD touch BD at P and Q respectively, then area of quadrilateral C_1PC_2Q (where C_1 and C_2 are incentres of triangle ABD and BCD respectively), is

(A) $3+\frac{\sqrt{15}}{2}$ sq. units (B) 3 sq. units

(C) $\frac{\sqrt{15}}{6}$ sq. units (D) 4 sq. units

Key. A

Sol.

In triangle ABD, we have $BP = \frac{6+4+8}{2} - 6 = 3$.

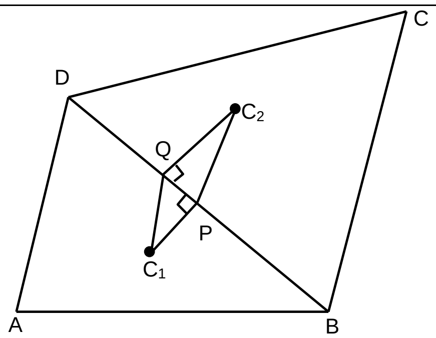
In triangle BCD, we have

$DQ = \frac{10+6+8}{2} - 10 = 2$.

$\Rightarrow PQ = 8 - (3 + 2) = 3$

$\Rightarrow$ area of trapezium $C_1PC_2Q = \frac{1}{2}(r_1 + r_2)\cdot PQ$,

where $r_1 = \frac{\sqrt{9\times3\times5\times1}}{9} = \frac{\sqrt{15}}{3}$ and $r_2 = \frac{\sqrt{12\times2\times6\times4}}{12} = 2$.



SMART ACHIEVERS LEARNING PVT. LTD.

$\Rightarrow$ area of quadrilateral $C_1PC_2Q = \frac{1}{2}\left(2+\frac{\sqrt{15}}{3}\right)\times 3 = 3+\frac{\sqrt{15}}{2}$ sq. units.

3. The area of the region whose boundaries are defined by the curves y = 2 cos x, y = 3 tan x and the y-axis, is

(A) $1+3\ln\left(\frac{2}{\sqrt{3}}\right)$ (B) $1 + \frac{3}{2}\ln 3 - 3\ln 2$

(C) $1 + \frac{3}{2}\ln 3 - \ln 2$ (D) $\ln 3 - \ln 2$

Key. B

Sol. Solving 2 cos x = 3 tan x we get, $2 - 2\sin^2 x = 3\sin x$

$\Rightarrow \sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}$.

Required area = $\int_0^{\pi/6}(2\cos x - 3\tan x)dx = 2\sin x - 3\ln\sec x\Big|_0^{\pi/6} = 1 - 3\ln 2 + \frac{3}{2}\ln 3$.

4. The area of the region bounded by the curve $y = x^2$ and $y = \sec^{-1}[-\sin^2 x]$, (where [.] denotes the greatest integer function), is

(A) $\pi\sqrt{\pi}$ (B) $\frac{4}{3}\pi\sqrt{\pi}$ (C) $\frac{2}{3}\pi\sqrt{\pi}$ (D) $\frac{1}{3}\pi\sqrt{\pi}$

Key. B

Sol. $[-\sin^2 x] = 0$ or -1 but $\sec^{-1}(0)$ is not defined.

$\Rightarrow \sec^{-1}[-\sin^2 x] = \sec^{-1}(-1) = \pi$.

The required area = $\int_{-\sqrt{\pi}}^{\sqrt{\pi}}(\pi - x^2)dx = \frac{4}{3}\pi\sqrt{\pi}$.

5. The range of the function $f(x) = \int_1^x |t|\,dt,\ x \in \left[\frac{-1}{2}, \frac{1}{2}\right]$ is

(A) $\left[\frac{3}{8}, \frac{5}{8}\right]$ (B) $\left[\frac{-5}{8}, \frac{3}{8}\right]$ (C) $\left[\frac{-3}{8}, \frac{5}{8}\right]$ (D) $\left[\frac{-5}{8}, \frac{-3}{8}\right]$

Key. D

Sol. If $0 \le x \le \frac{1}{2} \Rightarrow f(x) = \int_1^x t\,dt = \frac{x^2-1}{2}$

If $-\frac{1}{2} \le x \le 0 \Rightarrow f(x) = \int_1^0 t\,dt - \int_0^x t\,dt = -\left(\frac{x^2+1}{2}\right)$

$\Rightarrow f^1(x) > 0 \,\forall x \in \left[\frac{-1}{2}, \frac{1}{2}\right]$

$\therefore$ *Range of* $f(x)$ *is* $\left[f\left(\frac{-1}{2}\right), f\left(\frac{1}{2}\right)\right] \Rightarrow \left[\frac{-5}{8}, \frac{-3}{8}\right]$

6. $\int_0^1 Tan^{-1}\left(1 - x + x^2\right)dx =$ ________

SMART ACHIEVERS LEARNING PVT. LTD.

(A) $\ln 2$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{2}$ (D) $\frac{\pi}{2}-\ln 2$

Key. A

Sol. $\int_0^1 \tan^{-1}\left(1-x+x^2\right)dx=\int_0^1\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{x+(1-x)}{1-x(1-x)}\right)\right)dx$

$$=\frac{\pi}{2}-\int_0^1\left(\tan^{-1}(1-x)+\tan^{-1}x\right)dx$$

$$\Rightarrow \frac{\pi}{2}-2\int_0^1 \tan^{-1}(x)\,dx=\ln 2$$

7. $\int_0^{\frac{\pi}{4}}\left(\pi x-4x^2\right)\ln(1+Tanx)\,dx=$ ________

(A) $\frac{\pi^3}{192}$ (B) $\frac{\pi^3}{192}\ln 2$ (C) $\frac{\pi^2}{96}$ (D) $\frac{\pi^2}{96}\ln 2$

Key. B

Sol. Let $I=\int_0^{\frac{\pi}{4}} 4x\left(\frac{\pi}{4}-x\right)\ln(1+\tan x)\,dx=\int_0^{\frac{\pi}{4}} 4\left(\frac{\pi}{4}-x\right)x\ln\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx$

$$=4\ln 2\int_0^{\frac{\pi}{4}} x\left(\frac{\pi}{4}-x\right)dx-I$$

$$\Rightarrow I=2\ln 2\int_0^{\frac{\pi}{4}} x\left(\frac{\pi}{4}-x\right)dx$$

$$=\frac{\pi^3}{192}\ln 2$$

8. $\int_0^{2\pi} x\ln\left(\frac{3+\cos x}{3-\cos x}\right)dx=$ _____

(A) $\frac{\pi}{2}\ln 3$ (B) $\frac{\pi}{6}\ln 3$ (C) $\frac{\pi}{12}\ln 3$ (D) 0

Key. D

Sol.

$$I=\int_0^{2\pi}(2\pi-x)\ln\left(\frac{3+\cos x}{3-\cos x}dx\right)\Rightarrow 2I=2\pi\int_0^{2\pi}\ln\left(\frac{3+\cos x}{3-\cos x}\right)dx$$

$$\Rightarrow I=2\pi\int_0^{\pi}\ln\left(\frac{3+\cos x}{3-\cos x}\right)dx=2\pi\int_0^{\pi}\ln\left(\frac{3+\cos x}{3-\cos x}\right)dx=-I\Rightarrow I=0$$

SMART ACHIEVERS LEARNING PVT. LTD.

9. If a point P moves such that its distance from line y = $\sqrt{3}$ x – 7 is same as its distance from $\left(2\sqrt{3},\ -1\right)$, then area bounded by locus of P and the coordinate axes is (in sq. units)

(A) $\frac{\sqrt{3}}{2}$ (B) $2\sqrt{3}$ (C) 6 (D) $\frac{3\sqrt{3}}{2}$

Key. A

Sol. As point lise on the line. Locus of the point is straight line perpendicular to given line passing through $\left(2\sqrt{3},\ -1\right)$ i.e. $\frac{x}{\sqrt{3}}+y=1$

$\Rightarrow$ area of triangle = $\frac{\sqrt{3}\times 1}{2}$.

10. If A$\left(\frac{3}{\sqrt{2}},\ \sqrt{2}\right)$, B$\left(-\frac{3}{\sqrt{2}},\ \sqrt{2}\right)$, C$\left(-\frac{3}{\sqrt{2}},\ -\sqrt{2}\right)$ and D (3 cos θ, 2 sin θ) are four points, then the value of θ for which the area of quadrilateral ABCD is maximum, $\left(\frac{3\pi}{2}\le\theta\le 2\pi\right)$ is (in sq. units)

(A) $2\pi-\sin^{-1}\frac{1}{3}$ (B) $\frac{7\pi}{4}$ (C) $2\pi-\cos^{-1}\frac{3}{\sqrt{85}}$ (D) $\frac{\pi}{4}$

Key. B

Sol. Area of quadrilateral ABCD is maximum when area of ACD is maximum

$\Rightarrow$ distance of D from AC is maximum i.e. $(\cos\theta-\sin\theta)$ is maximum.

$\Rightarrow$ $\sqrt{2}\cos\left(\theta+\frac{\pi}{4}\right)$ is maximum

$\Rightarrow \theta = \frac{7\pi}{4}$.

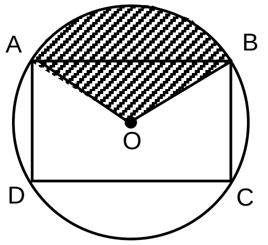
11. A square ABCD is inscribed in a circle of radius 4. A point P moves inside the circle such that d (P, AB) $\le$ min (d (P, BC), d (P, CD), d (P, DA)) where d (P, AB) is the distance of a point P from line AB. The area of region covered by moving point P is (in sq. units)

(A) 4π (B) 8π (C) 8π – 16 (D) $3\pi-4$

Key. A

Sol.

Shaded area is the required region = $\frac{\pi r^2}{4}=4\pi$.



SMART ACHIEVERS LEARNING PVT. LTD.

12. Let $I_1 = \int_{\sec^2 z}^{2-\tan^2 z} x f\left(x(3-x)\right)dx$ and let $I_2 = \int_{\sec^2 z}^{2-\tan^2 z} f\left(x(3-x)\right) dx$

where $'f'$ is a continuous function and $'z'$ is any real number, then $\frac{I_1}{I_2} =$

(A) $\frac{3}{2}$ (B) $\frac{1}{2}$ (C) 1 (D) $\frac{2}{3}$

Key. A

Sol. Conceptual

13. If f, g, h are continuous functions on [0, a] such that

$f(a-x) = f(x), g(a-x) = -g(x), \quad 3h(x) - 4h(a-x) = 5$ then $\int_0^a f(x)g(x)h(x)dx =$

a) 0 b) a c) a/2 d) 2a

Key. A

Sol. Conceptual

14. The value of $\int_{-2}^{2} \frac{\sin^2 x}{\left[\frac{x}{\pi}\right] + \frac{1}{2}} dx,$ where [x] is the greatest integer less than or equal to x, is

a) 1 b) 0 c) $4-\sin 4$ d) $4+\sin 4$

Key. B

Sol. Conceptual

15. $\int_0^{4/\pi} (3x^2.\sin\frac{1}{x} - x.\cos\frac{1}{x})dx =$

a) $\frac{8\sqrt{2}}{\pi^3}$ b) $\frac{24\sqrt{2}}{\pi^3}$ c) $\frac{32\sqrt{2}}{\pi^3}$ d) $\frac{32\sqrt{2}}{\pi}$

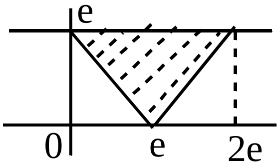
Key. C

Sol. Conceptual

16. The area bounded by the curves $f(x) = \begin{cases} x^{\frac{1}{\ln x}}, & x \neq 1 \\ e, & x = 1 \end{cases}$ and $y = |x-e|$ is

(A) $\frac{e^2}{2}$ (B) e^2 (C) $2e^2$ (D) 1

Key. B



Sol. $f(x) = \begin{cases} x^{\log_x^e} = e, & x \neq 1 \\ e, & x = 1 \end{cases}$

17. The area of the region bounded by the point $P(x, y)$ satisfying $\log_x \log_y x > 0$ and $\frac{1}{2} < x < 2$ is

SMART ACHIEVERS LEARNING PVT. LTD.

(A) $\frac{3}{4}$ (B) 1 (C) 2 (D) $\frac{7}{8}$

Key. D

Sol. (i) $\frac{1}{2} < x < 1$

a) $0 < y < 1 \Rightarrow x > y \wedge x < 1$

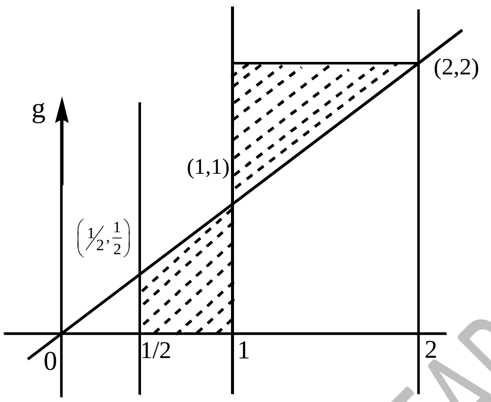
b) $y > 1 \Rightarrow x < y \wedge x > 1$

not possible

(ii) $1 < x < 2$

a) $0 < y < 1$

$x > y \wedge x < 1$



Not possible

b) y > 1

$x < y \wedge x > 1$

Area $= \frac{1}{2} - \frac{1}{2}.\frac{1}{2}.\frac{1}{2} + \frac{1}{2}$

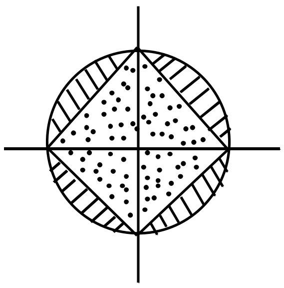
$= 1 - \frac{1}{8} = \frac{7}{8}$

18. Area of the region defined by $\left||x| - |y|\right| \geq 1$ and $x^2 + y^2 \leq 1$ is

(A) 1 (B) 2 (C) $\pi - 1$ (D) $2\pi - 1$

Key. C

Sol. $-1 \leq \left||x| - |y|\right| \leq 1$



$\left||x| - |y|\right| \leq 1 \wedge \left||x| - |y|\right| \geq -1$

Required area $= \pi(1)^2 = \pi$

19. If $I_n = \int \tan^n x dx$, then $I_0 + I_1 + 2(I_2 + ... + I_8) + I_9 + I_{10}$, is equal to

1) $\sum_{n=1}^{9} \frac{\tan^n x}{n}$ 2) $1+\sum_{n=1}^{8} \frac{\tan^n x}{n}$ 3) $\sum_{n=1}^{9} \frac{\tan^n x}{n+1}$ 4) $\sum_{n=2}^{10} \frac{\tan^n x}{n+1}$

Key. 1

Sol. We have $I_n = \int \tan^n x\, dx = \int \tan^{n-2} x(\sec^2 x - 1)dx = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$

i.e. $I_{n-2} + I_n = \frac{\tan^{n-1} x}{n-1}(n \geq 2)$

Thus, we have

$I_0 + I_1 + 2(I_2 + ... + I_8) + I_9 + I_{10}$

$=(I_0 + I_2) + (I_1 + I_3) + (I_2 + I_4) + (I_3 + I_5) + (I_4 + I_6) + (I_5 + I_7) + (I_6 + I_8) + (I_7 + I_9) + (I_8 + I_{10})$

.

$= \sum_{n=2}^{10} \frac{\tan^{n-1}}{n-1} = \sum_{n=1}^{9} \frac{\tan^n x}{n}$

20. $\int e^{\sin x}\left(\frac{x\cos^3 x - \sin x}{\cos^2 x}\right) dx$, is equal to

1) $e^{\sin x}(\tan x + x) + C$ 2) $e^{\sin x}(x - \sec x) + C$

3) $e^{\sin x}(\sec x + \tan x) + C$ 4)none of these

Key. 2

Sol. We have

$$I = \int e^{\sin x}\left(\frac{x\cos^3 x - \sin x}{\cos^2 x}\right) dx$$

$$= \int xe^{\sin x} \cos x dx - \int e^{\sin x}(\sec x \tan x) dx$$

$$= \left[xe^{\sin x} - \int e^{\sin x} dx\right] - \left[e^{\sin x} \sec x - \int e^{\sin x} dx\right]$$

$$= e^{\sin x}(x - \sec x) + C$$

SMART ACHIEVERS LEARNING PVT. LTD.

21. If $\int \frac{dx}{x\sqrt{1-x^3}} = a\ln\left(\frac{\sqrt{1-x^3}+b}{\sqrt{1-x^3}+1}\right)+k$, then

1) b=1, a =1
2) $b=-1,\ a=-\frac{1}{3}$
3) $b=1, a=-\frac{2}{3}$
4) None of these

Key. 2

Sol. $I=\int \frac{dx}{x\sqrt{1-x^3}}=\int \frac{x^2dx}{x^3\sqrt{1-x^3}}$

Put $1-x^3=t^2 \Rightarrow -3x^2dx=2tdt$

22. $\frac{\sin^3 x}{(\cos^4 x+3\cos^2 x+1)\tan^{-1}(\sec x+\cos x)}dx=$

1) $\tan^{-1}(\sec x+\cos x)+c$
2) $\log\left|\tan^{-1}(\sec x+\cos x)\right|+c$
3) $\frac{1}{(\sec x+\cos^2 x)^2}+c$
4) $\log|\sec x+\cos x|+c$

Key. 2

Sol. Put $Tan^{-1}(\sec x+\cos x)=f(x)$

$f^1(x)=\frac{\sin^3 x}{\cos^4 x+3\cos^2 x+1}$

$\therefore \int \frac{f^1(x)}{f(x)}=\log|f(x)|+c$

23. $\int\left(\sqrt{\sin x}+\sqrt{\cos x}\right)^{-4}dx=$

1) $\frac{2}{\left(\sqrt{\tan x}+1\right)^2}\left[\frac{1}{3\left(\sqrt{\tan x}+1\right)}-\frac{1}{2}\right]+c$

2) $\frac{2}{\left(\sqrt{\tan x}+1\right)^2}\left[\frac{1}{3\left(\sqrt{\tan x}+1\right)}+\frac{1}{2}\right]+c$

SMART ACHIEVERS LEARNING PVT. LTD.

3) $\frac{2}{\left(\sqrt{\tan x}+1\right)^{2}}\left[\frac{1}{\sqrt{3}\left(\sqrt{\tan x}+1\right)}+\frac{1}{2}\right]+c$

4) $\frac{2}{\left(\sqrt{\tan x}+1\right)^{2}}\left[\frac{1}{\sqrt{3}\left(\sqrt{\tan x}+1\right)}-\frac{1}{2}\right]+c$

Key. 1

Sol.

$$I=\int\left(\sqrt{\sin x}+\sqrt{\cos x}\right)^{-4}dx=\int\frac{dx}{\left[\sqrt{\cos x}\left(\sqrt{\tan x}+1\right)\right]^{4}}=\int\frac{1}{\cos^{2}x\left(\sqrt{\tan x}+1\right)^{4}}$$

$$=\int\frac{\sec^{2}x\,dx}{\left(\sqrt{\tan x}+1\right)^{4}}$$

$$Put\sqrt{\tan x}+1=y\Rightarrow\frac{1}{2\sqrt{\tan x}}\sec^{2}x\,dx=dy\Rightarrow\sec^{2}x\,dx=2\sqrt{\tan x}\,dy=2(y-1)dy$$

$$I=\int\frac{1}{y^{4}}.2(y-1)dy=2\int\left(\frac{1}{y^{3}}-\frac{1}{y^{4}}\right)dy=2\left[\frac{-1}{2y^{2}}+\frac{1}{3y^{3}}\right]+c=\frac{2}{y^{2}}\left[\frac{1}{3y}-\frac{1}{2}\right]$$

$$=\frac{2}{\left(\sqrt{\tan x}+1\right)^{2}}\left[\frac{1}{3\left(\sqrt{\tan x}+1\right)}-\frac{1}{2}\right]+c$$

24. $\int\sqrt{\tan x}\,dx=$

1) $\frac{1}{2\sqrt{2}}\left[2\tan^{-1}\left(\frac{\tan x-1}{\sqrt{2\tan x}}\right)+\log\left[\frac{\tan x-\sqrt{2\tan x}+1}{\tan x+\sqrt{2\tan x}+1}\right]\right]+c$

2) $\frac{1}{2\sqrt{2}}\left[2\tan^{-1}\left(\frac{\tan x-1}{2\tan x}\right)+\log\left[\frac{\tan x-\sqrt{2\tan x}+1}{\tan x+\sqrt{2\tan x}+1}\right]\right]+c$

3) $\frac{1}{2\sqrt{2}}\left[2\tan^{-1}\left(\frac{\tan x+1}{\sqrt{2}\tan x}\right)+\log\left[\frac{\tan^{2}x-2\tan x+1}{\tan x+2\tan x+1}\right]\right]+c$

4) $\frac{1}{5\sqrt{2}}\left[2\tan^{-1}\left(\frac{\tan x+1}{\sqrt{2}\tan x}\right)+\log\left[\frac{\tan^{2}x-2\tan x-1}{\tan x+2\tan x+1}\right]\right]+c$

Key. 1

Sol.

$$\frac{1}{2}\int2\sqrt{\tan x}\,dx=\frac{1}{2}\int\left(\sqrt{\tan x}+\sqrt{\cot x}\right)dx+\frac{1}{2}\int\left(\sqrt{\tan x}-\sqrt{\cot x}\right)dx$$

25. $\int\sqrt{x+\sqrt{x^{2}+2}}\,dx=$

1) $\left(\frac{x+\sqrt{x^2+2}}{3}\right)^{\frac{3}{2}}-2\left(x+\sqrt{x^2+2}\right)^{-\frac{1}{2}}+c$

2) $\frac{1}{3}\left(x+\sqrt{x^2+2}\right)^{\frac{3}{2}}-2\left(x+\sqrt{x^2+2}\right)^{\frac{-1}{2}}+c$

3) $\frac{2}{7}\left(x+\sqrt{x^2+2}\right)^{\frac{7}{2}}-2\left(x+\sqrt{x^2+2}\right)+c$

4) $\frac{2}{7}\left(x+\sqrt{x^2+2}\right)^{\frac{7}{2}}+2\left(x+\sqrt{x^2+2}\right)+c$

Key. 2

Sol. Put $x+\sqrt{x^2+2}=t$

26. If $\int \frac{2\cos x-\sin x+\lambda}{\cos x+\sin x-2}dx = A\ln\left|\cos x+\sin x-2\right|+Bx+C$ Then the ordered triplet A,B, λ is

1) $\left(\frac{1}{2},\frac{3}{2},-1\right)$ 2) $\left(\frac{3}{2},\frac{1}{2},-1\right)$ 3) $\left(\frac{1}{2},-1,-\frac{3}{2}\right)$ 4) $\left(\frac{3}{2},-1,\frac{1}{2}\right)$

Key. 2

Sol.

$$\frac{d}{dx}\left(A\,ln\left|\cos x+\sin x-2\right|+Bx+C\right)$$

$$= A\frac{\cos x-\sin x}{\cos x+\sin x-2}+B=\frac{A\cos x-A\sin x+B\cos x+B\sin x-2B}{\cos x+\sin x-2}$$

$\therefore \quad 2=A+B, -1=-A+B, \lambda=-2B$

$\therefore \quad A=3/2, B=1/2, \lambda=-1$

27. $\int \frac{x^8+4}{x^4-2x^2+2}dx=$

1) $\frac{x^5}{5}-\frac{2x^3}{3}+2x+C$ 2) $\frac{x^5}{5}-\frac{2x^3}{3}-2x+C$

3) $\frac{x^5}{5}+\frac{2x^3}{3}-2x+C$ 4) $\frac{x^5}{5}+\frac{2x^3}{3}+2x+C$

Key. 4

Sol.

$$\int \frac{\left(x^8+4+4x^4\right)-4x^4}{x^4-2x^2+2}dx=\int \frac{\left(x^4+2\right)^2-\left(2x^2\right)^2}{\left(x^4-2x^2+2\right)}dx$$

$$=\int \frac{\left(x^4+2-2x^2\right)\left(x^4+2+2x^2\right)}{\left(x^4-2x^2+2\right)}dx=\frac{x^5}{5}+\frac{2x^3}{3}+2x+C$$

SMART ACHIEVERS LEARNING PVT. LTD.

28. $\int \frac{\cos^4 x\,dx}{\sin^3 x\left(\sin^5 x+\cos^5 x\right)^{3/5}} = -\frac{1}{2}\left(1+\cot^A x\right)^B + C$

1) 5 2) $\frac{2}{5}$ 3) 2 4) 1

Key. 3

Sol.

$$I=\int \frac{\cos^4 x dx}{\sin^3 x\left(\sin^5 x+\cos^5 x\right)^{3/5}}$$

$$=\int \frac{\cos^4 x dx}{\sin^6 x\left(1+\cot^5 x\right)^{3/5}}$$

$$=\int \frac{\cot^4 x\,cosec^2 x\,dx}{\left(1+\cot^5 x\right)^{3/5}} \; put\; 1+\cot^5 x = t \qquad 5\cot^4 x\,cosec^2 x\,dx = -dt$$

29. If $f(x)=\sqrt{x}, g(x)=e^{x-1}$, and $\int fog(x)dx = A\; fog(x) + B\tan^{-1}\left(fog(x)\right)+C$, then $A+B$ is equal to

1) 1 2) 2 3) 3 4) 0

Key. 4

Sol.

$$fog(x)=\sqrt{e^x-1}$$

$$\therefore \qquad I=\int \sqrt{e^x-1}\,dx = \int \frac{2t^2}{t^2+1}dt \quad \left\{where\; \sqrt{e^x-1}=t\right\}$$

$$=2t-2\tan^{-1}t+C = 2\sqrt{e^x-1}-2\tan^{-1}\left(\sqrt{e^x-1}\right)+C = 2fog(x)-2\tan^{-1}\left(fog(x)\right)+C$$

$$\therefore \qquad A+B = 2+(-2)=0$$

30. If $\int \sin^{-1}x\cos^{-1}x dx = f^{-1}\left[\frac{\pi}{2}x - xf^{-1}(x) - 2\sqrt{1-x^2}\right]+\frac{\pi}{2}\sqrt{1-x^2}+2x+C$, then f(x) is equal to

1) sin *3x* 2) sin *2x* 3) sin *x* 4) sin *4x*

Key. 3

Sol.

$$\int \sin^{-1}x\cos^{-1}x dx = \int\left[\frac{\pi}{2}\sin^{-1}x - \left(\sin^{-1}x\right)^2\right]dx$$

$$\Rightarrow \qquad \frac{\pi}{2}\left(x\sin^{-1}x+\sqrt{1-x^2}\right)-\left(x\left(\sin^{-1}x\right)^2+\sin^{-1}x\sqrt{1-x^2}-x\right)+c \; By\; parts$$

$$\Rightarrow \qquad \sin^{-1}x\left[\frac{\pi}{2}x - x\sin^{-1}x - 2\sqrt{1-x^2}\right]+\frac{\pi}{2}\sqrt{1-x^2}+2x+c$$

$$\therefore \qquad f^{-1}(x)=\sin^{-1}x, f(x)=\sin x$$

SMART ACHIEVERS LEARNING PVT. LTD.

31. Let $F(x)=e^{\sin^{-1}x}\left(1-\frac{x}{\sqrt{1-x^2}}\right)dx$ and $F(0)=1,$ If $F(1/2)=\frac{k\sqrt{3}e^{x/6}}{\pi}$, then k =

1) $\frac{\pi}{4}$ 2) $\frac{\pi}{6}$ 3) $\frac{\pi}{2}$ 4) $\frac{\pi}{3}$

Key. 3

Sol.

$$F(x)=\int e^{\sin^{-1}x}\left(1-\frac{x}{\sqrt{1-x^2}}\right)dx=\int e^{\sin^{-1}x}\left(\frac{1}{\sqrt{1-x^2}}\sqrt{1-x^2}-\frac{x}{\sqrt{1-x^2}}\right)dx$$

$$F(x)=e^{\sin^{-1}x}\sqrt{1-x^2}+c$$

$$F(0)=1+c \qquad \Rightarrow \qquad c=0 \qquad (\because F(0)=1)$$

$$F(1/2)=e^{\pi/6}.\frac{\sqrt{3}}{2}=\frac{k\sqrt{3}}{\pi}e^{\pi/6}$$

$$\therefore \qquad k=\frac{\pi}{2}$$

32. $\int\left(\frac{x-1}{x+1}\right)\frac{dx}{\sqrt{x^3+x^2+x}}=2\tan^{-1}\sqrt{f(x)}+C$ then find $f(x)$.

1) $x+\frac{1}{x}+1$ 2) $x+\frac{1}{x}+2$ 3) $x-\frac{1}{x}+1$ 4) $x-\frac{1}{x}-2$

Key. 1

Sol.

$$I=\int\left(\frac{x-1}{x+1}\right)\frac{dx}{x\sqrt{x+1+\frac{1}{x}}}$$

$$so,\ I=\int\frac{(x-1)dx}{(x+1)x\sqrt{x+1+\frac{1}{x}}}=\int\frac{\left(1-\frac{1}{x}\right)\left(1+\frac{1}{x}\right)dx}{(x+1)\left(1+\frac{1}{x}\right)\sqrt{x+1+\frac{1}{x}}}=\int\frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}+2\right)\sqrt{x+\frac{1}{x}+1}}$$

$$Put\ x+1+\frac{1}{x}=t^2$$

$$\left(1-\frac{1}{x^2}\right)dx=2t\,dt$$

$= \int \frac{2t\,dt}{(t^2+1)t} = 2\tan^{-1}t + c = 2\tan^{-1}\left(\sqrt{x+\frac{1}{x}+1}\right)+c$

$x+\frac{1}{x}+1$ Ans.

33. $\int 2\cos 2x$ ln $(\tan x)\,dx$, is equal to

1) $\sin 2x \ln(\tan x) - 2x + C$
2) $\sin 2x \ln(\tan x) + 2x + C$
3) $\sin x \ln(\tan x) - x + C$
4) none of these

Key. 1

Sol. We have $I = \int 2\cos 2x \ln(\tan x)\,dx = \sin 2x \ln(\tan x) - \int \sin 2x.\frac{\sec^2 x}{\tan x}dx$

$= \sin 2x \ln(\tan x) - \int 2dx = \sin 2x \ln(\tan x) - 2x + C$

34. $\int \frac{\cos x + x\sin x}{x(x+\cos x)}dx =$

1) $\log\left|\frac{x}{x+\cos x}\right|+c$
2) $-\log\left|\frac{x}{x+\cos x}\right|+c$
3) $\log\left|\frac{x+\cos x}{2x}\right|+c$
4) $-\log\left|\frac{x+\cos x}{2x}\right|+c$

Key. 1

Sol. $\int \frac{x+\cos x + x\sin x - x}{x(x+\cos x)}dx$

35. $\int [1+\tan x \tan(x+\alpha)]dx$, is equal to

1) $\tan\alpha \ln\left|\frac{\sin(x+\alpha)}{\sin x}\right|+C$
2) $\cot\alpha \ln\left|\frac{\sin(x+\alpha)}{\sin x}\right|+C$
3) $\cot\alpha \ln\left|\frac{\sin x}{\sin(x+\alpha)}\right|+C$
4) $\cot\alpha \ln\left|\frac{\cos x}{\cos(x+\alpha)}\right|+C$

Key. 4

Sol. We have $\tan\alpha = \tan(x+\alpha-x) = \frac{\tan(x+\alpha)-\tan x}{1+\tan x\tan(x+\alpha)}$

Then, we have

$\int [1+\tan x\tan(x+\alpha)]dx = \int \cot\alpha[\tan(x+\alpha)-\tan x]dx$

SMART ACHIEVERS LEARNING PVT. LTD.

$= \cot\alpha\left[-\ln\left|\cos(x+\alpha)\right| + \ln\left|\cos x\right|\right] + C$

$= \cot\alpha \ln\left|\dfrac{\cos x}{\cos(x+\alpha)}\right| + C$

36. Let $x^2 \neq n\pi - 1, n \in N$. Then, the value of $\int x\sqrt{\dfrac{2\sin(x^2+1) - \sin 2(x^2+1)}{2\sin(x^2+1) + \sin 2(x^2+1)}}$ dx is equal to

1) $\log\left|\dfrac{1}{2}\sec(x^2+1)\right| + C$

2) $\log\left|\sec\left(\dfrac{x^2+1}{2}\right)\right| + C$

3) $\dfrac{1}{2}\log\left|\sec(x^2+1)\right| + C$

4) None of these

Key. 2

Sol. We have, $\int x\sqrt{\dfrac{2\sin(x^2+1) - \sin 2(x^2+1)}{2\sin(x^2+1) + \sin 2(x^2+1)}}dx$

$$= \int x\sqrt{\frac{2\sin(x^2+1) - 2\sin(x^2+1)\cos(x^2+1)}{2\sin(x^2+1) + 2\sin(x^2+1)\cos(x^2+1)}}dx$$

$$= \int x\sqrt{\frac{1-\cos(x^2+1)}{1+\cos(x^2+1)}}dx$$

$$= \int x\tan\left(\frac{x^2+1}{2}\right)dx$$

$$= \int \tan\left(\frac{x^2+1}{2}\right)d\left(\frac{x^2+1}{2}\right)$$

$$= \log\left|\sec\left(\frac{x^2+1}{2}\right)\right| + C$$

37. $\int \dfrac{dx}{\cos(2x)\cos(4x)}$ is equal to

1) $\dfrac{1}{2\sqrt{2}}\log\left|\dfrac{1+\sqrt{2}\sin 2x}{1-\sqrt{2}\sin 2x}\right| - \dfrac{1}{2}\left(\log\left|\sec 2x - \tan 2x\right|\right) + C$

SMART ACHIEVERS LEARNING PVT. LTD.

2) $\frac{1}{2\sqrt{2}}\log\left|\frac{1+\sqrt{2}\sin 2x}{1+\sqrt{2}\sin x}\right|-\frac{1}{2}\left(\log\left|\sec 2x-\tan 2x\right|\right)+C$

3) $\frac{1}{\sqrt{2}}\log\left|\frac{1+\sqrt{2}\sin 2x}{1-\sqrt{2}\sin 2x}\right|-\frac{1}{2}\left(\log\left|\sec 2x-\tan 2x\right|\right)+C$

4) None of these

Key. 1

Sol. $\int\frac{\sin\left(4x-2x\right)dx}{\sin\left(2x\right)\cos\left(2x\right)\cos\left(4x\right)}$

$=\int\frac{\sin\left(4x\right)dx}{\sin\left(2x\right)\cos\left(4x\right)}-\int\sec 2x\,dx$

$=2\int\frac{\cos 2x\,dx}{\cos 4x}-\frac{1}{2}\left(\log\left|\sec 2x-\tan 2x\right|\right)$

38. $\int\frac{1-7\cos^2 x}{\sin^7 x\cos^2 x}dx=\frac{f\left(x\right)}{\left(\sin x\right)^7}+C$, then f(x) is equal to

1) Sin x 2) Cos x 3) Tan x 4) Cot x

Key. 3

Sol. $\int\frac{1-7\cos^2 x}{\sin^7 x\cos^2 x}dx=\int\left(\frac{\sec^2 x}{\sin^7 x}-\frac{7}{\sin^7 x}\right)dx$

$=\int\frac{\sec^2 x}{\sin^7 x}dx-\int\frac{7}{\sin^7}dx=I_1+I_2$

Now, $I_1=\int\frac{\sec^2 x}{\sin^7 x}dx$

$=\frac{\tan x}{\sin^7 x}+7\int\frac{\tan x.\cos x}{\sin^8 x}dx$

$=\frac{\tan x}{\sin^7 x}-I_2$

$\therefore I_1+I_2=\frac{\tan x}{\sin^7 x}+C$

$\Rightarrow f\left(x\right)=\tan x$

SMART ACHIEVERS LEARNING PVT. LTD.

39. Integral of $\sqrt{1+2\cot x(\cot x+\operatorname{cosec} x)}$ with respect to x is:

1) $2\ln\cos\frac{x}{2}+c$ 2) $2\ln\sin\frac{x}{2}+c$

3) $\frac{1}{2}\ln\cos\frac{x}{2}+c$ 4) $\ln\sin x-\ln(\operatorname{cosec} x-\cot x)+c$

Key. 2

Sol. I = $\int\sqrt{1+2\operatorname{cosec} x\cot x+2\cot^2 x}$

= $\int\sqrt{\operatorname{cosec}^2 x+2\operatorname{cosec} x\cot x+\cot^2 x}\,dx$

= $\int(\operatorname{cosec} x+\cot x)\,dx$

40. Let $f(x)=\frac{1}{x}\ln\left(\frac{x}{e^x}\right)$ then its primitive with respect to x is

1) $\frac{1}{2}e^x-\ln x+C$ 2) $\frac{1}{2}\ln x-e^x+C$

3) $\frac{1}{2}\ln^2 x-x+C$ 4) $\frac{e^x}{2x}+C$

Key. 3

Sol. $\int\frac{1}{x}\ln\frac{x}{e^x}dx$ = $\int\frac{1}{x}(\ln x-\ln e^x)\,dx$

= $\int\frac{\ln x-x}{x}dx$ = $\left[\int\frac{1}{x}\ln x\,dx-\int\frac{1}{x}x\,dx\right]$ (put $\ln x = u$; $\frac{1}{x}dx=du$)

= $\int u\,du$ – $\int 1dx$ = $\frac{1}{2}\ln^2 x - x + C$]

41. Primitive of $f(x)=x.2^{\ln(x^2+1)}$ with respect to x is

1) $\frac{2^{\ln(x^2+1)}}{2(x^2+1)}+C$ 2) $\frac{(x^2+1)2^{\ln(x^2+1)}}{\ln 2+1}+C$ 3) $\frac{(x^2+1)^{\ln 2+1}}{2(\ln 2+1)}+C$ 4) $\frac{(x^2+1)^{\ln 2}}{2(\ln 2+1)}+C$

Key. 3

Sol. I = $\int x2^{\ln(x^2+1)}dx$ let $x^2 + 1 = t$; $x\,dx = \frac{dt}{2}$

Hence I = $\frac{1}{2}\int 2^{\ln t}dt$ = $\frac{1}{2}\int t^{\ln 2}dt$ = $\frac{1}{2}\cdot\frac{t^{\ln 2+1}}{\ln 2+1}$ + C = $\frac{1}{2}\cdot\frac{(x^2+1)^{\ln 2+1}}{\ln 2+1}$ + C $\Rightarrow$ (C)

42. Let g(x) be an antiderivative for f(x). Then $\ln\left(1+(g(x))^2\right)$ is an antiderivate for

1) $\frac{2f(x)g(x)}{1+(f(x))^2}$ 2) $\frac{2f(x)g(x)}{1+(g(x))^2}$ 3) $\frac{2f(x)}{1+(f(x))^2}$ 4) none

Key. 2

SMART ACHIEVERS LEARNING PVT. LTD.

Sol. Given $\int f(x)\,dx$ = g (x) $\Rightarrow$ g ' (x) = f (x)

now $\frac{d}{dx}\left(\ln(1+g^2(x)\right) = \frac{2g(x)g'(x)}{1+g^2(x)} = \frac{2f(x)g(x)}{1+g^2(x)}$ $\Rightarrow$ (B)

43. $\int \frac{\cos^3 x+\cos^5 x}{\sin^2 x+\sin^4 x}dx$ equals

1) $\sin x-6\tan^{-1}(\sin x)+c$

2) $\sin x-2\sin^{-1}x+c$

3) $\sin x-2(\sin x)^{-1}-6\tan^{-1}(\sin x)+c$

4) $\sin x-2(\sin x)^{-1}+5\tan^{-1}(\sin x)+c$

Key. 3

Sol. sin x = t ; I = $\int \frac{(1-t^2)(2-t^2)}{t^2(1+t^2)}$ dt = $\int \frac{(y-1)(y-2)}{y(1+y)}$ dy = 1 + $\frac{2(1-2y)}{y(y+1)}$; y = t^2

= 1 + 6 $\left[\frac{1}{3y}-\frac{1}{y+1}\right]$ = $\left(1+\frac{2}{t^2}-\frac{6}{1+t^2}\right)$dt

44. The evaluation of $\int \frac{px^{p+2q-1}-qx^{q-1}}{x^{2p+2q}+2x^{p+q}+1}dx$ is

1) $-\frac{x^p}{x^{p+q}+1}+C$ 2) $\frac{x^q}{x^{p+q}+1}+C$ 3) $-\frac{x^q}{x^{p+q}+1}+C$ 4) $\frac{x^p}{x^{p+q}+1}+C$

Key. 3

Sol. $\int \frac{px^{p+2q-1}-qx^{q-1}}{\left(x^{p+q}+1\right)^2}$ dx = $\int \frac{px^{p-1}-qx^{-q-1}}{\left(x^p+x^{-q}\right)^2}$ dx

taking x^q as x^{2q} common from Denominator and take it in N^r

45. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\left(x+\sqrt{x}\right)dx$ equals

1) $2e^{\sqrt{x}}\left[x-\sqrt{x}+1\right]+C$

2) $e^{\sqrt{x}}\left[x-2\sqrt{x}+1\right]$

3) $e^{\sqrt{x}}\left[x+\sqrt{x}\right]+C$

4) $e^{\sqrt{x}}\left[x+\sqrt{x}+1\right]+C$

Key. 1

Sol. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\left(x+\sqrt{x}\right)dx$; put x = t^2 ; dx = 2t dt

= $\int e^t(t^2+t)\,dt$ = e^t (At^2 + Bt + C) (Let)

Diffrentiate both the sides

$e^t(t^2+t) = e^t(2At+B)+(At^2+Bt+C)e^t$

On comparing coefficient we get

A = 1 ; B = – 1 ; C = 1]

SMART ACHIEVERS LEARNING PVT. LTD.

46. $\int \frac{e^{\tan^{-1}x}}{(1+x^2)}\left[\left(\sec^{-1}\sqrt{1+x^2}\right)^2+\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]dx$ (x > 0)

1) $e^{\tan^{-1}x}.\tan^{-1}x+C$
2) $\frac{e^{\tan^{-1}x}.\left(\tan^{-1}x\right)^2}{2}+C$
3) $e^{\tan^{-1}x}.\left(\sec^{-1}\left(\sqrt{1+x^2}\right)\right)^2+C$
4) $e^{\tan^{-1}x}.\left(\operatorname{cosec}^{-1}\left(\sqrt{1+x^2}\right)\right)^2+C$

Key. 3

Sol. note that $\sec^{-1}\sqrt{1+x^2} = \tan^{-1}x$; $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = 2\tan^{-1}x$ for x > 0

$I = \int \frac{e^{\tan^{-1}x}}{1+x^2}\left((\tan^{-1}x)^2+2\tan^{-1}x\right)dx$ put $\tan^{-1}x = t$

$= \int e^t(t^2+2t)\,dt = e^t.t^2 = e^{\tan^{-1}x}\left(\tan^{-1}x\right)^2 + C$

47. Let $f(x)=\frac{2\sin^2x-1}{\cos x}+\frac{\cos x(2\sin x+1)}{1+\sin x}$ then $\int e^x\left(f(x)+f'(x)\right)dx$ equals

(where c is the constant of integeration)

1) $e^x\tan x+c$ 2) $e^x\cot x+c$ 3) $e^x\operatorname{cosec}^2x+c$ 4) None of these

Key. 1

Sol. $\frac{\cos x(1+2\sin x)}{1+\sin x}-\frac{\cos^2x-\sin^2x}{\cos x}$

$=\frac{\cos^2x(1+2\sin x)-(1+\sin x)(\cos^2x-\sin^2x)}{\cos x(1+\sin x)} = \frac{-\sin x\cos^2x+\sin^3x}{\cos x(1+\sin x)}$

$=\frac{\sin x\cos^2x+\sin^2x(1+\sin x)}{\cos x(1+\sin x)} = \frac{\sin x(1-\sin x)+\sin^2x}{\cos x}$ = tan x

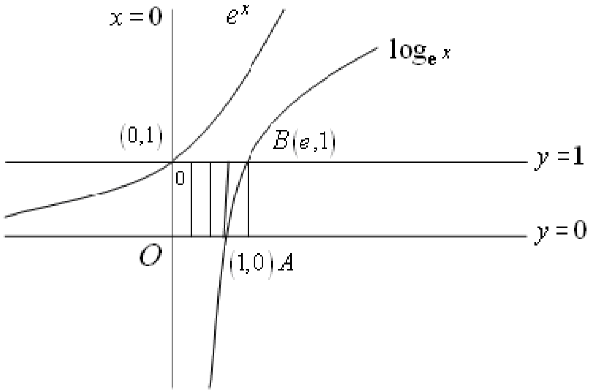
48. Area bounded by the curves $y=e^x,\ y=\log_e x$ and the lines $x=0,\ y=0,\ y=1$ is

A) $e^2+2sq.units$ B) $e+1sq.unit$
C) $e+2sq.units$ D) $e-1\,sq.unit$

Key: D

Hint:

SMART ACHIEVERS LEARNING PVT. LTD.



Area = Area of rectangle OABC $-\int_0^1 \log_e x$

49. The area of the loop of the curve $y^2 = x^4(x+2)$ is [in square units]

(A) $\frac{32\sqrt{2}}{105}$ (B) $\frac{64\sqrt{2}}{105}$ (C) $\frac{128\sqrt{2}}{105}$ (D) $\frac{256\sqrt{2}}{105}$

Key: D

Hint: Area = $2\int_{-2}^{0} y\,dx = 2\int_{-2}^{0} x^2\sqrt{x+2}\,dx = 4\sqrt{2}\int_0^2 \left(z^2-2\right)^2 z^2 dz (where \sqrt{x+2} = z)$

$= 4\left[\frac{z^7}{7} - \frac{4z^5}{5} + \frac{4z^3}{3}\right]_0^{\sqrt{2}}$

$= \frac{256\sqrt{2}}{105}$

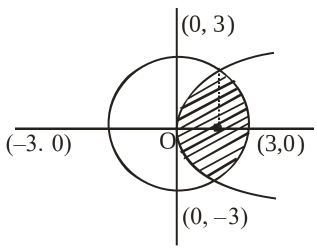
50. The area of the smaller portion enclosed by the curves $x^2+y^2=9$ and $y^2=8x$ is

A) $\frac{\sqrt{2}}{3} + \frac{9\pi}{4} - \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)$ B) $2\left(\frac{\sqrt{2}}{3} + \frac{9\pi}{4} - \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right)$

C) $2\left(\frac{\sqrt{2}}{3} + \frac{9\pi}{4} + \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right)$ D) $\frac{\sqrt{2}}{3} + \frac{9\pi}{4} + \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)$

Key: B

HINT :



$x^2 + y^2 = 9,$

$x^2 + 8x - 9 = 0$

$x = \frac{-8 \pm \sqrt{64+36}}{2}$

SMART ACHIEVERS LEARNING PVT. LTD.

$x=\frac{-8\pm 10}{2}-9,1$

$\boxed{x=1}$

Area enclosed $=2\left[\int_0^1 2\sqrt{2x}dx+\int_1^3\sqrt{9-x^2}\,dx\right]=2\left[2\sqrt{2}\int_0^1\sqrt{x}dx+\int_1^3\sqrt{9-x^2}\,dx\right]$

On simplifying we get

$=2\left[\frac{\sqrt{2}}{3}+\frac{9\pi}{4}-\frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right]$

51. The area of the region in the xy-plane defined by the inequalities $x-2y^2\ge 0$, $1-x-|y|\ge 0$ is

A) $\frac{1}{2}$ B) $\frac{1}{3}$ C) $\frac{1}{4}$ D) $\frac{7}{12}$

Key: D

Hint: Area $=2\int_0^{1/2}\sqrt{\frac{x}{2}}dx+\frac{1}{4}=\frac{7}{12}$

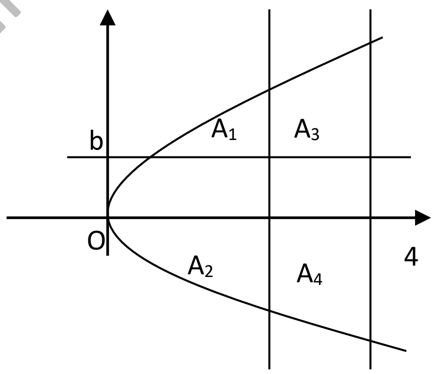
52. Area bounded by curve $y^2 = x$ and $x = 4$ is divided into 4 equal parts by the lines $x = a$ and $y = b$ then.

a) Area of each part = $\frac{8}{3}$ b) b = 0

c) $a=\sqrt{2}$ d) $a=(16)^{1/3}$

Key: D

Hint: Total area = $2\int_0^4\sqrt{x}\,dx=\frac{32}{3}$



Area of each part = 8/3

$A_3=A_4\Rightarrow\int_a^4\left(\sqrt{x}-b\right)dx=$

$\int_a^4\left(b+\sqrt{x}\right)dx=\frac{8}{3}\Rightarrow b=0$

$\int_a^4\sqrt{x}\,dx=\frac{8}{3}\Rightarrow a^3=16$

53. Area of the region in which point $p(x,y)$, $\{x>0\}$ lies; such that $y\le\sqrt{16-x^2}$ and $\left|\tan^{-1}\left(\frac{y}{x}\right)\right|\le\frac{\pi}{3}$ is

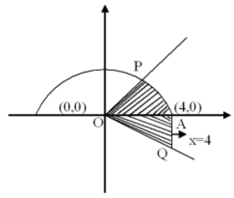
(A) $\left(\frac{16}{3}\pi\right)$ (B) $\left(\frac{8\pi}{3}+8\sqrt{3}\right)$ (C) $\left(4\sqrt{3}-\pi\right)$ (D) $\left(\sqrt{3}-\pi\right)$

Key: B

Hint: Required area is the area of shaded region (APOQ)

SMART ACHIEVERS LEARNING PVT. LTD.

= area of ΔOAQ + area of sector (OAP)



$$= \frac{1}{2}\times 4\times 4\sqrt{3} + \frac{\pi(4\times 4)}{6}$$

$$= \left(\frac{8\pi}{3} + 8\sqrt{3}\right)$$

54. Area bounded between the curves $y = \sqrt{4-x^2}$ and $y^2 = 3|x|$ is/are

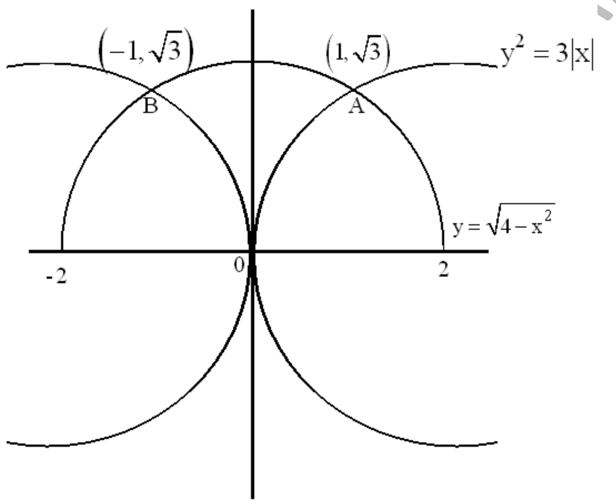
(A) $\frac{\pi-1}{\sqrt{3}}$ (B) $\frac{2\pi-1}{3\sqrt{3}}$ (C) $\frac{2\pi-\sqrt{3}}{3}$ (D) $\frac{2\pi-\sqrt{3}}{3\sqrt{3}}$

Key: C

Hint: Required area = $2\int_0^1\left(\sqrt{4-x^2} - \sqrt{3x}\right)dx$

$$= 2\left(\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right) - \frac{\sqrt{3}.2x^{3/2}}{3}\right)_0^1$$

$$= \frac{2\pi-\sqrt{3}}{3}$$



55. Let $f:[0,\infty)\to R$ be a continuous and strictly increasing function such that $f^3(x) = \int_0^x t.f^2(t)dt, \forall x > 0$. The area enclosed by $y = f(x)$, the x-axis and the ordinate at x=3, is

(A) 1 (B) $\frac{3}{2}$ (C) 2 (D) 3

Key: B

Hint: $f(x) = \frac{x^2}{6}$

$A=\frac{1}{6}\int_0^3 x^2dx=3/2$

56. Let f(x) = x + sin x. The area bounded by $y=f^{-1}(x), y=x,\ x\in[0,\pi]$ is

(a) 1 (b) 2 (c) 3

(d) canot be found because $f^{-1}(x)$ cannot be determined

Key: B

Hint: The curves given by y = x + sin x and $y = f^{-1}(x)$ are images of each other in the line y = x.

Hence required area = $\int_0^{\pi}((x+\sin x)-x)dx=-[\cos x]_0^{\pi}=2$

57. The area of the region bounded by the curves $|x + y| \le 2$, $|x - y| \le 2$ and $2x^2 + 6y^2 \ge 3$ is

(A) $\left(8+\frac{\sqrt{3}}{2}\pi\right)$ sq. units (B) $\left(8-\frac{\sqrt{3}}{2}\pi\right)$ sq. units

(C) $\left(4-\frac{3\sqrt{3}}{2}\pi\right)$ sq. units (D) $\left(8-\frac{3\sqrt{3}}{2}\pi\right)$ sq. units

Key : B

Sol :

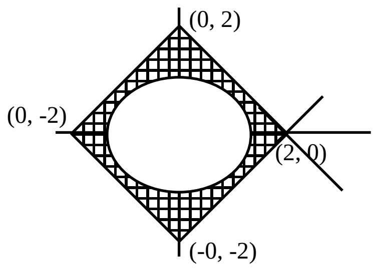
$2x^2 + 6y^2 \ge 3$ . . . . . . (1)

area of ellipse = $\pi\times\frac{\sqrt{3}}{\sqrt{2}}\times\frac{1}{\sqrt{2}}=\frac{\pi\sqrt{3}}{2}$

$|x + y| \le 2 \Rightarrow -2 \le (x + y) \le 2$ . . . . . .(2)

$|x - y| \le 2 \Rightarrow -2 \le (x - y) \le 2$ . . . . . . (3)

Required area = $\left(8-\frac{\pi\sqrt{3}}{2}\right)$ sq. units



58. If $A=\int_1^{\sin\theta}\frac{t\,dt}{1+t^2}dt, B=\int_1^{\operatorname{cosec}\theta}\frac{1}{t(1+t^2)}dt$ then $\begin{vmatrix} A & A^2 & B \\ e^{A+B} & B^2 & -1 \\ 1 & A^2+B^2 & -1 \end{vmatrix}=?$

a) $\sin\theta$ b) $\operatorname{cosec}\theta$ c) 0 d) 1

Key: C

SMART ACHIEVERS LEARNING PVT. LTD.

$$B=\int_1^{\operatorname{cosec}\theta}\frac{1}{t\left(1+t^2\right)}$$

Hint:

$$\text{let }\frac{1}{t}=u\Rightarrow B=\int_1^{\sin\theta}\frac{-udu}{1+u^2}$$

$$\Rightarrow A+B=0 \quad \Rightarrow A=-B$$

$$\therefore \begin{vmatrix} A & A^2 & -A \\ e^0 & A^2 & -1 \\ 1 & A^2+B^2 & -1 \end{vmatrix}=0$$

59. The value of $\int\limits_0^2 \frac{2x^3-6x^2+9x-5}{x^2-2x+5}dx$ equals

(A) 4 (B) 1

C) – 1 (D) None of the above

Key: D

Hint Make the substitution x – 1= t. It turns into an odd integral and so reduces to zero.

60. Let $I_n=\int\limits_0^{\infty} e^{-x}\left(\sin x\right)^n dx,\ n\in N,\ n>1$ then $\frac{I_{2008}}{I_{2006}}$ equals

(A) $\frac{2007\times 2006}{2008^2+1}$ (B) $\frac{2008\times 2007}{2008^2+1}$

(C) $\frac{2006\times 2004}{2008^2-1}$ (D) $\frac{2008\times 2007}{2008^2-1}$

Key: B

Hint In $=\int\limits_0^{\infty} e^{-x}\left(\sin x\right)^n dx$

$$=\left[\sin^n x\left(-e^{-x}\right)\right]_0^{\infty}+\int\limits_0^{\infty} n\sin^{n-1}x\cos x\, e^{-x}dx$$

$$=0+n\int\limits_0^{\infty}\left(\sin^{n-1}x\cos x\right)e^{-x}dx$$

$$=n\left[\left(\sin^{n-1}x\cos x\right)\left(-e^{-x}\right)\right]_0^{\infty}-n\int\limits_0^{\infty}\left\{-\sin^n x+(n-1)\sin^{n-2}x\cos^2 x\right\}\left(-e^{-x}\right)dx$$

$$=0+n\int\limits_0^{\alpha} e^{-x}\left\{-\sin^n x+(n-1)\sin^{n-2}x\,(1-\sin^2 x)\right\}dx$$

$$=n\int\limits_0^{\alpha} e^{-x}\left\{(n-1)\sin^{n-2}x-n\sin^n x\right\}dx$$

$$=n(n-1)I_{n-2}-n^2 I_n$$

SMART ACHIEVERS LEARNING PVT. LTD.

$we\ \ have\ \ \left(1+n^2\right)I_n = n(n-1)\,I_{n-2}$

$then\ \ \dfrac{I_n}{I_{n-2}} = \dfrac{n(n-1)}{n^2+1}$

61. A hyperbola passing through origin has 3x-4y-1=0 and 4x-3y-6=0 as its asymptotes. Then the equation of its transverse axis is

a) x-y-5=0 b)x+y+1=0 c) x+y-5=0 d)x-y-1=0

Key: C

Hint: Asymptotes are equally inclined to the axes of hyperbola
Find the bisector of the asymptotes which bisects the angle containing the origin.

62. If $\int_{\sin x}^{1} t^2.f(t)\,dt = 1-\sin x,\ \forall x \in \left(0,\dfrac{\pi}{2}\right)$ then the value of $f\left(\dfrac{1}{\sqrt{3}}\right)$ is

(A) $\dfrac{1}{\sqrt{3}}$ (B) $\sqrt{3}$ (C) $\dfrac{1}{3}$ (D) 3

Key: D

Hint: $\int_{\sin x}^{1} t^2.f(t)\,dt = 1-\sin x$

Differentiating both sides with respect to 'x'

$0-\sin^2 x.f(\sin x).\cos x = -\cos x \ \Rightarrow \cos x\left[1-\sin^2 x.f(\sin x)\right] = 0$

But $\cos x \neq 0$

So, $f(\sin x) = \dfrac{1}{\sin^2 x}$

$f\left(\dfrac{1}{\sqrt{3}}\right) = 3$

63. $\int_{-\pi}^{\pi} \dfrac{2x\left(1+\sin x\right)}{1+\cos^2 x}dx =$

(A) $\dfrac{\pi^2}{4}$ (B) π^2 (C) 0 (D) $\dfrac{\pi}{2}$

Key: B

Hint: $I = 4\int_0^{\pi} \dfrac{x\sin x}{1+\cos^2 x}dx = 4\int_0^{\pi} \dfrac{\left(\pi - x\right)\sin x}{1+\cos^2 x}dx$

$\therefore 2I = 4\pi\int_0^{\pi} \dfrac{\sin x}{1+\cos^2 x}dx = 8\pi\int_0^{\pi/2} \dfrac{\sin x}{1+\cos^2 x}dx$

$= 2\pi^2$

$\therefore I = \pi^2$

SMART ACHIEVERS LEARNING PVT. LTD.

64. Let $f:(0,\infty)\to R$ and F(x) = $\int_1^x f(t)dt$. If F(x²) = x² (1 + x) then f(4) equals

(A) 5/4 (B) 7 (C) 4 (D) 2

Key: C

Hint: $F'(x) = f(x)$

$F(x) = x(1+\sqrt{x}) = x + x^{3/2}$

$\therefore \; F'(x) = f(x) = 1 + \frac{3}{2}\sqrt{x}$

$\therefore$ f(4) = 4

65. If $f(x) = \int_0^x (1+t^3)^{-1/2} dt$ and g (x) is the inverse of f, then the value of $\frac{g''(x)}{g^2(x)}$ is

(A) 3/2 (B) 2/3 (C) 1/3 (D) 1/2

Key: A

Hint: $f(x) = \int_0^x (1+t^3)^{-1/2} dt$

i.e. $f[g(x)] = \int_0^{g(x)} (1+t^3)^{-1/2} dt$

i.e. $x = \int_0^{g(x)} (1+t^3)^{-1/2} dt$ [$\because$ g is inverse of f $\Rightarrow$ f[g(x)] = x]

Differentiating with respect to x, we have

$1 = (1+g^3)^{-1/2} . g'$

i.e. $(g')^2 = 1 + g^3$

Differentiating again with respect to x, we have

$2g'g'' = 3g^2g'$

gives $\frac{g''}{g^2} = \frac{3}{2}$

66. $\int_0^a \ln(\cot a + \tan x)dx$, where $a \in \left(0, \frac{\pi}{2}\right)$ is

a) a ln (sin a) b) – a ln (sin a) c) – a ln (cos a) d) none of these

Key: B

Hint: $I = \int_0^a \ln\frac{\cos(a-x)}{\sin a.\cos x}dx = \int_0^a \ln\frac{\cos x}{\sin a\cos(a-x)}dx$

adding $2I = \int_0^a \ln\frac{1}{\sin^2 a}dx = \int_0^a -2\ln\sin a \; dx = -2a\ln\sin a$

SMART ACHIEVERS LEARNING PVT. LTD.

67. If $f(x)=\int_1^x \frac{dt}{2+t^4}$, then

(A) $f(2)<\frac{1}{3}$ (B) $f(2)>\frac{1}{3}$

(C) $f(2)=\frac{1}{3}$ (D) $f(2)>1$

Key: A

Hint: $f'(x)=\frac{1}{2+x^4}$

By LMVT $f'(C)=\frac{f(2)-f(1)}{2-1}$ for some $c\in(1,2)$

$\Rightarrow f(2)=\frac{1}{2+c^4}$ as $f(1)=0$ $\Rightarrow 1<c<2 \Rightarrow 3<2+c^4<18 \Rightarrow f(2)<\frac{1}{3}$

68. Let $I_1=\int_0^{\pi/4} x^{2008}(\tan x)^{2008}dx$, $I_2=\int_0^{\pi/4} x^{2009}(\tan x)^{2009}dx$ and

$I_3=\int_0^{\pi/4} x^{2010}(\tan x)^{2010}dx$ Then the correct order sequence, is

(A) $I_2<I_3<I_1$ (B) $I_1<I_2<I_3$

(C) $I_3<I_1<I_2$ (D) $I_3<I_2<I_1$

Key: D

Hint:

69. .If f(x + y) = f(x) + f(y) + 2xy - 6 for all x, y$\in$R and f ′(0) = 2, then y = f(x) will be

(A) straight line (B) parabola

(C) ellipse (D) circle

Key: B

Hint: $f'(x)=\lim_{n\to 0}\frac{f(x+h)-f(x+0)}{h}$

$=\lim_{n\to 0}\frac{f(x)+f(h)+2xh-6-f(x)-f(0)+6}{h}$

$= 2x+\lim_{n\to 0}\frac{f(0+h)-f(0)}{h} = 2x+f'(0)$

f(x) = x^2 + 2x + C, but f(0) = 6

So, f(x) = x^2 + 2x + 6.

70. The value of $\int_3^6\left(\sqrt{x+\sqrt{12x-36}}+\sqrt{x-\sqrt{12x-36}}\right)dx$ is equal to

(A) $6\sqrt{3}$ (B) $4\sqrt{3}$ (C) $12\sqrt{3}$ (D) $2\sqrt{3}$

Key: A

SMART ACHIEVERS LEARNING PVT. LTD.

Hint: $\int_0^3 \left(\sqrt{(x+3)+2\sqrt{3}\sqrt{x}}+\sqrt{(x+3)-2\sqrt{3}\sqrt{x}}\right)dx$

$\int_0^3 \left(\left(\sqrt{x}+\sqrt{3}\right)+\left(\sqrt{3}-\sqrt{x}\right)\right)dx=\int_0^3 2\sqrt{3}\,dx=6\sqrt{3}$

71. Let $f:\left[0,\frac{\pi}{2}\right]\to[0,1]$ be a differentiable function such that $f(0)=0,\ f\left(\frac{\pi}{2}\right)=1$, then

(A) $f'(\alpha)=\sqrt{1-(f(\alpha))^2}$ for all $\alpha\in\left(0,\frac{\pi}{2}\right)$

(B) $f'(\alpha)=\frac{2}{\pi}$ for all $\alpha\in\left(0,\frac{\pi}{2}\right)$

(C) $f(\alpha)f'(\alpha)=\frac{1}{\pi}$ for at least one $\alpha\in\left(0,\frac{\pi}{2}\right)$

(D) $f'(\alpha)=\frac{8\alpha}{\pi^2}$ for at least one $\alpha\in\left(0,\frac{\pi}{2}\right)$

Key: A

Hint: Let : $\left[0,\frac{\pi}{2}\right]\to$ [0,1] be a ....

(A) Consider $g(x)=\sin^{-1}f(x)-x$

Since $g(0)=0, g\left(\frac{\pi}{2}\right)=0$

$\therefore$ There is at least one value of $\alpha\in\left(0,\frac{\pi}{2}\right)$ such that

$$g'(\alpha)=\frac{f'(\alpha)}{\sqrt{1-(f(\alpha))^2}}-1=0$$

i. e. $f'(\alpha)=\sqrt{1-(f(\alpha))^2}$ for atleast one value of α but may not be for all $\alpha\in\left(0,\frac{\pi}{2}\right)$

$\therefore$ flase

(B) Consider $g(x)=f(x)-\frac{2x}{\pi}$

Since $g(0)=0, g\left(\frac{\pi}{2}\right)=0$

$\therefore$ there is at least one value of $\alpha\in\left(0,\frac{\pi}{2}\right)$ such that

$$g'(\alpha)=f'(\alpha)-\frac{\pi}{2}=0$$

SMART ACHIEVERS LEARNING PVT. LTD.

i. e $f'(\alpha) = \frac{2}{\pi}$ for atleast one value of α but may not be for all $\alpha \in \left(0, \frac{\pi}{2}\right)$

$\therefore$ false

(C) Consider $g(x) = (f(x))^2 - \frac{2x}{\pi}$

Since $g(0) = 0, g\left(\frac{\pi}{2}\right) = 0$

$\therefore$ There is at least one value of $\alpha \in \left(0, \frac{\pi}{2}\right)$ such that

$g'(\alpha) = 2f(\alpha)f'(\alpha) - \frac{\pi}{2} = 0$

$\therefore f(\alpha)f'(\alpha) = \frac{1}{\pi}$

$\therefore$ True

(D) Consider $g(x) = f(x) - \frac{4x^2}{\pi^2}$

Since $g(0) = 0,\ g\left(\frac{\pi}{2}\right) = 0$

$\therefore$ there is at least one value of $\alpha \in \left(0, \frac{\pi}{2}\right)$ such that

$g'(\alpha) = f'(\alpha) - \frac{8\alpha}{\pi^2} = 0$

$\therefore f'(\alpha) = \frac{8\alpha}{\pi^2}$

$\therefore$ True

72. If x = a cos t, y = a sin t, then $\frac{d^2y}{dx^2}$ at t = $\frac{\pi}{4}$ is

(a) $\frac{a}{2\sqrt{2}}$ (b) $-\frac{a}{2\sqrt{2}}$ (c) $\frac{2\sqrt{2}}{a}$ (d) $-\frac{2\sqrt{2}}{a}$

Key: d

Hint: Clearly $x^2 + y^2 = a^2$ and $y(\pi/4) = a/\sqrt{2}$, $x(\pi/4) = a/\sqrt{2}$. Differentiating we get,

$2x + 2yy_1 = 0 \Rightarrow y_1 = -\frac{x}{y}$, so $y_1(\pi/4) = -1$.

Now $x + yy_1 = 0 \Rightarrow 1 + y_1^2 + yy_2 = 0$

$\Rightarrow y_2(\pi/4) = -\frac{1 + (y_1(\pi/4))^2}{y(\pi/4)} = \frac{-2\sqrt{2}}{a}$

73. If $z \neq 0$, then $\int_{x=0}^{100} \left[\arg|z|\right] dx$ is (where [.] denotes the greatest integer function)

SMART ACHIEVERS LEARNING PVT. LTD.

(A) 0 (B) 10
(C) 100 (D) not defined

Key : A

Sol : $\because |z|$ = real and positive, imaginary part is zero

$\therefore \ \arg|z| = 0$

$\Rightarrow \left[\arg|z|\right] = 0$

$\therefore \int_{x=0}^{100} \left[\arg|z|\right]dx = \int_{x=0}^{100} 0.dx = 0$

74. $\lim_{x \to \frac{\pi}{4}} \dfrac{\int_{2}^{\sec^2 x} f(t)\,dt}{x^2 - \frac{\pi^2}{16}}$ equals

(a) $\frac{8}{\pi} f(2)$ (b) $\frac{2}{\pi} f(2)$ (c) $\frac{2}{\pi} f\left(\frac{1}{2}\right)$ (d) $4f(2)$

Key: A

Hint: Required limits is of the form $\frac{0}{0}$, so it is equal to

$\lim_{x \to \frac{\pi}{4}} \dfrac{2\sec x \sec x \tan x f\left(\sec^2 x\right)}{2x} = \lim_{x \to \frac{\pi}{4}} \dfrac{\sec^2 x \tan x\ f\left(\sec^2 x\right)}{x} = \frac{8}{\pi} f(2)$

75. *Let* $f(x) = \int_0^1 |t - x| t\ dt$ for all real *x*. Then the minimum value of f is

a) $\frac{1}{2}$ b) $\frac{1}{3}\left(1 + \frac{1}{\sqrt{2}}\right)$ c) $\frac{1}{3}\left(1 - \frac{1}{\sqrt{2}}\right)$ d) $\frac{1}{6}$

Key: C

Hint: $f(x) = \begin{cases} \frac{1}{3} - \frac{x}{2} if \ \ x \le 0 \\ \frac{1}{3} + \frac{x^3}{3} - \frac{x}{2} if\ 0 \le x \le 1 \\ \frac{x}{2} - \frac{1}{3} if\ \ x \ge 1 \end{cases}$

f attains its minimum at x = $\frac{1}{\sqrt{2}}$

76. $\int \dfrac{(1 + \sqrt{\tan x})(1 + \tan^2 x)}{2\tan x} dx$ equal to

A) $\log \tan^2 x + \sqrt{\tan x} + c$ B) $\log \tan^2 x + \dfrac{1}{2\sqrt{\tan x}} + c$

SMART ACHIEVERS LEARNING PVT. LTD.

C) $\log|\tan x|+2\sqrt{\tan x}+c$ D) $\log|\tan|+\sqrt{\tan x}+c$

Key: A

Hint: $\int \frac{1}{2\sin x\cos x}dx+\frac{1}{2}\int\frac{\sqrt{\tan x}}{\sin x\cos x}dx$

$= \frac{1}{2}\int\frac{\sin^2 x+\cos^2 x}{\sin x\cos x}dx+\frac{1}{2}\int\frac{\sec^2 x}{\sqrt{\tan x}}dx = \log(\tan^2 x)+\sqrt{\tan x}+c$

77. $\int\frac{(2+\sec x)\sec x}{(1+2\sec x)^2}dx=$

a) $\frac{1}{2\text{cosec}x+\cot x}+C$ b) 2 cosec x +cot x+C c) $\frac{1}{2\text{cosec}x-\cot x}+C$ d) 2 cosec x – cot x + C

Key: A

Hint $I=\int\frac{(2\cos x+1)}{(2+\cos x)^2}dx=\int\frac{(2+\cos x)\cos x+\sin^2 x}{(2+\cos x)^2}dx$

$=\int\frac{\cos x}{2+\cos x}dx-\int\frac{-\sin^2 x}{(2+\cos x)^2}dx=\frac{\sin x}{2+\cos x}+c$

78. $\int\frac{\text{dx}}{(\text{x}-3)^{4/5}(\text{x}+1)^{6/5}}=$

A) $((\text{x}-3)(\text{x}+1))^{1/5}+\text{c}$ B) $\frac{5}{4}\left(\frac{\text{x}-3}{\text{x}+1}\right)^{1/5}+\text{c}$

C) $\left(\frac{\text{x}+1}{\text{x}-3}\right)^{1/5}+\text{c}$ D) $(\text{x}-3)^{6/5}(\text{x}+1)^{4/5}+\text{c}$

Key: B

Hint Put $\text{t}=\frac{\text{x}-3}{\text{x}+1}$

$\Rightarrow \text{dx}=\frac{(\text{x}+1)^2\,\text{dt}}{4}$

79. If the system of linear equations $x+y+z=6, x+2y+3z=14$ and $2x+5y+\lambda z=\mu$, $(\lambda,\mu\in R)$ has no solution, then

a) $\lambda\neq 8$ b) $\lambda=8,\mu\neq 36$ c) $\lambda=8,\mu=36$ d) None of these

Key: B

Hint $\begin{bmatrix}1 & 1 & 1 & 6\\ 1 & 2 & 3 & 14\\ 2 & 5 & \lambda & \mu\end{bmatrix}$ $R_2\to R_2-R_1$; $\text{R}_3\to R_3-2R_1$

SMART ACHIEVERS LEARNING PVT. LTD.

- $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & \lambda-2 & \mu-12 \end{bmatrix} R_3 \to R_3 - 3R_2$

- $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & \lambda-8 & \mu-36 \end{bmatrix}$

$\lambda - 8 = 0$ & $\mu - 36 \neq 0$

80. A man starts from the point P(3,-3) and reaches the point Q(0,2) after touching the line 2x+y=7 at R. The least value of PR+RQ is

a) $5\sqrt{2}$ b) $3\sqrt{2}$ c) $7\sqrt{2}$ d) $2\sqrt{2}$

Key: A

Hint P, Q lies on same side of the line find image of P w.r.t. line

81. Let $S(x) = \int \frac{dx}{e^x + 8e^{-x} + 4e^{-3x}}$, $R(x) = \int \frac{dx}{e^{3x} + 8e^{x} + 4e^{-x}}$ and $M(x) = S(x) - 2R(x)$. If $M(x) = \frac{1}{2}\tan^{-1}\left(f(x)\right) + c$ where c is an arbitrary constant then $f\left(\log_e 2\right) =$

A) $\frac{3}{2}$ B) $\frac{1}{2}$ C) $\frac{5}{2}$ D) $\frac{7}{2}$

Key: A

Hint: $M(x) = \int \frac{e^x\left(e^{2x} - 2\right)}{e^{4x} + 8e^{2x} + 4}dx \quad e^x = t \Rightarrow \int \frac{\left(t^2 - 2\right)dt}{t^4 + 8t^2 + 4} = \frac{1}{2}\tan^{-1}\left(\frac{t + 2/t}{2}\right) + c$

$= \frac{1}{2}\tan^{-1}\left(\frac{e^x + 2e^{-x}}{2}\right) + c$

82. If $\int\left(2 - 3\sin^2 x\right)\sqrt{\sec x}\,dx = 2f(x)\sqrt{g(x)} + c$ and f(x) is non constant function then

(A) $f^2(x) + g^2(x) = 1$ (B) $f^2(x) - g^2(x) = 1$ (C) $f(x)g(x) = 1$ (D) $f(x) = g(x)$

Key: A

Hint:

$\int \frac{2 - 3\sin^2 x}{\sqrt{\cos x}}dx = \int \frac{2\cos^2 x - \sin^2 x}{\sqrt{\cos x}}dx = 2\int(\cos x)\sqrt{\cos x}dx - \int \frac{\sin^2 x}{\sqrt{\cos x}}dx$

$= 2\sin x\sqrt{\cos x} \quad + c$

$\Rightarrow f(x) = \sin x, g(x) = \cos x$

83. $\int \frac{e^{\cot x}}{\sin^2 x}\left(2\ln \operatorname{cosec} x + \sin 2x\right)dx =$

SMART ACHIEVERS LEARNING PVT. LTD.

a) $-2e^{\cot x}\ln(\cos ecx)+c$ b) $e^{\cot x}\ln x+c$

c) $e^{\cot x}\ln(\cos ecx)+c$ d) $e^{\cot x}\ln(\sin x)+c$

Key: A

Hint:
$$t=\cot x\Rightarrow -\cos ec^2 xdx=dt\Rightarrow dx=\frac{-1}{1+t^2}dt$$
$$I=-\int e^t\left(\ln(1+t^2)+\frac{2t}{1+t^2}\right)dt=-e^t\ln(1+t^2)+c=-2e^{\cot x}\ln\cos ecx+c$$

84. If $\int\frac{\sec^2 x-2010}{\sin^{2010}x}dx=\frac{P(x)}{\sin^{2010}x}+C$, then value of $P\left(\frac{\pi}{3}\right)$ is

(A) 0 (B) $\frac{1}{\sqrt{3}}$

(C) $\sqrt{3}$ (D) None of these

Key: C

Hint: $\int\frac{\sec^2 x-2010}{\sin^{2010}x}dx$

$$=\int\sec^2 x(\sin x)^{-2010}-2010\int\frac{1}{(\sin x)^{2010}}dx=I_1-I_2$$

Applying by parts on I_1, we get

$$I_1=\frac{\tan x}{(\sin x)^{2010}}+2010\int\frac{\tan x\cos x}{(\sin x)^{2011}}dx=\frac{\tan x}{(\sin x)^{2010}}+2010\int\frac{dx}{(\sin x)^{2010}}$$

$$\Rightarrow I=I_1-I_2=\frac{\tan x}{(\sin x)^{2010}}=\frac{P(x)}{(\sin x)^{2010}}$$

$$P\left(\frac{\pi}{3}\right)=\tan\frac{\pi}{3}=\sqrt{3}$$

85. If $I=\int\frac{dx}{\sin\left(x-\frac{\pi}{3}\right)\cos x}$, then I equals

a) 2 log $\left|\sin x+\sin\left(x-\frac{\pi}{3}\right)\right|+C$ b) $2\log\left|\sin\left(x-\frac{\pi}{3}\right)\sec x\right|+C$

c) $2\log\left|\sin x-\sin\left(x-\frac{\pi}{3}\right)\right|+C$ d) None of these

Key: B

SMART ACHIEVERS LEARNING PVT. LTD.

Hint: $I=\frac{1}{\cos\left(\frac{\pi}{3}\right)}\int\frac{\cos\left(x-\left(x-\frac{\pi}{3}\right)\right)}{\sin\left(x-\frac{\pi}{3}\right)\cos x}dx$

86. $\int_0^1 \frac{x^6-x^3}{(2x^3+1)^3}dx$ is equal to

(A) 0 (B) $-\frac{1}{6}$

(C) $-\frac{1}{12}$ (D) $-\frac{1}{36}$

Key. D

Sol. $\int_0^1 \frac{\left(1-\frac{1}{x^3}\right)}{\left(2x+\frac{1}{x^2}\right)^3}dx$

and proceed

87. Suppose f, g are continuous and differentiable on [0,b], f', g' are non-negative on [0,b] and f is non constant with f(0) = 0, then the minimum value of $\int_0^a g(x)f'(x)dx+\int_0^b g'(x)f(x)dx$ on $a\in(0,b]$ is

(A) $f(a).g(a)$ (B) $f(b).g(b)$

(C) $f(a).g(b)$ (D) $f(b).g(a)$

Key. C

Sol. Integrating by parts the integral

$\int_0^a g(x)f'(x)dx+\int_0^b g'(x)f(x)dx$

$=g(x)f(x)\Big|_0^a-\int_0^a g'(x)f(x)dx\ +\int_0^b g'(x)f(x)dx$

$=f(a).g(a)+\int_a^b g'(x)f(x)dx$

$\geq f(a).g(a)+\int_a^b g'(x)f(x)dx=f(a).g(b)$

88. If f(x) be a real valued function, $f(x)+f(x+4)=f(x+2)+f(x+6)$, $g(x)=\int_x^{x+8}f(t)dt$. Then $g^1(x)$ is equal to

a) $f(x)$ b) $f(x+8)$ c) 8 d) 0

Key. D

Sol. Conceptual

89. Value of $\int_1^5\left(\sqrt{x+2\sqrt{x-1}}+\sqrt{x-2\sqrt{(x-1)}}\right)dx$ is

a) $\frac{8}{3}$ b) $\frac{16}{3}$ c) $\frac{32}{3}$ d) $\frac{34}{3}$

Key. D

Sol. $Q\ \sqrt{x+\left(2\sqrt{(x-1)}\right)}=\sqrt{\sqrt{(x-1)}^2+1^2+2\sqrt{(x-1)}}$

$$=\sqrt{(x-1)}+1$$

And $\sqrt{x-2\sqrt{(x-1)}}=\sqrt{\left(\sqrt{(x-1)}\right)^2+1^2-2\sqrt{(x-1)}}$

$$=\left|\sqrt{(x-1)}-1\right|$$

Then $\int_1^5\sqrt{x+2\sqrt{(x-1)}}+\sqrt{x-2\sqrt{(x-1)}}dx$

$$=\int_1^5\left(\sqrt{(x-1)}+1\right)+\int_1^5\left|\sqrt{(x-1)}-1\right|dx$$

$$=\int_1^5\left(\sqrt{(x-1)}+1\right)dx+\int_1^2\left(1-\sqrt{(x-1)}\right)dx+\int_2^5\left(\sqrt{(x-1)}-1\right)dx$$

$$=\int_0^4\left(\sqrt{x}+1\right)dx+\int_0^1\left(1-\sqrt{x}\right)dx+\int_1^4\left(\sqrt{x}-1\right)dx$$

$$=\left[\frac{2}{3}\left(x^{3/2}\right)+x\right]_0^4+\left[x-\frac{2}{3}x^{3/2}\right]_0^1+\left[\frac{2}{3}x^{3/2}-x\right]_1^4$$

$$=\left(\frac{16}{3}+4\right)+\left(1-\frac{2}{3}\right)+\left(\frac{16}{3}-4\right)+\left(\frac{2}{3}-1\right)=\frac{32}{3}$$

90. $\int_{-\pi/4}^{\pi/4}\frac{e^x.\sec^2 xdx}{e^{2x}-1}$ is equal to

a) 0 b) 2 c) e d) 2e

Key. A

Sol. Let $I=\int_{-\pi/4}^{\pi/4}\frac{e^x\sec^2 xdx}{e^{2x}-1}$

If $f(x)=\frac{e^x\sec^2 x}{e^{2x}-1}$

$\therefore$ $f(-x)=\frac{e^x\sec^2 x}{e^{2x}-1}$

$$=\frac{e^x\sec^2 x}{1-e^{2x}}$$

$$=-\frac{e^x\sec^2 x}{e^{2x}-1}$$

SMART ACHIEVERS LEARNING PVT. LTD.

$= -f(x)$

$\therefore$ I = 0 (f(x) is odd function)

91. Let $f(x) = \frac{1}{2}a_0 + \sum_{i=1}^{n} a_i \cos(ix) + \sum_{j=1}^{n} b_j \sin(jx)$, then $\int_{-\pi}^{\pi} f(x)\cos kx dx$ is equal to

a) a_k b) b_k c) πa_k d) πb_k

Key. C

Sol. Conceptual

92. Let $\int_0^x \left(\frac{bt\cos 4t - a\sin 4t}{t^2}\right)dt = \frac{a\sin 4x}{x}$, then a and b are given by

a) 1/4, 1 b) 2,2 c) -1, 4 d) 2, 4

Key. A

Sol. Since, $\int_0^x \left(\frac{bt\cos 4t - a\sin 4t}{t^2}\right)dt = \frac{a\sin 4x}{x}$

Differentiating both sides w.r.t. x

$\therefore \frac{bx\cos 4x - a\sin 4x}{x^2} = \frac{a\{4x\cos 4x - \sin 4x\}}{x^2}$

On comparing $b = 4a$

$a = 1/4$ and $b = 1$

93. If $f'''(x) = k$ in $[0,a]$, then $\int_0^a f(x)dx - \left\{xf(x) - \frac{x^2}{2!}f'(x) + \frac{x^3}{3!}f''(x)\right\}_0^a$ is

a) $-ka^4/12$ b) $ka^4/24$ c) $-ka^4/24$ d) $ka^4/12$

Key. C

Sol. Conceptual

94. If $f(x)$ is a differentiable function and $\int_0^{x^3} t^2 f(t)dt = \frac{3}{13}x^{13} + 5$ then $f\left(\frac{8}{27}\right) =$

A) 8 / 27 B) 16 / 27 C) 16 / 81 D) 8 / 9

Key. C

Sol. Diff. w.r.t. $x \Rightarrow f(x^3) = x^4$

95. $\int_{-2\pi}^{2\pi} \frac{\sin^6 x}{(\sin^6 x + \cos^6 x)(1+e^{-x})}dx =$

SMART ACHIEVERS LEARNING PVT. LTD.

A) 2π B) π C) $\pi/2$ D) 4π

Key. B

Sol. $\int_{-a}^{a} f(x)dx = \int_{0}^{a}\left(f(x)+f(-x)\right)dx$

96. Let $f(x), g(x), h(x)$ be continous in $[0, 2a]$ and satisfies

$f(2a-x)=f(x), g(2a-x)=g(x), h(x)+h(2a-x)=3,\ f(2a-x)g(2a-x)=f(x)g(x)$

then $\int_{0}^{2a} f(x)g(x)h(x)dx =$

A) $\int_{0}^{2a} f(x)g(x)dx$ B) $3\int_{0}^{a} f(x)g(x)dx$ C) $2\int_{0}^{a} f(x)g(x)dx$ D) $\int_{0}^{a} f(x)g(x)dx$

Key. B

Sol. $I = \int_{0}^{2a} f(x)g(x)h(x)dx = \int_{0}^{2a} f(2a-x)g(2a-x)h(2a-x)dx = \int_{0}^{2a} f(x)g(x)\left[3-h(x)\right]dx$

$I = \frac{3}{2}\int_{0}^{2a} f(x)g(x)dx = 3\int_{0}^{a} f(x)g(x)dx$

97. $\int \frac{\sqrt[3]{x^2}+\sqrt[6]{x}}{x\left(1+\sqrt[3]{x}\right)}dx =$

A) $\frac{1}{2}\tan^{-1}\left(\frac{x^6+x^{-6}}{2}\right)+C$ B) $\frac{x^{24}}{2}-\log(1+x^{24})+\tan^{-1}(x^3)+C$

C) $\frac{3}{2}x^{12}+6\tan^{-1}(x^6)+C$ D) $6\tan^{-1}(x^6)+3x^{12}-6\log_e\sqrt{1+x^{12}}+C$

Key. D

Sol. $x = t^6$

98. $\int \frac{dx}{x^{20}(1+x^{20})^{\frac{1}{20}}} =$

A) $-\frac{1}{19}\left(1+\frac{1}{x^{20}}\right)^{\frac{19}{20}}+C$ B) $\frac{1}{21}\left(1-\frac{1}{x^{20}}\right)^{\frac{19}{20}}+C$

C) $\frac{1}{19}\left(1-\frac{1}{x^{20}}\right)^{\frac{19}{20}}+C$ D) $-\frac{1}{21}\left(1+\frac{1}{x^{20}}\right)^{\frac{19}{20}}+C$

Key. A

SMART ACHIEVERS LEARNING PVT. LTD.

Sol. $\int \frac{dx}{x^{21}\left(\frac{1}{x^{20}}+1\right)^{\frac{1}{20}}}$, Put $\frac{1}{x^{20}}+1=t$

99. $\int \frac{\sin\left(\frac{\pi}{4}-x\right)dx}{2+\sin 2x} = A\tan^{-1}\left(f(x)\right)+B$, where A, B are constants then the range of $Af(x)$ is

A) $[-1,1]$ B) $[-\sqrt{2},\sqrt{2}]$ C) $[0,1]$ D) $[-1,0]$

Key. A

Sol. $\frac{1}{\sqrt{2}}\int \frac{d(\sin x+\cos x)}{(\sin x+\cos x)^2+1}$

100. The area bounded by the curves $y=2-|x-1|,\ y=\sin x;\ x=0$ and $x=2$ is

A) $1+2\cos^2 1$ B) $2+\sin^2 1$ C) $\frac{\pi}{2}$ D) $1+\log 2$

Key. A

Sol. Area $=\frac{3}{2}+(\cos 1-1)+3-\frac{3}{2}+(\cos 2-\cos 1)=2+\cos 2$

101. $\int_0^{\pi/2} \frac{\sin x}{1+\cos x+\sin x}dx =$

A) $\frac{\pi}{4}$ B) $\frac{\pi}{4}+\log\sqrt{2}$ C) $\frac{\pi}{4}-\log\sqrt{2}$ D) $\frac{\pi}{4}-\log 2$

Key. C

Sol. $I=\int_0^{\pi/2} \frac{\sin\frac{x}{2}}{\sin\frac{x}{2}+\cos\frac{x}{2}}dx = 2\int_0^{\pi/4} \frac{\sin x}{\sin x+\cos x}dx$

Let $\sin x = A(\sin x+\cos x)+B(\cos x-\sin x)$ $A-B=1$ and $A+B=0,\ A=\frac{1}{2},\ B=\frac{-1}{2}$

$I=2\times\frac{1}{2}\times\frac{\pi}{4}-2\times\frac{1}{2}\left[\log(\sin x+\cos x)\right]_0^{\pi/4}=\frac{\pi}{4}-\log\sqrt{2}$

SMART ACHIEVERS LEARNING PVT. LTD.

102. $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^3+6k^2+11k+5}{(k+3)!}=$

A) $\frac{2}{3}$ B) 1 C) $\frac{5}{3}$ D) $\frac{7}{3}$

Key. C

Sol. $k^3+6k^2+11k+5=(k+1)(k+2)(k+3)-1$

$\therefore \frac{k^3+6k^2+11k+5}{(k+3)!}=\frac{1}{k!}-\frac{1}{(k+3)!}$

103. $I_n=\int_0^{\pi/2}\cos^n x\cos(nx)\,dx,\ n\in N$ then $\sqrt{I_{2001}:I_{2002}}$ can be the eccentricity of

A) Parabola B) Ellipse C) Circle D) Hyperbola

Key. D

Sol. $I_{n+1}=\int_0^{\pi/2}\cos^{n+1}x\cos(n+1)x\,dx$

$=\int_0^{\pi/2}\cos^{n+1}(\cos nx\cos x-\sin nx\sin x)\,dx$

$=\int_0^{\pi/2}\cos^n x\cos nx\left(1-\sin^2 x\right)dx-\int_0^{\pi/2}\cos^{n+1}x.\sin nx.\sin x\,dx$

$\therefore I_{n+1}=I_n-I_{n+1}\Rightarrow 2I_{n+1}=I_n$

$\therefore I_n:I_{n+1}=2$

104. If $\int\frac{x\,dx}{\sqrt[2012]{\left(1+x^2\right)^{1012}\left(2+x^2\right)^{3012}}}=\frac{\alpha}{\beta}\left(1-f(x)\right)^{\frac{\beta}{2\alpha}}+k$ then which is true

A) $\alpha=503;\ \beta=500,\ f\left(\sqrt{2}\right)=\frac{1}{\beta-\alpha}$

B) $\alpha=503;\ \beta=250,\ f\left(\sqrt{2}\right)=\frac{1}{\alpha-\beta}$

C) $\alpha=503;\ \beta=500,\ f(1)=\frac{1}{\alpha-\beta}$

D) $\alpha=503;\ \beta=225,\ f\left(\sqrt{3}\right)=\frac{1}{\alpha-\beta}$

Key. C

SMART ACHIEVERS LEARNING PVT. LTD.

$$\int \frac{x}{\left(1+x^2\right)^2\left(\frac{2+x^2}{1+x^2}\right)^{\frac{3012}{2012}}}\,dx$$

Sol.

Let $\frac{2+x^2}{1+x^2}=t$ then $f(x)=\frac{1}{2+x^2}$ $f(1)=\frac{1}{3}=\frac{1}{\alpha-\beta}$

Where $\alpha=503:\beta=500$

105. $I_n=\int_0^1 x^n\tan^{-1}x\,dx.$ If $a_n\,I_{n+2}+b_n\,I_n=c_n\;\forall\; n\in N,\, n\ge 1$ then

A) $a_1,a_2,a_3.......$ are in A.P.

B) $b_1,b_2,b_3.......$ are in G.P.

C) $c_1,c_2,c_3.......$ are in H.P.

D) $a_1,a_2,a_3.......$ are in H.P.

Key. A

Sol. $I_n=\left(\frac{x^{n+1}}{n+1}\tan^{-1}x\right)_0^1-\int_0^1\frac{x^{n+1}}{n+1}.\frac{1}{1+x^2}dx$

$(n+1)I_n=\frac{\pi}{4}-\int_0^1\frac{x^{n+1}}{1+x^2}dx$

$(n+3)I_{n+2}=\frac{\pi}{4}-\int_0^1\frac{x^{n+3}}{1+x^2}dx$

$\therefore\;(n+1)I_n+(n+3)I_{n+2}=\frac{\pi}{2}-\frac{1}{n+2}$

$\therefore\;a_n=(n+3)\Rightarrow a_1,a_2,a_3......$ are in A.P.

$b_n=(n+1)\Rightarrow b_1,b_2.....$ are in A.P.

$c_n=\frac{\pi}{2}-\frac{1}{n+2}$ not in any progression.

106. $\int_0^1\frac{x^6-x^3}{(2x^3+1)^3}dx$ is equal to

(A) 0

(B) $-\frac{1}{6}$

(C) $-\frac{1}{12}$

(D) $-\frac{1}{36}$

SMART ACHIEVERS LEARNING PVT. LTD.

Key. D

Sol. $\int_0^1 \frac{\left(1-\frac{1}{x^3}\right)}{\left(2x+\frac{1}{x^2}\right)^3}dx$

and proceed

107. Suppose f, g are continuous and differentiable on [0,b], f', g' are non-negative on [0,b] and f is non constant with f(0) = 0, then the minimum value of $\int_0^a g(x)f'(x)dx + \int_0^b g'(x)f(x)dx$ on $a \in (0,b]$ is

(A) $f(a).g(a)$ (B) $f(b).g(b)$

(C) $f(a).g(b)$ (D) $f(b).g(a)$

Key. C

Sol. Integrating by parts the integral

$$\int_0^a g(x)f'(x)dx + \int_0^b g'(x)f(x)dx$$

$$= g(x)f(x)\Big|_0^a - \int_0^a g'(x)f(x)dx + \int_0^b g'(x)f(x)dx$$

$$= f(a).g(a) + \int_a^b g'(x)f(x)dx$$

$$\geq f(a).g(a) + \int_a^b g'(x)f(x)dx = f(a).g(b)$$

108. Let $I_1 = \int_{\sec^2 z}^{2-\tan^2 z} x\, f\left(x(3-x)\right)dx$ and let $I_2 = \int_{\sec^2 z}^{2-\tan^2 z} f\left(x(3-x)\right)dx$

where 'f' is a continuous function and 'z' is any real number, then $\frac{I_1}{I_2} =$

A) $\frac{3}{2}$ B) $\frac{1}{2}$ C) 1 D) $\frac{2}{3}$

Key. A

Sol.

$$I_1 = \int_{\sec^2 z}^{2-\tan^2 z} x\, f\left(x(3-x)\right)dx$$

$$I_2 = \int_{\sec^2 z}^{2-\tan^2 z} f\left(x(3-x)\right)dx$$

SMART ACHIEVERS LEARNING PVT. LTD.

$$2I_1 = \int_{\sec^2 z}^{2-\tan^2 z} (3-x) f((3-x)x)\,dx$$

$$2I_1 = \int_{\sec^2 z}^{2-\tan^2 z} 3f(x(3-x))\,dx$$

$$= 3\int_{\sec^2 z}^{2-\tan^2 z} ((x)(3-x))\,dx$$

$$2I_1 = 3I_2$$

$$\Rightarrow \frac{I_1}{I_2} = \frac{3}{2}$$

109. If $\int \frac{\sec^2 x - 2010}{\sin^{2010} x}dx = \frac{P(x)}{\sin^{2010} x} + C,$ then value of $P\left(\frac{\pi}{3}\right)$ is

A) 0 B) $\frac{1}{\sqrt{3}}$ C) $\sqrt{3}$ D) None of these

Key. C

Sol. $\int \frac{\sec^2 x - 2010}{\sin^{2010} x}dx$

$$= \int \sec^2 x(\sin x)^{-2010} - 2010\int \frac{1}{(\sin x)^{2010}}dx = I_1 - I_2$$

Applying, by parts on I_1, we get

$$I_1 = \frac{\tan x}{(\sin x)^{2010}} + 2010\int \frac{\tan x \cos x}{(\sin x)^{2011}}dx = \frac{\tan x}{(\sin x)^{2010}} + 2010\int \frac{dx}{(\sin x)^{2010}}$$

$$\Rightarrow I = I_1 - I_2 = \frac{\tan x}{(\sin x)^{2010}} = \frac{P(x)}{(\sin x)^{2010}}$$

$$P\left(\frac{\pi}{3}\right) = \tan\frac{\pi}{3} = \sqrt{3}$$

110. If $c > 0$ and the area of the region enclosed by the parabolas $y = x^2 - c^2$ *and* $y = c^2 - x^2$ is 576, then c =

a) 6 b) 4 c) 3 d) 8

Key: A

Hint: Area between the two parabolas = $4\int_0^C (c^2 - x^2)dx = \frac{8c^3}{3} = 576$

SMART ACHIEVERS LEARNING PVT. LTD.

only if c = 6

111. Let $f(x) = x^2 + 6x + 1$ and let R denote the set of points (x, y) in the XY-plane such that $f(x) + f(y) \le 0$ and $f(x) - f(y) \le 0$. Then the area of the region R is

A) 6π B) $3\pi + 2$ C) $2\pi + 8$ D) 8π

Key: D

Hint: $f(x) + f(y) \le 0 \Rightarrow (x+3)^2 + (y+3)^2 \le 16$

$f(x) - f(y) \le 0 \Rightarrow (x-y)(x+y+6) \le 0$

112. The quadrilateral formed by the lines $y = ax + c, y = ax + d, y = bx + c$ and $y = bx + d$ has area 18. The quadrilateral formed by the lines $y = ax + c, y = ax - d, y = bx + c$ and $y = bx - d$ has area 72. If a, b, c, d are positive integers then the least possible value of the sum $a + b + c + d$ is

A) 13 B) 14 C) 15 D) 16

Key: D

Hint: $\frac{(c-d)^2}{|a-b|} = 18$ and $\frac{(c+d)^2}{|a-b|} = 72$.

$a = 3, b = 1, d = 3, c = 9$ is a solution for which the minimum is attained.

113. Area of a square ABCD is 36 and side AB is parallel to the X-axis. Vertices A, B and C lie on the graphs of $y = \log_a x, y = 2\log_a x$ and $y = 3\log_a x$ respectively. Then $a =$

A) $3^{1/6}$ B) $\sqrt{3}$ C) $6^{1/3}$ D) $\sqrt{6}$

Key: A

Hint: Let $A = (p, \log_a p); B = (q, 2\log_a q),\ p, q > 0 \,\&\, a > 0, a \ne 1$, then $C = (q, 3\log_a q)$

$AB\ P\ X - \text{axis} \Rightarrow p = q^2 . |AB| = 6 \Rightarrow |p - q| = 6$ also $|\log_a q| = 6$.

114. The area bounded by the curve $\left(y - \text{Sin}^{-1}x\right)^2 = x - x^2$ s

a) $\frac{\pi}{4}$ b) $\frac{\pi}{2}$ c) π d) $\frac{\pi}{3}$

Key: A

Hint: The given curves on $y = \text{Sin}^{-1}x - \left(\sqrt{x - x^2}\right)$ & $y = \text{Sin}^{-1}x + \left(\sqrt{x - x^2}\right)$

$$\text{Requred area} = \int_0^1 2\sqrt{x - x^2}\,dx$$

115. Let F(x) = $\sin x \int_0^x \cos t\,dt + 2\int_0^x t\,dt + \cos^2 x - x^2$. Then area bounded by xF(x) and ordinate x = 0 and x = 5 with x-axis is

SMART ACHIEVERS LEARNING PVT. LTD.

(A) 16 (B) $\frac{25}{2}$ (C) $\frac{35}{2}$ (D) 25

Key: B

Hint: $F(x) = \sin x \int_0^x \cos t\, dt + 2\int_0^x t dt - x^2 + \cos^2 x$

$= \sin x (\sin t)_0^x + 2\left(\frac{t^2}{2}\right)_0^x - x^2 + \cos^2 x$

$= \sin^2 x + x^2 - x^2 + \cos^2 x = 1$

$A = \int_0^5 xF(x)dx = \int_0^5 (x)(1)dx = \left[\frac{x^2}{2}\right]_0^5 = \frac{25}{2}$

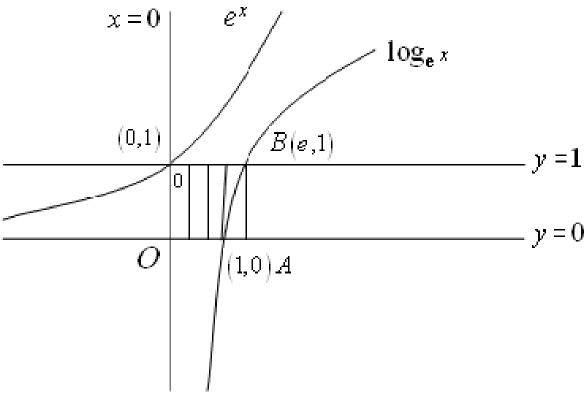
116. Area bounded by the curves $y = e^x$, $y = \log_e x$ and the lines $x = 0$, $y = 0$, $y = 1$ is

A) $e^2 + 2 sq.units$ B) $e + 1 sq.unit$

C) $e + 2 sq.units$ D) $e - 1\, sq.unit$

Key: D

Hint:



Area = Area of rectangle OABC $-\int_0^1 \log_e x$

117. The area of the loop of the curve $y^2 = x^4(x+2)$ is [in square units]

(A) $\frac{32\sqrt{2}}{105}$ (B) $\frac{64\sqrt{2}}{105}$ (C) $\frac{128\sqrt{2}}{105}$ (D) $\frac{256\sqrt{2}}{105}$

Key: D

Hint: Area = $2\int_{-2}^{0} y dx = 2\int_{-2}^{0} x^2\sqrt{x+2}dx = 4\sqrt{2}\int_0^2 \left(z^2-2\right)^2 z^2 dz (where \sqrt{x+2} = z)$

$= 4\left[\frac{z^7}{7} - \frac{4z^5}{5} + \frac{4z^3}{3}\right]_0^{\sqrt{2}}$

SMART ACHIEVERS LEARNING PVT. LTD.

$= \frac{256\sqrt{2}}{105}$

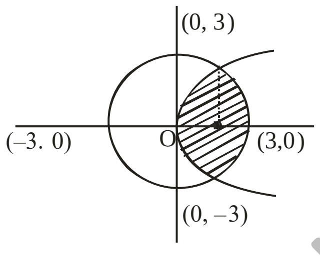
118. The area of the smaller portion enclosed by the curves $x^2+y^2=9$ and $y^2=8x$ is

A) $\frac{\sqrt{2}}{3}+\frac{9\pi}{4}-\frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)$ B) $2\left(\frac{\sqrt{2}}{3}+\frac{9\pi}{4}-\frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right)$

C) $2\left(\frac{\sqrt{2}}{3}+\frac{9\pi}{4}+\frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right)$ D) $\frac{\sqrt{2}}{3}+\frac{9\pi}{4}+\frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)$

Key: B

HINT :



$x^2+y^2=9,$

$x^2+8x-9=0$

$x=\frac{-8\pm\sqrt{64+36}}{2}$

$x=\frac{-8\pm10}{2}-9,1$

$\boxed{x=1}$

Area enclosed $=2\left[\int_0^1 2\sqrt{2x}dx+\int_1^3\sqrt{9-x^2}dx\right]=2\left[2\sqrt{2}\int_0^1\sqrt{x}dx+\int_1^3\sqrt{9-x^2}dx\right]$

On simplifying we get

$=2\left[\frac{\sqrt{2}}{3}+\frac{9\pi}{4}-\frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right]$

119. The area of the region in the xy-plane defined by the inequalities $x-2y^2\geq 0$, $1-x-|y|\geq 0$ is

A) $\frac{1}{2}$ B) $\frac{1}{3}$ C) $\frac{1}{4}$ D) $\frac{7}{12}$

Key: D

Hint: Area $=2\int_0^{1/2}\sqrt{\frac{x}{2}}dx+\frac{1}{4}=\frac{7}{12}$

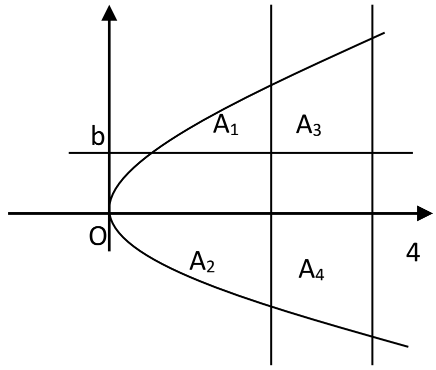
120. Area bounded by curve $y^2 = x$ and $x = 4$ is divided into 4 equal parts by the lines $x = a$ and $y = b$ then.

a) Area of each part = $\frac{8}{3}$ b) b = 0

SMART ACHIEVERS LEARNING PVT. LTD.

c) $a=\sqrt{2}$ d) $a=(16)^{1/3}$

Key: D



Hint: Total area = $2\int_0^4 \sqrt{x}\,dx=\frac{32}{3}$

Area of each part = 8/3

$A_3=A_4 \Rightarrow \int_a^4 \left(\sqrt{x}-b\right)dx=$

$\int_a^4 \left(b+\sqrt{x}\right)dx=\frac{8}{3} \Rightarrow b=0$

$\int_a^4 \sqrt{x}\,dx=\frac{8}{3} \Rightarrow a^3=16$

121. Area of the region in which point $p(x,y)$, $\{x>0\}$ lies; such that $y \le \sqrt{16-x^2}$ and $\left|\tan^{-1}\left(\frac{y}{x}\right)\right| \le \frac{\pi}{3}$ is

(A) $\left(\frac{16}{3}\pi\right)$ (B) $\left(\frac{8\pi}{3}+8\sqrt{3}\right)$ (C) $\left(4\sqrt{3}-\pi\right)$ (D) $\left(\sqrt{3}-\pi\right)$

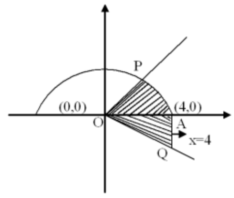
Key: B

Hint: Required area is the area of shaded region (APOQ)

= area of ΔOAQ + area of sector (OAP)

$=\frac{1}{2}\times 4\times 4\sqrt{3}+\frac{\pi(4\times 4)}{6}$

$=\left(\frac{8\pi}{3}+8\sqrt{3}\right)$



122. Area bounded between the curves $y=\sqrt{4-x^2}$ and $y^2=3|x|$ is/are

(A) $\frac{\pi-1}{\sqrt{3}}$ (B) $\frac{2\pi-1}{3\sqrt{3}}$ (C) $\frac{2\pi-\sqrt{3}}{3}$ (D) $\frac{2\pi-\sqrt{3}}{3\sqrt{3}}$

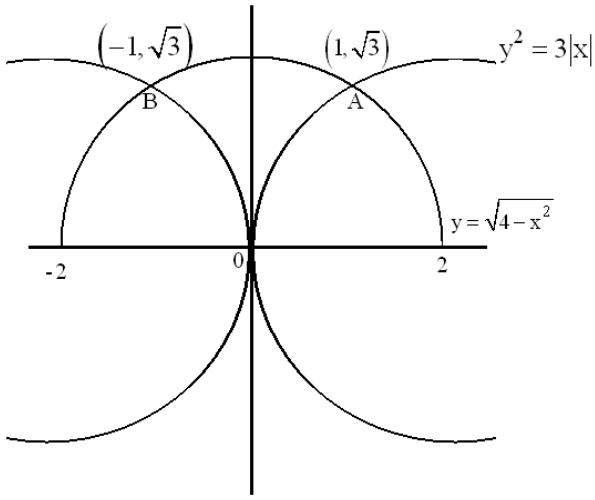
Key: C

Hint: Required area = $2\int_0^1 \left(\sqrt{4-x^2}-\sqrt{3x}\right)dx$

$=2\left(\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)-\frac{\sqrt{3}.2x^{3/2}}{3}\right)_0^1$

$=\frac{2\pi-\sqrt{3}}{3}$

SMART ACHIEVERS LEARNING PVT. LTD.



123. Let $f:[0,\infty)\to R$ be a continuous and strictly increasing function such that $f^3(x)=\int_0^x t.f^2(t)dt, \forall x>0$. The area enclosed by $y=f(x)$, the x-axis and the ordinate at x=3, is

(A) 1 (B) $\frac{3}{2}$ (C) 2 (D) 3

Key: B

Hint: $f(x)=\frac{x^2}{6}$

$A=\frac{1}{6}\int_0^3 x^2 dx = 3/2$

124. Let f(x) = x + sin x. The area bounded by $y=f^{-1}(x), y=x,\ x\in[0,\pi]$ is

(a) 1 (b) 2 (c) 3

(d) canot be found because $f^{-1}(x)$ cannot be determined

Key: B

Hint: The curves given by y = x + sin x and $y = f^{-1}(x)$ are images of each other in the line y = x.

Hence required area = $\int_0^{\pi}((x+\sin x)-x)dx = -[\cos x]_0^{\pi} = 2$

125. The area of the region bounded by the curves $|x + y| \le 2$, $|x - y| \le 2$ and $2x^2 + 6y^2 \ge 3$ is

(A) $\left(8+\frac{\sqrt{3}}{2}\pi\right)$ sq. units (B) $\left(8-\frac{\sqrt{3}}{2}\pi\right)$ sq. units

(C) $\left(4-\frac{3\sqrt{3}}{2}\pi\right)$ sq. units (D) $\left(8-\frac{3\sqrt{3}}{2}\pi\right)$ sq. units

Key : B

Sol :

$2x^2 + 6y^2 \ge 3$ . . . . (1)

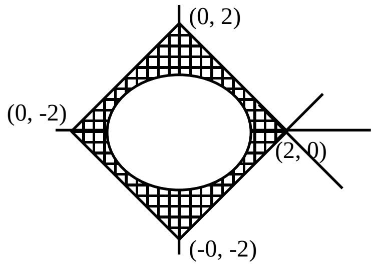
SMART ACHIEVERS LEARNING PVT. LTD.

area of ellipse = $\pi\times\frac{\sqrt{3}}{\sqrt{2}}\times\frac{1}{\sqrt{2}}=\frac{\pi\sqrt{3}}{2}$

$|x + y| \le 2 \Rightarrow -2 \le (x + y) \le 2$ . . . .(2)

$|x - y| \le 2 \Rightarrow -2 \le (x - y) \le 2$ . . . . (3)

Required area = $\left(8-\frac{\pi\sqrt{3}}{2}\right)$ sq. units



126. The curve $y=(|x|-1)\text{sgn}(x-1)$ divides $\frac{9x^2}{64}+\frac{4}{25}y^2=\frac{1}{\pi}$ in two parts having area A_1 and A_2 (where $A_1 < A_2$), then

a) $\frac{A_1}{A_2}=\frac{7}{13}$ b) $\frac{A_1}{A_2}=\frac{3}{7}$

c) $A_1=\frac{7}{3}$ d) $A_2=\frac{13}{7}$

Key: A

Sol : $A_1=\frac{10}{3}-1, A_2=\frac{10}{3}+1 \Rightarrow \frac{A_1}{A_2}=\frac{7}{13}$

127. Area bounded by the circle which is concentric with the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ and which passes through $\left(4,-\frac{9}{5}\right)$, the vertical chord common to both circle and ellipse on the positive side of x-axis is

a) $\frac{481}{25}\tan^{-1}\left(\frac{9}{20}\right)-\frac{36}{5}$ b) $2\tan^{-1}\left(\frac{9}{20}\right)$

c) $\frac{481}{25}\tan^{-1}\left(\frac{9}{20}\right)$ d) none of these

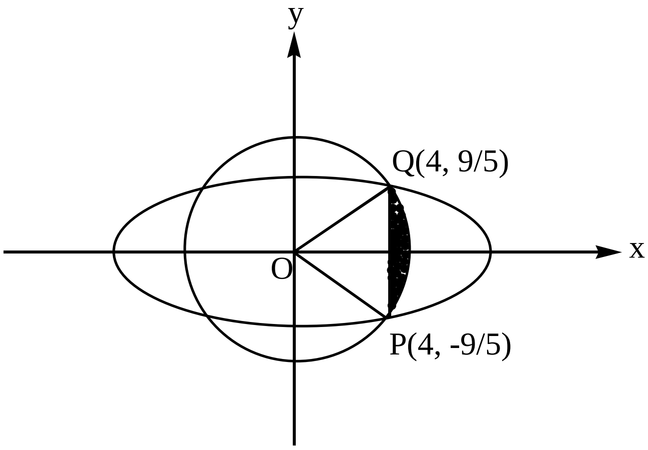
Key : a

Sol : As eccentricity of ellipse = $\frac{4}{5}$

Co-ordinate of foci = (4, 0), (-4, 0)

$\Rightarrow\left(4,-\frac{9}{5}\right)$ is one of the end point of latus – rectum

SMART ACHIEVERS LEARNING PVT. LTD.



$\Rightarrow$ required area is

$$\frac{1}{2\pi}\times\pi\times\left(4^2+\frac{9^2}{5^2}\right)\times 2\tan^{-1}\left(\frac{9}{20}\right)-\text{ area of }\Delta POQ$$

$$=\frac{481}{25}\tan^{-1}\left(\frac{9}{20}\right)-\frac{1}{2}\times 4\times\left(\frac{18}{5}\right)=\frac{481}{25}\tan^{-1}\left(\frac{9}{20}\right)-\frac{36}{5}$$

128. Area of the region in which point p(x, y), {x > 0} lies; such that $y\le\sqrt{16-x^2}$ and $\left|\tan^{-1}\left(\frac{y}{x}\right)\right|\le\frac{\pi}{3}$ is

a) $\left(\frac{16}{3},\pi\right)$

b) $\left(\frac{8\pi}{3}+8\sqrt{3}\right)$

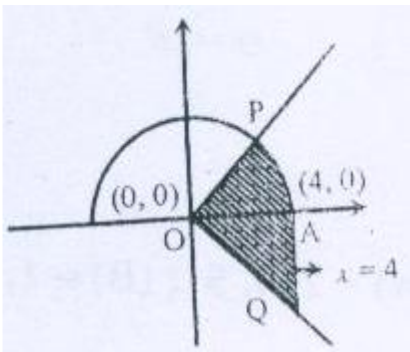
c) $\left(4\sqrt{3}-\pi\right)$

d) none of these

Key : b

Sol : Required area is the area of shaded region (APOQ)

= area of ΔOAQ + area of sector (OAP)



$$=\frac{1}{2}\times 4\times 4\sqrt{3}+\frac{\pi(4\times 4)}{6}$$

$=\left(\frac{8\pi}{3}+8\sqrt{3}\right)$

129. If $c>0$ and the area of the region enclosed by the parabolas $y=x^2-c^2$ *and* $y=c^2-x^2$ is 576, then c =

a) 6 b) 4 c) 3 d) 8

Key: A

Hint: Area between the two parabolas = $4\int_0^C\left(c^2-x^2\right)dx=\frac{8c^3}{3}=576$

only if c = 6

130. Let $f(x)=x^2+6x+1$ and let R denote the set of points (x,y) in the XY-plane such that $f(x)+f(y)\le 0$ and $f(x)-f(y)\le 0$. Then the area of the region R is

A) 6π B) $3\pi+2$ C) $2\pi+8$ D) 8π

Key: D

Hint: $f(x)+f(y)\le 0\Rightarrow (x+3)^2+(y+3)^2\le 16$

$f(x)-f(y)\le 0\Rightarrow (x-y)(x+y+6)\le 0$

131. The quadrilateral formed by the lines $y=ax+c, y=ax+d, y=bx+c$ and $y=bx+d$ has area 18. The quadrilateral formed by the lines $y=ax+c, y=ax-d, y=bx+c$ and $y=bx-d$ has area 72. If a,b,c,d are positive integers then the least possible value of the sum $a+b+c+d$ is

A) 13 B) 14 C) 15 D) 16

Key: D

Hint: $\frac{(c-d)^2}{|a-b|}=18$ and $\frac{(c+d)^2}{|a-b|}=72$.

$a=3, b=1, d=3, c=9$ is a solution for which the minimum is attained.

132. Area of a square ABCD is 36 and side AB is parallel to the X-axis. Vertices A, B and C lie on the graphs of $y=\log_a x, y=2\log_a x$ and $y=3\log_a x$ respectively. Then $a=$

A) $3^{1/6}$ B) $\sqrt{3}$ C) $6^{1/3}$ D) $\sqrt{6}$

Key: A

Hint: Let $A=(p,\log_a p); B=(q,2\log_a q),\ p,q>0\ \&\ a>0, a\ne 1$, then $C=(q,3\log_a q)$

$AB\,P\,X-\text{axis}\Rightarrow p=q^2.|AB|=6\Rightarrow |p-q|=6$ also $|\log_a q|=6$.

133. The area bounded by the curve $\left(y-\text{Sin}^{-1}x\right)^2=x-x^2$ s

a) $\frac{\pi}{4}$ b) $\frac{\pi}{2}$ c) π d) $\frac{\pi}{3}$

Key: A

SMART ACHIEVERS LEARNING PVT. LTD.

Hint: The given curves on $y = \text{Sin}^{-1}x - \left(\sqrt{x - x^2}\right)$ & $y = \text{Sin}^{-1}x + \left(\sqrt{x - x^2}\right)$

$$\text{Requred area} = \int_0^1 2\sqrt{x - x^2}\,dx$$

134. Let F(x) = $\sin x\int_0^x \cos t\,dt + 2\int_0^x t\,dt + \cos^2 x - x^2$. Then area bounded by xF(x) and ordinate x = 0 and x = 5 with x-axis is

(A) 16 (B) $\frac{25}{2}$ (C) $\frac{35}{2}$ (D) 25

Key: B

Hint: $F(x) = \sin x \int_0^x \cos t\,dt + 2\int_0^x t dt - x^2 + \cos^2 x$

$= \sin x(\sin t)_0^x + 2\left(\frac{t^2}{2}\right)_0^x - x^2 + \cos^2 x$

$= \sin^2 x + x^2 - x^2 + \cos^2 x = 1$

$A = \int_0^5 xF(x)dx = \int_0^5 (x)(1)dx = \left[\frac{x^2}{2}\right]_0^5 = \frac{25}{2}$

135. If $\int \frac{x\,dx}{\sqrt[2012]{\left(1+x^2\right)^{1012}\left(2+x^2\right)^{3012}}} = \frac{\alpha}{\beta}\left(1 - f(x)\right)^{\frac{\beta}{2\alpha}} + k$ then which is true

a) $\alpha = 503;\ \beta = 500,\ f\left(\sqrt{2}\right) = \frac{1}{\beta - \alpha}$

b) $\alpha = 503;\ \beta = 250,\ f\left(\sqrt{2}\right) = \frac{1}{\alpha - \beta}$

c) $\alpha = 503;\ \beta = 500,\ f(1) = \frac{1}{\alpha - \beta}$

d) $\alpha = 503;\ \beta = 225,\ f\left(\sqrt{3}\right) = \frac{1}{\alpha - \beta}$

Key. C

Sol. $\frac{2 + x^2}{1 + x^2} = t$

136. The area of the region in the xy-plane defined by the inequalities $x - 2y^2 \geq 0$, $1 - x - |y| \geq 0$ is

a) $\frac{1}{2}$ b) $\frac{1}{3}$ c) $\frac{1}{4}$ d) $\frac{7}{12}$

Key. D

Sol. Area $= 2\int_0^{1/2} \sqrt{\frac{x}{2}}dx + \frac{1}{4} = \frac{7}{12}$

137. The area bounded by $y = \sin^{-1} x, y = \cos^{-1} x$ and the x-axis is

1) $2+\sqrt{2}$ 2) $2-\sqrt{2}$ 3) $\sqrt{2}+1$ 4) $\sqrt{2}-1$

SMART ACHIEVERS LEARNING PVT. LTD.

Key. 4

Sol. By the graph

Required area $= \int_0^{1/\sqrt{2}} \sin^{-1} x dx + \int_{1/\sqrt{2}}^{1} \cos^{-1} x dx$

138. Let f be a real valued function satisfying $f\left(\frac{x}{y}\right) = f(x) - f(y)$ and $\underset{x\to 0}{Lt} \frac{f(1+x)}{x} = 3$. The area bounded by the curve y=f(x), the y-axis and the line y=3 is

1) 9e 2) 2e 3) 3e 4) none

Key. 3

Sol. $f^1(x) = \underset{h\to 0}{Lt} \frac{f(x+h) - f(x)}{h} = \underset{h\to 0}{Lt} \frac{f\left(1+\frac{h}{x}\right)}{\frac{h}{x}.x} = \frac{3}{x}$

139. Area enclosed by the curve y = f(x) defined parametrically as $x = \frac{1-t^2}{1+t^2}, y = \frac{2t}{1+t^2}$ is equal to

1) π sq.units 2) π /2 sq.unit 3) $\frac{3\pi}{4}$ sq.units 4) $\frac{3\pi}{2}$ sq.units

Key. 1

Sol. Clearly t can be any real number

Let t= $\tan\theta$

140. The maximum area of a rectangle whose two vertices lie on the x-axis and two on the curve y=3-|x| , $-3 \le x \le 3$ is

1) 9 2) 9/4 3) 3 4) 9/2

Key. 4

Sol. Take 2a and 3-a are the length of the sides of the rectangle

141. The area of the closed figure bounded by x= -1, x=2 and $y = \begin{cases} -x^2+2 & x \le 1 \\ 2x-1, & x > 1 \end{cases}$ and the x-axis is

1) 16 /3 sq. units 2) 10/3 sq. units 3) 13/3 sq. units 4) 7/3 sq. units

Key. 1

Sol. By the graph

$A = \int_{-1}^{1} \left(-x^2+2\right) + \int_{1}^{2} (2x-1) dx$

142. The value of the parameter ‘a’ such that the area bounded by $y = a^2x^2 + ax + 1$, positive coordinate axes and the line x=1 attains its least value, is equal to

1) $-\frac{1}{4}$ 2) $-\frac{1}{2}$ 3) $-\frac{3}{4}$ 4) -1

Key. 3

Sol. $A = \int_0^1 \left(a^2x^2 + ax + 1\right) dx$

SMART ACHIEVERS LEARNING PVT. LTD.

$=\frac{a^2}{3}+\frac{a}{2}+1$ which is minimum for a = -3/4

143. The area enclosed by the curves y= $\sqrt{4-x^2}$, $y \geq \sqrt{2}$ sin $\left(\frac{x\pi}{2\sqrt{2}}\right)$ and x-axis is divided by the y-axis in the ratio

1) $\frac{\pi^2-8}{\pi^2+8}$ 2) $\frac{\pi^2-4}{\pi^2+4}$ 3) $\frac{\pi-4}{\pi+4}$ 4) $\frac{2\pi^2}{2\pi+\pi^2-8}$

Key. 4

Sol. By the graph

Area of the left of y-axis is π ,Area of the right of y-axis = $\int_0^{\sqrt{2}}\left(\sqrt{4-x^2}-\sqrt{2}\sin\left(\frac{x\pi}{2\sqrt{2}}\right)\right)dx$

144. The area enclosed by the curves, $xy^2=a^2(a-x)$ and $(a-x)y^2=a^2x$ is

1) $(\pi-2)a^2 sq.units$ 2) $(4-\pi)a^2 sq.units$ 3) $\pi a^2/3 sq.units$ 4) $\frac{\pi a^2}{2}$ sq. units

Key. 1

Sol. By the graph, required area

= $2\int_0^a\left[a-\frac{a^3}{a^2+y^2}-\frac{a^3}{a^2+y^2}\right]dy$ (integrating along y-axis)

145. The area of the region bounded by the parabola $(y-2)^2=x-1$, the tangent to it at the point with the ordinate 3 and the x-axis is

1) 7 sq. units 2) 6 sq. units 3) 9 sq. units 4) 8 sq. units

Key. 3

Sol. Given parabola is $(y-2)^2=x-1$

Tangent at (2,3) is $y-3=\frac{1}{2}(x-2) \Rightarrow x-2y+4=0$

By the graph

$\int_0^3\left((y-2)^2+1\right)dy-\int_0^3(2y-4)dy$

146. Consider the region formed by the lines x = 0, y = 0, x = 2, y = 2. Area enclosed by the curves $y=e^x$ and y = 1n x, within this region, is being removed. Then, the area of the remaining region is

1) 2(ln2-1) sq. units 2) 2(1n2+1) sq. units

3) 2(21n2-1) sq. units 4) 2(2ln2+1)sq. units

Key. 3

Sol. By the graph

$2\int_0^{In2}(2-e^x)dx$

147. The area bounded by the curve f(x)=x+sinx and its inverse function between the ordinates x=0 and x= 2π is

1) 4 π sq. units 2) 8 π sq. units 3) 4 sq. units 4) 8 sq. units

SMART ACHIEVERS LEARNING PVT. LTD.

Key. 4

Sol. By the graph, required Area = 4A, where

$$A=\int_0^{\pi}(x+\sin x)dx-\int_0^{\pi}xdx$$

$$=\frac{\pi^2}{2}-\cos\pi+\cos 0-\frac{\pi^2}{2}=2 \text{ square units}$$

148. The area bounded by the curves y= $\sin^{-1}|\sin x|$ and $y=\left(\sin^{-1}|\sin x|\right)^2$, where $0\le x\le 2\pi$, is

1) $\frac{1}{3}+\frac{\pi^2}{4}$ sq.units
2) $\frac{1}{6}+\frac{\pi^3}{8}$ sq. units
3) $\frac{2}{3}+\frac{\pi^2(\pi-3)}{6}$ sq .units
4) $\frac{4}{3}+\frac{\pi^2(\pi-3)}{6}$ sq .units

Key. 4

Sol. $y=\sin^{-1}|\sin x|=\begin{cases} x, & 0\le x<\frac{\pi}{2} \\ \pi-x, & \frac{\pi}{2}\le x<\pi \\ x-\pi, & \pi\le x<\frac{3\pi}{2} \\ 2\pi-x, & \frac{3\pi}{2}\le x<2\pi \end{cases}$

By the graph

$4\int_0^1(x-x^2)dx+4\int_1^{\pi/2}(x^2-x)dx$

149. Area bounded by the curve $xy^2=a^2(a-x)$ and its asymptote is

1) $\pi a^2/2$ sq. units 2) πa^2 sq. units 3) 3 πa^2 sq. units 4) 4 πa^2 sq. units

Key. 2

Sol. By the graph, required area = $2\int_0^{\infty}\frac{a^3}{y^2+a^2}dy$

150. Consider two curves $C_1: y^2=4\left[\sqrt{y}\right]x$ and $C_2: x^2=4\left[\sqrt{x}\right]y$, where[.] denotes the greatest integer function. Then the area of the region enclosed by these two curves within the square formed by the lines x=1, y=1 ,x=4,y=4 is

1) 8/3 sq. units 2) 10/3 sq. units 3) 11/3 sq. units 4) 11/4 sq. units

Key. 2

Sol. By the graph

The required area

$$A=\int_1^2(2\sqrt{x}-1)dx+\int_2^4\left(2\sqrt{x}-\frac{x^2}{4}\right)dx$$

SMART ACHIEVERS LEARNING PVT. LTD.

SMART ACHIEVERS LEARNING PVT. LTD.

Definite, Indefinite Integration & Areas

Multiple Correct Answer Type

1. The area bounded by the function $f(x)=|\cos x|$ and $g(x)=\sin x$ between the ordinates x = 0 and $x=n\pi, n\in N,$ is equal to

(A) $4n\sqrt{2}$ if n is odd
(B) $2n\sqrt{2}$ if n is even
(C) $2\sqrt{2}(n+1)$ if n is odd
(D) $2\sqrt{2}(n+1)-4$ if n is odd

Key. B,D

Sol. Area bounded by $f(x)=|\cos x|, g(x)=\sin x.$

$$\int_0^{\pi}|f(x)-g(x)|dx = 4\int_0^{\pi/4}(\cos x-\sin x)dx = 4\left(\sqrt{2}-1\right)$$

$$\int_{\pi}^{2\pi}|f(x)-g(x)|dx = \int_{\pi}^{2\pi}(|\cos x|-\sin x)dx = 4$$

So if x is even $A=2x\sqrt{2}$

N is odd $A=2(n-1)\sqrt{2}+4\left(\sqrt{2}-1\right)$

$=2(n+1)\sqrt{2}-4$

2. If $I_1=\int_0^{\frac{\pi}{2}}\frac{\sin(\sin x)}{\sin x}dx,\ I_2=\int_0^{\frac{\pi}{2}}\frac{\sin x}{x}dx\ \ and\ I_3=\int_0^{\frac{\pi}{2}}\frac{\sin(\tan x)}{\tan x}dx$

then which of the following is true

(A) $I_1>I_3$
(B) $I_2>I_3$
(C) $I_1>I_2$
(D) $I_1<I_2$

Key. A,B,C

Sol. $0<x<\frac{\pi}{2}\Rightarrow\frac{\sin x}{x}$ *is* decreasing and $\sin x<x<\tan x$

$$\Rightarrow\frac{\sin(\sin x)}{\sin x}>\frac{\sin x}{x}>\frac{\sin(\tan x)}{\tan x}\Rightarrow I_1>I_2>I_3$$

3. If $f(x)$ is monotonic and differentiable function, then

$$\int_{f(a)}^{f(b)}2x\left(b-f^{-1}(x)\right)dx=\_\_\_$$

(A) $\int_a^b\left(f^2(x)-f^2(a)\right)dx$
(B) $\int_a^b\left(f^2(x)-f^2(b)\right)dx$

SMART ACHIEVERS LEARNING PVT. LTD.

(C) $\int_a^b f^2(x)\,dx+(a-b)\,f^2(b)$

(D) $\int_a^b f^2(x)\,dx+(a-b)\,f^2(a)$

Key. A,D

Sol. Let $f^{-1}(x)=y \Rightarrow x=f(y) \Rightarrow dx=f^1(y)\,dy$

$$I=\int_a^b 2f(y)(b-y)\,f^1(y)\,dy=b\int_a^b 2f(y)\,f^1(y)\,dy-\int_a^b 2yf(y)\,f^1(y)\,dy$$

$$=b\left(f^2(b)-f^2(a)\right)-bf^2(b)+af^2(a)+\int_a^b f^2(y)\,dy$$

$$=\int_a^b f^2(x)\,dx+(a-b)\,f^2(a)=\int_a^b\left(f^2(x)-f^2(a)\right)dx$$

4. If $u=\int_0^\infty \frac{dx}{x^4+7x^2+1}$ and $v=\int_0^\infty \frac{x^2dx}{x^4+7x^2+1}$ then

(A) u > v (B) u < v (C) u = v (D) $u=\frac{\pi}{6}$

Key. C,D

Sol. (C, D) $u=\int_0^\infty \frac{1}{x^4+7x^2+1}dx$ Put x = $\frac{1}{t}$, $dx=\frac{-1}{t^2}=dt$

$$u=\int_\infty^0 \frac{1}{\frac{1}{t^4}+\frac{7}{t^2}+1}\left(\frac{-dt}{t^2}\right)=\int_0^\infty \frac{t^2dt}{t^4+7t^2+1}=v$$

$\Rightarrow u=v$

So, $u+v=2y=\int_0^\infty \frac{1+x^2}{x^4+7x^2+1}dx$

$$\Rightarrow 2y=\int_0^\infty \frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x-\frac{1}{x}\right)^2+9}=\int_{-\pi/2}^{\pi/2}\frac{dt}{t^2+9}=\frac{\pi}{3}$$

5. If f(x) and g(x) be the two continuous functions on $x\in[a,b]$ and $F(x)=\max\{f(x),g(x)\}$ and $G(x)=\min\{f(x),g(x)\}$ the area bounded by $F(x)$ and G(x) between the lines x = a, x = b, and x –axis, is

SMART ACHIEVERS LEARNING PVT. LTD.

(A) $\int_a^b (f(x)-g(x))dx$ (B) $\int_a^b |f(x)-g(x)|dx$ (C) $\int_a^b |F(x)-G(x)|dx$ (D) none of these

Key. B,C

Sol. $|f(x)-g(x)|$ and $|f(x)-G(x)|$ represent the same area equal to $y=f(x)$ and $y=g(x)$.

6. If f(x) is an odd and periodic function with period T, then

(a) $\int_0^T f(x)\,dx = 0$ (b) $\int_0^{3T} f(x)\,dx = 0$

(c) $\int_a^{a+T} f(x)\,dx = \int_0^T f(x)\,dx$ for any a ∈ R (d) $\int_0^{T/2} f(x)\,dx = 0$

Key. A,B,C

Sol. (a) is correct.

Since $\int_0^T f(x)\,dx = \int_0^T f(T-x)\,dx$

$= \int_0^T f(-x)\,dx$ ($\because$ f(x) is periodic with period T)

$= -\int_0^T f(x)\,dx$ ($\because$ f is odd)

$\Rightarrow 2\int_0^T f(x)\,dx = 0$

$\Rightarrow$ (b) is also correct since

$\int_0^{3T} f(x)\,dx = 3\int_0^T f(x)\,dx$

7. If the area formed by a function y = f(x) between the ordinates x = 0 and x = 3, and x-axis is 27sq units then the function f(x) may be

(A) $3x^2$ (B) x^2+6 (C) $6|x-3|$ (D) $2|x-6|$

Key. A,B,C,D

Sol. Conceptual

8. If the Area bounded by

$y=(\sin x)^{\operatorname{cosec} x}$, $y=(\operatorname{cosec} x)^{\sin x}$, $y=(\sin x)^{\sin x}$ and $y=(\operatorname{cosec} x)^{\operatorname{cosec} x}$ between the ordinates x = 0, $x=\frac{\pi}{2}$ denoted by A_1, A_2, A_3 and A_4 then

(A) A_4 is the greatest (B) A_1 is the least (C) $A_2 > A_3$ (D) $A_2 < A_3$

Key. A,B,C

Sol. (A, B, C) for $x \in \left(0, \frac{\pi}{2}\right)$

$0 < \sin x < 1 < \operatorname{cosec} x < \infty$

So, $\sin x^{\operatorname{cosec} x} < \sin x^{\sin x} < \operatorname{cosec} x^{\sin x} < \operatorname{cosec} x^{\operatorname{cosec} x}$

SMART ACHIEVERS LEARNING PVT. LTD

$A_1 < A_3 < A_2 < A_4$

9. Given two functions f and g which are integrable on every interval and satisfy

(i) f is odd, g is even (ii) g(x) = f(x + 5), then

(A) f(x – 5) = g(x) (B) f(x – 5) = –g(x)

(C) $\int_0^5 f(t)\,dt = \int_0^5 g(5-t)\,dt$ (D) $\int_0^5 f(t)\,dt = -\int_0^5 g(5-t)\,dt$

Key. B,C

Sol. To test choice (a) and (b), we begin with computting g(x). Indeed g(x) = f(x + 5) (From (ii) in ques.)

$\Rightarrow$ g (– x) = f (– x + 5)

$\Rightarrow$ g (x) = –f (x – 5) (From (i) in question)

$\Rightarrow$ Choice (b) is true and choice (a) is ruled out. To test the choices (c) and (d), we compute

$I = \int_0^5 f(t)\,dt$

Indeed $I = \int_0^5 g(t-5)\,dt$ ($\because$ f(t) = g(t – 5) on replacing x by t – 5 in (ii))

$= \int_0^5 g(t-5)\,dt$ ($\because$ g is even)

$\Rightarrow$ Choice (c) is correct and choice (d) is false.

10. The value of $\int \frac{dx}{1+e\cos x}$ must be same as

(A) $\frac{1}{\sqrt{1-e^2}}\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{x}{2}\right)+c$ (e lies between 0 and 1)

(B) $\frac{2}{\sqrt{1-e^2}}\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{x}{2}\right)+c$, (e lies between 0 and 1)

(C) $\frac{1}{\sqrt{e^2-1}}\log\frac{e+\cos x+\sqrt{e^2-1}\sin x}{1+e\cos x}$ + c, (e is greater than 1)

(D) $\frac{2}{\sqrt{e^2-1}}\log\frac{e+\cos x+\sqrt{e^2-1}\sin x}{1+e\cos x}+c$, (e is greater than 1)

Key. B,C

Sol. $I=\frac{2}{1-e}\int\frac{dt}{t^2+\left(\frac{1+e}{1-e}\right)}$ (t = tanx/2)

if 0 < e < 1, $\frac{1+e}{1-e}>0$, So, (B) is correct

If e > 1, $\frac{1+e}{1-e}<0$ So, (C) is correct.

SMART ACHIEVERS LEARNING PVT. LTD.

11. If $\int \frac{3x+4}{x^3-2x-4}dx$ = log |x - 2| + k log f(x) + c, then
(A) k = -1/2 (B) f(x) = $x^2 + 2x + 2$
(C) f(x) = $|x^2 + 2x + 2|$ (D) k = 1/4

Key. A,B,C

Sol. We use partial fraction, we get

$$\frac{3x+4}{x^3-2x-4}=\frac{A}{x-2}+\frac{Bx+C}{x^2+2x+2}$$

Solving this equation, we get A = 1, B = -1 and C = -1

$$\therefore \int \frac{3x+4}{x^3-2x-4}dx=\int\frac{1}{x-2}dx-\int\frac{x+1}{x^2+2x+2}dx$$

$$= \log|x-2| - \frac{1}{2}\log|x^2+2x+2|+c$$

hence k = -1/2, so that f(x) = $|x^2 + 2x + 2| = x^2 + 2x + 2$ because $x^2 + 2x + 2 > 0$.

12. $\int \frac{dx}{\left(1+\sqrt{x}\right)^{2010}}=2\left[\frac{1}{\alpha\left(1+\sqrt{x}\right)^{\alpha}}-\frac{1}{\beta\left(1+\sqrt{x}\right)^{\beta}}\right]+c$ where $\alpha,\beta>0$ then

A) $|\alpha-\beta|=1$ B) $(\beta+2)(\alpha+1)=(2010)^2$ C) $\beta,\alpha,2010$ are in A.P D) $\alpha+1=\beta+2=2010$

Key. A,B,C,D

Sol.

$$\int\frac{dx}{\left(1+\sqrt{x}\right)^{2010}}=\int\frac{\sqrt{x}}{\sqrt{x}\left(1+\sqrt{x}\right)^{2010}}dx=2\int\frac{t-1}{t^{2010}}dt\left(t=1+\sqrt{x}\right)$$

$$=2\left[\frac{1}{2009.t^{2009}}-\frac{1}{2008.t^{2008}}\right]+c$$

$$\Rightarrow \alpha=2009,\quad \beta=2008$$

13. If x satisfies the equation $x^2\left(\int_0^1\frac{dt}{t^2+2t\cos\alpha+1}\right)-x\left(\int_{-3}^{3}\frac{t^2\sin 2t}{t^2+1}dt\right)-2=0$ $(0<\alpha<\pi)$, then the value of x is

a) $2\sqrt{\left(\frac{\sin\alpha}{\alpha}\right)}$ b) $-2\sqrt{\left(\frac{\sin\alpha}{\alpha}\right)}$ c) $4\sqrt{\left(\frac{\sin\alpha}{\alpha}\right)}$ d) $-4\sqrt{\left(\frac{\sin\alpha}{\alpha}\right)}$

Key. A,B

Sol. Let $f(x)=\int_0^{x^2}\left(\frac{t^2-5t+4}{2+e^t}\right)dt$

SMART ACHIEVERS LEARNING PVT. LTD.

$$f^1(x)=\left(\frac{x^4-5x^2+4}{2+e^{x^2}}\right)\times 2x$$

For extremum $f^1(x)=0$

14. The points of extremum of $\int_0^{x^2}\left(\frac{t^2-5t+4}{2+e^t}\right)dt$ are

a) $x=-2$ b) $x=1$ c) $x=0$ d) $x=-1$

Key. A,B,C,D

Sol. Conceptual

15. If $G(x,t)=\begin{cases}x(t-1), & \text{where } x\le t\\ t(x-1), & \text{where } t<x\end{cases}$ and if t is continuous function of x in $[0,1]$. Let $g(x)=\int_0^1 f(t)G(x,t)dt$, then

a) $g(0)=1$ b) $g(0)=0$

c) $g(1)=1$ d) $g^{11}(x)=f(x)$

Key. B,D

Sol. For x = 0

$$G(0,t)=0, t\ge 0$$

$$g(1)=\int_0^1 f(t)G(0,t)dt=\int_0^1 0.dt=0$$

And for x = 1

$$G(1,t)=0, t<1$$

$$g(1)=\int_0^1 f(t)G(1,t)dt=0$$

Also, $g(x)=\int_0^x f(t)G(x,t)dt+\int_x^1 f(t)G(x,t)dt$

$$=(x-1)\int_0^x tf(t)dt=x\int_x^1(t-1)f(t)dt$$

$$\therefore\ g^{11}(x)=xf(x)+\{0-(x-1)f(x)\}$$

$$=f(x)$$

16. $\int_{-1/2}^{1/2}\sqrt{\left\{\left(\frac{x+1}{x-1}\right)^2+\left(\frac{x-1}{x+1}\right)^2-2\right\}}dx$ is

SMART ACHIEVERS LEARNING PVT. LTD.

a) $4\ln\left(\frac{4}{3}\right)$ b) $4\ln\left(\frac{3}{4}\right)$ c) $-\ln\left(\frac{81}{256}\right)$ d) $\ln\left(\frac{256}{81}\right)$

Key. A,C,D

Sol. Let $I=\int_{-1/2}^{1/2}\sqrt{\left(\frac{x+1}{x-1}-\frac{x-1}{x+1}\right)}dx$

$$=\int_{-1/2}^{1/2}\left|\frac{4x}{x^2-1}\right|dx=-2\int_0^{1/2}\frac{4x}{\left(x^2-1\right)}dx$$

$$=-4\left\{\ln\left|x^2-1\right|\right\}_0^{1/2}=-4\ln\left(\frac{3}{4}\right)$$

$$=4\ln\left(\frac{4}{3}\right)=\ln\left(\frac{256}{81}\right)=-\ln\left(\frac{81}{256}\right)$$

17. Let $T>0$ be a fixed real number. Suppose $f(x)$ is a continuous function for all $x\in R,\ f(x+T)=f(x)$. If $I=\int_0^T f(x)dx$ then

A) $\int_5^{5+5T} f(x)dx=5I$ B) $\int_5^{5+5T} f(2x)dx=10I$

C) $\int_5^{5+5T} f(3x)dx=5I$ D) $\int_5^{5+5T} f(3x)dx=15I$

Key. A,C

Sol. $\int_a^{a+nT} f(x)dx=n\int_0^T f(x)dx$

18. Let $f(x)=\int_{\pi^2/4}^{x^2}\frac{\sin x}{1+\cos^2\sqrt{t}}dt$ then

A) $f^1\left(\frac{\pi}{2}\right)=\pi$ B) $f^1\left(-\frac{\pi}{2}\right)=\pi$

C) $f^1\left(\frac{3\pi}{2}\right)=-3\pi$ D) $f^1\left(\pi\right)=\int_{\pi^2}^{\pi^2/4}\frac{dx}{1+\cos^2\sqrt{x}}$

Key. A,B,C,D

Sol. $f^1(x)=\sin x\times\frac{2x}{1+\cos^2 x}+\cos x\int_{\pi^2/4}^{x^2}\frac{1}{1+\cos^2\sqrt{t}}dt$

19. The triangle formed by the lines $x+y=0,\ 3x+y-4=0$ and $x+3y-4=0$ is

SMART ACHIEVERS LEARNING PVT. LTD.

A) isosceles B) scalene C) acute angled D) obtuse angled

Key. A,D

Sol. Conceptual

20. The value of $\int_{1/e}^{\tan x}\frac{t\,dt}{1+t^2}+\int_{1/e}^{\cot x}\frac{dt}{t\left(1+t^2\right)}$ is

A) $\frac{1}{2+\tan^2 x}$ B) 1 C) $\frac{\pi}{4}$ D) $\frac{2}{\pi}\int_{-1}^{1}\frac{dt}{1+t^2}$

Key. B,D

Sol. Let $I=\int_{1/e}^{\cot x}\frac{dt}{t\left(1+t^2\right)}$

Put $t=\frac{1}{z}$

$\therefore \quad dt=-\frac{1}{z^2}dz$

$$\therefore \quad I=\int_{e}^{\tan x}\frac{-\frac{1}{z^2}dz}{\frac{1}{z}\left(1+\frac{1}{z^2}\right)}=\int_{\tan x}^{e}\frac{z\,dz}{\left(z^2+1\right)}$$

$$=\int_{\tan x}^{e}\frac{t\,dt}{\left(1+t^2\right)}$$

$$\therefore \int_{1/e}^{\tan x}\frac{t\,dt}{\left(1+t^2\right)}+\int_{1/e}^{\cot x}\frac{dt}{t\left(1+t^2\right)}$$

$$=\int_{1/e}^{\tan x}\frac{t}{\left(1+t^2\right)}dt+\int_{\tan x}^{e}\frac{t}{\left(1+t^2\right)}dt$$

$$=\int_{1/e}^{e}\frac{t\,dt}{\left(1+t^2\right)}=\frac{1}{2}\left[ln\left(1+t^2\right)\right]_{1/e}^{e}$$

$$=\frac{1}{2}\left\{In\left(1+e^2\right)-ln\left(1+\frac{1}{e^2}\right)\right\}$$

$$=\frac{1}{2}\left(ln\,e^2\right)=1.$$

Also, $\frac{2}{\pi}\int_{-1}^{1}\frac{dt}{1+t^2}=\frac{4}{\pi}\int_{0}^{1}\frac{dt}{1+t^2}=\frac{4}{\pi}.\tan^{-1}1$

$$=\frac{4}{\pi}.\frac{\pi}{4}=1.$$

21. If the area bounded by the curve $\mathrm{y}=\mathrm{x}-\mathrm{x}^2$ and the line y = mx equals $\frac{9}{2}$. Then the value of m is

SMART ACHIEVERS LEARNING PVT. LTD.

(A) -4 (B) -2
(C) 2 (D) 4

Key. B

Sol. Conceptual

22. $\int_0^1 \frac{2x^2+3x+3}{(x+1)(x^2+2x+2)}dx$ is

(A) $\frac{\pi}{4}+2\ln 2-\tan^{-1}2$ (B) $\frac{\pi}{4}+2\ln 2-\tan^{-1}\frac{1}{3}$

(C) $2\ln 2-\cot^{-1}3$ (D) $-\frac{\pi}{4}+\ln 4+\cot^{-1}2$

Key. A,C,D

Sol. $\int \frac{2x^2+3x+3}{(x+1)(x^2+2x+3)}dx=\int \frac{A}{x+1}+\frac{Bx+C}{x^2+2x+3}$ and Proceed

23. $\int_a^b \operatorname{sgn} x\,dx=$

(A) $|b|-|a|$ (B) $(b-a)\operatorname{sgn}(b-a)$
(C) $b\operatorname{sgn} b-a\operatorname{sgn} a$ (D) $|a|-|b|$

Key. A,C

Sol. Conceptual

24. Let f be a differentiable function such that $f'(x)=f(x)+\int_0^2 f(x)dx$, $f(0)=\frac{4-e^2}{3}$, then f(x) is

(A) $e^x-\left(\frac{e^2-1}{3}\right)$ (B) $e^x-\frac{(e^2-2)}{3}$

(C) $e^x+\frac{e-[\arg|z-1|]}{3}$

(D) $e^x-\frac{e^2-1}{[\arg(-|z|)]}$ (where [.] greatest integer & z is a complex number)

Key. A,D

Sol. $f'(x)=f(x)+k$

$\int \frac{f'(x)}{f(x)+k}=\int dx$

$\log(f(x)+k)=x+C$

$f(x)=k_1e^x-k$

$f(0)=k_1-k=\frac{4-e^2}{3}$ ….. (1)

$k=\int_0^2 f(x)dx$

SMART ACHIEVERS LEARNING PVT. LTD.

$3k = k_1(e^2 - 1)$ ….. (2)

Solving (1) and (2), we get

$$f(x) = e^x - \left(\frac{e^2 - 1}{3}\right)$$

25. Suppose f is a function that satisfies the equation $f(x+y) = f(x) + f(y) + x^2y + xy^2$ for all real numbers x and y. If $\lim_{x\to 0}\frac{f(x)}{x} = 1$, then

a) $f(x) > 0$ *for* $x > 0$ *and* $f(x) < 0$ *for* $x < 0$

b) $f'(0) = 1$

c) $f''(0) = 1$

d) $f'''(x)$ is a constant for all real x

Key: A,B,D

Hint Observe that f (0) = 0 and

$$f'(0) = \lim_{h\to 0}\frac{f(h)}{h} = 1$$

Also $f'(x) = \underset{h\to 0}{Lt}\frac{f(x+h) - f(x)}{h}$

$= \underset{h\to 0}{Lt}\frac{f(h) + x^2h + xh^2}{h}$

$= x^2 + 1$

Hence, $f(x) = \frac{x^3}{3} + x$

26. *Let* $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} x + 2x^2\sin\frac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

then,

a) f is differentiable for all x , but $f'(x)$ is not continuous at x = 0

b) $f'(0) = 1$

c) f is increasing at x = 0

d) f is not increasing in any neighbourhood of '0'

Key: A,B,C,D

Hint: $f'(x) = 1 + 4x\sin\frac{1}{x} - 2\cos\left(\frac{1}{x}\right)\ (x \neq 0)$

= 1 (x = 0)

In every neighbourhood of '0', the derivative $f'(x)$ assumes both positive and negative values.

SMART ACHIEVERS LEARNING PVT. LTD.

for example, $f'\left(\frac{1}{2n\pi}\right) = -1 < 0$ *and*

$$f'\left(\frac{2}{(4n+1)\pi}\right) = 1 + \frac{8}{(4n+1)\pi} > 0 \text{ for } n \in N$$

27. If $\int_0^x \left(x - [x] - \frac{1}{2}\right) dx = f(x).g(x)$ where $[x]$ and $\{x\}$ are integral and fractional parts of x, respectively

a) $f(x) = \frac{\{x\}}{2}$

b) $g(x) = (\{x\} - 1)$

c) $f(x) = \frac{[x]}{2}$

d) $g(x) = ([x] - 1)$

Key: A,B

Hint $I = \int_0^{[x]+\{x\}} \left(\{x\} - \frac{1}{2}\right) dx = \int_0^{[x]} \left(\{x\} - \frac{1}{2}\right) dx + \int_{[x]}^{[x]+\{x\}} \left(\{x\} - \frac{1}{2}\right) dx$

$$= [x] \int_0^1 \left(\{x\} - \frac{1}{2}\right) dx + \int_0^{\{x\}} \left(\{x\} - \frac{1}{2}\right) dx$$

$$= [x] \int_0^1 \left(x - \frac{1}{2}\right) dx + \int_0^{\{x\}} \left(x - \frac{1}{2}\right) dx$$

$$= [x] \left(\frac{x^2}{2} - \frac{x}{2}\right)_0^1 + \left(\frac{x^2}{2} - \frac{x}{2}\right)_0^{\{x\}}$$

$$= 0 + \frac{\{x\}(\{x\} - 1)}{2} = \frac{\{x\}}{2}(\{x\} - 1)$$

28. Consider the function

$$f(x) = x^2 \left|\cos\frac{\pi}{2x}\right|, \text{ if } x \neq 0$$
$$0 \quad , \text{ if } x = 0 \text{ then}$$

A) $f^1(0) = 0$

B) $f^1(0)$ does not exist

C) $f^1\left(\frac{1}{3}\right)$ does not exist

D) $f^1(x)$ does not exist whenever x is in the form of $\frac{1}{2n+1}$ where $n \in Z$

Key: A,C,D

Hint: Total No. of ways $= n^n$

29. Let f(x) = $\int_0^1 |t - x|$ dt then

a) f(x) is a constant function

b) f(x) is continuous for all $x \in R$

SMART ACHIEVERS LEARNING PVT. LTD.

c) f(x) is differentiable for all $x \in R$ d) f(x) is not differentiable at x = 0, 1

Key: B, C

Hint: $f(x)=\begin{cases}\int_0^1 (t-x)\,dt & x\le 0\\ \int_0^x (x-t)\,dt+\int_x^1 (t-x)\,dt & 0<x<1\\ \int_0^1 (x-t)\,dt & x\ge 1\end{cases}$ $f(x)=\begin{cases}\frac{1}{2}-x & x\le 0\\ \frac{1}{2}-x+x^2 & 0<x<1\\ x-\frac{1}{2} & x\ge 1\end{cases}$

Which is continuous and differentiable for all x.

30. Let I(λ) = $\int_0^1 \frac{dx}{1+x^{\lambda}}$ (λ∈R, λ > 1) then I(λ) lie in

(A) (1, ∞) (B) (0, λ)

(C) $\left(\frac{\lambda}{1+\lambda},1\right)$ (D) none of these

Key: B or C

Hint: $1 - x^{\lambda} < \frac{1}{1+x^{\lambda}} < 1$

$\Rightarrow \int_0^1 (1-x^{\lambda})\,dx < \int_0^1 \frac{dx}{1+x^{\lambda}} < \int_0^1 dx$

$\frac{\lambda}{\lambda+1}$ < I(λ) < I

31. If f : R → R, f(x) is a differentiable function such that $\{f(x)\}^2 = e^2 + \int_0^x \left[\{f(t)\}^2 + \{f'(t)\}^2\right]dt$, ∀ x ∈ R. The value of f(1) can take is/are

(A) e^2 (B) $-e^2$ (C) 1 (D) –1

Key: A, B

Hint: $2f(x)\,f'(x) = \{f(x)\}^2 + \{f'(x)\}^2$

$\Rightarrow [f(x) - f'(x)]^2 = 0$

$\Rightarrow f(x) = f'(x)$

$\Rightarrow f(x) = Ae^x = \pm e^{(x+1)}$

$\Rightarrow f(1) = \pm e^2$.

32. Let $S_n = \lim_{n\to\infty}\left(\frac{1}{n^6}+\frac{32}{n^6}+.....+\frac{1}{n}\right)$ and $T_n = \lim_{n\to\infty}\left(\frac{1}{n^6}+\frac{32}{n^6}+.....+\frac{(n-1)^5}{n^6}\right)$, then which of the following is/ are true ?

(A) $S_n \to \frac{1}{6}^{+}$ (B) $\left(S_n + T_n\right) < \frac{1}{3}$

SMART ACHIEVERS LEARNING PVT. LTD.

(C) $\left(S_n + T_n\right) > \frac{1}{3}$ (D) $T_n \to \frac{1}{6}^{-}$

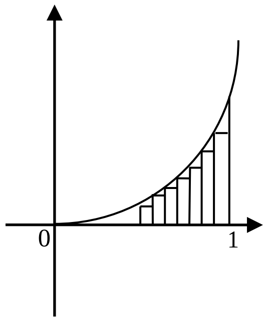
Key: A, B

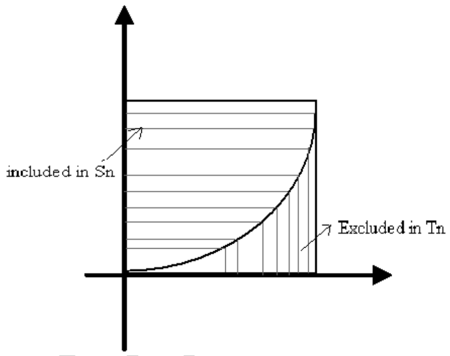
Hint: $S_n = \lim_{n\to\infty}\left(\frac{1}{n^6} + \frac{32}{n^6} + \frac{243}{n^6} + \ldots\ldots\frac{1}{n}\right)$

$$T_n = \lim_{n\to\infty}\left(\frac{1}{n^6} + \frac{32}{n^6} + \frac{243}{n^6} + \ldots\ldots\frac{(n-1)^5}{n^6}\right)$$

$$S_n = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{r^5}{n^6} = \int_0^1 x^5 dx = \left.\frac{x^6}{6}\right|_0^1 = \frac{1}{6}$$

$$\Rightarrow S_n = \left(\frac{1}{6}\right)^{+}; T_n = \left(\frac{1}{6}\right)^{-}$$





But since x^5 is concare up ward the area included in S_n for any two consecutive values of r is more than area excluded in T_n for same values of r

$$\Rightarrow S_n - \frac{1}{6} > \frac{1}{6} - T_n$$

$$\Rightarrow S_n + T_n > \frac{1}{3}$$

33. If the area bounded by the curve $y = x - x^2$ and the line y = mx equals $\frac{9}{2}$. Then the value of m is

(A) -4 (B) -2

(C) 2 (D) 4

Key. B

Sol. Conceptual

34. $\int_0^1 \frac{2x^2 + 3x + 3}{(x+1)(x^2 + 2x + 2)} dx$ is

(A) $\frac{\pi}{4} + 2\ln 2 - \tan^{-1} 2$ (B) $\frac{\pi}{4} + 2\ln 2 - \tan^{-1}\frac{1}{3}$

SMART ACHIEVERS

(C) $2\ln 2-\cot^{-1}3$ (D) $-\frac{\pi}{4}+\ln 4+\cot^{-1}2$

Key. A,C,D

Sol. $\int\frac{2x^2+3x+3}{(x+1)(x^2+2x+3)}dx=\int\frac{A}{x+1}+\frac{Bx+C}{x^2+2x+3}$ and Proceed

35. $\int_a^b \operatorname{sgn} x\,dx=$

(A) $|b|-|a|$ (B) $(b-a)\operatorname{sgn}(b-a)$

(C) $b\operatorname{sgn} b-a\operatorname{sgn} a$ (D) $|a|-|b|$

Key. A,C

Sol. Conceptual

36. Let f be a differentiable function such that $f'(x)=f(x)+\int_0^2 f(x)dx$, $f(0)=\frac{4-e^2}{3}$, then f(x) is

(A) $e^x-\left(\frac{e^2-1}{3}\right)$ (B) $e^x-\frac{(e^2-2)}{3}$

(C) $e^x+\frac{e-[\arg|z-1|]}{3}$

(D) $e^x-\frac{e^2-1}{[\arg(-|z|)]}$ (where [.] greatest integer & z is a complex number)

Key. A,D

Sol. $f'(x)=f(x)+k$

$\int\frac{f'(x)}{f(x)+k}=\int dx$

$\log(f(x)+k)=x+C$

$f(x)=k_1e^x-k$

$f(0)=k_1-k=\frac{4-e^2}{3}$ ….. (1)

$k=\int_0^2 f(x)dx$

$3k=k_1(e^2-1)$ ….. (2)

Solving (1) and (2), we get

$f(x)=e^x-\left(\frac{e^2-1}{3}\right)$

37. Let $I_n=\int_{-\pi}^{\pi}\frac{\sin nx}{\sin x}dx; n\in N$ then

A) $I_{n+2}=I_n$ B) $\sum_{m=1}^{20} I_{2m+1}=20\pi$

C) $I_{2m}=0$ where $m=1,2,3,.....$ D) $I_{n+1}=I_n$

SMART ACHIEVERS LEARNING PVT. LTD.

Key. A,B,C

Sol. $I_{n+2}-I_n=\int_{-\pi}^{\pi}\frac{\sin(n+2)x-\sin nx}{\sin x}dx=\int_{-\pi}^{\pi}2\cos(n+1)dx=0$

$\Rightarrow I_1=I_3=I_5=....$

$I_{2m}=2\int_0^{\pi}\frac{\sin 2mx}{\sin x}dx=0\ \left(\because f(\pi-x)=-f(x)\right)$

38. Let $I_{m,n}=\int\frac{\sin^m x}{\cos^n x}dx(n\neq 1)$ and $I_{m,n}=f(n)\frac{\sin^{m-1}x}{\cos^{n-1}x}+g(m,n)I_{m-2,n-2}$ then

A) $f(n)=\frac{1}{n-1}$ B) $f(n)=\frac{1}{1-n}$

C) $g(m,n)=(m-1)f(n)$ D) $g(m,n)=(1-m)f(n)$

Key. A,D

Sol. $I_{m,n}=\int\frac{\sin^m x}{\cos^n x}dx=\int\underset{u}{\sin^{m-1}x}\,\underset{v}{\frac{\sin x}{\cos^n x}}dx$

39 If $y=f(x)=\int_0^x (x-t)^6\sin t\,dt$ then the true statements among the following are

A) $y''+y=x^6$ B) $y'(0)=0$ C) $y''-y=x^6$ D) $y'(0)=1$

Key. A,B

Sol. $f(x)=\int_0^x\left[x-(x-t)\right]^6\sin(x-t)dt=\int_0^x t^6(\sin x\cos t-\cos x\sin t)dt$

$=\sin x\int_0^x t^6\cos t\,dt-\cos x\int_0^x t^6\sin t\,dt$

40 The absolute value of $\int_{10}^{19}\frac{\sin x}{1+x^8}dx$ is

A. Less than 10^{-7} B. More than 10^{-7}

C. Less than 10^{-6} D. None of these

Key. A,C

Sol. Since, $|\sin x|\le 1$ for $x\ge 10$, then

$\left|\frac{\sin x}{1+x^8}\right|\le\frac{1}{|1+x^8|}$

But $10\le x\le 19$

SMART ACHIEVERS LEARNING PVT. LTD.

$\therefore 1+x^8 > x^8 \geq 10^8 \Rightarrow \frac{1}{1+x^8} < \frac{1}{x^8} \leq \frac{1}{10^8}$

Or $\frac{1}{|1+x^8|} \leq \frac{1}{10^8}$

From Eqs.(i) and (ii) we get

$\left|\frac{\sin x}{1+x^8}\right| \leq 10^{-8}$

$\left|\int_{10}^{19} \frac{\sin x}{1+x^8}\right| \leq 10^{-8}$

$\therefore \left|\int_{10}^{19} \frac{\sin x}{1+x^8}\,dx\right| \leq \int_{10}^{19} 10^{-8}\,dx$

$=(19-10)\times 10^{-8}$

$=9\times 10^{-8}$

$=(10-1)\times 10^{-8}$

$=10^{-7}-10^{-8} < 10^{-7}$

Hence, $\left|\int_{10}^{19} \frac{\sin x}{1+x^8}\,dx\right| < 10^{-7}$

41. The value of $\int_{a}^{a+\pi/2} (\sin^4 x+\cos^4 x)dx$ is

A. Independent of a B. $a\left(\frac{\pi}{2}\right)^2$ C. $3\pi/8$ D. $\frac{3}{8}\pi a^2$

Key. A,C

Sol. Let $f(x)=\sin^4 x+\cos^4 x$, it is easy to show that $f(x)$ is periodic with period $\pi/2$

$\therefore \int_{a}^{a+\pi/2} (\sin^4 x+\cos^4 x)dx$

$=\int_{0}^{\pi/2} (\sin^4 x+\cos^4 x)dx$ (by property)

$=\frac{3.1}{4.2}.\frac{\pi}{2}+\frac{3.1}{4.2}.\frac{\pi}{2}$ (by wall's formula)

SMART ACHIEVERS LEARNING PVT. LTD.

$= \frac{3\pi}{8}$

42. If $I = \int_0^1 \sqrt{(1+x^3)}dx,$ then

A. $I < 1$

B. $I \neq \sqrt{5}/2$

C. $I < \sqrt{7}/2$

D. None of these

Key. A,C

Sol. Since, $I = \int_0^1 \sqrt{(1+x^3)}dx$

$\because 0 < x < 1$

$\therefore x^3 < x$

Or $\sqrt{(1+x^3)} < \sqrt{(x+1)}$

$\int_0^1 \sqrt{(1+x^3)}dx < \int_0^1 \sqrt{(1+x)}dx$

$\Rightarrow I < \frac{2}{3}[(1+x)^{3/2}]_0^1$

$\Rightarrow I < \frac{2}{3}(2\sqrt{2}-1)$

Hence, $I < 1$ and $I < \sqrt{7}/2$

43. If $\frac{2x}{\pi} < \sin x < x$ for $0 < x < \frac{\pi}{2}$, then the value of the integral $\int_0^{\pi/2} \frac{\sin x}{x} dx$ is

A. > 1

B. < 1

C. $> \frac{\pi}{2}$

D. $< \frac{\pi}{2}$

Key. A,D

Sol. $\because 0 < x < \frac{\pi}{2}$

$\therefore \frac{2}{\pi} < \frac{\sin x}{x} < 1$

$\int_0^{\pi/2} \frac{2}{\pi} dx < \int_0^{\pi/2} \frac{\sin x}{x} dx < \int_0^{\pi/2} 1.dx$

$\Rightarrow 1 < \int_0^{\pi/2} \frac{\sin x}{x} dx < \frac{\pi}{2}$

SMART ACHIEVERS LEARNING PVT. LTD.

44. The value of the integral $\int_0^{\pi/4} \frac{dx}{(a^2\cos^2 x + b^2\sin^2 x)}$ is

A. $\frac{1}{ab}\tan^{-1}\left(\frac{b}{a}\right)(a>0, b>0)$

B. $\frac{1}{ab}\tan^{-1}\left(\frac{b}{a}\right)(a<0, b<0)$

C. $\frac{\pi}{4}(a=1, b=1)$

D. None of these

Key. A,B,C

Sol. Let $I = \int_0^{\pi/4} \frac{\sec^2 x dx}{a^2 + b^2\tan^2 x}$

Put $b \tan x = t \Rightarrow \sec^2 x dx = \frac{dt}{b}$

$\therefore I = \frac{1}{b}\int_0^b \frac{dt}{a^2+t^2} = \frac{1}{ab}\left\{\tan^{-1}\left(\frac{t}{a}\right)\right\}_0^b$

$= \frac{1}{ab}\tan^{-1}(\frac{b}{a})$

For $a = b = 1$

$I = \tan^{-1} 1 = \frac{\pi}{4}$.

45. If $I = \int \frac{5x^8 + 7x^6}{\left(x^2+1+2x^7\right)^2} dx$ then I is equal to

A) $\frac{x^7}{2x^7+x^2+1} + C$

B) $\frac{x^5}{x^2+1+2x^7} + C$

C) $\frac{-1}{2x^7+x^2+1} + C$

D) $\frac{p(x)}{q(x)}, \deg p(x) = \deg q(x) = 7$

Key. A,D

Sol. We can write

$$I = \int \frac{5x^8+7x^6}{x^{14}\left(\frac{1}{x^5}+\frac{1}{x^7}+2\right)^2} dx$$

$$= \int \frac{\frac{5}{x^6}+\frac{7}{x^8}}{\left(\frac{1}{x^5}+\frac{1}{x^7}+2\right)^2} dx$$

Put $t = \frac{1}{x^5}+\frac{1}{x^7}+2$, so that

$$I = \int \frac{-dt}{t^2} = \frac{1}{t} + C$$

SMART ACHIEVERS LEARNING PVT. LTD.

$= \dfrac{x^7}{x^2 + 1 + 2x^7} + C$.

46. If $I_1 = \int_0^1 2^{x^2} dx, I_2 = \int_0^1 2^{x^3} dx, I_3 = \int_1^2 2^{x^2} dx$ and $I_4 = \int_1^2 2^{x^3} dx$, then

A) $I_1 > I_2$ B) $I_2 > I_1$ C) $I_3 > I_4$ D) $I_4 > I_3$

Key. A,D

Sol. For $0 < x < 1$

We know that $x^2 > x^3$ and for $1 < x < 2$

We have $x^3 > x^2$.

Thus, $2^{x^2} > 2^{x^3}$ for $x \in (0,1)$

And $2^{x^3} > 2^{x^2}$ for $x \in (1,2)$

So, $I_1 > I_2$ *and* $I_3 < I_4$.

47. Let $I_a = \int_0^\infty \dfrac{e^{-ax} - e^{-x}}{x} dx (\text{for } a \geq 1)$ *and* $I_b = \int_0^\infty \dfrac{e^{-bx} - e^{-x}}{x}$ $(\text{for } b \geq 1)$ then

a) I_a, I_b are both finite b) $I_a = \ln a$ c) $I_a - I_b = \ln \dfrac{a}{b}$ d) $I_a = e^{-a} - \dfrac{1}{e}$

Key. A,B,C

$\dfrac{d}{da}(I_a) = \dfrac{1}{a} \Rightarrow I_a = \ln a + c \quad$ *but* $I_1 = 0$

Sol. $\therefore C = 0$

Hence $I_a = lna$, $I_b = \ln b$ *are finite*

48. Let f(x) be strictly monotonic differentiable function on [0,1]such that f(0)=1,f(1) =-1 then $2\int_{-1}^{1} x(f^{-1}(x) - 1)dx =$

a) $\int_0^1 f^2(x)dx - 1$ b) $\int_0^1 f(x)dx$ exists c) $1 + \int_0^1 f^2(x)dx$ d) is positive

Key. A,B

Sol. Let $f^{-1}(x) = t \Rightarrow x = f(t) \Rightarrow dx = f^1(t)dt$

$\therefore 2\int_{-1}^{1} x(f^{-1}(x) - 1)dx = 2\int_{1}^{0} f(t)(t-1) f^1(t)dt = \int_0^1 f^2(t)dt - 1$ by integration by parts

Also f(x) is differentiable => f(x) is continuous

$\therefore \int_0^1 f(x)dx$ exists

49. If $f(2-x) = f(2+x)$, $f(4-x) = f(4+x)$ and $f(x)$ is a function for which $\int_0^2 f(x)dx = 5$ then $\int_0^{50} f(x)dx =$

(A) 125 (B) $\int_{-4}^{46} f(x)dx$ (C) $\int_1^{51} f(x)dx$ (D) $\int_2^{52} f(x)dx$

Key. A,B,D

SMART ACHIEVERS LEARNING PVT. LTD.

Sol. $f(2-x)=f(2+x)$, $f(4-x)=f(4+x)$

$\Rightarrow f(4-x)=f(2+(2-x))=f(2-(2-x))=f(x)$

$\Rightarrow f(4+x)=f(4-x)=f(x)$

$\Rightarrow f(x)$ is a periodic function with period 4

$$\int_0^{50} f(x)dx=\int_0^{48} f(x)dx+\int_0^{2} f(x)dx$$

$$=12\int_0^{4} f(x)dx+\int_0^{2} f(x)dx \qquad =12\left[\int_0^{2} f(x)dx+\int_2^{4} f(x)dx\right]+5$$

$$=12\left[5+\int_0^{2} f(4-x)dx\right]+5 \qquad =12[5+5]+5$$

= 125

50. Let 'e' be the eccentricity of a hyperbola and $f(e)$ be the eccentricity of its conjugate hyperbola, then

$$\int_1^3 \underbrace{f(f(f......f(e)))}_{\text{(n times)}}de=$$

(A) $2\sqrt{2}$ if 'n'is even (B) $2\sqrt{2}$ if 'n' is odd (C) 4, if 'n' is even (D) 4, if 'n' is odd

Key. B,C

Sol. Let e_1, e_2 are eccentricities of hyperbola and its conjugate

$$\Rightarrow \frac{1}{e_1^2}+\frac{1}{e_2^2}=1 \qquad \frac{1}{e_2^2}=\frac{e_1^2-1}{e_1^2}$$

$$e_2=\frac{e_1}{\sqrt{e_1^2-1}}$$

$$f(e)=\frac{e}{\sqrt{e^2-1}}$$

$$\Rightarrow f(f(e))=f\left(\frac{e}{\sqrt{e^2-1}}\right)=\frac{\frac{e}{\sqrt{e^2-1}}}{\sqrt{\frac{e^2}{e^2-1}-1}}=e$$

$$\therefore \quad \underbrace{f(f(f........f(e))....)}_{\text{(n times)}}=\begin{cases}\frac{e}{\sqrt{e^2-1}} & if\ 'n'\ is\ odd\\ e & if\ 'n'\ is\ even\end{cases}$$

51. If $x>0$ and $\int_0^x [x]dx=[x]\left(\frac{A}{2}+B\right)$ where $[.]$ denotes g.I.f., then

(A) $A=[x]-1$ (B) B = $x-[x]$ (C) $A=[x]+1$ (D) $B=x+[x]$

Key. A,B

Sol. Let $x=n+f$, $\quad n\in I^+$, $\quad 0<f<1$

SMART ACHIEVERS LEARNING PVT. LTD.

$\Rightarrow \int_0^x [x]dx = \int_0^n [x]dx + \int_n^{n+f} [x]dx$

$= \frac{n(n-1)}{2} + \int_n^{(n+f)} n\,dx \qquad = \frac{n(n-1)}{2} + n(x)_n^{n+f} \qquad = \frac{n(n-1)}{2} + n\,f$

$= [x]\left(\frac{[x]-1}{2} + x - [x]\right)$

52. Let $f:(0,\infty) \to R$ and $F(x) = \int_0^x f(t)\,dt$. If $F(x^2) = x^2(1+x)$, then

(A) $f(4) = 1$ (B) $f(1) = 4$ (C) $f(4) = 7$ (D) $f(1) = \frac{5}{2}$

Key. D

Sol. $F'(x) = f(x)$

$F'(x^2)2x = 2x(1+x) + x^2 \qquad f(x) = \frac{2x(1+x)+x^2}{2x} = \frac{2(1+x)+x}{2}$

53. $\int \frac{\cos 8x - \cos 7x}{1 + 2\cos 5x} = A\sin 3x + B\sin 2x + c$. Then

(A) $A = \frac{1}{2}$ (B) $B = 2$ (C) $A = \frac{1}{3}$ (D) $B = -\frac{1}{2}$

Key. C,D

Sol. $3A\cos 3x + 2B\cos 2x = \frac{\cos 8x - \cos 7x}{1 + 2\cos 5x}$

$\Rightarrow 3A\cos 3x + 2B\cos 2x + 3A(\cos 8x + \cos 2x) + 2B(\cos 7x + \cos 3x)$
$= \cos 8x - \cos 7x$

Comparing, $3A = 1$, $2B = -1$

54. If $\int \frac{e^{2x} - 2e^x}{e^{2x}+1}dx = A\ln\left(e^{2x}+1\right) + B\tan^{-1}\left(e^x\right) + c$, then

(A) $A = \frac{1}{2}$ (B) $A = -2$ (C) $B = -2$ (D) $B = \frac{1}{2}$

Key. A,C

Sol. $\int \frac{e^{2x}}{e^{2x}+1}dx - 2\int \frac{e^x}{e^{2x}+1}dx = \frac{1}{2}\ln\left(1+e^{2x}\right) - 2\tan^{-1}\left(e^x\right) + c$

55. If $\int \sqrt{x + \sqrt{x^2+2}}\,dx = A\left\{x + \sqrt{x^2+2}\right\}^{3/2} + \frac{B}{\sqrt{x + \sqrt{x^2+2}}} + c$, then

(A) $A = \frac{1}{3}$ (B) $B = -2$ (C) $A = \frac{2}{3}$ (D) $B = -1$

Key. A,B

Sol. $x + \sqrt{x^2+2} = t \Rightarrow x^2 + 2 = t^2 + x^2 - 2tx \qquad \Rightarrow \qquad x = \frac{1}{2}\left(t - \frac{2}{t}\right)$

So $\int \sqrt{x + \sqrt{x^2+2}}\,dx = \frac{1}{2}\int t^{1/2}\left(1 + \frac{2}{t^2}\right)dt$

$I = \frac{1}{2}\int t^{1/2}dt + \int t^{-3/2}dt = \frac{1}{3}t^{3/2} - \frac{2}{\sqrt{t}} + c$

SMART ACHIEVERS LEARNING PVT. LTD.

56. If $\int \frac{\ln}{x^3} dx = A\frac{\ln x}{x^2} + \frac{B}{x^2} + c$, then

(A) $A = \frac{1}{2}$ (B) $\frac{1}{4}$ (C) $A = -\frac{1}{2}$ (D) B = $-\frac{1}{4}$

Key. C,D

Sol. $\int \frac{\ln x}{x^3} dx = (\ln x)\left(-\frac{1}{2x^2}\right) + \int \frac{1}{x}.\frac{1}{2x^2} dx = \left(-\frac{1}{2}\right)\frac{\ln x}{x^2} + \left(-\frac{1}{4}\right)\frac{1}{x^2} + c$

57. If $I = \int \frac{5x^8 + 7x^6}{\left(x^2 + 1 + 2x^7\right)^2} dx$ then I is equal to

(A) $\frac{x^7}{2x^7 + x^2 + 1} + C$ (B) $\frac{x^5}{x^2 + 1 + 2x^7} + C$

(C) $\frac{-1}{2x^7 + x^2 + 1} + C$ (D) $\frac{p(x)}{q(x)}, \deg p(x) = \deg q(x) = 7$

Key. A,D

Sol. We can write

$$I = \int \frac{5x^8 + 7x^6}{x^{14}\left(\frac{1}{x^5} + \frac{1}{x^7} + 2\right)^2} dx$$

$$= \int \frac{\frac{5}{x^6} + \frac{7}{x^8}}{\left(\frac{1}{x^5} + \frac{1}{x^7} + 2\right)^2} dx$$

Put $t = \frac{1}{x^5} + \frac{1}{x^7} + 2$, so that

$$I = \int \frac{-dt}{t^2} = \frac{1}{t} + C$$

$$= \frac{x^7}{x^2 + 1 + 2x^7} + C.$$

58. If $I_1 = \int_0^1 2^{x^2} dx, I_2 = \int_0^1 2^{x^3} dx, I_3 = \int_1^2 2^{x^2} dx$ and $I_4 = \int_1^2 2^{x^3} dx$, then

(A) $I_1 > I_2$ (B) $I_2 > I_1$

(C) $I_3 > I_4$ (D) $I_4 > I_3$

Key. A,D

Sol. For $0 < x < 1$

We know that $x^2 > x^3$ and for $1 < x < 2$

We have $x^3 > x^2$.

Thus, $2^{x^2} > 2^{x^3}$ for $x \in (0,1)$

And $2^{x^3} > 2^{x^2}$ for $x \in (1,2)$

So, $I_1 > I_2$ *and* $I_3 < I_4$.

59. The value of $\int_{1/e}^{\tan x} \frac{t\,dt}{1+t^2} + \int_{1/e}^{\cot x} \frac{dt}{t\left(1+t^2\right)}$ is

SMART ACHIEVERS LEARNING PVT. LTD.

(A) $\dfrac{1}{2+\tan^2 x}$ (B) 1

(C) $\dfrac{\pi}{4}$ (D) $\dfrac{2}{\pi}\int_{-1}^{1}\dfrac{dt}{1+t^2}$

Key. B,D

Sol. Let $I=\int_{1/e}^{\cot x}\dfrac{dt}{t\left(1+t^2\right)}$

Put $t=\dfrac{1}{z}$

$\therefore \quad dt=-\dfrac{1}{z^2}dz$

$$\therefore \quad I=\int_{e}^{\tan x}\frac{-\frac{1}{z^2}dz}{\frac{1}{z}\left(1+\frac{1}{z^2}\right)}=\int_{\tan x}^{e}\frac{z\,dz}{\left(z^2+1\right)}$$

$$=\int_{\tan x}^{e}\frac{t\,dt}{\left(1+t^2\right)}$$

$$\therefore \int_{1/e}^{\tan x}\frac{t\,dt}{\left(1+t^2\right)}+\int_{1/e}^{\cot x}\frac{dt}{t\left(1+t^2\right)}$$

$$=\int_{1/e}^{\tan x}\frac{t}{\left(1+t^2\right)}dt+\int_{\tan x}^{e}\frac{t}{\left(1+t^2\right)}dt$$

$$=\int_{1/e}^{e}\frac{t\,dt}{\left(1+t^2\right)}=\frac{1}{2}\left[ln\left(1+t^2\right)\right]_{1/e}^{e}$$

$$=\frac{1}{2}\left\{In\left(1+e^2\right)-ln\left(1+\frac{1}{e^2}\right)\right\}$$

$$=\frac{1}{2}\left(ln\,e^2\right)=1.$$

Also, $\dfrac{2}{\pi}\int_{-1}^{1}\dfrac{dt}{1+t^2}=\dfrac{4}{\pi}\int_{0}^{1}\dfrac{dt}{1+t^2}=\dfrac{4}{\pi}.\tan^{-1}1$

$$=\frac{4}{\pi}.\frac{\pi}{4}=1.$$

60. If $\int\dfrac{\mathrm{dx}}{\cos^3 x-\sin^3 x}=A\tan^{-1}(\sin x+\cos x)+B\ln\left|\dfrac{\cos x+\sin x-\sqrt{2}}{\cos x+\sin x+\sqrt{2}}\right|+C$, then

(A) $A=2/3$ (B) $B=-2/3\sqrt{2}$

(C) $A=-2/3$ (D) $B=-1/3\sqrt{2}$

Key. A,D

Sol. $$\int\frac{\mathrm{dx}}{\cos^3 x-\sin^3 x}=\int\frac{\mathrm{dx}}{(\cos x-\sin x)\left(1+\frac{\sin 2x}{2}\right)}=\int\frac{(\cos x-\sin x)\mathrm{dx}}{(1-\sin 2x)\left(1+\frac{\sin 2x}{2}\right)}$$

Let $\cos x+\sin x=t$

SMART ACHIEVERS LEARNING PVT. LTD.

$$= \int \frac{2dt}{(2-t^2)(t^2+1)} = \frac{2}{3}\tan^{-1}(\sin x + \cos x) - \frac{1}{3\sqrt{2}}\ln\left|\frac{\cos x + \sin x - \sqrt{2}}{\cos x + \sin x + \sqrt{2}}\right| + C$$

61. If $\int \frac{dx}{5+4\cos x} = K\tan^{-1}\left(m\tan\frac{x}{2}\right) + c$ then

a) $K = 1$

b) $K = \frac{2}{3}$

c) $m = \frac{1}{3}$

d) $m = \frac{2}{3}$

Key. B,C

Sol. Put $\tan\frac{x}{2} = t$ then $\int \frac{dx}{5+4\cos x} = 2\int \frac{dt}{5\left(1+t^2\right) + 4\sqrt{1-t^2}}$

$$= 2\int \frac{dt}{t^2+9} = \frac{2}{3}\tan^{-1}\frac{t}{3} + c$$

$$= \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right) + c$$

$\therefore\ K = \frac{2}{3}$ and $M = \frac{1}{3}$

62. If $\int \frac{\sin x}{\sin\left(x - \frac{\pi}{4}\right)} dx = A\left[f(x) + \log\left|\sin x - \cos x\right|\right] + c$ then

a) $A = \sqrt{2}$

b) $A = \frac{1}{\sqrt{2}}$

c) $f(x) = \sin x$

d) $f(x) = x$

Key. B,D

Sol. $\int \frac{\sin x}{\sin\left(x - \frac{\pi}{4}\right)} dx = \int \frac{\sin\left(u + \frac{\pi}{4}\right)}{\sin u} du$ (put $u = x - \frac{\pi}{4}$)

$$= \int \frac{\sin u\cos\frac{\pi}{4} + \cos u\sin\frac{\pi}{4}}{\sin u} du$$

$$= \frac{1}{\sqrt{2}}u + \frac{1}{\sqrt{2}}\log\left|\sin u\right| + c$$

$$= \frac{1}{\sqrt{2}}\left(x + \log\left|\sin x - \cos x\right|\right) + c$$

SMART ACHIEVERS LEARNING PVT. LTD.

$\therefore \quad A = \frac{1}{\sqrt{2}}, f(x) = x$

63. If $f^1(x) = \frac{1}{-x+\sqrt{x^2+1}}$ and $f(0) = -\left(\frac{1+\sqrt{2}}{2}\right)$ then $f(1)$ is equal to

a) $-\frac{1}{2}\log\left(\sqrt{2}-1\right)$ b) 1

c) $1+\sqrt{2}$ d) $\frac{1}{2}\log\left(1+\sqrt{2}\right)$

Key. A,D

Sol. $f(x) = \int\left(x+\sqrt{x^2+1}\right)dx$

$$= \frac{x^2}{2} + \frac{x}{2}\sqrt{x^2+1} + \frac{1}{2}\log\left|x+\sqrt{x^2+1}\right| + c$$

64. If $J = \int \frac{5x^8+7x^6}{\left(x^2+1+2x^7\right)^2}dx$ then J equals to

a) $\frac{x^7}{2x^7+x^2+1}+c$ b) $\frac{x^5}{x^2+1+2x^7}+c$

c) $\frac{-1}{2x^7+x^2+1}+c$ d) $\frac{p(x)}{q(x)}, \deg = \left(p(x)\right) = \deg = q(x) = 7$

Key. A,D

Sol. $J = \int \frac{5x^8+7x^6}{x^{14}\left(\frac{1}{x^5}+\frac{1}{x^7}+2\right)}dx$

$$= \int \frac{\frac{5}{x^6}+\frac{7}{x^8}}{\left(\frac{1}{x^5}+\frac{1}{x^7}+2\right)}dx$$

Put $t = \frac{1}{x^5}+\frac{1}{x^7}+2$

65. $\int \sin^8 x \cos^6 x \; dx =$

(a) $\frac{\sin^9 x.\cos^5 x}{14} + \frac{5}{14}\int \sin^8 x.\cos^4 x \; dx$

SMART ACHIEVERS LEARNING PVT. LTD.

(b) $-\frac{\sin^7 x.\cos^7 x}{14}+\frac{7}{14}\int \sin^6 x.\cos^6 x\ dx$

(c) $\frac{\sin^7 x.\cos^9 x}{14}+\frac{5}{14}\int \sin^8 x.\cos^4 x\ dx$

(d) $-\frac{\sin^7 x.\cos^7 x}{14}+\frac{5}{14}\int \sin^6 x.\cos^6 x\ dx$

Key. A,B

Sol. Conceptual

66. If $\int \frac{\sin x}{\sin 4x}dx = A\ \ln\left|\frac{1-\sin x}{1+\sin x}\right| + B\ln\left|\frac{1-\sqrt{2}\sin x}{1+\sqrt{2}\sin x}\right| + c$

(a) $A=-\frac{1}{8}$ (b) $A=\frac{1}{8}$ (c) $B=\frac{1}{4\sqrt{2}}$ (d) $B=-\frac{1}{4\sqrt{2}}$

Key. B,D

Sol. $\int \frac{\sin x}{4\sin x.\cos x.\cos 2x}dx = \frac{1}{4}\int \frac{\cos xdx}{(1-\sin^2 x)(1-2\sin^2 x)}(t=\sin x)$

$= \frac{1}{4}\int \frac{dt}{(1-t^2)(1-2t^2)} = \frac{1}{4}\int\left(\frac{2}{1-2t^2}-\frac{1}{1-t^2}\right)dt$

67. $\int \frac{1}{(\sin x+2\cos x)(\cos x+2\sin x)}dx=$

(a) $\frac{1}{3}\ln\left|\frac{2\tan x+1}{\tan x+2}\right|+c$ (b) $\frac{1}{3}\ln\left|\frac{\tan x+2}{\tan x+1}\right|+c$

(c) $\frac{1}{3}\ln\left|\frac{\sin x+\cos x}{2\sin x+\cos x}\right|+c$ (d) $\frac{1}{3}\ln\left|\frac{2\sin x+\cos x}{\sin x+2\cos x}\right|+c$

Key. A,D

Sol. $I=\int \frac{1}{3\sin x}\ \frac{2.(2\sin x+\cos x)-(\sin x+2\cos x)}{(\sin x+2\cos x)(\cos x+2\sin x)}dx$

$=\int\left(\frac{2\sec^2 x}{3\tan^2 x+6\tan x}-\frac{\sec^2 x}{3\tan x+6\tan^2 x}\right)dx$ =

$\frac{2}{3}\int\frac{dt}{t^2+2t}-\frac{1}{3}\int\frac{dt}{t+2t^2}$ $(\because \tan x=t)$

68. If $\int f(x)\sin x.\cos x\ dx=\frac{1}{2(b^2-a^2)}\ \ln|f(x)|+c$ *then* $\frac{1}{f(x)}$ can be

(a) $a^2\sin^2 x+b^2\cos^2 x$ (b) $-a^2\cos^2 x-b^2\sin^2 x$

(c) $\frac{1}{2}\left(b^2-a^2\right)\cos 2x$ (d) $\frac{1}{2}\left(a^2-b^2\right)\cos 2x$

Key. A,B,C

Sol. f(x).sinx.cosx = $\frac{1}{2(b^2-a^2)}\ \frac{f^{|}(x)}{f(x)^2}$ $\Rightarrow 2(b^2-a^2)\sin x.\cos x=\frac{f^{|}(x)}{f(x)^2}$

$\int 2b^2\sin x.\cos xdx-\int 2a^2\sin x\cos xdx=\int\frac{f^{|}(x)}{f(x)}dx$

sinx = t, in Ist and cosx = t in IInd → $b^2\sin^2 x+a^2\cos^2 x=-\frac{1}{f(x)}$

SMART ACHIEVERS LEARNING PVT. LTD.

Let cosx = t is ist, sinx = t in IInd $\Rightarrow -b^2\cos^2 x - a^2\sin^2 x = -\frac{1}{f(x)}$

$$\int (b^2-a^2)\sin 2x dx = \int \frac{f^{|}(x)dx}{f(x)^2} \Rightarrow -\frac{1}{2}(b^2-a^2)\cos 2x = -\frac{1}{f(x)}$$

69. The area of the region containing the points satisfying $|y| + ½ \le e^{-|x|}$ and max $(|x|, |y|) \le 2$. is
(A) 2 – ln 4 sq. unit (B) ln $(e^2/4)$ sq. unit
(C) 2 + ln 4 sq. unit (D) ln $(e^2.4)$ sq. unit

Key. A,B

Sol. Required area is equal to $4\int_0^{\ln 2}\left(e^{-x}-\frac{1}{2}\right)dx$

70. The area bounded by the curve y = f(x), x-axis and the ordinates x = 1 and x = b is (b – 1) sin (3b + 4), then
(A) f(x) = (x – 1) cos (3x + 4)
(B) f(x) = 0 has atleast one root (1/2, 2)
(C) f(x) = sin (3x + 4) + 3 (x – 1) cos (3x + 4)
(D) f(x) = 0 has atleast one root in (1, 2)

Key. B,C,D

Sol. Clearly $\int_4^b f(x)dx = (b-1)\sin(3b+4)$

Now apply Leibnitz theorem

SMART ACHIEVERS LEARNING PVT. LTD.

SMART ACHIEVERS LEARNING PVT. LTD.

Definite, Indefinite Integration & Areas

Assertion Reasoning Type

(A) STATEMENT–1 is True, STATEMENT–2 is True; STATEMENT–2 is a correct explanation for STATEMENT–1

(B) STATEMENT–1 is True, STATEMENT–2 is True; STATEMENT–2 is NOT a correct explanation for STATEMENT–1

(C) STATEMENT–1 is True, STATEMENT–2 is False

(D) STATEMENT–1 is False, STATEMENT–2 is True

1. Statement-1: If $I_n = \int \cot^n x\,dx$ *then* $5(I_6 + I_4) = -\cot^5 x$

   Because

   Statement-2: If $I_n = \int \cot^n x\,dx$ *then* $I_n = -\dfrac{\cot^{n-1} x}{n} - I_{n-2}$, *where* $n \ge 2$.

Key. C
Sol. Conceptual

2. Statement-1: If $\int \dfrac{1}{f(x)}dx = 2\ln|f(x)| + C$, *then* $f(x) = \dfrac{x}{2}$

   Because

   Statement-2: When $f(x) = \dfrac{x}{2}$ *then* $\int \dfrac{1}{f(x)}dx = \int \dfrac{2}{x}dx = 2\ln|x| + C$

Key. D
Sol. Conceptual

3. Statement-1: The function $F(x) = \int \dfrac{x}{(x-1)(x^2+1)}dx$ is discontinuous at x=1

   Statement-2: If $f(x)$ is discontinuous at x=a then $F(x) = \int f(x)dx$ is also discontinuous at x=a.

Key. C
Sol. Conceptual

4. Consider the following statements

   Statement-1: $f(x) = \cos x$ *and* $g(x) = -\sin x$ are both periodic functions.

   Because

   Statement-2: If $f(x)$ is periodic then its antiderivative function is also a periodic function.

Key. C
Sol. Conceptual

5. Statement – 1 : $\int_0^1 e^{\sin x}\,dx = \lambda$, then $\int_0^{200} e^{\sin x}\,dx = 200\lambda$

Statement – 2 : $\int_0^{na} f(x)dx = n\int_0^{a} f(x)dx,\ n \in I$ and f(x) is a periodic function with period equal to 'a'

Key. D

Sol. Conceptual

6. Statement – 1 : $\int_0^{1/4} \frac{dx}{1+(\cot 2\pi x)^{\sqrt{2}}} = \frac{1}{8}$

Statement – 2 : If $a+b=\pi/2$, then $\int_a^b \frac{f(\sin x)}{f(\sin x)+f(\cos x)}dx = b-a$

Key. C

Sol. Conceptual

7. Statement – 1: $\int_{-2}^{2} \log\left(\frac{1+x}{1-x}\right)dx = 0$

Statement – 2 : If 'f' is an odd function $\int_{-a}^{a} f(x)dx = 0$

Key. A

Sol. Conceptual

8. Statement – 1 : $\int_{-1}^{1} \frac{\sin x - x^2}{3-|x|}$ is same as $\int_0^1 \frac{-2x^2}{3-|x|}dx$

Statement – 2: $\frac{\sin x}{3-|x|}$ is an odd function

Key. A

Sol. Conceptual

9. Let $f(x)$ be a continuous periodic function with period T and let $I = \int_a^{a+3T} f(x)dx$, then

Statement - 1: I is independent of 'a', because

Statement - 2: $\frac{dI}{da} = 0$.

Key. A

Sol. Conceptual

SMART ACHIEVERS LEARNING PVT. LTD.

10. Let $f(x)=e^{\sin^2 x}\cos^3 2nx, n\in N$ then,

Statement - 1: $\int\limits_0^{2\pi} f(x)dx = 4\int\limits_0^{\pi/2} f(x)dx$, because

Statement - 2: $f(x)$ is an even function.

Key. C

Sol. Conceptual

11. Let $I(x)=\int e^{x^2}\left(\dfrac{x^3-2x^2+x}{(1+x^2)^2}\right)dx$,

Statement - 1: $I(x)=\dfrac{e^{x^2}}{x^2+1}+C$, because

Statement - 2: $\int \dfrac{(x-1)^2}{(1+x^2)^2}dx=\tan^{-1}x+\dfrac{1}{1+x^2}+C$

Key. D

Sol. Conceptual

12. Let $\int \dfrac{3e^x-4e^{-x}}{e^x+2e^{-x}}dx = Ax+B\ln\left(1+2e^{-2x}\right)+C$ then,

Statement - 1: $A+B=\dfrac{7}{2}$, because

Statement - 2: $\int \dfrac{dx}{1+e^{2x}}=-\dfrac{1}{2}\ln\left(1+e^{-2x}\right)+C$.

Key. B

Sol. Conceptual

13. f : [1, 13]R $\rightarrow$ be a integrable function with $f''(x)>0\ \forall x\in R$

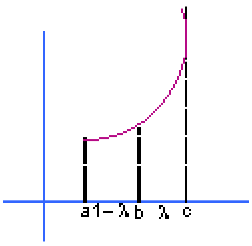
STATEMENT-1 : $\int\limits_1^3 f(x)dx+\int\limits_{11}^{13} f(x)dx \ge \int\limits_5^9 f(x)dx$

STATEMENT-2 : If a < b < c and $f''(x)>0$ then $f(a-b+c)\le f(a)+f(c)-f(b)$

Key: A

Hnt: Let $b=\lambda a+(1-\lambda)c$

$\Rightarrow$ $f(b)\le \lambda f(a)+(1-\lambda)f(c)$ -------- (i)



Now, $a-b+c=(a+c)-(\lambda a+(1-\lambda)c)$

SMART ACHIEVERS LEARNING PVT. LTD.

$\Rightarrow \quad (1-\lambda)a+\lambda c$

$f(a-b+c)\le(1-\lambda)f(a)+\lambda f(c)$ ---------- (ii)

On adding Eqs. (i) and (ii), we get

$F(a-b+c)+f(b)\le f(a)+f(c)$

So statement II is true.

Now, let c = a + 10 and b = a + 4

$\Rightarrow \quad f(a+6)+f(a+4)\le f(a)+f(a+10)$

On integrate both sides in $a\in[1,3]$

$$\int_1^3 f(a+6)da+\int_1^3 f(a+4)da\le\int_1^3 f(a)da+\int_1^3 f(a+10)da$$

$$\Rightarrow \int_7^9 f(x)dx+\int_5^7 f(x)dx\le\int_1^3 f(x)dx+\int_{11}^{13} f(x)dx$$

$$\Rightarrow \int_5^9 f(x)dx\le\int_1^3 f(x)dx+\int_{11}^{13} f(x)dx.$$

Hence (a) is the correct answer.

14. STATEMENT–1: $\displaystyle\int_1^\infty \frac{x^2-2}{x^3\sqrt{x^2-1}}dx=0$

STATEMENT–2: $f(x)=\dfrac{x^2-2}{x^3\sqrt{x^2-1}}$ is an odd function

Key: B

Hint: $I=\displaystyle\int_1^\infty \frac{x^2-2}{x^3\sqrt{x^2-1}}dx=\int_1^\infty \frac{x^2}{x^3\sqrt{x^2-1}}dx-2\int_1^\infty \frac{dx}{x^3\sqrt{x^2-1}}$

Put $x=\sec\theta$

$dx=\sec\theta\tan\theta d\theta$

$$\therefore I=\int_0^{\frac{\pi}{2}}\frac{\sec\theta.\tan\theta d\theta}{\sec\theta.\tan\theta}-2\int_0^{\frac{\pi}{2}}\frac{\sec\theta\tan\theta d\theta}{\sec^3\theta.\tan\theta} \qquad =[\theta]_0^{\frac{\pi}{2}}-2\int_0^{\frac{\pi}{2}}\cos^2\theta d\theta \quad =\frac{\pi}{2}-\frac{\pi}{2}=0$$

15. Let T > 0 be a fixed real number. Suppose $f(x)$ is a continuous function for all $x\in R,\ f(x+T)=f(x)$ then

STATEMENT -1: $\displaystyle\int_6^{6+6T} f(4x)dx=6\int_0^T f(x)dx$

STATEMENT -2: $\displaystyle\int_0^{24T} f(x)dx=24\int_0^T f(x)dx$

Key: A

Hint: $\displaystyle\int_a^{a+nT} f(x)dx=\int_0^{nT} f(x)dx=n\int_0^T f(x)dx$

SMART ACHIEVERS LEARNING PVT. LTD.

16. STATEMENT–1: The inequality $\int_0^1 \frac{dx}{1+x^n} > 1-\frac{1}{n}$ is true for $n \in N$.

STATEMENT–2: $\int_0^1 \frac{x^{n-1}-x^n+x^{2n-1}}{1+x^n}dx > 0, \ \forall \ n \in N$.

Key: A

Hint: $\because \int_0^1 \frac{x^{n-1}-x^n+x^{2n-1}}{1+x^n} dx > 0$

$$\Rightarrow \int_0^1 \left\{\frac{1}{1+x^n} - \left(1-x^{n-1}\right)\right\} dx > 0$$

$$\Rightarrow \int_0^1 \frac{dx}{1+x^n} > \int_0^1 \left(1-x^{n-1}\right)dx$$

$$\Rightarrow \int_0^1 \frac{dx}{1+x^n} > 1-\frac{1}{n}; \ \forall \ n \in N.$$

17. STATEMENT- 1: $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{dx}{(1+10^x)(1+x^2)} = \frac{\pi}{3}$

STATEMENT 2: $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

Key: C

Hint: Conceptual

18. Statement-1: If $F(x) = \int_1^x \frac{\ln t}{1+t+t^2}dt$, then $F(x) = -F(1/x)$.

Statement-2: If $F(x) = \int_1^x \frac{\ln t}{t+1}dt$, then F(x) + F(1/x) = (1/2) (ln x)^2.

Key: D

Hint: $F(1/x) = \int_1^{1/x} \frac{\ln t}{1+t+t^2}dt = \int_1^x \frac{\ln(1/u)}{1+\frac{1}{u}+\frac{1}{u^2}}\left(-\frac{du}{u^2}\right) \qquad \left(u = \frac{1}{t}\right)$

$$= \int_1^x \frac{\ln u}{u^2+u+1}du = F(x)$$

So statement-1 is not true.

If F(x) = $\int_1^x \frac{\ln t}{t+1}dt$ then

SMART ACHIEVERS LEARNING PVT. LTD.

$$F(x)+F(1/x)=\int_1^x \frac{\ln t}{t+1}dt+\int_1^{1/x}\frac{\ln t}{t+1}dt=\int_1^x \frac{\ln t}{t+1}dt+\int_1^x \frac{\ln(1/u)}{1+\frac{1}{u}}\left(-\frac{du}{u^2}\right)$$

$$=\int_1^x \ln t\left(\frac{1}{t+1}+\frac{1}{t^2+1}\right)dt-\int_1^x \frac{\ln t}{t+1}\left(1+\frac{1}{t}\right)dt=\int_1^x \frac{\ln t}{t}dt=\frac{(\ln x)^2}{2}$$

19. Statement - 1: If $I_n=\int_0^1 (1-x^5)^n dx$ then $\frac{I_{10}}{I_{11}}=\frac{56}{55}$.

Statement - 2: If $y(x)=\int_{\frac{\pi^2}{16}}^{x^2}\frac{\cos x.\cos\sqrt{\theta}}{1+\sin^2\sqrt{\theta}}d\theta$ then $\left(\frac{dy}{dx}\right)_{x=\pi}=2\pi$.

Key: A

Hint Conceptual Question

20. Statement- I: $\int_0^{\pi/2}\left(\sin^6 x+\cos^6 x\right)dx$ lies in the interval $\left(\frac{\pi}{8},\frac{\pi}{2}\right)$.

Statement – II: $\sin^6 x+\cos^6 x$ is periodic with period $\pi/2$.

Key. B

Sol. $\because \sin^6 x+\cos^6 x=\left(\sin^2 x\right)^3+\left(\cos^2 x\right)^3$

$$=\left(\sin^2 x+\cos^2 x\right)^3-3\sin^2 x\cos^2 x\left(\sin^2 x+\cos^2 x\right)$$

$$=1-3\sin^2 x\cos^2 x$$

$$=1-\frac{3}{4}\sin^2 2x=\frac{5}{8}+\frac{3}{8}\cos 4x \quad \left(\therefore period=\frac{2\pi}{4}=\frac{\pi}{2}\right)$$

$\therefore$ Least and greatest value of $\sin^6 x+\cos^6 x$ are $\frac{1}{4}$ and 1

Hence,

$$\left(\frac{\pi}{2}-0\right)\times\frac{1}{4}<\int_0^{\pi/2}\left(\sin^6 x+\cos^6 x\right)dx<\left(\frac{\pi}{2}-0\right)\times 1$$

$$\Rightarrow \quad \frac{\pi}{8}<\int_0^{\pi/2}\left(\sin^6 x+\cos^6 x\right)dx<\frac{\pi}{2}.$$

21. Statement- I: If $f(x)=x-x^2+1$ and $g(x)=\max\{f(t):0\le t\le x\}$, then $\int_0^1 g(x)dx=\frac{29}{24}$.

Statement – II: $f(x)$ is increasing in $\left(0,\frac{1}{2}\right)$ and decreasing in $\left(\frac{1}{2},1\right)$.

Key. A

Sol. Conceptual

22. STATEMENT 1. : If root of $ax^2+bx+c=0$ are non real and $c=\int_0^{\sin^2 x}\sin^{-1}\sqrt{x}dx+\int_0^{\cos^2 x}\cos^{-1}\sqrt{x}dx$ then 'a' must be positive.

because

SMART ACHIEVERS LEARNING PVT. LTD.

STATEMENT 2. : If roots of quadratic equation $ax^2 + bx + c = 0$ are non-real then 'a' and 'c' have same sign.

Key. A

Sol. Let $f(x) = \int_0^{\sin^2 x} \sin^{-1}\sqrt{x}dx + \int_0^{\cos^2 x} \cos^{-1}\sqrt{x}dx$

$f'(x) = 0$

$\Rightarrow$ f(x) = constant.

And $f\left(\frac{\pi}{4}\right) = \int_2^{1/2} \frac{\pi}{2}dx = \frac{\pi}{4}$

$\Rightarrow$ $c = \frac{\pi}{4}$

as roots of $g(x) = ax^2 + bx + c = 0$ are non real

$\Rightarrow$ D < 0

And g(0) = c > 0

$\Rightarrow$ a > 0

23. Statement–1 : If $2f(x) + f(-x) = \frac{1}{x}\sin\left(x - \frac{1}{x}\right)$, then value of $I = \int_{1/e}^{e} f(x)dx = 0$

Statement–2 : If $f(2a - x) = -f(x)$, then $\int_0^{2a} f(x)dx = 0$

Key. B

Sol. $2f(x) + f(-x) = 2f(-x) + f(x) \Rightarrow f(x) = f(-x)$

$f(x) = \frac{1}{3x}\sin\left(x - \frac{1}{x}\right)$

$I = \int_{1/e}^{e} \frac{1}{3x}\sin\left(x - \frac{1}{x}\right)dx = -\int_{1/e}^{e} \frac{1}{3t}\sin\left(t - \frac{1}{t}\right)dx = -I \quad \Rightarrow \quad I = 0$

24. Let f(x) and g(x) be continuous functions of x in (a, b) and f(x) < g(x) < 0 $\forall$ x$\in$ (a, b) then

STATEMENT–1

Area of the region bounded by y = f(x), x–axis, x = a and x = b is less than the area of the region bounded by y = g(x), x–axis, x = a and x = b.

because

STATEMENT–2

$|f(x)| > |g(x)| \quad \forall x \in (a, b)$

Key. D

Sol. Clearly |f(x)| > |g(x)|

So, area $A_1 = \int_a^b |f(x)|dx > \int_a^b |g(x)|dx = A_2$

25. If $f(x) = \frac{e^x}{1+e^x}$

Statement – 1 : $I_1 = \int_{f(-a)}^{f(a)} xg(x(1-x))dx$ and $I_2 = \int_{f(-a)}^{f(a)} g(x(1-x))dx$ then the value of $\frac{I_2}{I_1}$ is 2.

Because

SMART ACHIEVERS LEARNING PVT. LTD.

Statement – 2 : Here $f(a)+f(-a)=1$ and also by the property

$$\int_p^q f(x)\,dx=\int_p^q f(p+q-x)\,dx$$.

Key. A

Sol. $2f'(x)=\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}+\frac{f(x-h)-f(x)}{-h}\right)$

$2f'(x)=f(x).\lim_{h\to 0}\left(\frac{f(h)-f(0)}{h}+\frac{f(-h)-f(0)}{-h}\right)$

$\Rightarrow 2f'(x)=2f(x)\frac{\pi}{2} \Rightarrow f(x)= e^{\frac{\pi}{2}x}+c$

Put x = 0 in the given relation

$\Rightarrow f(y)=f(-y)$ which is not possible or

$f(0)=1 \Rightarrow f(0)=e^c=1 \Rightarrow c=0$

$\therefore\ f(x)=e^{\frac{\pi}{2}x}$

26. Statement – 1 : y – axis divides the area bounded by $y=1+x+|x|$ and $y=\alpha-x^2$ in two equal parts for all $\alpha>1$.

Because

Statement – 2 : In any quadratic equation $ax^2+bx+c=0\ \ a\neq 0$, roots are imaginary if $b^2-4ac<0$.

Key. D

Sol. Coordinates of D and C are $\left(-1+\sqrt{\alpha},\ -1+2\sqrt{\alpha}\right)$ and $\left(-\sqrt{\alpha-1},1\right)$ respectively According to the given condition

$\int_{-\sqrt{\alpha-1}}^{0}\left(\alpha^2-x^2-1\right)dx=\int_0^{-1+\sqrt{\alpha}}\left(\alpha-x^2-1-2x\right)dx$

$\Rightarrow 144\alpha^2-105\alpha+25=0$ since roots of this equation are imaginary

$\Rightarrow$ there is no value of α.

27. Statement – 1 : The value of the definite integral $\int_{1+2\pi}^{1+5\frac{\pi}{2}}\left(\sin^{-1}(\cos x)+\cos^{-1}(\sin x)\right)dx$ is equal to $\frac{\pi^2}{4}$.

Because

Statement – 2 : The period of the function $\sin^{-1}x+\cos^{-1}x$ is $\frac{\pi}{2}$.

Key. C

Sol. The period of the function $\sin^{-1}(\cos x)+\cos^{-1}(\sin x) is\ 2\pi$

$\therefore \int_0^{\frac{\pi}{2}}\sin^{-1}(\cos x)+\cos^{-1}(\sin x)$

SMART ACHIEVERS LEARNING PVT. LTD.

$= \therefore \int_0^{\frac{\pi}{2}} (\pi - 2x)\, dx$

$= \frac{\pi^2}{4}.$

28. Let $\int \frac{x + (\cos^{-1} 3x)^2}{\sqrt{1-9x^2}} dx = A\sqrt{1-9x^2} - B(\cos^{-1} 3x)^3 + c$, where c is a constant of integration, then

STATEMENT–1

A + B = 0

because

STATEMENT–2

$A = \frac{2}{9}$ and $B = -\frac{2}{9}$

Key. C

Sol. Put $3x = \cos\theta \Rightarrow 3dx = -\sin\theta\, d\theta$

$\therefore -\frac{1}{3} \int \frac{\frac{\cos\theta}{3} + \theta^2}{\sin\theta} . \sin\theta\, d\theta = -\frac{1}{3} \int \left(\frac{1}{3}\cos\theta + \theta^2 \right) d\theta$

$= -\frac{1}{9} \sin\theta - \frac{1}{9} \theta^3 + c$

$= -\frac{1}{9}\sqrt{1-9x^2} - \frac{1}{9}(\cos^{-1} 3x)^3 + c$

Clearly $A = -\frac{1}{9}\ \&\ B = \frac{1}{9}$

A + B = 0.

29. Statement – 1 : $\int_1^{\infty} \frac{x^2 - 2}{x^3\sqrt{x^2 - 1}} dx = 0$

Because

Statement – 2 : $f(x) = \frac{x^2 - 2}{x^3\sqrt{x^2 - 1}}$ is an odd function.

Key. B

Sol. Put $x = \sec\theta$

$I = \int_0^{\pi/2} d\theta - 2\int_0^{\pi/2} \frac{\sec\theta\tan\theta}{\sec^2\theta\tan\theta} d\theta$

$= \frac{\pi}{2} - 2.\frac{1}{2}\frac{\pi}{2} = 0$

30. STATEMENT -1: $\int_0^{1/4} \frac{dx}{1 + (\cot 2\pi x)^{\sqrt{2}}} = \frac{1}{4}$

because

STATEMENT-2: If $a + b = \frac{\pi}{2}$, then $\int_a^b \frac{f(\sin x)}{f(\sin x) + f(\cos x)} dx = \frac{b-a}{2}$

Key. D

SMART ACHIEVERS LEARNING PVT. LTD.

Sol. $I = \frac{1}{2\pi}\int_0^{\pi/2} \frac{(\sin x)^{\sqrt{2}}}{(\sin x)^{\sqrt{2}} + (\cos x)^{\sqrt{2}}} dx = \frac{1}{8}$

$I_1 = \int_a^b \frac{f(\sin x)}{f(\sin x) + f(\cos x)} = \int_a^b \frac{f(\sin x(a+b-x))}{f(\cos x) + f(\sin x)}$

$2I_1 = \int_a^b 1 dx = b - a$

$I_1 = \frac{b-a}{2}$

31. Statement – 1 : $\int e^{3x}(3\sin x + \cos x)dx = e^{3x}\sin x + c$

Because

Statement – 2 : Antiderivative of a periodic function is a periodic function.

Key. C

Sol. Put 3x = t

$\frac{1}{3}\int e^t\left(3\sin\frac{t}{3} + \cos\frac{t}{3}\right)dt = \sin\frac{t}{3} + c = \sin x + c$

Statement-2 is false

32. Statement I : If $n > 2$ then $\int_0^{\infty} \frac{dx}{1+x^n} dx = \int_0^1 \frac{x^{n-2}}{1+x^n} dx$

Statement II : If $n > 2$ then $\int_0^{\infty} \frac{1}{1+x^n} dx > \frac{\pi}{4}$

Key. C

Sol. Put $x = \frac{1}{t}$ in 1st integral of statement -1 , for statement -2 , $x^n \ge x^2 \quad \forall x \ge 1, n > 2$

33. Statement I : For x> 0 , if f(x) = $\int_1^x \frac{\ln t}{1+t} dt$ then f(e) + $f\left(\frac{1}{e}\right) = \frac{1}{2}$

Statement II : For f(x) given in statement 1 , f(x) + $f\left(\frac{1}{x}\right) = \frac{(\ln x)^2}{2}$

Key. A

Sol. $f(x) + f\left(\frac{1}{x}\right) = \int_1^x \frac{\ln t}{1+t} dt + \int_1^{\frac{1}{x}} \frac{\ln t}{1+t} dt$

$= \int_1^x \frac{\ln t}{1+t} dt + \int_1^x \frac{\ln t}{t} dt - \int_1^x \frac{\ln t}{1+t} dt$

$= \frac{(\ln x)^2}{2}$

34. Statement - 1: $I_1, I_2, I_3, ...$ is an increasing sequence

Statement - 2: $\ln$ x is an increasing function

Key. C

Sol. We know that if $1 < x < e$ then $0 < \ln x < 1$, hence $(\ln x)^n$ is decreases as 'n' increases

$\Rightarrow I_n$ is decreasing sequence

SMART ACHIEVERS LEARNING PVT. LTD.

Where as $\log x$ is increasing function.

35. Statement - 1: $\int_0^\infty f\left(x^n + x^{-n}\right)\frac{\log x}{1+x^2}dx = 0$

Statement - 2: Given function is an odd function

Key. C

Sol. Put $\log x = t \quad \Rightarrow \; e = e^t \; \Rightarrow \; dx = e^t\, dt$

$$I = \int_{-\infty}^{\infty} f\left(e^{nt} + e^{-nt}\right)\frac{t}{1+e^{2t}}e^t dt$$

$$= \int_{-\infty}^{\infty} \frac{f\left(e^{nt}+e^{-nt}\right)}{e^t + e^{-t}} t\, dt$$

= 0 ($\because$ function is odd)

36. Statement - 1: $\int \frac{1}{4e^{-x} - 9e^x} dx = \frac{1}{6}\log\left|\frac{2+3e^x}{2-3e^x}\right| + C$

Statement - 2: $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$

Key. D

Sol. $\int \frac{e^x}{2^2 - (3e^x)^2} dx = \frac{1}{12}\log\left|\frac{2+3e^x}{2-3e^x}\right| + C$

37. Statement - 1: If $a > 0, b^2 - 4ac < 0$, then the value of $\int \frac{dx}{ax^2+bx+c}$ will be of the type $\lambda Tan^{-1}\left(\frac{x+A}{B}\right) + C$ where A, B, C, λ are constants.

Statement - 2: If $a > 0, b^2 - 4ac < 0$, then $ax^2 + bx + c$ can be written as sum of two squares.

Key. A

Sol. Since $\int \frac{1}{x^2+a^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

38. Statement- I: $\int_0^{\pi/2}\left(\sin^6 x + \cos^6 x\right)dx$ lies in the interval $\left(\frac{\pi}{8}, \frac{\pi}{2}\right)$.

Statement – II: $\sin^6 x + \cos^6 x$ is periodic with period $\pi/2$.

Key. B

Sol. $\because \sin^6 x + \cos^6 x = \left(\sin^2 x\right)^3 + \left(\cos^2 x\right)^3$

$$= \left(\sin^2 x + \cos^2 x\right)^3 - 3\sin^2 x\cos^2 x\left(\sin^2 x + \cos^2 x\right)$$

$$= 1 - 3\sin^2 x\cos^2 x$$

$$= 1 - \frac{3}{4}\sin^2 2x = \frac{5}{8} + \frac{3}{8}\cos 4x \quad \left(\therefore period = \frac{2\pi}{4} = \frac{\pi}{2}\right)$$

$\therefore$ Least and greatest value of $\sin^6 x + \cos^6 x$ are $\frac{1}{4}$ and 1

Hence,

SMART ACHIEVERS LEARNING PVT. LTD.

$$\left(\frac{\pi}{2}-0\right)\times\frac{1}{4}<\int_0^{\pi/2}\left(\sin^6 x+\cos^6 x\right)dx<\left(\frac{\pi}{2}-0\right)\times 1$$

$$\Rightarrow \qquad \frac{\pi}{8}<\int_0^{\pi/2}\left(\sin^6 x+\cos^6 x\right)dx<\frac{\pi}{2}.$$

39. Statement- I: If $f(x)=x-x^2+1$ and $g(x)=\max\{f(t):0\le t\le x\}$, then $\int_0^1 g(x)dx=\frac{29}{24}$.

Statement – II: $f(x)$ is increasing in $\left(0,\frac{1}{2}\right)$ and decreasing in $\left(\frac{1}{2},1\right)$.

Key. A
Sol. Conceptual

40. STATEMENT–1

The area bounded by y = x(x – 1)^2, the y–axis and the line y = 2 is $\int_0^2 (x(x-2)^2-2)\,dx$ is equal to $\frac{10}{3}$.

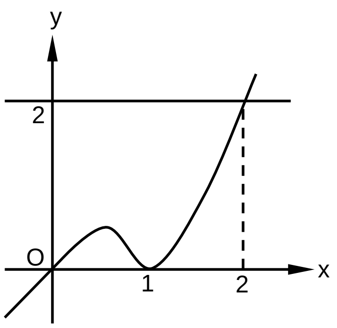
because

STATEMENT–2

The curve y = x(x – 1)^2 is intersected by y = 2 at x = 2 only and for 0 < x < 2, the curve y = x(x – 1)^2 lies below the line y = 2.

Key. A

Sol. Solving y = x(x – 1)^2 and y = 2, we get x = 2. Hence y = x(x – 1)^2 intersects the line y = 2 at x = 2 only.
Statement – II is true because of above and the graphs of y = 2 and y = x(x – 1)^2.
Statement – I is obviously true and it is because of statement – II.
Hence (A) is the correct answer.



41. Statement - 1: The area bounded by the curves $y=x^2+2x-3$ and $y=\lambda x+1$ is least if $\lambda=2$.

Statement - 2: The area bounded by the curve $y=x^2+2x-3$ and $y=\lambda x+1$ is $\left\{(\lambda-2)^2+16\right\}^{3/2}$ sq. units.

Key. C

Sol. Conceptual

42. Statement 1 : $\int \frac{\sqrt{1+\sqrt[3]{x}}}{\sqrt[3]{x^2}}dx = 2\left(1+x^{1/3}\right)^{3/2}+c$

Statement 2 : $\int \frac{dx}{x^3\sqrt[5]{x+\frac{1}{x}}} = \sqrt{x+\frac{1}{x}}+c$

Key. C

Sol. Conceptual

43. Statement 1: $\int \frac{dx}{(x+1)^2\sqrt{x^2+2x+2}} = -\frac{\sqrt{x^2+2x+2}}{x+1}+c$

Statement 2 :

$$\int\sqrt{\left(x^2-1\right)^3}dx = \frac{1}{8}x\left(2x^2-1\right)\sqrt{x^2-1}-\frac{1}{2}x\sqrt{x^2-1}+\frac{3}{8}\log\left(x+\sqrt{x^2-1}\right)+c$$

Key. B

Sol. Conceptual

44. Statement 1: $\int \frac{dx}{\left(x^2+1^2\right)^2} = \frac{x}{2\left(x^2+1\right)}+\frac{1}{2}\tan^{-1}x+c$

Statement 2: If $I_n = \int \frac{dx}{\left(x^2+a^2\right)^n}$ then $I_{n+1} = \frac{1}{2na^2}.\frac{x}{\left(x^2+a^2\right)n}+\frac{2n-1}{2n}\frac{1}{a^2}I_n$

where $n \in N$

Key. D

Sol. Conceptual

45. Statement 1: $\int\left(x^{\sin x-1}.\sin x + x^{\sin x}.\cos x.\ln x\right)dx = x^{\sin x}+c$

Statement 2: $\int \frac{x\cos x+1}{\sqrt{2x^3e^{\sin x}+x^2}}dx = \ln\left|\frac{t-1}{t+1}\right|+c$ where $t^2 = 2xe^{\sin x}+1$

Key. B

Sol. Conceptual

46. ASSERTION (A): $\int \frac{e^x\left(2-x^2\right)}{(1-x)\sqrt{1-x^2}}\, dx = e^x\sqrt{\frac{1-x}{1+x}}+c$

REASON (R): $\int e^x\left(f(x)+f'(x)\right)dx = e^x f(x)+c$

Key. D

Sol. $\int \frac{e^x\left(2-x^2\right)}{(1-x)\sqrt{1-x^2}}\, dx = \int \frac{e^x\left(1+\left(1-x^2\right)\right)}{(1-x)\sqrt{1-x^2}}\, dx$

SMART ACHIEVERS LEARNING PVT. LTD.

$$= \int e^x \left(\frac{1}{(1-x)\sqrt{1-x^2}} + \frac{1+x}{\sqrt{1-x^2}} \right) dx$$

$$= \int e^x \left[\frac{1}{(1-x)^{3/2}(1+x)^{1/2}} + \sqrt{\frac{1+x}{1-x}} \right] dx$$

$$= e^x \sqrt{\frac{1+x}{1-x}} + c \ dx$$

47. ASSERTION (A): $\int \frac{dx}{\tan x + \cot x + \sec x + csex} = \frac{1}{2}(\sin x - \cos x - x) + c$

REASON (R): $\int f^n(x) f'(x) \ dx = \frac{f^{n+1}(x)}{n+1} + c$

Key. B

Sol. $\int \frac{dx}{\tan x + \cot x + \sec x + csex} = \int \frac{\sin x \cos x}{1 + \sin x + \cos x} dx = \int \frac{\sin x}{\sec x + \tan x + 1} dx$

Rationalize the denominator.

48. ASSERTION (A): $f\left(\frac{3x-4}{3x+4}\right) = x + 2$ then $\int f(x) dx = \frac{2}{3}x - \frac{8}{3}\ln|1-x| + c$

REASON (R): $\int \frac{3x-4}{3x+4} dx = x - \frac{8}{3}\log|3x+4| + c$

Key. B

Sol. Let $\frac{3x-4}{3x+4} = t \quad \Rightarrow \frac{6x}{8} = \frac{t+1}{1-t} \quad \Rightarrow x = \frac{4}{3}\left(\frac{1+t}{1-t}\right) + 2$

49. Statement - I : $\sum_{r=0}^{12} \int \cot^{2r} x \, dx = -\sum_{r=1}^{6} \frac{\cot^{4r-1} x}{4r-1} + c$

Statement – II : $\int \cot^n x \, dx + \int \cot^{n-2} x.dx = -\frac{\cot^{n-1} x}{n-1} + c$

Key. D

Sol. $\int \cot^n x dx = \int \cot^{n-2} x.\cot^2 x \, dx = \int \cot^{n-2} x.\cos ec^2 x \, dx - \int \cot^{n-2} x \, dx$

$$\int \cot^n x = \frac{-\cot^{n-1} x}{n-1} - \int \cot^{n-2} x \, dx$$

$I = \int \left(\cot^{24} x + \cot^{22} + ...+1\right) dx = \left(I_{24} + I_{22}\right) + \left(I_{20} + I_{18}\right) + ...+ \left(I_4 + I_2\right) + \int 1 dx$

$= \frac{-\cot^{23} x}{23} - \frac{\cot^{19} x}{19} - .... - \frac{\cot^3 x}{3} + x + c$

50. Statement - I : $\int \frac{x^2+1}{x\sqrt{x^2+2x-1}\sqrt{1-x^2-x}} dx = 2\sin^{-1}\sqrt{x - \frac{1}{x} + 2} + c$

Statement - II : $\int \frac{dt}{\sqrt{1-t^2}} = \sin^{-1} t + c$

Key. A

Sol. $t=\sqrt{x-\frac{1}{x}+2};\ dt=\frac{1}{2\sqrt{x-\frac{1}{x}+2}}.\left(1+\frac{1}{x^2}\right)dx \Rightarrow I=\int\frac{2dt}{\sqrt{1-t^2}}=2\sin^{-1}\sqrt{x+2-\frac{1}{x}}+c$

51. Statement - I : $\int\frac{e^{\cot x}}{\sin^2 x}\left(2\ln\cos ecx+\sin 2x\right)dx$ = $2e^{\cot x}\ln\sin x+c$

Statement - II : $\int e^x\left(f(x)+f^{|}(x)\right)dx=e^x f(x)$+c

Key. A

Sol. $t=\cot x\Rightarrow dt=-\cos ec^2x\,dx \quad I=\int\frac{e^{\cot x}}{\sin^2 x}\left(\ln\cos ec^2x+\frac{2\tan x}{1+\tan^2 x}\right)dx$

$$= -\int e^t\left(\ln(1+t^2)+\frac{\frac{2}{t}}{1+\frac{1}{t^2}}\right)dt \ = -e^t\ln|1+t^2|+c=-2e^{\cot x}\ln|\cos ecx|+c$$

$=2e^{\cot x}\ln\sin x+c$

52. Let F(x) be an indefinite integral of $\sin^2 x$

Statement - I : The function F(x) satisfies F(x+π) = F(x) for all x ∈ R.

Statement - II : $\sin^2(\pi+x)=\sin^2 x \ \ for\ all\ \ x\in R$

Key. D

Sol. F(x) = $\int\sin^2 x dx=\frac{1}{4}\left(2x-\sin 2x\right)+k \Rightarrow$ F(x+π) – F(x) = $\frac{1}{2}\pi\neq 0$

SMART ACHIEVERS LEARNING PVT. LTD.

Definite, Indefinite Integration & Areas

Comprehension Type

Passage: 1

Using the properties of differentiation and integration. Simplify the following.

1. Let g(x) be an anti derivative for f(x). Then $\ell n\left[1+(g(x))^2\right]+C$ is an anti derivative for

A) $\frac{f(x)g(x)}{1+g^2(x)}$ B) $\frac{-f(x)g(x)}{1+g^2(x)}$

C) $-\frac{2f(x)g(x)}{1+g^2(x)}$ D) $\frac{2f(x)g(x)}{1+g^2(x)}$

Key. D
Sol. Conceptual

2. A function f satisfying $f^1(\sin x)=\cos^2 x$ for all x and f(1)=1 is

A) x B) x^2 C) $f(x)=x-\frac{x^3}{3}+\frac{1}{3}$ D) $2-x^2$

Key. C
Sol. Conceptual

3. If

$$x=f^{11}(t)\cos t+f^1(t)\sin t;\ \ y=-f^{11}(t)\sin t+f^1(t)\cos t\ then\ \int\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]^{1/2}dt=$$

A) $f(t)+f^{11}(t)$ B) $f(t)+f^1(t)$

C) $f(t)-f^1(t)$ D) $f^1(t)+f^{11}(t)$

Key. A
Sol. Conceptual

Passage: 2

Integrals of class of functions following a definite pattern can be found by the method of reduction and recursion. Reduction formulas make it possible to reduce an integral dependent on the index n>0, called the order of the integrals, to an integral of the same type with a smaller index. Integration by parts helps us to derive reduction formulas

4. If $I_n=\int\frac{dx}{(x^2+a^2)^n}\ then\ I_{n+1}+\frac{1-2n}{2n}.\frac{1}{a^2}I_n$ is equal to

A) $\frac{x}{(x^2+a^2)^n}$ B) $\frac{1}{2na^2}\frac{1}{(x^2+a^2)^{n-1}}$ C) $\frac{1}{2na^2}\frac{x}{(x^2+a^2)^n}$ D) $\frac{1}{2na^2}\frac{x}{(x^2+a^2)^{n-1}}$

Key. C

SMART ACHIEVERS LEARNING PVT. LTD.

Sol. Conceptual

5. If $I_{n,-m}=\int \frac{\sin^n x}{\cos^m x}dx$ *then* $I_{n,-m}+\frac{n-1}{m-1}I_{n-2,2-m}$ is equal to

A) $\frac{\sin^{n-1} x}{\cos^{m-1} x}$

B) $\frac{1}{(m-1)}\frac{\sin^{n-1} x}{\cos^{m-1} x}$

C) $\frac{1}{(n-1)}\frac{\sin^{n-1} x}{\cos^{m-1} x}$

D) $\frac{n-1}{m-1}\frac{\sin^{n-1} x}{\cos^{m-1} x}$

Key. B

Sol. Conceptual

6. The value of $I_m=\int\left(x^{3m}+x^{2m}+x^m\right)\left(2x^{2m}+3x^m+6\right)^{1/m}dx$ is

A) $\frac{1}{6(m+1)}\left(2x^{2m}+3x^m+6\right)^{(m+1)/m}$

B) $\frac{1}{(m+1)}\left(2x^{2m}+3x^m+6\right)^{(m+1)/m}$

C) $\frac{1}{6(m+1)}\left(2x^{3m}+3x^{2m}+6x^m\right)^{(m+1)/m}$

D) $\frac{1}{6(m+1)}\left(x^{3m}+x^{2m}+x^m\right)^{(m+1)/m}$

Key. C

Sol. Conceptual

Passage – 3

Two real valued differentiable functions $f(x),g(x)$ satisfy the following conditions:

1) $f'(x)=\frac{f(x)-g(x)}{3}$

2) $g'(x)=\frac{2(g(x)-f(x))}{3}$

3) $f(0)=5$

4) $g(0)=-1$ then

7. $\int\frac{1}{f(x)}dx=$

A) $x-ln(f(x))+c$

B) $\frac{1}{2}(x-ln\, f(x))+c$

C) $\frac{1}{3}(x-ln\, f(x))+c$

D) $\frac{1}{x-ln\, f(x)}+c$

Key. C

8. $\int\frac{1}{g(x)}dx=$

A) $x-ln(g(x))+c$

B) $\frac{1}{2}(x-ln|g(x)|)+c$

SMART ACHIEVERS LEARNING PVT. LTD.

C) $\frac{1}{3}\left(x-ln\left|g\left(x\right)\right|\right)+c$ D) $\frac{1}{x-ln\left|g\left(x\right)\right|}+c$

Key. C

9. $\int\frac{g\left(x\right)}{f\left(x\right)}dx=$

A) $x+3ln\,f\left(x\right)+c$ B) $x-3ln\,f\left(x\right)+c$

C) $3ln\,f\left(x\right)-x+c$ D) $3\left(x-ln\,f\left(x\right)\right)+c$

Key. B

Sol. 7. $\int\frac{1}{f\left(x\right)}dx=\int\frac{1}{2e^{x}+3}dx=\int\frac{e^{-x}}{2+3e^{-x}}dx=\frac{-1}{3}ln\left(2+3e^{-x}\right)+c$

$=\frac{1}{3}\left(x-ln\,f\left(x\right)\right)+c$

8. $\int\frac{1}{g\left(x\right)}dx=\int\frac{1}{3-4e^{x}}dx=\int\frac{e^{-x}}{3e^{-x}-4}dx=\frac{-1}{3}ln\left|3e^{-x}-4\right|+c$

$=\frac{1}{3}ln\left(x-ln\left|g\left(x\right)\right|\right)+c$

9. $\int\frac{3-4e^{x}}{3+2e^{x}}\,dx=\int\left(1-\frac{3\left(2e^{x}\right)}{3+2e^{x}}\right)dx=x-3ln\left(f\left(x\right)\right)+c$

Passage – 4

If $\int\frac{dx}{(1+x^2)\sqrt{1-x^2}}=A\sin^{-1}\{\sqrt{f(x)}\}+c$

10. The value of A is

(A) 1 (B) $\frac{1}{\sqrt{2}}$

(C) $-\frac{1}{\sqrt{2}}$ (D) none of these

Key. C

11. The f(x) is

(A) $\frac{1+x^2}{1-x^2}$ (B) $\frac{1-x^2}{1+x^2}$

SMART ACHIEVERS LEARNING PVT. LTD.

(C) $\sqrt{\frac{1-x^2}{1+x^2}}$ (D) $\sqrt{1-x^2}$

Key. B

12. $\int f(x)dx$ is

(A) $2\tan^{-1}x - x + c$ (B) $2\tan^{-1}x + x + c$

(C) $\tan^{-1}x + 2x + c$ (D) none of these

Key. A

Sol. 10. $\int \frac{dx}{(1+x^2)(\sqrt{1-x^2})}$

$= \int \frac{dx}{x^3\left(1+\frac{1}{x^2}\right)\sqrt{\frac{1}{x^2}-1}}$

Put $\frac{1}{x^2} - 1 = t^2 \Rightarrow -\frac{2}{x^3}dx = 2t\,dt$

$\Rightarrow -\int \frac{tdt}{(t^2+2)t} = -\frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}} = -\frac{1}{\sqrt{2}}\tan^{-1}\frac{\sqrt{1-x^2}}{\sqrt{2}x}$

$= -\frac{1}{\sqrt{2}}\sin^{-1}\sqrt{\frac{1-x^2}{1+x^2}}$

$\therefore$ A = $-\frac{1}{\sqrt{2}}$

11. f(x) = $\frac{1-x^2}{1+x^2}$

12. $\int f(x)\,dx = \int \frac{1-x^2}{1+x^2}dx$

$= \int\left(\frac{2}{1+x^2} - 1\right)dx$

$= 2\tan^{-1}x - x + c$

SMART ACHIEVERS LEARNING PVT. LTD.

Paragraph – 5

Let $I_n = \int \tan^n x dx, I_n^1 = \int \cot^n x dx, n \in N$

$$f(x,n) = I_n + I_{n-2} (n \neq 1), g(x,n) = I_n^1 + I_{n-2}^1 (n \neq 1), f(0,n) = g\left(\frac{\pi}{2}, n\right) = 0$$

13. $\underset{x\to 0}{Lt}\frac{f(x,2n+1)}{x^{2n}g(x,2n+1)} =$

A) 0 B) 1 C) -1 D) 2

Key. A

14. If $I_0 + I_1 + 2(I_2 + I_3 + ... + I_{2006} + I_{2009}) + 3I_{2007} + I_{2008} = \sum_{r=1}^{2008} a_r \tan^r x$ then, $a_{2008} =$

A) 1 / 2007 B) 2 / 2007 C) 1 / 1004 D) 1 / 2008

Key. C

15. If $I_0^1 + I_1^1 + 2\left(I_2^1 + I_3^1 + ... + I_{18}^1\right) + I_{19}^1 + I_{20}^1 = \sum_{r=1}^{19} a_r \cot^r x$ then $a_{10} =$

A) 1/10 B) -1/10 C) 1/9 D) -2/9

Key. B

Sol. 13. $f(x, 2n+1) = \dfrac{\tan^{2n} x}{2n}, g(x, 2n+1) = -\dfrac{\cot^{2n} x}{2n}$

14. $I_n + I_{n-2} = \dfrac{\tan^{n-1} x}{n-1}$

15. $I_n^1 + I_{n-2}^1 = -\dfrac{\cot^{n-1} x}{n-1}$

Paragraph – 6

Let $f(x)$ defined in $[a, b]$ has discontinuities $C_1, C_2, C_3, ..., C_n$ such that $a < C_1 < C_2 < .... < C_n < b$ then

$$\int_a^b f(x)dx = \int_a^{C_1} f(x)dx + \int_{C_1}^{C_2} f(x)dx + ... + \int_{C_{n-1}}^{C_n} f(x)dx + \int_{C_n}^{b} f(x)dx$$

16. $\int_{-1}^{1} [2x-3]dx =$ (where [.] is greatest integer function)

A) -7 B) -9 C) 5 D) 11/2

Key. A

17. $\int_0^{50\pi} [\tan^{-1} x]dx =$ (where [.] is greatest integer function)

A) $\tan 1 + 50\pi$ B) $-\tan 1$ C) $50\pi - \tan 1$ D) $20\pi - 2\tan 1$

Key. C

18. $\int_{\pi/2}^{3\pi/2} [\sin x]dx =$ (where [.] is greatest integer function)

A) $-\pi$ B) $\pi/2$ C) $-\pi/2$ D) π

Key. C

Sol. 16. $\int_{-1}^{-1/2} -5dx + \int_{-1/2}^{0} -4dx + \int_0^{1/2} -3dx + \int_{1/2}^{1} -2dx$

17. $\int_0^{\tan 1} 0dx + \int_{\tan 1}^{50\pi} 1dx$

SMART ACHIEVERS LEARNING PVT. LTD.

18. $\int_{\pi/2}^{\pi} 0dx + \int_{\pi}^{3\pi/2} -1dx$

Paragraph – 7

Let $f(x)$ be a differentiable function satisfying $(x-y)f(x+y)-(x+y)f(x-y)$

$= 4xy(x^2-y^2)$ for all $x, y \in R$. If $f(1)=1$ then

19. The function f at $x=0$ attains

(A) Local maximum (B) Local minimum

(C) Point of inflexion (D) None of these

20. The value of $\int_{-1}^{2} f(x)dx$ is

(A) 0 (B) $\frac{1}{4}$

(C) $\frac{11}{4}$ (D) $\frac{15}{4}$

21. The area of the region bounded by the curves $y=f(x)$ and $y=x^2$ is

(A) $\frac{1}{4}$sq.units (B) $\frac{1}{12}$sq.units

(C) $\frac{7}{12}$sq.units (D) $\frac{11}{12}$sq.units

Sol. 19. (c) 20. (d) 21. (b)

$$(x-y)f(x+y)-(x+y)f(x-y)=4xy(x^2-y^2)$$

$$\Rightarrow \frac{f(x+y)}{x+y}-\frac{f(x-y)}{x-y}=4xy$$

So; $$\frac{f(x+h)}{x+h}-\frac{f(x-h)}{x-h}=4xh$$

$$\Rightarrow \left[\frac{f(x+h)-f(x)+f(x)}{x+h}\right]-\left[\frac{f(x-h)-f(x)+f(x)}{x-h}\right]=4xh$$

$$\Rightarrow \left[\frac{f(x+h)-f(x)}{x+h}\right]-\left[\frac{f(x-h)-f(x)}{x-h}\right]=4xh+f(x)\left[\frac{1}{x-h}-\frac{1}{x+h}\right]$$

SMART ACHIEVERS LEARNING PVT. LTD.

$$\Rightarrow\left[\frac{f(x+h)-f(x)}{x+h}\right]-\left[\frac{f(x-h)-f(x)}{x-h}\right]=4xh+\frac{2hf(x)}{x^2-h^2}$$

$$\Rightarrow\left[\frac{f(x+h)-f(x)}{h}\right].\left(\frac{1}{x+h}\right)+\left[\frac{f(x-h)-f(x)}{-h}\right]\left(\frac{1}{x-h}\right)=4x+\frac{2f(x)}{x^2-h^2}$$

Taking limit of both sides as $h\to 0$ ***;***

$$\frac{f'(x)}{x}+\frac{f'(x)}{x}=4x+\frac{2f(x)}{x^2}\quad\Rightarrow\frac{f'(x)}{x}=2x+\frac{f(x)}{x^2}$$

$$f'(x)=2x^2+\frac{f(x)}{x}$$

$\Rightarrow f'(x)+\left[\frac{-f(x)}{x}\right]=2x^2$ ***, which is a linear differential equation***

Integrating factor $=e^{-\int\frac{1}{x}dx}=\frac{1}{x}$

Its solution is

$$f(x).\frac{1}{x}=\int 2x^2\times\frac{1}{x}+c$$

$$\Rightarrow\frac{f(x)}{x}=x^2+c\Rightarrow f(x)=x^3+cx$$

But $f(1)=1\ \Rightarrow c=0$

So, $f(x)=x^3$

$f'(x)=3x^2$ ***and*** $f''(x)=6x$

Here; $f''(x)=0\Rightarrow x=0$

So, at $x=0, f(x)$ ***has point of inflexion***

$$\int_{-1}^{2}x^3dx=\int_{-1}^{1}x^3dx+\int_{1}^{2}x^3dx=0+\frac{1}{4}\left[x^4\right]_1^2=\frac{15}{4}$$

The area of the region bounded by the curves $y=f(x)$ ***and*** $y=x^2$ ***is***

$$\int_0^1\left(x^2-x^3\right)dx=\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1=\frac{1}{3}-\frac{1}{4}=\frac{1}{12}\text{sq.units}$$

Paragraph – 8

Let $f(x)$ be defined as $f(x)=\text{maximum}\{a,b,c\}$ where

$$a=\lim_{n\to\infty}\lim_{\alpha\to1^+}\frac{\alpha^n|\sin x|+|\cos x|\alpha^{-n}}{\alpha^n+\alpha^{-n}}\qquad b=\lim_{n\to\infty}\lim_{\alpha\to1^-}\frac{\alpha^n|\sin x|+|\cos x|\alpha^{-n}}{\alpha^n+\alpha^{-n}}$$

SMART ACHIEVERS LEARNING PVT. LTD.

$c = \lim_{n\to\infty} \frac{\pi}{4n}\left[1+\cos\frac{\pi}{2n}+....+\cos\frac{(n-1)\pi}{2n}\right]$, then

22. The range of $f(x)$ is

A) $[0,1]$ B) $\left[\frac{1}{2},1\right]$ C) $\left[\frac{1}{2},2\right]$ D) $\left[\frac{1}{\sqrt{2}},1\right]$

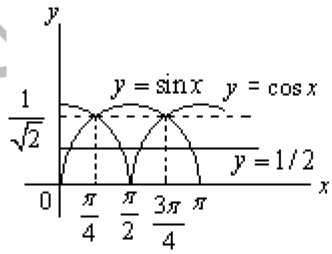
23. The function is differentiable for

A) $R-\left\{(2n+1)\frac{\pi}{4}\right\}$ B) $R-\{n\pi\}$ C) R D) none

24. The area bounded by $f(x)=\max\{a,b,c\}$, x-axis, $x=0$ and $x=\pi$ is (in sq. units)

A) $\sqrt{2}$ B) $1/\sqrt{2}$ C) $2\sqrt{2}$ D) $4\sqrt{2}$

Sol. 22. (d)
23. (a)
24. (c)



$a=|\sin x|, b=|\cos x|, c=\frac{1}{2}$

So, $f(x)=$ maximum $\left\{|\sin x|,|\cos x|,\frac{1}{2}\right\}$

Range of $f(x)=\left[\frac{1}{\sqrt{2}},1\right]$

Differentiable for $x\in R-\left\{(2n+1)\frac{\pi}{4}\right\}$

Required area = $2\left\{\int_0^{\pi/4}\cos x\,dx+\int_{\pi/4}^{\pi/2}\sin x\,dx\right\}=2\sqrt{2}$ sq.units

Paragraph – 9

A curve passing through origin is such that slope of tangent at any point is reciprocal of sum of co-ordinate of point of tangency.

25. Slope of tangent at ordinate ln3.

a) 1 b) $\frac{1}{3}$ c) $\frac{1}{2}$ d) –2

26. Area bounded by curve and the abscissa $y=0$ and $y=1$ is

a) $e-\frac{1}{2}$ b) $e-\frac{3}{2}$ c) $e-\frac{5}{2}$ d) e+1

27. $\int_{-1}^{[|\sin\alpha|+|\cos\alpha|]} xe^{-y}d\left(e^{y}\right)$ where $[.]$ represents integer function, is equal to

a) $e-e^{-1}-\frac{1}{3}$ b) $e-e^{-1}-2$ c) $e-\frac{1}{e}$ d) $e+e^{-1}+\frac{1}{3}$

SMART ACHIEVERS LEARNING PVT. LTD.

Sol. 25. (C) (i) $\frac{dy}{dx}=\frac{1}{x+y}$

$Let \quad y+x=v$

$\therefore \quad \frac{dy}{dx}+1=\frac{dv}{dx}$

$\Rightarrow \frac{dv}{dx}-1=\frac{1}{v}$

$\Rightarrow \frac{dv}{dx}=\frac{v+1}{v}$

$\Rightarrow dv\left(\frac{v+1-1}{v+1}\right)=dx$

$\Rightarrow v-\ln(v+1)=dx$

$\Rightarrow y+x-\ln(x+y+1)=x+c$

$at \quad x=0, y=0,$

$c=0$

$y=\ln(x+y+1)$

$x=e^y-y-1$

$\frac{dy}{dx}=\frac{1}{e^y-1}$

$\Rightarrow \frac{dy}{dx}|_{\ln 3}=\frac{1}{e^{\ln 3}-1}=\frac{1}{2}$

26. (B) $Area=\int_0^1\left(e^y-y-1\right)dy$

$=e-\frac{1}{2}-1=e-\frac{3}{2}$

27. (B) $\int_{-1}^{[|\sin\alpha|+|\cos\alpha|]} xe^{-y}d\left(e^y\right)$

Let $e^y=z, \Rightarrow e^y\frac{dy}{dz}=1 \Rightarrow dy=e^{-y}dz$

$1\le|\sin\alpha|+|\cos\alpha|\le\sqrt{2}$

Since $\therefore \left[|\sin\alpha|+|\cos\alpha|\right]=1$

$\Rightarrow \int_{-1}^{1}\left(e^y-y-1\right)dy=e-e^{-1}-2$

Paragraph – 10

If a curve is given by its parametric equation in the form $x=f(t), y=g(t)$ and suppose the derivatives $f^1(t)$ and $g^1(t)$ are continuous functions on the interval $[t_1,t_2]$. If t_1 and t_2 are the values of parameter '*t*' corresponding respectively to the initial and final position in which

SMART ACHIEVERS LEARNING PVT. LTD.

the curve can be described as a contour in the positive direction (i.e., figure remains left) then the area described by the curve

$$A = -\int_{t_1}^{t_2} g(t) f^1(t)dt = -\int_{t_1}^{t_2} f(t)g^1(t)dt = \frac{1}{2}\int_{t_1}^{t_2} (xg^1(t)dt - yf^1(t))dt .$$

Answer the following questions

28. Area enclosed by the curve $x = a\cos^3 t, y = a\sin^3 t, 0 \le t \le 2\pi$

A) $\frac{3}{2}a^2\pi$ B) $\frac{3}{4}a^2\pi$ C) $\frac{3}{8}a^2\pi$ D) $\frac{3}{16}a^2\pi$

Key. C

29. Area of the loop of the curve $x = a(1-t^2), y = at(1-t^2), -1 \le t \le 1$ must be

A) $\frac{2}{15}a^2$ B) $\frac{4}{15}a^2$ C) $\frac{8}{15}a^2$ D) $\frac{a^2}{5}$

Key. C

30. The area of the curve $x = 2\cos t, y = 2\sin t$ must be

A) 4π B) 2π C) 8π D) 16π

Key. A

Sol. 28. $x = a\cos^3 t, y = a\sin^3 t, 0 \le +t \le 2\pi$

$$(xg^1(t) - yf^1(t)) = a^2(\cos^3 t.3\sin^2 t + \cos t + \sin^3 t.3\cos^2 t + \sin t) = 3a^2\cos^2 t + \sin^2 t$$

$$\Rightarrow \text{Area of the curve } = \frac{1}{2}\int_0^{2\pi} (xg^1(t) - yf^1(t))dt = \frac{1}{2}\int_0^{2\pi} 3a^2\cos^2 t\sin^2 t = \frac{3}{8}a^2\pi$$

29. The curve is described when t varies from -1 to 1. Indeed when $t = -1$ both x and y are zero. When $t = 0, x = a, y = 0$. Thus is the adjoining figure upper portion is described when $0 \le t \le 1$ and the lower portion is described when $-1 < t < 0$

Required are $= -\int_{-1}^{1} g(r) f^1(t)dt = -\int_{-1}^{1} at(1-t^2)(-2at)dt = \frac{8a^2}{15}$.

Paragraph – 11

Let $A_r(r \in N)$ be the area of the bounded region whose boundary is defined by $\left(6\pi^3 r y^2 - x\right)\left(6e^2 y - x\right) = 0$ then

31. $\sqrt{A_1}, \sqrt{A_2}, \sqrt{A_3}, ...$ are in

A) A.P. B) G.P. C) H.P. D) A.G.P.

Key. C

Sol. $\sqrt{A_r} = \left(\frac{e}{\pi}\right)^3 \frac{1}{r} \Rightarrow \sqrt{A_1}\sqrt{A_2}\sqrt{A_3}........$ are in H.P.

$$= \sqrt{A_r\ A_{r+1}\ A_{r+2}} = \left(\frac{e}{\pi}\right)^9 \frac{1}{r(r+1)(r+2)} = \frac{1}{2}\left(\frac{e}{\pi}\right)^9 \left[\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}\right]$$

SMART ACHIEVERS LEARNING PVT. LTD.

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\sqrt{A_r A_{r+1} A_{r+2}} = \frac{1}{2}\left(\frac{e}{\pi}\right)^9 \frac{1}{1.2} = \frac{1}{4}\left(\frac{e}{\pi}\right)^9 = e^{\frac{-1}{\sqrt{A_r}}} = e^{-\left(\frac{\pi}{e}\right)^3 r} = \left(e^{\frac{\pi^3}{e^3}}\right)^{-r} = k^{-r}$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n} k^{-r} = \frac{1}{k-1} = (k-1)^{-1} = \left(e^{\frac{\pi^3}{e^3}} - 1\right)^{-1}$$

32. $\lim_{n\to\infty}\left[\sqrt{A_1 A_2 A_3} + \sqrt{A_2 A_3 A_4} + \sqrt{A_3 A_4 A_5}.......n\ terms\right]$ is

A) $\left(\frac{e}{\pi}\right)^9$ B) $\frac{1}{2}\left(\frac{e}{\pi}\right)^9$ C) $\frac{1}{3}\left(\frac{e}{\pi}\right)^9$ D) $\frac{1}{4}\left(\frac{e}{\pi}\right)^9$

Key. D

33. $\lim_{n\to\infty}\left[e^{\frac{-1}{\sqrt{A_1}}} + e^{\frac{-1}{\sqrt{A_2}}} + e^{\frac{-1}{\sqrt{A_3}}} + ...n\ terms\right]$ is

A) $e^{\frac{-\pi^3}{e^3}}$ B) $\left(e^{\frac{\pi^3}{e^3}} - 1\right)^{-1}$

C) $e^{\frac{2\pi^3}{e^3}}$ D) $\left(e^{\frac{\pi^3}{e^3}} - 1\right)^{-2}$

Key. B

Paragraph – 12

Let $A_r\,(r \in N)$ be the area of the bounded region whose boundary is defined by $(6\pi^3 r y^2 - x)\ (6e^2 y - x) = 0$ then

34. $\sqrt{A_1}, \sqrt{A_2}, \sqrt{A_3},$ .... are in

A) A.P. B) G.P. C) H.P. D) A.G.P

35. $\lim_{n\to\infty}\left[\sqrt{A_1A_2A_3} + \sqrt{A_2A_3A_4} + \sqrt{A_3A_4A_5}.............n\ \text{terms}\right]$ is

A) $\left(\frac{e}{\pi}\right)^9$ B) $\frac{1}{2}\left(\frac{e}{\pi}\right)^9$ C) $\frac{1}{3}\left(\frac{e}{\pi}\right)^9$ D) $\frac{1}{4}\left(\frac{e}{\pi}\right)^9$

36. $\lim_{n\to\infty}\left[e^{-\frac{1}{\sqrt{A_1}}} + e^{-\frac{1}{\sqrt{A_2}}} + e^{-\frac{1}{\sqrt{A_3}}} + ...n\ \text{terms}\right]$ is

SMART ACHIEVERS LEARNING PVT. LTD.

A) $e^{-\frac{\pi^3}{e^3}}$ B) $\left(e^{\frac{\pi^3}{e^3}}-1\right)^{-1}$ C) $e^{-\frac{2\pi^3}{e^3}}$ D) $\left(e^{\frac{\pi^3}{e^3}}-1\right)^{-2}$

Key: 34-C, 35-D, 36-B

Hint: 34 - 36

$\sqrt{A_r}=\left(\frac{e}{\pi}\right)^3\frac{1}{r}\Rightarrow\sqrt{A_1}\sqrt{A_2}\sqrt{A_3}$ ...... are in H.P.

= $\sqrt{A_rA_{r+1}A_{r+2}}=\left(\frac{e}{\pi}\right)^9\frac{1}{r(r+1)(r+2)}=\frac{1}{2}\left(\frac{e}{\pi}\right)^9\left[\frac{1}{r(r+1)}-\frac{1}{(r+1)(r+2)}\right]$

= $\lim_{n\to\infty}\sum_{r=1}^{n}\sqrt{A_rA_{r+1}A_{r+2}}=\frac{1}{2}\left(\frac{e}{\pi}\right)^9\frac{1}{1.2}=\frac{1}{4}\left(\frac{e}{\pi}\right)^9$ = $e^{-\frac{1}{\sqrt{A_r}}}=e^{-\left(\frac{\pi}{e}\right)^3r}=\left(e^{\frac{\pi^3}{e^3}}\right)^{-r}=k^{-r}$

= $\lim_{n\to\infty}\sum_{r=1}^{n}k^{-r}=\frac{1}{k-1}=(k-1)^{-1}=\left(e^{\frac{\pi^3}{e^3}}-1\right)^{-1}$

Paragraph – 13

Let the curves $S_1:y=x^2,\ S_2:y=-x^2,\ S_3:y^2=4x-3$

37. Area bounded by the curves S_1,S_2,S_3 is

(A) $\frac{4}{3}sq.u$ (B) $\frac{8}{3}sq.u$ (C) $\frac{1}{6}sq.u$ (D) $\frac{1}{3}sq.u$

38. Area bounded by the curves S_1,S_3 and the line $x=3$ is

(A) $\frac{13}{3}sq.u$ (B) $\frac{5}{4}sq.u$ (C) $\frac{8}{3}sq.u$ (D) $\frac{7}{4}sq.u$

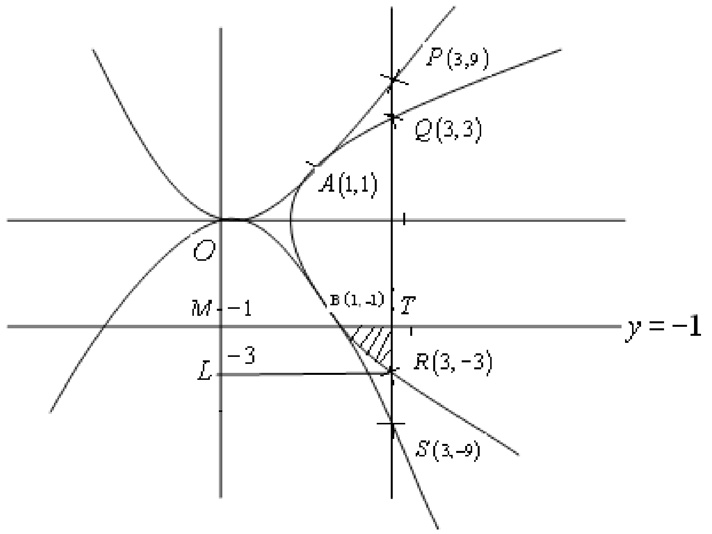
39. Area bounded by the curve $S_3,\ y\le -1$ *and the line* $x=3$ is

(A) $\frac{7}{3}sq.u$ (B) $\frac{11}{3}sq.u$ (C) $\frac{9}{2}sq.u$ (D) $\frac{13}{4}sq.u$

Key: D-A-A

Hint: 37, 38, 39

SMART ACHIEVERS LEARNING PVT. LTD.



37. Area OAB

$$=2\int\left(\frac{y^2+3}{4}-\sqrt{y}\right)dy$$

$$=\frac{1}{3}$$

38. Area $APQ=\int_1^3\left(x^2-\sqrt{(4x-3)}\right)dx$

39. Area BTR = Area of rectangle LMRT

$-Area\ \ LMRB$

$$=6-\int_{-3}^{-1}\frac{y^2+3}{4}dy$$

Paragraph – 14

If f(x) is a differentiable function wherever it is continuous nd $f'(c_1)=f'(c_2)=0$.

$f''(c_1).f''(c_2)<0,\ f(c_1)=5,\ f(c_2)=0$ and ($C_1<C_2$).

40. If f(x) is continuous in [c_1, c_2] and $f''(c_1)-f''(c_2)>0,$ then minimum number of roots of

$f'(x)=0$ in $[c_1-1, c_2+1]$ is

a) 2 b) 3 c) 4 d) 5

41. If f(x) is continuous in [c_1,c_2] and $f''(c_1)-f''(c_2)<0$, then minimum number of roots of $f'(x)=0$ in $[c_1-1, c_2+1]$ is

a) 1 b) 2 c)3 d) 4

SMART ACHIEVERS LEARNING PVT. LTD.

42. If f(x) is continuous in [c_1,c_2] and $f''(c_1) - f''(c_2) > 0,$ then minimum number of roots of f(x)=0 in $[c_1 - 1, c_2 + 1]$ is

a) 2 b) 3 c) 4 d) 5

Key: C-B-A

Hint 40. $f''(C_1) > f''(C_2)$

$f''(C_1) > 0 \; \& f''(C_2) < 0$

$\Rightarrow$ At x=C_1 f has local minimum &
At x=C_2 f has local maximum

41. $f''(C_1) < f''(C_2)$

$f''(C_1) < 0 \;\; \& \;\; f''(C_2) > 0$

$\Rightarrow$ At x=C_1 f has local maximum &
At x=C_2 f has local minimum

42. Conceptual

Paragraph – 15

Let *m,n* be two positive real numbers and define $f(n) = \int_0^{\infty} x^{n-1} e^{-x} dx$ and

$g(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$.

It is known that *f(n),* for *n>0* is finite and *g(m,n)=g(n,m)* for *m,n>0*
Answer the following .

43. $\int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx =$

(A) *g(m,n)* (B) *g(m-1,n)* (C) *g(m-1,n-1)* (D) *g(m,n-1)*

44. $\int_0^1 x^m \left(\log_e \frac{1}{x} \right)^n dx =$

(A) $\frac{f(n+1)}{(m+1)^n}$ (B) $\frac{f(n)}{(m+1)^{n+1}}$

(C) $\frac{f(n+1)}{(m+1)^{n+1}}$ (D) $g(m+1, n+1)$

SMART ACHIEVERS LEARNING PVT. LTD.

45. $\int_0^1 \frac{x^{m-1}+x^{n-1}}{\left(1+x\right)^{m+n}}dx$

(A) *g(n,m)* (B) *g(m-1,n+1)*

(C) *g(m-1,n-1)* (D) *g(m+1,n-1)*

Key: A-C-A

Hint:

43. Putting $\log_e \frac{1}{x} = t \Rightarrow x = e^{-t} = \int_0^1 x^m \left(\log_e \frac{1}{x}\right)^n dx$

$$= \int_0^\infty e^{-mt} t^n \left(-e^{-t}\right) dt$$

$$= \int_0^\infty t^n e^{-(m+1)t} dt$$

$$= \frac{1}{m+1} n+1 \int_0^\infty t^n e^{-t} dt$$

$$= \frac{f\left(n+1\right)}{\left(m+1\right)^{n+1}}$$

44. $I = \int_0^1 \frac{x^{m-1}+x^{n-1}}{\left(1+x\right)^{m+n}}dx = \int_0^1 \frac{x^{m-1}}{\left(1+x\right)^{m+n}}dx + \int_0^1 \frac{x^{n-1}}{\left(1+x\right)^{m+n}}dx$

$= I_1 + I_2$

In I_2 put $x = \frac{1}{t}$ then $I_2 = \int_\infty^1 \frac{\frac{1}{t^{n-1}}}{\left(1+\frac{1}{t}\right)^{m+n}} \left(\frac{-dt}{t^2}\right)$

$= \int_\infty^1 \frac{x^{m-1}}{\left(1+x\right)^{m+n}}dx$

$$\therefore I = \int_0^1 \frac{x^{m-1}}{\left(1+x\right)^{m+n}}dx + \int_1^\infty \frac{x^{m-1}}{\left(1+x\right)^{m+n}}dx = \int_0^\infty \frac{x^{m-1}}{\left(1+x\right)^{m+n}}dx = g(m,n)$$

Paragraph – 16

A series of the form *f(x)=* $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ is called Fourier series where a_0, $a_1, a_2, \ldots.. a_n$, b_1, b_2, …b_n are constants and the se coefficients are evaluated for $x \in [\alpha, \alpha + 2\pi]$ using the formula $a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)dx, a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\, dx$ and $b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\ dx$ Let us consider f(x) =x+x^2 for -$\pi \le x \le \pi,$ then

SMART ACHIEVERS LEARNING PVT. LTD.

46. The value of a_0 is

A) $\frac{2\pi^2}{3}$ B) $\frac{4\pi^3}{5}$ C) π D) $\frac{3\pi^2}{2}$

47. The value of a_n is

A) $\frac{4(-1)^n}{\pi^2}$ B) $\frac{4}{\pi^2}$ C) $\frac{2(-1)^n}{\pi^2}$ D) $\frac{2}{\pi^2}$

48. The approximate value of $\frac{\pi^2}{6}$ is given by

A) $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+....\infty$ B) $2-\frac{1}{2^2}+\frac{2}{3}-\frac{2}{4^3}+....\infty$

C) $2+\frac{1}{2}+\frac{2}{3}+\frac{2}{4^2}+....\infty$ D) $2-\frac{2}{3^2}+\frac{2}{5^2}-\frac{2}{7^2}+....\infty$

Key: A-A-A

Hint:

46. $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} x+x^2 dx=\frac{2}{\pi}x\frac{\pi^3}{3}=\frac{2\pi^2}{3}$

47. $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(x+x^2\right)\cos nx\,dx=\frac{4}{\pi^2}(-1)^n$

48. $x+x^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}.\cos nx+2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$

Here $b_n=\frac{2(-1)^{n+1}}{n}$

Putting **x =** π

$\pi+\pi^2=\frac{\pi^2}{3}+4\left[1+\frac{1}{2^2}+\frac{1}{3^2}+...\right]$

Putting **x =** $-\pi$

$-\pi+\pi^2=\frac{\pi^2}{3}+4\left[1+\frac{1}{2^2}+\frac{1}{3^2}+...\right]$

Adding $\pi^2=\frac{\pi^2}{3}+4\left[1+\frac{1}{2^2}+\frac{1}{3^2}+...\right]$

$\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+........$

Paragraph – 17

Differentiate **I** w.r.t. the parameter within the sign of integrals taking variable of the integrand as constant. Now, evaluate the integral so obtained as usual as a function of the parameter and then integrate the result to get **I**. Constant of integration can be computed by giving some arbitrary values to the parameter and the corresponding value of **I**.

SMART ACHIEVERS LEARNING PVT. LTD.

49. The value of $\int_0^1 \frac{x^a - 1}{\log x}$ dx is

a) log(a-1) b) log(a+1) c) a log (a+1) d) None of these

50. The value of $\int_0^{\frac{\pi}{2}} \log(\sin^2\theta + k^2\cos^2\theta)d\theta$, where $k \geq 0$, is

a) $\pi\log(1+k) + \pi\log 2$
b) $\pi\log(1+k)$
c) $\pi\log(1+k) - \pi\log 2$
d) $\pi\log(1+k) - \log 2$

51. If $\int_0^{\pi} \frac{dx}{(a-\cos x)} = \frac{\pi}{\sqrt{a^2-1}}$, then the value of $\int_0^{\pi} \frac{dx}{(\sqrt{10}-\cos x)^3}$ is

a) $\frac{\pi}{81}$ b) $\frac{7\pi}{81}$ c) $\frac{7\pi}{162}$ d)None of these

KEY : B-C-C

HINT: 49.

$$let\ I(a) = \int_0^1 \frac{x^a - 1}{\log x}.dx$$

Diff w.r.t. a keeping x as constant

$$\frac{d}{da}I(a) = \int_0^1 x^a dx$$

$$= \frac{1}{a+1}$$

Integrating both sides w.r.t a

I(a)=log(a+1)+C

I(0)=0 $\Rightarrow$ C=0

50. let f(K)= $\int_0^{\frac{\pi}{2}} \ln(\sin^2\theta + k^2\cos^2\theta)d\theta$

$f'(K) = \frac{\pi}{1+K}$ & F(1)=0

$f(K) = \pi\log(1+k) - \pi\log 2$

51. $\int_0^{\pi} \frac{dx}{a-\cos x} = \frac{\pi}{\sqrt{a^2-1}}$

Diff w.r.t. a on both sides treating x as constant

SMART ACHIEVERS LEARNING PVT. LTD.

$$-\int_0^{\pi}\frac{dx}{(a-\cos x)^2}=\frac{-\pi a}{(a^2-1)^{\frac{3}{2}}}$$

Again diff w.r.t. a tertiary x as a constant

$$2\int_0^{\pi}\frac{dx}{(a-\cos x)^3}=\frac{\pi(1+2a^2)}{(a^2-1)^{\frac{5}{2}}}$$

Paragraph – 18

If f(x) is discontinuous and not having same definition between a and b, then we must break the interval such that f(x) becomes continuous and having same definition in the breaking intervals. Now, if f(x) is discontinuous at $x=c(a<c<b)$, then $\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx$ and also if f(x) discontinuous x = a in (0, 2a) , then we can write $\int_0^{2a} f(x)dx=\int_0^a\{f(a-x)+f(a+x)\}dx$

52. $\int_0^{10}\left[\frac{x^2+2}{x^2+1}\right]dx$ (where [.] denotes the greatest integer function) is equal to

a) 0 b) 2 c) 5 d) 10

Key. D

53. $\int_0^1 \sin([x]+[2x])dx$ (where [.] denotes the greatest integer function) is equal to

a) $\sin 1$ b) $\sin\left(\frac{3}{2}\right)$ c) $\frac{\sin 1}{2}$ d) $\frac{\sin 2}{3}$

Key. C

54. $\int_{-1}^1 [|x|]\,d\left(\frac{1}{1+e^{-1/x}}\right)$ (where [.] denotes the greatest integer function) is equal to

a) -3 b) -2 c) -1 d) 0

Key. D

Sol. 52. $\because \frac{x^2+2}{x^2+1}=1+\frac{1}{x^2+1}$

$=1+\text{proper fraction}$

$\therefore \left[\frac{x^2+2}{x^2+1}\right]=1$

Then, $\int_0^{10}\left[\frac{x^2+2}{x^2+1}\right]dx=\int_0^{10}1.dx=10$

53. $\int_0^1 \sin([x]+[2x])dx=\int_0^{1/2}\sin([x]+[2x])dx+\int_{1/2}^1 \sin([x]+[2x]dx)$

SMART ACHIEVERS LEARNING PVT. LTD

$$=0+\int_{1/2}^{1}\sin(0+1)dx$$

$$=\frac{\sin 1}{2}$$

54. $\int_{-1}^{1}[|x|]d\left(\frac{1}{1+e^{-1/x}}\right)=\int_{-1}^{0}[-x]d\left(\frac{1}{1+e^{-1/x}}\right)+\int_{0}^{1}[x]d\left(\frac{1}{1+e^{-1/x}}\right)$

$$=0+0=0$$

Paragraph – 19

Given : $\int_a^b f(x)d\alpha(x)+\int_a^b \alpha(x)df(x)=\alpha(b)f(b)-\alpha(a)f(a)$

55. $\int_0^3 (x^2+1)d[x]$ (where [.] denotes the greatest integer function) is equal to

a) 3 b) $\frac{9}{2}$ c) 17 d) $\frac{27}{2}$

Key. C

56. $\int_{1/4}^{3/4}[x]d[2x]$ (where [.] denotes the greatest integer function) is equal to

a) 0 b) 1 c) 2 d) 3

Key. D

57. $\int_0^6 (x^2+[x])d|3-x|$ (where [.] denotes the greatest integer function) is equal to

a) 36 b) 72 c) 63 d) 126

Key. C

Sol. 55. Here, $f(x)=x^2+1, \alpha(x)=[x]$

$$\therefore \int_0^3 (x^2+1)d[x]+\int_0^3 [x]2xdx=[3](9+1)-0$$

$$\Rightarrow \int_0^3 (x^2+1)d[x]=30-2\int_0^3 x[x]dx$$

$$=30-2\left\{\int_0^1 0.dx+\int_1^2 xdx+\int_2^3 2xdx\right\}$$

$$=30-2\left\{0+\frac{3}{2}+5\right\}$$

$$=30-3-10=17$$

56. $\int_0^6 (x^2+[x]d|3-x|)=\int_0^3 (x^2+[x])d(3-x)+\int_3^6 (x^2-[x])d(x-3)$

SMART ACHIEVERS LEARNING PVT. LTD.

$=-\int_0^3 x^2dx+\int_3^6 x^2dx-\int_0^3[x]dx+\int_3^6[x]dx$

$=-\left\{\frac{x^3}{3}\right\}_0^3+\left\{\frac{x^3}{3}\right\}_3^6-\int_0^3[x]dx+\int_0^3[x+3]dx$

$=-9+(72-9)-\int_0^3[x]dx+\int_0^3[x]dx+3\int_0^3 1dx$

$= 72-18+9$

$=72-9=63$

Paragraph – 20

For evaluating $\int_a^b \max\{f_1(x),f_2(x),....f_n(x)\}dx$

or $\int_a^b \min\{g_1(x),g_2(x),....g_n(x)\}dx$

Compare the functions involved and accordingly split the given interval of integration.

58. $\int_{-1}^1 \max\{x,x^3\}dx$ is equal to

a) $1/2$ b) $3/2$ c) 1/4 d) ¾

Key. C

59. $\int_0^\pi \max\{\sin x,\cos x\}dx$ is equal to

a) $1-\frac{1}{\sqrt{2}}$ b) $1+\frac{1}{\sqrt{2}}$

c) $\sqrt{2}-1$ d) $\sqrt{2}+1$

Key. D

60. $\int_0^\pi \min\{2\sin x,1-\cos x,1\}dx$ is equal to

a) $\frac{\pi}{6}+2-\sqrt{3}$ b) $\frac{5\pi}{6}+1-\sqrt{3}$

c) $\frac{\pi}{3}+2-\sqrt{3}$ d) $\frac{\pi}{6}+1+\sqrt{3}$

Key. B

Sol. Conceptual

Paragraph – 21

Let $f(x)=\int\frac{dx}{e^x+8e^{-x}+4e^{-3x}}, g(x)=\int\frac{dx}{e^{3x}+8e^x+4e^{-x}}, h(x)=\int\frac{dx}{e^{2x}+4e^{-2x}+8}$.

SMART ACHIEVERS LEARNING PVT. LTD.

61. $f(x)-2g(x)=$

A) $\frac{1}{2}\log\left|\frac{e^x+2e^{-x}-2}{e^x+2e^{-x}+2}\right|+C$

B) $\frac{1}{4\sqrt{3}}\log\left|\frac{e^x-2e^{-x}-2\sqrt{3}}{e^x+2e^{-x}+2\sqrt{3}}\right|+C$

C) $\frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{e^x-2e^{-x}}{2\sqrt{3}}\right)+C$

D) $\frac{1}{2}\tan^{-1}\left(\frac{e^x+2e^{-x}}{2}\right)+C$

Key. D

Sol. $f(x)-2g(x)=\int\frac{e^x(e^{2x}-2)}{e^{4x}+8e^{2x}+4}dx$ put $e^x=t$

$$=\int\frac{1-\frac{2}{t^2}}{\left(t+\frac{2}{t}\right)^2+4}dt=\frac{1}{2}\tan^{-1}\left(\frac{t+\frac{2}{t}}{2}\right)+C$$

62. $f(x)+2g(x)=$

A) $\frac{1}{4\sqrt{3}}\log\left|\frac{e^x-2e^{-x}-2\sqrt{3}}{e^x+2e^{-x}+2\sqrt{3}}\right|+C$

B) $\frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{e^x-2e^{-x}}{2\sqrt{3}}\right)+C$

C) $\frac{1}{2}\tan^{-1}\left(\frac{e^x+2e^{-x}}{2}\right)+C$

D) $\frac{1}{4}\log\left|\frac{e^x+2e^{-x}-2}{e^x+2e^{-x}+2}\right|+C$

Key. B

Sol. $f(x)+2g(x)=\int\frac{e^x(e^{2x}+2)}{e^{4x}+8e^{2x}+4}dx$ put $e^x=t$

$$=\int\frac{1+\frac{2}{t^2}}{\left(t-\frac{2}{t}\right)^2+12}dt=\frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{t-\frac{2}{t}}{2\sqrt{3}}\right)+C$$

63. Let $h(x)=\frac{1}{8\sqrt{3}}\log\left|\frac{f(x)-2\sqrt{3}}{f(x)+2\sqrt{3}}\right|+C$, then $f\left(\log_e^2\right)=$

A) 1 B) 8 C) 4 D) 10

Key. B

Sol. $h(x)=\frac{1}{2}\int\frac{dt}{t^2+8t+4}$ where $t=e^{2x}$

Paragraph – 22

If $f(x)$ and $g(x)$ be two functions, such that $f(a)=g(a)=0$ and f and g are both differentiable at everywhere in some neighborhood of point a except possibly 'a'

SMART ACHIEVERS LEARNING PVT. LTD.

Then $\lim\limits_{x\to a}\frac{f(x)}{g(x)}=\lim\limits_{x\to\infty}\frac{f^1(x)}{g^1(x)}$, provided $f^1(a)$ and $g^1(a)$ are not both zero

64. The value of $\lim\limits_{x\to 0}\frac{\int_0^{x^2}\sin\sqrt{t}dt}{x^3}$ is

A. 0 **B.** $2/9$ **C.** $1/3$ **D.** $2/3$

Key. D

Sol. $\lim\limits_{x\to 0}\frac{\int_0^{x^2}\sin\sqrt{t}dt}{x^3}$

$=\lim\limits_{x\to 0}\frac{\sin x.2x}{3x^2}$

$=\frac{2}{3}\lim\limits_{x\to 0}\frac{\sin x}{x}$

$=\frac{2}{3}.1=\frac{2}{3}$

65. The value of $\lim\limits_{x\to\infty}\frac{\left\{\int_0^x e^{t^2}d\right\}^2}{\int_0^x e^{2t^2}dt}$ is

A. $1/3$ **B.** $2/3$

C. 1 **D.** None of these

Key. D

Sol. $\lim\limits_{x\to\infty}\frac{\left\{\int_0^x e^{t^2}dt\right\}^2}{\int_0^x e^{2t^2}dt}$

$=\lim\limits_{x\to\infty}\frac{\left(2\int_0^x e^{t^2}dt\right)(e^{x^2})}{e^{2x^2}}$

$=\lim\limits_{x\to\infty}\frac{2\int_0^x e^{t^2}dt}{e^{x^2}}$

$=\lim\limits_{x\to\infty}\frac{2e^{x^2}}{2xe^{x^2}}=\lim\limits_{x\to\infty}\frac{1}{x}=0$

66. The value of $\lim\limits_{t\to 0}\frac{\int_0^t xdx}{t\sin t}$ is

SMART ACHIEVERS LEARNING PVT. LTD.

A. $\frac{1}{4}$ **B.** $\frac{1}{3}$ **C.** $\frac{1}{2}$ **D.** 0

Key. C

Sol. $\lim_{t\to 0}\frac{\int_0^t xdx}{t\sin t}=\lim_{t\to 0}\frac{t^2/2}{t\sin t}=\frac{1}{2}\lim_{t\to 0}\frac{t}{\sin t}=\frac{1}{2}.1=\frac{1}{2}$

Paragraph – 23

$$\int_a^b f(x)d\alpha(x)+\int_a^b \alpha(x)df(x)=\alpha(b)f(b)-\alpha(a)f(a)$$

67. $\int_0^3 (x^2+1)d[x]$ (where [.] denotes the greatest integer function) is equal to

A. 3 **B.** $\frac{9}{2}$ **C.** 17 **D.** $\frac{27}{2}$

Key. C

68. $\int_{-2}^3 [|x|]d|x|$ (where [.] denotes the greatest integer function) is equal to

A. 0 **B.** 1
C. 2 **D.** None of these

Key. C

Sol. 67. Here, $f(x)=x^2+1,\alpha(x)=[x]$

$$\therefore \int_0^3 (x^2+1)d[x]+\int_0^3 [x]2xdx=[3](9+1)-0$$

$$\Rightarrow \int_0^3 (x^2+1)d[x]=30-2\int_0^3 x[x]dx$$

$$=30-2\{\int_0^1 0.dx+\int_1^2 xdx+\int_2^3 2xdx\}$$

$$=30-2\{0+\frac{3}{2}+5\}$$

$$=30-3-10$$

$$=17$$

68. $\int_{-2}^3 [|x|]d|x|=\int_{-2}^0 [-x]d(-x)+\int_0^3 [x]dx$

SMART ACHIEVERS LEARNING PVT. LTD.

$$= -\int_{-2}^{0}[-x]dx + \int_{0}^{3}[x]dx$$

$$= \int_{2}^{0}[x]dx + \int_{0}^{3}[x]dx \quad \text{(by property)}$$

$$= -\int_{0}^{2}[x]dx + \int_{0}^{2}[x]dx + \int_{2}^{3}[x]dx$$

$$= \int_{2}^{3}[x]dx = \int_{2}^{3}2.dx = 2(3-2)$$

$$= 2$$

Paragraph – 24

Let $f : R \to R$ be a differentiable function such that $f(x) = x^2 + \int_0^x e^{-t} f(x-t)dt$.

69. $f(x)$ increases for

A. $x > 1$ **B.** $x < -2$ **C.** $x > 2$ **D.** None of these

KEY. B

70. $y = f(x)$ is

A. Injective but not surjective **B.** surjective but not injective
C. Bijective **D.** Neither injective nor surjective

KEY. B

71. The value $\int_0^1 f(x)dx$ is

A. $\frac{1}{4}$ **B.** $-\frac{1}{12}$ **C.** $\frac{5}{12}$ **D.** $\frac{12}{7}$

KEY. C

SOL. 69TO 71. $f(x) = x^2 + \int_0^x e^{-t} f(x-t)dt$ (1)

$$= x^2 + \int_0^x e^{-(x-t)} f(x-(x-t))dt$$

$$[Using \int_a^b f(x)dx = \int_a^b f(a+b-x)dx]$$

$$= x^2 + e^{-x}\int_0^x e^t f(t)dt(2)$$

SMART ACHIEVERS LEARNING PVT. LTD.

Diff. w.r.t. x, we get

$\Rightarrow f^1(x) = 2x - e^{-x}\int_0^x e^t f(t)dt + e^{-x}e^x f(x)$

$= 2x - e^{-x}\int_0^x e^t f(t)dt + f(x)$

$= f^1(x) = 2x + x^2$ [using equation (2)]

$\Rightarrow f(x) = \frac{x^3}{3} + x^2 + c$ [from equation (1)]

Also $f(0) = 0$

$\Rightarrow f(x) = \frac{x^3}{3} + x^2$

$\Rightarrow f^1(x) = x^2 + 2x$

$\Rightarrow f^1(x) = 0$ has real roots, hence $f(x)$ is non-monotonic.

Hence, $f(x)$ is many-one, but range is R, hence surjective.

$\int_0^1 f(x)\,dx = \int_0^1 (\frac{x^3}{3} + x^2)dx$

$\left[\frac{x^4}{12} + \frac{x^3}{3}\right]_0^1$

$= \frac{1}{12} + \frac{1}{3} = \frac{5}{12}$

Paragraph – 25

$f(x) = \sin x + \int_{-\pi/2}^{\pi/2} (\sin x + t\cos x) f(t)dt$

72. The range of $f(x)$ is

A. $\left[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right]$ **B.** $\left[-\frac{\sqrt{5}}{3}, \frac{\sqrt{5}}{3}\right]$ **C.** $\left[-\frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{2}\right]$ **D.** None of these

KEY. B

73. $f(x)$ is not invertiable for

A. $x \in \left[-\frac{\pi}{2} - \tan^{-1} 2, \frac{\pi}{2} - \tan^{-1} 2\right]$ **B.** $x \in \left[\tan^{-1}\frac{1}{2}, \pi + \tan^{-1}\frac{1}{2}\right]$

C. $x \in \left[\pi + \cot^{-1} 2, 2\pi + \cot^{-1} 2\right]$ **D.** None of these

KEY. D

74. The value of $\int_0^{\pi/2} f(x)dx$ is

A. 1 **B.** -2 **C.** -1 **D.** 2

SMART ACHIEVERS LEARNING PVT. LTD.

KEY. C

$$f(x)=\sin x+\sin x\int_{-\pi/2}^{\pi/2} f(t)+\cos x\int_{-\pi/2}^{\pi/2} tf(t)dt$$

SOL. 72 TO 74.

$$=\sin x(1+\int_{-\pi/2}^{\pi/2} f(t)dt)+\cos x\int_{-\pi/2}^{\pi/2} tf(t)dt$$

$$=A\sin x+B\cos x$$

$$A=1+\int_{-\pi/2}^{\pi/2} f(t)dt$$

Thus,

$$=1+\int_{-\pi/2}^{\pi/2}(A\sin t+B\cos t)dt$$

$$=1+2B\int_{0}^{\pi/2}\cos tdt$$

$$\Rightarrow A=1+2B$$

$$B=\int_{-\pi/2}^{\pi/2} tf(t)dt$$

$$=\int_{-\pi/2}^{\pi/2} t(A\sin t+B\cos t)dt$$

$$=2A\int_{0}^{\pi/2} t\sin tdt$$

$$=2A[-t\cos t+\sin t]_0^{\pi/2}$$

$$\Rightarrow B=2A(2)$$

From equation (1) and (2), we get

$A=-1/3, B=-2/3$

$$\Rightarrow f(x)=-\frac{1}{3}(\sin x+2\cos x)$$

Thus, the range of $f(x)$ is $\left[-\frac{\sqrt{5}}{3},\frac{\sqrt{5}}{3}\right]$

$$f(x)=-\frac{1}{3}(\sin x+2\cos x)$$

$$=-\frac{\sqrt{5}}{3}\sin(x+\tan^{-1}2)$$

$$=-\frac{\sqrt{5}}{3}\cos\left(x-\tan^{-1}\frac{1}{2}\right)$$

$f(x)$ is invertible if $-\frac{\pi}{2}\le x+\tan^{-1}2\le\frac{\pi}{2}$

$$\Rightarrow -\frac{\pi}{2}-\tan^{-1}2\le x\le\frac{\pi}{2}-\tan^{-1}2$$

Or $0\le x-\tan^{-1}\frac{1}{2}\le\pi$

SMART ACHIEVERS LEARNING PVT. LTD.

$\Rightarrow \tan^{-1}\frac{1}{2} \le x \le \pi + \tan^{-1}\frac{1}{2}$

Or $\pi \le x - \tan^{-1}\frac{1}{2} \le 2\pi$

$\Rightarrow x \in [\pi + \cot^{-1} 2, 2\pi + \cot^{-1} 2]$

$\int_0^{\pi/2} f(x)dx = -\frac{1}{3}\int_0^{\pi/2} (\sin x + 2\cos x)dx$

$= -\frac{1}{3}[-\cos x + 2\sin x]_0^{\pi/2}$

$= -1$

Paragraph – 26

Let $f(x)$ be a continuous function defined on the closed interval $[a,b]$, then

$$\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{1}{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)dx.$$

75. The value of $\lim_{n\to\infty}\left\{\frac{1}{n} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + ...+\frac{1}{8n}\right\}$ is

A) 5/4 B) 3/4 C) 5/8 D) 3/8

Key. D

76. The value of $\lim_{n\to\infty}\left\{\prod_{r=1}^{n}\left(1+\frac{r}{n}\right)\right\}^{1/n}$ is

A) 3/e B) 4/e C) 1/e D) 2/e

Key. B

77. The value of $\lim_{n\to\infty}\left\{\tan\left(\frac{\pi}{2n}\right)\tan\left(\frac{2\pi}{2n}\right)\tan\left(\frac{3\pi}{2n}\right)........\tan\left(\frac{n\pi}{2n}\right)\right\}^{1/n}$ is

A) 1 B) 2 C) 3 D) not defined

Key. A

Sol. 75. $\because \lim_{n\to\infty}\left\{\frac{1}{n} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + ...+\frac{1}{8n}\right\}$

$= \lim_{n\to\infty}\left\{\frac{n^2}{(n+0)^3} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + ...+\frac{n^2}{(n+n)^3}\right\}$

$= \lim_{n\to\infty}\sum_{r=0}^{n}\frac{n^2}{(n+r)^3}$

$= \lim_{n\to\infty}\sum_{r=0}^{n}\frac{1}{n\left(1+\frac{r}{n}\right)^3}$

$= \int_0^1 \frac{dx}{(1+x)^3}$

SMART ACHIEVERS LEARNING PVT. LTD.

$$= -\left\{\frac{1}{2(1+x)^2}\right\}_0^1$$

$$= -\left\{\frac{1}{8}-\frac{1}{2}\right\} = \frac{1}{2}-\frac{1}{8} = \frac{3}{8}.$$

76. Let $P = \lim_{n\to\infty}\left\{\prod_{r=1}^{n}\left(1+\frac{r}{n}\right)\right\}^{1/n}$

$$\therefore In\, P = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n} In\left(1+\frac{r}{n}\right)$$

$$= \int_0^1 In(1+x)\,dx$$

$$= \left[x\, In(1+x)\right]_0^1 \int_0^1 \frac{x}{1+x}\,dx$$

$$= \ln 2 - \int_0^1\left(1-\frac{1}{1+x}\right)dx$$

$$= \ln 2 - \left\{x - \ln(1+x)\right\}_0^1$$

$$= \ln 2 - \{1 - \ln 2\}$$

$$= \ln 4 - \ln e$$

$$= \ln(4/e)$$

$$\therefore P = 4/e.$$

Paragraph – 27

Let n be a non-negative integer and , Let $I_n = \int x^n\sqrt{a^2-x^2}\,dx\ (a>0)$ we can find relation among I_n, I_{n-1}, I_{n-2}. It can be observed that I_1 is elementary integration whose value is $-\frac{1}{3}(a^2-x^2)^{3/2}$.

If $I_n = -\frac{x^{n-1}(a^2-x^2)^{3/2}}{A} + a^2 B I_{n-2}$ where A and B are constants. Then

78. A must be equal to

(A) n + 1 (B) n – 1

(C) n + 2 (D) n

Key. C

79. B must be equal to

(A) $\frac{n+1}{n+2}$ (B) $\frac{n+2}{n+1}$

(C) $\frac{n}{n+2}$ (D) $\frac{n-1}{n+2}$

Key. D

80. The value of $\int_0^a x^4\sqrt{a^2-x^2}\,dx$ must be

(A) $\frac{\pi a^6}{32}$ ***(B)*** $\frac{\pi a^4}{16}$

SMART ACHIEVERS LEARNING PVT. LTD.

(B) $\frac{\pi a^4}{64}$ (D) $\frac{\pi a^6}{16}$

Key. A

Sol. 78,79, 80

$$I_n = \int x^n \sqrt{a^2 - x^2}dx$$
$$= \int x^{n-1}\left(x\sqrt{a^2 - x^2}\right)dx$$

apply integration by parts

$$I_n = -\frac{1}{3}(a^2 - x^2)^{3/2}x^{n-1} + \frac{n-1}{3}\int x^{n-2}(a^2 - x^2)^{3/2}dx$$
$$= -\frac{1}{3}(a^2 - x^2)^{3/2}x^{n-1} + \frac{n-1}{3}\int\left(a^2x^{n-2}\sqrt{a^2 - x^2} - x^n\sqrt{a^2 - x^2}\right)dx$$
$$I_n = -\frac{1}{3}(a^2 - x^2)^{3/2}x^{n-1} + \frac{n-1}{3}a^2I_{n-2} - \frac{n-1}{3}I_n$$

$\Rightarrow$ $I_n = \frac{-(a^2 - x^2)^{3/2}x^{n-1}}{n+2} + \left(\frac{n-1}{n+2}\right)a^2I_{n-2}$

$\Rightarrow$ a = n + 2, $B = \frac{n-1}{n+2}$

if $I_n = \int_0^a x^n \sqrt{a^2 - x^2}dx$

$\Rightarrow$ $I_n = \left(\frac{n-1}{n+2}\right)a^nI_{n-2}$

$\Rightarrow$ $I_4 = \frac{3}{6}a^2I_2$

$\Rightarrow$ $I_2 = \frac{1}{4}a^2I_0$

$\Rightarrow$ $I_0 = \frac{\lambda a^2}{4}$

$\Rightarrow$ $I_4 = \frac{3}{6}\times\frac{1}{4}\times\frac{\pi}{4}a^6 = \frac{\pi a^6}{32}$

Paragraph – 28

A series of the form $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ is called fourier series where a_0, $a_1, a_2, \ldots a_n$, b_1, b_2 , ...b_n are constants and the se coefficients are evaluated for $x \in [\alpha, \alpha + 2\pi]$ using the formula $a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)dx, a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx$ and $b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx$ Let us consider f(x) =x+x^2 for $-\pi \le x \le \pi$, then

81. The value of a_0 is

a) $\frac{2\pi^2}{3}$ b) $\frac{4\pi^3}{5}$ c) π d) $\frac{3\pi^2}{2}$

SMART ACHIEVERS LEARNING PVT. LTD.

Key. A

82. The value of a_n is

a) $\frac{4(-1)^n}{n^2}$ b) $\frac{4}{n^2}$ c) $\frac{2(-1)^n}{n^2}$ d) $\frac{2}{n^2}$

Key. A

83. The approximate value of $\frac{\pi^2}{6}$ is given by

a) $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+....\infty$ b) $2-\frac{1}{2^2}+\frac{2}{3}-\frac{2}{4^3}+....\infty$

c) $2+\frac{1}{2}+\frac{2}{3}+\frac{2}{4^2}+....\infty$ d) $2-\frac{2}{3^2}+\frac{2}{5^2}-\frac{2}{7^2}+....\infty$

Key. A

Sol. 81. $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} x+x^2dx=\frac{2}{\pi}x\frac{\pi^3}{3}=\frac{2\pi^2}{3}$

82. $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(x+x^2\right)\cos nx\,dx=\frac{4}{\pi^2}(-1)^n$.

83. $x+x^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}.\cos nx+2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$

Here $b_n=\frac{2\left(-1\right)^{n+1}}{n}$

Putting **x =** π

$$\pi+\pi^2=\frac{\pi^2}{3}+4\left[1+\frac{1}{2^2}+\frac{1}{3^2}+...\right]$$

Putting **x =** $-\pi$

$$-\pi+\pi^2=\frac{\pi^2}{3}+4\left[1+\frac{1}{2^2}+\frac{1}{3^2}+...\right]$$

Adding $\pi^2=\frac{\pi^2}{3}+4\left[1+\frac{1}{2^2}+\frac{1}{3^2}+...\right]$

$$\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+........$$

SMART ACHIEVERS LEARNING PVT. LTD.

Paragraph – 29

Let f(x) , is a quadratic expression with positive integral coefficients such that for every $x_1,x_2\in R, x_1<x_2$, $\int_{x_1}^{x_2} f(x)dx>0$. Let $g(t)=f''(t)\ f(t)$ and $g(0)=12$.If f(x) = 0 has no real roots, then

84. The possible number of such quadratic expression are

a) 4 b) 8 c)12 d) 16

Key. D

85. The minimum value of f(1) is

a) 4 b) 6 c) 8 d) 10

Key. B

86. The maximum value of f(1) is

a) 5 b) 7 c) 9 d) 11

Key. D

Sol. 84. 85. 86 . Let $f(x)=ax^2+bx+c$

Given $\int_\alpha^\beta f(x)dx>0 \, for \;\; all \;\; x>0$

So $f(x)=0$ has no real roots.

Now $g(0)=f''(0)f(0)=2ac \;\; \Rightarrow ac=6$.

∵ a and c are the integers, so the possible values are a = 6, c = 1, a = 1, c = 6, a = 2 , c = 3 , a= 3, c = 2 Again $b^2<4ac \;\; or \;\; b^2<24$. So 'b' can be 1, 2, 3, 4

∴ in all $4\times4=16$ such quadratic are possible Now $f(1)=a+b+c$,then

Max. value of $f(1)=7+4=11$

And min $f(1)=5+1=6$.

Paragraph – 30

Let A_r ($r\in N$) be the area of the bounded region whose boundary is defined by $(6\pi^3 ry^2 - x)(6e^2 y - x) = 0$. Then

87. $\sqrt{A_1}, \sqrt{A_2}, \sqrt{A_3}, \ldots$ are in

(A) A.P. (B) G.P.

(C) H.P. (D) none of these

Key. C

88. $\lim\limits_{n\to\infty}\left[\sqrt{A_1A_2A_3}+\sqrt{A_2A_3A_4}+\sqrt{A_3A_4A_5}+\ldots\ldots \text{ n terms}\right]$ is

(A) $\left(\frac{e}{\pi}\right)^9$ (B) $\frac{1}{2}\left(\frac{e}{\pi}\right)^9$

(C) $\frac{1}{3}\left(\frac{e}{\pi}\right)^9$ (D) $\frac{1}{4}\left(\frac{e}{\pi}\right)^9$

Key. D

89. $\lim\limits_{n\to\infty}\left[e^{-\frac{1}{\sqrt{A_1}}}+e^{-\frac{1}{\sqrt{A_2}}}+e^{-\frac{1}{\sqrt{A_3}}}+\ldots\ldots \text{n terms}\right]$, is

(A) $e^{-\frac{\pi^3}{e^3}}$ (B) $\left(e^{\frac{\pi^3}{e^3}}-1\right)^{-1}$

(C) $e^{-\frac{2\pi^3}{e^3}}$ (D) $\left(e^{\frac{\pi^3}{e^3}}-1\right)^{-2}$

Key. B

SMART ACHIEVERS LEARNING PVT. LTD.

Sol. 87. $\sqrt{A_r} = \left(\frac{e}{\pi}\right)^3 . \frac{1}{r}$, So

$\sqrt{A_1}, \sqrt{A_2}, \sqrt{A_3}, \ldots$ are in H.P.

88. $\sqrt{A_r . A_{r+1} . A_{r+2}} = \left(\frac{e}{\pi}\right)^9 . \frac{1}{r(r+1)(r+2)} = \frac{1}{2}\left(\frac{e}{\pi}\right)^9 . \left[\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}\right]$

So, $\lim_{n\to\infty} \sum_{r=1}^{n} \sqrt{A_r . A_{r+1} . A_{r+2}} = \frac{1}{2}\left(\frac{e}{\pi}\right)^9 . \frac{1}{1.2} = \frac{1}{4}\left(\frac{e}{\pi}\right)^9$

89. $e^{-\frac{1}{\sqrt{A_r}}} = e^{-\left(\frac{\pi}{e}\right)^3 r} = \left(e^{\frac{\pi^3}{e^3}}\right)^{-r} = k^{-r}$

So, $\lim_{n\to\infty} \sum_{r=1}^{n} k^{-r} = \frac{\frac{1}{k}}{1-\frac{1}{k}} = \frac{1}{k-1} = (e^{\frac{\pi^3}{e^3}} - 1)^{-1}$

Paragraph – 31

Consider the functions defined implicitly by the equation $y^3 - 3y + x = 0$ on various intervals in the real line.

If $x \in (-\infty, -2) \cup (2, \infty)$, the equation implicitly defines a unique real valued differentiable function y = f(x).

If $x \in (-2, 2)$, the equation implicitly defines a unique real valued differentiable function y = g(x) satisfying g(0) = 0.

90. If $f(-10\sqrt{2}) = 2\sqrt{2}$ then $f''(-10\sqrt{2}) =$

(A) $\frac{4\sqrt{2}}{7^3.3^2}$ (B) $-\frac{4\sqrt{2}}{7^3.3^2}$

(C) $\frac{4\sqrt{2}}{7^3.3}$ (D) $-\frac{4\sqrt{2}}{7^3.3}$

Key. B

91. The area of the region bounded by the curve y = f(x), x-axis, and the lines x = a and x = b, where $-\infty < a < b < -2$, is

(A) $\int_a^b \frac{x}{3\{(f(x))^2 - 1\}} dx + bf(b) - af(a)$ (B) $-\int_a^b \frac{x}{3\{(f(x))^2 - 1\}} dx + bf(b) - af(a)$

(C) $\int_a^b \frac{x}{3\{(f(x))^2 - 1\}} dx - bf(b) + af(a)$ (D) $-\int_a^b \frac{x}{3\{(f(x))^2 - 1\}} dx - bf(b) + af(a)$

Key. A

92. $\int_{-1}^{1} g'(x) dx =$

(A) $2g(-1)$ (B) 0

(C) $-2g(1)$ (D) $2g(1)$

Key. D

SMART ACHIEVERS LEARNING PVT. LTD.

Sol. 90. Differentiating $y^3 - 3y + x = 0$ w.r.t. x,

$$3y^2\frac{dy}{dx} - 3\frac{dy}{dx} + 1 = 0 \quad \text{or} \quad (3y^2 - 3)\frac{dy}{dx} + 1 = 0 \qquad .....(i)$$

When $x = -10\sqrt{2},\ y = f(-10\sqrt{2}) = 2\sqrt{2}$ and $\frac{dy}{dx} = f'(-10\sqrt{2})$.

$\therefore \quad \{3(2\sqrt{2})^2 - 3\}f'(-10\sqrt{2}) + 1 = 0 \quad \therefore f'(-10\sqrt{2}) = \frac{-1}{21}$

Again, differentiating (i) w.r.t. x,

$$6y\left(\frac{dy}{dx}\right)^2 + (3y^2 - 3)\frac{d^2y}{dx^2} = 0$$

When $x = -10\sqrt{2},\ y = f(-10\sqrt{2}) = 2\sqrt{2},\ \frac{dy}{dx} = f'(-10\sqrt{2}) = \frac{-1}{21}, \frac{d^2y}{dx^2} = f''(-10\sqrt{2})$.

$\therefore \quad 6.2\sqrt{2}.\left(\frac{-1}{21}\right)^2 + 3(2\sqrt{2})^2 -) f''(-10\sqrt{2}) = 0$

or $\quad \frac{12\sqrt{2}}{21^2} + 21.f''(-10\sqrt{2}) = 0$

$\therefore \quad f''(-10\sqrt{2}) = -\frac{12\sqrt{2}}{(21)^3} = \frac{-4\sqrt{2}}{7^3.3^2}$

91. Area $= \int_a^b f(x)dx = [f(x).x]_a^b - \int_a^b x.f'(x)dx$

$= bf(b) - af(a) - \int_a^b x.\frac{-1}{3y^2 - 3}dy,$ by (i) above

$= \int_a^b \frac{x}{3\{f(f(x))^2 - 1\}}dx + bf(b) - af(a)$

92. $g'(x) = \frac{dy}{dx} = \frac{-1}{3\{(f(x))^2 - 1\}}$; $\quad \therefore$ g ′(x) is an even function.

$\therefore \quad \int_{-1}^{1} g'(x)dx = 2\int_0^1 g'(x)dx = 2[g(x)]_0^1 = 2g(1) - 2g(0) = 2g(1).$

Paragraph – 32

Consider a polynomial of $f(x)$ which satisfies the following conditions

$(i)\ f(x) = \left(f'(x)\right)^2\ \forall x \quad (ii)\int_0^1 f(x)\,dx = \frac{19}{12} \quad (iii)\ f'(0) > 0$

93. The function $f(x)$ can be

(A) a linear function
(B) a quadratic function
(C) a cubic function
(D) any polynomial of even degree

Key. B

94. The value of $f'(0)$ is

(A) 0 (B) $\frac{1}{4}$ (C) $\frac{1}{2}$ (D) 1

SMART ACHIEVERS LEARNING PVT. LTD.

Key. D

95. The function $f(x)$ is

(A) Even function (B) Odd function

(C) Neither even nor odd (D) May be either even or odd

Key. C

Sol. 93. On differentiating a polynomial of n^{th} degree, we get another polynomial of $(n-1)^{th}$ degree.

$$\Rightarrow f(x)=\left(f'(x)\right)^2 = n=2(n-1) \Rightarrow n=2$$

94. Let $f(x)=ax^2+bx+c$ $\quad f'(x)=2ax+b$

$$\Rightarrow f'(0)>0 \quad \Rightarrow b>0$$

Also $f(x)=\left(f'(x)\right)^2 \Rightarrow \left(ax^2+bx+c\right)=(2ax+b)^2$

$$\Rightarrow ax^2+bx+c=4a^2x^2+4abx+b^2\forall x$$

$$\Rightarrow a=4a^2 \quad b=4ab \quad c=b^2 \quad \Rightarrow a=\frac{1}{4} \; (\because \; b\neq 0)$$

$$\int_0^1 f(x)dx=\frac{19}{12}$$

$$\Rightarrow \frac{a}{3}+\frac{b}{2}+c=\frac{19}{12} \quad \Rightarrow b=1 \quad c=1$$

$$\Rightarrow f'(0)=b=1$$

95. $f(x)=\frac{x^2}{4}+x+1$

Paragraph – 33

Let $f(x)=\sin 2x \sin\left(\frac{\pi}{2}\cos x\right)$ and $g(x)=\frac{f(x)}{2x-\pi}$ then answer the following

96. $\int_0^{\pi} g(x)dx=$

(A) 0 (B) $\frac{8}{\pi}$ (C) $\frac{8}{\pi^2}$ (D) $\frac{16}{\pi^2}$

Key. A

97. $\int_0^{\pi} x\,g(x)dx=$

(A) 0 (B) $\frac{8}{\pi}$ (C) $\frac{8}{\pi^2}$ (D) $\frac{16}{\pi^2}$

Key. C

98. $\int_0^{\pi} x^2\,g(x)dx=$

SMART ACHIEVERS LEARNING PVT. LTD.

(A) 0 (B) $\frac{8}{\pi}$ (C) $\frac{8}{\pi^2}$ (D) $\frac{16}{\pi^2}$

Key. B

Sol. 96. $I = \int_0^{\pi} \frac{\sin 2x \sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi} dx$

$$= \int_0^{\pi} \frac{\sin(2\pi - 2x)\sin\left(\frac{\pi}{2}\cos(\pi - x)\right)}{2(\pi - x) - \pi} dx$$

$$= \int_0^{\pi} \frac{\sin 2x \sin\left(\frac{\pi}{2}\cos x\right)}{(\pi - 2x)}$$

$$= -I \qquad \Rightarrow I = 0$$

97. $I = \int_0^{\pi} \frac{x \sin 2x \sin\left(\frac{\pi}{2}\cos x\right)}{(2x-\pi)} dx$

$$= \int_0^{\pi} \frac{(\pi - x)\sin(2\pi - 2x)\sin\left(\frac{\pi}{2}\cos(\pi - x)\right)}{2\pi - 2x - \pi} dx$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin 2x \sin\left(\frac{\pi}{2}\cos x\right)}{\pi - 2x} dx$$

$$I + I = \int_0^{\pi} \sin 2x \sin\left(\frac{\pi}{2}\cos x\right) dx$$

$$2I = \frac{16}{\pi^2} \quad \Rightarrow \quad I = \frac{8}{\pi^2}$$

Paragraph – 34

We know that for a continuous function f in $[a,b]$

$$\lim_{n\to\infty} \sum_{r=1}^{n} h\, f(a + rh) = \int_a^b f(x)\,dx, \quad h = \frac{b-a}{n} ......(i)$$

On putting $a = 0,\ b = 1$ $(1) \Rightarrow \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx$. This formula enables us to evaluate limits of the form $\lim_{n\to\infty}\left[\phi_1(n) + \phi_2(n) + ... + \phi_n(n)\right]$. To evaluate this limit we express the r^{th} term as $\phi_r(n) = \frac{1}{n} f\left(\frac{r}{n}\right)$ and then replace $\frac{r}{n}$ by x, $\frac{1}{n}$ by dx. And

$$\lim_{n\to\infty} \sum_{r=1}^{n} by \int_0^1 Also \lim_{n\to\infty} \sum_{r=1}^{kn} \frac{1}{n} f\left(\frac{r}{n}\right) = \int_0^k f(x)\,dx$$

SMART ACHIEVERS LEARNING PVT. LTD.

99. $\lim_{n\to\infty}\left[\frac{1^2}{1^3+n^3}+\frac{2^2}{2^3+n^3}+........+\frac{1}{2n}\right]=$

(A) $\frac{1}{3}\log 2$ (B) $\frac{1}{2}\log 2$ (C) $\log 2$ (D) $1/2$

Key. A

100. $\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right).......\left(1+\frac{n}{n}\right)\right]^{1/n}=$

(A) $\frac{2}{e}$ (B) $\frac{e}{2}$ (C) $\frac{e}{4}$ (D) $\frac{4}{e}$

Key. D

101. $\lim_{n\to\infty}\left[\frac{1}{n^2}\sec^2\left(\frac{1}{n^2}\right)+\frac{2}{n^2}\sec^2\left(\frac{2^2}{n^2}\right)+.........+\frac{1}{n}\sec^2 1\right]=$

(A) $\tan 1$ (B) $\frac{1}{2}\tan 1$ (C) $\frac{1}{2}\sec 1$ (D) $\frac{1}{2}cse1$

Key. B

Sol. 99-101. $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{r^2}{r^3+n^3}$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\frac{(r/n)^2}{\left(\frac{r}{n}\right)^3+1}$$

$$=\int_0^1\frac{x^2}{x^3+1}dx$$

$$=\frac{1}{3}\log 2$$

Paragraph – 35

Integrals of class of functions following a definite pattern can be found by the method of reduction and recursion. If the integral of member of the class having positive integral power can be found then the other members of the class can be found by successive application of the recursion formula. In case of definite integrals, the limits can be put in the portion which has been integrated. While deriving a recursion formula, we integrate by parts and try to bring back the original integral.

For example, let $I_n=\int(\sin x)^n dx$

Then $I_n=\int(\sin x)^{n-1}\sin x dx$

$$=(\sin x)^{n-1}(-\cos x)-\int(n-1)(\sin x)^{n-2}\cos x(-\cos x)dx$$

$$=-(\sin x)^{n-1}\cos x+(n-1)\int(\sin x)^{n-2}\cos^2 x dx$$

$$=-(\sin x)^{n-1}\cos x+(n-1)\int(\sin x)^{n-2}\left(1-\sin^2 x\right)dx$$

$$=-(\sin x)^{n-1}\cos x+(n-1)I_{n-2}-(n-1)I_n$$

SMART ACHIEVERS LEARNING PVT. LTD.

$\Rightarrow \quad nI_n = -(\sin x)^{n-1}\cos x + (n-1)I_{n-2}$

$\Rightarrow \quad I_n = -\dfrac{(\sin x)^{n-1}\cos x}{n} + \dfrac{n-1}{n}I_{n-2}$

Which is the required reduction formula. If I_o and I_1 are known all I_n's can be determined.

Indeed

$I_o = x + c$ and $I_1 = -\cos x + c$

Key.

102. If $I_{m,n} = \int x^m (1-x)^n dx$ then $I_{m,n} - \dfrac{(1-x)^n . x^{m+1}}{m+1} =$

(A) $\dfrac{n}{m+1}I_{m+1,n-1}$ (B) $\dfrac{m}{n+1}I_{m+1,n-1}$ (C) $\dfrac{m}{m+n+1}I_{m+1,\,n-1}$ (D) none of these

Key. A

103. If $I_{m,n} = \int x^m (\ln x)^n dx$ where m and n are positive integers, then $I_{m,n} - \dfrac{x^{m+1}}{m+1}(\ln x)^n =$

(A) $\dfrac{n}{m+1}I_{m,n-1}$ (B) $\dfrac{m}{n+1}I_{m,n-1}$ (C) $-\dfrac{n}{m+1}I_{m,n-1}$ (D) none of these

Key. C

104. If $I_{n-2,n} = \int \cos^{n-2} x \sin nx dx$ where n > 2 be a positive integer then $I_{n-2,n} + \dfrac{\cos^{n-2} x \cos nx}{2(n-1)} =$

(A) $-\dfrac{n-2}{2(n-1)}I_{n-3,n-1}$ (B) $\dfrac{n-2}{2(n-1)}I_{n-3,n-1}$ (C) $\dfrac{n-2}{n-1}I_{n-3,n-1}$ (D) none of these

Key. B

Sol. 102. (A)

$$I_{m,n} = \int_0^1 \underset{I}{x^m} \underset{II}{(1-x)^n} = (1-x)^n \frac{x^{m+1}}{m+1} + \int \frac{n}{m+1}(1-x)^{n-1}.x^{m+1}dx$$

$$I_{m,n} = \frac{(1-x)^n x^{m+1}}{m+1} + \frac{n}{m+1} I_{m+1,n-1}$$

103. (C)

$$I_{m,n} = \int x^m.(\ln x)^n = (\ln x)^n \frac{x^{m+1}}{m+1} - \int \frac{n(\ln x)^{n-1}}{x}.\frac{x^{m+1}}{m+1}dx$$

$$= \frac{(\ln x)^n x^{m+1}}{m+1} - \frac{n}{m+1}\int (\ln x)^{n-1}.x^m dx$$

SMART ACHIEVERS LEARNING PVT. LTD.

$$I_{m,n} - \frac{x^{m+1}}{m+1}(\ln x)^n = -\frac{n}{m+1} I_{m,n-1}$$

104. (B)

$I_{n-2,\,n} = \int (\cos x)^{n-3}.(\cos x.\sin nx)\,dx$

$$\Rightarrow \quad I_{n-2,n} = \frac{1}{2}\left[I_{n-3,n+1} + I_{n-3,n-1}\right] \qquad (1)$$

$$I_{n-2,n} = -\cos^{n-2} x \frac{\cos nx}{n} \int (n-2)\cos^{n-3} x \sin x.\frac{\cos nx}{n} dx$$

$$I_{n-2,n} = -\frac{\cos^{n-2} x \cos nx}{n} - \frac{n-2}{n}\int \cos^{n-3} x \sin x \cos nx dx$$

$$= -\frac{\cos^{n-2} x \cos nx}{n} - \frac{n-2}{2n}\int \cos^{n-3} x.\{\sin(n+1)x - \sin(n-1)x\} dx$$

$$= -\frac{\cos^{n-2} x \cos nx}{n} - \frac{n-2}{2n}\left[I_{n-3,n+1}\right] + \frac{n-2}{2n}.I_{n-3,n-1} \qquad (2)$$

Adding (1) and (2), we get

$$2I_{n-2,n} = -\frac{\cos^{n-2} x \cos nx}{n} + I_{n-3,n+1}\left\{\frac{1}{2} - \frac{n-2}{2n}\right\} + I_{n-3,n-1}\left\{\frac{1}{2} + \frac{n-2}{2n}\right\}$$

$$= \frac{-\cos^{n-2} x \cos nx}{n} + \frac{I_{n-3,n+1}}{n} + I_{n-3,n-1}\left(\frac{n-1}{n}\right)$$

$$nI_{n-2,n} = -\frac{\cos^{n-2} x \cos nx}{2} + \frac{I_{n-3,n+1}}{2} + \frac{(n-1)I_{n-3,n-1}}{2} \qquad (3)$$

Now, (3) – (1) , we get

$$I_{n-2,n}(n-1) = -\frac{\cos^{n-2} x.\cos nx}{2} + I_{n-3,n-1}\left(\frac{n-1}{2} - \frac{1}{2}\right)$$

$$(n-1)I_{n-2,n} = -\frac{\cos^{n-2} x.\cos nx}{2} + \left(\frac{n-2}{2}\right)I_{n-3,n-1}$$

$$I_{n-2,n} = -\frac{\cos^{n-2} x \cos nx}{2(n-1)} + \frac{n-2}{2(n-1)} I_{n-3,n-1}$$

Paragraph – 36

Let $y^2 = 3x^2 + 2x + 1$, $I_n = \int \frac{x^n}{y} dx$, then $\alpha I_{10} + \beta I_9 + \gamma I_8 = x^9 y$.

105. $\alpha =$

(A) 30 (B) 19 (C) 10 (D) 9

Key. A

106. $\beta =$

(A) 30 (B) 19 (C) 10 (D) 9

Key. B

107. $\gamma =$

(A) 30 (B) 19 (C) 10 (D) 9

SMART ACHIEVERS LEARNING PVT. LTD.

Key. D

Sol. 105-107. $\alpha x^{10} + \beta x^9 + \gamma x^8 - y^2 9x^8 = x^9 y \dfrac{dy}{dx}$ $\left(\because\ y\dfrac{dy}{dx} = 3x+1\right)$.

$\alpha x^{10} + \beta x^9 + \gamma x^8 = (3x^2 + 2x + 1)\, 9x^8 + x^9 (3x + 1);$

$\alpha = 30;\ \beta = 19,\ \gamma = 9.$

Paragraph – 37

Let $f(x)$ be a continuous function defined on the closed interval $[a,b]$, then

$$\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{1}{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx.$$

108. The value of $\lim_{n\to\infty}\left\{\frac{1}{n}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+............+\frac{1}{8n}\right\}$ is

(A) 5/4 (B) 3/4 (C) 5/8 (D) 3/8

Key. D

109. The value of $\lim_{n\to\infty}\left\{\prod_{r=1}^{n}\left(1+\frac{r}{n}\right)\right\}^{1/n}$ is

(A) 3/e (B) 4/e (C) 1/e (D) 2/e

Key. B

110. The value of $\lim_{n\to\infty}\left\{\tan\left(\frac{\pi}{2n}\right)\tan\left(\frac{2\pi}{2n}\right)\tan\left(\frac{3\pi}{2n}\right)........\tan\left(\frac{n\pi}{2n}\right)\right\}^{1/n}$ is

(A) 1 (B) 2 (C) 3 (D) not defined

Key. A

Sol. 108. (D)

$$\because \lim_{n\to\infty}\left\{\frac{1}{n}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+..........+\frac{1}{8n}\right\}$$

$$=\lim_{n\to\infty}\left\{\frac{n^2}{(n+0)^3}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+...........+\frac{n^2}{(n+n)^3}\right\}$$

$$=\lim_{n\to\infty}\sum_{r=0}^{n}\frac{n^2}{(n+r)^3}$$

$$=\lim_{n\to\infty}\sum_{r=0}^{n}\frac{1}{n\left(1+\frac{r}{n}\right)^3}$$

$$=\int_0^1\frac{dx}{(1+x)^3}$$

$$=-\left\{\frac{1}{2(1+x)^2}\right\}_0^1$$

$$==-\left\{\frac{1}{8}-\frac{1}{2}\right\}=\frac{1}{2}-\frac{1}{8}=\frac{3}{8}.$$

SMART ACHIEVERS LEARNING PVT. LTD.

109. (B)

Let $P=\lim\limits_{n\to\infty}\left\{\prod_{r=1}^{n}\left(1+\frac{r}{n}\right)\right\}^{1/n}$

$$\therefore \ln P=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\ln\left(1+\frac{r}{n}\right)$$

$$=\int_0^1 \ln(1+x)\,dx$$

$$=\left[x\ln(1+x)\right]_0^1\int_0^1\frac{x}{1+x}\,dx$$

$$=\ln 2-\int_0^1\left(1-\frac{1}{1+x}\right)dx$$

$$=\ln 2-\left\{x-\ln(1+x)\right\}_0^1$$

$$=\ln 2-\{1-\ln 2\}$$

$$=\ln 4-\ln e$$

$$=\ln(4/e)$$

$\therefore P=4/e$.

110. (A)

Paragraph – 38

If $\int u(x).v(x)dx=u(x)\int v(x)dx-\int\left\{\left(\frac{d}{dx}u(x)\right)\int v(x)dx\right\}dx$, then

111. $\int \sin x\log\left(\cot\frac{x}{2}\right)dx$ is equal to

(A) $-\cos x\log\left(\cot\frac{x}{2}\right)-\log\left(\sin\frac{x}{2}\right)+\log\left(\sec\frac{x}{2}\right)+C$

(B) $-\sin x\log\left(\cot\frac{x}{2}\right)+\log\left(\tan\frac{x}{2}\right)+C$

(C) $-\cos x\log\left(\tan\frac{x}{2}\right)-\log\left(\sin\frac{x}{2}\right)-\log\left(\sec\frac{x}{2}\right)+C$

(D) $-\sin x\log\left(\cot\frac{x}{2}\right)+\cos x\log\left(\tan\frac{x}{2}\right)+C$

Key. A

112. $\int 3^x\left(g'(x)+g(x).\ln 3\right)dx$ is equal to

(A) $3^x g(x)+C$

(B) $3^x\ln 3g(x)+C$

(C) $\frac{3^x g(x)}{\ln 3}+C$

(D) $3^x+g(x)+C$

Key. A

SMART ACHIEVERS LEARNING PVT. LTD.

113. $\int \cos 2\theta \log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)d\theta$ is equal to

(A) $\frac{1}{2}\sin 2\theta \log\left|\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right| - \frac{1}{2}\log|\cos 2\theta| + C$

(B) $\frac{1}{2}\sin 2\theta \log\left|\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right| + \frac{1}{2}\log|\cos 2\theta| + C$

(C) $\frac{1}{2}\sin 2\theta \log\left|\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right| + \frac{1}{2}\log|\sin 2\theta| + C$

(D) $\frac{1}{2}\sin 2\theta \log\left|\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right| - \frac{1}{2}\log|\sin 2\theta| + C$

Key. B

Sol. 111. (A)

$$\int \sin x \log\left(\cot\frac{x}{2}\right)dx = \log\cot\frac{x}{2}(-\cos x) - \int \frac{1}{\cot\frac{x}{2}}\left(-\text{cosec}^2\frac{x}{2}\right).\frac{1}{2}(-\cos x)dx$$

$$= -\cos x\log\left(\cot\frac{x}{2}\right) - \int\frac{\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}{2\cos\frac{x}{2}\sin\frac{x}{2}}dx = -\cos x\log\left(\cot\frac{x}{2}\right) - \int\frac{1}{2}\cot\frac{x}{2}dx + \int\frac{1}{2}\tan\frac{x}{2}dx$$

$$= -\cos x\log\left(\cot\frac{x}{2}\right) - \log\left|\sin\frac{x}{2}\right| + \log\left|\sec\frac{x}{2}\right| + C$$

112. (A)

$$\int 3^x\left(g'(x)+g(x)\ln 3\right)dx = 3^x\int g'(x)dx - \int\left(3^x\ln 3\int g'(x)dx\right)dx + \int g(x)3^x\ln 3dx = 3^x g(x) + C$$

113. (B)

$$\int \cos 2\theta\log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)d\theta$$

$$= \log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)\int\cos 2\theta d\theta - \int\left\{\left(\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right)\frac{(\cos\theta-\sin\theta)^2+(\cos\theta+\sin\theta)^2}{(\cos\theta-\sin\theta)^2}\int\cos 2\theta d\theta\right\}d\theta$$

$$= \frac{\sin 2\theta}{2}\log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right) - \int\tan 2\theta d\theta = \frac{\sin 2\theta}{2}\log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right) + \frac{1}{2}\log|\cos 2\theta| + C$$

Paragraph – 39

If f(x) is a periodic function with period T, then

(a) $\int_a^{a+nT} f(x)dx = n\int_0^T f(x)dx,$ where $n \in I$

(b) $\int_{mT}^{nT} f(x)dx = (n-m)\int_0^T f(x)dx,$ where $n, m \in I$

SMART ACHIEVERS LEARNING PVT. LTD.

(c) $\int_{a+nT}^{b+nT} f(x)dx = \int_a^b f(x)dx$, where $n \in I$

114. If f(x) is given to be an odd function in $\left[-\frac{T}{2}, \frac{T}{2}\right]$ and has a period equal to T, then $\int_a^x f(y)\,dy$ has a fundamental period of

(A) T (B) 2T
(C) 3T (D) 4T

Key. A

115. $\int_0^{n\pi+v} |\sin x|dx$ is equal to (where n is positive integer and $0 \le v < \pi$)

(A) $2n+1-\cos v$ (B) $2n+1+\cos v$
(C) $n+1-\cos v$ (D) $n+1+\cos v$

Key. A

116. Let T > 0 be a real number, suppose f is a continuous function such that for all x $\in$ R, if f(x+T) = f(x), If $I = \int_0^T f(x)dx$, then the value of $\int_3^{3+3T} f(2x)dx$ is

(A) $\frac{3}{2}I$ (B) $2I$
(C) $3I$ (D) $6I$

Key. C
Sol. Conceptual

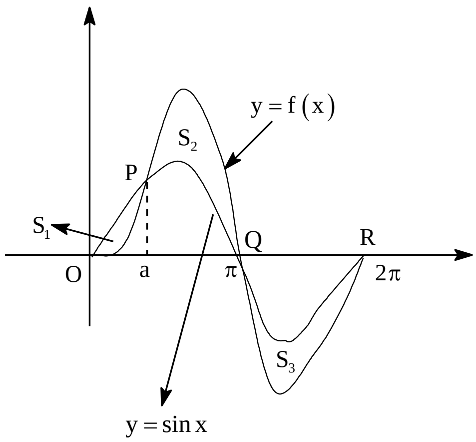
Paragraph – 40

In the adjacent figure, the graphs of $y = f(x)$ and y = sin x are given. The graphs of two equations intersect at $P(a, f(a)), Q(\pi, 0)$ and $R(2\pi, 0)$.

Let S_1, S_2 and S_3 represent areas bounded by the curves $y = f(x)$ and y = sin x between x = 0 and x = a; between x = a and $x = \pi$ and between $x = \pi$ and $x = 2\pi$ respectively.

It is given that $S = 1 - \sin t + (t-1)\cos t \ \forall \ t \le a$ represents the area bounded between $y = f(x)$ and $y = \sin x$

SMART ACHIEVERS LEARNING PVT. LTD.



117. The value of a is _____

a) $\frac{\pi}{4}$ b) $\frac{\pi}{3}$ c) 1 d) $\frac{\sqrt{3}}{2}$

Key. C

118. The value of $S_2 - S_1$ is _____

a) π b) $\pi - 1 + \sin 1$ c) $1 - \sin 1$ d) $\pi - 2$

Key. D

119. The area S_3 is equal to___

a) $3\pi - 2$ b) $2(\pi - 1)$ c) $\pi - 2$ d) $4 - \pi$

Key. A

Sol. 117-119. $S = 1 - \sin t + (t-1)\cos t = \int_0^1 \{\sin x - f(x)\} dx$

Different w.r.t t we get

$f(t) = t\sin t \Rightarrow f(x) = x\sin x$

Now solving $f(x) = x\sin x$ and $y = \sin x$, we get $x = 1 \text{ or } n\pi, n \in I \Rightarrow a = 1$

Thus $S_1 = 1 - \sin 1$

Also $S_2 = \int (x\sin x - \sin x) dx = [-x\cos x + \sin x + \cos x]_1^\pi = \pi - 1 - \sin 1$

$\therefore S_2 - S_1 = \pi - 2$

Further,

$$S_3 = \int_\pi^{2\pi} (\sin x - x\sin x) dx = 3\pi - 2$$

Paragraph – 41

A curve $y = f(x)$ passes through (2,0) and slope of tangent at any point $P(x, y)$ on the curve is $\frac{(x+1)^2 + y - 3}{x+1}$, then

120. The curve is

SMART ACHIEVERS LEARNING PVT. LTD.

a) a parabola b)a circle c) an ellipse d) a hyperbola

Key. A

121. Area bounded between $y=|f(x)|$, x-axis and $|x|=3$

a) 20 b) 21 c) $\frac{62}{3}$ d) $\frac{52}{3}$

Key. C

122. The number of points at which $y=x|f(x)|$ is not differentiable is

a) 1 b) 2 c) 0 d) 3

Key. A

Sol. **120 to 122**

Given $\frac{dy}{dx}-\frac{y}{x+1}=x+1-\frac{3}{x+1}\Rightarrow y\left(\frac{1}{x+1}\right)=\int\left(1-\frac{3}{(x+1)^2}\right)dx$

$\Rightarrow y=(x+1)(x+c)+3$ *But* $(2,0)$ lies on this curve

$\therefore$ c=-3 .Hence curve is $y=x^2-2x$, a parabola

Area bounded by $y=|x^2-2x|,\ x-axis,|x|=3\ is$

$$=\int_{-3}^{0}(x^2-2x)dx+\int_{0}^{2}(2x-x^2)dx+\int_{2}^{3}(x^2-2x)dx=62/3.$$

Paragraph – 42

If the integrand is a rational function of x and fractional power of a linear fractional function of the form $\frac{ax+b}{cx+d}$ then rationalization of the integral is affected by the substitution $\frac{ax+b}{cx+d}=t^m$, where m is l.c.m of fractional powers of $\frac{ax+b}{cx+d}$

123. If $I=\int\frac{(2x-3)^{1/2}}{(2x-3)^{1/3}+1}dx=$

$3\left[\frac{1}{7}(2x-3)^{7/6}-\frac{1}{5}(2x-3)^{5/6}+\frac{1}{3}(2x-3)^{1/2}-(2x-3)^{1/6}+g(x)\right]+C$, then g(x) is equal to

a) $\tan^{-1}(2x-3)^{1/6}$ b) $\tan^{-1}(2x-3)^{1/6}$

c) $3\tan^{-1}(2x-3)^{1/6}$ d) $4(2x-3)^{1/6}$

Key. A

124. If $I=\int\frac{dx}{\sqrt[4]{(x-1)^3+(x+2)^5}}=A\sqrt{\frac{x-1}{x+2}}+C$ then A is equal to

a) $\frac{1}{3}$ b) $\frac{2}{3}$ c) $\frac{3}{4}$ d) $\frac{4}{3}$

Key. D

SMART ACHIEVERS LEARNING PVT. LTD.

125. If $I=\int \frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}}=K\sqrt[3]{\frac{1+x}{1-x}}+c$ then K is equal to

a) $\frac{2}{3}$ b) $\frac{3}{2}$ c) $\frac{1}{3}$ d) $\frac{1}{2}$

Key. B

Sol. 125. Put $\frac{x+2}{x-1}=t$

126. Put $\frac{x+1}{1-x}=t^3$

Paragraph – 43

We can derive reduction formulas for the integral of the form $\int \sin^n x dx, \int \tan^n x dx, \int \sec^n x dx$ etc. using integration by parts. In turn these reduction formulas can be used to compute integrals of higher powers of $\sin x, \tan x$ etc.

127. If $\int \sin^5 x dx = -\frac{1}{5}\sin^4 x \cos x + A\sin^2 x \cos x - \frac{8}{15}\cos x + c$ then A is equal to

a) $-\frac{2}{15}$ b) $-\frac{3}{5}$

c) $-\frac{4}{15}$ d) $-\frac{1}{15}$

Key. C

128. If $\int \tan^6 x dx = \frac{1}{5}\tan^5 x + A\tan^3 x + \tan x - x + c$ then A is equal to

a) $\frac{1}{3}$ b) $\frac{2}{3}$ c) $\frac{-2}{3}$ d) $-\frac{1}{3}$

Key. D

129. If $\int \sec^6 x dx = \frac{1}{5}\tan^5 x + A\tan^3 x + \tan x + c$ then A is equal to

a) $\frac{1}{3}$ b) $\frac{2}{3}$ c) $-\frac{1}{3}$ d) $-\frac{2}{3}$

Key. B

Sol. 127. $\sin^5 x = (1-\cos^2 x)^2 \sin x$

128. $\tan^6 x = (\sec^2 x - 1)\tan^4 x$

129. $\sec^6 x = \sec^4 x \sec^2 x$

Paragraph – 44

Integrals of class of functions following a definite pattern can be found by method of reduction and recursion. Reduction formulas make it possible to reduce an integral dependent on the

SMART ACHIEVERS LEARNING PVT. LTD.

index n > 0, called the order of integral, to an integral of the same type with a smaller index. Integration by parts helps us to derive reduction formulas

130. If $I_n = \int \frac{dx}{\left(x^2+a^2\right)^n}$ then $I_{n+1} + \frac{1-2n}{2n}\frac{1}{a^2}I_n$, is equal to

a) $\frac{x}{\left(x^2+a^2\right)^n}$ b) $\frac{1}{2na^2}\frac{1}{\left(x^2+a^2\right)^{n-1}}$ c) $\frac{1}{2na^2}\frac{x}{\left(x^2+a^2\right)^n}$ d) $\frac{1}{2na^2}\frac{x}{\left(x^2+a^2\right)^{n-1}}$

Key. C

131. If $I_{n,-m} = \int \frac{\sin^n x}{\cos^m x}dx$ then $I_{n,m}\frac{n-1}{m-1}I_{n-2,2-m}$ is equal to

a) $\frac{\sin^{n-1} x}{\cos^{m-1} x}$ b) $\frac{1}{m-1}\frac{\sin^{n-1} x}{\cos^{m-1} x}$

c) $\frac{1}{n-1}\frac{\sin^{n-1} x}{\cos^{m-1} x}$ d) $\frac{n-1}{m-1}\frac{\sin^{n-1} x}{\cos^{m-1} x}$

Key. B

132. If $u_n = \int \frac{x^n}{\sqrt{ax^2+bx+c}}dx$ then $(n+1)au_{n+1} + (2n+1)bu_n + ncu_{n-1}$ is equal to

a) $x^{n-1}\sqrt{ax^2+bx+c}$ b) $\frac{x^{n-2}}{\sqrt{ax^2+bx+c}}$

c) $\frac{x^n}{\sqrt{ax^2+bx+c}}$ d) $x^n\sqrt{ax^2+bx+c}$

Key. D

Sol. 130. $I_{n,-m} = \frac{\sec^{n-1} x}{(m-1)\cos^{m-1} x} - \frac{n-1}{m-1}\int \frac{\sec^{n-2} x}{\cos^{m-2} x}dx$

$= \frac{\sec^{m-1} x}{(m-1)\cos^{m-1} x} - \left(\frac{n-1}{m-1}\right)I_{n-2,2-m}$

131. Consider

$U_{n+1} = \int \frac{x^{n+1}}{\sqrt{ax^2+bx+c}}dx$

$= \frac{1}{2a}\int \frac{x^n(2ax+2b)-2bx^n}{\sqrt{ax^2+bx+c}}dx$

$= \frac{1}{2a}\int \frac{x^n(2ax+2b)}{\sqrt{ax^2+bx+c}}dx - \frac{b}{a}u_n$

$= I_n - \frac{b}{a}u_n$ (1)

SMART ACHIEVERS LEARNING PVT. LTD.

$$I_n = \frac{1}{x}\int \frac{x^n(2ax+2b)}{\sqrt{ax^2+bx+c}}$$

$$= \frac{1}{2a}\int x^n 2\sqrt{ax^2+bx+c}dx - \int nx^{n-1}2\sqrt{ax^2+bx+c}dx$$

$$= \frac{x^n}{9}\sqrt{ax^2+bx+c} - \frac{n}{a}\int \frac{x^{n-1}(ax^2+2bx+c)}{\sqrt{ax^2+bx+c}}$$

$$\Rightarrow aI_n = x^n\sqrt{ax^2+bx+c} - nau_{n+1}$$

Putting this value in (1)

$$au_{n+1} = x^n\sqrt{ax^2+bx+c} - nau_{n+1} - 2bnu_n - ncu_{n-1} - bu_n$$

$$\Rightarrow (n+1)au_{n+1} + (2n+1)bu_n + ncu_{n-1} = x^n\sqrt{ax^2+bx+c}$$

Paragraph – 45

If u and v are two functions in 'x' , then $\int uv\,dx = u\int v\,dx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx$.

In applying the above rule, care has to be taken in the selection of first function '*u*' and selection of second function '*v*' usually we use the following order for the first function Inverse, logarithmic, algebraic , trigonometric, exponent.

133. $\int \frac{x-\sin x}{1-\cos x}\,dx =$

a) $-x\cot\frac{x}{2}+c$ b) $\cot\frac{x}{2}+c$ c) $-\cot\frac{x}{2}+c$ d) $x\tan\frac{x}{2}+c$

Key. A

134. $\int \sec^{-1}\sqrt{x}\,dx =$

a) $x\sec^{-1}\sqrt{x} - \log(1+x)+c$ b) $\sec^{-1}\sqrt{x} - \tan^{-1}\sqrt{x}+c$

c) $x\sec^{-1}\sqrt{x} - \sqrt{x-1}+c$ d) $x\tan^{-1}\sqrt{x} - \sec^{-1}\sqrt{x}+c$

Key. C

135. $\int \sin^{-1}\sqrt{\frac{x}{a+x}}dx =$

a) $a\left[\sqrt{\frac{x}{a}}\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{\frac{x}{a}} + \tan^{-1}\frac{x}{a}\right]+c$ b) $a\left[\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{\frac{x}{a}} + \tan^{-1}\frac{x}{a}\right]+c$

c) $a\left[\frac{x}{a}\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{\frac{x}{a}} + \tan^{-1}\sqrt{\frac{x}{a}}\right]+c$ d) $\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{\frac{x}{a}}+c$

Key. C

Sol. 133. $\int \frac{x-\sin x}{1-\cos x}\,dx = \int \frac{x}{1-\cos x}dx - \int \frac{\sin x}{1-\cos x}dx = \frac{1}{2}\int x\,\mathrm{cs}e^2\frac{x}{2} - \int \cot\frac{x}{2}dx$

134. Put $x = \sec^2\theta$

135. Put $x = a\tan^2\theta \Rightarrow dx = 2a\tan\theta\sec^2\theta$

Paragraph – 46

Some times it is useful to write the integral as a sum of two related integrals which can be evaluated by making suitable substitutions

Algebraic Twins: $\int \frac{2x^2}{x^4+1}dx = \int \frac{x^2+1}{x^4+1}dx + \int \frac{x^2-1}{x^4+1}dx$

$$\int \frac{2}{x^4+1}dx = \int \frac{x^2+1}{x^4+1}dx - \int \frac{x^2-1}{x^4+1}dx$$

$$\int \frac{2x^2}{x^4+1+kx^2}dx,\quad \int \frac{2}{x^4+1+kx^2}dx$$

Trigonometric Twins: $\int \sqrt{\tan x}\,dx, \int \sqrt{\cos x}\,dx, \int \frac{1}{\sin^4 x+\cos^4 x}dx, \int \frac{1}{\sin^6 x+\cos^6 x}dx$

136. $\int \frac{x^2-1}{x^4+x^2+1}dx =$

a) $\frac{1}{4}\log\left|\frac{x^2-x-1}{x^2+x+1}\right|+c$

b) $\frac{1}{2}\log\left|\frac{x^2+x+1}{x^2-x+1}\right|+c$

c) $\frac{1}{4}\log\left|\frac{x^2+x+1}{x^2-x+1}\right|+c$

d) $\frac{1}{2}\log\left|\frac{x^2-x+1}{x^2+x+1}\right|+c$

Key. D

137. $\int \frac{1}{\cos^6 x+\sin^6 x}dx =$

a) $\tan^{-1}(\tan x-\cot x)+c$

b) $\log|\tan x-\cot x|+c$

c) $\sin^{-1}(\tan x-\cot x)+c$

d) $\cos^{-1}(\tan x-\cot x)+c$

Key. A

138. $\int\left(\sqrt{\tan x}+\sqrt{\cot x}\right)dx =$

a) $\tan^{-1}\left(\sqrt{\tan x}+\sqrt{\cot x}\right)+c$

b) $\sin^{-1}(\sin x+\cos x)+c$

c) $\sqrt{2}\sin^{-1}(\sin x+\cos x)+c$

d) $\sqrt{2}\sin^{-1}(\sin x-\cos x)+c$

Key. D

Sol. 136. $\int \frac{x^2\left(1-\frac{1}{x^2}\right)}{x^2\left(x^2+1+\frac{1}{x^2}\right)}dx = \int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x^2}\right)^2-1}dx$ Put $x+\frac{1}{x}=t$

137. $\int\left(\sqrt{\tan x}+\sqrt{\cot x}\right)dx = \int \frac{\sin x+\cos x}{\sqrt{\cos x\sin x}}dx$ Put $\sin x-\cos x=t$

Paragraph – 47

Integrals of the form $\int f(x), \sqrt{ax^2+bx+c}$ are calculated with the help of Euler's substitution.

SMART ACHIEVERS LEARNING PVT. LTD.

$\sqrt{ax^2+bx+c}=t\pm x\sqrt{a}\ \ if\ \ a>0$

$\sqrt{ax^2+bx+c}=tx\pm\sqrt{c}\ \ if\ \ c>0$

$\sqrt{ax^2+bx+c}=(x-\alpha)t\ \ if\ \ ax^2+bx+c=a(x-\alpha)(x-\beta)$

138. $\int \frac{\left(x+\sqrt{1+x^2}\right)^{15}}{\sqrt{1+x^2}}dx=$

(a) $\frac{\left(x+\sqrt{1+x^2}\right)^{16}}{16}+C$ (b) $\frac{\left(x-\sqrt{1+x^2}\right)^{15}}{15}+C$

(c) $\frac{\left(x+\sqrt{1-x^2}\right)^{15}}{15}+C$ (d) $\frac{\left(1+\sqrt{1-x^2}\right)^{15}}{15}+C$

Key. ***C***

139. $\int \frac{dx}{(x-1)\sqrt{-x^2+3x-2}}=$

(a) $-2\sqrt{\frac{x-2}{1-x}}+c$ (b) $-2\sqrt{\frac{x-2}{x-1}}+c$ (c) $2\sqrt{\frac{1-x}{x-2}}+c$ (d) $\sqrt{\frac{x-1}{x-2}}+c$

Key. ***A***

140. $\int \frac{dx}{x\sqrt{x^2+2x-1}}=$

(a) $\frac{1}{\sqrt{x^2+2x-1}}+c$ (b) $2\tan^{-1}\left(x+\sqrt{x^2+2x-1}\right)+c$

(c) $\frac{1}{2\sqrt{3}}\ln\left|\frac{x^2+2x-1}{x^2-2x+1}\right|+c$ (d) $\sin^{-1}\left(x+\sqrt{x^2+2x-1}\right)+c$

Key. ***B***

Sol. 138. $\sqrt{1+x^2}=t-x\Rightarrow x+\sqrt{1+x^2}=t\Rightarrow 1+\frac{x}{\sqrt{1+x^2}}dx=dt\ \Rightarrow\frac{dx}{\sqrt{1+x^2}}=\frac{dt}{t}\ \Rightarrow$

$I=\int t^{14}dt=\frac{t^{15}}{15}+c$

139. Let $-x^2+3x-2=t(x-2)\Rightarrow t=\sqrt{\frac{1-x}{x-2}}\Rightarrow x=\frac{2t^2+1}{t^2+1}\Rightarrow dx=\frac{2t\,dt}{(t^2+1)^2}\ \Rightarrow$

$I=\int\frac{2}{t^2}dt=-\frac{2}{t}+c$

140. $t=x+\sqrt{x^2+2x-1}\ \Rightarrow\ \ x=\frac{t^2+1}{2(t+1)}\Rightarrow\frac{dx}{dt}=\frac{1}{2}\frac{t^2+2t-1}{(1+t)^2}\ \Rightarrow\ I=2\int\frac{dt}{(1+t^2)}=2\tan^{-1}t$

Paragraph – 48

Repeated application of integration by parts give us, the reduction formula if the integrand is depend on a natural number n.

141. If $I_n=\int\cos nx.\cos ecx.dx\ then\ I_n-I_{n-2}=$

(a) $\frac{2\sin(n-1)x}{n-1}$ (b) $\frac{2\cos(n-1)x}{n-1}$

SMART ACHIEVERS LEARNING PVT. LTD.

(c) $\frac{2\cos(n-1)x}{n}$ (d) $\frac{2\cos(n+1)x}{n}$

Key. **B**

142. If $I_{m,n}=\int x^m(1-x)^{n-1}dx$ then $I_{m,n}-\frac{n-1}{m+n}\;I_{(m,n-1)}=$

(a) $\frac{x^m(1-x)^{n-2}}{m+n}$ (b) $\frac{x^{m+1}(1-x)^{n-1}}{m+n}$

(c) $\frac{x^{m-1}(1-x)^{n-1}}{m+n}$ (d) $\frac{x^{m-1}(1-x)^{n-2}}{m+n-2}$

Key. **B**

143. If $I_n=\int\frac{x^n}{\sqrt{ax^2+2bx+c}}dx$ then $(n+1)a\,I_{n+1}+(2n+1)b\,I_n+nc\;I_{n-1}$ is equal to

(a) $x^{n-1}\sqrt{ax^2+2bx+c}$ (b) $\frac{x^{n-2}}{\sqrt{ax^2+2bx+c}}$

(c) $x^n\sqrt{ax^2+2bx+c}$ (d) $\frac{x^n}{\sqrt{ax^2+2bx+c}}$

Key. **C**

Sol. 141. Let $P=x^{m+1}(1-x)^{n-1}$

$$\frac{dP}{dx}=-(n-1).x^{m+1}.(1-x)^{n-2}+(m+1)(1-x)^{n-1}.x^m$$

$$=(n-1).(1-x-1)x^m(1-x)^{n-2}+(m+1)\frac{d}{dx}I_{m,n}$$

$$=(n-1).(1-x)^{n-1}.x^m-x^m(n-1)(1-x)^{n-2}+(m+1)\frac{d}{dx}I_{m,n}$$

$$\frac{dP}{dx}=(m+n)\frac{d}{dx}I_{m,n}-(n-1)\frac{d}{dx}I_{m,n-1}$$

Integrating

$$x^{m+1}(1-x)^{n-1}=(m+n)I_{m,n}-(n-1).I_{m,n-1}$$

143. $I_{n+1}=\frac{1}{2a}\int\frac{x^n(2ax+2b)-2bx^n}{\sqrt{ax^2+2bx+c}}dx=\frac{1}{2a}\int\frac{x^n(2ax+2b)}{\sqrt{ax^2+2bx+c}}-\frac{b}{a}I_n=U_n-\frac{b}{a}I_n$

$$U_n=\frac{1}{2a}\left(x^n\,2\sqrt{ax^2+bx+c}-\int n\,x^{n-1}2\sqrt{ax^2+2bx+c}\;dx\right)$$

$$aU_n=x^n\sqrt{ax^2+bx+c}-naI_{n+1}-2bnI_n-nc\,I_{n-1}$$

Paragraph – 49

Consider $\int\frac{x^3+3x^2+2x+1}{\sqrt{x^2+x+1}}dx=\left(ax^2+bx+c\right)\sqrt{x^2+x+1}+\lambda\int\frac{dx}{\sqrt{x^2+x+1}}$ then

SMART ACHIEVERS LEARNING PVT. LTD.

144. Value of b =

(a) $\frac{13}{12}$ (b) $\frac{12}{5}$ (c) $\frac{1}{3}$ (d) 1

Key. A

145. Value of c =

(a) $\frac{-13}{6}$ (b) $\frac{-7}{24}$ (c) $\frac{-10}{7}$ (d) 1

Key. B

146. Value of λ

(a) $\frac{1}{16}$ (b) $\frac{10}{3}$ (c) $\frac{10}{7}$ (d) 1

Key. A

Sol. 144 to 146

differentiating both side

$$\frac{x^3+3x^2+2x+1}{\sqrt{x^2+x+1}}=\frac{(ax^2+bx+c)(2x+1)}{2\sqrt{x^2+x+1}}+(2ax+b)\sqrt{x^2+x+1}+\lambda\frac{1}{\sqrt{x^2+x+1}}$$

$$=\frac{6ax^3+(5a+4b)x^2+(4a+3b+2c)x+(c+2b+2\lambda)}{2\sqrt{x^2+x+1}}$$

comparing coefficient of x on both sides.

SMART ACHIEVERS LEARNING PVT. LTD.

Paragraph – 50

Let $f(x)=\frac{\sin 2x\sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}$

147. $\int_0^{\pi} f(x)dx$ =

A) 0 B) $\frac{8}{\pi}$ C) $\frac{8}{\pi^2}$ D) $\frac{16}{\pi^2}$

Key. A

148. $\int_0^{\pi} x\ f(x)dx$ =

A) 0 B) $\frac{8}{\pi}$ C) $\frac{8}{\pi^2}$ D) $\frac{16}{\pi^2}$

Key. C

149. $\int_0^{\pi} x^2 f(x)dx$ =

A) 0 B) $\frac{8}{\pi}$ C) $\frac{8}{\pi^2}$ D) $\frac{16}{\pi^2}$

Key. B

Sol. **(147 – 149)**

Let $I = \int_0^a f(x)dx$

If $f(a-x) = -f(x)$ then $I = 0$

Also $I = \int_0^a f(a-x)dx$

SMART ACHIEVERS LEARNING PVT. LTD.

SMART ACHIEVERS LEARNING PVT. LTD.

Definite, Indefinite Integration & Areas

Integer Answer Type

1. Let $f(x)$ be differentiable function such that $f(x)=x^2+\int_0^x e^{-t}f(x-t)\,dt$ then $6f(1)=$ __

Key. 8

Sol. $f(x)=x^2+\int_0^x e^{-t}f(x-t)\,dt=x^2+e^{-x}\int_0^x e^{t}f(t)\,dt$

$\Rightarrow f^1(x)=2x-e^{-x}\left(e^x\left(f(x)-x^2\right)\right)+e^{-x}.e^x f(x)$

$\Rightarrow f^|(x)=2x+x^2\Rightarrow f(x)=\frac{x^3}{3}+x^2+K$

$But,\quad f(0)=0\ \Rightarrow k=0$

$\therefore\ f(1)=\frac{4}{3}$

2. If the value of definite integral $\int_1^a x.a^{-\left[\log_a^x\right]}dx$ where $a>1, and\,[.]$ denotes the greatest integer, is $\frac{e-1}{2}$ then the value of $5[a]$ is ____

Key. 5

Sol. Let $\log_a^x=t\Rightarrow a^t=x\Rightarrow dx=a^t\log_e^a$

$\therefore I=\ln a\int_0^1 a^t.a^{-(t)}.a^t dt=\ln a\int_0^1 a^{2t}dt=\frac{a^2-1}{2}=\frac{e-1}{2}\Rightarrow a=\sqrt{e}$

3. If $I=\int_0^{\pi}x\left(\sin^2(\sin x)+\cos^2(\cos x)\right)dx$, then $[I]=$ ____, where $[.]$ denotes the greatest integer function

Key. 4

Sol.

$$I=\int_0^{\pi}(\pi-x)\left(\left(\sin^2(\sin x)\right)+\cos^2(\cos x)\right)dx\Rightarrow 2I=2\pi\int_0^{\frac{\pi}{2}}\left(\sin^2(\sin x)+\cos^2(\cos x)\right)dx$$

$$\Rightarrow I=\pi\int_0^{\frac{\pi}{2}}\left(\sin^2(\sin x)\right)+\cos^2(\cos x)\,dx=\pi\int_0^{\frac{\pi}{2}}\left(\sin^2(\cos x)+\cos^2(\sin x)\right)dx$$

SMART ACHIEVERS LEARNING PVT. LTD.

$$\Rightarrow 2I = \pi\int_0^{\frac{\pi}{2}} 2dx \Rightarrow I = \frac{\pi^2}{2}$$

4. If $\int\limits_{\cos x}^{1} t^2 f(t)dt = 1 - \cos x \ \forall \ x \in \left(0, \frac{\pi}{2}\right)$, then $\left[f\left(\frac{\sqrt{3}}{4}\right)\right] = ([.]$ denotes the greatest integer function.)

Key. 5

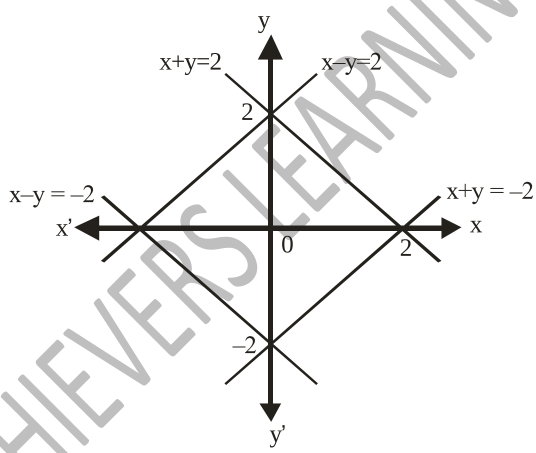
Sol. Conceptual

5. Area bounded by $2 \ge \max.\{|x-y|, |x+y|\}$ is k sq. units then k =

Key. 8

Sol. $2 \ge \max.\{|x-y|, |x+y|\}$

$\Rightarrow \quad |x-y| \le 2 \text{ and } |x+y| \le 2,$ which forms a square of diagonal length 4 units.



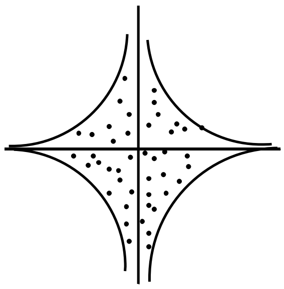
$\Rightarrow$ The area of the region is $\frac{1}{2} \times 4 \times 4 = 8 \text{ sq. units}$

This is equal to the area of the square of side length $2\sqrt{2}$.

6. The area bounded by the curves $y = \ln x, y = ln|x|, y = |\ln x|, y = |\ln|x||$ is

Key. 4

Sol. Area $= 4\int_0^1 |\ln x| dx$



$= -4\int_0^1 \ln x dx$

$= -4\left[x \ln x - x\right]_0^1$

SMART ACHIEVERS LEARNING PVT. LTD.

= 4

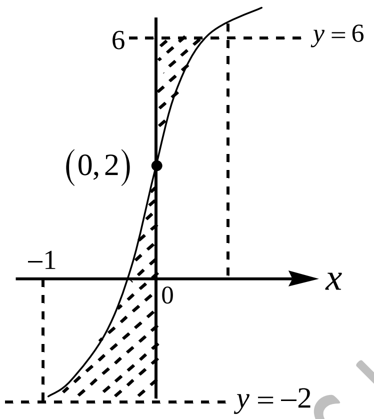
7. Let $f(x) = x^3 + 3x + 2$ and g(x) is the inverse of it. The area bounded by g(x), the x-axis and the ordinates at x= -2 and x= 6 is $\frac{m}{n}$ where $m, n \in N$ & $G.C.D\ of\ (m,n) = 1$ then $m - 2 =$

Key. 7

Sol. The required area will be equal to area enclosed by $y = f(x)$, the y-axis between the abscissa at $y = -2$ and $y = 6$

Required area

$= \int_0^1 \{6 - f(x)\} dx + \int_{-1}^0 [f(x) - (-2)] dx$



$\frac{9}{2} = \frac{m}{n} \Rightarrow m - n = 7$

8. The integral $\int_{\pi/4}^{5\pi/4} (|\cos t| \sin t + |\sin t| \cos t)\, dt$ has the value equal to

Key. 0

Sol. I = $\int_{\pi/4}^{\pi/2} 2 \sin t \cos t \, dt + \int_{\pi/2}^{\pi} \{(-\sin t \cos t) + (\sin t \cos t)\} dt + \int_{\pi}^{5\pi/4} -2 \sin t \cos t \, dt$

$= \int_{\pi/4}^{\pi/2} \sin 2t \, dt - \int_{\pi}^{5\pi/4} \sin 2t \, dt$

= 0.

9. Let 'f' is a differentiable function such that $f'(x) = f(x) + \int_0^2 f(x) dx, f(0) = \frac{4 - e^2}{3}$ then the value of $[f(2)]$ where [.] denotes the greatest integer $\le x$ is.

Key. 5

Sol. Given $f'(x) = f(x) + A$ where $A = \int_0^2 f(x) dx$

Solving – (1)

SMART ACHIEVERS LEARNING PVT. LTD.

$$f(x)=\lambda\left(e^{x}-1\right)+\frac{4-e^{2}}{3}$$

$$\because \int_{0}^{2} f(x)\,dx = A \Rightarrow \lambda = 1 \text{ and } A=\frac{e^{2}-1}{3}$$

$$\therefore f(x)=e^{x}-1+\frac{4-e^{2}}{3}=e^{x}-\frac{1}{3}\left(e^{2}-1\right)$$

$$f(2)=\frac{2e^{2}+1}{3} \qquad \therefore \left[f(2)\right]=5$$

10. If the area bounded by the curves $y = -x^2 + 6x - 5$, $y = -x^2 + 4x - 3$ and the line $y = 3x - 15$ is $\frac{73}{\lambda}$, then the value of λ is

Ans: 6

Hint: Area = $\left|\int_{1}^{5}(6x-x^{2}-5)\,dx-\int_{1}^{3}(4x-x^{2}-3)\,dx\right|$

$+\left|\int_{3}^{4}(4x-x^{2}-3)dx\right|+\left|\int_{4}^{5}(3x-15)dx\right|=\frac{73}{6}$

11. The minimum area bounded by the function $y= f(x)$ and $y = \alpha x + 9$ ($\alpha\in R$) where f satisfies the relation $f(x + y) = f(x) + f(y) + y\sqrt{f(x)}$ $\forall x,y\in R$ and $f'(0) = 0$ is 9A, value of A is

Ans: 1

Hint: $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x+0)}{h} \Rightarrow \lim_{h\to 0}\frac{f(x)+f(h)+h\sqrt{f(x)}-f(x)-f(0)-0\sqrt{f(x)}}{h}$

$\Rightarrow \lim_{h\to 0}\left(\frac{f(h)-f(0)}{h-0}\right)+\sqrt{f(x)}$

$f'(x) = \sqrt{f(x)}$

$\int\frac{f'(x)}{\sqrt{f(x)}}dx=\int dx$

$2\sqrt{f(x)} = x + c$

$f(x) = \frac{x^{2}}{4}$

when $\alpha = 0$ area is minimum

required minimum area = $2\int_{0}^{9}2\sqrt{y}\,dy$

SMART ACHIEVERS LEARNING PVT. LTD.

$\Rightarrow\ 4\left(\frac{y^{3/2}}{3/2}\right)_0^9$ = 72 sq. unit.

12. Let R = {x, y : $x^2 + y^2 \le 144$ and sin(x + y) ≥ 0}. And S be the area of region given by R, then find S/9π.

Ans: 8

Hint: $x^2 + y^2 \le 144$ and $\sin(x + y) \ge 0 \Rightarrow 2n\pi \le x + y \le (2n + 1)\pi$; n∈I

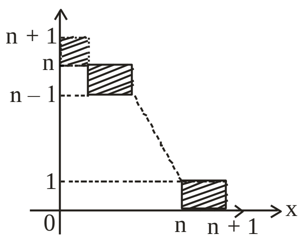
Hence we get the area

$S = \frac{\pi.144}{2} \Rightarrow \frac{S}{9\pi} = 8$

13. If the area bounded by [x] + [y] = n and y = k; n, $k \in N$ and $k \le (n+1)$ and [.] is greatest integer function, in the first quadrant, is n + r, then find r.

Ans: 1

Hint: Area = *n* + 1



SMART ACHIEVERS LEARNING PVT. LTD.

14. Let α, β be roots of the quadratic equation $18x^2 - 9\pi x + \pi^2 = 0$, where $\alpha < \beta$. Also $f(x) = x^2$ and $g(x) = \cos x$. If the area bounded by the curve y = (fog) (x), the vertical lines x = α, x = β and x-axis is $\frac{\pi}{\lambda}$, then find the sum of the digit in λ

Ans: 3

Hint: Let α, β be roots of the ……….

$(fog)(x) = f(\cos x) = \cos^2 x$

$\alpha, \beta : 18x^2 - 9\pi x + \pi^2 = 0 \Rightarrow \alpha = \frac{\pi}{6}, \beta = \frac{\pi}{3}$

Area = $\int_{\pi/6}^{\pi/3} \cos^2 x\ dx = \frac{\pi}{12}$

15. The minimum area bounded by the function y= f(x) and y = αx + 9 (α∈R) where f satisfies the relation f(x + y) = f(x) + f(y) + y $\sqrt{f(x)}$ ∀x,y∈R and f′(0) = 0 is 9A, value of A is

Key: 8

Hint: $f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x+0)}{h} \Rightarrow \lim_{h\to 0}\frac{f(x) + f(h) + h\sqrt{f(x)} - f(x) - f(0) - 0\sqrt{f(x)}}{h}$

$\Rightarrow \lim_{h\to 0}\left(\frac{f(h) - f(0)}{h - 0}\right) + \sqrt{f(x)}$

$f'(x) = \sqrt{f(x)}$

$\int \frac{f'(x)}{\sqrt{f(x)}} dx = \int dx$

$2\sqrt{f(x)} = x + c$

$f(x) = \frac{x^2}{4}$

when $\alpha = 0$ area is minimum

required minimum area = $2\int_0^9 2\sqrt{y}\, dy$

$\Rightarrow 4\left(\frac{y^{3/2}}{3/2}\right)_0^9$ = 72 sq. unit.

16. Given that

$\int_1^y \sec^{-1} x\, dx = \lambda$ then $\int_{-y}^{-1} \sec^{-1} x - \tan^{-1}\left(\sqrt{x^2-1}\right)dx + \int_1^y \sec^{-1} x - \tan^{-1}\left(\sqrt{x^2-1}\right)dx = \_\_\_ (|y| \ge 1)$

equals to $\pi(y-a) - b\lambda$ then a+b=___

Ans: 3

$\int_{-y}^{y} \sec^{-1} x - \tan^{-1}\left(\sqrt{x^2-1}\right)dx = \int_{-y}^{-1} \sec^{-1} x\left(\pi - \sec^{-1} x\right)dx$

$+\int_1^y \sec^{-1} x - \sec^{-1} x\, dx$

Hint: $= 2\int_{-y}^{-1} \sec^{-1} x\, dx - \int_{-y}^{-1} \sec^{-1} x$

$= 2\int_1^y \left(\pi - \sec^{-1} x\right)dx - \pi(y-1) = \pi(y-1) - 2\lambda$

$\therefore a + b = 3$

17. $Let\ I_n = \int_0^{\pi/2} (\sin x + \cos x)^n\, dx\ (n \ge 2)$. Then the value of $nI_n - 2(n-1)I_{n-2} is$

Ans: 2

Hint $I_n = \int_0^{\pi/2} (\sin x + \cos x)^{n-1} (\sin x - \cos x)' dx$

$= 2 + (n-1)\int_0^{\pi/2} (\sin x + \cos x)^{n-2} (\cos x - \sin x)^2\, dx$

$= 2 + (n-1)\int_0^{\pi/2} (\sin x + \cos x)^{n-2} \left[2 - (\sin x + \cos x)^2\right] dx$

$= 2 + 2(n-1)I_{n-2} - (n-1)I_n$

$\Rightarrow nI_n - 2(n-1)I_{n-2} = 2$

18. If $\lim_{x\to 0} \int_0^x \frac{t^2 dt}{(x - \sin x)\sqrt{a+t}} = 1$ then the value of a is …………………..

Key: 4

SMART ACHIEVERS LEARNING PVT. LTD.

Hint:

$$\underset{x\to 0}{\text{lt}}\frac{\int_0^x \frac{t^2}{\sqrt{a+t}}dt}{x-\sin x}=\underset{x\to 0}{\text{lt}}\frac{x^2}{\sqrt{a+x}\left(1-\cos x\right)}=\underset{x\to 0}{\text{lt}}\frac{x^2}{\sqrt{a+x}\ \left(2\sin^2\frac{x}{2}\right)}$$

$$=\frac{2}{\sqrt{a}}=1\Rightarrow a=4$$

19. Given $\int_0^{\pi/2}\ln\sin x dx=\frac{\pi}{2}\ln\frac{1}{2}$ and $\int_0^{\pi/2}\left(\frac{x}{\sin x}\right)^2 dx=\frac{k.\pi}{2}\ln 2$ then k = ………………

Key: 2

Hint:

$$I=\int_0^{\pi/2}x^2\cos ec^2 x dx=\left[-x^2\cot x\right]_0^{\pi/2}+\int_0^{\pi/2}2x\cot x dx$$

$$=0+\underset{x\to 0^+}{Lt}\frac{x^2}{\tan x}+2\left[x\ln\sin x\right]_0^{\pi/2}-2\int_0^{\pi/2}\ln\sin x dx$$

$$=0-2\underset{x\to 0^+}{Lt}\ x\ln\sin x-2\frac{\pi}{2}\ln\frac{1}{2}=\pi\ln 2$$

$$\left(\because\ \underset{x\to 0^+}{Lt}\frac{\ln\sin x}{1/x}=\underset{x\to 0^+}{Lt}\frac{\frac{1}{\sin x}.\cos x}{-\frac{1}{x^2}}=0\right)$$

20. Find the value of $\dfrac{29\int_0^1\left(1-x^4\right)^7 dx}{4\int_0^1\left(1-x^4\right)^6 dx}$

Key: 7

Hint: $l=\int_0^1\left(1-x_1^4\right)^7\ -\underset{\pi}{1}\,dx$

$$=\left[x\left(1-x^4\right)^7\right]_0^1+7\times 4\int_0^1 x\left(1-x^4\right)^6 x^3 dx$$

$$=-28\int_0^1\left(1-x^4\right)^6 dx+28\int_0^1\left(1-x^4\right)^6 dx=-28I+28\int_0^1\left(1-x^4\right)^6 dx$$

$$29I-28\int_0^1\left(1-x^4\right)dx$$

SMART ACHIEVERS LEARNING PVT. LTD.

$$\frac{29\int_0^1\left(1-x^4\right)^7 dx}{4\int_0^1\left(1-x^4\right)^7 dx}=7$$

21. If $f(x)=a\cos(\pi x)+b, f'\left(\frac{1}{2}\right)=\pi$ and $\int_{1/2}^{3/2} f(x)dx=\frac{2}{\pi}+1$, then find the value of, $-\frac{12}{\pi}\left(\sin^{-1}a+\cos^{-1}b\right)$

Key: 6

Hint: $f'(x)=-\alpha\pi\sin(\pi x)$

$$\Rightarrow f'\left(\frac{1}{2}\right)=-a\pi\sin\frac{\pi}{2}=-a\pi=\pi\Rightarrow a=-1$$

$$\int(a\cos\pi x+b)dx=\left(\frac{a\sin\pi x}{\pi}+bx\right)_{1/2}^{3/2}=\left(\frac{-a}{\pi}+\frac{3b}{2}\right)-\left(\frac{a}{\pi}+\frac{b}{2}\right)$$

$$\Rightarrow\frac{2a}{\pi}+b=\frac{2}{\pi}+1\Rightarrow b=1$$

So, $\frac{-12}{\pi}\left(\sin^{-1}(-1)+\cos^{-1}1\right)=\frac{-12}{\pi}\left(-\frac{\pi}{2}+0\right)=6$

22. Evaluate $\left[\int_0^{\pi}\frac{dx}{1+2\sin^2 x}\right]$ where [.] denotes greatest integer function

Key: 1

Hint: $\int_0^{\pi}\frac{dx}{1+2\sin^2 x}=2\int_0^{\pi/2}\frac{dx}{1+2\sin^2 x}=2\int_0^{\pi/2}\frac{\sec^2 x\,dx}{1+3\tan^2 x}$

Put tan x = t = $2\int_0^{\infty}\frac{dt}{1+\left(\sqrt{3}t\right)^2}$

$$=\frac{2}{\sqrt{3}}\left[\tan^{-1}\sqrt{3}t\right]_0^{\infty}=\frac{\pi}{\sqrt{3}}$$

$$\left[\frac{\pi}{\sqrt{3}}\right]=1$$

23. Let $F(x)$ be a non-negative continuous function defined on R such that $F(x)+F\left(x+\frac{1}{2}\right)=3$ and the value of $\int_0^{1500}F(x)dx$ is $\frac{9000}{\lambda}$. Then the numerical value of λ is

Key: 4

Hint: We have $F(x)+F\left(x+\frac{1}{2}\right)-3$ ......(i)

Replace x by $x+\frac{1}{2}$ in (I), we get $F\left(x+\frac{1}{2}\right)+F(x+1)=3$....(2)

$\therefore$ From (1) and (2), we get $F(x)=F(x+1)$ ....(3) $\Rightarrow$ F(x) is periodic function.

Now consider $I=\int_0^{1500} F(x)dx\int_0^1 F(x)dx=1500\left|\int_0^{\frac{1}{2}} F(x)dx+\int_{\frac{1}{2}}^{1} F(x)dx\right|$ $\left(\begin{matrix}\text{Using property}\\ \text{of periodic}\\ \text{function}\end{matrix}\right)$

Put $x+y+\frac{1}{2}m2^{nd}$ integral, we get

$$I=1500\left|\int_0^{\frac{1}{2}} F(x)dx+\int_0^{\frac{1}{2}} F\left(y+\frac{1}{2}\right)dy\right|-1500\int_0^{\frac{1}{2}}\left(F(x)+F\left(x+\frac{1}{2}\right)\right)dx=1500\int_0^{\frac{1}{2}} 3dx\,|\text{Using}(1)|$$

Hence I = $1500(3)\left(\frac{1}{2}\right)=750\times3=2250$

24. Let $f(x)=\frac{4^x}{4^x+2}, I_1=\int_{f(1-a)}^{f(a)} xf(x(1-x))dx$ and $I_2=\int_{f(1-a)}^{f(a)} f(x(1-x))dx$ where $f(a)>f(1-a)$ then the value of $\frac{I_2}{I_1}$ is

Key: 2

Hint: $f(x)+f(1-x)=1$

$I_1=\int_{f(1-x)}^{f(x)} x+(x(-x))dx$

$x+f(1-x)+f(x)=1$

$f(1-x)$

25. If $I=\int_{-\frac{\pi}{2}}^{2\pi}\sin^{-1}(\sin x)dx$ then $-\frac{16}{\pi^2}I=$

Key: 2

Hint: $I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} xdx+\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}(\pi-x)dx+\int_{\frac{3\pi}{2}}^{2\pi}(x-2\pi)dx=\frac{-\pi^2}{8}$

SMART ACHIEVERS LEARNING PVT. LTD.

26. The value of $\int_{-1}^{1}\left[x\left[1+\sin\pi x\right]+1\right]dx$ is ([.] denote the greatest integer function)

Key : 2

Sol : Let $I=\int_{-1}^{1}\left[x\left[1+\sin\pi x\right]+1\right]dx=\int_{-1}^{1}\left[x\left[1+\sin\pi x\right]+1\right]dx+\int_{1}^{0}\left[x\left[1+\sin\pi x\right]+1\right]dx$

Now, $-1<x<0\Rightarrow\left[1+\sin\pi x\right]=0$

and $0<x<1\Rightarrow\left[1+\sin\pi x\right]=1$

$\therefore\ I=\int_{-1}^{0}1.dx+\int_{0}^{1}\left[x+1\right]dx$

$=\int_{-1}^{0}1.dx+\int_{0}^{1}\left[x\right]dx+\int_{0}^{1}1.dx$

$=\left[0-(-1)\right]+0+(1-0)$ = 2

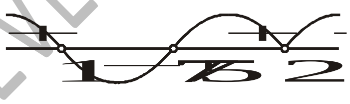
27. The function $f(x)=\int_{1}^{x}\left\{2(t-1)(t-2)^3+3(t-1)^2(t-2)^2\right\}dt$ attains its maximum at x is equal to

Key : 1

Sol : $f'(x)=2(x-1)(x-2)^3+3(x-1)^2(x-2)^2$

$=(x-1)(x-2)^2\left\{2(x-2)+3(x-1)\right\}$

$=(x-1)(x-2)^2(5x-7)$



sign change of $f'(x)$ from +ve to –ve at x = 1

$\therefore$ maximum at x = 1.

28. If $\lim_{n\to\infty}\frac{\sum_{r=1}^{n}\sqrt{r}\sum_{r=1}^{n}\frac{1}{\sqrt{r}}}{\sum_{r=1}^{n}r}=\frac{k}{3}$ then find k.

ANS: 8

Hint: $\lim_{n\to\infty}\frac{\sum_{r=1}^{n}\sqrt{r}\sum_{r=1}^{n}\frac{1}{\sqrt{r}}}{\frac{n(n+1)}{2}}\quad\lim_{n\to\infty}\frac{\sum_{r=1}^{n}\sqrt{\frac{r}{n}}\sum_{r=1}^{n}\frac{1}{\sqrt{\frac{r}{n}}}}{\frac{n^2}{2}\left(1+\frac{1}{n}\right)}$

$=2\times\int_{0}^{1}\sqrt{x}\,dx\int_{0}^{1}\frac{1}{\sqrt{x}}dx$

SMART ACHIEVERS LEARNING PVT. LTD.

$$= 2\times\left[\frac{x^{3/2}}{\frac{3}{2}}\right]_0^1\left[\frac{x^{1/2}}{\frac{1}{2}}\right]_0^1$$

$$= 2\times\frac{2}{3}\times 2 = \frac{8}{3}$$

$$\therefore\ k = 8$$

29. $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^{1/a\left\{n^{a-\frac{1}{a}}+k^{a-\frac{1}{a}}\right\}}}{n^{a+1}}$ is equal to

Key : 1

Sol : $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^{\frac{1}{a}}\left\{n^{a-\frac{1}{a}}+k^{a-\frac{1}{a}}\right\}}{n^{a+1}}$

$$= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}.\left\{\left(\frac{k}{n}\right)^{1/a}+\left(\frac{k}{n}\right)^{a}\right\}$$

$$= \int_0^1\left(x^{1/a}+x^a\right)dx$$

$$= \left\{\frac{x^{(1/a)+1}}{\frac{1}{a}+1}+\frac{a^{a+1}}{a+1}\right\}_0^1$$

$$= \frac{a}{a+1}+\frac{1}{a+1} = 1$$

30. If $\int \sin 4x\ e^{\tan^2 x}dx = c - A\cos^4 x.e^{\tan^2 x}$ then A=___

Ans: 2

Hint:

$$I = 4\int \sin x\cos x\left(\frac{1-\tan^2 x}{1+\tan^2 x}\right)e^{\tan^2 x}dx = 4\int \tan x\sec^2 x\cos^6 x\left(1-\tan^2 x\right)e^{\tan^2 x}$$

$$t = \tan^2 x$$

$$\Rightarrow I = 2\int\frac{(1-t)e^t}{(1+t)^3}dt = \frac{-2e^t}{(1+t)^2}+c$$

$$= -2\cos^4 x e^{\tan^2 x}+c$$

31. If $\int\frac{x^4+1}{x(x^2+1)^2}dx = A\ ln\ |x| + \frac{B}{1+x^2} + c$, where c is the constant of integration then A + B is :

Ans: 2

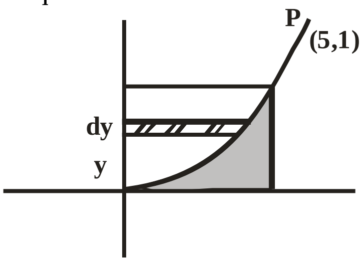
Hint: Add and subtract $2x^2$ in the numerator]

32. Let y = g(x) be the inverse of a bijective mapping f : R → R defined as f (x) = $3x^3 + 2x$. The area bounded by the graph of g(x), the x-axis and the coordinate at x = 5 is 'A' then the value of (4A –7) is

Key. 6

Sol. Inverse of $y = 3x^3 + 2x$ is $x = 3y^3 + 2y$

Required area



$$A = \int_0^1 5 - (3y^3 + 2y)dy = \left[5y - \left(\frac{3y^4}{4} + y^2\right)\right]_0^1$$

$$= 5 - \left(\frac{3}{4} + 1\right) = \frac{20-7}{4} = \frac{13}{4}$$

$4A = 13$

$\therefore \quad 4A - 7 = 6$

33. A point P(x, y) moves is such a way that [x + y + 1] = [x] (where [.] greatest integer function) and x ∈ (0, 2). Then the area representing all the possible positions of P equals

Key. 2

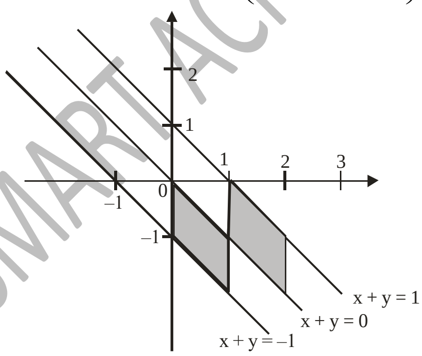
Sol. If x ∈ (0, 1)

Then $-1 \le x + y < 0$

And if x ∈ [1, 2)

$0 \le x + y < 1$

Required area $= 4\left(\frac{1}{2}.1.\sqrt{2}\sin\frac{\pi}{4}\right) = 2$ sq. units



34. If $f(0) = 1, f(2) = 3, f'(2) = 5$

then the value of the definite integral $\int_0^1 x f''(2x)dx$ is

Key. 2

Sol. $\int \underset{I}{x} \underset{II}{f''(2x)} dx = x\dfrac{f'(2x)}{2} - \int \dfrac{f'(2x)}{2} dx$

$= x\dfrac{f'(2x)}{2} - \dfrac{f(2x)}{4}$

$\int_0^1 x f''(2x) dx = \left| \dfrac{xf'(2x)}{2} - \dfrac{f(2x)}{4} \right|_0^1$

$= \left(\dfrac{f'(2)}{2} - \dfrac{f(2)}{4} \right) - \left(0 - \dfrac{f(0)}{4} \right)$

$\left(\dfrac{5}{2} - \dfrac{3}{4} \right) - \left(0 - \dfrac{1}{4} \right) = 2$

35. A point P(x, y) moves is such a way that [x + y + 1] = [x] (where [.] greatest integer function) and $x \in (0, 2)$. Then the area representing all the possible positions of P equals

Key. 2

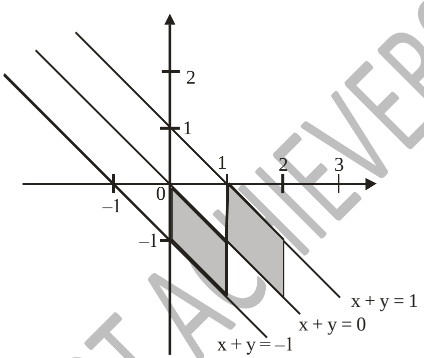
Sol. If $x \in (0, 1)$

Then $-1 \le x + y < 0$

And if $x \in [1, 2)$

$0 \le x + y < 1$

Required area $= 4\left(\dfrac{1}{2}.1.\sqrt{2}\sin\dfrac{\pi}{4} \right) = 2$ sq. units



36. If $\int_0^{x^2(1+x)} f(t)dt = x,$ then the value of $10f(2)$ must be

Key. 2

Sol. Differentiating both sides w.r.t. x, then

$f(x^2(1+x)) \times (2x + 3x^2) = 1$

At $x = 1 \Rightarrow f(2) = \dfrac{1}{5} \therefore \; 10f(2) = 2$

37. If $I = \int_0^2 x[2x]dx,$ where [.] denotes the greatest integer function, then the value of $\dfrac{4}{17}I =$

Key. 1

SMART ACHIEVERS LEARNING PVT. LTD.

Sol. $I=\int_{0}^{2} x[2x]dx$

$=\int_{0}^{1/2} 0.dx+\int_{1/2}^{1} xdx+\int_{1}^{3/2} 2xdx+\int_{3/2}^{2} 3xdx$

$=0+[\frac{x^2}{2}]_{1/2}^{1}+[x^2]_{1}^{3/2}+[\frac{3x^2}{2}]_{3/2}^{2}$

$=\frac{1}{2}(1-\frac{1}{4})+(\frac{9}{4}-1)+\frac{3}{2}(4-\frac{9}{4})$

$=\frac{3}{8}+\frac{5}{4}+\frac{21}{8}$

$=\frac{34}{8}=\frac{17}{4}$

$\therefore \frac{4}{17}I=1$

38. If $\int_{0}^{\pi/2} \sin^8 x\cos^4 xdx=\frac{k\pi}{2048}$ then the value of k must be

Key. 7

Sol. $\int_{0}^{\pi/2} \sin^8 x\cos^4 xdx=\frac{(7.5.3.1)(3.1)}{12.10.8.6.4.2}.\frac{\pi}{2}$

(by Wallis' formula) $=\frac{7\pi}{2048}\Rightarrow k=7$

39. If $I=\int_{0}^{1} x(1-x)^{49}dx,$ then the value of $5100I=$

Key. 2

Sol. $\because I=\int_{0}^{1} x(1-x)^{49}dx$

$=\int_{0}^{1}(1-x)(1-(1-x))^{40}dx$

$=\int_{0}^{1}(1-x)x^{49}dx=\int_{0}^{1}(x^{49}-x^{50})dx$

$=[\frac{x^{50}}{50}-\frac{x^{51}}{51}]_{0}^{1}$

SMART ACHIEVERS LEARNING PVT. LTD.

$$=\frac{1}{50}-\frac{1}{51}=\frac{1}{2550}\Rightarrow 5100I$$

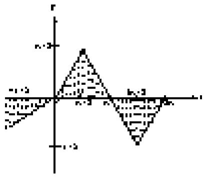
$$\Rightarrow 5100\frac{1}{2550}=2$$

40. If $I=\int_{-\pi/2}^{2\pi}\sin^{-1}(\sin x)dx=\frac{-\pi^2}{k}$ then $k=$

Key. 8

Sol. $I=\int_{-\pi/2}^{0}\sin^{-1}(\sin x)dx+\int_{0}^{\pi}\sin^{-1}(\sin x)dx+\int_{\pi}^{2\pi}\sin^{-1}(\sin x)dx$

=Area of shaded region



$$=-\left(\frac{1}{2}\times\frac{\pi}{2}\times\frac{\pi}{2}\right)+\left(\frac{1}{2}\times\pi\times\frac{\pi}{2}\right)-\left(\frac{1}{2}\times\pi\times\frac{\pi}{2}\right)$$

$$=-\frac{\pi^2}{8}$$

Since, $k=8$

41. If $f(x)=\int_a^x\frac{1}{f(x)}dx$ and $\int_a^1\frac{1}{f(x)}dx=\sqrt{2}$, then the value of $f(2)$ =

Key. 2

Sol. Since, $f(x)=\int_a^x\frac{1}{f(x)}dx$

Differentiating both sides w.r.t. x, then

$$f^1(x)=\frac{1}{f(x)}\Rightarrow 2f(x)f^1(x)=2$$

Integrating both sides, then

$$(f(x))^2=2x+c$$

$$\therefore f(x)=\sqrt{(2x+c)}$$

SMART ACHIEVERS LEARNING PVT. LTD.

But $\int_0^1 \frac{1}{f(x)}dx = \sqrt{2}$

And $f(1) = \int_a^1 \frac{1}{f(x)}dx = \sqrt{2}$

$\Rightarrow \sqrt{(2+c)} = \sqrt{2}$

$\therefore c = 0$

Then, $f(x) = \sqrt{2x}$

42. If $\int_0^{\pi} x\sin^5 x\cos^6 xdx = \frac{k\pi}{693}$ then k =

Key. 8

Sol. Let

$$I = \int_0^{\pi} x\sin^5 x.\cos^6 xdx = \int_0^{\pi}(\pi - x).\sin^5(\pi - x).\cos^6(\pi - x)dx = \int_0^{\pi}(\pi - x)\sin^5 x.\cos^6 xdx$$

$$= \int_0^{\pi}\pi\sin^5 x.\cos^6 xdx - \int_0^{\pi} x\sin^5 x\cos^6 xdx$$

$$\Rightarrow 2I = \pi.2\int_0^{\pi/2}\sin^5 x.\cos^6 xdx \Rightarrow I = \pi\left[\frac{4}{11}.\frac{2}{9}.\frac{1}{7}\right] = \frac{8\pi}{693}.$$

43. If $\int_0^{\pi/4}\tan^5 xdx = \frac{1}{k}\log 2 - \frac{1}{4}$ then k =

Key. 2

Sol. $\int_0^{\pi/4}\tan^5 xdx = \frac{1}{4} - \frac{1}{2} + \frac{1}{2}\log 2 = \frac{1}{2}\log 2 - \frac{1}{4}$

44. If $I = \int_{-\pi/2}^{\pi/2}\sqrt{\cos x - \cos^3 x}dx = \frac{k}{3}$ then $k =$

Key. 4

Sol.

$$I = \int_{-\pi/2}^{\pi/2}\sqrt{\cos x}\,|\sin x|\,dx = 2\int_{-\pi/2}^{\pi/2}\sqrt{\cos x}\,|\sin|\,xdx = 2\int_0^{\pi/2}\sqrt{\cos x}\sin xdx = \left[-\frac{4}{3}(\cos x)^{3/2}\right]_0^{\pi/2} = \frac{4}{3}$$

45. The value of $\int_{-1}^{3}\{|x-2| + [x]\}dx,$ where $[x]$ denotes the greatest integer less than or equal to x is

Key. 7

Sol. $\int_{-1}^{3}\{|x-2| + [x]\}dx$

SMART ACHIEVERS LEARNING PVT. LTD.

$$=\int_{1}^{0}\{|x-2|+[x]\}\,dx+\int_{0}^{1}\{|x-2|+[x]\}\,dx+\int_{1}^{2}\{|x-2|+[x]\}dx+\int_{2}^{3}\{|x-2|+[x]\}\,dx$$

$$=\int_{-1}^{0}(2-x-1)dx+\int_{0}^{1}(2-x+0)dx+\int_{1}^{2}(2-x+1)dx+\int_{2}^{3}(x-2+2)dx$$

$$=\left[x-\frac{x^2}{2}\right]_{-1}^{0}+\left[2x-\frac{x^2}{2}\right]_{0}^{1}+\left[3x-\frac{x^2}{2}\right]_{1}^{2}+\left[\frac{x^2}{2}\right]_{2}^{3}=-\left(-1-\frac{1}{2}\right)+\left(2-\frac{1}{2}\right)+\left(6-2\right)-\left(3-\frac{1}{2}\right)+\frac{9}{2}-2=7$$

46. $I=3\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{\cos x-\cos^3 x}\,dx$ is

KEY. 4

SOL. Given integral

$$I=\int_{-\pi/2}^{\pi/2}\sqrt{[\cos x(1-\cos^2 x)]}dx \;=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{(\cos x\sin^2 x)}dx$$

$$I=\int_{-\pi/2}^{\pi/2}\sqrt{(\cos x)}\,|\sin x|\,dx \qquad (1)$$

Now $|\sin x|=\{\begin{matrix}-\sin x, if\ -\pi/2\le x<0\\ \sin x, if\ 0<x\le\pi/2\end{matrix}$

From (1), we have

$$I=\int_{-\pi/2}^{0}\sqrt{(\cos x)}(-\sin x)dx+\int_{0}^{\pi/2}\sqrt{(\cos x)}\sin x\,dx$$

Putting $\cos x=t,-\sin x\,dx=dt$, we get

$$I=\int_{0}^{1}t^{1/2}dt-\int_{1}^{0}t^{1/2}dt=2\int_{0}^{1}t^{1/2}dt$$

$$=2\times(\frac{2}{3})[t^{3/2}]_0^1$$

$$=\frac{4}{3}$$

47. If $\int_{0}^{\pi/2}x^n\sin x\,dx=(3/4)(\pi^2-8)$, then the value of n is _____

KEY. 3

SOL. Let $I_n=\int_{0}^{\pi/2}x^n\sin x\,dx$

Integrating by parts choosing $\sin x$ as the second function, we get

SMART ACHIEVERS LEARNING PVT. LTD.

$$I_n = [x^n(-\cos x)]_0^{\pi/2} - \int_0^{\pi/2} nx^{n-1}(-\cos x)dx$$

$$= 0 - n\int_0^{\pi/2} x^{n-1}\cos x\, dx$$

Again integrating by parts,

$$I_n = n[x^{n-1}\sin x]_0^{\pi/2} - n(n-1)\int_0^{\pi/2} x^{n-2}\sin x\, dx$$

$$\Rightarrow I_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}$$

R.H.S. contains π^2. So putting $n=3$, we get

$$I_3 = 3\left(\frac{\pi}{2}\right)^2 - 3\times 2I_1$$

$$= \frac{3\pi^2}{4} - 6\int_0^{\pi/2} x\sin x\, dx$$

$$= \frac{3\pi^2}{4} - 6[x(-\cos x) + \sin x]_0^{\pi/2}$$

$$= \frac{3\pi^2}{4} - 6(1) = \frac{3}{2}(\pi^2 - 8)$$, which is true.

Hence, $n=3$.

48. If $I_n = \int_0^{\pi} x^n \sin x\, dx$ and $I_5 + 20I_3 = \pi^k$, then the value of k is _____

KEY. 5

SOL.
$$I_n = \int_0^{\pi} x^n \sin x\, dx$$

$$= [-x^n\cos x]_0^{\pi} + n\int_0^{\pi} x^{n-1}\cos x\, dx$$

$$= \pi^n + n[x^{n-1}\sin x]_0^{\pi} - n(n-1)\int_0^{\pi} x^{n-2}\sin x\, dx$$

$$\Rightarrow I_n = \pi^n + n\times 0 - n(n-1)I_{n-2}$$

Putting $n=5$, we get

$$I_5 = \pi^5 - 20I_3$$

$$\Rightarrow I_5 + 20I_3 = \pi^5$$

49. If $I = \int_{-1}^{1}(1+x)^{1/2}(1-x)^{3/2}\, dx$, then the value of $\sec^3(I/2)$ is _____

KEY. 8

SOL. Using property $\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$

$$I = \int_{-1}^{1} (1-x)^{1/2}(1+x)^{3/2}\,dx$$

$$\Rightarrow 2I = \int_{-1}^{1} (1+x)^{1/2}(1-x)^{1/2}[(1-x)+(1+x)]dx$$

$$\Rightarrow 2I = 2\int_{-1}^{1} \sqrt{1-x^2}\,dx$$

$$\Rightarrow I = 2\int_{0}^{1} \sqrt{1-x^2}\,dx (x = \sin\theta) \Rightarrow dx = \cos\theta d\theta$$

$$\Rightarrow I = 2\int_{0}^{\pi/2} \cos^2\theta\, d\theta = \frac{\pi}{2}$$

50. The value of $\int_{-\pi/4}^{\pi/4} [(x^9 - 3x^5 + 7x^3 - x + 1)/\cos^2 x]dx$ is ____

KEY. 2

SOL. $f(x) = \dfrac{x^9 - 3x^5 + 7x^3 - x}{\cos^2 x} + \sec^2 x$

$= \sec^2 x(x^9 - 3x^5 + 7x^3 - x) + \sec^2 x$

$\Rightarrow \int_{-\pi/4}^{\pi/4} f(x)dx = \int_{-\pi/4}^{\pi/4} \sec^2 x\, dx$

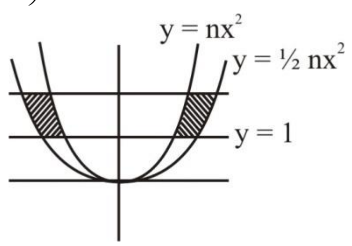
$= 2\int_{0}^{\pi/4} \sec^2 x\, dx$

$= 2\tan x |_0^{\pi/4} = 2$

51. Find natural number *n* so that area bounded by y = nx², $y = \frac{1}{2}nx^2$ and $y^2 - 4y + 3 = 0$ is greatest.

Key. 1

Sol. $\text{Area} = \int_1^3 \left(\sqrt{\frac{2y}{n}} - \sqrt{\frac{y}{n}}\right)dy$



is greatest when n is least

52. $f:[0, 5] \to R$, $y = f(x)$ such that $f''(x) = f''(5-x) \forall x \in [0,5]$ $f'(0) = 1$ and $f'(5) = 7$, then evaluate $\int_1^4 f'(x)dx - 4$.

SMART ACHIEVERS LEARNING PVT. LTD.

Key. 8

Sol. $\int_1^4 f'(x)dx = \left[xf'(x)\right]_1^4 - \int_1^4 xf''(x)dx$

$I = \int_1^4 x f''(x)dx = \int_1^4 (5-x)f''(5-x)dx$

$= 5\int_1^4 f''(x)dx - I$

$I = \frac{5}{2}\left[f'(4) - f'(1)\right]$

So, $\int_1^4 f'(x)dx = \frac{3}{2}\left[f'(4) + f'(1)\right]$

Now, $f''(x) = f''(5-x) \Rightarrow f'(x) = -f'(5-x) + c$

$f'(0) + f'(5) = c \Rightarrow c = 8$

so $f'(x) + f'(5-x) = 8 \Rightarrow f'(4) + f'(1) = 8$

SMART ACHIEVERS LEARNING PVT. LTD.

Definite, Indefinite Integration & Areas

Matrix-Match Type

1. List–I below gives values of integrals involving parameters A and B while List–II gives values of these parameters for which the results given are correct. Match the integrals in List–I with the values of parameters A, B in List–II so that the given result is correct.

Column –I		Column –II	
(A)	$\int \frac{e^{2x}-2e^{x}}{e^{2x}+1}dx$ = Aln $(e^{2x}+1)$ + Btan$^{-1}(e^{x})$ + c	(p)	$A = -\frac{1}{2}, B=-\frac{1}{4}$
(B)	$\int \sqrt{x+\sqrt{x^2+2}}\,dx$ =A $\left\{x+\sqrt{x^2+2}\right\}^{3/2}+\frac{B}{\sqrt{x+\sqrt{x^2+2}}}$ +c	(q)	$A = \frac{1}{3}, B=-2$
(C)	$\int \frac{\cos 8x-\cos 7x}{1+2\cos 5x}dx$ = Asin3x + Bsin2x + c	(r)	$A = \frac{1}{2}, B=-2$
(D)	$\int \frac{\ln x}{x^3}dx = A\frac{\ln x}{x^2}+\frac{B}{x^2}$ + c	(s)	$A = \frac{1}{3}, B=-\frac{1}{2}$

Key. A–R;B–Q; C–S;D–P

Sol. Conceptual

2.

Column I		Column II	
(A)	$\int \frac{\cos x}{\cos 3x}dx$	(p)	$-\frac{1}{\tan x+2\sec x}+c$
(B)	$\int \frac{\cos^3 x}{(1+\sin^2 x)^2}dx$	(q)	$-\frac{1}{2\sqrt{3}}\ln\left\|\frac{\sqrt{3}\tan x-1}{\sqrt{3}\tan x+1}\right\|+c$
(C)	$\int \frac{dx}{4\sin^2 x+\cos^2 x}$	(r)	$\frac{1}{2}\tan^{-1}(2\tan x)+c$
(D)	$\int \frac{1+2\sin x}{(2+\sin x)^2}dx$	(s)	$(\sin x + \operatorname{cosec} x)^{-1} + c$

Key. A–Q;B–S; C–R;D–P

Sol. (A) $\int \frac{\cos x}{\cos 3x}dx = \int \frac{dx}{4\cos^2 x-3}$

(B) $\int \frac{(1-\sin^2 x)\cos x}{(1+\sin^2 x)^2}dx$

(C) $\int \frac{\sec^2 x\,dx}{1+4\tan^2 x}$

SMART ACHIEVERS LEARNING PVT. LTD.

(D) $\int\left[\dfrac{(2\sin x+1+3)-3}{(2+\sin x)^2}\right]dx$

3. Match the following:

Column -I		Column -II	
(A)	$\int\sqrt{\operatorname{cosec}x+1}\,dx,\ x\in\left(0,\frac{\pi}{2}\right)$	(p)	$\frac{1}{\sqrt{2}}\cos^{-1}\left(\frac{\cos x}{\sqrt{2}}\right)+c$
(B)	$\int\frac{\sin x}{\sqrt{3-\cos 2x}}dx,\ x\in\left(0,\frac{\pi}{4}\right)$	(q)	$\sin^{-1}(\sin x\cos x)+c$
(C)	$\int\frac{\cos x}{\sqrt{\cos 2x}}dx,\ x\in\left(0,\frac{\pi}{4}\right)$	(r)	$\frac{-1}{\sqrt{2}}\sin^{-1}\left(\sqrt{\cos 2x}\right)+c$
(D)	$\int\frac{\cos 2x}{\sqrt{\cos^2 x+\sin^4 x}}dx$	(s)	$\sin^{-1}(2\sin x-1)+c$

Key. A–S;B–P; C–R;D–Q

Sol. A) $\int\sqrt{\operatorname{cosec}x+1}\,dx : \int\frac{\sqrt{1+\sin x}}{\sqrt{\sin x}}dx=\int\frac{\cos x}{\sqrt{\sin x(1-\sin x)}}dx=\int\frac{dt}{\sqrt{t(1-t)}}(t=\sin x)$

$=\sin^{-1}(2\sin x-1)+c$

B) $\int\frac{\sin x}{\sqrt{3-\cos 2x}}dx=\frac{1}{\sqrt{2}}\int\frac{\sin x\,dx}{\sqrt{2-\cos^2 x}}=\frac{1}{\sqrt{2}}\cos^{-1}\left(\frac{\cos x}{\sqrt{2}}\right)+c$

C) $\int\frac{\cos x}{\sqrt{\cos 2x}}dx=\frac{1}{\sqrt{2}}\int\frac{\sin 2x}{\sqrt{1-\cos 2x}\sqrt{\cos 2x}}dx=-\frac{1}{\sqrt{2}}\int\frac{1}{\sqrt{1-t^2}}dt\left(1=\sqrt{\cos 2x}\right)$

$=-\frac{1}{\sqrt{2}}\sin^{-1}\left(\sqrt{\cos 2x}\right)+c$

D) $\int\frac{\cos 2x}{\cos^2 x+\sin^4 x}dx=\int\frac{2\cos 2x}{\sqrt{4-\sin^2 2x}}dx=\sin^{-1}\left(\frac{\sin 2x}{2}\right)+c$

4. For $0<x<1$, Match the following:

Column -I		Column -II	
(A)	$\int\frac{dx}{\left(1-\sqrt{x}\right)\sqrt{1-x}}=$	(p)	$2\left(\frac{\sqrt{x}+1}{\sqrt{1-x}}\right)+c$
(B)	$\int\frac{dx}{\left(1+\sqrt{x}\right)\sqrt{1-x}}=$	(q)	$2\left(\frac{1-\sqrt{x}}{\sqrt{1-x}}+\sin^{-1}\sqrt{x}\right)+c$
(C)	$\int\frac{dx}{\left(1-\sqrt{x}\right)\sqrt{x-x^2}}=$	(r)	$2\left(\frac{\sqrt{x}-1}{\sqrt{1-x}}\right)+c$

SMART ACHIEVERS LEARNING PVT. LTD.

(D) $\displaystyle\int \frac{dx}{\left(1+\sqrt{x}\right)\sqrt{x-x^2}} =$ (s) $2\left(\frac{1+\sqrt{x}}{\sqrt{1-x}} - \sin^{-1}\sqrt{x}\right)+c$

Key. A–R;B–Q; C–P;D–R

Sol. Put $\sqrt{x} = \sin\theta,\ x = \sin^2\theta, \sqrt{1-x} = \cos\theta,\ dx = \sin 2\theta d\theta$ etc.

5. Match the definite integrals in column I with their values in column II.

Column – I		Column – II	
(A)	$\displaystyle\int_0^{\pi} x\left(\sin^2(\sin x)+\cos^2(\cos x)\right)dx$	(P)	$\dfrac{\pi^2}{16}$
(B)	$\displaystyle\int_0^{\pi^2/4}\left(2\sin\sqrt{x}+\sqrt{x}\cos\sqrt{x}\right)dx$	(Q)	$\dfrac{\pi^2}{2}$
(C)	$\displaystyle\left\lvert\frac{\pi}{\ln 2}\int_{-\pi/4}^{\pi/4}\ln\left(\sqrt{1+\sin 2x}\right)dx\right\rvert$	(R)	$\dfrac{\pi^2}{4}$
(D)	$\displaystyle\int_0^{\pi}\frac{8x^3\cos^4 x\sin^2 x}{\pi^2-3\pi x+3x^2}dx$	(S)	$\dfrac{\pi^2}{8}$

KEY : A-Q, B-Q, C-R, D-R

Hint: (A) $I = \displaystyle\int_0^{\pi} x\left(x(\sin^2(\sin x)+\cos^2(\cos x)\right)dx$

$\Rightarrow I = \displaystyle\int_0^{\pi}(\pi - x)\left(\sin^2(\sin x)+\cos^2(\cos x)\right)dx$

Adding $2I = \pi\displaystyle\int_0^{\pi}\left(\sin^2(\sin x)+\cos^2(\cos x)\right)dx$

$2I = 2\pi\displaystyle\int_0^{\pi/2}\left(\sin^2(\sin x)+\cos^2(\cos x)\right)dx$

$I = \pi\displaystyle\int_0^{\pi/2}\left(\sin^2(\sin x)+\cos^2(\cos x)\right)dx$

Also $I = \pi\displaystyle\int_0^{\pi/2}\left[\sin^2(\cos x)+\cos^2(\sin x)\right]dx$

Adding $2I = \pi\displaystyle\int_0^{\pi/2} 2dx = 2\pi.\pi/2 = \pi^2$

$I = \pi^2/2$

B) Let $f(x) = 2\sin\sqrt{x}$

SMART ACHIEVERS LEARNING PVT. LTD.

Then $x\ f^{'}(x)=x.\dfrac{2\cos\sqrt{x}}{\sqrt{x}}=\sqrt{x}\cos\sqrt{x}$

$$I=\int_0^{\pi^2/4}\left(f(x)+x\,f^{'}(x)\right)dx=\left[x\ f(x)\right]_0^{\pi^2/4}$$

$$=2\times\pi^2/4\times\sin\pi/2=\pi^2/2$$

C) $\displaystyle\int_{-\pi/8}^{\pi/8}\ln\sqrt{1+\sin^2 x}\ dx=\int_{-\pi/4}^{\pi/4}\ln\left|\sin x+\cos x\right|dx$

$$=\int_{-\pi/4}^{\pi/4}\ln\left|\sqrt{2}\ \sin(x-\pi/4)\right|dx$$

$$=\int_0^{\pi/4}\ln\left|\sqrt{2}\ \sin t\right|dt$$

$$=\ln\sqrt{2}\int_0^{\pi/2}dt+\int_0^{\pi/2}\ln\sin t\ dt$$

$$=\pi/4\ln 2\ -\pi/2\ \ln\ 2=-\pi/4\ \ln 2$$

$$\therefore\ \frac{\pi}{\ln 2}\int_{-\pi/4}^{\pi/4}\ln\sqrt{1+\sin 2x}\,dx=\left(\frac{\pi}{\ln 2}\right)\left(\frac{-\pi}{4}\ \ln 2\right)=\frac{-\pi^2}{4}$$

D) $\displaystyle I=\int_0^{\pi}\frac{x^3\cos^4 x\sin^2 x}{\pi^2-3\pi x+3x^2}\ dx$

$$I=\int_6^{\pi}\frac{(\pi-x)^3\cos^4 x\sin^2 x}{\pi^2-3\pi x+3x^2}$$

Adding $\displaystyle 2I=\pi\int_0^{\pi}\cos^4 x\sin^2 x\ dx$

$$I=\frac{\pi}{2}.\ 2\int_0^{\pi/2}\cos^4 x\sin^2 x\,dx=\pi\int_0^{\pi/2}\cos^4 x\sin^2 x\ dx$$

$$=\pi\,\frac{(3.1)1}{6.4.2}\ \pi/2=\pi^2/32$$

6. Match the definite integrals in Column – I with their values in Column – II.

Column I	Column II
(A) $\displaystyle\int_0^{\pi/2} f(\sin 2x).\sin x\,dx$	(p) $\displaystyle\pi\int_0^{\pi/2} f(\cos x)dx$
(B) $\displaystyle\int_0^{\pi/2} f(\sin 2x).\cos x\,dx$	(q) $\displaystyle\int_0^{\pi/4}\sqrt{2}f(\cos 2x)\cos x\,dx$

SMART ACHIEVERS LEARNING PVT. LTD.

(C) $\int_0^{\pi} x.f\left(\sin x\right)dx$

(r) $\frac{1}{\sqrt{2}}\int_{\frac{7\pi}{4}}^{\frac{9\pi}{4}} f\left(\cos 2x\right)\cos x\,dx$

(D) $\int_0^{\frac{\pi}{2}} x.f\left(\sin 2x\right)dx$

(s) $\frac{\pi}{4}\int_0^{\frac{\pi}{2}} f\left(\cos x\right)dx$

Key: A→q,r; B→q,rC→p; D→s

Hint: A) $I = \int_0^{\frac{\pi}{2}} f\left(\sin 2x\right)\sin x\,dx$.......................(1)

$$= \int_0^{\frac{\pi}{2}} f\left(\left(\sin\pi - 2x\right)\right)\cos x\,dx = \int_0^{\frac{\pi}{2}} f\left(\sin 2x\right)\cos x\,dx \text{................}(2)$$

$$2I = \int_0^{\frac{\pi}{2}} f\left(\sin 2x\right)\left\{\sin x + \cos x\right\}dx$$

$$I = \frac{1}{2}\int_0^{\frac{\pi}{2}} f\left(\sin 2x\right)\sqrt{2}\cos\left(x - \frac{\pi}{4}\right)dx = \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}} f\left(\sin 2x\right)\cos\left(x - \frac{\pi}{4}\right)dx$$

Let $x - \frac{\pi}{4} = t \Rightarrow dx - dt$

$$I = \frac{1}{\sqrt{2}}\int_{-\left(\frac{\pi}{4}\right)}^{\frac{\pi}{4}} f\left(\cos 2t\right)\cos t\,dt = \sqrt{2}\int_0^{\frac{\pi}{4}} f\left(\cos 2t\right)\cos t\,dt$$

$$= \frac{1}{\sqrt{2}}\int_{2\pi-(\pi/4)}^{2\pi+(\pi/4)} f\left(\cos 2t\right)\cos t\,dt = \frac{1}{\sqrt{2}}\int_{7\pi/4}^{9\pi/4} f\left(\cos 2t\right)\cos t\,dt$$ [Since 2π is period]

C) $I = \int_0^{\pi} xf\left(\sin x\right)dx \Rightarrow I = \int_0^{\pi}\left(\pi - x\right)f\left(\sin x\right)dx$

$$\Rightarrow 2I = \int_0^{\pi} \pi.f\left(\sin x\right)dx \Rightarrow I = \frac{\pi}{2}\int_0^{\pi} f\left(\sin x\right)dx$$

SMART ACHIEVERS LEARNING PVT. LTD.

$$\Rightarrow \; I = \pi \int_0^{\frac{\pi}{2}} f(\sin x)\,dx \Rightarrow \; 1 = \pi \int_0^{\frac{\pi}{2}} f(\cos x)\,dx$$

D) $$I = \int_0^{\frac{\pi}{2}} x.f(\sin 2x)\,dx = \int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2} - x\right) f(\sin 2x)\,dx$$

$$\Rightarrow \; 2I = \int_0^{\frac{\pi}{2}} \frac{\pi}{2} f(\sin 2x)\,dx \Rightarrow \; I = \int_0^{\frac{\pi}{2}} \frac{\pi}{4} f(\sin 2x)\,dx$$

Put 2x =1

$$I = \int_0^{\frac{\pi}{2}} \frac{\pi}{4} f(\sin t)\frac{dt}{2} = \frac{\pi}{8}\int_0^{\pi} f(\sin t)\,dt$$

$$= \frac{\pi}{4}\int_0^{\frac{\pi}{2}} f(\sin t)\,dt = \frac{\pi}{4}\int_0^{\frac{\pi}{2}} f(\cos t)\,dt$$

7. Column I (family of curves) — Column II (order of the differential equation of family of curves)

Column I (family of curves)	Column II (order of the differential equation of family of curves)
(A) Circle passing through a given point	(p) 1
(B) family of ellipse having major axis along a given line	(q) 2
(C) family of parabola having fixed length of latus rectum	(r) 3
(D) family of hyperbola having fixed eccentricity	(s) 4

Key: (A-q), (B-r), (C-r), (D-s)

8. Match the definite integrals in Column – I with their values in Column – II.

Column I	Column II
(A) $\int_0^{\pi} x\left(\sin^2(\sin x)+\cos^2(\cos x)\right)dx$	(p) $\frac{\pi^2}{16}$
(B) $\int_0^{\pi^2/4} \left(2\sin\sqrt{x}+\sqrt{x}\cos\sqrt{x}\right)dx$	(q) $\frac{\pi^2}{2}$
(C) $\left\lvert \frac{\pi}{\ln 2}\int_{-\pi/4}^{\pi/4} \ln\left(\sqrt{1+\sin 2x}\right)dx \right\rvert$	(r) $\frac{\pi^2}{4}$
(D) $\int_0^{\pi} \frac{8x^3\cos^4 x\sin^2 x}{\pi^2-3\pi x+3x^2}dx$	(s) $\frac{\pi^2}{8}$

SMART ACHIEVERS LEARNING PVT. LTD.

(t) $\frac{\pi^2}{32}$

Key: A→q; B→qA→r; B→r

Hint: (A) $I=\int_0^{\pi} x\left(x(\sin^2(\sin x)+\cos^2(\cos x)\right)dx$

$\Rightarrow I=\int_0^{\pi}(\pi-x)\left(\sin^2(\sin x)+\cos^2(\cos x)\right)dx$

ADDING $2I=\pi\int_0^{\pi}\left(\sin^2(\sin x)+\cos^2(\cos x)\right)dx$

$2I=2\pi\int_0^{\pi/2}\left(\sin^2(\sin x)+\cos^2(\cos x)\right)dx$

$I=\pi\int_0^{\pi/2}\left(\sin^2(\sin x)+\cos^2(\cos x)\right)dx$

ALSO $I=\pi\int_0^{\pi/2}\left[\sin^2(\cos x)+\cos^2(\sin x)\right]dx$

ADDING $2I=\pi\int_0^{\pi/2}2dx=2\pi.\pi/2=\pi^2$

$I=\pi^2/2$

B) LET $f(x)=2\sin\sqrt{x}$

THEN $x\ f^{'}(x)=x.\frac{2\cos\sqrt{x}}{\sqrt{x}}=\sqrt{x}\cos\sqrt{x}$

$I=\int_0^{\pi^2/4}\left(f(x)+x\,f^{'}(x)\right)dx=\left[x\ f(x)\right]_0^{\pi^2/4}$

$=2\times\pi^2/4\times\sin\pi/2=\pi^2/2$

C) $\int_{-\pi/8}^{\pi/8}\ln\sqrt{1+\sin^2 x}\ dx=\int_{-\pi/4}^{\pi/4}\ln\left|\sin x+\cos x\right|dx$

$=\int_{-\pi/4}^{\pi/4}\ln\left|\sqrt{2}\ \sin(x-\pi/4)\right|dx$

$=\int_0^{\pi/4}\ln\left|\sqrt{2}\ \sin t\right|dt$

$=\ln\sqrt{2}\int_0^{\pi/2}dt+\int_0^{\pi/2}\ln\sin t\ dt$

$=\pi/4\ln 2\ -\pi/2\ln 2=-\pi/4\ \ln 2$

SMART ACHIEVERS LEARNING PVT. LTD.

$$\therefore \frac{\pi}{\ln 2}\int_{-\pi/4}^{\pi/4} \ln\sqrt{1+\sin 2x}\,dx = \left(\frac{\pi}{\ln 2}\right)\left(\frac{-\pi}{4}\ln 2\right) = \frac{-\pi^2}{4}$$

D) $I = \int_0^{\pi} \frac{x^3\cos^4 x\sin^2 x}{\pi^2 - 3\pi x + 3x^2}\,dx$

$$I = \int_6^{\pi} \frac{(\pi - x)^3\cos^4 x\sin^2 x}{\pi^2 - 3\pi x + 3x^2}$$

ADDING $2I = \pi\int_0^{\pi}\cos^4 x\sin^2 x\,dx$

$$I = \frac{\pi}{2}.\,2\int_0^{\pi/2}\cos^4 x\sin^2 x\,dx = \pi\int_0^{\pi/2}\cos^4 x\sin^2 x\,dx$$

$$= \pi.\frac{(3.1)}{6.4.2}.\frac{\pi}{2} = \frac{\pi^2}{32}$$

9.

	Column I		Column II
(A)	The value of $\int_{\alpha}^{\pi/2-\alpha}\frac{d\theta}{1+\cot^n\theta}$ Where, $0<\alpha<\frac{\pi}{2}, n>0$ is	(P)	$\frac{\pi}{2}$
(B)	The value of $\int_{-\pi}^{\pi}\frac{\sin^2 x}{1+\alpha^x}dx,\ \alpha > 0$ is	(Q)	$\frac{\pi}{4}-\alpha$
(C)	The value of $\int_{\alpha}^{2\pi-\alpha}\frac{x\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx$ is	(R)	$2\pi^2 - 2\pi\alpha$
(D)	$I = \int_{\tan^{-1}\alpha}^{\cot^{-1}\alpha}\frac{\tan x}{\tan x+\cot x}dx$	(S)	dependent of α

Key : A→ Q, S B→ P; C→ S D→ S (FINAL KEY)

Sol : (A) $I = \int_{\alpha}^{\pi/2-\alpha}\frac{\sin^n x}{\sin^n x+\cos^n x}dx$ ...(i)

$$I = \int_{\alpha}^{\pi/2-\alpha}\frac{\sin^n\left(\frac{\pi}{2}-\alpha\right)}{\sin^n\left(\frac{\pi}{2}-\alpha\right)+\cos^n\left(\frac{\pi}{2}-\alpha\right)}dx$$

$= \int_{\alpha}^{\pi/2-\alpha}\frac{\cos^n\alpha}{\cos^n\alpha+\sin^n\alpha}$ ...(ii)

Adding Eqs. (i) and (ii), then

SMART ACHIEVERS LEARNING PVT. LTD.

$$2I = \int_{\alpha}^{\pi/2-\alpha} 1dx = \frac{\pi}{2} - 2\alpha$$

$\therefore$ $I = \frac{\pi}{4} - \alpha$

(B) $I = \int_{-\pi}^{\pi} \frac{\sin^2 x}{1+\alpha^x} dx$ ... (i)

$$I = \int_{-\pi}^{\pi} \frac{\sin^2 (0-x)}{1+\alpha^{0-x}} dx$$

$\int_{-\pi}^{\pi} \frac{\alpha^x \sin^2 x}{1+\alpha^x} dx$ ...(ii)

Adding Eqs. (i) and (ii), then

$$2I = \int_{-\pi}^{\pi} \sin^2 x dx = 2\int_{0}^{\pi} \sin^2 x dx$$

$\Rightarrow$ $I = \int_{0}^{\pi} \sin^2 x dx = 2\int_{0}^{\pi/2} \sin^2 x dx$

$$2.\frac{1}{2}.\frac{\pi}{2} = \frac{\pi}{2}$$

(C) $I = \int_{\alpha}^{2\pi-\alpha} \frac{x \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}$ ... (i)

$$= \int_{\alpha}^{2\pi-\alpha} \frac{(2\pi - x)\sin^{2n}(2\pi - x)}{\sin^{2n}(2\pi - x) + \cos^{2n}(2\pi - x)} dx$$

$= \int_{\alpha}^{2\pi-\alpha} \frac{(2\pi - x)\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx$ ... (ii)

Adding Eqs. (i) and (ii), then

$$2I = 2\pi \int_{\alpha}^{2\pi-\alpha} 1.dx = 2\pi(2\pi - \alpha - \alpha)$$

$\therefore$ $1 = 2\pi^2 - 2\pi\alpha$

(D) Let $I = \int_{\tan^{-1}\alpha}^{\cot^{-1}\alpha} \frac{\tan x}{\tan x + \cot x} dx$ ... (i)

$= \int_{\tan^{-1}\alpha}^{\cot^{-1}\alpha} \frac{\cot x}{\cot x + \tan x} dx$ ... (ii) (by property)

($\because \tan^{-1}\alpha + \cot^{-1}\alpha = \pi/2$)

Adding Eqs. (i) and (ii), we get

$$2I = \int_{\tan^{-1}\alpha}^{\cot^{-1}\alpha} 1dx = \cot^{-1}\alpha - \tan^{-1}\alpha$$

$$= \left(\frac{\pi}{2} - \tan^{-1}\alpha\right) - \tan^{-1}\alpha$$

or $I = \left(\frac{\pi}{4} - \tan^{-1}\alpha\right)$

SMART ACHIEVERS LEARNING PVT. LTD.

10. The anti-derivative of

Column I		Column II	
(A)	$\dfrac{\sec x}{(\sec x+\tan x)^2}$	(P)	$\log\left\lvert\dfrac{\sin x-2}{\sin x-1}\right\rvert+C$
(B)	$\dfrac{\cos x}{(\sin x-1)(\sin x-2)}$	(Q)	$-\dfrac{\cos^2 x}{2(1+\sin x)^2}+C$
(C)	$\sin^{-1}\dfrac{2x}{1+x^2}$, $\lvert x\rvert<1$	(R)	$2x\tan^{-1}x-\log(1+x^2)+C$
(D)	$\sqrt{\tan x}+\sqrt{\cot x}$	(S)	$\sqrt{2}\tan^{-1}\left(\dfrac{\tan x-1}{\sqrt{2\tan x}}\right)+C$

Key: A-Q, B - P, C - R, D - S

Hint: (A) $\int\dfrac{\sec x}{(\sec x+\tan x)^2}dx = \int\dfrac{\sec x(\sec x+\tan x)}{(\sec x+\tan x)^3}dx=\int\dfrac{dt}{t^3}$ (t = sec x + tan x)

$= -\dfrac{1}{2}(\sec x+\tan x)^{-2}+C$

(B) $\int\dfrac{\cos x}{(\sin x-1)(\sin x-2)}dx = \int\dfrac{dt}{(t-1)(t-2)}$ (t = sin x)

$= \int\left(\dfrac{1}{t-2}-\dfrac{1}{t-1}\right)dt=\log\dfrac{\lvert\sin x-2\rvert}{\lvert\sin x-1\rvert}+C$

(C) $\int\sin^{-1}\dfrac{2x}{1+x^2}dx = 2\int t\sec^2 t\,dt$ (x = tan t)

$= 2[t\tan t - \log\lvert\sec t\rvert + C]$

$= 2x\tan^{-1}x - \log(1+x^2) + C$

(D) Putting $\tan x = y^2$, so that $dx = 2y\,dy/(1+y^4)$, we have

$\int\left(\sqrt{\tan x}+\sqrt{\cot x}\right)dx = \int\left(y+\dfrac{1}{y}\right)\dfrac{2y}{1+y^4}dy=2\int\dfrac{y^2+1}{1+y^4}dy$

$=2\int\dfrac{1+1/y^2}{y^2+1/y^2}dy=2\int\dfrac{1+1/y^2}{(y-1/y)^2+2}$ $dy=2\int\dfrac{du}{u^2+2}$ where $u = y - 1/y$

$=2\dfrac{1}{\sqrt{2}}\tan^{-1}\dfrac{u}{\sqrt{2}}+C=\sqrt{2}\tan^{-1}\left(\dfrac{\tan x-1}{\sqrt{2\tan x}}\right)+C$

11. Column I | Column II

(A) Let $f(x)=\int x^{\sin x}(1+x\cos x\log_e x+\sin x)dx$ and $f\left(\dfrac{\pi}{2}\right)=\dfrac{\pi^2}{4}$ then $f(\pi)=$ ___ (p) Rational

(B) Let $g(x)=\int \frac{1+2\cos x}{(\cos x+2)^2}dx$ and $g(0)=0$ then $g\left(\frac{\pi}{2}\right)=$ ___ (q) Irrational

(C) If real numbers x and y satisfy $(x+5)^2+(y-12)^2=14^2$ then minimum value of $\sqrt{x^2+y^2}$ is ___ (r) Integral

(D) Let $k(x)=\int \frac{(x^2+1)dx}{\sqrt[3]{x^3+3x+6}}$ and $k(-1)=\frac{1}{\sqrt[3]{2}}$ then $k(-2)=$ ______ (s) Prime

Key: $(a\to q, b\to p, c\to p,r, d\to p,r,s)$

Hint: a) $f(x)=x^{\sin x}$ and use

$$\int (f(x)+x\,f^{|}(x))dx = x\,f(x)+c$$

$$\text{and } f\left(\frac{\pi}{2}\right)=\frac{\pi^2}{4}\Rightarrow c=0\Rightarrow f(\pi)=\text{irrational}$$

b)

$$\int \cos x\frac{1}{(\cos x+2)}dx+\int\frac{\sin^2 x}{(\cos x+2)^2}dx=\frac{\sin x}{2+\cos x}+c$$

$$g(0)=0\Rightarrow c=0 \quad \therefore g\left(\frac{\pi}{2}\right)=\frac{1}{2}\ (\text{rational})$$

c)

$$\left.\begin{array}{l}\text{Let } x+5=14\cos\theta\\ y-12=14\sin\theta\end{array}\right\}\Rightarrow x^2+y^2=365+28(12\sin\theta-5\cos\theta)$$

$$\therefore \min \text{ of } \sqrt{x^2+y^2}=\sqrt{365-28\times 13}=1$$

d)

$$t=x^3+3x+6$$

$$k(x)=\int t\,dt=\frac{t^2}{2}+c$$

$$k(-1)=\frac{1}{2}(2)^{2/3}+c\Rightarrow c=0 \quad \therefore k(-2)=2$$

12. Match the following

Column – I Column – II

SMART ACHIEVERS LEARNING PVT. LTD.

a)	If $\int x^2 d(\tan^{-1}x)dx = x - f(x) + c$ then $f(1)$ is equal to	p)	$\frac{1}{8}$
b)	If sw $= a\log\left\|\cos\frac{x}{2} - \sin\frac{x}{2}\right\| + c$ then a is equal to $(0 < x < \pi/2)$	q)	-2
c)	If $\int x^2 e^{2x}dx = e^{2x}f(x) + c$, then the minimum value of $f(x)$ is	r)	0
d)	If $\int \frac{(x^4+1)}{x(x^2+1)^2}dx = a\log\|x\| + \frac{b}{x^2+1} + c$ then $a-b$ is equal to	s)	$\frac{\pi}{4}$

Key: a) s b) q c) p d) r

Hint: a) $\int x^2.\frac{1}{1+x^2}dx = x - \tan^{-1}x + C \Rightarrow f(x) = \tan^{-1}x \Rightarrow f(1) = \tan^{-1}1 = \pi/4$

b) $\int \sqrt{1+2\tan x(\tan x + \sec x)}dx = \int (\sec x + \tan x)dx$

c) $\int x^2 e^{2x}dx = \frac{1}{2}x^2.e^{2x} - \int 2x.\frac{1}{2}e^{2x}dx = \frac{1}{2}x^2e^{2x} - \left[x.\frac{1}{2}e^{2x} - \frac{1}{4}e^{2x}\right]$

$= \left(\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}\right)e^{2x} + C$

d) $\int \frac{(x^4+1)}{x(x^2+1)^2}dx = \int \frac{\left((x^2+1)^2 - 2x^2\right)dx}{x(x^2+1)^2} = \int \frac{1}{x}dx - 2\int \frac{xdx}{(x^2+1)^2} = \log|x| + \frac{1}{x^2+1} + C$

13. Match the following:

$[x] \rightarrow$ greatest integer less than or equal to $'x'$

	Column -I		Column -II
(A)	$25\int_0^{\pi/4}\left(\tan^6(x-[x]) + \tan^4(x-[x])\right)dx =$	(p)	4
(B)	$I_1 = \int_{-2}^{2}\frac{x^6+3x^5+7x^4}{x^4+2}dx$ $I_2 = \int_{-3}^{1}\frac{2(x+1)^2+11(x+1)+14}{(x+1)^4+2}dx$ then the $I_1 + I_2 =$	(q)	2

SMART ACHIEVERS LEARNING PVT. LTD.

(C) $\frac{2}{\pi}\int_{-1}^{3}\left(\tan^{-1}\frac{x}{x^2+1}+\tan^{-1}\frac{x^2+1}{x}\right)dx =$ (r) $\frac{100}{3}$

(D) $f(x)=\frac{2-x}{\pi}\cos\pi(x+3)+\frac{1}{\pi^2}\sin\pi(x+3)$ where $0<x<4$. The number of points of local extrema are (s) 5

(t) $\frac{102}{3}$

Key. A – s ; B – r ; C – q ; D – q

Sol. A) $I = 25\int_0^{\pi}\left(\tan^6 x+\tan^4 x\right)dx = 25\int_0^{\pi/4}\left(\tan^4 x\sec^2 x\right)dx$

B) Let $x+1=t$ in I_2 then $I_1+I_2=\int_{-2}^{2}\left(x^2+7\right)dx=\frac{100}{3}$

C) Let $I=\int_{-1}^{0}\frac{-\pi}{2}dx+\int_0^3\frac{\pi}{2}dx=\pi$

D) $f'(x)=0$ at $x=2,1,3$ but $x=2$ is point of inflexion.

SMART ACHIEVERS LEARNING PVT. LTD.

14. Match the following:

Column I	Column -II
(A) $\sum_{n=1}^{\infty}\frac{n}{(2n-1)^2(2n+1)^2}=$	(p) $\frac{1}{4}$
(B) $\sum_{n=1}^{\infty}\frac{n-\sqrt{n^2-1}}{\sqrt{n(n+1)}}=$	(q) $\frac{1}{8}$
(C) $\sum_{n=1}^{\infty}\frac{2n+1}{n^2(n+1)^2}=$	(r) 1
(D) $\sum_{n=1}^{\infty}\frac{1}{\left(\sqrt{n}+\sqrt{n+1}\right)\sqrt{n(n+1)}}=$	(s) $\frac{1}{2}$

(t) ∞

Key. A – q ; B – r ; C – r ; D – r

Sol. A) $\frac{1}{8}\sum_{n=1}^{\infty}\left[\frac{1}{(2n-1)^2}-\frac{1}{(2n+1)^2}\right]=\frac{1}{8}$

B) $\sum_{n=1}^{\infty}\left(\sqrt{\frac{n}{n+1}}-\sqrt{\frac{n-1}{n}}\right)=1$

C) $\sum_{n=1}^{\infty}\left(\frac{1}{n^2}-\frac{1}{(n+1)^2}\right)=1$

D) $\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n(n+1)}}=\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)=1$

15. If $(1+x+x^2)^n = a_0+a_1x+a_2x^2+.....+a_{2n}x^{2n}, then$

	Column-I		Column-II
A)	$a_0^2-a_1^2+a_2^2-a_3^2+.........+a_{2n}^2$ is equals	p)	$\frac{1}{2}(3^n+a_n)$
B)	$a_0a_2-a_1a_3+a_2a_4......+a_{2n-2}a_{2n}$ is equals	q)	$\frac{1}{2}(3^n-1)$
C)	$a_0+a_1+a_2+......+a_n$ is equals	r)	a_{n+1}
D)	$a_2+a_4+......+a_{2n}$ is equals	s)	a_n

Key: A) s B) r C) p D) q

Hint. A) $a_0^2-a_1^2+a_2^2..........+a_{2n}^2=$ coefficient $x^{2n} in (1+x+x^2)^n(x^2-x+1)^n$

B) $a_0a_2-a_1a_3+.......+a_{2n-2}a_{2n}=$ coefficient x^{-2} in $(1+x+x^2)^n\left(1-\frac{1}{x}+\frac{1}{x^2}\right)^n$

C) $a_r=a_{2n-r}$

D) let f(x) $=(1+x+x^2)^n$

$a_0+a_2+a_4......+a_{2n}=\frac{f(1)+f(-1)}{2}$

$a_0=1$

16. Match the following:-

	Column – I		Column – II
(A)	Area bounded by $y = x^2 + 2$ and $y = 2\|x\| - \cos \pi x$ is equal to A, then 3A equals	(p)	4
(B)	The value of $2\cot(\cot^{-1}3 + \cot^{-1}7 + \cot^{-1}13 + \cot^{-1}21)$	(q)	$\frac{8}{3}$
(C)	Tangent and normal at the ends A and C of focal chord AC of parabola $y^2 = 4x$ intersect at B and D, then minimum area of ABCD is	(r)	3
(D)	Number of integral values of ordered pair (α, β) for which the area common to $x^2 + y^2 - 2\alpha^2 x - 2\beta^2 y + c = 0$ and its image in $x + y = 1$ is maximum is	(s)	8

Key: A – S, B – R, C – S, D – P

Hint: (A) Solving two curves

$x^2 + 2 = 2|x| - \cos\pi x$

$x^2 - 2|x| + 2 = -\cos\pi x$

$(|x| - 1)^2 + 1 = -\cos\pi x$

Clearly solution is $x = \pm 1$

Required area $= \int_{-1}^{1} (x^2 + 2 - 2|x| + \cos \pi x)dx = \frac{8}{3}$

(B)

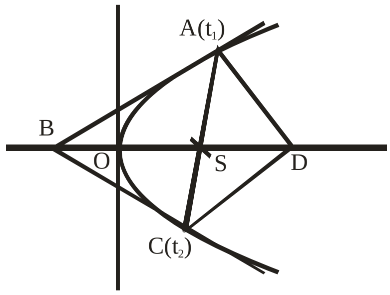
$$\cot^{-1}3 + \cot^{-1}7 = \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{7}}{1-\frac{1}{21}}\right) = \tan^{-1}\left(\frac{1}{2}\right)$$

$$\cot^{-1}13 + \cot^{-1}21 = \tan^{-1}\frac{\left(\frac{1}{13}+\frac{1}{21}\right)}{1-\frac{1}{13}\times\frac{1}{21}} = \tan^{-1}\frac{1}{8}$$

$$\Rightarrow \quad \cot\left(\cot^{-1}3 + \cot^{-1}7 + \cot^{-1}13 + \cot^{-1}21\right)$$

$$\Rightarrow \quad \cot\left(\tan^{-1}\frac{2}{3}\right) = \frac{3}{2}$$

(C) $t_1 t_2 = -1$



Clearly ABCD is a rectangle

Co-ordinate of B(t_1t_2, ($t_1 + t_2$))

$AB = |(t_2 - t_1)|\sqrt{1+t_1^2}$

$BC = (t_2 - t_1)\sqrt{1+t_2^2}$

Area = AB × BC = $(t_2 - t_1)^2\sqrt{(1+t_1^2)(1+t_2^2)}$

$= \left(t_1 + \frac{1}{t_1}\right)^3$

Least value = 8.

(D) For area to be max. centre $(\alpha_1^2\ \beta^2)$

Should lie on x + y = 1 ⇒ $\alpha^2 + \beta^2 = 1$

As $\alpha, \beta \in I$

$\alpha^2 = 0$ and $\alpha^2 = 1$

⇒ (0, ±1)

or $\alpha^2 = 1$ and $\beta^2 = 0$

⇒ (±1, 0)

SMART ACHIEVERS LEARNING PVT. LTD.

17. Match the following

a) $\int \frac{1}{1+\sin x+\cos x}dx =$ p) $\frac{1}{2}\log\left|\tan\left(\frac{x}{2}+\frac{\pi}{12}\right)\right|+c$

b) $\int \frac{1}{\sqrt{3}\sin x+\cos x}dx =$ q) $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{\sqrt{3}}\right)+c$

c) $\int \frac{1}{1-2\sin x}dx =$ r) $\log\left|1+\tan\frac{x}{2}\right|+c$

d) $\int \frac{1}{2+\cos x}dx =$ s) $\frac{1}{\sqrt{3}}\log\left|\frac{\tan\frac{x}{2}-2-\sqrt{3}}{\tan\frac{x}{2}-2+\sqrt{3}}\right|+c$

Key. $a \to r, b \to p, c \to s, d \to q$

Sol. a) put $\tan\frac{x}{2} = t$

c) put $\tan\frac{x}{2} = t$

d) put $\tan\frac{x}{2} = t$

18. If $\frac{\pi}{2} < x < \pi$ then match the following

a) $\int\sqrt{1+\cos x}\ dx =$ p) $2\left(\sin x/2 - \cos x/2\right)+c$

b) $\int\sqrt{1-\cos x}\ dx =$ q) $-2\left(\sin x/2 + \cos x/2\right)+c$

c) $\int \sqrt{1+\sin x}\ dx =$ r) $2\sqrt{2}\sin x/2 + c$

d) $\int \sqrt{1-\sin x}\ dx =$ s) $-2\sqrt{2}\cos x/2 + c$

Key. $a \to r, b \to s,\ c \to p,\ d \to q$

19. If $\int \frac{1+x^{2/3}}{1+x}dx = Ax^{1/A} + B\log\left|x^{1/3}+1\right| + C\log\left|x^{2/3}-x^{1/3}+1\right| + \frac{1}{D}\tan^{-1}\left(D\left(1-2x^{1/3}\right)\right) + K$

where K is arbitrary constant then match the following:

a) A= p) $+\frac{1}{\sqrt{3}}$

b)B= q) 1/2

c) 1/C= r) 2

d)D= s) 3/2

Key. $a \to s, b \to r,\ c \to r,\ d \to p$

Sol. Put $x=t^3$

20. A function F is defined by $F(x) = \int_1^x \frac{e^t}{t}dt \quad \forall x > 0$. Now express the functions in Column – I in terms of F

Column – I	**Column – II**
a) $\int_1^x \frac{e^t}{t+2}dt$	p) $F(x) - \frac{e^x}{x} + e$
b) $\int_1^x \frac{e^{3t}}{t}dt$	q) $xe^{\frac{1}{x}} - e - F\left(\frac{1}{x}\right)$
c) $\int_1^x \frac{e^t}{t^2}dt$	r) $e^{-2}[F(x+2) - F(3)]$
d) $\int_1^x e^{\frac{1}{t}}dt$	s) $F(3x) - F(3)$

Key. a) r; b) s; c) p; d) q

Sol. a) put $t+2=z$
b) put $3t=z$
c) integrate by part
d) put $1/t=z$

SMART ACHIEVERS LEARNING PVT. LTD.

SMART ACHIEVERS LEARNING PVT. LTD.