Binomial Theorem

Single Correct Answer Type

1. If 7 divides $32^{32^{32}}$, the remainder is

A) 1 B) 0 C) 4 D) 6

Key. C

Sol. $32 = 2^5 \Rightarrow (32)^{32} = \left(2^5\right)^{32}$

$= 2^{160} = (3-1)^{160} = 3m+1, m \in N$

$\therefore (32)^{32^{32}} = (32)^{3m+1} = 2^{5(3m+1)}$

$2^{3(5m+1)}\, 2^2 = 4.8^{5m+1}$

$4(7+1)^{5m+1} = 4(7n+1), n \in N = 28n+4$

$\therefore$ When 7 divides $(32)^{32^{32}}$ remainder =4

2. If {x} represents the fractional part of x, then $\left\{\frac{5^{200}}{8}\right\}$ is

A) $\frac{1}{4}$ B) $\frac{1}{8}$ C) $\frac{3}{8}$ D) $\frac{5}{8}$

Key. B

Sol. $\frac{5^{200}}{8} = \frac{\left(5^2\right)^{100}}{8} = \frac{(1+24)^{100}}{8}$

$= \frac{1 + {}^{100}C_1.24 + {}^{100}C_2(24)^2 + ... + {}^{100}C_{100}(24)^{100}}{8}$

$= \frac{1}{8} + \text{integer} \Rightarrow \left\{\frac{5^{200}}{8}\right\} = \frac{1}{8}$

3. For n>3 ,1.2 ${}^nC_r - 2.3\ {}^nC_{r-1} + ... + (-1)^r (r+1)(r+2)$ is

A) ${}^{n-3}C_r$ B) 2. ${}^{n-3}C_r$ C) ${}^{n+3}C_{r+1}$ D) ${}^{n-2}C_r$

Key. B

Sol. We have $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ...$

$+ {}^nC_{r-1}\ x^{r-1} + {}^nC_r\ x^r + ... + {}^nC_nx^n$ .....(1)

and $(1+x)^{-3} = 1 - {}^3C_1x + {}^4C_2x^2 ...... +$

$(-1)^{r-1}\ {}^{r+1}C_{r-1}\ x^{r-1} + (-1)^r$

$^{r+2}C_r\ x^r + ...to\infty$ ....(2)

Multiply (1) and (2) , we get

$(1+x)^{n-3} = \left({}^nC_0 + {}^nC_1x + {}^nC_2\ x^2 + ... + {}^nC_n\ x^n\right)\left(1 - {}^3C_1x + {}^4C_2\ x^2 - .....\infty\right)$

Clearly , the coefficient of x^r from the product in R.H.S is

$1.{}^nC_r - {}^3C_1.{}^nC_{r-1} + {}^4C_2.{}^nC_{r-2} - ...... + (-1)^r\ {}^{r+2}C_r.{}^nC_0$

$= {}^nC_r - 3{}^nC_{r-1} + \frac{4.3}{2.1}\ {}^nC_{r-2} - \frac{5.4}{2.1}\ {}^nC_{r-3}$

$+ ...... + (1)^r \frac{(r+2)(r+1)}{2.1}$

$= \frac{1}{2}\left[1.2\ {}^nC_r - 2.3\,{}^nC_{r-1} + 4.3\ {}^nC_{r-2} - ...... + (-1)r(r+2)(r+1)\right]$

$\therefore$ Required series = $2\times$ coefficient of x^r in $(1+x)^{n-3} = 2.{}^{n-3}C_r$

4. The coefficient of x^{50} in the expansion of

$(1+x)^{1000} + 2x(1+x)^{999} + 3x^2(1+x)^{998} + ... + 1001\ x^{1000}$

A) $1000_{C_{50}}$ B) $1001_{C_{50}}$ C) $1002_{C_{50}}$ D) 2^{1001}

Key. C

Sol. Let, $S = (1+x)^{1000} + 2x(1+x)^{999} + 2x(1+x)^{999} + 3x^2(1+x)^{998} + ... + 1001\ x^{1000}$

$\frac{x}{1+x}S = x(1+x)^{999} + +2x^2(1+x)^{998} + ...$

$+1000_x{}^{1000} + \frac{1001x^{1001}}{1+x}$

Subtract above equations,

$\left(1 - \frac{x}{1+x}\right)S = (1+x)^{1000} + (1+x)^{999} +$

$x^2(1+x)^{998} + ... + x^{1000} - \frac{1001x^{1001}}{1+x}$

$\Rightarrow S = (1+x)^{1001} + x(1+x)^{1000} + x^2(1+x)^{999}$

$+ .... + x^{1000}(1+x) - 100]\ x^{1001}$

$= \frac{(1+x)^{1001}\left[\left(\frac{x}{1+x}\right)^{1001} - 1\right]}{\frac{x}{1+x} - 1} - 1001\ x^{1001}$

[sum of G.P]

$= (1+x)^{1002} - x^{1002} - 1002\ x^{1001}$

$\therefore$ coefficient of x^{50} in S= coefficient of x^{50} in $\left[(1+x)^{1002}-x^{1002}-1002x^{1001}\right]={}^{1002}C_{50}$

5. The coefficient of the term independent of x in the expansion of $\left(\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right)^{10}$

A) 70 B) 112 C) 105 D) 210

Key. D

Sol. Given expression $=\frac{\left(x^{1/3}\right)^3+(1)^3}{x^{2/3}-x^{1/3}+1}-\frac{(x-1)}{x^{1/2}\left(x^{1/2}-1\right)}$

$=\frac{\left(x^{1/3}+1\right)\left(x^{2/3}-x^{1/3}+1\right)}{\left(x^{2/3}-x^{1/3}+1\right)}-\frac{\left(x^{1/2}+1\right)\left(x^{1/2}-1\right)}{x^{1/2}\left(x^{1/2}-1\right)}$

$=\left(x^{1/3}+1\right)-\left(1+x^{-1/2}\right)=x^{1/3}-x^{-1/2}$

$\Rightarrow\left(\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right)^{10}$

$=\left(\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right)^{10}$

$=\left(x^{1/3}-x^{-1/2}\right)^{10}$

T_{r+1} *in* $\left(x^{1/3}-x^{-1/2}\right)^{10}$ is

${}^{10}C_r\left(x^{1/3}\right)^{10-r}.(-1)^r.\left(x^{-1/2}\right)^r$

$=(-1)^r\ {}^{10}C_r\ x^{\left(\frac{10-r}{3}-\frac{r}{2}\right)}$ which is independent of x

If $\left(\frac{10-r}{3}-\frac{r}{2}\right)=0\Rightarrow r=4$

Hence required coefficient $={}^{10}C_4(-1)^4=210$

6. Let n be an odd natural number greater than 1. Then the number of zeros at the end of the sum 99^n+1 is

(A) 3 (B) 4 (C) 2 (D) None of these

Key. C

Sol. $1+99^n=1+(100-1)^n=1+$
$\left\{{}^nC_0 100^n-{}^nC_1.100^{n-1}+.........-{}^nC_n\right\}$

Because n is odd $=100\left\{{}^nC_0.100^{n-1}-{}^nC_1.100^{n-2}+.........-\right.$
$\left.{}^nC_{n-2}.100+{}^nC_{n-1}\right\}$

= 100$\times$integer whose units place is different from 0

$\left[\because {}^nC_{n-1}=n, has\ odd\ digit\ at\ unit\ place\right]$

$\therefore$ There are two zeros at the end of the sum 99^n+1

7. If $C_0,C_1,C_2,............C_n$ are the Binomial coefficients in the expansion $(1+x)^n$. 'n' being even, then $C_0+(C_0+C_1)+(C_0+C_1+C_2)+..........(C_0+C_1+C_2+....+C_{n-1})=$ is equal to

(A) $n2^n$ (B) $n.2^{n-1}$ (C) $n.2^{n-3}$ (D) $n.2^{n-2}$

Key. B

Sol. Sum $=\left\{C_0+\left(c_1+c_2+......+c_{n-1}\right)\right\}+=\left\{\left(c_0+c_1\right)+\left(c_0+c_1+.......+c_{n-2}\right)\right\}+$
$\left\{\left(c_0+c_1+c_2\right)+\left(c_0+c_1+......+c_{n-3}\right)\right\}+.......\ to\left(\frac{n}{2}\right)$

Terms = $=\left(c_0+c_1+......c_n\right)\times\frac{n}{2}=n.2^{n-1}$

8. If $\sum_{r=0}^{n}\left\{\frac{{}^nC_{r-1}}{{}^nC_r+{}^nC_{r-1}}\right\}^3=\frac{25}{24}$, then n is equal to

(A) 3 (B) 4 (C) 5 (D) 6

Key. C

Sol. Let $t_r=\frac{{}^nC_{r-1}}{{}^nC_r+{}^nC_{r-1}}=\frac{{}^nC_{r-1}}{{}^{n+1}C_r}=\frac{{}^nC_{r-1}}{\frac{n+1}{r}\ {}^nC_{r-1}}$

$\therefore l_r=\frac{r}{n+1}$

Now,

$S=\sum_{r=0}^{n}\left\{l_r\right\}^3\Rightarrow S=\sum_{r=0}^{n}\frac{r^3}{(n+1)^3}=\frac{1}{(n+1)^3}\sum_{r=0}^{n}r^3$

$\Rightarrow S=\frac{1}{(n+1)^3}\left\{\frac{n(n+1)}{2}\right\}^2\Rightarrow S=\frac{n^2}{4(n+1)}$

Now, $S=\frac{25}{24}$ (given) which is only possible for 5

9. If m and n are any two odd positive integers with n < m, then the largest positive integer which divides all numbers of the form, $m^2 - n^2$ can be

(A) 4 (B) 6 (C) 8 (D) 9

Key. C

Sol. Let $m = 2k - 1$ *and* $n = 2p - 1, p < k$

Then $m^2 - n^2 = (m+n)(m-n)$

$= (2k + 2p - 2)(2k - 2p) = 4(k + p - 1)(k - p)$

Further if k and p both even, then k-p is even but k+p-1 is odd

If k and p both odd then k-p is even but k+p-1 is odd. If one is even and other odd then k-p is odd but k+p-1 is even. Thus in every case $(k - p)(k + p - 1)$ even

$\therefore m^2 - n^2$ is divisible by $4 \times 2 = 8$. Hence , $m^2 - n^2$ is divisible by 8 or any multiple of 8. The largest integer among the given options is 8

10. The number of terms in $(a_1 + a_2 + a_3 + a_4)^3$ is

(A) 64 (B) 81 (C) 30 (D) 20

Key: D

Hint Any term of $(a_1 + \alpha_2 + a_3 + a_4)^3$ is of the form $a_1^{\alpha} . a_2^{\beta} . a_3^{\gamma} . a_4^{\delta}$.

where $\alpha + \beta + \gamma + \delta = 3, \alpha, \beta, \gamma, \delta \in \{0, 1, 2, 3\}$

Thus number of terms is 20.

11. The value of ${}^{12}C_2 + {}^{13}C_3 + {}^{14}C_4 + \ldots\ldots\ldots\ldots + {}^{999}C_{989}$ is

(A) ${}^{1000}C_{11} - 12$ (B) ${}^{1000}C_{11} + 12$ (C) ${}^{999}C_{11} - 12$ (D) ${}^{1000}C_{989}$

Key: A

Hint Since ${}^{10}C_0 + {}^{11}C_1 + {}^{12}C_2 + {}^{13}C_3 + \ldots\ldots\ldots + {}^{999}C_{989}$

$= {}^{1000}C_{989} = {}^{1000}C_{11}$

$\left(\text{Since , } {}^{10}C_0 = {}^{11}C_0 \text{ and } {}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r\right)$

So, ${}^{12}C_2 + {}^{13}C_3 + {}^{14}C_4 + \ldots\ldots + {}^{999}C_{989} = {}^{1000}C_{11} - 12$

12. The sum $S_n = \sum_{k=0}^{n} (-1)^K . {}^{3n}C_k$, where $n = 1, 2, \ldots\ldots\ldots\ldots$ is

(A) $(-1)^n . {}^{3n-1}C_{n-1}$ (B) $(-1)^n . {}^{3n-1}C_n$ (C) $(-1)^n . {}^{3n-1}C_{n+1}$ (D) None of these

Key: B

Hind: $S_n = {}^{3n}C_0 - {}^{3n}C_1 + {}^{3n}C_2 + \ldots\ldots\ldots + (-1)^n . {}^{3n}C_n$

But ${}^{3n}C_0 = {}^{3n-1}C_0$

$-{}^{3n}C_1 = -{}^{3n-1}C_0 - {}^{3n-1}C_1$

${}^{3n}C_2 = {}^{3n-1}C_1 + {}^{3n-1}C_2$

$-{}^{3n}C_3 = -{}^{3n-1}C_2 - {}^{3n-1}C_3$

....

$(-1)^n . {}^{3n}C_n = (-1)^n . {}^{3n-1}C_{n-1} + (-1)^n . {}^{3n-1}C_n$

On adding we get $S_n = (-1)^n . {}^{3n-1}C_n$

13. The Coefficient of x^9 in $\left(x - {}^{21}C_0\right)\left(x - {}^{21}C_1\right)\left(x - {}^{21}C_2\right)........\left(x - {}^{21}C_{10}\right) is$

(A) $2^{40} - \frac{1}{2}\,{}^{42}C_{20}$ (B) $2^{39} - \frac{1}{2}\,{}^{42}C_{21}$ (C) $2^{40} - {}^{42}C_{20}$ (D) $2^{39} - \frac{1}{4}\,{}^{42}C_{21}$

Key: D

Hint: Coefficient x^{10} =sum of products of ${}^{20}C_0, {}^{20}C_1............{}^{20}C_{10}$. Taking '2' at a time.

14. $\sum_{K=1}^{10} \frac{(-1)^{K-1}}{K}.\left(10_{C_K}\right) =$

(A) $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+---+\frac{1}{11}$ (B) $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+---+\frac{1}{10}$

(C) $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+---+\frac{1}{9}$ (D) $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+---+\frac{1}{12}$

Key: B

Hint: Required value is $\frac{10_{C_1}}{1} - \frac{10_{C_2}}{2} + \frac{10_{C_3}}{3} ---- \frac{10_{C_{10}}}{10}$

To find which, consider $(1-x)^{10} = 10_{C_0} - 10_{C_1}x + 10_{C_2}x^2 ------ + 10_{C_{10}}x^{10}$

$$\Rightarrow \frac{(1-x)^{10}-1}{x} = -\left[10_{C_1} - 10_{C_2}x + ------ + 10_{C_{10}}x^9\right]$$

$$\Rightarrow \int_0^1 \frac{1-(1-x)^{10}}{x}dx = \int_0^1 \left[10_{C_1} - 10_{C_2}x + ---- + 10_{C_{10}}x^9\right]dx$$

$$= 10_{C_1} - \frac{10_{C_2}}{2} + \frac{10_{C_3}}{3} ----- \frac{10_{C_{10}}}{10}$$

To find LHS consider $I_n = \int_0^1 \frac{1-(1-x)^n}{x}dx \Rightarrow I_{n+1} - I_n = \int_0^1 (1-x)^n\,dx = \frac{1}{n+1}$

$$\therefore I_{n+1} = \frac{1}{n+1} + I_n$$

$$\therefore I_{10} = \int_0^1 \frac{1-(1-x)^{10}}{x}dx = \frac{1}{10} + I_9 = \frac{1}{10} + \frac{1}{9} + I_8 \approx ---$$

$$= \frac{1}{10} + \frac{1}{9} + \frac{1}{8} + ---- + 1$$

15. The sum $\sum_{k=1}^{\infty}\frac{6^k}{\left(3^k-2^k\right)\left(3^{k+1}-2^{k+1}\right)}=$

(A) 1 (B) 2 (C) 3 (D) 4

Key: B

Hint: $\frac{6^k}{\left(3^k-2^k\right)\left(3^{k+1}-2^{k+1}\right)}=\frac{3^k-2^k}{\left(3^k-2^k\right)\left(3^{k+1}-2^{k+1}\right)}$

$$=\frac{3^k}{3^k-2^k}-\frac{3^{k+1}}{3^{k+1}-2^{k+1}}$$

$$\sum_{k=1}^{\infty}\frac{3^k}{3^k-2^k}-\frac{3^{k+1}}{3^{k+1}-2^{k+1}}=\frac{3}{3-2}-\underset{n\to\infty}{Lt}\frac{3^2}{3^n-2^n}$$

$$=3-1=2$$

16. The value of the expression ${}^{10}C_0 10^9-{}^{10}C_1 9^9+{}^{10}C_2 8^9..........-{}^{10}C_9$ *is*

A) $\lfloor 9$ B) $\lfloor 10$ C) $9\lfloor 10$ D) 0

Key: D

Hint: Given expression = No.of on to functions from a set of 9 elements to a set of 10 elements = 0

17. $\sum_{r=0}^{n}\frac{n-3r+1}{n-r+1}\frac{{}^nC_r}{2^r}$ is equal to

a) $\frac{1}{2^n}$ b) $\frac{1}{3^n}$ c) $\frac{1}{4^n}$ d) $\frac{1}{2^n}+1$

Key: A

Hint: $S=\sum_{r=0}^{n}\left(1-\frac{2r}{n-r+1}\right)\frac{{}^nC_r}{2^r}=\sum_{r=0}^{n}\frac{{}^nC_r}{2^r}-\sum_{r=0}^{n}\frac{{}^nC_{r-1}}{2^{r-1}}=\frac{1}{2^n}$

18. The sum of all the coefficient of those terms in the expansion of $(a + b + c + d)^8$ which contains b but not c

(A) 6305 (B) 6561 (C) 256 (D) 4^8

Key: A

Hint: Sum of the coefficients of the terms not containing c is 3^8 and of the term not containing b and c both is 2^8, so required sum = $3^8 - 2^8$.

19. If $\lfloor 15=2^{P_1}3^{P_2}5^{P_3}7^{P_4}11^{P_5}13^{P_6}$ then $\sum_{r=1}^{6}P_r$ is

(A) 24 (B) 23 (C) 22 (D)21

Key: A

Hint: $15!=2^{11}\times3^6\times5^3\times7^2\times11^1\times13^1$

$$\therefore\sum_{r=1}^{6}P_r=11+6+3+2+1+1=24$$

20. The number of the positive integer pairs (x,y) such that $\frac{1}{x}+\frac{1}{y}=\frac{1}{2007}$ where $x<y$ is

(A) 5 (B) 6 (C) 7 (D) 8

Key: C

Hint: $\frac{1}{x}+\frac{1}{y}=\frac{1}{2007}$

$\Rightarrow (x+y)2007 = xy$

$\Rightarrow xy - 2007x - 2007y = 0$

$(x-2007)(y-2007) = 2007^2 = 3^4 \times 223^2$

The number of pairs is equal to the number of divisors of 2007^2 that is $(4+1)\times(2+1)=15$

Since $x < y$, so required number of pairs = 7

21. The greatest integer less than or equal to $\left(\sqrt{2}+1\right)^6$ is

(a) 194 (b) 195 (c) 196 (d) 197

Key: d

Hint: Let $\left(\sqrt{2}+1\right)^6$ = 1 + F, where I is an integer and 0 < F <1.

Let f = $\left(\sqrt{2}-1\right)^6$. We have

$\sqrt{2}-1=\frac{1}{\sqrt{2}+1} \Rightarrow 0<\sqrt{2}-1<1 \Rightarrow$ 0 < f < 1.

Also 1 + F + f = $\left(\sqrt{2}+1\right)^6+\left(\sqrt{2}-1\right)^6$

= $2[{}^6C_0 . 2^3 + {}^6C_2 . 2^2 + {}^6C_4 .2 + {}^6C_6]$

= 2(8 + 60 + 30 + 1) = 198

Hence F + f = 198 – I is an integer. But 0 < F + f < 2.

Therefore F + f = 1, and thus, I = 197.

22. The coefficient of x^6 in the expansion of $\left(1+\frac{x}{\lfloor 1}+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\frac{x^4}{\lfloor 4}+\frac{x^5}{\lfloor 5}\right)^2$ is

(A) $\frac{2}{15}$ (B) $\frac{4}{15}$ (C) $\frac{31}{360}$ (D) $\frac{2}{45}$

Key: C

Hint: Coefficient of x^6 is $\frac{1}{\lfloor 5}\frac{1}{\lfloor 1}+\frac{1}{\lfloor 4}\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}\frac{1}{\lfloor 3}+\frac{1}{\lfloor 2}\frac{1}{\lfloor 4}+\frac{1}{\lfloor 1}\frac{1}{\lfloor 5}$

$=\frac{1}{\lfloor 6}\left[{}^6C_1+{}^6C_2+{}^6C_3+{}^6C_4+{}^6C_5\right]$

$=\frac{1}{\lfloor 6}\left(2^6-2\right)=\frac{31}{360}$

23. Coefficient of x^{2009} in $(1 + x + x^2 + x^3 + x^4)^{1001} (1 - x)^{1002}$ is

(A) 0 (B) 4. ${}^{1001}C_{501}$

(C) -2009 (D) none of these

Key: A

Hint: $(1 + x + x^2 + x^3 + x^4)^{1001} (1 - x)^{1002}$

= $(1 - x)(1 - x^5)^{1001}$, so all the powers of x will be of the 5m or 5m + 1 ($m \in I$)

So coff. of x^{2009} will be 0

24. If $\left(x^2+2x+4\right)^n = \sum_{r=0}^{2n} a_r x^r$, then $\sum_{r=n}^{2r}\left(\frac{a_{2n-r}}{a_r}\right)$ is

a) $2^{2n+1}-1$ b) $\frac{4^{n+1}-1}{3}$

c) 4^{n-1} d) $\frac{4^{n-1}+1}{2}$

Key : b

Sol : Put x = 2x

$$\left(4x^2+4x+4\right)^n = \sum_{r=0}^{2n} a_r 2^r x^r$$

Put $x = \frac{1}{x}$

$$\left[1+2x+(2x)^2\right]^n = \sum_{r=0}^{2n} a_r\, x^{2n-r}$$

Put 2x = y

$$\left(1+y+y^2\right)^n = \sum_{r=0}^{2n} a_r \frac{y^{2n-r}}{2^{2n-r}}$$

Equating coefficient of $x^r \frac{1}{4^n} a_r 2^r = a_{2n-r} 2^{-r}$

$$\Rightarrow \frac{a_{2n-r}}{a_r} = \frac{1}{4n}4^r$$

$$\sum_{r=n}^{2n}\frac{a_{2n-r}}{a_n} = \sum_{r=n}^{2n}\frac{1}{4n}4r = \left(\frac{4^{n+1}-1}{3}\right)$$

25. Coefficient of x^6 in $\left((1+x)\left(1+x^2\right)^2\left(1+x^3\right)^3.....\left(1+x^n\right)^n\right)$ is

a) 26 b) 28 c) 30 d) 35

Key : b

Sol : The coefficient of x^6 in the given expression = coefficient of x^6 in

$\left(1+{}^6C_1x^6\right)\left(1+{}^5C_1x^5\right)\left(1+{}^4C_1x^4\right)\left(1+{}^3C_1x^3+{}^3C_2x^6\right)\left(1+{}^2C_1x^2+{}^2C_2x^4\right)(1+x)$

= coefficient of x^6 in $\left(1+6x^6+5x^5+4x^4\right)\left(1+2x^2+3x^3+x^4+6x^5+3x^6\right)(1+x)$

= coefficient of x^6 in $\left(11x^5+17x^6\right)\ (1+x)$ = 28

26. The coefficient of x^2 in $(1+x)(1+2x)(1+4x).....\left(1+2^n x\right)$ is

a) $\frac{\left(2^n+1\right)\left(2^n+2\right)}{3}$ b) $\frac{\left(2^{n+1}-1\right)\left(2^{n+1}-2\right)}{3}$

c) $\frac{\left(2^{n+1}+1\right)\left(2^{n+1}+2\right)}{3}$ d) $\frac{\left(2^{n+1}+1\right)\left(2^{n+1}-2\right)}{5}$

Key : b

Sol : Coefficient of $x^2=\frac{1}{2}\left[\left(1+2+2^2......+2^n\right)^2\right]-\left[1^2+2^2+....\left(2^n\right)^2\right]$

$$=\frac{1}{2}\left[\left(\frac{2^{n+1}-1}{1}\right)^2-\left(\frac{\left(2^2\right)^{n+1}-1}{3}\right)\right]$$

$$=\frac{1}{2}\left[\left(2^{n+1}-1\right)\left[\left(2^{n+1}-1\right)-\frac{2^{n+1}+1}{3}\right]\right]$$

$$=\frac{\left(2^{n+1}-1\right)\left(2^{n+1}-2\right)}{3}$$

27. Let $n\in N$ and $n<\left(5\sqrt{3}+8\right)^4$. Then the greatest value of n is

a) 77040 b) 77042

c) 77041 d) none of these

Key : c

Sol : Suppose $x=\left(5\sqrt{3}+8\right)^4$

= [x] + [x]

$0<5\sqrt{3}-8<1$

$0<\left(5\sqrt{3}-8\right)^4<1$

Let $F=\left(5\sqrt{3}-8\right)^{4}$

$0 < F < 1$

$\left[x\right]+\left\{x\right\}+F=\left(5\sqrt{3}+8\right)^{4}+\left(5\sqrt{3}-8\right)^{4}$

$={}^{4}C_{0}\left(5\sqrt{3}\right)^{4}+{}^{4}C_{1}\left(5\sqrt{3}\right)^{3}(8)+....+\left({}^{4}C_{0}\left(5\sqrt{3}\right)^{4}-{}^{4}C_{1}\left(5\sqrt{3}\right)^{3}(8)+...\right)$

$=2.\left[C_{0}\left(5\sqrt{3}\right)^{4}+C_{2}(8)^{2}\left(5\sqrt{3}\right)^{2}+C_{4}(8)^{4}\right]$

$=2\left[625(9)+(6)64(25)(3)+(64)(64)\right]$

$=2\left[5625+28800+4096\right]=77042$

$\left\{x\right\}$ + F must be an integer

Also $0<\left\{x\right\}+F<2$

$\Rightarrow\left\{x\right\}+f=1$

$\Rightarrow\left[x\right]=77042-1=77041$

28. If $P_{k}=\dfrac{1-x^{k+1}}{1-x}$, the number of terms in the product $P_{1}.P_{2}....P_{n}$ is

a) $\dfrac{n(n+1)}{2}$

b) $\dfrac{n^{2}-n}{2}$

c) $\dfrac{n^{2}+n-2}{2}$

d) $\dfrac{n^{2}+n+2}{2}$

Key ; D

Sol : $P_{k}=\dfrac{1-x^{k+1}}{1-x}$

$$P_{1}P_{2}P_{3}.....P_{n}=\frac{\left(1-x^{2}\right)\left(1-x^{3}\right)\left(1-x^{4}\right).....\left(1-x^{n+1}\right)}{\left(1-x^{n}\right)}$$

No. of terms = 1 + max. power of x = $1+\dfrac{n(n+1)}{2}=\dfrac{n^{2}+n+2}{2}$

29. The value of $a_{0}+a_{2}+a_{4}+....$ is

a) $\frac{2^n-1}{2}$ b) $\frac{2^n+1}{2}$

c) $\frac{(n-1)!}{2}$ d) $\frac{(n+1)!}{2}$

Key: d

Sol : $(1+x)(1+x+x^2)....(1+x+...+x^n)=a_0+a_1x+a_2x^2+a_3x^3+....$

Put, x = 1,

$2\times3\times4\times...(n-1)=a_0+a_1+a_2+a_3+....$

$0=a_0-a_1+a_2-a_3+...$

Put x = -1,

$(n+1)!=2[a_0+a_2+...]\Rightarrow a_0+a_2+....=\frac{(n+1)!}{2}$

30. The coefficient of abc^3de^2 in the expansion of $(a+b+c+d+e)^8$ is equal to

a) 3630 b) 3600

c) 3360 d) none of these

KEY : c

Sol : Coefficient of $abc^3de^2 is \frac{8!}{3!2!}=3360$

31. If each coefficient in the expansion of the expression $x(1+x)^n (n\in N)$ in powers of 'x' is divided by the exponent of corresponding power, then the sum of the values thus obtained is equal to

A) $\frac{2^n}{n+1}$ B) $\frac{2^n-1}{n+1}$ C) $\frac{2^n+1}{n+1}$ D) $\frac{2^{n+1}-1}{n+1}$

Key. D

Sol. $\frac{c_0}{1}+\frac{c_1}{2}+\frac{c_2}{3}+.....+\frac{c_{n-1}}{n}+\frac{c_n}{n+1}=\frac{2^{n+1}-1}{n+1}$

32. If $x^{15}-x^{13}+x^{11}-x^9+x^7-x^5+x^3-x=7$ then

(A) x^{16} is 15 (B) x^{16} is less than 15

(C) x^{16} greater than 15 (D) Nothing can be said about x^{16}

Key. C

Sol. $x^{15}-x^{13}+x^{11}-x^9+x^7-x^5+x^3-x$

$= x(x^8+1)(x^4+1)(x^2-1)$

$x^{16}-1=(x^8+1)(x^4+1)(x^2+1)(x^2-1)$

$= \frac{7}{x}(x^2+1)=7\left[\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2+2\right]>14$

$\therefore x^{16}>15$

33. The value of the expression $\sum_{0\le i}\sum_{<j\le n}(-1)^{i+j-1}\ {}^nC_i.{}^nC_j =$

1) ${}^{2n-1}C_n$ 2) ${}^{2n}C_n$ 3) ${}^{2n+1}C_n$ 4) none

Key: 1

Hint: Let the required value be S

$$\sum_{i=0}^{n}\sum_{j=0}^{n}(-1)^{i+j-1}\ {}^nC_i{}^nC_j=\sum_{i=0}^{n}(-1)^{2i-1}\left({}^nC_i\right)^2+2S$$

$$0 \quad =-\sum_{i=0}^{n}\left({}^nC_i\right)^2+2S$$

$$S={}^{2n-1}C_{n-1}={}^{2n-1}C_n$$

34. If $A_{(i,j)}$ be the co-efficient of $a^ib^jc^{2010-i-j}$ in the expansion of $(a+b-c)^{2010}$ then

(A) $A_{i,\,i}$ is defined for $i \le 1010$ (B) $A_{i,\,j} = A_{j,\,i}$
(C) $A_{2i,\,3i}$ is defined for $i \le 405$ (D) $A_{0,\,1} = 2000$

Key. B

Sol. Clearly, $A_{i,j}=\frac{2010!}{i!\ j!\ (2010-i-j)!}$

$A_{j,i}=\frac{2010!}{j!i!(2010-i-j)!}$

Hence, $A_{i,j}=A_{j,i}$

35. If n > 3 and $a, b \in R$, then the value of $ab-n(a-1)(b-1)+\frac{n(n-1)}{1.2}(a-2)$ $(b-2)-.....+(-1)^n(a-n)(b-n)$ is equal to

(A) a^n+b^n (B) $\frac{a^n-b^n}{a-b}$

(C) $(ab)^n$ (D) 0

Key. D

Sol. $T_{k+1}=(-1)^{k}\ .{}^nC_k(a-k)(b-k)$

$=(-1)^{k}.{}^nC_k\left[ab-k(a+b)+k^2\right]$

Thus, the sum of series in (i)

$=\sum_{k=0}^{n}(-1)^k.{}^nC_k\left[ab-k(a+b)+k^2\right]$

$= ab\sum_{k=0}^{n}(-1)^k \cdot {}^nC_k - (a+b)\sum_{k=0}^{n}(-1)^k k.{}^nC_k + \sum_{k=0}^{n}(-1)^k k^2.{}^nC_k$

We know that

$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ... + {}^nC_nx^n ...(ii)$

Differentiating both sides w.r.t. x, we get

$n(1+x)^{n-1} = 1.\ {}^nC_1 + 2.{}^nC_2x + 3.\ {}^nC_3x^2 + ... + n.{}^nC_nx^{n-1} ....(iii)$

Multiplying it by x we get

$n.x(1+x)^{n-1} = 1.{}^nC_1x + 2.\ {}^nC_2x^2 + 3.\ {}^nC_3x^3 + ... + n.{}^nC_nx_n$

Differentiating w.r.t. x, we get

$n(1+x)^{n-1} + n(n-1)x(1+x)^{n-2}$

$= 1^2.{}^nC_1 + 2^2.{}^nC_2x + ... + n^2.{}^nC_nx^{n-1}$ ...(iv)

Putting x = –1 in (ii), (iii) and (iv), we get

$\sum_{k=0}^{n}(-1)^k.{}^nC_k = 0$

$\sum_{k=0}^{n}(-1)^k k.{}^nC_k = 0$

$\sum_{k=0}^{n}(-1)^k k.^2\ {}^nC_k = 0$

Thus, the sum of the series is 0

36. The expression $\frac{1}{\sqrt{4x+1}}\left[\left[\frac{1+\sqrt{4x+1}}{2}\right]^7 - \left[\frac{1-\sqrt{4x+1}}{2}\right]^7\right]$ is a polynomial in x of degree

(A) 7 (B) 5 (C) 4 (D) 3

Key. D

Sol. put = y and expand

37. Co-efficient of α^t in the expansion of,

$(\alpha + p)^{m-1} + (\alpha + p)^{m-2}(\alpha + q) + (\alpha + p)^{m-3}(\alpha + q)^2 + ...(\alpha + q)^{m-1}$

where $\alpha \neq -q$ and $p \neq q$ is :

(A) $\frac{{}^mC_t\left(p^t - q^t\right)}{p-q}$ (B) $\frac{{}^mC_t\left(p^{m-t} - q^{m-t}\right)}{p-q}$

(C) $\frac{{}^mC_t\left(p^t + q^t\right)}{p-q}$ (D) $\frac{{}^mC_t\left(p^{m-t} + q^{m-t}\right)}{p-q}$

Key. B

Sol. E = $(\alpha + p)^{m-1}$

$\Rightarrow$ *co-efficient of* α^t = = =

38. Let $\left(5+2\sqrt{6}\right)^n = p + f$ where $n \in N$ and $p \in N$ and $0 < f < 1$ then the value of, $f^2 - f + pf - p$ is :

(A) a natural number
(B) a negative integer
(C) a prime number
(D) are irrational number

Key. B

Sol. Ans is – 1

39. The coefficient of x^4 of in the expansion $(1+5x+9x^2+......\infty)(1+x^2)^{11}$ is

(A) ${}^{11}C_2+4\,{}^{11}C_1+3$
(B) ${}^{11}C_2+3\,{}^{11}C_1+4$
(C) $3\,{}^{11}C_2+4\,{}^{11}C_1+3$
(D) 171

Key. D

40. The number of distinct terms in the expansion of $\left(x+y^2\right)^{13}+\left(x^2+y\right)^{14}$ is

A) 27
B) 29
C) 28
D) 25

Key. C

Sol. To get common terms in both the expansions

$x^{r_1}\left(y^2\right)^{13-r_1}=\left(x^2\right)^{r_2}\left(y\right)^{14-r_2}$

$r_1=2r_2$ & $26-2r_1=14-r_2$

$r_1=8;\ r_2=4$

$\therefore$ Only one term is common.

41. If $A_{(i,j)}$ be the co-efficient of $a^i b^j c^{2010-i-j}$ in the expansion of $(a+b-c)^{2010}$ then

(A) $A_{i,\,i}$ is defined for $i \le 1010$
(B) $A_{i,\,j} = A_{j,\,i}$
(C) $A_{2i,\,3i}$ is defined for $i \le 405$
(D) $A_{0,\,1} = 2000$

Key. A

Sol. Clearly, $A_{i,j}=\dfrac{2010!}{i!\,j!\left(2010-i-j\right)!}$

$A_{j,i}=\dfrac{2010!}{j!i!\left(2010-i-j\right)!}$

Hence, $A_{i,j}=A_{j,i}$

42. The value of $\dfrac{{}^{11}C_0}{1}+\dfrac{{}^{11}C_1}{2}+\dfrac{{}^{11}C_2}{3}+.......+\dfrac{{}^{11}C_{11}}{12}$ will be

A) $\frac{1}{12}\left(2^{12}-1\right)$ B) $\frac{1}{12}\left(2^{11}-1\right)$ C) $\frac{1}{12}\left(2^{11}+1\right)$ D) None of these

Key. A

Sol. Using ${}^{n}C_{k}=\frac{n}{k}\cdot{}^{n-1}C_{k-1}$

For $0\le k\le 11$

$\frac{{}^{11}C_{k}}{k+1}=\frac{{}^{12}C_{k+1}}{12}$

So, given expression is

$$\Rightarrow \frac{1}{12}\sum_{k=0}^{11}{}^{12}C_{k+1}\Rightarrow \frac{1}{12}\left[\sum_{k=0}^{11}{}^{12}C_{k}-{}^{12}C_{0}\right]\Rightarrow \frac{1}{12}\left(2^{12}-1\right)$$

43. $\sum_{m=1}^{n}\left(\sum_{k=1}^{m}\left(\sum_{p=k}^{m}{}^{n}C_{m}\cdot{}^{m}C_{p}\cdot{}^{p}C_{k}\right)\right)=$

a) $3^{n}-2^{n}$ b) $4^{n}-3^{n}$ c) $3^{n}+2^{n}$ d) $4^{n}-1$

Key: B

Hint: $\sum_{m=1}^{n}{}^{n}C_{m}\left(\sum_{k=1}^{m}\left(\sum_{p=k}^{m}\frac{m!}{p!(m-p)!}\cdot\frac{p!}{k!(p-k)!}\right)\right)$

$=\sum_{m=1}^{n}{}^{n}C_{m}\left(\sum_{k=1}^{m}\left(\sum_{p=k}^{m}{}^{m-k}C_{p-k}\right)\frac{m!}{k!(m-k)!}\right)$

$=\sum_{m=1}^{n}{}^{n}C_{m}\left(\sum_{k=1}^{m}2^{m-k}\cdot{}^{m}C_{k}\right)$

$=\sum_{m=1}^{n}{}^{n}C_{m}\left((1+2)^{m}-2^{m}\right)=\sum_{m=1}^{n}\left({}^{n}C_{m}3^{m}-{}^{n}C_{m}2^{m}\right)$

$=(1+3)^{n}-1-(1+2)^{n}+1=4^{n}-3^{n}$

44. The value of $2000_{C_2}+2000_{C_5}+2000_{C_8}+......+2000_{C_{2000}}=?$

a) $\frac{2^{1999}-1}{3}$ b) $\frac{2^{1999}+1}{3}$ c) $\frac{2^{2000}+1}{3}$ d) $\frac{2^{2000}-1}{3}$

Key: D

Hint $(1+x)^{n}=n_{C_0}+n_{C_1}x+n_{C_2}x^{2}+......+n_{C_n}x^{n}$

Put x = 1, w, w^{2} and add

$\Rightarrow C_{2}+C_{5}+C_{8}+......=\frac{1}{3}\left\{2^{n}+(-1)^{n}\left(w^{2n+1}+w^{n+2}\right)\right\}$

$=\frac{1}{3}\left\{2^{2000}+(-1)^{2000}\left(w^{4001}+w^{2002}\right)\right\}$

$= \frac{2^{2000}-1}{3}$

45. $3^{10}\ {}^{20}C_0\ {}^{20}C_{10} - 3^9\ {}^{20}C_1\ {}^{19}C_9 + ... + {}^{20}C_{10}\ {}^{10}C_0$ equals (${}^nC_r$ denote coefficient of x^r in $(1 + x)^n$.)

(A) ${}^{10}C_{10}.3^{10}$ (B) ${}^{20}C_{10}.3^{10}$

(C) ${}^{20}C_{10}.2^{10}$ (D) ${}^{20}C_{10}\ {}^{10}C_8$

Key. C

Sol. Coefficient of x^{10} in $\left[{}^{20}C_0(3+x)^{20} - {}^{20}C_1(3+x)^{19} + ... + {}^{20}C_{10}(3+x)^{10}\right]$

$= x^{10}$ in $(3 + x - 1)^{20}$

$= x^{10}$ in $(2 + x)^{20}$

$= 2^{10}\ {}^{20}C_{10}$

46. Number of ways, 3 persons having 6 one rupee coins, 7 one rupee coins, 8 one rupee coins respectively donate 10 one rupee coin collectively is ……….

a) 29 b) 83 c) 44 d) 47

Key. D

Sol. Coeff of $x^{10}\left(1+x+x^2+-----+x^6\right)\left(1+x+--+x^7\right)\left(1+x+---+x^8\right)$ is

47. $\sum_{r=1}^{n}\sum_{p=0}^{r-1} {}^nC_r.\ {}^rC_p.2^p$ is equal to

a) $4^n - 3^n + 1$ b) $4^n - 3^n - 1$

c) $4^n - 3^n + 2$ d) $4^n - 3^n$

Key. D

Sol. $\sum_{r=1}^{n} nC_r\left((1+2)^r - 2^r\right) = \sum_{r=1}^{n} nC_r 2^r - \sum_{r=1}^{n} nC_r 2^r$

$=\left(4^n-1\right)-\left(3^n-1\right) = 4^n - 3^n$

48. The value of $\binom{50}{6}-\binom{5}{1}\binom{40}{6}+\binom{5}{2}\binom{30}{6}-\binom{5}{3}\binom{20}{6}+\binom{5}{4}\binom{10}{6}$ where $\binom{n}{r}$ denotes ${}^nC_r$, is

(A) 15625 (B) 0

(C) 1000000 (D) 2250000

Key. D

Sol. ${}^{50}C_6 - {}^5C_1\ {}^{40}C_6 + {}^5C_2\ {}^{30}C_6 - {}^5C_3\ {}^{20}C_6 + {}^5C_4\ {}^{10}C_6$ = coefficient of x^6 in $[{}^5C_0 (1 + x)^{50} - {}^5C_1 (1 + x)^{40} + {}^5C_2 (1 + x)^{30} - {}^5C_3 (1 + x)^{20} + {}^5C_4 (1 + x)^{10} - {}^5C_5 (1 + x)^0]$ = coefficient x^6 in $[(1 + x)^{10} - 1]^5$
= coefficient of x^6 in $({}^{10}C_1\ x + {}^{10}C_2\ x^2 + ...)^5 = {}^5C_1 ({}^{10}C_2) ({}^{10}C_1)^4 = 2250000$.

49. The value of the expression ${}^{10}C_0 10^9 - {}^{10}C_1 9^9 + {}^{10}C_2 8^9 .... - {}^{10}C_9\ is$

a) $|\underline{9}$ b) $|\underline{10}$ c) $9|\underline{10}$ d) 0

Key. D

Sol. Given expression = No.of on to functions from a set of 9 elements to a set of 10 elements = 0

50. The term independent of x and y in the expansion of

$[(\sqrt{x}+\frac{1}{\sqrt{x}})^2+(\sqrt{y}+\frac{1}{\sqrt{y}})^2+(\sqrt{xy}+\frac{1}{\sqrt{xy}})^2-4]^n$ is

a) $(\sum_{r=0}^{n}{}^nc_r)^2$ b) $\sum_{r=0}^{n}({}^nc_r)^2$ c) $(\sum_{r=0}^{n}{}^nc_r)^3$ d) $\sum_{r=0}^{n}({}^nc_r)^3$

Key. D

Sol. The given expression can be written as $(1+x)^n.(1+y)^n.(1+\frac{1}{xy})^n$. The constant term is clearly $C_0^3+C_1^3+-----+C_n^3$ where $c_r={}^nc_r$.

51. The sum of the rational terms in the expansion of $\left(\sqrt{2}+\sqrt[5]{3}\right)^{10}$ is :

a) 41 b) 42 c) 39 d) 45

Key. A

Sol. $T_{r+1}=10C_r\frac{10-r}{2^2}.3^{\frac{r}{5}}$

This is rational , if $\frac{10-r}{2}$ and $\frac{r}{5}$ are integers.

$\therefore$ There are only two rational terms

Namely $10C_0\left(\sqrt{2}\right)^{10}\left(3^{\frac{1}{5}}\right)^0$ and $10C_{10}\left(\sqrt{2}\right)^0\left(3^{\frac{1}{5}}\right)^{10}$

$\therefore$ sum = 32 + 9 = 41

52. The values of 'r' such that $(100)C_r\left(\frac{1}{5^8}\right)^{100-r}\left(\frac{1}{2^6}\right)^r$ is rational is :

a) 84 b) 85 c) 86 d) 42

Key. A

Sol. Direct verification in sufficient.

53. If $\left(1+x\right)^n=a_0+a_1x+a_2x^2+.....+a_nx^n$ in which a_{n-3},a_{n-2},a_{n-1} are in AP, then

a) a_1,a_2,a_3 are in GP b) a_1,a_2,a_3 are in HP c) $n=7$ d) $n=14$

Key. C

Sol. $a_{n-3}=a_3,a_{n-2}=a_2,a_{n-1}=a_1\left(\because nC_r=nC_{n-r}\right)$

$\therefore$ (A) is correct.

a_1,a_2,a_3 are in AP $\Rightarrow n,\frac{n\left(n-1\right)}{2},\frac{n\left(n-1\right)\left(n-2\right)}{6}$ are in AP.

$$\frac{n+\frac{n\left(n-1\right)\left(n-2\right)}{6}}{2}=\frac{n\left(n-1\right)}{2}$$

$6n+n^3-3n^2+2n=6n^2-6n$

$n^3-9n^2+14n=0$, $\left(n-7\right)\left(n-2\right)=0$

$\therefore$ n = 7.

54. The coefficient of $a^8b^6c^4$ in the expansion of $\left(a+b+c\right)^{18}$ is :

a) $18C_4\times14C_6$ b) $18C_{10}\times10C_8$

c) $18C_8 \times 12C_8$ d) $18C_{14} \times 14C_7$

Key. A

Sol. $(a+b+c)^{18} = [a+(b+c)]^{18}$

$= 18C_0 a^{18} + 18C_1 a^{17}(b+0^{71} + 18C_2 a^{16}(b+c)^2 + ... + 18C_{10} a^8 (b+c)^{10} + ...$

$\therefore$ coefficient of $a^8 b^6 c^4 = 18C_{10} \times 10C_6$

55. In the expansion of $(1 + x)(1 + x + x^2) ..... (1 + x + x^2 + ... + x^{2n})$, the sum of coefficients is

(A) 1 (B) (2n)!
(C) (2n)! + 1 (D) (2n + 1)!

Key. D

Sol. Let $(1 + x)(1 + x + x^2) ..... (1 + x + ... + x^{2n}) = a_0 + a_1x + a_2x^2 + ...$

Put x = 1, we get

$a_0 + a_1 + a_2 + ... = 2.3.4 ..... (2n + 1) = (2n + 1)!$

56. If x = $\left(\sqrt{3}+\sqrt{2}\right)^6$, then the largest integer not exceeding x is

(A) 729 (B) 971
(C) 969 (D) 970

Key. C

Sol. $x = (5 + 2\sqrt{6})^3$

Let x = k + f $k \in N$, 0 < f < 1

Consider: $5 - 2\sqrt{6} = \dfrac{1}{5+2\sqrt{6}} \Rightarrow 0 < 5 - 2\sqrt{6} < 1 \Rightarrow 0 < (5 - 2\sqrt{6})^n < 1$

Let $F = (5 - 2\sqrt{6})^n$, 0 < F < 1

$k + f + F = (5 + 2\sqrt{6})^3 + (5 - 2\sqrt{6})^3 = 2[5^3 + {}^3C_2\, 5. (2\sqrt{6})^2]$

= 2[125 + 360] = 970.

$\Rightarrow$ f + F is a integer, but 0 < f + F < 2

$\Rightarrow$ f + F = 1

$\Rightarrow$ k = 969.

57. The coefficient of x^{53} in the expansion of $\sum_{m=0}^{100} {}^{100}C_m (x-3)^{100-m} 2^m$ is

A) ${}^{100}C_{47}$ B) ${}^{100}C_{53}$ C) $-{}^{100}C_{53}$ D) $-{}^{100}C_{100}$

Key. C

Sol. $(x-3)^{100} + {}^{100}C_1 (x-3)^{99} 2^1 + ... + {}^{100}C_{100} 2^{100}$

$= \{(x-3)+2\}^{100} = (x-1)^{100} = (1-x)^{100} = {}^{100}C_{53}(-1)^{53} = -{}^{100}C_{53}$

58. If 7 divides $32^{32^{32}}$, the reminder is

A) 1 B) 0 C) 4 D) 81

Key. C

Sol. $32^{32^{32}} = 3m+1, m \in I^+$

$32^{32^{32}} = (2^5)^{3m+1} = 2^{15m+5} = 28n + 5$

59. The number of irrational terms in the expansion of $\left(5^{\frac{1}{3}}+2^{\frac{1}{4}}\right)^{100}$ is

A) 94 B) 92 C) 93 D) 91

Key. B

Sol. $\left(5^{\frac{1}{3}}+2^{\frac{1}{4}}\right)^{100}=\left(2^{\frac{1}{4}}+5^{\frac{1}{3}}\right)^{100}$

$T_{r+1}={}^{100}C_r\left(2^{\frac{1}{4}}\right)^{100-r}.\left(5^{\frac{1}{3}}\right)^{r}={}^{100}C_r\,2^{25-\frac{r}{4}}.5^{\frac{r}{3}}$

For rational terms, 'r' should be divisible by 12.

$\therefore$ No. of rational terms = 9

$\therefore$ No. of irrational terms = 101-9=92

60. The greatest integer less than or equal to $(\sqrt{3}+1)^6$ is

A) 416 B) 414 C) 417 D) 415

Key. D

Sol. $(\sqrt{3}+1)^6+(\sqrt{3}-1)^6=2\left[{}^6C_0(\sqrt{3})^6+{}^6C_2(\sqrt{3})^4+{}^6C_4(\sqrt{3})^2+{}^6C_6\right]=416$

Let $(\sqrt{3}+1)^6=I+F$ where I is integral part and F is fractional part

Let $(\sqrt{3}-1)^6=G$

$0<F<1;0<G<1\Rightarrow 0<F+G<2\Rightarrow F+G=1$

$I+F+G=416\Rightarrow I+1=416\Rightarrow I=415$

61. $2\,{}^{10}C_0+\frac{2^2}{2}\,{}^{10}C_1+\frac{2^3}{3}\,{}^{10}C_2+......+\frac{2^{11}}{11}\,{}^{10}C_{10}=$

A) $2^{11}-1/11$ B) $3^{11}-1/11$ C) $2^{11}-2/11$ D) $4^{11}-1/11$

Key. B

Sol. Conceptual

62. If the fourth term in the expansion of $\left(2+\frac{3}{8}x\right)^{10}$ has the maximum numerical value then the range of x contains.

(a) $-\frac{64}{21}<x<-2$ (b) $1<x<\frac{64}{21}$ (c) $-2<x<\frac{64}{21}$ (d) $-\frac{64}{21}<x<2$

Key. A

Sol. Conceptual

63. If n is an odd natural number then $\sum_{r=0}^{n}\frac{(-1)^r}{n_{C_r}}$ is equal to

(a) 0 (b) 1 / n (c) $\frac{n}{2^n}$ (d) $n.2^n$

Key. A

Sol. Conceptual

64. If $(1+x)^n=\sum_{r=0}^{n}n_{c_r}x^r$ then $C_0^{\ 2}+\frac{C_1^{\ 2}}{2}+\frac{C_2^{\ 2}}{3}+.......+\frac{C_n^{\ 2}}{n+1}=$

(a) $\frac{\lfloor 2n}{(\lfloor n)^2}$ (b) $\frac{\lfloor 2n+1}{(\lfloor n+1)^2}$ (c) $\frac{\lfloor 2n-1}{(\lfloor n-1)^2}$ (d) $\frac{\lfloor n}{(\lfloor n-1)^2}$

Key. B

Sol. Conceptual

65. In the expansion of $\left(\sqrt[3]{\frac{a}{b}}+\sqrt[3]{\frac{b}{\sqrt{a}}}\right)^{21}$ the term containing same powers of a and b is

(a) 11th (b) 13th (c) 12th (d) 6th

Key. B

Sol. $T_{r+1}=21_{C_r}a^{\frac{42-3r}{6}}b^{\frac{2r-21}{3}}$

$42-3r=4r-42 \Rightarrow r=12$

66. The coefficient of x^4 of the expansion $(1+5x+9x^2+.......\infty)(1+x^2)^{11}$ is

(a) $11_{C_2}+4.11_{C_1}+3$ (b) $11_{C_2}+3.11_{C_1}+4$ (c) $3.11_{C_2}+4.11_{C_1}+3$ (d) 171

Key. D

Sol. Co-efficient of x^4

= $(1+5x+9x^2+13x^3+17x^4+......)(1+11x^2+11_{C_2}x^4+.....)$

$=11_{C_2}+99+17$

67. The coefficient of x^{n-2} in the polynomial $(x-1)(x-2)(x-3).....(x-n)$ is

(a) $\frac{n(n^2+2)(3n+1)}{24}$ (b) $\frac{n(n^2-1)(3n+2)}{24}$ (c) $\frac{n(n^2+1)(3n+4)}{24}$ (d) None

Key. B

Sol. $T_2=(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)......(x-\alpha_n)$

$=x^n-\sum\alpha_i x^{n-1}+\alpha_1\alpha_2 x^{n-2}+...............$

68. If $1,\propto_1,\propto_2,\propto_3,\propto_4$ are the fifth roots of unity then $\sum_{i=1}^{4}\frac{1}{2-\propto_i}=$

(a) $\frac{51}{31}$ (b) $\frac{49}{31}$ (c) $\frac{25}{32}$ (d) $\frac{25}{16}$

Key. B

Sol. we know that $z^5-1=(z-1)(z-\alpha_1)(z-\alpha_2)(z-\alpha_3)(z-\alpha_4)$

Take log on both sides, diff.w.r.t. z and put z = 2.

69 If a_1 and a_2 be the coefficients of x^n in the expansion of $(1+x)^{2n}$ and $(1+x)^{2n-1}$ respectively, then

1. $a_2=2a_1$ **2.** $a_1=2a_2$ **3.** $a_1=a_2$ **4.** None of these

Key. **2**

Sol. Consider T_{r+1} in $(1+x)^{2n}$ $\therefore$ $T_{r+1}={}^{2n}C_r x^r$

$a_1=$ Coefficient of $x^n={}^{2n}C_n$

$$=\frac{2n!}{n!n!}=\frac{2n(2n-1)!}{n(n-1)!(n)!}=\frac{2(2n-1)!}{(n-1)!(n!)}$$

Again coefficient of T_{r+1} in $(1+x)^{2n-1}$ is ${}^{2n-1}C_r$

$a^2=$ Coefficient of x^n in $(1+x)^{2n-1}$

$$={}^{2n-1}C_n=\frac{(2n-1)!}{n!(n-1)!}$$

$$a_2=\frac{1}{2}\frac{2(2n-1)!}{(n-1)!n!}=\frac{1}{2}a_1$$

$\therefore\ 2a_2=a_1$

70. If $C_0, C_1, C_2.....$ are binomial coefficients in the expansion $\sum_{r=0}^{n} C_r x^r$, then value of the expression (series)

$$\frac{2C_0}{1}+\frac{3C_1}{2}+\frac{4C_2}{3}+\frac{5C_3}{4}+......+\text{ is}$$

1. $\frac{2^n+1}{n+1}$ **2.** $\frac{2^n-1}{n+1}$ **3.** $\frac{2^n(n+3)-1}{n+1}$ **4.** $\frac{2^n(n+2)-1}{n+1}$

Key. **3**

Sol. Given

$$(1+x)^n=C_0+C_1x+c_2x^2+.....+C_nx^n$$

Integrating both sides with respect to x, we get

$$\frac{(1+x)^{n+1}}{n+1}=\frac{C_0x}{1}+\frac{C_1x^2}{2}+\frac{C_2x^3}{3}+.....+\frac{C_nx^{n+1}}{n+1}+k$$

Putting $x=0$,

We get $k=\frac{1}{n+1}$

$$\therefore\quad \frac{(1+x)^{n+1}-1}{n+1}$$

$$= \frac{C_0 x}{1} + \frac{C_1 x^2}{2} + \frac{C_2 x^3}{3} + ... + \frac{C_n x^{n+1}}{n+1}$$

Multiplying with x both sides

$$\frac{x(1+x)^{n+1} - x}{n+1} = \frac{C_0 x^2}{1} + \frac{C_1 x^3}{2} + \frac{C_2 x^4}{3} + ... + \frac{C_n x^{n+1}}{n+1}$$

Differentiating with respect to x

$$\frac{(n+1)x(1+x)^n + (1+x)^{n+1} - 1}{n+1}$$

$$= \frac{2C_0 x}{1} + \frac{3C_1 x^2}{2} + \frac{4C_2 x^3}{3} + ... + \frac{(n+2)C_n x^{n+1}}{n+1}$$

Now putting $x = 1$ both sides, we get

$$\frac{2^{n+1} + (n+1)2^n - 1}{n+1}$$

$$= \frac{2C_0}{1} + \frac{3C_1}{2} + \frac{4C_2}{3} + ... + \frac{(n+2)C_n}{n+1}$$

$$\frac{2^n(n+3) - 1}{n+1} = \frac{2C_0}{1} + \frac{3C_1}{2} + \frac{4C_2}{3} + ... + \frac{(n+2)C_n}{n+1}$$

71. In the expansion of $(1+x)^{70}$,the sum of coefficients of odd powers of x is

1. 0 **2.** 2^{69} **3.** 2^{70} **4.** 2^{71}

Key. **2**

Sol. Fact. The sum of the coefficients of odd powers in the expansion of $(1+x)^n$ = sum of the coefficients of even powers in $(1+x)^n$

$= 2^{n-1}$

$2^{70-1} = 2^{69}$

72. Number of irrational terms in the expansion of $\left(\sqrt[5]{2}+\sqrt[10]{3}\right)^{60}$ are

1. 54 **2.** 61 **3.** 30 **4.** 31

Key. **1**

Sol. Given $\left(\sqrt[5]{2}+\sqrt[10]{3}\right)^{60}$ = $\left(2^{\frac{1}{5}}+3^{\frac{1}{10}}\right)^{60}$

Now L.C.M. of 5 and 10 is 10

$\therefore$ Number of rational terms let us writes $T_{r+1}={}^{60}C_r\left(2^{\frac{1}{5}}\right)^{60-r}\left(3^{\frac{1}{10}}\right)^{r}$

$={}^{60}C_r\,2^{12-\frac{r}{5}}3^{\frac{r}{10}}$

As $0\le r\le 60$

$\therefore$ $r=0,10,20,30,40,50,60$

$\therefore$ Number of rational terms is 7

$\therefore$ Number of irrational terms equals to

Total number of terms - Number of rational terms

$=61-7=54$

73. If $C_0,C_1,C_2,....,C_n$ are Binomial Coefficients, such that $S_n=\sum_{r=0}^{n}\frac{1}{{}^nC_r}$ and $t_n=\sum_{r=0}^{n}\frac{r}{{}^nC_r}$ then $\frac{t_n}{S_n}$ equals

1. $\frac{n}{2}$ **2.** $\frac{n(n+1)}{2}$ **3.** $\frac{n+1}{2}$ **4.** None of these

Key. **1**

Sol. Given $t_n=\sum_{r=o}^{n}\frac{r}{C_r}=\sum_{r=0}^{n}\frac{n-(n-r)}{C_{n-r}}$ $\left(\because {}^nC_r={}^nC_{n-r}\right)$

$$=\sum_{r=0}^{n}\frac{n}{C_{n-r}}-\sum_{r=0}^{n}\frac{n-r}{C_{n-r}}$$

$$=nS_n-\left[\frac{n}{C_n}+\frac{n-1}{C_{n-1}}+....+\frac{1}{C_1}+0\right]$$

$$t_n=nS_n-\sum_{r=0}^{n}\frac{r}{C_r}$$

$$t_n=nS_n-t_n$$

$$2t_n=nS_n$$

$$\therefore\ \frac{t_n}{S_n}=\frac{n}{2}$$

74. If $(1+x)^n=\sum_{r=0}^{n}C_rx^r$, then the value of $3C_1+7C_2+\ 11C_3+....+(4n-1)C_n$ is

1. $(4n-1)2^n$ **2.** $(2n-1)2^n$ **3.** $(2n-1)2^n+1$ **4.** $(4n-1)2^n-1$

Key. 3

Sol. Let $S=3C_1+7C_2+11C_3+....+(4n-1)C_n$

Let us write

$$T_r=(4r-1)C_r$$

$$T_r=4rC_r-C_r$$

$=4r\frac{n}{r}C_{r-1}-C_r$ using ${}^nC_r=\frac{n}{r}.{}^{n-1}C_{r-1}$ \

$$\therefore\quad \sum_{r=1}^{n}T_r=4n\sum_{r=1}^{n}C_{r-1}-\sum_{r=1}^{n}C_r$$

$$=4\ n\,.\,2^{n-1}\left(2^n-1\right)$$

$$=2n.2^n-2^n+1$$

$S=2^n(2n-1)+1$ By using

$(1+x)^{n-1} = C_0 + C_1x + C_2x^2 + ... + C_{n-1}x^{n-1}$

$2^{n-1} = C_0 + C_1 + C_2 + ... + C_{n-1}$

75. Middle term in the expansion of $(1-3x+3x^2-x^3)^{2n}$ is

1. $\frac{(6n)!x^n}{(3n)!(3n)!}$ **2.** $\frac{(6n)!.x^{3n}}{(3n)!}$ **3.** $\frac{(6n)!}{(3n)!(3n)!}(-x)^{3n}$ **4.** None of these

Key. 3

Sol. $(1-3x+3x^2-x^3)^{2n} = (1-x)^{6n}$

Concept: Index $= 6n$ which is even so most middle term

is $\left(\frac{6n}{2}+1\right)^{th}$ $i.e., (3n+1)^{th}$ term is middle term,

$T_{3n+1} = {}^{6n}C_{3n}(-x)^{3n} = \frac{6n!}{3n!3n!}(-x)^{3n}$

76. Let n be an odd natural number greater than 1. Then the number of zeros at the end of the sum $99^n + 1$ is

(A) 3 (B) 4 (C) 2 (D) None of these

Key. C

Sol. $1+99^n = 1+(100-1)^n = 1+$
$\{{}^nC_0 100^n - {}^nC_1.100^{n-1} + ... - {}^nC_n\}$
Because n is odd $= 100\{{}^nC_0.100^{n-1} - {}^nC_1.100^{n-2} + ... -$
${}^nC_{n-2}.100 + {}^nC_{n-1}\}$
= 100 $\times$ integer whose units place is different from 0
$[\because {}^nC_{n-1} = n, has\ odd\ digit\ at\ unit\ place]$
$\therefore$ There are two zeros at the end of the sum $99^n + 1$

77. If $C_0, C_1, C_2, ...C_n$ are the Binomial coefficients in the expansion $(1+x)^n$. 'n' being even, then $C_0 + (C_0+C_1) + (C_0+C_1+C_2) + ...(C_0+C_1+C_2+....+C_{n-1}) =$ is equal to

(A) $n2^n$ (B) $n.2^{n-1}$ (C) $n.2^{n-3}$ (D) $n.2^{n-2}$

Key. B

Sol. Sum $= \{C_0 + (c_1+c_2+......+c_{n-1})\} + = \{(c_0+c_1)+(c_0+c_1+.......+c_{n-2})\} +$

$\left\{\left(c_0+c_1+c_2\right)+\left(c_0+c_1+......+c_{n-3}\right)\right\}+.......\ to\ \left(\frac{n}{2}\right)$

Terms = $=\left(c_0+c_1+......c_n\right)\times\frac{n}{2}=n.2^{n-1}$

78. The positive integral values of n such that
$1.2^1+2.2^2+3.2^3+4.2^4+5.2^5+....+n.2^n=2^{(n+10)}+2$ is

(A) 313 (B) 513 (C) 413 (D) 613

Key. B

Sol.

$2^1+2^2+2^3+...+2^n=2^{n+1}-2$

$2^2+2^3+...2^n=2^{n+1}-2^2$

$2^3+...+2^n=2^{n+1}-2^3$

... ...

$+2^n=2^{n+1}-2$

$=n\left(2^{n+1}\right)-\left(2^{n+1}-2\right)$

$=2^{n+1}\left(n-1\right)+2$

Given that $2^{n+1}\left(n-1\right)+2=2^{2+10}+2$

$\Rightarrow\left(n-1\right)2^{n+1}=2^{n+10}$

$\Rightarrow n-1=2^9$

$\Rightarrow n=2^9+1=513$

79. If $\sum_{r=0}^{n}\left\{\frac{{}^nC_{r-1}}{{}^nC_r+{}^nC_{r-1}}\right\}^3=\frac{25}{24}$, then n is equal to

(A) 3 (B) 4 (C) 5 (D) 6

Key. C

Sol. Let $t_r=\frac{{}^nC_{r-1}}{{}^nC_r+{}^nC_{r-1}}=\frac{{}^nC_{r-1}}{{}^{n+1}C_r}=\frac{{}^nC_{r-1}}{\frac{n+1}{r}\ {}^nC_{r-1}}$

$\therefore l_r=\frac{r}{n+1}$

Now,

$S=\sum_{r=0}^{n}\left\{l_r\right\}^3\Rightarrow S=\sum_{r=0}^{n}\frac{r^3}{\left(n+1\right)^3}=\frac{1}{\left(n+1\right)^3}\sum_{r=0}^{n}r^3$

$\Rightarrow S = \frac{1}{(n+1)^3}\left\{\frac{n(n+1)}{2}\right\}^2 \Rightarrow S = \frac{n^2}{4(n+1)}$

Now, $S = \frac{25}{24}$ (given) which is only possible for 5

80. If m and n are any two odd positive integers with n < m, then the largest positive integer which divides all numbers of the form, $m^2 - n^2$ can be

(A) 4 (B) 6 (C) 8 (D) 9

Key. C

Sol. Let $m = 2k-1$ *and* $n = 2p-1, p < k$

Then $m^2 - n^2 = (m+n)(m-n)$

$= (2k+2p-2)(2k-2p) = 4(k+p-1)(k-p)$

Further if k and p both even, then k-p is even but k+p-1 is odd

If k and p both odd then k-p is even but k+p-1 is odd. If one is even and other odd then k-p is odd but k+p-1 is even. Thus in every case $(k-p)(k+p-1)$ even

$\therefore m^2 - n^2$ is divisible by $4 \times 2 = 8$. Hence , $m^2 - n^2$ is divisible by 8 or any multiple of 8. The largest integer among the given options is 8

81. If $C_0, C_1, C_2, ..., C_n$ denote the binomial coefficients in the expansion of $(1+x)^n$, then $\sum_{r=0}^{n} (-1)^r \, {}^nC_r \frac{1 + r\log_e 10}{(1+\log_e 10^n)^r}$ is equal to

A) 0 B) 1 C) 2 D) 3

Key. A

Sol. Let $\log_e 10 = x$

$= \sum_{r=0}^{n} (-1)^r \, {}^nC_r \frac{1+rx}{(1+nx)^r}$

$= \left(1 - \frac{1}{1+nx}\right)^n - \frac{nx}{1+nx}\left(1 - \frac{1}{1+nx}\right)^{n-1} = 0$

82. In the expansion of $\left(\sqrt[3]{\frac{a}{b}} + \sqrt[3]{\frac{b}{\sqrt{a}}}\right)^{21}$ the term containing same powers of a and b is

A) 11th B) 13th C) 12th D) 6th

Key. B

Sol. $t_{r+1} = {}^{21}C_r \left(\frac{a}{b}\right)^{\frac{21-r}{3}} \frac{b^{r/3}}{a^{r/6}} = {}^{21}C_r \, a^{\frac{42-3r}{6}} \, b^{\frac{2r-21}{3}}$

$\therefore$ 42 – 3r = 4r – 42 i.e. r = 12

$\therefore$ 13th term contains same powers of a and b

83. The coefficient of x^4 of in the expansion $(1+5x+9x^2+....)(1+x^2)^{11}$ is

A) ${}^{11}C_2 + 4\,{}^{11}C_1 + 3$ B) ${}^{11}C_2 + 3\,{}^{11}C_1 + 4$ C) $3\,{}^{11}C_2 + 4\,{}^{11}C_1 + 3$ D) 171

Key. D

Sol. Coefficient of x^4 is $\left(1+5x+9x^2+...\right)\left(1+x^2\right)^{11}$

$=\left(1+5x+9x^2+...\right)\left(1+11x^2+{}^{11}C_2\left(x^2\right)^2+...\right)$

$=\left(1+5x+9x^2+13x^3+17x^4+...\right)\left(1+11x^2+{}^{11}C_2x^4+...\right)$

Coefficient of x^4 is ${}^{11}C_2+9+11+17=55+99+17=171$

84. If $\left(1+x+x^2\right)^{25}=a_0+a_1x+a_2x^2+...+a_{50}x^{50}$, then $a_0+a_2+a_4+...+a_{50}$ is

A) even
B) odd and of the form 3n
C) odd and of the form (3n – 1)
D) odd and of the form (3n + 1)

Key. A

Sol. putting x = 1 and – 1 and adding

$a_0+a_2+...+a_{50}=\frac{3^{25}+1}{2}=\frac{(1+2)^{25}+1}{2}$

$=\frac{{}^{25}C_0+{}^{25}C_1 2+{}^{25}C_2.2^2+...+{}^{25}C_{25}.2^{25}+1}{2}$

$=\frac{2\left[1+{}^{25}C_1+{}^{25}C_2.2+...+{}^{25}C_{25}.2^{24}\right]}{2}=2\left[13+{}^{25}C_2+...+{}^{25}C_{25}.2^{23}\right]$ is an even integer

85. The co-efficient of x^{n-2} in the polynomial (x – 1) (x – 2) (x – 3) … (x – n) is

A) $\frac{n\left(n^2+2\right)(3n+1)}{24}$
B) $\frac{n\left(n^2-1\right)(3n+2)}{24}$
C) $\frac{n\left(n^2+1\right)(3n+4)}{24}$
D) none of these

Key. B

Sol. $E=\left(x-\alpha_1\right)\left(x-\alpha_2\right)\left(x-\alpha_3\right)...\left(x-\alpha_n\right)$ wher $\alpha_1=1,\ \alpha_2=2$ etc

$=x^n-\left(\sum\alpha_1\right)x^{n-1}+\left(\sum\alpha_1\alpha_2\right)x^{n-2}+...$

Hence co-efficient of x^{n-2} = sum of all the products of the first 'n' natural numbers taken two at a time $=\frac{(1+2+3+...+n)^2-\left(1^2+2^2+...+n^2\right)}{2}=\frac{n\left(n^2-1\right)(3n+2)}{24}$

86. The remainder when 27^{40} is divided by 12 is

A) 3 B) 7 C) 9 D) 11

Key. C

Sol. $27^{40}=3^{120}$

$3^{119}=(4-1)^{119}={}^{119}C_0 4^{119}-{}^{119}C_1 4^{118}+{}^{119}C_2 4^{117}-{}^{119}C_3 4^{116}+...+{}^{119}C_{118}4-1$

$\therefore$ $3^{119}=4k-1$

$\therefore$ $3^{120}=12k-3=12(k-1)+9$

$\therefore$ The required remainder is 9

87. If $\sum_{r=0}^{2n}a_r(x-1)^r=\sum_{r=0}^{2n}b_r(x-2)^r$ and $b_r=(-1)^{r-n}$ for all $r\ge n$, then $a_n=$

A) ${}^{2n+1}C_{n-1}$ B) ${}^{3n}C_n$ C) ${}^{2n+1}C_n$ D) 0

Key. C

Sol. Let x – 1 = t, then

$$\sum_{r=0}^{2n} a_r t^r = \sum_{r=0}^{2n} b_r (t-1)^r$$

$\therefore \quad a_n =$ coefficient of t^n in $\sum_{r=0}^{2n} b_r (t-1)^r$

= coefficient of t^n b in $\left(b_0 + b_1(t-1) + ... + b_n(t-1)^n + b_{n+1}(t-1)^{n+1} + ... + b_{2n}(t-1)^{2n}\right)$

$= b_n\, {}^nC_0 + b_{n+1}\, {}^{n+1}C_1(-1)^1 + b_{n+2}\, {}^{n+2}C_2(-1)^2 + ... + b_{2n}\, {}^{2n}C_n(-1)^n$

$= (-1)^{n-n}.{}^nC_0 + (-1)^{n+1-n+1}, {}^{n+1}C_1 + ... + (-1)^{2n-n+n}\, {}^{2n}C_n$

$= {}^nC_0 + {}^{n+1}C_1 + {}^{n+2}C_2 + ... + {}^{2n}C_2 + ... + {}^{2n}C_n = {}^{2n}C_n = {}^{2n+1}C_{n+1}$

$= {}^{2n+1}C_n$

88. In the expansion of $\left(7^{\frac{1}{3}} + 11^{\frac{1}{9}}\right)^{6561}$ total number of terms free from radical signs is

A) 729 B) 730 C) 731 D) none of these

Key. B

Sol. $T_{r+1} = {}^{6561}C_r\, 7^{\frac{6561-r}{3}}\, 11^{r/9}$

The term is free from radical sign, if r is multiple of 9 and 6561 – r is a multiply of 3

i.e. r = 0, 9, 18, 27, … 6561. These are 730 in number,

89. The last two digits of the number $(23)^{14}$ are

A) 01 B) 03 C) 09 D) None of these

Key. C

Sol. $(23)^{14} = (529)^7 = (530-1)^7$

$= {}^7C_0(530)^7 - {}^7C_1(530)^6 + ... - 7C_5(530)^2 + {}^7C_6 530 - 1$

$= {}^7C_0(530)^7 - {}^7C_1(530)^6 + ... + 3710 - 1$

= 100 m + 3709

$\therefore$ last two digits are 09

90. $\sum\sum_{0 \le i < j \le n} \left(C_i + C_j\right)^2 =$ ____

a) $(n-1).{}^{2n}C_n + 2^{2n}$ b) ${}^{2n}C_n + 2^{2n}$ c) ${}^{2n}C_n - (n+1)2^n$ d) None

Key. A

Sol. $\sum\sum\left(C_i + C_j\right)^2 = \sum\sum C_i^2 + C_j^2 + 2C_iC_j = n\left(C_0^2 + C_1^2 + ... + C_n^2\right) + 2\sum\sum C_iC_j$ is

$$= n.{}^{2n}C_n + 2\left(\frac{2^{2n} - {}^{2n}C_n}{2}\right)$$

Since $\left(C_0 + C_1 + ... + C_n\right)^2 = C_0^2 + C_1^2 + ... + C_n^2 + 2\sum\sum C_iC_j$

$2^{2n} = {}^{2n}C_n + 2\sum\sum C_iC_j$

91. The value of $\frac{1}{\lfloor 1 \lfloor 15} + \frac{1}{\lfloor 3 \lfloor 13} + \frac{1}{\lfloor 5 \lfloor 11} + \frac{1}{\lfloor 7 \lfloor 9}$ is

a) $\frac{2^{14}}{\lfloor 15}$ b) $\frac{2^{15}}{\lfloor 16}$ c) $\frac{2^{10}}{\lfloor 15}$ d) $\frac{2^{13}}{\lfloor 15}$

Key. C

Sol. Multiply and divide by 16!

92. The term independent of x and y in the expansion of

$$\left[\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2+\left(\sqrt{y}+\frac{1}{\sqrt{y}}\right)^2+\left(\sqrt{xy}+\frac{1}{\sqrt{xy}}\right)^2-4\right]^n \text{ is}$$

a) $\left(\sum_{r=0}^{n} {}^nC_r\right)^2$ b) $\sum_{r=0}^{n}\left({}^nC_r\right)^2$ c) $\left(\sum_{r=0}^{n} {}^nC_r\right)^3$ d) $\sum_{r=0}^{n}\left({}^nC_r\right)^3$

Key. D

Sol. It can be simplified as $(1+x)^n(1+y)^n\left(1+\frac{1}{xy}\right)^n$

The constant term is $C_0^3+C_1^3+.....+C_n^3$

93. If $C_r={}^nC_r$, then $(C_0-C_2+C_4-C_6+...)^2+(C_1-C_3+C_5-C_7+...)^2$ is

a) 2^{2n} b) 2^n c) 2^{n^2} d) $2^{\frac{n+1}{2}}$

Key. B

Sol. Put $x=i$ in the expansion of $(1+x)^n$ we get

$(C_0-C_2+C_4-C_6.....)+i(C_1-C_3+C_5....)=2^{n/2}\left[\cos n\pi/4+i\sin n\pi/4\right]$ Take modulus both sides and square it

94. The coefficient of x^3 in $\left(2x-\frac{3}{x^2}\right)^9$ is

a) 41472 b) $2^8.3^5$ c) $2^8.3^4$ d) 44172

Key. A

Sol. r = 2 , coefficient $={}^9C_2\,(2)^7(-3)^2=(2)^9(3)^4$

95. If the coefficient of x in $\left(x^2+\frac{k}{x}\right)^5$ is 270, then the value of k is

a) 2 b) 3 c) 4 d) 5

Key. B

Sol. r = 3 , ${}^5C_3\,k^3=270$,k = 3

96. In the expansion of $\left(2+\frac{x}{3}\right)^n$, coefficient of x^7 and x^8 are equal. Then the value of n is

a) 49 b) 50 c) 55 d) 56

Key. C

Sol. ${}^nC_7\,\frac{2^{n-7}}{3^7}={}^nC_8\,\frac{2^{n-8}}{3^8}$, n = 55.

97. Value of the numerically greatest term in the expansion of $\left(1+\frac{1}{\sqrt{3}}\right)^{20}$ is

a) ${}^{20}C_7 \frac{1}{27}$ b) ${}^{20}C_8 \frac{1}{27}$

c) ${}^{20}C_8 \frac{1}{27\sqrt{3}}$ d) ${}^{20}C_7 \frac{1}{27\sqrt{3}}$

Key. D

Sol. $\frac{21}{\sqrt{3}+1}=7g......, T_{7+1}= {}^{20}C_7 \frac{1}{27\sqrt{3}}$.

98. The coefficients of 9th, 10th and 11th terms in the expansion of $(1+x)^n$ are in A.P. then the Value of n is

a) 23 b) 7 c) 4 d) 21

Key. A

Sol. r = 9, $n^2-(4\times 9+1)n+4\times 9^2-2=0$, n = 14 or 23.

99. The number of terms in the expansion of $(p+q+r+s)^n$ is

a) $\frac{n(n+1)}{2}$ b) $\frac{(n+1)(n+2)(n+3)}{6}$

c) $\frac{n(n+1)(2n+1)}{6}$ d) $\frac{(n+1)^2 n^2}{4}$

Key. B

Sol. Standard formulae.

100. The number of irrational terms in the expansion of $\left(\sqrt{3}+\sqrt[4]{5}\right)^{124}$ is

a) 125 b) 32 c) 93 d) 34

Key. C

Sol. No of rational terms are $\left[\frac{124}{4}\right]+1$ = 32,

101. If a_1, a_2, a_3, a_4 are any 4 consecutive binomial coefficients of the expansion of $(1+x)^n$

Respectively, then $\frac{a_1}{a_1+a_2}, \frac{a_2}{a_2+a_3}, \frac{a_3}{a_3+a_4}$ are in

a) A.G.P b) G.P c) H.P d) A.P

Key. D

Sol. $\frac{1}{1+3}, \frac{3}{3+3}, \frac{3}{3+1} = \frac{1}{4}, \frac{3}{6}, \frac{3}{4} = \frac{3}{12}, \frac{6}{12}, \frac{9}{12}$, A.P.

102. If the sum of the coefficients in the expansion $(x - 2y+3z)^n$ is 128, then the greatest coefficient in $(1+x)^n$ is

a) 25 b) 35 c) 45 d) 31

Key. B

Sol. Put $x = y = z = 1$ then $2^n = 128$, n=7, $7_{C_3} = \frac{7.6.5}{1.2.3} = 35$

103. If n = 2009, then N = $2009^n - 1982^n - 1972^n + 1945^n$ is divisible by

a) 658 b) 1977 c) 1988 d) 2009

Key. B

Sol. Since n is odd $x^n + y^n$ has divisor x+y.

104. If $C_0, C_1 \ldots C_{10}$ are the binomial coefficient in the expansion of $(1+x)^{10}$, then

$2C_0 + \frac{2^2}{2}C_1 + \frac{2^3}{3}.C_2 + \ldots + \frac{2^{11}}{11}.C_{10}$

a) $\frac{2^{11}}{11}$ b) $\frac{2^{11}-1}{11}$ c) $\frac{3^{11}}{11}$ d) $\frac{3^{11}-1}{11}$

Key. D

Sol. $\int_0^2 (1+x)^{10}\,dx = 10_{C_0}2 + \frac{2^2 10_{C_1}}{2} + \ldots\ldots\ldots \frac{2^{11}10_{C_{10}}}{11} = \frac{3^{11}-1}{11}$

105. If $(1+px+x^2)^n = \sum_{r=0}^{2n} a_r x^r$, then $\sum_{r=0}^{2n}(2r+1)a_r =$

a) $(p+2)^n$ b) $(2p+1)(p+2)^2$

c) $(2n+1)(p+2)^n$ d) $(p+2)^{n+1}$

Key. C

Sol. $(2\sum_{r=0}^{2n} ra_r + \sum_{r=0}^{2n} a_r = 2n(2+p)^{n-1}(P+2) + (2+P)^n$

$= (2n+1)(P+2)^n$

106. Let $(x^3 + \alpha x^2 + 2x - 5)^{19}(x^2 + \beta x - 41)^8(x^4 - x^3 + x - 7)^6 =$

$x^{97} + 391x^{96} + a_{95}x^{95} + a_{94}x^{94} + \ldots + a_1 x + a_0$ be an identity, where

$\alpha, \beta, a_{95}, a_{94}, \ldots, a_1, a_0$ are integers. If $\alpha + \beta < 10$, then the smallest possible value of α is

a) 7 b) 8 c) 31 d) 23

Key. C

Sol. It will be an identity even if we replace x by $\frac{1}{y}$ and considering numerator alone.

Differentiating on both sides with respect to y, at y =0 we get $19\alpha + 8\beta = 397,\ \alpha + \beta = 10 - k$ where k is positive integer. Put $\beta = 10 - \alpha - k$ in first equation we get $11\alpha - 8k = 317$

$\therefore\ \alpha = 31$

107. The number of different terms in the expansion of

$(1+x)^{2009} + (1+x^2)^{2008} + (1+x^3)^{2007}$ is

a) 3683 b) 4017 c) 4018 d) 4352

Key. B

Sol. $(1+x)^{2009}$ has 2010 terms in total. $(1+x^2)^{2008}$ has a constant, even power of x starting from 2 to 4016 but already even powers of x from 2 to 2008 were enumerated in $(1+x)^{2009}$. The remaining terms containing even powers of x are from 2010 to 4016. They are 1004 in number. In $(1+x^3)^{2007}$ has a constant, multiples of 3 as powers of x. Even multiples of 3 from 6 to 4014 were already enumerated in above expansions. The remaining even multiples of 3 from 4020 to 6018 which are 334 in number. Odd multiples of 3 as powers of x from 3 to 2007 were enumerated in above expansions and the remaining from 2013 to 6021 are to be enumerated. They are 669 in number.

$\therefore$ the number of terms in the expansion = 2010 + 1004 + 669 + 334 = 4017.

108. The term independent of x and y in the expansion of

$[(\sqrt{x}+\frac{1}{\sqrt{x}})^2+(\sqrt{y}+\frac{1}{\sqrt{y}})^2+(\sqrt{xy}+\frac{1}{\sqrt{xy}})^2-4]^n$ is

A) $(\sum_{r=0}^{n} {}^nc_r)^2$ B) $\sum_{r=0}^{n}({}^nc_r)^2$ C) $(\sum_{r=0}^{n} {}^nc_r)^3$ D) $\sum_{r=0}^{n}({}^nc_r)^3$

Key. D

Sol. The given expression can be written as $(1+x)^n.(1+y)^n.(1+\frac{1}{xy})^n$. The constant term is clearly $C_0^3+C_1^3+----+C_n^3$ where $c_r={}^nc_r$.

109. The sum of the rational terms in the expansion of $\left(\sqrt{2}+\sqrt[5]{3}\right)^{10}$ is :

A) 41 B) 42 C) 39 D) 45

Key. A

Sol. $T_{r+1}=10C_r\,\frac{10-r}{2^2}.3^{\frac{r}{5}}$

This is rational , if $\frac{10-r}{2}$ and $\frac{r}{5}$ are integers.

$\therefore$ There are only two rational terms

Namely $10C_0\left(\sqrt{2}\right)^{10}\left(3^{\frac{1}{5}}\right)^0$ and $10C_{10}\left(\sqrt{2}\right)^0\left(3^{\frac{1}{5}}\right)^{10}$

$\therefore$ sum = 32 + 9 = 41

110. If $(1+x)^n=a_0+a_1x+a_2x^2+.....+a_nx^n$ in which a_{n-3},a_{n-2},a_{n-1} are in AP, then

A) a_1,a_2,a_3 are in GP B) a_1,a_2,a_3 are in HP

C) $n=7$ D) $n=14$

Key. C

Sol. $a_{n-3}=a_3,a_{n-2}=a_2,a_{n-1}=a_1\left(\because nC_r=nC_{n-r}\right)$

$\therefore$ (A) is correct.

$a_,,a_2,a_3$ are in AP $\Rightarrow n,\frac{n(n-1)}{2},\frac{n(n-1)(n-2)}{6}$ are in AP.

$$\frac{n+\frac{n(n-1)(n-2)}{6}}{2}=\frac{n(n-1)}{2}$$

$$6n+n^3-3n^2+2n=6n^2-6n$$
$$n^3-9n^2+14n=0,\ (n-7)(n-2)=0$$

$\therefore$ n = 7.

111. If x = $\left(2+\sqrt{3}\right)^n$, then the value of $x-x^2+x[x]$ where [.] denotes the greatest integer function, is equal to

A) 1 B) 2 C) 2^{2n} D) 2^n

Key. A

Sol. $x-x^2+x[x]=x-x(x-[x])=x(1-\{x\})$

Now $x+x_1$ = even integer where $x_1=\left(2-\sqrt{3}\right)^n$ clearly $x_1 \varepsilon (0,1)\ \forall$ n ε N.

$\therefore$ $\{x\}+x_1$= integer

$\Rightarrow$ $\{x\}+x_1=1$

112. In the binomial expansion of $\left(\sqrt{x}+\dfrac{1}{2\sqrt[4]{x}}\right)^n$, $n\in N$ the coefficients of first, second and third terms form an A.P. The number of rational terms in the expansion is (Assume that x is a rational number and $\sqrt{x}$, $\sqrt[4]{x}$ are irrational)

A) 1 B) 2 C) 3 D) 4

Key. C

Sol. $$\left(\sqrt{x}+\frac{1}{2\sqrt[4]{x}}\right)^n=\sum_{r=0}^{n}{}^nC_r\left(\sqrt{x}\right)^{n-r}.\left(\frac{1}{2\sqrt[4]{x}}\right)^r=\sum_{r=0}^{n}{}^nC_r.\frac{1}{2^r}x^{\frac{2n-3r}{4}}$$

1, $\dfrac{n}{2}$, $\dfrac{n(n-1)}{8}$ are in A.P. $\Rightarrow$ $n=8$

113. $\left(|\underline{1}+2|\underline{2}+3|\underline{3}+.......+2009|\underline{2009}\right)+1$

A) $2|\underline{2010}$ B) $|\underline{2010}$ C) $|\underline{2011}$ D) $2|\underline{2011}$

Key. B

Sol. $|\underline{1}+2|\underline{2}+3|\underline{3}+..........+n|\underline{n}=|\underline{n+1}-1$

114. The sum of coefficients of the terms of degree 'm' in the expansion of $(1+x)^n(1+y)^n(1+z)^n$ is

(A) $\left({}^nC_r\right)^3$ (B) $3\left({}^nC_r\right)$ (C) ${}^nC_{3r}$ (D) ${}^{3n}C_m$

Key. D

Sol. Putting y = z = x we get $(1+x)^{3n}$ coeff $x^m={}^{3n}C_m$

115. If C_r denotes ${}^nC_r$ then the value of $\dfrac{C_0}{2}-\dfrac{C_1}{3}+\dfrac{C_2}{4}-\dfrac{C_3}{5}+------(-1)^n\dfrac{C_n}{n+2}=$

(A) $\dfrac{1}{n+1}$ (B) $\dfrac{1}{n+2}$ (C) $\dfrac{1}{n(n+1)}$ (D) $\dfrac{1}{(n+1)(n+2)}$

Key. D

Sol. Req sum = $\displaystyle\int_0^1 x(1-x)^n\,dx=\frac{1}{(n+1)(n+2)}$

116. If the 4[th] term in the expansion of $\left(2+\frac{3x}{8}\right)^{10}$ has the maximum numerical value, then 'x' lies in the interval

(A) $\left(\frac{-64}{21},-2\right)\cup\left(2,\frac{64}{4}\right)$ (B) $\left(\frac{-60}{23},-2\right)\cup\left(2,\frac{64}{23}\right)$

(C) $\left(\frac{-64}{21},2\right)$ (D) $\left(-2,\frac{-64}{21}\right)$

Key. A or B

Sol. $\left|\frac{t_3}{t_4}\right|<1 \ \ and \left|\frac{t_5}{t_4}\right|<1$

i.e., $\left|\frac{2}{x}\right|<1; \ \left|\frac{21}{64}x\right|<1$

$\therefore x\in\left(\frac{-64}{21},-2\right)\cup\left(2,\frac{64}{21}\right)$

117. The Value of $\sum_{r=0}^{15}{}^{15}C_r\left(r-\frac{15}{2}\right)^2$ is

A) $2^{10}.15$ B) $2^{12}.15$ C) $2^{13}.15$ D) $2^{15}.15$

Key. C

Sol. $\sum_{r=0}^{15}{}^{15}C_r.r^2-15\sum_{r=0}^{15}r.{}^{15}C_r+\frac{225}{4}\times 2^{15}$

$\sum_{r=0}^{15}r^2.{}^{15}C_r=15\sum_{r=1}^{15}[r-1+1]\times{}^{14}C_{r-1}=15.2^{14}+15.14.2^{13}=2^{15}.60$

$\therefore$ Required Sum $=\frac{225}{4}\times 2^{15}+2^{15}.60-225.2^{14}$

118. The Value of $\frac{1}{\lfloor 1\lfloor 15}+\frac{1}{\lfloor 3\lfloor 13}+\frac{1}{\lfloor 5\lfloor 11}+\frac{1}{\lfloor 7\lfloor 9}$ is

A) $\frac{2^{14}}{\lfloor 15}$ B) $\frac{2^{15}}{\lfloor 16}$ C) $\frac{2^{10}}{\lfloor 15}$ D) $\frac{2^{13}}{\lfloor 15}$

Key. C

Sol. Let S be the required Sum. Then we have $2S\times\angle 16={}^{16}C_1+{}^{16}C_3+{}^{16}C_5+{}^{16}C_7+...+{}^{16}C_{15}$

Binomial Theorem

Multiple Correct Answer Type

1. The largest coefficient in the expansion of $(4+3x)^{25}$ is

A) ${}^{25}C_{11}3^{25}\left(\frac{4}{3}\right)^{14}$ B) ${}^{25}C_{11}4^{25}\left(\frac{3}{4}\right)^{11}$ C) ${}^{25}C_{14}4^{14}3^{11}$ D) ${}^{25}C_{14}4^{11}3^{14}$

Key. A,B,C

Sol. We have $(4+3x)^{25}=4^{25}\left(1+\frac{3}{4}x\right)^{25}$

Let $(r+1)^{th}$ is term is the term having greatest coefficient

$\Rightarrow$ coefficient of $T_{r+1}\geq$ coefficient of T_r

$\Rightarrow 4^{25}\left\{{}^{25}C_r\left(\frac{3}{4}\right)^r\right\}\geq 4^{25}\left\{{}^{25}C_{r-1}\left(\frac{3}{4}\right)^{r-1}\right\}$

$\Rightarrow \frac{{}^{25}C_r}{{}^{25}C_{r-1}}\left(\frac{3}{4}\right)\geq 1\Rightarrow \frac{25-(r-1)}{r}\frac{3}{4}\geq 1$

$\Rightarrow 75-3r+3\geq 4r$

$\therefore r\leq \frac{78}{7}\leq 11.142$

But , r is an integer , hence r = 11

2. Let $R=\left(8+3\sqrt{7}\right)^{20}$ *and* $[R]=$ the greater integer less than or equal to R. Then

A) [R] is even B) [R] is odd

C) $R-[R]=1-\frac{1}{\left(8+3\sqrt{7}\right)^{20}}$ D) $R+R[R]=1+R^2$

Key. B,C,D

Sol. $R=[R]+f=\left(8+3\sqrt{7}\right)^{20}$

$={}^{20}C_0\,8^{20}+{}^{20}C_1 8^{19}\left(3\sqrt{7}\right)+.......,$

Where $0<f<1$

Let $F=\left(8-3\sqrt{7}\right)^{20}={}^{20}C_0$

$8^{20}-{}^{20}C_1 8^{19}\left(3\sqrt{7}\right)+.....$ where 0<F<1

$\because [R]+f+F=2\left[{}^{20}C_0.8^{\ 20}+{}^{20}C_2\ 8^{20}+{}^{20}C_2 8^{18}\left(3\sqrt{7}\right)\right]^2$

$+......]=$ an even integer

$\therefore [R]=$ an even integer -1= an odd integer

Also, $R-[R]=f=1-F=1-\left(8-3\sqrt{7}\right)^{20}$

$= 1 - \frac{1}{\left(8+3\sqrt{7}\right)^{20}}$

Again $RF = 1 \Rightarrow R(1-f) = 1 \Rightarrow R\{1 - R + [R]\} = 1$

There fore $R + R[R] = 1 + R^2$

3. A value of r satisfying the equation ${}^{69}C_{3r-1} - {}^{69}C_{r^2} = {}^{69}C_{r^2-1} - {}^{69}C_{3r}$ is

(A) 1 (B) 2 (C) 3 (D) 7

Key. C,D

Sol. Since ,

${}^{69}C_{3r-1} + {}^{69}C_{3r} = {}^{69}C_{r^2-1} + {}^{69}C_{r^2}$ (given)

$\Rightarrow {}^{70}C_{3r} = {}^{70}C_{r^2}$

$\Rightarrow$ either $3r = r^2$ *or* $r^2 + 3r = 70$

i.e. either $r = 0, 3$ *or* $r = 7, -10$

But the given equation is not defined for

R=0, -10

Hence , r=3 or 7

4. Which of the following is correct?

(a) $(101)^{100} > (100)^{101}$ (B) $(26)^{25} < (25)^{26}$

(C) $(300)^{299} < (299)^{300}$ (D) $(198)^{199} < (199)^{198}$

Key. B,C

Sol. We note that each option is of the form

$(n+1)^n > or < n^{n+1}$

Now, $\frac{(n+1)^n}{n^{n+1}} = \frac{1}{n}\left[1+\frac{1}{n}\right]^n$

But $\left[1+\frac{1}{n}\right]^n = 1 + n.\frac{1}{n} + \frac{n(n+1)}{2!}\left(\frac{1}{n}\right)^2$

$+ \frac{n(n+1)(n+2)}{3!}\left(\frac{1}{n}\right)^3 + \ldots\ldots\ldots$

$= 1 + 1 + \frac{1}{2!}\left(1+\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \ldots\ldots$

Clearly , $\left(1+\frac{1}{n}\right)^n > 2$ …….(1)

Also $\left(1+\frac{1}{n}\right)^n < 2 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots\ldots.$

$$\infty = 2 + \frac{\frac{1}{2}}{1-\frac{1}{2}} = 3,\ provided\ n > 3$$

$$\therefore 2 < \left(1+\frac{1}{n}\right)^n < 3$$

$$\Rightarrow \frac{2}{n} < \frac{1}{n}\left(1+\frac{1}{n}\right)^n < \frac{3}{n} < 1,\ for\ n > 3$$

$$\therefore \frac{(n+1)^n}{n^{n+1}} < 1 \Rightarrow (n+1)^n < n^{n+1}\ \ for\ n > 3$$

Which is satisfied only by the option (b) and (c)

5. the value of $^{n}C_0 + {}^{n+1}C_1 + {}^{n+2}C_2 + ... + {}^{n+k}C_k$ is equal to

(A) $^{n+k+1}C_k$ (B) $^{n+k+1}C_{n+1}$ (C) $^{n+k}C_{n+1}$ (D) none of these

Key. A,B

Sol. The given expression is the coefficient of x^n in $(1+x)^n + (1+x)^{n+1} + ... + (1+x)^{n+k}$

$$\Rightarrow co\ efficient\ of\ \ x^n\ \ in\ (1+x)^n \left\{\frac{(1+x)^{k+1}-1}{1+x-1}\right\}$$

$$= \frac{(1+x)^n (1+x)^{k+1} - (1+x)^n}{x}$$

$\Rightarrow$ Coefficient of x^{n+1} in

$(1+x)^{n+k+1} - (1+x)^n$ $i.e$ $^{n+k+1}C_{n+1}$

Or $^{n+k+1}C_k$

6. Which of the following is/are true

A) $5^6 - {}^5C_1.4^6 + {}^5C_2.3^6 - {}^5C_3.2^6 + {}^5C_4.1^6 = {}^6C_2.\underline{|5}$

B) $6^5 - {}^6C_1.5^5 + {}^6C_2.4^5 - {}^6C_3.3^5 + {}^6C_4.2^5 - {}^6C_1.1^5 = 0$

C) $6^6 - {}^6C_1.5^6 + {}^6C_2.4^6 - {}^6C_3.3^6 + {}^6C_4.2^6 - {}^6C_5.1^6 = 720$

D) $6^5 - {}^6C_1.5^5 + {}^6C_2.4^5 - {}^6C_3.3^5 + {}^6C_4.2^5 - {}^6C_5.1^5 = {}^5C_2.\underline{|6}$

Key. A,B,C

Sol. a) Number of onto functions from a set containing 6 elements to a set containing 5 elements $= {}^6C_2.\underline{|5}$

b) Number of onto functions from a set containing 5 elements to a set containing 6 elements $= 0$

c) Number of onto function from a set containing 6 elements to a set containing 6 elements $= \underline{|6} = 720$

7. If $1+(1+x)+(1+x)^2+.....+(1+x)^n = \sum_{k=0}^{n} a_k x^k$, then

(A) $\sum_{k=0}^{n} a_k = 2^{n+1}$ (B) $a_{n-2} = \frac{n(n+1)}{2}$

(C) $a_p > a_{p-1}$ for $p < \frac{n}{2}$ (D)

$(a_9)^2 - (a_8)^2 = {}^{n+2}C_{10}\left({}^{n+1}C_{10} - {}^{n+1}C_9\right)$

Key: B,C,D

Hint: $a_k = {}^{n+1}C_{k+1}$

8. If $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_{2n}x^{2n}$, then

(A) $a_0 - a_2 + a_4 - a_6 + \ldots\ldots = 0$, if n is odd

(B) $a_1 - a_3 + a_5 - a_7 + \ldots\ldots = 0$, if n is even

(C) $a_0 - a_2 + a_4 - a_6 + \ldots\ldots = 0$, if $n = 4p$, $p \in I^+$

(D) $a_1 - a_3 + a_5 - a_7 + \ldots\ldots = 0$, if $n = 4p + 1$, $p \in I^+$

Key : A, B

Sol : $\because (1+x+x^2)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n}x^{2n}$

Putting $x = i (i = \sqrt{-1})$

Then, we get

$$(1+i+i^2)^n = (a_0 - a_2 + a_4 - a_6 + ...) + i(a_1 - a_3 + a_5 - a_7 + ...)$$

$\Rightarrow \quad i^n = (a_0 - a_2 + a_4 - a_6 + ...) + i(a_1 - a_3 + a_5 - a_7 + ...)$

If n is odd, then $Re(i^n) = 0$

$\Rightarrow \quad a_0 - a_2 + a_4 - a_6 + ... = 0$

If n is even, then $Im(i^n) = 0$

$\Rightarrow \quad a_1 - a_3 + a_5 - a_7 + ... = 0$

9. Let $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 .... + a_{2n}x^{2n}$, where n is odd integer, if

$S_1 = a_0 + a_4 + a_8 + ...$

$S_2 = a_1 + a_5 + a_9 + ...$

$S_3 = a_2 + a_6 + a_{10} + ...$

$S_4 = a_3 + a_7 + a_{11} + ...$

then

a) $S_1 = S_3$ b) $S_2 = S_4$

c) $S_2 + S_4 = 0$ d) either $S_1 = S_2 = S_3$ or $S_1 = S_3 = S_4$

Key : a, b

Sol : $\left(1 + x + x^2\right)^n = a_0 + a_1x + a_2x^2 + ... + a_2nx^{2n}$

Put $x = i - i$

$i^n = a_0 + a_1i - a_2 + a_3i + a_4 + a_5i +$ ….. (1)

$\left(-i^n\right) = a_0 + a_1i - a_2 + a_3i + a_4 - a_5i +$ …..(2)

(1) + (2) $\Rightarrow 2\left(a_0 - a_2 + a_4 - a_6 + ...\right) = 0 \ \left(\therefore n\,is\,odd\right)$

$\Rightarrow a_0 + a_4 + a_8 + ... = a_2 + a_6 + ... \Rightarrow S_1 = S_3$

Multiply 2 by I and equating real points

$a_1 - a_3 + a_5 - a_7 + ... = 0$

$a_1 + a_5 + a_9 + ... = a_3 + a_7 + ... \Rightarrow S_2 = S_4$

10. If n is a positive integer and $\left(3\sqrt{3} + 5\right)^{2n+1} = \alpha + \beta$ where α is an integer and $0 < \beta < 1$ then

a) α is an even integer

b) $\left(\alpha + \beta\right)^2$ is divisible by 2^{2n+1}

c) The integer just below $\left(3\sqrt{3} + 5\right)^{2n+1}$ divisible by 3

d) α is divisible by 10

Key ; a, b, d

Sol : Given $\left(3\sqrt{3} + 5\right)^{2n+1} = \alpha + \beta$ …. (1)

Q $0 < \beta < 1$

Let $\beta' = \left(3\sqrt{3} - 5\right)^{2n+1}$ …. (2)

Q $0 < \beta' < 1$

From (1) & (2)

$\alpha + \beta - \beta' = \left(3\sqrt{3} + 5\right)^{2n+1} - \left(3\sqrt{3} - 5\right)^{2n+1}$

$= 2\left[\,{}^{2n+1}C_1\left(3\sqrt{3}\right)^{2n} 5 + {}^{2n+1}C_3\left(3\sqrt{3}\right)^{2n-2}\left(5\right)^3 + ... + {}^{2n+1}C_{2n+1}5^{2n+1}\right]$

$\alpha+\beta-\beta'=10\{1\}$

But $-1<\beta-\beta'<1$

$\therefore \beta-\beta'$ is an integer

$\therefore \beta-\beta'=0$

$\therefore \alpha=101$

$\therefore \alpha$ divisible by 10.

$\Rightarrow (\alpha+\beta)^2=\left[\left(3\sqrt{3}+5\right)^2\right]^{2n+1}=\left(52+30\sqrt{3}\right)^{2n+1}=2^{2n+1}\left(26+15\sqrt{3}\right)^{2n+1}$

$\therefore (\alpha+\beta)^2$ divisible by 2^{2n+1}

11. (L-2)If the term independent of x in the expansion of $\left(\sqrt{x}+\frac{k}{x^2}\right)^{10}$ is 405, then a value of k may be

a) 3 b) -3 c) 2 d) -2

Key : a, b

Sol : The term independent of x in $\left(\sqrt{x}+\frac{k}{x^2}\right)^{10}$

Is $10_{c_2}\left(\sqrt{x}\right)^8.\left(\frac{k}{x^2}\right)^2=405$

$\Rightarrow 45k^2=405$

$\Rightarrow k^2=9 \ \Rightarrow K=\pm 3$

12. (L-1) $(a\alpha x+a^2y+1)^{2009}$ is a polynomial in x and y if the sum of the co-efficients vanishes for some real value of a. Then possible value of α is / are

a) -2 b) – 3 c) 1 d) 2

Key : a, b, d

Sol : $x=y=1\Rightarrow\left(a\lambda+a^2+1\right)^{2009}=0 \ \Rightarrow a^2+a\lambda+1=0$ if a is real then

$b^2-4ac\ge 0\Rightarrow \lambda^2-4\ge 0 \ \therefore \lambda=-2,-3,2$

13. The number $\frac{5^k+3}{4}(k\in N)$, when divided by 10, may leave remainder

(A) 2 (B) 6 (C) 7 (D) 8

Key: A,C

Hint. $\frac{5^k+3}{4}=\frac{5^k-5+8}{4}=\frac{5(5^{k-1}-1)}{5-1}+2=5+5^2+5^3+...(k-1)\text{ terms}+2$

So if k – 1 = even, the last digit is 2

If k –1 = odd, the last digit is 5 + 2 = 7.

14. For r = 0, 1, ..., 10, let A_r, B_r and C_r denote, respectively, the coefficient of x^r in the expansion of $(1 + x)^{10}$, $(1 + x)^{20}$ and $(1 + x)^{30}$, then $\sum_{r=1}^{10} A_r (B_{10} B_r - C_{10} A_r)$ is equal to

(A) $C_{10} - B_{10}$ (B) 0

(C) $A_{10} (B_{10}^2 - C_{10} A_{10})$ (D) $B_{10} - C_{10}$

Key. A

Sol. $\sum_{r=1}^{10} A_r (B_{10}B_r - C_{10}A_r) = \sum_{r=1}^{10} {}^{10}C_r \; ({}^{20}C_{10}\; {}^{20}C_r - {}^{30}C_{10}\; {}^{10}C_r)$

$= {}^{20}C_{10} \sum_{r=1}^{10} {}^{10}C_{10-r}\, {}^{20}C_r - {}^{30}C_{10} \sum_{r=1}^{10} {}^{10}C_r\, {}^{10}C_{10-r}$

$= {}^{20}C_{10}\, ({}^{30}C_{10} - 1) - {}^{30}C_{10}\, ({}^{20}C_{10} - 1) = {}^{30}C_{10} - {}^{20}C_{10} = C_{10} - B_{10}$

15. If 1, α_1, α_2, ... α_{n-1} are the distinct nth roots of unity, then

(A) $(1 + \alpha_1)(1 + \alpha_2) \ldots (1 + \alpha_{n-1}) = \dfrac{1+(-1)^n}{2}$

(B) $(1 + \alpha_1)(1 + \alpha_2) \ldots (1 + \alpha_{n-1}) = \dfrac{1-(-1)^n}{2}$

(C) $\left|\cos\dfrac{\pi}{n}\cos\dfrac{2\pi}{n}\cos\dfrac{3\pi}{n}\ldots\cos\dfrac{(n-1)\pi}{n}\right| = \dfrac{1+(-1)^n}{2^n}$

(D) $\cos4° \cos8° \cos12° \ldots \cos88° = \dfrac{1}{2^{22}}$

Key. B,D

Sol. $(z - 1)(z - \alpha_1)(z - \alpha_2) \ldots (z - \alpha_{n-1}) = z^n - 1$

putting z = –1, we get, $(1 + \alpha_1)(1 + \alpha_2) \ldots (1 + \alpha_{n-1}) = \dfrac{1-(-1)^n}{2}$

Now, $|1 + \alpha_n| = |1 + \cos\dfrac{2k\pi}{n} + i\sin\dfrac{2k\pi}{n} = 2\left|\cos\dfrac{k\pi}{n}\right|$

for n = 45, $|1 + \alpha_n| = 2\left|\cos\dfrac{k\pi}{45}\right| = 2|\cos(4k)°|$

as $\prod_{k=1}^{44} (1+2k| = \dfrac{1-(-1)^{45}}{2} = 1$

$\Rightarrow 2^{44} (\cos4° \cos8° \cos12° \ldots \cos176°) = 1 \Rightarrow \cos4° \cos8° \cos12° \ldots \cos88° = \dfrac{1}{2^{22}}$

16. If $\left(1+2x+3x^2\right)^{10} = a_0 + a_1x + a_2x^2 + ----+ a_{20}x^{20}$ then

a) $a_6 = 12$ b) $a_2 = 210$

c) $a_1 = 20$ d) $a_4 = 8085$

Key. B,C,D

Sol. $\left(1+x\left(2+3x\right)\right)^{10} = 1 + 10c_1x\left(2+3x\right) + ----+ 10c_{10}x^{10}\left(2+3x\right)^{10}$

17. If $\left(1+ax+bx^2\right)^4 = a_0 + a_1x + a_2x^2 + ... + a_8x^8$ where $a, b, a_0, a_1, a_2.....a_8 \in R$ such that $a_0 + a_1 + a_2 \neq 0$ and $\begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_1 & a_0 \\ a_2 & a_0 & a_1 \end{vmatrix} = 0$ then

a) $a_0 = a_1 \neq a_2$ b) $a_1 = a_2 = 1$ c) $a_0 = a_1 = a_2 = 4a$ d) $b = \frac{5}{32}$.

Key. B,C,D

Sol. $x = 0 : a_0 = 1$

Differentiations $4\left(1+ax+bx^2\right)^3 (a+2bx) = a_1 + 2a_2x + ...$

Let $x = 0$ $4a = a_1$

The value of the determinant is $-\left(a_0^3 + a_1^3 + a_2^3 - 3a_0a_1a_2\right) = 0$

$\Rightarrow \because a_0 + a_1 + a_2 \neq 0$ the $a_0 = a_1 = a_2 = 4a$

Differentially again and putting $x = 0$

$a = \frac{1}{4} : b = \frac{5}{32}$

18. The value of $\sum_{k=0}^{7} \left[\frac{\binom{7}{k}}{\binom{14}{k}} \sum_{r=k}^{14} \binom{r}{k}\binom{14}{r} \right]$, where $\binom{n}{r}$ denotes ${}^{n}C_{r}$, is

(A) 6^7 (B) greater than 7^6

(C) 8^7 (D) greater than 7^8

Key. A,B

Sol. $\sum_{k=0}^{7} \left(\frac{{}^7C_k}{{}^{14}C_k} \sum_{r=k}^{14} {}^rC_k \,.\, {}^{14}C_r \right) = \sum_{k=0}^{7} \left(\frac{{}^7C_k}{\lfloor 14} \times \lfloor k \lfloor 14-k \sum_{r=k}^{14} \frac{\lfloor r}{\lfloor k \lfloor r-k} \cdot \frac{\lfloor 14}{\lfloor r \lfloor 14-r} \right)$

$= \sum_{k=0}^{7} \left({}^7C_k \sum_{r=k}^{14} {}^{14-k}C_{r-k} \right) = \sum_{k=0}^{7} {}^7C_k \,.\, 2^{14-k} = 2^{14} \sum_{k=0}^{7} {}^7C_k \left(\frac{1}{2}\right)^k$

$= 2^{14} . \left(1+\frac{1}{2}\right)^7 = 6^7 > 7^6$

19. $\sum_{K=0}^{n} {}^nc_k \left({}^{n+1}c_{K+1} + {}^{n+1}c_{K+2} + ----- + {}^{n+1}c_{n+1}\right)$ is equal to

a) ${}^{2n+1}c_0 + {}^{2n+1}c_1 + ----- + {}^{2n+1}c_n$ b) 4^n

c) ${}^{2n+1}c_0 + {}^{2n+1}c_1 + ----- + {}^{2n+1}c_{2n+1}$ d) 2^{2n+1}

Key. A,B

Sol. Consider the product of the expansions $(1+x)^n . \left(1+\frac{1}{x}\right)^{n+1} = \frac{(1+x)^{2n+1}}{x^{n+1}}$. The given expression is sum of the coeffts of negative powers of x in this product.

$\therefore$ It is equal to ${}^{2n+1}c_0 + ----- + {}^{2n+1}c_n = 2^{2n}$

20. The value of $c_0^2+3.c_1^2+5.c_2^2+----$ to (n+1) terms where $c_r={}^nc_r$, is

a) ${}^{2n-1}c_{n-1}$ b) $(2n+1).{}^{2n-1}c_n$

c) $2(n+1).{}^{2n-1}c_n$ d) ${}^{2n-1}c_n+(2n+1)\,{}^{2n-1}c_{n-1}$

A) Both A and R are true and R is the correct explanation of A

B) Both A and R are true but R is not the correct explanation of A

C) A is true, R is false D) A is false, R is true

Key. C,D

Sol. Let $S=c_0^2+3.c_1^2+5.c_2^2+---+(2n+1).c_n^2$

$\Rightarrow S=(2n+1).c_n^2+------------+c_0^2$

Adding $2S=(2n+2).(c_0^2+c_1^2----+c_n^2)$

$\Rightarrow S=(n+1).{}^{2n}c_n$

$\Rightarrow S=(n+1).\dfrac{2n}{n}\,{}^{2n-1}c_{n-1}=2(n+1)\,{}^{2n-1}c_n$

$\therefore$ c is correct and a, b are not correct.

d is also correct, because ${}^{2n-1}c_n+(2n+1).{}^{2n-1}c_{n-1}=2(n+1).{}^{2n-1}c_n$

21. If $(1+2x+3x^2)^{10}=a_0+a_1x+a_2x^2+-----+a_{20}x^{20}$,then

a) $a_1=20$ b) $a_2=210$ c) $a_{19}=20.3^9$ d) $a_{20}=2^2.3^7.7$

Key. A,B,C

Sol. $\left(1+2x+3x^2\right)^{10}=a_0+a_1x+a_2x^2+------+a_{20}x^{20}$. Differentiate *w. r. t x*.

$10\left(1+2x+3x^2\right)^9(2+6x)=a_1+2a_2x+3a_3x^2+------+20.a_{20}x^{19}$

Put *x* = 0 : we get $a_1=20$

Again differentiate *w. r. t x*

$10\left(1+2x+3x^2\right)^9.6+90(1+2x+3x^2)^8(2+6x)^2$

$=2.a_2+6.a_3x+-----+20.19.a_{20}x^{18}$

Put *x* = 0; we get $2.a_2=60+360\Rightarrow a_2=210$

Replace *x* by $\dfrac{1}{x}$ in the original expansion

$(1+\dfrac{2}{x}+\dfrac{3}{x^2})^{10}=\dfrac{(x^2+2x+3)^{10}}{x^{20}}=a_0+\dfrac{a_1}{x}+\dfrac{a_2}{x^2}+---+\dfrac{a_{20}}{x^{20}}$

$\Rightarrow(x^2+2x+3)^{10}=a_0.x^{20}+a_1x^{19}+---+a_{19}x+a_{20}$

Put *x* = 0, we get $a_{20}=3^{10}$.

Differentiate *w. r. t. x*

$10(x^2+2x+3)^9(2x+2)=20.a_0.x^{19}+---+a_{19}$

Put *x* = 0, we get $a_{19}=20.3^9$

22. The number $\dfrac{5^k+3}{4}(k\in N)$, when divided by 10, may leave remainder

(A) 2 (B) 6

(C) 7 (D) 8

Key. A,C

Sol. $\frac{5^k+3}{4}=\frac{5^k-5+8}{4}=\frac{5(5^{k-1}-1)}{5-1}+2=5+5^2+5^3+\ldots(k-1)\text{ terms}+2$

So if k – 1 = even, the last digit is 2

If k –1 = odd, the last digit is 5 + 2 = 7.

23. If the third term in the expansion of $\left(\frac{1}{x}+x^{\log_{10}x}\right)^5$ is 1000 then x is equal to

A) 100 B) 10 C) 1 D) $1/\sqrt{10}$

Key. A,D

Sol. $T_3={}^5C_2\frac{1}{x^3}.\left(x^{\log_{10}x}\right)^2=10x^{-3+2\log_{10}x}=1000$

$\therefore x^{-3+2\log_{10}x}=100\Rightarrow x=100,\frac{1}{\sqrt{10}}$

24. The number $101^{100}-1$ is divisible by

(a) 100 (b) 1000 (c) 10000 (d) 100000

Key. A,B,C

Sol. $(101)^{100}=10^4[1+10_{C_2}+.......+100^{98}]$

25. Let $a_n=\left(1+\frac{1}{n}\right)^n$ then for each $n\in N$

(a) $a_n\ge 2$ (b) $a_n<3$ (c) $a_n<4$ (d) $a_n<2$

Key. A,B,C

Sol. $a_n=2+\frac{1}{\lfloor 2}\left(1-\frac{1}{n}\right)+\frac{1}{\lfloor 3}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+.....+\frac{1}{\lfloor n}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right).....\left(1-\frac{n-1}{n}\right)$

Take $a_n\le 1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+.......+\frac{1}{\lfloor n}$

$\le 1+1+\frac{1}{2}+\frac{1}{2^2}+........+\frac{1}{2^{n-1}}$

$a_n\le 3-\frac{1}{2^{n-1}}<3\forall n\ge 1$

$\Rightarrow a_n<4\forall n\ge 1$

26. If $\left(4+\sqrt{15}\right)^n=I+f$ where n is odd natural number, I is an integer and $0<f<1$ then

(a) I is a natural number (b) I is an even integer (c) $(I+f)(1-f)=1$ (d) $(I+f)(I-f)=1$

Key. A,C

Sol. $I+f=\left(4+\sqrt{15}\right)^n$

Let $\delta = \left(4-\sqrt{15}\right)^n \quad 0<\delta<1$

$I+f = n_{c_0}4^n + n_{c_1}4^{n-1}\sqrt{15} + n_{c_2}4^{n-2}.15..............$

$(1+f)(1-f)=1$

27. If 13_{C_r} is denoted by C_r then the value of $C_1+C_5+C_7+C_9+C_{11}$ is equal to

(a) $2^{12}-287$ (b) $2^{12}-165$ (c) $2^{12}-C_3-C_{13}$ (d) $2^{12}-C_2-C_{13}$

Key. A,C

Sol. Conceptual

28. A value of r satisfying the equation ${}^{69}C_{3r-1}-{}^{69}C_{r^2}={}^{69}C_{r^2-1}-{}^{69}C_{3r}$ is

(A) 1 (B) 2 (C) 3 (D) 7

Key. C,D

Sol. Since ,

${}^{69}C_{3r-1}+{}^{69}C_{3r}={}^{69}C_{r^2-1}+{}^{69}C_{r^2}$ (given)

$\Rightarrow {}^{70}C_{3r}={}^{70}C_{r^2}$

$\Rightarrow$ either $3r=r^2$ *or* $r^2+3r=70$

i.e. either $r=0,\ 3$ *or* $r=7,-10$

But the given equation is not defined for

R=0, -10

Hence , r=3 or 7

29. Which of the following is correct?

(a) $(101)^{100}>(100)^{101}$ (B) $(26)^{25}<(25)^{26}$

(C) $(300)^{299}<(299)^{300}$ (D) $(199)^{198}=(198)^{199}$

Key. B,C

Sol. We note that each option is of the form

$(n+1)^n > or < n^{n+1}$

Now, $\dfrac{(n+1)^n}{n^{n+1}}=\dfrac{1}{n}\left[1+\dfrac{1}{n}\right]^n$

But $\left[1+\dfrac{1}{n}\right]^n=1+n.\dfrac{1}{n}+\dfrac{n(n+1)}{2!}\left(\dfrac{1}{n}\right)^2$

$+\dfrac{n(n+1)(n+2)}{3!}\left(\dfrac{1}{n}\right)^3+.........$

$=1+1+\dfrac{1}{2!}\left(1+\dfrac{1}{n}\right)+\dfrac{1}{3!}\left(1-\dfrac{1}{n}\right)\left(1-\dfrac{2}{n}\right)+......$

Clearly , $\left(1+\frac{1}{n}\right)^n > 2$ ....(1)

Also $\left(1+\frac{1}{n}\right)^n < 2+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+.......$

$$\infty = 2+\frac{\frac{1}{2}}{1-\frac{1}{2}} = 3,\ provided\ n>3$$

$$\therefore 2 < \left(1+\frac{1}{n}\right)^n < 3$$

$$\Rightarrow \frac{2}{n} < \frac{1}{n}\left(1+\frac{1}{n}\right)^n < \frac{3}{n} < 1,\ for\ n>3$$

$$\therefore \frac{(n+1)^n}{n^{n+1}} < 1 \Rightarrow (n+1)^n < n^{n+1}\ \ for\ n>3$$

Which is satisfied only by the option (b) and (c)

30. the value of ${}^{n}C_0 + {}^{n+1}C_1 + {}^{n+2}C_2 + ... + {}^{n+k}C_k$ is equal to

(A) ${}^{n+k+1}C_k$ (B) ${}^{n+k+1}C_{n+1}$ (C) ${}^{n+k}C_{n+1}$ (D) none of these

Key. A,B

Sol. The given expression is the coefficient of x^n in $(1+x)^n + (1+x)^{n+1} + ... + (1+x)^{n+k}$

$$\Rightarrow co\ efficient\ of\ \ x^n\ in\ (1+x)^n \left\{\frac{(1+x)^{k+1}-1}{1+x-1}\right\}$$

$$= \frac{(1+x)^n (1+x)^{k+1} - (1+x)^n}{x}$$

$\Rightarrow$ Coefficient of x^{n+1} in

$(1+x)^{n+k+1} - (1+x)^n$ $i.e\ \ {}^{n+k+1}C_{n+1}$

Or ${}^{n+k+1}C_k$

31. Let an expression E be given by $E=(1+x)^n(1+y)^n(1+z)^n$ then

A) number of dissimilar terms in E will be $(n+1)^3$

B) number of dissimilar terms in E will be n^3

C) coefficient of n^r in E is $\left({}^{n}C_r\right)^3$

D) Sum of coefficient in E is 2^{3n}

Key. A,D

Sol. Each individual expansion will have $n+1$ terms.

Put $x=y=z=1$

32. Let $a_n = \left(1+\frac{1}{n}\right)^n$. Then for each $n \in N$

A) $a_n \ge 2$ B) $a_n < 3$ C) $a_n < 4$ D) $a_n < 2$

Key. A,B,C

Sol. We have $a_1 = 2$ and for $n \ge 2$.

$$a_n = \left(1+\frac{1}{n}\right)^n = {}^nC_0 + {}^nC_1\left(\frac{1}{n}\right) + {}^nC_2\left(\frac{1}{n}\right)^2 + ... + {}^nC_r\left(\frac{1}{n}\right)^r + ... + {}^nC_n\left(\frac{1}{n}\right)$$

$$= 1 + 1 + \frac{n(n+1)}{2!}\frac{1}{n^2} + ... + \frac{n(n-1)...(n-r+1)}{r!}\frac{1}{n^r} + ... + \frac{n(n-1)...2.1}{n!}\frac{1}{n^n} \qquad ...(i)$$

$$= 2 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + ... + \frac{1}{r!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)...\left(1-\frac{r-1}{n}\right) + ...$$

$$+ \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)...\left(1-\frac{n-1}{n}\right)$$

Hence from (i), $a_n \ge 2$ for all $n \in N$. Also

$$a_n \le 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + ... + \frac{1}{r!} + ... + \frac{1}{n!}$$

For $2 \le r \le n$, we have r! = 1. 2. 3… $r \ge 2^{r-1}$. Thus,

$$a_n \le 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^{r-1}} + ... + \frac{1}{2^{n-1}}$$

$$= 1 + \frac{1-(1/2^n)}{1-(1/2)} = 1 + 2\left(1-\frac{1}{2^n}\right) = 3 - \frac{1}{2^{n-1}}$$

$$a_n \le 3 - \frac{1}{2^{n-1}} < 3 \qquad \forall \quad n \ge 1 \quad \Rightarrow \quad a_n < 4 \ \forall\ n \ge 1$$

33. If $\left(4+\sqrt{5}\right)^n = 1 + f$, where n is an odd natural number, I is an integer and 0 < f < 1, then

A) I is natural number B) I is an even integer

C) (I + f) (I – f) = 1 D) $I - f = \left(4-\sqrt{5}\right)^n$

Key. A,C,D

Sol. $I + f = \left(4+\sqrt{5}\right)^n$

Let $g = \left(4-\sqrt{15}\right)^n$, then 0 < g < 1

$$I + f = {}^nC_0\ 4^n + {}^nC_1 4^{n-1}\sqrt{5} + {}^nC_2 4^{n-2} 15 + {}^nC_3 4^{n-3}\left(\sqrt{15}\right)^3 + ...$$

$$g = {}^nC_0 4^n - {}^nC_1 4^{n-1}\sqrt{15} + {}^nC_2.\ 4^{n-2}.15 - {}^nC_3 4^{n-3}\left(\sqrt{15}\right)^3 + ...$$

$\therefore$ $I = f + g = 2\left({}^nC_0 4^n + {}^nC_2 4^{n-2}.15 + ...\right) =$ even integer

Q $0 < f + g < 2$ $\Rightarrow$ $f + g = 1$ $\Rightarrow$ $1 - f = g$

Thus I is an odd integer

1 – f = g = $\left(4-\sqrt{15}\right)^n$

(I + f) (1 – f) = (I + f). g = 1

34. $\sum_{k=0}^{n} {}^nC_k\left({}^{n+1}C_{k+1} + {}^{n+1}C_{k+2} + --- + {}^{n+1}C_{n+1}\right)$ is equal to

a) ${}^{2n+1}C_0 + {}^{2n+1}C_1 + ---- + {}^{2n+1}C_n$ b) 4^n

c) ${}^{2n+1}C_0 + {}^{2n+1}C_1 + ---- + {}^{2n+1}C_{2n+1}$

d) 2^{2n+1}

Key. A,B

Sol. Consider the product of the expansions $(1+x)^n.\left(1+\frac{1}{x}\right)^{n+1} = \frac{(1+x)^{2n+1}}{x^{n+1}}$. The given expression is sum of the coefficients of negative powers of x in this product.

$\therefore$ it is equal to ${}^{2n+1}C_0 + ---- + {}^{2n+1}C_n = 2^{2n}$

35. The value of $C_0^2 + 3.C_1^2 + 5.C_2^2 + ----\text{to } (n+1)$ terms where $C_r = {}^nC_r$, is

a) ${}^{2n-1}C_{n-1}$ b) $2n+1.{}^{2n-1}C_n$

c) $2(n+1).{}^{2n-1}C_n$ d) ${}^{2n-1}C_n + (2n+1)^{2n-1}C_{n-1}$

Key. C,D

Sol. Let $S = C_0^2 + 3.C_1^2 + 5.C_2^2 + ... + (2n+1).C_n^2$

$\Rightarrow S = (2n+1).C_n^2 + -------- + C_0^2$

Adding 2S = $(2n+2).(C_0^2 + C_1^2 ---- + C_n^2)$

$\Rightarrow S = (n+1).{}^{2n}C_n$

$\Rightarrow S = (n+1).\frac{2n}{n}\,{}^{n+1}C_{n-1} = 2(n+1)^{2n-1}C_n$

$\therefore$ c is correct and a, b are not correct.

d is also correct, because ${}^{2n-1}C_n + (2n+1).{}^{2n-1}C_{n-1} = 2(n+1).{}^{2n-1}C_n$

36. If $(1+2x+3x^2)^{10} = a_0 + a_1x + a_2x^2 + ----- + a_{20}x^{20}$, then

a) $a_1 = 20$ b) $a_2 = 210$ c) $a_{19} = 20.3^9$ d) $a_{29} = 2^2.3^7.7$

Key. A,B,C

Sol. $(1+2x+3x^2)^{10} = a_0 + a_1x + a_2x^2 + ... + a_{20}x^{20}$. Differentiate w.r.t x.

$10(1+2x+3x^2)^9(2+6x) = a_1 + 2a_2x + 3a_3x^2 + ... + 20.a_{20}x^{19}$

Put $x=0$: we get $a_1 = 20$

Again differentiate w.r.t x

$10(1+2x+3x^2)^9.6 + 90(1+2x+3x^2)^8(2+6x)^2 = 2.a_2 + 6.a_3.x + ... + 20.19.a_{20}x^{18}$

Put $x=0$; we get $2.a_2 = 60 + 360 \Rightarrow a_2 = 210$

Replace x by $\frac{1}{x}$ in the original expansion

$$\left(1+\frac{2}{x}+\frac{3}{x^2}\right)^{10} = \frac{(x^2+2x+3)^{10}}{x^{20}} = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + ... + \frac{a_{20}}{x^{20}}$$

$\Rightarrow (x^2+2x+3)^{10} = a_0.x^{20} + a_1.x^{19} + ... + a_{19}x + a_{20}$

Put x = 0, we get $a_{20} = 3^{10}$

Differentiate w.r.t x

$$10\left(x^2+2x+3\right)^9\left(2x+2\right)=20.a_0.x^{19}+...+a_{19}$$

Put $x=0$, we get $a_{19}=20.3^9$

37. In the expansion of $\left(\sqrt[4]{3}+\sqrt[3]{4}\right)^{136}$

a) Number of rational terms is 12
b) Number of irrational terms is 125
c) Number of total terms is 137
d) Number of rational terms is 0

Key. A,B,C

Sol. $\left[\frac{136}{12}\right]+1$ = Number of rational terms is 12, Number of irrational terms is 125

Number of total terms is 137.

38. If n is an even integer then the value of

$$\frac{2}{n!}+\frac{1}{1!(n-1)!}+\frac{1}{2!(n-2)!}+\frac{1}{3!(n-3)!}+......+\frac{1}{(n-1)!1!}$$ is

a) $\frac{2^n}{(n!)}$
b) $\frac{2^n}{(n-1)!}$
c) $\frac{2^{n+1}}{2(n!)}$
d) $\frac{2^{n-1}}{(n-1)!}$

Key. A,C

Sol. $\frac{1}{n!}\left(c_o+c_1+c_2+........c_n\right)=\frac{2^n}{n!}$

39. Sum of the products of the binomial coefficients of $(1+x)^n$ taken two at a time is

a) $2^n-{}^{2n}C_n$
b) $2^{2n-1}-\frac{1}{2}\left({}^{2n}C_n\right)$
c) $\frac{1}{2}\left(2^n-{}^{2n}C_n\right)$
d) $\frac{1}{2}\left(4^n-{}^{2n}C_n\right)$

Key. B,D

Sol. $\frac{\left(C_o+C_1+.......C_n\right)^2-\left(C_o^2+C_1^2+.......+C_n^2\right)}{2}$ = $2^{2n-1}-\frac{1}{2}\left({}^{2n}C_n\right)$

= $\frac{1}{2}\left(4^n-{}^{2n}C_n\right)$ = $\frac{1}{2}\left(2^{2n}-{}^{2n}C_n\right)$.

40. The constant term in the expansion of $\left(54+54x+108x^3\right)\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^9$ is

a) $\frac{{}^9c_3}{4}-\frac{{}^9c_2}{9}$
b) $\frac{{}^9c_3}{216}+\frac{{}^9c_2}{486}$
c) $\frac{17}{54}$
d) 17

Key. A,D

Sol. $1\times 9_{c_6}\times\left(\frac{3}{2}\right)^3\left(\frac{-1}{3}\right)^6+0+2\times 9_{c_7}\times\left(\frac{3}{2}\right)^2\left(\frac{-1}{3}\right)^7$ = $\frac{{}^9c_3}{216}-\frac{{}^9c_2}{486}$ = $\frac{17}{54}$

41. $\sum_{K=0}^{n} {}^{n}c_k\left({}^{n+1}c_{K+1} + {}^{n+1}c_{K+2} + ----+ {}^{n+1}c_{n+1}\right)$ is equal to

A) ${}^{2n+1}c_0 + {}^{2n+1}c_1 + ----+ {}^{2n+1}c_n$ B) 4^n

C) ${}^{2n+1}c_0 + {}^{2n+1}c_1 + ----+ {}^{2n+1}c_{2n+1}$ D) 2^{2n+1}

Key. A,B

Sol. Consider the product of the expansions $(1+x)^n.\left(1+\frac{1}{x}\right)^{n+1} = \frac{(1+x)^{2n+1}}{x^{n+1}}$. The given expression is sum of the coeffts of negative powers of x in this product.

$\therefore$ It is equal to ${}^{2n+1}c_0 + ----+ {}^{2n+1}c_n = 2^{2n}$

42. The value of $c_0^2 + 3.c_1^2 + 5.c_2^2 + ----$ to (n+1) terms where $c_r = {}^{n}c_r$, is

A) ${}^{2n-1}c_{n-1}$ B) $(2n+1).{}^{2n-1}c_n$

C) $2(n+1).{}^{2n-1}c_n$ D) ${}^{2n-1}c_n + (2n+1)\,{}^{2n-1}c_{n-1}$

Key. C,D

Sol. Let $S = c_0^2 + 3.c_1^2 + 5.c_2^2 + ---+ (2n+1).c_n^2$

$\Rightarrow S = (2n+1).c_n^2 + ------------+ c_0^2$

Adding $2S = (2n+2).(c_0^2 + c_1^2 ----+ c_n^2)$

$\Rightarrow S = (n+1).{}^{2n}c_n$

$\Rightarrow S = (n+1).\frac{2n}{n}\,{}^{2n-1}c_{n-1} = 2(n+1)\,{}^{2n-1}c_n$

$\therefore$ c is correct and a, b are not correct.

d is also correct, because ${}^{2n-1}c_n + (2n+1).{}^{2n-1}c_{n-1} = 2(n+1).{}^{2n-1}c_n$

43. If $(1+2x+3x^2)^{10} = a_0 + a_1x + a_2x^2 + ----+ a_{20}x^{20}$,then

A) $a_1 = 20$ B) $a_2 = 210$ C) $a_{19} = 20.3^9$ D) $a_{20} = 2^2.3^7.7$

Key. A,B,C

Sol. Conceptual

44. Let S = $c_0^2 + 1.\frac{c_1^2}{c_0} + 2.\frac{c_2^2}{c_1} + 3.\frac{c_3^2}{c_2} + ... + n.\frac{c_n^2}{c_{n-1}}$ where $c_r = n_{c_r}$ then

A) 2^n divides 'S' B) n+2 divides S + n

C) 2^{n-1} divides S + n D) S is a positive integer.

Key. B,C,D

Sol. $r\frac{\left(n_{c_r}\right)^2}{n_{c_{r-1}}}$ = (n – r + 1)

$\sum_{r=1}^{n} r\frac{c_r^2}{c_{r-1}} = \sum_{r=1}^{n}\left\{(n+1)-r\right\}c_r = (n+1)\left(2^n-1\right) - n2^{n-1}$

= $(n+2)2^{n-1} - (n+1)$

S = $c_0^2 + \sum_{r=1}^{n} r.\frac{c_r^2}{c_{r-1}} = (n+2)2^{n-1} - n$

S + n = $(n+2)2^{n-1}$ which is divisible by

(n+2) and 2^{n-1} also, s is a positive integer.

45. If $(1+x)^{2n} = a_0 + a_1 x + a_2 x^2 + ... + a_{2n} x^{2n}$ then

A) $a_0 + a_2 + a_4 + ... = 1/2(a_0 + a_1 + a_2 + a_3 + ...)$ (B) $a_{n+1} < a_n$

(C) $a_{n-3} = a_{n+3}$ (D) $a_{n+1} > a_n$

Key. A,B,C

Sol. $a_0 + a_1 + a_2 + ... = 2^{2n}$ and $a_0 + a_1 + a_4 + ... = 2^{2n-1}$

$a_n = {}^{2n}C_n =$ the greatest coefficient, being the middle coefficient

$a_{n-3} = {}^{2n}C_{n-3} = {}^{2n}C_{2n-(n-3)} = {}^{2n}C_{n+3} = a_{n+3}$

Binomial Theorem

Assertion Reasoning Type

A) Both Statement - 1 and Statement - 2 are true and Statement - 2 is the correct explanation of
Statement - 1

B) Both Statement - 1 and Statement - 2 are true but Statement - 2 is not the correct explanation
of Statement - 1

C) Statement - 1 is true, Statement - 2 is false

D) Statement - 1 is false, Statement - 2 is true

1. Statement - 1: If q = $\frac{1}{3}$ and $p+q=1$, then $\sum_{r=0}^{15} r\,{}^{15}C_r\,p^r q^{15-r} = 15\times\frac{1}{3} = 5$

Statement -2 : if p + q = 1, 0 <p < 1, then $\sum_{r=0}^{n} r\,{}^{n}C_r\,p^r q^{n-r} = np$

Key. D

Sol. Conceptual

2. Statement 1: $2^{4n} - 2^n(7n+1)$ is divisible by square of 14 $\forall\ n \in N$

Statement 2: $(1+x)^n = 1 + {}^nC_1 x + ... + {}^nC_n x^n\ \forall\ n \in N$

Key. A

Sol. $2^{4n} - 2^n(7n+1) = 7^2 - 2^n\left({}^nC_2 + {}^nC_3.7^3 + ...\right) = 14^2 \times a$ a positive integer.

3. Statement – 1: Number of terms in the expansion of $(x+y+z)^{10}$ is 11.

Statement – 2: The number of terms in the expansion of $(x+y)^n$ $n \in N$ is $(n+1)$.

Key: D

Hint: Conceptual Question

4. STATEMENT-1: The integral part of $\left(8+3\sqrt{7}\right)^{20}$ is an even integer.

STATEMENT–2: $\left(8+3\sqrt{7}\right)^{20} + \left(8-3\sqrt{7}\right)^{20}$ is an even integer.

Key: D

Hint: Let $\left(8+3\sqrt{7}\right)^{20} = I + f,$ where f = fractional part and I = integral part

Also let $\left(8-3\sqrt{7}\right)^{20} = g$ then $0 < g < I$

Here

$$I + f + g = \left(8+3\sqrt{7}\right)^{20} + \left(8-3\sqrt{7}\right)^{20} = 2\left[8^{20} + {}^{20}C_2.8^{18}.\left(3\sqrt{7}\right)^2 + ... + {}^{20}C_{20}\left(3\sqrt{7}\right)^{20}\right]$$

$\Rightarrow I + f + g$ = even integer

But $0 < f + g < 2$

So, $I + 1 = \text{even Integer} \ (\because f + g = 1)$

$\Rightarrow I = \text{odd Integer}$

5. STATEMENT–1 The middle term in the expansion of $\left(1+3x+3x^2+x^3\right)^{2n}$ is ${}^{6n}C_{3n+1} \, . \, x^{3n+1}$

STATEMENT–2 $\left(1+3x+3x^2+x^3\right)^{2n} = \left\{(1+x)^3\right\}^{2n} = (1+x)^{6n}$

KEY : D

HINT : $\left\{(1+x)^3\right\}^{2n} = (1+x)^{6n}$

The middle term $= T_{3n+1} = {}^{6n}C_{3n}.x^{3n}$

6. (L-3)Statement-1 : The largest exponent of 2 which divides the number N= $2^{2008} + 10^{2008}$ is 2009.

Statement-2 : $5^n + 1$ is divisible by 2 but not divisible by 4, $\forall n \in N$

Key : A

Sol : N = $2^{2008} + 2^{2008} \times 5^{2008}$

= $2^{2008}[1+ 5^{2008}]$

= $2^{2008} [1 + (1 + 4)^{2008}]$

= $2^{2008} [(1+1)+{}^{2008}C_1 . 4 + {}^{2008}C_2 . 4^2 + \ldots\ldots + {}^{2008}C_{2008} . 4^{2008}]$

= $2^{2009} [1 + 2. {}^{2008}C_1 + \ldots\ldots.]$

= $2^{2009} \underbrace{[1+2m], m \in N}_{\text{odd} \Rightarrow \text{not divisible by 2}}$

Hence highest exponent of 2 = 2009

7. (L-2)Consider the sequence $\frac{{}^nC_0}{1.2}, \frac{{}^nC_1}{2.3}, \frac{{}^nC_2}{3.4}, \ldots\ldots$

Statement – 1 : For $n = 12$, 6^{th} term of sequence is greatest.

Statement – 2 : The general term of sequence is $T_r = \frac{{}^nC_{r-1}}{r.(r+1)}$ (for $r = 1, 2, \ldots, n$)

Key : a

Sol : $T_r = \frac{{}^nC_{r-1}}{r(r+1)} = \frac{n!}{(r+1)!(n-r+1)!} = \frac{(n+2)!}{(n+1)(n+2)(r+1)!(n-r+1)!}$

$= \frac{1}{(n+1)(n+2)} \, {}^{(n+2)}C_{r+1}$

If n is even then it is maximum for $r+1=\frac{n+2}{2} \Rightarrow r=\frac{n}{2}$

$\therefore$ for n = 12, r = 6

$\therefore$ 6^{th} term is greatest

8. (L-2)If $x=2+5i$ and $2\left(\frac{1}{1!9!}+\frac{1}{3!7!}\right)+\frac{1}{5!5!}=\frac{2^a}{b!}$. Then

Statement-1 : The value of $\left(x^3-5x^2+33x-19\right)$ is equal to a + b

Statement-2 : Sum of odd binomial coefficient in the expansion of $(a+b)^n$ is 2^{n-1}

Key : D

Sol : Put x = 2 + 5i in given equations.

9. Statement – I : Any positive integral power of $\left(\sqrt{2}-1\right)$ can be expressed as $\sqrt{N}-\sqrt{N-1}$ for some natural number N > 1.

Statement – II : Any positive integral power of $\sqrt{2}-1$ can be expressed as $A+B\sqrt{2}$, where A and B are integers.

Key ; b

Sol : The statement-I is correct since $\left(\sqrt{2}-1\right)^2=3-2\sqrt{2}=\sqrt{9}-\sqrt{8}$

$\left(\sqrt{2}-1\right)^3=5\sqrt{2}-7=\sqrt{50}-\sqrt{49}$ (This can be proved by induction)

The statement-2 is also correct, since any integral power of $\left(\sqrt{2}-1\right)$ will have rational and irrational part.

The irrational part will have only one surd $\sqrt{2}$. Thus $\left(\sqrt{2}-1\right)^n=A+B\sqrt{2}$, where A and B are integers.

But the Statement-2 does not explain the Statement-I.

III. (L-3)If $P_k=1+x+....+x^k, k\in N$ and terms of the product $P_1P_2P_3.....P_n$ obtained are arranged in increasing power of x as $P_1.P_2.P_3....P_n=a_0+a_1x+a_2x^2+....$

10. If n is an even +ve integer, then

STATEMENT–1

${}^{2n}C_1 + {}^{2n}C_3 + {}^{2n}C_5 + ... {}^{2n}C_{n-1} = 2^{n-1}$

because

STATEMENT–2

${}^{2n}C_1 + {}^{2n}C_3 + {}^{2n}C_5 + .. + {}^{2n}C_{2n-1} = 2^{2n-1}$

Key. D

Sol. II is correct. The answer of LHS of I is $\frac{1}{2}\left(2^{\mathbf{2n-1}}\right)=2^{2n-2}$.

11. Statement 1 : The coefficient of x^{50} in the expansion of

$(1+x)^{100}+2x(1+x)^{999}+3x^2(1+x)^{998}+---+1001x^{1000}$ is $1002C_{50}$

Statement 2 : If $C_0, C_1, C_2, ----, C_n$ be the binomial coefficients then

$$\frac{(C_0+C_1)(C_1+C_2)------(C_{n-1}+C_n)}{C_0C_1C_2----C_n}=\left(1+\frac{1}{n}\right)^n$$

Key. C

Sol. I) $\frac{S}{1+x}=x(1+x)^{999}+2x^2(1+x)^{998}+---+1001\frac{x^{1001}}{(1+x)}$

$\therefore S=x(1+x)^{1000}+2x^2(1+x)^{1000}+---+1001x^{1001}$

Subtraction $x=(1+x)^{1002}-x^{1001}(1+x)-1001x^{1001}$

Coefficient of x^{50} in =1002 C_{50}

II) $\frac{(C_0+C_1)(C_1+C_2)----(C_{n-1}+C_n)}{C_0, C_1---C_n}=\frac{(n+1)^n}{\lfloor n}$

$(1+x)^5=C_0+C_1x+C_2x^2+---+C_5x^5$

12. STATEMENT-1 : Numerically greatest term in (1 + [|sin x| + |cos x|])^{20} where [.] denotes greatest integer less than equal to x is T_{11} (11th term).
because

STATEMENT-2 : If $\left|\frac{x}{a}\right|=1$ then numerically greatest term in (a + x)^n is its middle term.

Key. A

Sol. |sin x| + |cos x| is periodic with fundamental period $=\frac{\pi}{2}$

$\Rightarrow$ range of |sin x| + |cos x| is $[1,\sqrt{2}]$

$\Rightarrow$ [|sin x| + |cos x|] = 1

$\Rightarrow$ numerically greatest term in the given binomial is its middle term i.e. t_{11}.

13. STATEMENT-I: If $(2x+3y)^{12}=T_1+T_2+---+T_{13}$ where T_1, T_2,-----T_{13} are terms of binomial expansion when $x=\frac{1}{3}$ and $y=\frac{1}{2}$,then $\sum_{r=1}^{12}\text{sgn}(T_r-T_{r+1})=-4$.

STATEMENT-II : $T_1<T_2<----<T_8<T_9=T_{10}>T_{11}>T_{12}>T_{13}$

Key. D

Sol. $\left(\frac{2}{3}+\frac{3}{2}\right)^{12}=\left(\frac{2}{3}\right)^{12}\left(1+\frac{9}{4}\right)^{12}$

$$\frac{(n+1)|x|}{|x|+1}=\frac{13\times\frac{9}{4}}{\frac{9}{4}+1}=9$$

$\therefore$ $T_9 = T_{10}$ are greatest terms and $T_1<T_2<$ ------ $< T_8<T_9 = T_{10}>T_{11}>T_{12}>T_{13}$

$$\sum_{r=1}^{12}\text{sgn}(T_r-T_{r+1})=(-1)\times 8+0+1\times 3=-5$$

14. STATEMENT-I: ${}^{30}c_{15}-1$ is divisible by 31

STATEMENT-II : If n is a prime, then ${}^{n}c_r$ is divisible by n for r = 1,2,3,----,n-1 and

Key. D

Sol. $${}^{n}c_r=\frac{n(n-1)----(n-r+1)}{1.2.3.---r}$$

${}^{n}c_r$ is a positive integer and hence all the factors of the denominator must cancel out with some of the factors in the numerator. But n, being prime, remains without cancellation. Hence ${}^{n}c_r$ is divisible by *n* for *r* = 1,2,---,*n-1*.

Now ${}^{n}c_r+{}^{n}c_{r-1}={}^{n+1}c_r$ is well known.

$\Rightarrow {}^{n}c_r={}^{n+1}c_r-{}^{n}c_{r-1}$

Using this formula again and again, we can show that

${}^{30}c_{15}={}^{31}c_{15}-{}^{31}c_{14}+{}^{31}c_{13}---+{}^{31}c_3-{}^{31}c_2+{}^{31}c_1-{}^{30}c_0$

$\Rightarrow {}^{30}c_{15}+1={}^{31}c_{15}-{}^{31}c_{14}+---+{}^{31}c_1$

On the R.H.S each term is divisible by 31

$\therefore {}^{30}c_{15}+1$ is divisible by 31.

15. STATEMENT-1: The term independent of x in the expansion of $\left(x+\frac{1}{x}+2\right)^m$ is $\frac{\angle 4m}{(\angle 2m)^2}$

STATEMENT-2: The coefficient of x^k in the expansion of $(1+x)^n$ is nC_k

Key. D

Sol. $T_{r+1}=10C_r(-k)^2x^5-\frac{5r}{2}$

$\therefore 5-\frac{5r}{2}=0\Rightarrow r=2$

$\therefore 10C_2k^2=405,\ \therefore k^2=9,\ \therefore k=\pm 3$

16. STATEMENT -1: The number of ways of writing 1400 as a product of two positive integers is 12

STATEMENT-2: $1400=2^3\times 5^2\times 7$

Key. A

Sol. $\left(x+\frac{1}{x}+2\right)^m=\frac{(x+1)^{2m}}{x^m}$

$\therefore$ Term independent of $x=\frac{2mC_mx^m}{x^m}=(2m)C_m$

17. STATEMENT -1: If n is a positive integer less than 20, then $\angle n\ \angle(20-n)$ is minimum when n = 10

STATEMENT-2: $(2m)C_r$ is maximum when r = m.

Key. D

Sol. $20C_n = \dfrac{\angle 20}{\angle n \angle 20-n}$

$\angle n \angle 20-n$ is minimum $\Rightarrow 20C_n$ is maximum

$\therefore n = 10$

18. Statement - 1: ${}^{30}C_0\,{}^{30}C_{10} + {}^{30}C_1\,{}^{30}C_{11} + {}^{30}C_2\,{}^{30}C_{12} + ... + {}^{30}C_{20}\,{}^{30}C_{30} = {}^{30}C_{10}$.

Statement - 2: ${}^{m}C_0\,{}^{n}C_K + {}^{m}C_1\,{}^{n}C_{K-1} + ... + {}^{m}C_K\,{}^{n}C_0 = {}^{(m+n)}C_K$.

Where $m, n, K \in N$ and ${}^{p}C_q = 0\ \ \forall p < q$

Key. A

Sol. Conceptual

19. Assertion (A) : Coefficient of x^{51} in the expansion of $(x-1)(x^2-2)(x^3-3).......(x^{10}-10)$ is – 1

Reason (R) : Coefficient of $x^{\frac{n(n+1)}{2}-4}$ in the expansion of $(x-1)(x^2-2)(x^3-3)......(x^n-n)$ is $-4+(-1)(-3)=-1$

Key. A

Sol. Conceptual

20. Assertion (A) : If $a \in N$ then the coefficients of x^3 and x^a in $(1+3x+3x^2+x^3)(1+x)^a$ are equal

Reason (R) : If p and q be positive integers then the co-efficients of x^p and x^q in the expansion of $(1+x)^{p+q}$ will be equal

Key. A

Sol. Conceptual

21. Assertion (A) : Co-efficient of x^5 in $(1-x)^{-8}$ is 13_{C_5}

Reason (R) : Co-efficient of x^m in $(1-x)^{-n}$ is $n+m-1_{C_m}$

Key. D

Sol. Conceptual

22. Assertion (A) : If n is odd prime then integral part of $\left(\sqrt{5}+2\right)^n - 2^{n+1}$ is divisible by 20 n

Reason (R) : If n is prime then $n_{c_1}, n_{c_2}, n_{c_3}.........n_{c_{n-1}}$ must be divisible by n

Key. A

Sol. $\left(\sqrt{5}+2\right)^n = N + f$ where n is an integer and $0 < f < 1$

$\left(\sqrt{5}-2\right)^n = f'$ then $0 < f' < 1$

23. Assertion: The term independent of x in the expansion of $\left(x+\frac{1}{x}+2\right)^{2m}$ is $\frac{(4m)!}{(2m)!(2m)!}$

Reason: The coefficient of x^a in the expansion of $(1+x)^n$ is ${}^nC_a$.

1)Assertion is True, Reason is True and Reason is correct explanation of Assertion
2)Assertion is True, Reason is True but Reason is not correct explanation of Assertion
3)Assertion is True, Reason is False
4)Assertion is False, Reason is True

Key. 1

Sol. $\left(x+\frac{1}{x}+2\right)^{2m} = \left(\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2\right)^{2m} = \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^{4m}$

$\therefore\ T_{r+1} = {}^{4m}C_r x^{\frac{4m-r}{2}} x^{\frac{-r}{2}} = {}^{4m}C_r x^{2m-r}$

$\therefore$ Term independent of x in $\left(x+\frac{1}{x}+2\right)^{2m}$ is ${}^{4m}C_{2m} = \frac{(4m)!}{(2m)!(2m)!}$ followed by the reason (R)

24. Statement – 1: Coefficient of $a^2b^3c^4$ in the expansion of $(a+b+c)^8$ is $\frac{8!}{2!3!4!}$

Statement – 2: Coefficient of $a^\alpha b^\beta c^\gamma$, where $\alpha+\beta+\gamma=n$, in the expansion of $(a+b+c)^n$ is $\frac{n!}{\alpha!\beta!\gamma!a}$.

Key. D

Sol. Q $\quad (a+b+c)^n = \sum \frac{n!}{p!.q!.r!}.a^pb^qc^r,\ p+q+r=n$

In statement – 1 p + q + r exceeds n

25. Statement – 1: If $q=\frac{1}{3}$ and p + q = 1, then $\sum_{r=0}^{15} r\,{}^{15}C_r p^r q^{15-4} = 15\times\frac{1}{3}=5$

Statement – 2: If p + q = 1, 0 < p < 1, then $\sum_{r=0}^{n} r\,{}^{n}C_r p^r\ q^{n-r} = np$

Key. D

Sol. $\sum_{r=0}^{n} r\,{}^nc_r p^r q^{n-r} = np\sum_{r=0}^{n}\ {}^{n-1}C_{r-1}p^{r-1}q^{n-r} = np(q+p)^{n-1} = np$

26. Statement – 1: Coefficient of x^{51} in the expension of $(x-1)(x^2-2)(x^3-3)...(x^{10}-10)$ is – 1

Statement – 2: Coefficient of $x^{\frac{n(n+1)}{2}-4}$, $n\geq 4, n\in N$, in the expansion of $(x-1)(x^2-2)(x^3-3)...(x^n-n)$ is – 4 + (- 1) (- 3) = - 1

Key. A

Sol. If $n\geq 4$, then term contain ing $x^{\frac{n(n+1)}{2}-4}$ is $(-4)x^{1+2+3+5+6+7+...+n} + (-1)(-3)x^{2+4+5+6+...+n}$

∴ coefficient of $x^{\frac{n(n+1)}{2}-4}$ is – 4 (- 1) (- 3) = - 1

Statement – 2 is true and it explains statement – 1

27. Statement – 1: If $r \geq 40$, then greatest possible value of $^{40}C_0, {}^{60}C_r + {}^{40}C_1\ {}^{60}C_{r-1}.... {}^{40}C_{40}, {}^{60}C_{r-40}$ is $^{100}C_{50}$

Statement – 2: The greatest value of $^{2n}C_r$ occurs at r = n,

Key. A

Sol. The number of ways of selecting committee of r persons among 40 women and 60 men $= {}^{100}C_r$

The will assume greatest value at r = 50

28. Statement – 1: If $x = {}^nC_{n-1} + {}^{n+1}C_{n-1} + {}^{n+2}C_{n-1} + ... + {}^{2n}C_{n-1}$, then $\frac{x+1}{2n+1}$ is integer

Statement – 2: $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$ and ^nC_r is divisible by n if n and r are co-prime.

Key. A

Sol. $1 + x = {}^nC_n + {}^nC_{n-1} + {}^{n+1}C_{n-1} + ... + {}^{2n}C_{n-1} = {}^{2n+1}C_n$

Since 2n + 1 and n are coprime for every natural number n.

∴ $^{2n+1}C_n$ is divisible by 2n + 1

∴ $\frac{x+1}{2n+1}$ is an integer

29. **Statement 1:** If $(2x+3y)^{12} = T_1 + T_2 + --- + T_{13}$ where $T_1, T_2, ---T_{13}$ are terms of binomial expansion when $x = \frac{1}{3}$ and $y = \frac{1}{2}$, then $\sum_{r=1}^{12} sgn(T_r - T_{r+1}) = -4$. (sgn indicates signum function)

Statement 2: $T_1 < T_2 < --- < T_8 < T_9 = T_{10} > T_{11} > T_{12} > T_{13}$

Key. D

Sol. $\left(\frac{2}{3}+\frac{3}{2}\right)^{12} = \left(\frac{2}{3}\right)^{12}\left(1+\frac{9}{4}\right)^{12}$

$$\frac{(n+1)|x|}{|x|+1} = \frac{13\times\frac{9}{4}}{\frac{9}{4}+1} = 9$$

$\therefore T_9 = T_{10}$ are greatest terms and $T_1 < T_2 < .... < T_8 < T_9 = T_{10} > T_{11} > T_{12} > T_{13}$

$$\sum_{r=1}^{12} sgn(T_r - T_{r+1}) = (-1)\times 8 + 0 + 1\times 3 = -5$$

30. **Statement 1:** $^{30}C_{15} - 1$ is divisible by 31

Statement 2: If n is a prime, then ^nC_r is divisible by n for $r = 1,2,3,.....,n-1$ and $^nc_r + {}^nc_{r-1} = {}^{n+1}c_r$

Key. D

Sol. $^nc_r = \frac{n(n-1)----(n-r+1)}{1.2.3.---r}$

^nc_r is a positive integer and hence all the factors of the denominator must cancel out with some of the factors in the numerator. But n, being prime, remains without cancellation. Hence ^nc_r is divisible by n for $r = 1, 2, ---, n-1$.

Now $^nC_r + ^nC_{r-1} = ^{n+1}C_r$ is well known.

$\Rightarrow ^nC_r = ^{n+1}C_r - ^nC_{r-1}$

Using this formula again and again, we can show that

$^{30}C_{15} = ^{31}C_{15} - ^{31}C_{14} + ^{31}C_{13} - - - + ^{31}C_3 - ^{31}C_2 + ^{31}C_1 - ^{30}C_0$

$\Rightarrow ^{30}C_{15} + 1 = ^{31}C_{15} - ^{31}C_{14} + - - - + ^{31}C_1$

On the R.H.S each term is divisible by 31

$\therefore$ $^{30}c_{15} + 1$ is divisible by 31.

31. Let T_r stands for r th term in the expansion of $(1+x)^{10}$, where $x = 2/3$.

Statement 1: $\frac{T_{r+1}}{T_r} < 1$ has 5 solutions for r

Statement 2: $\frac{T_{r+1}}{T_r} > 1$ for at least one integer r.

Key. D

Sol. $\left(\frac{Tr+1}{Tr} = \frac{10_{C_r}x}{10_{C_r} - 1} = \frac{11-r}{r}.\frac{2}{3} < 1\right.$, r = 5, 6, 7, 8, 9, 10. Statement 1 is false. Statement 2 is true).

32. Let n be a positive integer.

Statement 1: If $\left(1-x+x^2\right)^n = \sum_{r=0}^{2n} a_r x^r$, then $\sum_{r=0}^{n} a_{2r} = \frac{\left(1+3^n\right)}{2}$

Statement 2: Sum of the coefficients of even powers of x in $(f(x))^n$ where f (x) is a polynomial is $\frac{\left(f(1)\right)^n + \left(f(-1)\right)^n}{2}$

Key. A

Sol. Statements 1,2 are true. 2 explains 1.

33. Statement-1: The number of non negative integral solutions of 2x + y + z = 21 is 132

Statement-2: For $n \in N, \left(n^2\right)!$ is divisible by $(n!)^n$.

Key. B

Sol. Statement I : Req. number of solutions

$= \sum_{x=0}^{10} {}^{21-x+2-1}C_{2-1} = 132.$

Statement II : Is true by the definition of n!.

34. Statemnet-1: The Harmonic mean of $^{2n+1}C_r$ and $^{2n+1}C_{r+1}$ is $^{2n}C_r$

Statement-2: $\sum_{r=1}^{2n-1}(-1)^{r-1}\frac{r}{^{2n}C_r} = \frac{n}{n+1}$

Key. D

Sol. Statement I is false.

Binomial Theorem

Comprehension Type

Paragraph – 1

Let n be positive integer such that $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx_n$ then

1. The value of $\sum\sum_{0\le r<s\le n}(r+s)(C_r + C_s + C_rC_s)$, is

(A) $n^2.2^n - \frac{n}{2}\left[2^{2n} - {}^{2n}C_n\right]$ (B) $n^2.2^n + \frac{n}{2}\left[2^{2n} - {}^{2n}C_n\right]$

(C) $n^2.2^n - \frac{n}{2}\left[2^{2n} + {}^{2n}C_n\right]$ (D) $n^2.2^n + \frac{n}{2}\left[2^n - 1\right]$

Key. B

2. The value of $\sum_{r=0}^{n}\sum_{s=0}^{n}\sum_{t=0}^{n}\sum_{u=0}^{n}(1)$, is

(A) ${}^nC_4$ (B) ${}^{n+1}C_4$ (C) ${}^{4n+1}C_4$ (D) $(n+1)^4$

Key. D

3. The value of $\sum\sum_{0\le i<j\le n} i.{}^nC_j$ is equal to

(A) $n(n+1).2^{n-3}$ (B) $n^2.2^{n-3}$ (C) $n(n-1).2^{n-3}$ (D) None of these

Key. C

Paragraph – 2

If $(1+x+x^2)^{100} = \sum_{r=0}^{200} a_r x^r$

4. Which of the following is true

(A) $a_{28} = a_{72}$ (B) $a_{56} = a_{144}$ (C) $a_{200} = a_{300}$ (D) None of these

Key: B

Hint: $\sum_{r=0}^{200} a_r\left(\frac{1}{x}\right)^r = \left(1+\frac{1}{x}+\frac{1}{x^2}\right)^{100} = \frac{1}{x^{200}}(x^2+x+1)$

$\Rightarrow \sum_{r=0}^{200} a_r x^{200-r} = (x^2+x+1)^{100} = \sum_{r=0}^{200} a_r x^r = \sum_{r=0}^{200} a_{200-r} x^{200-r}$

Equating the coefficients of x^{200-r} we get $a_r = a_{200-r}$

5. $a_0 + a_1 + a_2 + ... + a_{99}$ is equal to

(A) $\frac{3^{99} - a_{99}}{2}$ (B) $\frac{3^{101} - a_{99}}{2}$ (C) $\frac{3^{100} - a_{100}}{2}$ (D) None of these

Key: C

Hint: Put x = 1

$a_0 + a_1 + a_2 + ... + a_{200} = 3^{100}$

But $a_r = a_{200-r}$

$\therefore 2(a_0 + ... + a_{99}) + a_{100} = 3^{100}$

$a_0 + a_1 + ... + a_{99} = \frac{1}{2}(3^{100} - a_{100})$

6. $37a_{37}$ is equal to

(A) $64a_{36} + 165a_{35}$ (B) $64a_{35} + 148a_{36}$ (C) $56a_{32} + 168a_{22}$ (D) None of these

Key: A

Hint: Differentiating we get

$$100(1+2x)(1+x+x^2)^{99} = \sum_{r=0}^{200} ra_r x^{r-1}$$

Paragraph – 3

Let $(1 + x)^m = C_0 + C_1x + C_2x^2 + ... + C_mx^m$

where $C_r = {}^mC_r = \frac{m!}{r!(m-r)!}$

$$\sum_{0\le i}\sum_{<j\le n} f(i, j) = \sum_{0\le i<j\le n} f(n-i, n-j)$$, if f(i, j) = f(j, i)

7. $\sum_{0\le i}\sum_{<j\le m}$ (i + j) C_i C_j is equal to

(A) m (2^{m-1}) (B) $m(2^{2m-1} - {}^{2m-1}C_{m-1})$ (C) $m(2^{m-1} - {}^{2m-1}C_{m-1})$ (D) none of these

Key: B

Hint: $\sum_{0\le i<j\le m} (i+j)C_iC_j$ = n $\sum_{0\le i<j\le m} C_iC_j$

Now $(\Sigma C_i)^2 = \Sigma C_i^2 + 2\Sigma\Sigma(C_iC_j \Rightarrow \sum_{0\le i<j\le m} (i+j)C_iC_j = m\,[2^{2m-1} - {}^{2m-1}C_{m-1}]$

8. $\sum_{0\le i}\sum_{<j\le m} i.C_j$ is equal to

(A) $m(m - 1)\, 2^{m-2}$ (B) $m(m - 1)\, 2^{m-3}$ (C) 2^{2m} (D) $2^{2m} - {}^{2m}C_m$

Key: B

Hint: $\sum_{0\le i<}\sum_{j\le m} iC_j$ = $(C_2 + C_3 + ... + C_m) + 2.\,[C_3 + C_4 + ... + C_m)$

$+ 3(C_4 + C_5 + ... + C_m) + (m-1)\, C_m$

$= \sum_{r=2}^{m} \frac{r(r-1)}{2} C_r = m(m-1)\, 2^{m-3}$

9. $\sum_{0\le i}\sum_{<j\le m} C_i^2$ is equal to

(A) $({}^{2m}C_m)^2$ (B) $(2^{2m} - {}^{2m}C_m)^2$ (C) $m\, {}^{2m-1}C_{m-1}$ (D) none of these

Key: C

Hint: $\sum_{0\le i<}\sum_{j\le m} C_i^{\,2} = \sum_{r=0}^{m}(m-r)C_r^{\,2}$

$= \sum_{r=0}^{m} r.C_r^{\,2}$ = m $^{2m-1}C_{m-1}$

Paragraph – 4.

(L-3) Given that $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$, where x is complex number and $C_0, C_1, ..., C_n$ are constants. Then,

10. The value of $C_1 + C_4 + C_7 + ...$ will be

a) $\frac{1}{3}\left[2^n + 2\cos(n-2)\frac{\pi}{4}\right]$

b) $\frac{1}{3}\left[2^n + 2\cos(n-2).\frac{\pi}{3}\right]$

c) $\frac{1}{3}\left[2^n - 2\cos(n-2)\frac{\pi}{4}\right]$

d) $\frac{1}{3}\left[2^n - 2\cos(n-2).\frac{\pi}{3}\right]$

Key : b

Sol : $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + ...$

Multiply both sides by x^2

$x^2(1+x)^n = C_0x^2 + C_1x^3 + C_2x^4 + C_3x^5 + C_4x^6 + C_5x^7 + C_6x^8 + C_7x^9 + ...$ ……(1)

Now put x = 1, ω, ω^2 in (1) we get,

$1.2^n = C_0 + C_1 + C_2 + C_3 + C_4 + C_5 + C_6 + C_7 + ...$

$\omega^2(1+\omega)^n = C_0\omega^2 + C_1 + C_2\omega + C_3\omega^2 + C_4 + C_5\omega + C_6\omega^2 + C_7 + ...$

$\omega\left(1+\omega^2\right)^n = C_0\omega + C_1 + C_2\omega^2 + C_3\omega + C_4 + C_5\omega^2 + C_6\omega + C_7 + ...$

Then add we get,

$2^n + \omega^2(1+\omega)^n + \omega\left(1+\omega^2\right)^n = 3(C_1 + C_4 + C_7 + ...)$

$\therefore 3(C_1 + C_4 + C_7 + ...) = 2^n + e^{\frac{4\pi i}{3}}.e^{\frac{n\pi i}{3}} + e^{\frac{2\pi i}{3}}.e^{\frac{-n\pi i}{3}}$

$$\left[\begin{array}{l} Q \quad \omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{\frac{2\pi i}{3}} \\ \omega^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2} = e^{\frac{4\pi i}{3}} \\ 1+\omega = \frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{\frac{i\pi}{3}} \\ 1+\omega^2 = \frac{1}{2} - i\frac{\sqrt{3}}{2} = e^{-\frac{i\pi}{3}} \end{array}\right]$$

$$= 2^n + e^{-\frac{2\pi i}{3}}.e^{\frac{n\pi i}{3}} + e^{\frac{2\pi i}{3}} + e^{\frac{n\pi i}{3}}$$

$$= 2^n + e^{\frac{(n-2)\pi i}{3}} + e^{-\frac{(n-2)\pi i}{3}} = 2^n + 2\cos\frac{n-2}{3}\pi$$

$$\therefore C_1 + C_4 + C_7 + ... = \frac{1}{3}\left[2^n + 2\cos\frac{n-2}{3}\pi\right]$$

11. (L-3)The value of $C_0 + C_3 + C_6 + ...$ will be

a) $\frac{1}{3}\left(2^n + 2\cos\frac{n\pi}{3}\right)$ b) $\frac{1}{3}\left(2^n - 2\cos\frac{n\pi}{3}\right)$ c) $\frac{1}{3}\left(2^n + 2\cos\frac{n\pi}{4}\right)$ d) $\frac{1}{3}\left(2^n - 2\cos\frac{n\pi}{4}\right)$

Key : a

Sol : $\left(1+x^n\right) = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + ...$

Put x = 1, ω, ω^2 and adding we get,

$$3\left(C_0 + C_3 + C_6 + ...\right) = 2^n + \left(-\omega^2\right)^n + \left(-\omega\right)^n$$

$$\left[Q\, 1 + \omega + \omega^2 = 0\right]$$

$$-\omega = \frac{1}{2} - i\frac{\sqrt{3}}{2} = \cos\frac{\pi}{3} - i\sin\frac{\pi}{3}$$

$$-\omega^2 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$$

$$\therefore C_0 + C_3 + C_6 + ... = \frac{1}{3}\left[2^n + \cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3} + \cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right] = \frac{1}{3}\left[2^n + 2\cos\frac{n\pi}{3}\right]$$

12. The value of $C_2 + C_5 + C_8 + ...$ will be

a) $\frac{1}{3}\left[2^n + 2^2\cos(n-2)\frac{\pi}{3}\right]$ b) $\frac{1}{3}\left[2^n - 2\cos(n-2)\frac{\pi}{3}\right]$

c) $\frac{1}{3}\left[2^n - 2^2\cos(n-2)\frac{\pi}{3}\right]$ d) $\frac{1}{3}\left[2^n + 2\cos(n+2)\frac{\pi}{3}\right]$

Key ; d

Sol : $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + C_5x^5 + C_6x^6 + C_7x^7 + C_8x^8 + ...$

Multiply by x

$$x(1+x)^n = C_0x + C_1x^2 + C_2x^3 + C_3x^4 + C_4x^5 + C_5x^6 + C_6x^7 + C_7x^8 + C_8x^9 + ...$$

Put x = $1, \omega, \omega^2$ we get

$$1.2^n = C_0 + C_1 + C_2 + C_3 + C_4 + C_5 + C_6 + C_7 + C_8 + ...$$

$$\omega(1+\omega)^n = C_0\omega + C_1\omega^2 + C_2 + C_3\omega + C_4\omega^2 + C_5 + C_6\omega + C_7\omega^2 + C_8 + ...$$

$$\omega^2(1+\omega^2)^n = C_0\omega^2 + C_1\omega + C_2 + C_3\omega^2 + C_4\omega + C_5 + C_6\omega^2 + C_7\omega + C_8 + ...$$

Adding these we get,

$$3.(C_2 + C_5 + C_8 + ...) = 2^n + e^{\frac{2\pi i}{3}}.e^{\frac{in\pi}{3}} + e^{\frac{4\pi i}{3}}.e^{-\frac{n\pi i}{3}}$$

$$= 2^n + e^{\frac{(n+2)\pi i}{3}} + e^{-\frac{(n+2)\pi i}{3}}$$

$$= 2^n + 2\cos\frac{(n+2)\pi}{3}$$

$$\therefore C_2 + C_5 + C_8 + ... = \frac{1}{3}\left[2^n + 2\cos(n+2)\frac{\pi}{3}\right]$$

Paragraph – 5.

(L-2) Integration plays a vital role in proving identities involving binomial coefficients whose algebraic method of proving is in genera cumbersome and requires the help of mathematical induction. If we apply integration techniques, several binomial identities are easily proved.

For instance $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - .... + \frac{(-1)^n C_n}{n+1} = \int_0^1 (1-x)^n dx = \frac{1}{n+1}$, where

$C_r = {}^nC_r, r = 0,1,2,.....n$

13. If $S_1 = \sum_{k=0}^{n} \frac{(-1)^k \ {}^nC_k}{k+m+1}, S_2 = \sum_{k=0}^{m} \frac{(-1)^k \ {}^mC_k}{k+n+1}$ where m > n then

a) $S_2 = \frac{m+n}{m-n} S_1$ b) S_2 = - S_1 c) $S_2 = S_1$ d) $S_1 = S_2 = 0$ for all m and n

Key : c

Sol : $S_1 = \int_0^1 x^m (1-x)^n dx; S_2 = \int_0^1 x^n (1-x)^m dx \Rightarrow S_1 = S_2$

14. The value of binmial series $C_1 - \frac{C_2}{2} + \frac{C_3}{3} - .... + \frac{(-1)^{n-1} C_n}{n}$ must be equal to

a) $1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$ b) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... + \frac{(-1)^{n-1}}{n}$

c) $1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2}$ d) $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ... + \frac{(-1)^{n-1}}{n^2}$

Key : a

Sol : Given series = $\int_0^1 \frac{1-(1-x)^n}{x} dx = \int_0^1 \frac{1-x^n}{1-x} dx$

$= 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$

15. The value of series $C_0 - \frac{C_1}{3} + \frac{C_2}{5} - \frac{C_3}{7} + ... + + \frac{(-1)^n C_n}{2n+1}$ must be equal

a) $\frac{2^{2n}(2n+1)!}{(n!)^2}$ b) $\frac{2^{2n}(n!)^2}{(2n+1)!}$ c) $2^{2n} (n!)^2$ d) $\frac{2n}{2n+1}$

Key : b

Sol : Given series = $\int_0^1 (1-x^2)^n dx = \int_0^{\pi/2} \cos^{2n+1}\theta \, d\theta$

$= \frac{2n}{2n+1} . \frac{2n-2}{2n-1} . \frac{2n-4}{2n-3} .... \frac{2}{3} = \frac{2^{2n}.(n!)^2}{(2n+1)!}$

Paragraph – 6

Consider a G.P. with first term $(1+x)^n, (|x| < 1)$, common ratio $\left(\frac{1+x}{2}\right)$ and number of terms (n+1). Let 'S' be sum of all the terms of the G.P., then

16. 'S' equals

(A) $\frac{1}{2^n}\left[2^{n+1}(1+x)^n - (1+x)^{2n+1}\right] . \lim_{m\to\infty} \sum_{k=0}^{m} x^k$

(B) $\frac{1}{2^n}\left[2^{n+1}(1+x)^{n+1} - (1+x)^{2n}\right] . \lim_{m\to\infty} \sum_{k=0}^{m} x^k$

(C) $\frac{1}{2^n}\left[2^{n}(1+x)^{n+1} - (1+x)^{2n+1}\right] . \lim_{m\to\infty} \sum_{k=0}^{m} x^k$

(D) $\frac{1}{2^n}\left[2^n(1+x)^n-(1+x)^{2n}\right].\lim_{m\to\infty}\sum_{k=0}^{m}x^k$

Key. A

Sol. S = $\frac{(1+x)^n\left[1-\left(\frac{1+x}{2}\right)^{n+1}\right]}{1-\left(\frac{1+x}{2}\right)}=\frac{1}{2^n}\frac{(1+x)^n.[2^{n+1}-(1+x)^{n+1}]}{(1-x)}$

= $\frac{1}{2^n}[2^{n+1}(1+x)^n-(1+x)^{2n+1}]\ .\ \frac{1}{1-x}$

= $\frac{1}{2^n}[2^{n+1}(1+x)^n-(1+x)^{2n+1}]\ .\ (1+x+x^2+\ldots\infty)$

17. The coefficient of x^n in 'S' is

(A) 2^n (B) 2^{n+1}

(C) 2^{2n} (D) 2^{2n+1}

Key. A

Sol. The coefficient of x^n in S = $\frac{1}{2^n}\left[2^{n+1}\sum_{r=0}^{n}{}^nC_r-\sum_{r=0}^{n}{}^{2n+1}C_r\right]$

= $\frac{1}{2^n}[2^{n+1}.2^n-\frac{1}{2}2^{2n+1}]$

= $\frac{1}{2^n}\left[2.2^{2n}-2^{2n}\right]=\frac{2^{2n}}{2^n}=2^n$

18. $\sum_{r=0}^{n}{}^{n+r}C_r\left(\frac{1}{2}\right)^r$ equals

(A) $(3/4)^n$ (B) 1

(C) 2^n (D) 3^n

Key. C

Sol. $\sum_{r=0}^{n}{}^{n+r}C_r\left(\frac{1}{2}\right)^r={}^nC_n\left(\frac{1}{2}\right)^0+{}^{n+1}C_n\left(\frac{1}{2}\right)^2+{}^{n+2}C_n\left(\frac{1}{2}\right)^2+\ldots+{}^{24}C_n\left(\frac{1}{2}\right)^n$

= coefficient of x^n in

$\left[(1+x)^n\left(\frac{1}{2}\right)^0+(1+x)^{n+1}\left(\frac{1}{2}\right)^1+(1+x)^{n+2}\left(\frac{1}{2}\right)^2+\ldots+(1+x)^{2n}\left(\frac{1}{2}\right)^n\right]$

= coefficient of x^n in S = 2^n

Paragraph – 7

Let $(1+x)^5=\sum_{r=0}^{5}C_rx^r$ when $C_r=5C_r$. Now answer the following questions.

19. $\sum_{0\le r<s\le 5}\sum C_r.C_s=$

a) 283 b) 386 c) 397 d) 287

Key. B

20. $\sum_{0\le r<s\le 5}\sum\left(C_r-C_s\right)^2=$

a) 488 b) 772 c)740 d) 284

Key. A

21. $\sum_{0\le r<s\le 5}\sum (r+s).C_rC_s =$

a) 1435 b) 1935 c) 1930 d) 1415

Key. C

Sol. 19. $\sum_{0\le r}\sum_{<r\le 5} C_r.C_s = \left(\sum_{r=0}^{5} C_r\right)^2 - \sum_{r=0}^{5} C_r^2$

$= 2^{10} - 10C_5$

20. $\sum_{0\le r}\sum_{<r\le 5}(C_r - C_s)^2 = 5\left(C_0^2 + C_1^2 + ---- + C_5^2\right) - 2\sum_{0\le r}\sum_{<r\le 5} C_r.C_s$

$= 1260 - 772 = 488$

21. $S = \sum_{0\le r}\sum_{<r\le 5}(r+s)C_rC_s$

$2S = 10\sum_{0\le r}\sum_{<r\le 5} C_rC_s = 3860$

S = 1930

Paragraph – 8

Consider the expansion of $(a + b + c + d)^{10}$

22. The number of terms in the expansion is

a) ${}^{14}c_4$ b) ${}^{14}c_3$ c) ${}^{13}c_4$ d) ${}^{13}c_3$

Key. D

223. The greatest coefficient in the expansion is

a) 25200 b) 14400 c) 22500 d) 32400

Key. A

24. The number of terms in the expansion having the greatest coefficient is

a) 2 b) 4 c) 6 d) 8

Key. C

Sol. 22. No.of terms in the expansion of $(x_1 + x_2 + --- + x_r)^n = {}^{n+r-1}c_n$

$\therefore$ No.of terms in $(a+b+c+d)^{10} = {}^{10+4-1}c_{10} = {}^{13}c_{10} = {}^{13}c_3$

23. Greatest coefficient = $\dfrac{10!}{2!2!3!3!} = 25200$

Greatest coefficient occurs in 6 terms. They are $a^2b^2c^3d^3$, $a^2b^3c^2d^3$, $a^2b^3c^3d^2$, $a^3b^2c^2d^3$, $a^3b^2c^3d^2$ and $a^3b^3c^2d^2$.

Paragraph – 9

If n is a positive integer, then $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$ where

$C_r = nC_r = \dfrac{\angle n}{\angle r\angle(n-r)}$. Answer the questions 15, 16 and 17.

25. The value of $nC_0 - nC_2 + nC_4 - nC_6 + ...$ is

a) $2^{\frac{n}{2}}\cos\dfrac{n\pi}{2}$ b) $2^{\frac{n}{2}}\sin\dfrac{n\pi}{2}$

c) $2^{\frac{n}{2}}\cos\frac{n\pi}{4}$ d) $2^{\frac{n}{2}}\sin\frac{n\pi}{4}$

Key. C

26. $\left(nC_0 - nC_2 + nC_4 - nC_6 + ...\right)^2 + \left(nC_1 - nC_3 + nC_5 - nC_7 + ...\right)^2$ is

a) 2^{2n} b) 2^n c) 2^{n^2} d) $2^{\frac{n+1}{2}}$

Key. B

27. $nC_0 + nC_4 + nC_8 + ...$ is equal to

a) $2^{\frac{n}{2}}\cos\frac{n\pi}{8}$ b) $2^{\frac{n}{2}}\sin\frac{n\pi}{8}$ c) $2^n + 2^{\frac{n}{2}}\cos\frac{n\pi}{4}$ d) $2^{n-2} + 2^{\frac{n-2}{2}}\cos\frac{n\pi}{4}$

Key. D

Sol. 25. LLL, AA, P, RE

No. of selections = coefficient of x^4 in the

expansion of $\left(1+x+x^2+x^3\right)\left(1+x+x^2\right)\left(1+x\right)^3$ which is not 15.

26. put x = i in the expansion and equate real and imaginary parts.

27. In problem 15, when we put x = i. Then $\left(nC_0 - nC_2 + nC_4 - nC_6\right) + i\left(nC_1 - nC_3 + nC_5...\right)$

$= 2^{\frac{n}{2}}\left[\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right]$

Take modulus both sides and square it.

Use $C_0 + C_2 + C_4 + C_6 + ... = 2^{n-1}$

Paragraph – 10

Let $(1+x)^5 = \sum_{r=0}^{5} C_r x^r$ where $C_r = {}^5C_r$.

28. $\sum\sum_{0\le r<s\le 5} C_r C_s =$

A) 283 B) 386 C) 397 D) 287

Key. B

29. $\sum\sum_{0\le r<s\le 5} (C_r - C_s)^2 =$

A) 488 B) 772 C) 740 D) 284

Key. A

30. $\sum\sum_{0\le r<s\le n} (r+s) C_r C_s =$

A) 1435 B) 1935 C) 1930 D) 1415

Key. C

Sol. 28. $(1+x)^5 = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + C_5 x^5$

$(C_0 + C_1 + C_2 + C_3 + C_4 + C_5)^2 = (C_0^2 + C_1^2 + C_2^2 + C_3^2 + C_4^2 + C_5^2) + 2\sum\sum_{0\le r<s\le 5} C_r C_s$

$(2^5)^2 = {}^{10}C_5 + 2\sum\sum_{0\le r<s\le 5} C_r C_s$

$\therefore \sum\sum_{0\le r<s\le 5} C_r C_s = \frac{1}{2}\left(2^{10} - {}^{10}C_5\right) = \frac{1}{2}(1024 - 252) = 386$

29.

$$\therefore \sum_{0\le r<s\le 5}\sum (C_r - C_s)^2 = (C_0 - C_1)^2 + (C_0 - C_2)^2 + ... + (C_0 - C_5)^2$$
$$+(C_1 - C_2)^2 + ... + (C_1 - C_5)^2 + (C_2 - C_3)^2 + (C_2 - C_4)^2 + (C_2 - C_5)^2$$
$$+(C_3 - C_4)^2 + (C_3 - C_5)^2 + (C_4 - C_5)^2$$
$$= 5\left(C_0^2 + C_1^2 + C_2^2 + C_3^2 + C_4^2 + C_5^2\right) - 2\sum_{0\le r<s\le 5}\sum C_r C_s = 1260 - 772 = 488$$

30. Let

$$S = \sum_{0\le r<s\le 5}\sum (r+s)C_r C_s = \sum_{0\le r<s\le 5}\sum (5-r+5-s)C_{5-r}C_{5-s} = \sum_{0\le r<s\le 5}\sum (10)C_r C_s - \sum_{0\le r<s\le n}\sum (r+s)C_r C_s$$
$$\therefore 2S = 10\sum_{0\le r<s\le 5}\sum C_r C_s = 3860$$
$$\therefore S = 1930$$

Paragraph – 11

If n is any rational number

$(1+x)^n = 1 + \frac{n}{\lfloor 1}x + \frac{n(n-1)}{\lfloor 2}x^2 + \frac{n(n-1)(n-2)}{\lfloor 3}x^3 + ... + \frac{n(n-1)...(n-r+1)}{\lfloor r}x^r + ...\infty$ where $-1 < x < 1$

31. Co-efficient of x^2 in the expansion $\frac{4+x}{\sqrt{1+4x}}$ is

(a) 22 (b) 11 (c) – 7 (d) – 74

Key. A

Sol. $\frac{4+x}{\sqrt{1+4x}} = (4+x)(1-2x+6x^2-20x^3.....)$

$= 4 - 7x + 22x^2 - 74x^3 + ...$

32. The $(r+1)^{th}$ term in the expansion of $(1-2x)^{-2/3}$

(a) $\frac{2.5.8.........(3r-1)}{\lfloor r}\left(\frac{2x}{3}\right)^r$ (b) $\frac{2.5.8.........(3r-1)}{\lfloor r}x^r$

(c) $\frac{2.5.8......(3r-1)}{\lfloor r}\left(\frac{3x}{2}\right)^r$ (d) $\frac{2.5.8......(3r+1)}{\lfloor r}x^r$

Key. A

Sol. Conceptual

33. If $y = 2x + 3x^2 + 4x^3 + ...\infty$ then x =

(a) $1 + \frac{1}{\sqrt{1+y}}$ (b) $1 - \frac{1}{\sqrt{1+y}}$ (c) $1 + \sqrt{1+y}$ (d) $\sqrt{1+y} - 1$

Key. B

Sol. Conceptual

Paragraph – 12

If n is a positive integer and $a_1a_2a_3.........a_m \in C$ then

$(a_1+a_2+a_3+......a_m)^n = \sum \frac{\lfloor n}{\lfloor n_1 \lfloor n_2 ......\lfloor n_m} a_1^{n_1} a_2^{n_2} a_3^{n_3} ....a_m^{n_m}$ where $n_1n_2n_3.......n_m$ are all non-negative integers subject to the condition $n_1+n_2+n_3+......+n_m = n$

34. The co-efficient of $a^3b^4c^5$ in the expansion of $(bc+ca+ab)^6$ is

(a) 60 (b) 50 (c) 40 (d) 120

Key. A

Sol. $a^3b^4c^5 = (ab)^x(bc)^y(ca)^z$

$= a^{z+x}b^{x+y}c^{y+z}$

$\Rightarrow z+x=3, x+y=4, y+z=5$

$\Rightarrow x+y+z=6$

Put $x=1, y=3, z=2$

35. Find the greatest co-efficient in the expansion of $(a+b+c+d)^{15}$ is

(a) $\frac{\lfloor 15}{(\lfloor 3)^3(\lfloor 4)^3}$ (b) $\frac{\lfloor 15}{(\lfloor 3)(\lfloor 4)^3}$ (c) $\frac{\lfloor 15}{\lfloor 3 \lfloor 4}$ (d) $\frac{\lfloor 15}{(\lfloor 3)^2 \lfloor 4}$

Key. B

Sol. put $n=15, m=4$

36. The total number of terms in the expansion $(a+b+c)^{10}$ are

(a) 65 (b) 66 (c) 67 (d) 68

Key. B

Sol. $10+3-1c_{10} = 12_{c_{10}} = 66$

Paragraph – 13

If $\forall n \in N \left(\sum_{r=0}^{2} x^r\right)^n = a_0+a_1x+a_2x^2+......+a_{2n}x^{2n}$ where $a_1a_2......a_{2n} \in Z$

37. $\sum_{r=0}^{2n} \frac{r}{a_r} =$

(a) $n\sum_{r=0}^{2n}\frac{1}{a_r}$ (b) $2n\sum_{r=0}^{2n}\frac{1}{a_r}$ (c) $\frac{3n}{2}\sum_{r=0}^{2n}\frac{1}{a_r}$ (d) none of these

Key. A

Sol. Since $a_r = a_{2_{n-r}}$

$S = \sum_{r=0}^{2n}\frac{r}{a_r} = \sum_{r=0}^{2n}\frac{2n-r}{a_r}$

$S = n\sum_{r=0}^{2n}\frac{1}{a_r}$

38. If $a=\sum_{k=1}^{[n/2]} n_{C_{2k}}\, 2k_{C_k}$ where [ ] denotes greatest integer function then $(a_n - a)$ is equal to

(a) 0 (b) 1 (c) – 1 (d) 2

Key. B

Sol. Since $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n}x^{2n}$

Dividing both sides by x^n and comparing

Constant terms we have

$a_n = n_{c_0} + n_{c_2}.2_{c_1} + n_{c_4}.4_{c_2} + ...$

= $a+1$

39. If $a_0{}^2 - a_1{}^2 + a_2{}^2 - a_3{}^2 + ... + a_{2n}{}^2 = ka_n$ then k =

(a) 1 (b) 2 (c) 1 / 2 (d) 0

Key. A

Sol. $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n}x^{2n}$

Replacing by $-\frac{1}{x}$ we have

$\left(1-\frac{1}{x}+\frac{1}{x^2}\right)^n = a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} + ... + \frac{a_{2n}}{x^{2n}}$

Now $a_0{}^2 - a_1{}^2 + a_2{}^2 - a_3{}^2 + ... + a_{2n}{}^2 =$ coefficient of x in $\left(1+x+x^2\right)^n\left(1-\frac{1}{x}+\frac{1}{x^2}\right)^n$

$= a_n$

Paragraph – 14

Let n be a positive integer such that

$\left(1+x+x^2\right)^n = a_0 + a_1x + a_2x^2 + ... + a_2x^{2n}$ then

40. The value of $a_0 + a_1 + a_2 + ... + a_{n-1}$ is

a) $\frac{3^n}{2}$ b) $\frac{1}{2}\left(3^n - a_n\right)$ c) $\frac{a_n}{2}$ d) $3.a_n^n$

Key. B

41. The value of $a_0^2 - a_1^2 + a_2^2 - a_3^2 + ... + a_{2n}^2$ is

a) a_n b) $2a_n$ c) $3a_n$ d) $4a_n$

Key. A

42. The value of $a_0a_{2r} - a_1a_{2r+1} + a_2a_{2r+2} - ...... + a_{2n-2r}a_m$ is

a) a_{r+1} b) a_{r+2} c) a_{n+r+1} d) a_{n+r}

Key. D

Sol. 40. $a_r = a_{2n-r}, a \le r \le 2n$

$\Rightarrow 2\left(a_0 + a_1 + a_2 + ... + a_{n-1}\right) + a_n = 3^n$

41. $\left(1+x+x^2\right)^n = a_0 + a_1x + a_2x^2 + ...$

$\left(x^2-x+1\right)^n=a_0x^{2n}-a_1x^{2n-1}+a_2x^{2n-2}+.....$

Comparing coefficient of x^{2n} in $\left[\left(x^2+1\right)^2-x^2\right]^n=\left(x^4+x^2+1\right)^n$

Let $x^2=t$

$\Rightarrow t^n \text{ in}\left(1+t+t^2\right)^n=a_n$

42. Compare the coefficient of x^{2n-2r} in $\left(1+x^2+x^4\right)^n=a_{n-r}=a_{n+r}$.

Paragraph – 15

Consider the identity $\left(1+x\right)^{6m}=\sum_{r=0}^{6m}{}^{6m}C_r.\,x^r$. By putting different values of x on both sides, we can get summation of several series involving binomial coefficients. For example, by putting $x=\frac{1}{2}$ we get $\sum_{r=0}^{6m}{}^{6m}C_r\frac{1}{2r}=\left(\frac{3}{2}\right)^{6m}$.

43. The value of $\sum_{r=0}^{6m}(-1)^r\,{}^{6m}C_{2r}$ is

A) 2^{3m} B) 0 if m is odd C) -2^{3m} if m is even D) None of these

Key. B

Sol. $\sum_{r=0}^{6m}{}^{6m}C_r2^{r/2}$ put $x=\sqrt{2}$

$=\left(1+\sqrt{2}\right)^{6m}=\left(3+2\sqrt{2}\right)^{3m}$

44. The value of $\sum_{r=0}^{3m}(-1)^r\,{}^{6m}C_{2r}$ is

A) 2^{3m} B) 0 if m is odd

C) -2^{3m} if m is even D) None of these

Key. B

Sol. $\sum_{r=0}^{3m}(-1)^r\,{}^{6m}C_{2r}=\left(\sqrt{2}\right)^{6m}\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)^{6m}+\left(\sqrt{2}\right)^{6m}\left(\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right)^{6m}$

$=2^{3m}.\,2\cos\frac{3m\pi}{2}=\begin{cases}0 & \text{if} \quad m\,\text{is}\,\text{odd}\\(-1)^{\frac{m}{2}}2^{3m+1} & \text{if} \quad m\,\text{is}\,\text{even}\end{cases}$

45. The value of $\sum_{r=1}^{3m}(-3)^{r-1}\,{}^{6m}C_{2r-1}$ is

A) 0 B) 1 C) depends on m D) None of these

Key. A

Sol. $\sum_{r=1}^{3m}(-3)^{r-1}\,{}^{6m}C_{2r-1}$

$=\frac{1}{\sqrt{3}i}\left\{\sqrt{3}i\,{}^{6m}C_1+\left(\sqrt{3}i\right)^3\,{}^{6m}C_3+\left(\sqrt{3}i\right)^5\,{}^{6m}C_5+...+\left(\sqrt{3}i\right)^{6m-1}\,{}^{6m}C_{6m-1}\right\}$

$$\left(1+\sqrt{3}i\right)^{6m} = {}^{6m}C_0 + \sqrt{3}i\ {}^{6m}C_1 + \left(\sqrt{3}i\right)^2\ {}^{6m}C_2 + \left(\sqrt{3}i\right)^3\ {}^{6m}C_3 + ...$$

$$\left(1-\sqrt{3}i\right)^{6m} = {}^{6m}C_0 - \sqrt{3}i\ {}^{6m}C_1 + \left(\sqrt{3}i\right)^2\ {}^{6m}C_2 - \left(\sqrt{3}i\right)^3\ {}^{6m}C_3 + ...$$

$$\therefore \left(1+\sqrt{3}i\right)^{6m} - \left(1-\sqrt{3}i\right)^{6m} = 2\left[\sqrt{3}i\ {}^{6m}C_1 + \left(\sqrt{3}i\right)^3\ {}^{6m}C_3 + ...\right]$$

$$\therefore \text{ given expression } = \frac{2^{6m}}{2\sqrt{3}i}\left(\cos 2m\pi + i\sin 2m\pi - \cos 2m\pi + i\sin 2m\pi\right) = 0$$

Paragraph – 16

If $\left(1+x\right)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$ …(i)

Then sum of the series $C_0 + C_k + C_{2k} + ...$ can be obtained by putting all the roots of the equation $x^k - 1 = 0 \Rightarrow x = \pm 1$ in (i)

x = 1 $\quad C_0 + C_1 + C_2 + C_3 + ... = 2^n$

x = - 1 $\quad C_0 - C_1 + C_2 - C_3 + ... = 0$

$2\left(C_0 + C_2 + C_4 + ...\right) = 2^n$

$\therefore \quad C_0 + C_2 + C_4 + ... = 2^{n-1}$

46. Values of x, we should substitute in (i) to get the sum of the series $C_0 + C_3 + C_6 + C_9 ...$, ARE

A) 1, - 1, ω B) $\omega, \omega^2, \omega^3$ C) $\omega, \omega^2 - 1$ D) None of these

Key. B

47. If n is a multiple of 3, then $C_0 + C_3 + C_6 + ...$ is equal to

A) $\frac{2^n + 2}{3}$ B) $\frac{2^n - 2}{3}$

C) $\frac{2^n + 2(-1)^n}{3}$ D) $\frac{2^n - 2(-1)^n}{3}$

Key. C

48. Sum of values of x, which we should substitute in (i) to give the sum of the series $C_0 + C_4 + C_8 + C_{12} + ...$, is

A) 2 B) 2 (1 + i) C) 2(1 – i) D) 0

Key. D

Sol. (46-48)

$x^3 - 1$

$x = 1, \omega, \omega^2$ or $x = \omega, \omega^2, \omega^3$

$x = 1 \ :\quad C_0 + C_1 + C_2 + C_3 + C_4 + ... = 2^n$

$x = \omega \ :\quad C_0 + C_1\omega + C_2\omega^2 + C_3\omega^3 + C_4\omega^4 + ... = \left(1+\omega\right)^n$

$x = \omega^2 \ :\quad C_0 + C_1\omega^2 + C_2\omega^4 + C_3\omega^6 + ... = \left(1+\omega^2\right)^n$

$3\left(C_0 + 0 + 0 + C_3 + 0 + 0 + C_6 + ...\right) = 2^n + \left(-\omega^2\right)^n + \left(-\omega\right)^n = 2^n + \left(-1\right)^n + \left(-1\right)^n$

$\therefore \quad C_0 + C_3 + C_6 + ... = \frac{2^n + 2(-1)^n}{3}$

$x^2 - 1 = 0 \quad \Rightarrow \quad x = \pm\ 1, \pm\ i$

$\therefore$ Sum of values of x = 1 + (- 1) + I + (- 1) = 0

Paragraph – 17

Let $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$ when $C_r = {}^nC_r$, then

49. $\sum_{r=1}^{n}(-1)^{r-1}\frac{r}{r+1}C_r =$

a) $\frac{1}{n}$ b) 0 c) $\frac{1}{n-1}$ d) $\frac{1}{n+1}$

Key. D

50. $\sum_{r=0}^{n}\frac{C_r}{r+1} =$

a) $\frac{2^{n+1}}{n}$ b) $\frac{2^{n+1}+1}{n+1}$ c) $\frac{2^{n+1}-1}{n+1}$ d) $\frac{2^{n+1}-1}{n}$

Key. C

51. $\sum_{r=0}^{n}{}^nC_r\cos(r+1)x =$

a) $2^n\cos^n\frac{x}{2}\cos\frac{nx}{2}$ b) $2^n\cos^n\frac{x}{2}\cos\frac{(n+1)}{2}x$

c) $2^n\cos^n\frac{x}{2}\cos\frac{(n+2)}{2}x$ d) $2^n\cos^n\frac{x}{2}\cos\frac{(n-1)}{2}x$

Key. C

Sol. 49. $\left(\frac{r}{r+1}\right)C_r = \left(1-\frac{1}{r+1}\right)C_r$

$$\sum_{r=1}^{n}(-1)^{r-1}\frac{r}{r+1}C_r = (C_1 - C_2 + C_3....) - \frac{1}{n+1}\left({}^{n+1}C_2 + {}^{n+1}C_3 + ...\right)$$

$$= C_0 - \frac{1}{n+1}\left[-{}^{n+1}C_0 + {}^{n+1}C_1\right] = \frac{1}{n+1}$$

50. $C_0 + C_1x + C_2x^2 + ... + C_nx^n = (1+x)^n$

Integrating $\int_0^1 \left(C_0 + C_1x + ... + C_nx^n\right)dx = \int_0^1 (1+x)^n dx = \frac{2^{n+1}-1}{n+1}$

51. $\sum_{r=0}^{n}{}^nC_r = \text{Re}\sum_{r=0}^{n}{}^nC_re^{i(r+1)x} = \text{Re}\left(1+e^{ix}\right)^n e^{ix}$

$$= \text{Re}(1+\cos x + \sin x)^n(\cos x + i\sin x)$$

$$= \text{Re}\left(2^n\cos^n x/2\left(\cos nx/2 + i\sin nx/2\right)(\cos x + i\sin x)\right)$$

$$= 2^n\cos^n x/2\left[\cos\frac{nx}{2}\cos x - \sin\frac{nx}{2}\cos x\right]$$

$$= 2^n\cos^n x/2\cos\left(\frac{n+2}{2}\right)x$$

Paragraph – 18

If $(1+x)^n = C_ox^o + C_1x + C_2x^2 + ... + C_nx^n$ **then**

52. Value of $\dfrac{(C_0+C_1)(C_1+C_2)(C_2+C_3)+..........(C_{n-1}+C_n)}{C_1C_2........C_{n-1}}$ is

a) $\dfrac{(n+1)^n}{n!}$ b) $\dfrac{n+1}{n!}$

c) $\dfrac{(n+1)^{n-1}}{n!}$ d) $\dfrac{(n+1)^n}{n}$

Key. A

53. Value of $C_0+(C_0+C_1)+.........+(C_0+C_1+C_2+.........+C_{n-1})$ is

a) $(n+1).2^n$ b) $n.2^{n-1}$ c) $(n.+1)2^{n-1}$ d) $n.2^n$

Key. B

54. Value of $\dfrac{C_0}{1}+\dfrac{C_1}{2}+\dfrac{C_2}{3}+......+\dfrac{C_n}{n+1}$ is

a) $\dfrac{n-1}{n+1}$ b) $\dfrac{n+1}{n-1}$ c) $\dfrac{2^{n+1}-1}{n+1}$ d) $\dfrac{2^n-1}{n+1}$

Key. C

Sol. 52. Standard problem, $\dfrac{(n+1)^n}{n!}$

53. $n\times C_0+(n-1)\times C_1+.........+1\times(C_{n-1})=n.2^{n-1}$.

54. $\int_0^1(1+x)^n\,dx=\dfrac{2^{n+1}-1}{n+1}$.

Paragraph – 19

If n is a positive integer $n_1+n_2+n_3+.........+n_m=n$ and

$$(a_1+a_2+a_3+........a_m)^n=\sum\left(\frac{n!}{n_1!n_2!n_3!......n_m!}\right)a_1^{n_1}.a_2^{n_2}.a_3^{n_3}...............a_m^{n_m}$$

Where $n_1+n_2+n_3+.........n_m$ are all non-negative integers

55. The number of distinct terms in the expansion of $(x_1+x_2+x_3+.........+x_n)^4$ is

a) $\dfrac{n^4+6n^3+11n^2+6n}{24}$ b) ${}^nC_4$

c) ${}^{(n+1)}C_4$ d) ${}^{(n+2)}C_4$

Key. A

56. The coefficient of x^2y^3 in the expansion of $(1+x+y)^{10}$ is

a) 2320 b) 2520 c) 2420 d) 2620

Key. B

57. The coefficient of $a^{10}b^{7}c^{3}$ in the expansion of $(bc+ca+ab)^{10}$ is

a) 40 b) 100 c) 120 d) 80

Key. C

Sol. 55. Standard formula = ${}^{(n+4-1)}C_{n-1} = {}^{(n+3)}C_4$.

56. $\frac{10!}{2!3!5!}$ = 2520.

57. Standard formula

Paragraph – 20

If $\left(1+x+x^2\right)^{25} = \sum_{r=0}^{50} a_r x^r$ *then*

58. $\sum_{r=0}^{16} a_{3r} =$

a) $3^{25}-1$ b) 3^{24} c) $\frac{3^{24}+1}{3}$ d) $\frac{3^{24}-1}{3}$

Key. B

59. $\sum_{r=0}^{12} a_{4r} =$

a) $\frac{3^{25}+1}{4}$ b) $\frac{3^{25}-1}{4}$ c) $\frac{3^{24}-1}{2}$ d) $\frac{3^{24}-1}{3}$

Key. A

60. $\sum_{r=0}^{8} a_{6r} =$

a) $\frac{3^{25}-2^{25}+1}{6}$ b) $\frac{3^{25}+2^{25}-1}{6}$ c) $\frac{3^{25}-2^{25}-1}{6}$ d) $\frac{3^{25}+2^{25}+1}{6}$

Key. D

Sol. 58 to 60

(Put x = 1, ω, ω^2, where ω is complex cube root of unity and add the result to get (14)

Put x = 1, - 1, i, - i, where i^2 = -1, and add the result to get (15)

Put x = 1, -1, ω, $-\omega$, ω^2, $-\omega^2$, where ω is complex cube root of unity and add the result to get (16).

Binomial Theorem

Integer Answer Type

1. The sum of last 3 digits of 3^{100} is

Key. 1

Sol. We have

$$3^{100}=\left(3^4\right)^{25}$$
$$=\left(81\right)^{25}$$
$$=\left(80+1\right)^{25}$$
$$={}^{25}C_0.\left(80\right)^{25}+{}^{25}C_1.\left(80\right)^{24}+....+{}^{25}C_{22}\left(80\right)^3+{}^{25}C_{23}\left(80\right)^2+{}^{25}C_{24}\left(80\right)+{}^{25}C_{25}$$
$$=10^3\left[{}^{25}C_0\,8^{25}\times10^{22}+{}^{25}C_1\times8^{24}\times10^{21}+.....+{}^{25}C_{22}\times8^3\right]$$
$$+\frac{25\times24}{2}\times\left(80\right)^2+25\times80+1$$
$$=10^3m+1920000+2000+1,\ where\ m\in N$$
$$=10^3\left(m+1920+2\right)+1$$
$$\Rightarrow 3^{100}-1=10^3\left(m+1922\right)$$
$\Rightarrow 3^{100}-1$ is divisible by 1000

Thus, last three digits of 3^{100} are 001, last two digits of 3^{100} are 01 and the last digit of 3^{100} is 1

2. If $n\in N\ and\ C_k={}^nC_k$, and $\sum_{k=1}^{n}k^3\left(\frac{{}^nC_k}{{}^nC_{k-1}}\right)^2=\frac{n\left(n+1\right)^2\left(n+2\right)}{3p}$ then p is

Key. 4

Sol. We have

$$\frac{{}^nC_k}{{}^nC_{k-1}}=\frac{n-k+1}{k}$$
$$\therefore \sum_{k=1}^{n}k^3\left(\frac{{}^nC_k}{{}^nC_{k-1}}\right)^2$$
$$=\sum_{k=1}^{n}k^3\left(\frac{n-k+1}{k}\right)^2$$
$$=\sum_{k=1}^{n}k\left\{\left(n+1\right)-k\right\}^2$$
$$=\sum_{k=1}^{n}k\left\{\left(n+1\right)^2-2\left(n+1\right)k+k^2\right\}$$
$$=\sum_{k=1}^{n}\left[k\left(n+1\right)^2-2\left(n+1\right)k^2+k^3\right]$$
$$=\left(n+1\right)^2\left(\sum_{k=1}^{n}k\right)-2\left(n+1\right)\left(\sum_{k=1}^{n}k^2\right)+\left(\sum_{k=1}^{n}k^3\right)$$

$$=(n+1)^2\left\{\frac{n(n+1)}{2}\right\}-2(n+1)\frac{n(n+1)(2n+1)}{6}+\left\{\frac{n(n+1)}{2}\right\}^2$$

$$=\frac{n(n+1)^2}{2}\left\{(n+1)-\frac{2}{3}(2n+1)+\frac{n}{2}\right\}$$

..... (i) $=\dfrac{n(n+1)^2}{2}\times\dfrac{6(n+1)-4(2n+1)+3n}{6}$

$$=\frac{n(n+1)^2(n+2)}{12}$$

3. If $C_0, C_1, C_2 \ldots C_n$ denote binomial coefficient in the expansion of $(1+x)^n$ and given a=b=2 and n=10 , then
$aC_O+(a+b)C_1+(a+2b)C_2+\ldots..+(a+nb)C_n=4K.2^9$ where k is ____

Key. 6

Sol. We have

$$aC_0+(a+b)C_1+(a+2b)C_2+\ldots.+(a+nb)C_n$$

$$\sum_{r=0}^{n}(a+rb)\,{}^nC_r$$

$$=a\left(\sum_{r=0}^{n}{}^nC_r\right)+b\left(\sum_{r=0}^{n}r\,{}^nC_r\right)$$

$$=a\left(\sum_{r=0}^{n}{}^nC_r\right)+b\left(\sum_{r=1}^{n}r\,\frac{n}{r}\,{}^{n-1}C_{r-1}\right)$$

$$=a\left(\sum_{r=0}^{n}{}^nC_r\right)+bn\left(\sum_{r=1}^{n}{}^{n-1}C_{r-1}\right)$$

$$=a,\,2^n+bn\,2^{n-1}\quad\left[\because\sum_{r=0}^{n}{}^nC_r=2^n,\sum_{r=1}^{n}{}^{n-1}C_{r-1}=2^{n-1}\right]$$

$$=(2a+bn)2^{n-1}$$

4. Given ${}^8C_1x(1-x)^7+2.{}^8C_2x^2(1-x)^6+3.{}^8C_3x^3(1-x)^5+\ldots..+8.x^8=ax+b, Find\ a+b$

Key. 8

Sol. ${}^8C_1x(1-x)^7+2.{}^8C_2\ x^2(1-x)^6$

$+3.{}^8C_3x^3(1-x)^5+\ldots..+n.{}^8C_8\ x^8=8x$

Solution we have ,

${}^nC_1\ x(1-x)^{n-1}+2.\ {}^nC_2x^2(1-x)^{n-2}$

$+3.{}^nC_3x^3(1-x)^{n-3}+\ldots..+n.\ {}^nC_nx^n$

$$=\sum_{r=1}^{n} r.{}^{n}C_r x^r (1-x)^{n-r}$$

$$=\sum_{r=1}^{n} r.\frac{n}{r} n-1\ C_{r-1}\ x^r (1-x)^{n-r}$$

$$=n\sum_{r=1}^{n} {}^{n-1}C_{r-1}\ x.\ x^{r-1}(1-x)^{(n-1)-(r-1)}$$

$$=nx\sum_{r=1}^{n} {}^{n-1}C_{r-1}\ x^{r-1}(1-x)^{(n-1)-(r-1)}$$

$$=nx\left[x+(1-x)\right]^{n-1}$$

=nx .=8x(n=8)

a + b = 8

5. Let $\alpha_n=\sum_{r=0}^{n}\frac{1}{n_{C_r}}$, $Let\ \beta_n=\sum_{r=0}^{n}\frac{r}{n_{C_r}}$ for n = 10, find β_n/α_n

Key. 5

Sol. We have $\alpha_n=\sum_{r=0}^{n}\frac{1}{{}^{n}C_r}$: n=10

$$=\left\{\sum_{r=0}^{(n/2)-1}\left(\frac{1}{{}^{n}C_r}\right)+\frac{1}{{}^{n}C_{n-r}}\right\}+\frac{1}{{}^{n}C/2}$$

$$=\left\{\sum_{r=0}^{(n/2)-1}\frac{2}{{}^{n}C_r}\right\}+\frac{1}{{}^{n}C_{n/2}}$$

$$=2\left\{\sum_{r=0}^{(n/2)-1}\frac{2}{{}^{n}C_r}\right\}+\frac{1}{{}^{n}C_{n/2}} \quad \text{......(ii)}$$

$$\therefore \sum_{r=0}^{n}\frac{r}{{}^{n}C_r}$$

$$=\sum_{r=0}^{(n/2)-1}\left\{\frac{r}{{}^{n}C_r}+\frac{n-r}{{}^{n}C_r}\right\}+\frac{n/2}{{}^{n}C_{n/2}}$$

$$=\left\{\sum_{r=0}^{n/2-1}\frac{n}{{}^{n}C_r}\right\}+\frac{n}{2.{}^{n}C_{n/2}}$$

$$=\frac{n}{2}\left[\left\{\sum_{r=0}^{n/2-1}\frac{1}{{}^{n}C_r}\right\}+\frac{1}{{}^{n}C_{n/2}}\right]=\frac{n}{2}\alpha_n \quad \text{[Using (ii)]}$$

Hence ,

$$\sum_{r=0}^{n}\frac{r}{{}^{n}C_r}=\frac{n}{2}\alpha_n\ for\ all\ n\in N$$

6. If $\sum_{r=0}^{n}(-1)^r\frac{{}^{n}C_r}{{}^{(r+2)}C_r}=\frac{k}{n+2}$, find k

Key. 2

Sol. We have

$$\sum_{r=0}^{n}(-1)^{r}\frac{{}^{n}C_{r}}{{}^{n+2}C_{r}}$$

$$=\sum_{r=0}^{n}(-1)^{r}\frac{n!}{(n-r)!r!}\times\frac{2!r!}{(r+2)!}$$

$$=2\sum_{r=0}^{n}(-1)^{r}\frac{n!}{(n-r)!(r+2)!}$$

$$=\frac{2}{(n+1)(n+2)}\sum_{r=0}^{n}(-1)^{r}\frac{(n+2!)}{\{(n+2)-(r+2)\}!(r+2)!}$$

$$=\frac{2}{(n+1)(n+2)}\sum_{r=0}^{n}(-1)^{r+2}\ {}^{n+2}C_{r+2}$$

$$=\frac{2}{(n+1)(n+2)}\sum_{s=2}^{n+2}(-1)^{S}\ {}^{2}C_{s}$$

$$=\frac{2}{(n+1)(n+2)}\left[\left(\sum_{s=0}^{n+1}(-1)^{s}\ {}^{n+2}C_{s}\right)-\left({}^{n+2}C_{0}-{}^{n+2}C_{1}\right)\right]$$

$$=\frac{2}{(n+1)(n+2)}\left[0-\{1-(n+2)\}\right]$$

$$=\frac{2}{n+2}$$

7. $s=a+(a+d)+(a+2d)+......+(a+nd)$ and

$A=a+(a+d)^{n}C_{1}+(a+2d)n_{c_2}+.....+(a+nd)n_{c_n}$ then

$(n+1)A=k^{n}S\ where\ k=$ __________

Key. 2

Sol. We have

$$s=a+(a+d)+(a\_2d)+......+(a+nd)$$

$$\Rightarrow s=\frac{n+1}{2}\left[2a+(n+1-1)d\right]$$

$$\Rightarrow s=\frac{n+1}{2}(2a+nd)$$

Now,

$$A=a+(a+d)\ {}^{n}C_{1}+(a+2d)\ {}^{n}C_{2}+.....+(a+nd)^{n}C_{n}$$

$= \sum_{r=0}^{n} (a+rd)\ {}^nC_r$

$= a\left(\sum_{r=0}^{n} {}^nC_r\right) + d\left(\sum_{r=0}^{n} r.\ {}^nC_r\right)$

$= a.2^n + d.n\ 2^{n-1} \qquad \left[\because \sum_{R=0}^{N} {}^rC_r = 2^n, \sum_{r=0}^{n} r\ {}^nC_r = n\ 2^{n-1}\right]$

$= (2a+nd)2^{n-1}$

$= \left\{\frac{n+1}{2}(2a+nd)\right\}\left\{\frac{2}{n+1} \times 2^{n-1}\right\}$

$= \frac{s.2^n}{n+1}$

$\therefore (n+1)A = 2^n S$

8. Consider two polynomials $f(x)$ and $g(x)$ as $g(x) = \sum_{r=0}^{200} \alpha_r x^r$ and $f(x) = \sum_{r=0}^{200} \beta_r x^r$.

Given (i) $\beta_r = 1 \forall r \geq 100$, (ii) $f(x+1) = g(x)$. Let $A = \sum_{r=100}^{200} \alpha_r$. Find the remainder when A is divided by 15.

Key: 1

Hint: $\sum_{r=0}^{200} \alpha_r x^r = \sum_{r=0}^{200} \beta_r (r+x)^r$

$\alpha_0 + \alpha_1 x + \alpha_2 x^2 + ... + \alpha_{200} x^{200}$

$= \beta_0 + \beta_1(1+x) + ... + \beta_{200}(1+x)^{200}$

Equating coefficient of x^{100}, we get $\alpha_{100} = {}^{100}C_{100} + {}^{101}C_{100} + ... {}^{200}C_{100} = {}^{201}C_{101}$

Similarly we cant find $\alpha_{100} ..... \alpha_{200}$

$\sum_{r=100}^{200} \alpha_r = {}^{201}C_{101} + {}^{201}C_{102} + ... {}^{201}C_{201}$

$A = 2^{200}$

When A is divided by 15 remainder is 1.

9. (L-1)Coefficient of x^6 in $\left((1+x)(1+x^2)^2(1+x^3)^3....(1+x^n)^n\right)$ is 4k. The numerical value of k is

Key : 7

Hint : The coefficient of x^6 in the given expression = coefficient of x^6 in

$(1+{}^6C_1x^6)(1+{}^5C_1x^5)(1+{}^4C_1x^4)(1+{}^3C_1x^3+{}^3C_2x^6)(1+{}^2C_1x^2+{}^2C_2x^4)(1+x)$

= coefficient of x^6 in $\left(1+6x^6+5x^5+4x^4\right)\left(1+2x^2+3x^3+x^4+6x^5+3x^6\right)(1+x)$

= coefficient of x^6 in $\left(11x^5+17x^6\right)(1+x)$

= 28

10. $(1+x)(1+x+x^2)(1+x+x^2+x^3)\ldots\ldots(1+x+x^2+\ldots..+x^{100})$

When written in the ascending power of x then (the highest exponent of x) -5045 is

Key. 5

Sol. Highest exponent of $x = 1 + 2 + 3 + \ldots.. + 100 = \frac{100(101)}{2} : 5050$

11. Find the Coefficient of x^{103} in $(1 + x + x^2 + x^3 + x^4)^{199} (x - 1)^{201}$.

Key. 0

Sol. Coefficient of x^{103} in $(1 + x + x^2 + x^3 + x^4)^{199} (x - 1)^{201}$

Coefficient of x^{103} in $-(1 - x)^2 (1 - x^5)^{199}$

Coefficient of x^{103} in $-(1 - 2x + x^2) (1 - {}^{199}C_1 x^5 + {}^{199}C_2 x^{10} - {}^{199}C_3 x^{15} + \ldots..)$

Coefficient of $x^{103} = 0$

12. Given ${}^8C_1x(1-x)^7+2.{}^8C_2x^2(1-x)^6+3.{}^8C_3x^3(1-x)^5+\ldots..+8.x^8=ax+b, Find\ a+b$

Key. 8

Sol. ${}^8C_1x\left(1-x\right)^7+2.{}^8C_2\ x^2\left(1-x\right)^6$

$+3.{}^8C_3x^3\left(1-x\right)^5+\ldots..+n.{}^8C_8\ x^8=8x$

Solution we have ,

${}^nC_1\ x\left(1-x\right)^{n-1}+2.\ {}^nC_2x^2\left(1-x\right)^{n-2}$

$+3.{}^nC_3x^3\left(1-x\right)^{n-3}+\ldots..+n.\ {}^nC_nx^n$

$=\sum_{r=1}^{n} r.{}^nC_rx^r\left(1-x\right)^{n-r}$

$=\sum_{r=1}^{n} r.\ \frac{n}{r}n-1\ C_{r-1}\ x^r\left(1-x\right)^{n-r}$

$=n\sum_{r=1}^{n} {}^{n-1}C_{r-1}\ x.\ x^{r-1}\left(1-x\right)^{(n-1)-(r-1)}$

$=nx\sum_{r=1}^{n} {}^{n-1}C_{r-1}\ x^{r-1}\left(1-x\right)^{(n-1)-(r-1)}$

$=nx\left[x+\left(1-x\right)\right]^{n-1}$

=nx .=8x(n=8)

a + b = 8

13. Let $\alpha_n=\sum_{r=0}^{n}\frac{1}{n_{C_r}}, Let\ \beta_n=\sum_{r=0}^{n}\frac{r}{n_{C_r}}$ for n = 10, find β_n/α_n

Key. 5

Sol. We have $\alpha_n = \sum_{r=0}^{n} \frac{1}{{}^nC_r}$: n=10

$$= \left\{ \sum_{r=0}^{(n/2)-1} \left(\frac{1}{{}^nC_r} \right) + \frac{1}{{}^nC_{n-r}} \right\} + \frac{1}{{}^nC/2}$$

$$= \left\{ \sum_{r=0}^{(n/2)-1} \frac{2}{{}^nC_r} \right\} + \frac{1}{{}^nC_{n/2}}$$

$$= 2\left\{ \sum_{r=0}^{(n/2)-1} \frac{2}{{}^nC_r} \right\} + \frac{1}{{}^nC_{n/2}} \quad \text{......(ii)}$$

$$\therefore \sum_{r=0}^{n} \frac{r}{{}^nC_r}$$

$$= \sum_{r=0}^{(n/2)-1} \left\{ \frac{r}{{}^nC_r} + \frac{n-r}{{}^nC_r} \right\} + \frac{n/2}{{}^nC_{n/2}}$$

$$= \left\{ \sum_{r=0}^{n/2-1} \frac{n}{{}^nC_r} \right\} + \frac{n}{2.{}^nC_{n/2}}$$

$$= \frac{n}{2}\left[\left\{ \sum_{r=0}^{n/2-1} \frac{1}{{}^nC_r} \right\} + \frac{1}{{}^nC_{n/2}} \right] = \frac{n}{2}\alpha_n \quad \text{[Using (ii)]}$$

Hence ,

$$\sum_{r=0}^{n} \frac{r}{{}^nC_r} = \frac{n}{2}\alpha_n \ \textit{for all} \ n \in N$$

14. If $\sum_{r=0}^{n} (-1)^r \frac{{}^nC_r}{{}^{(r+2)}C_r} = \frac{k}{n+2}$, find k

Key. 2

Sol. We have

$$\sum_{r=0}^{n} (-1)^r \frac{{}^nC_r}{{}^{n+2}C_r}$$

$$= \sum_{r=0}^{n} (-1)^r \frac{n!}{(n-r)!r!} \times \frac{2!r!}{(r+2)!}$$

$$= 2\sum_{r=0}^{n} (-1)^r \frac{n!}{(n-r)!(r+2)!}$$

$$= \frac{2}{(n+1)(n+2)} \sum_{r=0}^{n} (-1)^r \frac{(n+2!)}{\{(n+2)-(r+2)\}!(r+2)!}$$

$$= \frac{2}{(n+1)(n+2)} \sum_{r=0}^{n} (-1)^{r+2}\ {}^{n+2}C_{r+2}$$

$$= \frac{2}{(n+1)(n+2)} \sum_{s=2}^{n+2} (-1)^{S} \ {}^{2}C_s$$

$$= \frac{2}{(n+1)(n+2)} \left[\left(\sum_{s=0}^{n+1} (-1)^s \ {}^{n+2}C_s \right) - \left({}^{n+2}C_0 - {}^{n+2}C_1 \right) \right]$$

$$= \frac{2}{(n+1)(n+2)} \left[0 - \{1 - (n+2)\} \right]$$

$$= \frac{2}{n+2}$$

15. $s = a + (a+d) + (a+2d) + ... + (a+nd)$ and

$A = a + (a+d)^n C_1 + (a+2d) n_{c_2} + ... + (a+nd) n_{c_n}$ then

$(n+1)A = k^n S$ *where* $k =$ __________

Key. 2

Sol. We have

$$s = a + (a+d) + (a \_ 2d) + ... + (a+nd)$$

$$\Rightarrow s = \frac{n+1}{2} \left[2a + (n+1-1) d \right]$$

$$\Rightarrow s = \frac{n+1}{2} (2a + nd)$$

Now,

$$A = a + (a+d) \ {}^nC_1 + (a+2d) \ {}^nC_2 + ... + (a+nd)^n \ C_n$$

$$= \sum_{r=0}^{n} (a+rd) \ {}^nC_r$$

$$= a \left(\sum_{r=0}^{n} {}^nC_r \right) + d \left(\sum_{r=0}^{n} r. \ {}^nC_r \right)$$

$$= a.2^n + d.n \ 2^{n-1} \qquad \left[\because \sum_{R=0}^{N} {}^rC_r = 2^n, \sum_{r=0}^{n} r \ {}^nC_r = n \ 2^{n-1} \right]$$

$$= (2a + nd) 2^{n-1}$$

$$= \left\{ \frac{n+1}{2} (2a + nd) \right\} \left\{ \frac{2}{n+1} \times 2^{n-1} \right\}$$

$$= \frac{s.2^n}{n+1}$$

$$\therefore (n+1) A = 2^n S$$

16. The sum of the series $3.{}^{2007}C_0 - 8.{}^{2007}C_1 + 13\,{}^{2007}C_2 - 18.{}^{2007}C_3 + ...$ upto 2008 terms is K, then K is

Key. 0

Sol. Series $= \sum_{r=0}^{2007} (-1)^r (5r+3)\ {}^{2007}C_r = 0 \Rightarrow K = 0$

17. If n is a +ve integer greater than 3 such that $(1+x^2)^2(1+x)^n = \sum_{K=0}^{n+4} a_K x^K$ and a_1, a_2, a_3 are in AP then maximum value of n is ________.

Key. 4

Sol. $(1+x^2)^2(1+x)^n = \sum_{K=0}^{n+4} a_K x^K$

$a_1 = nC_1,\ 2 + {}^nC_2 = a_2$

${}^nC_3 + 2.{}^nC_1 = a_3$

But a_1, a_2, a_3 are in AP

$(n-2)(n-3)(n-4) = 0 \Rightarrow n = 2,3,4$

Maximum value of n is 4.

18. If C_r is a binomial co-efficient in the expansion of $(1+x)^n$, find the value of $\sum_{i=1}^{n}\sum_{j=1}^{n} (i+j)C_iC_j$.

Sol. Note: $C_0 + C_1 + C_2 + ... + C_n = 2^n$ &$1.C_1 + 2.C_2 + ... + n.C_n = n\,2^{n-1}$

$\sum_{i=1}^{n}\sum_{j=1}^{n}(i+1)C_iC_j = \Sigma\Sigma i\ C_iC_j + \Sigma\Sigma j\ C_i C_j$

$= \sum_{i=1}^{n} iC_i.\left(\sum_{j=1}^{n} C_j\right) + \sum_{i=1}^{n} jC_j.\left(\sum_{j=1}^{n} C_i\right) = \left(2^n - 1\right)2\Sigma iC_i = \left(2^n-1\right)2n.2^{n-1} = n\,2^n\left(2^n - 1\right)$

19. Let $\left(1+x^3\right)^n = \sum_{r=0}^{n} a_r x^r (1-x)^{3n-2r}$, $n > 2$, then find the value of n so that a_1, a_2, a_3 are in A.G.P.

Sol. If $\left(1-x^3\right)^n = \sum_{r=0}^{n} a_r x^r (1-x)^{3n-2r}$

$\Rightarrow \left[(1-x)\left(1+x+x^2\right)\right]^n = (1-x)^n \sum_{r=0}^{n} a_r x^r \left[(1-x)^2\right]^n = \sum_{r=0}^{n} a_r x^r \left[(1-x)^2\right]^{n-r}$

$\Rightarrow \sum_{r=0}^{n} {}^nC_r (3x)^r \left[(1-x)^2\right]^{n-r} = \sum_{r=0}^{n} a_r x^r \left[(1-x)^2\right]^{n-r}$

Comparing the coefficients of like power of x on both sides, we get $a_r = {}^nC_r.\,3^r$.

$\therefore$ $a_1 = 3.\ {}^nC_1,\ a_2 = 9.\ {}^nC_2$ and $a_3 = 27.\ {}^nC_3$

$\therefore$ a_1, a_2, a_3 are in A.G.P. iff ${}^nC_1,\ {}^nC_2,\ {}^nC_3$ are in A.P.

$\therefore$ n = 7 Ans.

20. Prove that ${}^nC_0\ {}^nC_1\ {}^nC_2 ... {}^nC_n < \dfrac{(n-1)!\left(2^{n+1} - n - 2\right)^n}{n^{n-1}(n+1)^{n-1}}$.

Sol. $\left[\frac{1}{(n+1)!}\left({}^nC_0.{}^nC_1.{}^nC_2....{}^nC_n\right)\right]^{\frac{1}{n}}=\left[\frac{1}{2}{}^nC_1.\frac{1}{3}{}^nC_2\frac{1}{4}{}^nC_3...\frac{1}{n+1}{}^nC_n\right]^{\frac{1}{n}}$

$$\le\frac{\frac{1}{2}{}^nC_1+\frac{1}{3}{}^nC_2+\frac{1}{4}{}^nC_3+...+\frac{1}{n+1}{}^nC_n}{n}=\frac{\sum_{r=1}^{n}{}^{n+1}C_{r+1}}{n(n=1)}=\frac{2^{n+1}-n-2}{n(n+1)}$$

$$\therefore\quad {}^nC_0{}^nC_1{}^nC_2...{}^nC_n<\frac{(n+1)!\left(2^{n+1}-n-2\right)^n}{n^n(n+1)^n}=\frac{(n-1)!\left(2^{n+1}-n-2\right)^n}{n^{n-1}(n+1)^{n-1}}$$

21. If $\left(1+2x+2x^2\right)^n=a_0+a_1x+a_2x^2+...+a_{2n}x^{2n}$ where n is even, then find the value of $a_0a_{2n}-a_1a_{2n-1}+...+a_{2n}a_0$.

Sol. $\left(1+2x+2x^2\right)^n=a_0+a_1xa_2x^2+...+a_{2n}x^{2n}$ …(i)

Replace x by – x

$\left(1-2x+2x^2\right)^n=a_0-a_1x+a_2x^2-...+a_{2n}x^{2n}$ …(ii)

$\therefore$ $a_0a_{2n}-a_1a_{2n-1}...$ = coefficient of x^{2n} in the product of RHS of (i) and (ii)

= coefficient of x^{2n} in the product of LHS

= coefficient of x^{2n} in $\left[\left(1+2x^2\right)^2-(2x)^2\right]^n$

i.e. in $\left(1+4x^4\right)^n$

= coefficient of y^n in $\left(1+4y^2\right)^n$ where $x^2=y$

$={}^nC_{\frac{n}{2}}.2^n$ (n is even)

22. If the middle term of $\left(\frac{1}{x}+x\sin x\right)^{10}$ is equal to $7\frac{7}{8}$, then find the value of x.

Sol. There are 11 terms in the expansion $\therefore$ 6^{th} term is the middle term.

$T_6={}^{10}C_5\left(\frac{1}{x}\right)^5(x\sin x)^5$

${}^{10}C_5(\sin x)^5=\frac{63}{8}$

$(\sin x)^5=\frac{63}{8}\times\frac{1}{252}=\frac{1}{32}$

$\Rightarrow$ $\sin x=\frac{1}{2}$

$\therefore$ $x=n\pi+(-1)^n\frac{\pi}{6},\ n\in I$ Ans.

23. Find the last three digits of 17^{256}

Sol. We have $17^2=289=290-1$

Now $17^{256}=\left(17^2\right)^{128}=(290-1)^{128}$

$={}^{128}C_0(290)^{128}-{}^{128}C_1(290)^{127}+{}^{128}C_2(290)^{126}-.....$

$-{}^{128}C_{125}(290)^3+{}^{128}C_{126}(290)^2-{}^{128}C_{127}(290)+1$

$$= 1000\,m + \frac{(128)(127)}{2}(290)^2 - (128)(290) + 1$$

Where m is a positive integer.

= 1000 m + (128) (290) [(127) (145) – 1] + 1
= 1000 m + (128) (290) (18414) + 1
= 1000 m + 683527680 + 1
= 1000 [m + 683527] + 680 + 1
= 1000 [m + 683527] + 681

Thus, the last three digits of 17^{256} are 681. Ans.

24. Find sum of rational coefficients in expansion of $\left(\sqrt[3]{5}x + \sqrt{3}y + z\right)^6$

Sol. Any term of the expansion is of the form $\frac{6!}{a!b!c!}\left(5^{1/3}x\right)^a\left(3^{1/2}y\right)^b z^c$

z^c, a, b, c non-negative integers and a + b + c = 6. For rational coefficients ‘a’ must be multiple of 3 and b must be multiple of 2.
The following are the possibilities

Sum of coefficients $= \frac{6!}{6!} + \frac{6!}{2!4!}.3 + \frac{6!}{4!2!}3^2 + \frac{6!}{6!}3^3 + \frac{6!}{3!3!}5 + \frac{6!}{3!2!}.5.3 + \frac{6!}{6!}5^2 = 1233$ Ans.

25. Find the algebraically second largest term in the expansion of $(3-2x)^{15}$ at $x = \frac{4}{3}$.

Sol. $(3-2x)^{15}$ at $x = \frac{4}{3}$

$$r \le \frac{16|2x|}{3+|2x|} = \frac{16 \times 2 \times \frac{4}{3}}{3 + \frac{8}{3}} = \frac{128}{17}$$

r = 7

$\therefore$ $t_{r+1} = t_8 = {}^{15}C_7 3^8 (-2x)^7 = -{}^{15}C_7 3^8 \left(\frac{8}{3}\right)^7$ is numerically greatest term.

$\therefore$ t_7 and t_9 are positive terms

$t_7 = {}^{13}C_6 3^9 (-2x)^6 = {}^{15}C_6 3^9 (8/3)^6 = {}^{15}C_6 8^6.3^3$

and $t_9 = {}^{15}C_8 3^7 (-2x)^8 = {}^{15}C_8.\ 3^7 (8/3)^8 = {}^{15}C_8 8^8.3^{-1}$

$\therefore$ $t_9 t_7 = \frac{15!}{8!7!}\frac{8^8}{3} - \frac{15!}{8!6!}8^6.3^3 = \frac{15!}{8!6!}\frac{8^6}{3}\left[\frac{64}{7} - \frac{3^4}{9}\right] > 0$

$\therefore$ $t_7 < t_9$

$t_{11} = {}^{15}C_{10}\frac{8^{10}}{3^5}$

$\therefore$ $t_7 - t_{11} = 10935 - 4096 > 0$

$\therefore$ $t_7 > t_{11}$

Thus $t_9 > t_7 > t_{11}$

Hence t_7 is the second largest term.

26. Coefficient of x^{203} in $(x-1)(x^2-2)(x^3-3)......(x^{20}-20)$ is k then sum of the digits of k is

Key. 4

Sol. (x–1) (x^2–2) (x^3–3) .. (x^n–n)

Highest power of x = 1+2+3+…..+20 = 210 we require, coefficient of x^{210-7}

$\Rightarrow$ either, we should leave

X^7–7, (x–1) (x^6–6), (x^2–2) (x^5–5), (x^3–3) (x^4–4), (x–1) (x^2–2) (x^{4-}–4)

$\Rightarrow$ required coefficient is 12 + 10+ 6 – 7 – 8 = 13

27. If the coefficient of x^7 in $\left(ax^2+\frac{1}{bx}\right)^{11}$ and the coefficient of x^{-7} in $\left(ax-\frac{1}{bx^2}\right)^{11}$ are equal if b = $\frac{1}{9}$ then the value of a is

Key. 9

Sol. ab = 1

28. Let f(n) = $\left(\sqrt{2}+1\right)^n$, 'n' being an odd positive integer [.] the greatest integer function then the value of 'n' for which $\left[f(x)\right]=82$ is.

Key. 5

Sol. $\left(\sqrt{2}+1\right)^n-\left(\sqrt{2}-1\right)^n=2\left\{{}^nC_1\left(\sqrt{2}\right)^{n-1}+{}^nC_3\left(\sqrt{2}\right)^{n-3}+...+{}^nC_n\right\}$

= integer

$\Rightarrow \left(\sqrt{2}-1\right)^n$ is the fractional part of $\left(\sqrt{2}-1\right)^n$

$\therefore$ $[f_{(n)}]$ = $\left[\left(\sqrt{2}+1\right)^n\right]=\left(\sqrt{2}+1\right)^n-\left(\sqrt{2}-1\right)^n$ is satisfied for n = 5

29. If α and β are the roots of equation x^2 + 4x + p = 0 where p = $\sum_{r=0}^{n} n_{C_r}\frac{1+rx}{(1+nx)^r}(-1)^r$ then the value of | α-β | is ……

Key. 4

Sol. p = $\sum_{r=0}^{n} n_{C_r}\cdot\left(\frac{-1}{1+nx}\right)^r+\sum_{r=0}^{n} n_{C_r}\frac{(-1)^r rx}{(1+nx)^r}$

= $\left(1-\frac{1}{1+nx}\right)^n+x\sum_{r=1}^{n}{}^{n-1}C_{r-1}\frac{(-1)^r x}{(1+nx)^r}$

= $\left(\frac{nx}{1+nx}\right)^n-\left(\frac{nx}{1+nx}\right)\left(1-\frac{1}{1+nx}\right)^{n-1}$

= 0

$\therefore |\alpha-\beta|=4$

30. ${}^{n}C_{o}\,{}^{4n}C_{n} - {}^{n}C_{1}\,{}^{(4n-3)}C_{n} + {}^{n}C_{2}\,{}^{4n-6}C_{n} - {}^{n}C_{3}\,{}^{4n-9}C_{n} + \ldots +$ to n terms is equal to $(1+k)^{n}$ then the value of k is

Key. 2

Sol. Coefficient x^n in $\left\{ {}^{n}C_{o}(1+x)^{4n} - {}^{n}C_{1}(1+x)^{4n-3} + {}^{n}C_{2}(1+x)^{4n-6} \ldots + (-1)^{n}\,{}^{n}C_{n}(1+x)^{n} \right\}$

$\Rightarrow$ Coefficient x^n in $(1+x)^{n}\left\{(1+x)^{3}-1\right\}^{n} \Rightarrow 3^{n}$

31. If the sum of the coefficient in the expansion of $\left(\alpha x^{2}-2x+1\right)^{35}$ is equal to the sum of the coefficients in the expansion of $(x-\alpha y)^{35}$,then the value of α is.

Key. 1

Sol. Conceptual

32. The value of $99^{50} - 99.98^{50} + \frac{99.98}{1.2}(97)^{50} - \ldots\ldots + 99$ is

Key. 0

Sol. ${}^{99}C_{0}99^{50} - {}^{99}C_{1}(99-1)^{50} + {}^{99}C_{2}(99-2)^{50} - \ldots . + {}^{99}C_{98}(99-98)^{50} - {}^{99}C_{99}(99-99)^{50}$

$= 99^{50}\left[{}^{99}C_{0} - {}^{99}C_{1} + {}^{99}C_{2} - \ldots . + {}^{99}C_{98} - {}^{99}C_{99}\right] + {}^{50}C_{1}99^{49}\left[{}^{99}C_{1} - 2.{}^{99}C_{2} + 3.{}^{99}C_{3} - \ldots\right] + \ldots . = 0$

33. At which x > 0 the 6th term in the expansion of the binomial $\left(\sqrt{2^{\log\left(10-3^{x}\right)}} + \sqrt[5]{2^{(x-2)\log 3}}\right)^{m}$ is equal to 21, if it is known that the binomial coefficient of the 2nd, 3rd and 4th term in the expansion represent respectively the 1st, 3rd and 5th terms of an A.P (the symbol log stands for logarithm to the base 10).

Key. 2

Sol. $2.{}^{m}C_{2} = {}^{m}C_{1} + {}^{m}C_{3}$

$\Rightarrow m = 7$

$\Rightarrow 21 = {}^{7}C_{5}\left[\sqrt{2^{\log\left(10-3^{x}\right)}}\right]^{7-5} \times \left[\sqrt[5]{2^{(x-2)\log 3}}\right]^{5}$

$\Rightarrow x = 2$

34. When 32^{33} is divided by 34, it leaves the remainder 3k + 5 then the value of k is _____.

Key. 9

Sol. $32^{33} = 2^{165} = 2 \times 16^{41} = 2 \times (17-1)^{41} = 2 \times (17k-1) = 34k - 34 + 32$

So the remainder is 32.

35. If the sum of all the coefficients of the terms in the expansion of $(x + y + z + w)^{6}$ which contain x but not y, is 95t then the value of t is _____.

Key. 7

Sol. The sum of coefficients of terms not containing y = 3^{6}

The sum of coefficients of terms not containing both x & y = 2^{6}

So the required number = $3^6 - 2^6 = 665$.

Binomial Theorem

Matrix-Match Type

1. Match the following:

Column-I	**Column-II**
A) $\sum\limits_{0\le i<j\le n}(i+j)(C_i.C_j)$ is	p) $n.2^n$
B) $\sum\limits_{0\le i<j\le n}\sum\left({}^nC_i+{}^nC_j\right)$ is equal to	q) $(n+1)^{2n}C_n$
C) $\sum\limits_{0\le i<j\le n}\sum i\,{}^nC_j$ is equal to	r) $n(n-1)2^{n-3}$
D) $\sum\limits_{r=0}^{n} r\,{}^nC_r$ is equal to	s) $\frac{n}{2}.\left[2^{2n}-{}^{2n}C_n\right]$

Key. A-S;B-P;C-R;D-S

Sol. A) $S=\sum\limits_{o\le i<j\le n}(i+j)(ci.cj)=\sum\limits_{o\le i<j\le n}(n-i+n-j)(ci.cj)$

$2S=2n\sum\limits_{o\le i<j\le n}(ci.cj)$

$S=n\sum ci.cj=\frac{n}{2}\left[2^{2n}-2n_{C_n}\right]\quad\left[\because\sum\limits_{o\le i<j\le n}(ci.cj)=\frac{1}{2}\left[2^{2n}-2n_{c_n}\right]\right]$

B) $\sum\sum\limits_{0\le i<j\le n}\left({}^nC_i+{}^nC_j\right)=\sum\sum\limits_{0\le i<j\le n}{}^nC_i+\sum\sum\limits_{0\le i<j\le n}{}^nC_j$

$=n.2^{n-1}+\sum\limits_{j=1}^{n}j.\,{}^nC_j=n.2^{n-1}+n.2^{n-1}=n.2^n$

C) $\sum\sum\limits_{0\le i<j\le n}i\,{}^nC_j=\sum\limits_{j=1}^{n}{}^nC_j\left(0+1+2+.....+j-1\right)$

$=\sum\limits_{j=1}^{n}{}^nC_j\frac{j(j-1)}{2}=\frac{1}{2}\sum\limits_{j=1}^{n}j^2\,{}^nC_j-\frac{1}{2}\sum\limits_{j=1}^{n}j\,{}^nC_j$

$=\frac{1}{2}(n+1)n.2^{n-2}-\frac{1}{2}n.2^{n-1}=n(n-1).2^{n-3}$

2. Match the following:

	Column -I		Column -II
(A)	Number of distinct terms in the expansion of $(x+y-z)^{16}$ is	(p)	2^{12}
(B)	Number of terms in the expansion of $\left(x+\sqrt{x^2-1}\right)^6+\left(x-\sqrt{x^2-1}\right)^6$ is	(q)	97
(C)	The number of irrational terms in $\left(\sqrt[8]{5}+\sqrt[6]{2}\right)^{100}$ is	(r)	4
(D)	The sum of numerical coefficients in the expansion of $\left(1+\frac{x}{3}+\frac{2y}{3}\right)^{12}$ is	(s)	153

Key. (A) $\to$(s); (B) $\to$(r); (C) $\to$(q); (D) $\to$(p)

Sol. (A) $^{(16+3-1)}C_{3-1}={}^{18}C_2=\frac{18\times17}{2}=153$

(B) $\frac{n}{2}+1=\frac{6}{2}+1=4$

(C) $\frac{100}{24}=4+F$ (T_1,T_{25},T_{49},T_{73} are rationals)

Total number of terms rational terms = Irrational terms $\Rightarrow 101-4=97$

(D) put $x=y=z \Rightarrow \left(1+\frac{1}{3}+\frac{2}{3}\right)^{12}=2^{12}$

3. If $(1+x+x^2)^n=a_0+a_1x+a_2x^2+.....+a_{2n}x^{2n}, then$

	Column-I		Column-II
A)	$a_0^2-a_1^2+a_2^2-a_3^2+.........+a_{2n}^2$ is equals	p)	$\frac{1}{2}(3^n+a_n)$
B)	$a_0a_2-a_1a_3+a_2a_4......+a_{2n-2}a_{2n}$ is equals	q)	$\frac{1}{2}(3^n-1)$
C)	$a_0+a_1+a_2+......+a_n$ is equals	r)	a_{n+1}
D)	$a_2+a_4+......+a_{2n}$ is equals	s)	a_n

Key: A) s B) r C) p D) q

Hint. A) $a_0^2-a_1^2+a_2^2..........+a_{2n}^2=$coefficient $x^{2n} in (1+x+x^2)^n(x^2-x+1)^n$

B) $a_0a_2-a_1a_3+.......+a_{2n-2}a_{2n}=$ coefficient x^{-2} in $(1+x+x^2)^n\left(1-\frac{1}{x}+\frac{1}{x^2}\right)^n$

C) $a_r=a_{2n-r}$

D) let f(x) = $(1+x+x^2)^n$

$$a_0 + a_2 + a_4 ...... + a_{2n} = \frac{f(1)+f(-1)}{2}$$

$a_0 = 1$

4. Consider the binomial expansion of $(1+2x)^n$ (n is a positive integer) in which the sum of the coefficients is 6561. $Let\,(1+2x)^n = R = I + f$ $where\ I\ is$ the largest integer not exceeding R and $0 < f < 1$.

	Column I		Column II
(A)	If r th term in the expansion is the greatest term, then r cannot exceed	(p)	3
(B)	If i th term is having the greatest coefficient, then i can be	(q)	4
(C)	The number of integral terms in the expansion when $x = \sqrt[3]{3}$ is less than	(r)	5
(D)	For $x = \frac{1}{\sqrt{2}}$, the value of $R(1-f)$ is less than	(s)	6
		(t)	7

KEY :	A-r, s, t	B-s ,t	C-q, r, s, t	D-p, q, r, s, t

Hint We first observe that n = 8

(A) r = 5

(B) $8_{C_r}.2^r \ge 8_{C_{r-1}} 2^{r-1} \Leftrightarrow \frac{9-r}{r} \ge \frac{1}{2} \Leftrightarrow r \le 6$

But for r = 6, $8_{C_6}.2^6 = 8_{C_5}.2^5$

$\therefore T_6$ and T_7 are the terms whose coefficients have greatest value (equal to 7 (2^8)).

(C) $T_{r+1} = n_{C_r}.2^r\ x^{r/3}$ is an integer only if r = 0,3 &6. So the number of integral terms is 3

(D) $x = \frac{1}{\sqrt{2}} \Rightarrow R = \left(\sqrt{2}+1\right)^8 = 1 - \left(\sqrt{2}-1\right)^8 + \left(\sqrt{2}+1\right)^8 + \left(\sqrt{2}-1\right)^8 - 1$

$= I + f$ $where$ $f = 1 - \left(\sqrt{2}-1\right)^8$

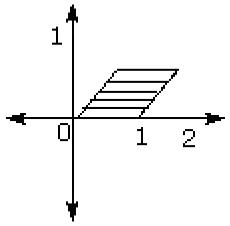
$\Rightarrow R(1-f) = \left(\sqrt{2}+1\right)^8 \left(\sqrt{2}-1\right)^8 = 1$

5.

	Column-I		Column-II
A)	Area enclosed by y=[x] and y={x} where [.] & {.}represent greatest integer and fractional part functions	p)	$\frac{32}{5}$ sq.unit
B)	Area bounded by the curves $y^2 = x^3 and \|y\| = 2x$	q)	1 sq.unit
C)	The smaller area included between the curves $\sqrt{x} + \sqrt{\|y\|} = 1$ and $\|x\| + \|y\| = 1$	r)	12 sq.unit
D)	Area enclosed by $[\|x\|] + [\|y\|] = 2$ Where [.] denotes greatest integer function.	s)	$\frac{2}{3}$ Sq.unit

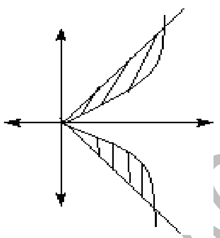
Key: A) q B) p C) s D) r

Hint. A) area =bh=1×1=1

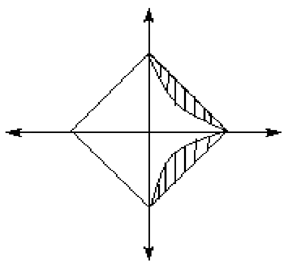


B) $y^2 = x^3$ and $|y| = 2x$

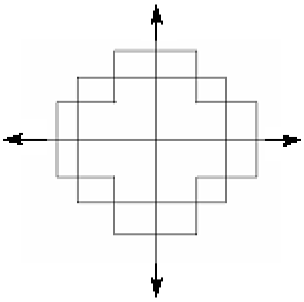
Area= $2\int_0^4 (2x - x^{\frac{3}{2}})dx$



C) $2\left[\frac{1}{2} - \int_0^1 (1 - \sqrt{2x} + x)dx\right]$



D) 4(3)=12



6. Match the following: -

	Column – I		Column – II
(A)	${}^mC_1\ {}^nC_m - {}^mC_2\ {}^{2n}C_m + {}^mC_3\ {}^{3m}C_m + \ldots$ is	(p)	The coefficient of x^m in the expansion of $\left(1-(1+x)^n\right)^m$
(B)	${}^nC_m + {}^{n-1}C_m + {}^{n-2}C_m + \ldots + {}^mC_m$ is	(q)	The coefficient of x^m in $\frac{(1+x)^{n+1}}{x}$
(C)	$C_0C_n + C_1C_{n-1} + \ldots + C_nC_0$ is	(r)	The coefficient of x^{n+1} in $(1+x)^{2n}$
(D)	$2^m\ {}^nC_0 - 2^{m-1}\ {}^nC_1\ {}^{n-1}C_{m-1} + \ldots + (-1)^m\ {}^nC_m\ {}^{n-m}C_0$ is	(s)	The coefficient of x^m in the expansion of $(1+x)^n$
		(t)	The coefficient of x^n in $(1+x)^{2n}$

Key. A $\rightarrow$p; B $\rightarrow$q; C $\rightarrow$ t; D $\rightarrow$ s

Sol. A) $\left({}^mC_1\ {}^nC_m - {}^mC_2\ {}^{2n}C_m + {}^mC_3\ {}^{3n}C_m - \ldots(-1)^{m-1m}\ C_m\ {}^{mn}C_m\right)$

= Coefficient of x^m in the expansion of

$\left({}^mC_1(1+x)^n - {}^mC_2(1+x)^{2n} + {}^mC_3(1+x)^{3n} \ldots + (-1)^{m-1}\ {}^mC_m.(1+x)^{mn}\right)$

= Coefficient of x^m in the expansion of

$$\left({}^mC_0 - \left[{}^mC_0 - {}^mC_1(1+x)^n + {}^mC_2(1+x)^{2n} + {}^mC_3(1+x)^{3m} + \ldots + (-1)^m\ {}^mC_m(1+x)^{mn}\right]\right)$$

= Coefficient of x^m in the expansion of $\left(1-\left(1-(1+x)^n\right)^m\right)$

= Coefficient of x^m in the expansion of $\left(1-(1+x)^n\right)^m$

B) ${}^nC_m + {}^{n-1}C_m + {}^{n-2}C_m + \ldots + {}^mC_m$ is the coefficient of x^m in the expansion of

$(1+x)^n + (1+x)^{n-1}(1+x)^{n-2} + \ldots + (1+x)^m$

$= (1+x)^m\left[1+(1+x)+(1+x)^2+\ldots+(1+x)^{n-m}\right]$

$$= (1+x)^m\left(\frac{1-(1+x)^{n-m+1}}{1-(1+x)}\right) = \frac{(1+x)^{n+1}-(1+x)^m}{x}$$

Thus the given expression is equal to the coefficient of x^m in the expansion of $\frac{(1+x)^{n+1}}{x}$

C) $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + \ldots + {}^nC_nx^n - A$

$\left(1+x\right)^{n}={}^{n}C_{0}+{}^{n}C_{1}x+{}^{n}C_{1}x+{}^{n}C_{2}x^{2}+{}^{n}C_{3}x^{3}+...+{}^{n}C_{n}x^{n}$

Multiplying eq. A and B and equating coefficients of x^{n} on both the sides.

Coefficient of x^{n} in the expansion of

$\left(1+x\right)^{n}\left(x+x\right)^{n}={}^{n}C_{0}\,{}^{n}C_{n}+{}^{n}C_{1}\,{}^{n}C_{n-1}+{}^{n}C_{2}\,{}^{n}C_{n-2}+...+{}^{n}C_{n}\,{}^{n}C_{0}$

$\therefore$ Coefficient of x^{n} in the expansion of $\left(1+x\right)^{2n}={}^{2n}C_{n}$

D) $2^{m}\,{}^{n}C_{m}=$ Coefficient of x^{m} in the expansion of $\left(1+2x\right)^{n-1}$

$2^{m-1}\,{}^{n-1}C_{m-1}=$ Coefficient of x^{m-1} in the expansion of $\left(1+2x\right)^{n-1}$

= Coefficient of x^{m} in the expansion of $x\left(1+2x\right)^{n-1}$

$\therefore$ given expression = coefficient of x^{m} in the expansion of

${}^{n}C_{0}\left(1+2x\right)^{n}-{}^{n}C_{1}x\left(1+2x\right)^{n-1}+{}^{n}C_{2}x^{2}\left(1+2x\right)^{n-2}-...$

7. Match the following: -

Column – I		Column – II	
(A)	If (r + 1) th term is the first negative term in the expansion of $\left(1+x\right)^{7/2}$, then the value of r(where 0 < x < 1) is	(p)	Divisible by 2
(B)	The coefficient of y in the expansion of $\left(y^{2}+1/y\right)^{5}$ is	(q)	Divisible by 5
(C)	If the second term in the expansion $\left(a^{\frac{1}{13}}+\frac{a}{\sqrt{a^{-1}}}\right)^{n}$ is $14a^{5/2}$, then the value of n is	(r)	Divisible by 10
(D)	The coefficient of x^{4} in the expression $\left(1+2x+3x^{2}+4x^{3}+...\text{up to }\infty\right)^{1/2}$ is $c, c\in N$, then c + 1 (where $\lvert x\rvert<1$) is	(s)	a prime number
		(t)	a composite number

Key. A $\to$ q,s; B $\to$ p,q,r,t; C $\to$ p,t; D $\to$ P,S

Sol. A) $T_{r+1}\dfrac{\frac{7}{2}\left(\frac{7}{2}-1\right)\left(\frac{7}{2}-2\right)...\left(\frac{7}{2}-r+1\right)x^{r}}{r!}$

First negative term if $\frac{7}{2}-r+1<0$ i.e. $r>\frac{9}{2}$

Hence r = 5

B) $T_{r+1}={}^{5}C_{r}\left(y^{2}\right)^{5-r}\left(\frac{1}{y}\right)^{r}={}^{5}C_{r}\,y^{10-3r}$

$\therefore$ $10-3r-r$ $\Rightarrow$ $r=3$

So, coefficient of $y={}^{5}C_{3}=10$

C)

$T_{2}=14a^{5/2}\Rightarrow {}^{n}C_{1}\left(a^{1/13}\right)^{n-1}\left(a^{3/2}\right)=14a^{5/2}\Rightarrow n\,a^{\frac{n-1}{13}+\frac{3}{2}}=14a^{5/2}\Rightarrow n=14$

D) $\left(1+2x+3x^{2}+4x^{3}+...\right)^{1/2}$

$=\left[(1-x)^{-2}\right]^{1/2}=(1-x)^{-1}=1+x+x^2+...+x^n+...$

Hence, coefficient of $x^4=1$ $\therefore$ $c=1$, hence c + 1 = 2

8. Match the following: -

	COLUMN – 1		COLUMN - 2
A	The number of terms in the expansion of $(2x-y+4z)^{12}$ is	P	2
B	Suppose that f(0)=0, $f'(0)=2$ and let g(x) = f(-x +f(f(x))). The value of $g'(0)$ is (g being differentiable)	Q	10
C	If p,q are positive integers, then ${}^pc_q=0$ for p<q. m,n,k are positive integers then the least value of k for which $\sum_{i=0}^{k} 20_{C_i}10_{C_{k-i}}$ is maximum	R	6
D	Let n=2010. The least positive integer k such that $kn^2(n^2-1)(n^2-2^2)...(n^2-(n-1)^2)=\lfloor r$ for some positive integer r, is	S	91

Key. A-S; B-R; C-Q;D-P

Sol. (a) The number of terms in the expansion of $(2x-y+4z)^{12}={}^{14}c_2=91$

(b) $g'(0)=f'(0)\left(-1+f'(0)f'(0)\right)=6$

(c) the sum= 30_{C_k}, it increases as $0\le k\le 10$ and for k>10, 10_{C_k} give 0. Hence the terms in the sum will be equal to 0 for k>10.

$\therefore$ sum will be maximum at k=10.

(d) $k(n-1)(n+1)(n+2)(n-2).....(2n-1)(1)n^2=$

$1(2)(3)......(n-1)n(n+1)......(2n-1)(nk)=\lfloor r \Rightarrow \lfloor 2n-1(kn)=\lfloor r\Rightarrow k=2$

9. Match the following: -

	COLUMN – 1		COLUMN - 2
A	The number of integer values of x for which 4th term is numerically greatest term in the expansion of $\left(2+\frac{3x}{8}\right)^{10}$ is	P	197

B	If[x] = greatest integer less than or equal to x, then $\left[\left(\sqrt{2}+1\right)^{6}\right]$ equals	Q	1
C	The number of prime factors of the coefficient of x^8 in $\left(1-2x+3x^2-4x^3+5x^4-6x^5+7x^6\right)^6$ is	R	0
D	e number of real roots of the equation x+sin(x) = 1 is	S	4

Key. A-S; B-P; C-S;D-Q

Sol. (a) $3 \le \frac{(n+1)\left|\frac{3x}{16}\right|}{1+\left|\frac{3x}{16}\right|} = \frac{11\left|\frac{3x}{16}\right|}{1+\left|\frac{3x}{16}\right|} < 4, \left]\frac{-64}{21},-2\right] \cup \left[2,\frac{64}{21}\right[$

The required values of x are -2, -3, 2, 3.

(b) $\left(\sqrt{2}+1\right)^6 = I+F,\ 0<F<1,\ 0<f=\left(\sqrt{2}-1\right)^6<1,$

$1+F+f=2\left({}^6c_0(8)+{}^6c_2(4)+{}^6c_4(2)+1\right)=198$

$0<F+f<2\ \ F+f=198-i=\text{integer}=1$

$i=197$

(c) Coefficient of $x^8 \text{in} \left(1+8x^7+7x^8\right)^6 (1+x)^{-12} = 75048\ = 2^3\ 3^1 (59)(53)$

(d) $x+\sin(x)$ is a bijection, hence x+sin x = 1 has only one solution.

10. Suppose $\sum_{r=0}^{2n} a_r (x-2)^r = \sum_{r=0}^{2n} b_r (x-3)^r$. for $k \ge n$, match the values of b_n in column with the corresponding a_k of column - I

Column I		Column II	
(A)	$a_k = 1$	(P)	0
(B)	$a_k = \left({}^kC_n\right)^{-1}$	(Q)	**n+1**
(C)	$a_k = 0$	(R)	${}^{2n+1}C_n$
(D)	$a_k = \frac{(k-n)!}{k!}$	(S)	$\frac{n+1}{n!}$

Key. (A-r), (B-q), (C-p), (D-s)

Sol. $b_n = coeff\ (x-3)^n\ in \sum_0^{2n} br(x-3)^n = \sum_0^{2n} a_r \left(1+(x-3)\right)^r$

$= \sum_0^{2n} ar\left(1+{}^rC_1(x-3)+{}^rC_2(x-3)^2+\ldots\ldots+{}^rC_r(x-3)^r\right)$

$= an.{}^nC_n + a_{n+1}\,{}^{n+1}C_n + a_{n+2}\,{}^{n+2}C_n + \ldots\ldots\ldots + a_{2n}\,{}^{2n}C_n$

If $a_k = 1\ for\ k \ge n\ then$

$b_n = {}^nC_n + {}^{n+1}C_n + ... + {}^{2n}C_n = {}^{2n+1}C_n$

If $a_k = \left({}^kC_n\right)^{-1}$ $then\, b_n = \frac{1}{{}^nC_n}.{}^nC_n + \frac{1}{{}^{n+1}C_n}.{}^{n+1}C_n + ... + \frac{{}^{2n}C_n}{{}^{2n}C_n} = n+1$

If $a_k = 0$ $for\, k \geq n\, then \quad b_n = 0$

If $a_k = \frac{(k-n)!}{k!}$ $then$

$b_n = \sum_{k=n}^{2n} a_k.{}^kC_n = \sum_{k=n}^{2n} \frac{(k-n)!}{k!} \times \frac{k!}{(k-n)!n!} = \frac{n+1}{n!}$