Properties of Triangles

Single Correct Answer Type

1. If a,b,c be the sides of a triangle ABC and the roots of the equation $a(b-c)x^2 + b(c-a)x + c(a-b) = 0$ are equal, then $\sin^2\left(\frac{A}{2}\right), \sin^2\left(\frac{B}{2}\right), \sin^2\left(\frac{C}{2}\right)$ are in

(A)AP (B) GP (C) AGP (D) HP

Key. D

Sol. $\because$ $a(b-c) + b(c-a) + c(a-b) = 0$

$\therefore$ x = 1 is a root of the equation

$a(b-c)x^2 + b(c-a)x + c(a-b) = 0$

Then, other root = 1 ($\because$ roots are equal)

$\therefore$ $\alpha \times \beta = \frac{c(a-b)}{a(b-c)}$

$\Rightarrow$ ab – ac = ca – bc

$\therefore$ $b = \frac{2ac}{a+c}$

$\therefore$ a, b, c are in HP

Then, $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP.

$\Rightarrow$ $\frac{s}{a}, \frac{s}{b}, \frac{s}{c}$ are in AP

$\Rightarrow$ $\frac{s}{a}-1, \frac{s}{b}-1, \frac{s}{c}-1$ are in AP.

$\Rightarrow$ $\frac{(s-a)}{a}, \frac{(s-b)}{b}, \frac{(s-c)}{c}$ are in AP.

Multiplying in each by $\frac{abc}{(s-a)(s-b)(s-c)}$

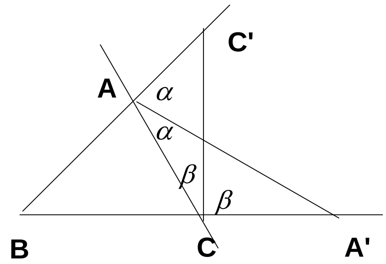
Then $\frac{bc}{(s-b)(s-c)}, \frac{ca}{(s-c)(s-a)}, \frac{ab}{(s-a)(s-b)}$ are in AP.

$\Rightarrow$ $\frac{(s-b)(s-c)}{bc}, \frac{(s-c)(s-a)}{ca}, \frac{(s-a)(s-b)}{ab}$ are in HP.

Or $\sin^2\left(\frac{A}{2}\right), \sin^2\left(\frac{B}{2}\right), \sin^2\left(\frac{C}{2}\right)$ are in HP

2. Given in $\Delta ABC : AB = 1cm : AC = 2\,cm$ The lengths of external angular bisectors of angles A & C are equal. ie., $AA' = CC'$. If $BC \neq 1$ then BC = ________________

In the given figure $\alpha = 90^0 - \frac{A}{2}$ and $\beta = 90^0 - \frac{C}{2}$



(a) $\frac{1+\sqrt{15}}{2}$ (b) $\frac{1+\sqrt{13}}{2}$ (c) $\frac{1+\sqrt{17}}{2}$ (d) $\frac{1+\sqrt{19}}{2}$

Key. C

Sol. Length of external angular bisector of angle A is $\frac{2bc}{|b-c|}\sin\frac{A}{2}$. Length of external angular bisector of angle C is $\frac{2ab}{|a-b|}\sin\frac{C}{2}$

3. In Δ ABC, the bisector of the angle A meets the side BC at D and the circumscribed circle at E, then DE equals

(A) $\frac{a^2 \sec\frac{A}{2}}{2(b+c)}$ (B) $\frac{a^2 \sin\frac{A}{2}}{2(b+c)}$

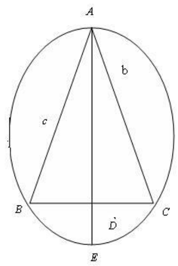
(C) $\frac{a^2 \cos\frac{A}{2}}{2(b+c)}$ (D) $\frac{a^2 \operatorname{cosec}\frac{A}{2}}{2(b+c)}$

Key. A

Sol. AD.DE = BD.DC

$$DE = \frac{BD.DC}{AD} = \frac{\left(\frac{ac}{b+c}\right)\left(\frac{ab}{b+c}\right)}{\frac{2bc}{b+c}.\cos\frac{A}{2}}$$

$$= \frac{a^2}{2(b+c)}\sec\frac{A}{2}$$



4. In Δ ABC, If A – B = 120° and R = 8r, then the value of $\frac{1+\cos C}{1-\cos C}$ equals

(All symbols used have their usual meaning in a triangle)

(A) 12 (B) 15 (C) 21 (D) 31

Key. B

Sol. $\frac{r}{R} = \cos A + \cos B + \cos C - 1$

$$\frac{1}{8} = 2\cos\frac{A+B}{2} + \cos\frac{A-B}{2} - 1 + \cos C$$

$$\Rightarrow \frac{1}{8} = \sin\frac{C}{2} - 2\sin^2\frac{C}{2}$$

$$\Rightarrow \sin\frac{C}{2} = \frac{1}{4} \quad \therefore \cos C = 1 - \frac{1}{8} = \frac{7}{8}$$

5. In a $\Delta ABC, if\ A = 30^0\ and\ \frac{b}{c} = \frac{2+\sqrt{3}+\sqrt{2}-1}{2+\sqrt{3}-\sqrt{2}+1}$, then the measure of $\angle C, is$

A) $67\frac{1^0}{2}$ B) $22\frac{1^0}{2}$ C) $52\frac{1^0}{2}$ D) $97\frac{1^0}{2}$

Key. C

Sol. use $\frac{b-c}{b+c}\cot\frac{A}{2} = \tan\left(\frac{B-C}{2}\right)$; and $B+C = 150^0$

$$\frac{b}{1+\sqrt{3}+\sqrt{2}} = \frac{c}{3+\sqrt{3}-\sqrt{2}} \Rightarrow \frac{b+c}{4+2\sqrt{3}} = \frac{b-c}{2\sqrt{2}-2} \Rightarrow \frac{b+c}{b-c} = \frac{\sqrt{3}+2}{\sqrt{2}-1}$$

$\therefore \frac{b-c}{b+c} = \frac{\sqrt{2}-1}{2+\sqrt{3}}$ which gives $\frac{\mathrm{b-c}}{\mathrm{b+c}}\cot 15° = \tan 22\frac{1°}{2}$

$$B - C = 45^0; B + C = 150^0$$

6. In $\Delta ABC, if \cos A+\sin A-\frac{2}{\cos B+\sin B}=0, then \frac{a+b}{c}$ is equal to

A) $\sqrt{2}$ B) 1 C) $\frac{1}{\sqrt{2}}$ D) $2\sqrt{2}$

Key. A

Sol. given $(\cos A+\sin A)(\cos B+\sin B)=2$

$\cos(A-B)+\sin(A+B)=2$

$\Rightarrow \cos(A-B)=1; \sin(A+B)=1$

$A=B;\ A+B=\frac{\pi}{2}\ \Rightarrow C=\frac{\pi}{2}$

$\therefore \frac{a+b}{c}=\sqrt{2}$

7. In a ΔABC, $\sum \sin\frac{A}{2}=\frac{6}{5}$ and $\sum II_1=9$ where I_1, I_2, I_3 are external and I is incentre, then circum radius R=

A) $\frac{15}{2}$ B) $\frac{15}{4}$ C) $\frac{15}{8}$ D) $\frac{1}{3}$

Key. C

Sol. $\sum II_1=\sum 4R\sin\frac{A}{2}\Rightarrow 9=4R\times\frac{6}{15}\Rightarrow R=\frac{45}{24}=\frac{15}{8}$

8. Let there exist a unique point P inside a ΔABC such that $\angle PAB=\angle PBC=\angle PCA=\alpha$

If PA=x, PB=y, PC=z, Δ=area of ΔABC and a,b,c are the sides opposite to the angles A,B,C respectively, then tan α is equal to

A) $\frac{a^2+b^2+c^2}{4\Delta}$ B) $\frac{a^2+b^2+c^2}{2\Delta}$ C) $\frac{2\Delta}{a^2+b^2+c^2}$ D) $\frac{4\Delta}{a^2+b^2+c^2}$

Key. D

Sol. $\cot A+\cot B+\cot c=\cot\alpha\Rightarrow \tan\alpha=\frac{4\Delta}{a^2+b^2+c^2}$

9. In a triangle ABC with usual notations, if $r=1, r_1=7$ and R = 3,then the triangleABC is

A) equilateral B) acute angled which is not equilateral

C) obtuse angled D) right angled

Key. D

Sol. $r_1 - r = 4R\sin^2\frac{A}{2} \Rightarrow \sin^2\frac{A}{2} = \frac{1}{2} \Rightarrow A = \frac{\pi}{2}$

10. In a triangle ABC, $a:b:c = 4:5:6$. The ratio of the radius of the circumcircle to that of the incircle is

A) 15/4 B) 11/5 C) 16/7 D) 16/3.

Key. C

Sol. $\frac{a}{4} = \frac{b}{5} = \frac{c}{6}$ use $\Delta rs = \frac{abc}{4R}$

11. In triangle ABC , $\frac{s-a}{\Delta} = \frac{1}{8}, \frac{s-b}{\Delta} = \frac{1}{12}, \frac{s-c}{\Delta} = \frac{1}{24}$ then b=

1) 16 2) 20 3) 24 4) 28

Key. 1

Sol.
$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$$
$$b = \sqrt{(r_2 - r)(r_1 + r_3)}$$

12. If in a triangle ABC, $\frac{s - r_2}{r_2} = \sqrt{2}$ then $\frac{a^2 + c^2 - b^2}{2ac} =$

1) $\frac{1}{\sqrt{2}}$ 2) $-\frac{1}{\sqrt{2}}$ 3) $\frac{\sqrt{3}}{2}$ 4) $-\frac{\sqrt{3}}{2}$

Key. 1

Sol. $\Rightarrow r_2(\sqrt{2}+1) = s \Rightarrow \tan\frac{B}{2} = \sqrt{2} - 1$

13. ABCD is a quadrilateral, AB=a, BC=b , CD=c , DA=d , is inscribed to a circle and circumscribed to another circle. Then the value $\tan^2\frac{A}{2}$ =

1) $\frac{ad}{bc}$ 2) $\frac{ab}{cd}$ 3) $\frac{bc}{ad}$ 4) $\frac{ac}{bd}$

Key. 3

Sol. $\cos A = \frac{ad-bc}{ad+bc} = \frac{1-\frac{bc}{ad}}{1+\frac{bc}{ad}}$

14. In a triangle ABC, C=60^0 and R=16 then I I_3=

1) 30 2) 31 3) 32 4) 34

Key. 3

Sol. I I_3 = 4R sin $\frac{C}{2}$

15. In a triangle ABC, r = 2, $\angle B = 60^0$ and $\angle C = 90^0$ then r_1 =

1) $\sqrt{3}$ 2) $2\sqrt{3}$ 3) $3\sqrt{3}$ 4) $4\sqrt{3}$

Key. 2

Sol. $r_1 = r.\cot\frac{B}{2}.\cot\frac{C}{2}$

16. If a,b,c are the sides of a triangle, then the minimum value of $\frac{2a}{b+c-a}+\frac{2b}{c+a-b}+\frac{2c}{a+b-c}$ is

1) 3 2) 6 3) 8 4) 1/8

Key. 1

Sol. $\frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c} = -3+\left(\frac{s}{s-a}+\frac{s}{s-b}+\frac{s}{s-c}\right) \geq -3+9=6$

$$\left(\because (x_1+x_2+x_3)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}\right) \geq 9\right)$$

17. If x,y,z are the distances of the vertices of triangle ABC from its orthocenter then x+y+z=

1) 2 (R + r) 2) 2 (R-r) 3) 2R-r 4) 2R + r

Key. 1

Sol. X =2 R cos A , y=2R cos B, z=2R cos C

18. If in a triangle the ex-radii r_1, r_2, r_3 are in the ratio 1:2:3, then their sides are in the ratio :

1) 5:8:9 2) 1:2:3 3) 3:5:7 4) 1:5:9

Key. 1

Sol. $r_1 : r_2 : r_3 = 1 : 2 : 3$, $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{11}{6}$

$a : b : c = \sqrt{(r_1 - r)(r_2 + r_3)} : \sqrt{(r_2 - r)(r_1 + r_3)} : \sqrt{(r_3 - r)(r_1 + r_2)}$

19. If length of the sides of a triangle ABC are 3,4 and 5 cm, then distance between its orthocentre and circumcentre is

1) 2.5 c.m. 2) 2 c.m. 3) 1.5 c.m. 4) 8

Key. 1

Sol. $O^1 = R\sqrt{1 - 8\cos A \cos B \cos C} = R = 2.5$

20. If length of the sides of a triangle ABC are 3,4 and 5 cm, then distance between its incentre and circumcentre is

1) $\frac{\sqrt{3}}{2}$ 2) $\frac{\sqrt{5}}{2}$ 3) $\frac{1}{2}$ 4) $\frac{1}{\sqrt{2}}$

Key. 2

$$OI = \sqrt{R^2 - 2Rr}, R = 5/2, r = \frac{\frac{1}{2} \times 3 \times 4}{6} = 1$$

Sol.

21. If P is a point on the altitude AD of the triangle ABC such that $\angle DBP = \frac{B}{3}$, then AP is equal to

A) $2a\sin\frac{C}{3}$ B) $2b\sin\frac{C}{3}$ C) $2c\sin\frac{B}{3}$ D) $2c\sin\frac{C}{3}$

Key. C

Sol. $\angle DBP = \frac{B}{3}$

$\angle DBP = \frac{B}{3}$

$\angle ABP = \frac{2B}{3}$

$\frac{AP}{\sin\frac{2B}{3}} = \frac{c}{\sin\left(90+\frac{B}{3}\right)} \Rightarrow AP = C\left(2\sin\frac{B}{3}\right)$

22. In triangle ABC, if B = 90^0 then $\cos^{-1}\left(\frac{R}{r_1+r_3}\right) =$

1) $\frac{\pi}{6}$ 2) $\frac{\pi}{4}$ 3) $\frac{\pi}{3}$ 4) $\frac{2\pi}{3}$

Key. 3

Sol. $r_1 + r_3 = 4R\cos^2\frac{B}{2}$

23. A circle is inscribed in an equilateral triangle of side 6 units. The area of any square inscribed in this circle is

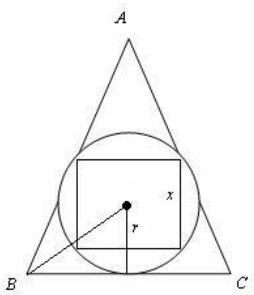
1) 6 2) 36 3) 9 4) 72

Key. 1

Sol. Let r be radius of in circle and x be side of the square

$r = \sqrt{3}$

$\sqrt{2}x = 2\sqrt{3} \Rightarrow x^2\,\frac{4\times3}{2} = 6$



24. If the area of triangle ABC is $b^2 - (c-a)^2$, then $\tan B =$

1) $\frac{3}{4}$ 2) $\frac{1}{4}$ 3) $\frac{8}{15}$ 4) $\frac{15}{8}$

Key. 3

Sol. $\Delta = b^2 - (c-a)^2 = b^2 - c^2 - a^2 + 2ac$

$= 2ac\left(1 - \frac{a^1 + c^2 - b^2}{2ac}\right) = 2ac(1 - \cos B)$

$\frac{abc}{4R} = 2ac.2\sin^2\frac{B}{2} \Rightarrow \tan\frac{B}{2} = \frac{1}{4}$

$\therefore \tan B = \frac{2/4}{1 - 1/16} = \frac{8}{15}$

25. If in a triangle ABC, (r_2-r_1) (r_3-r_1) =2r_2r_3, then the triangle is :

1) Right angled 2) Isosceles

3) Equilateral 4) Right angled Isosceles

Key. 1

Sol. $\left(\frac{\Delta}{s-b} - \frac{\Delta}{s-a}\right)\left(\frac{\Delta}{s-c} - \frac{\Delta}{s-a}\right) = 2\frac{\Delta}{s-b}\frac{\Delta}{s-c}$

$(b-a)(c-a) = 2(s-a)^2$

$\Rightarrow 2(b-a)(c-a) = (b+c-a)^2$

$\Rightarrow b^2 + c^2 = a^2$

26. If r_1,r_2,r_3 are exradii of any triangle then $r_1r_2 + r_2r_3 + r_3r_1$ is equal to :

1) Δ/r 2) Δ^2/r^2 3) r/Δ 4) r^2/Δ^2

Key. 2

Sol. $r_1 = \frac{\Delta}{s-a}, r_2 = \frac{\Delta}{s-b}, r_3 = \frac{\Delta}{s-c}$

27. If in a triangle ABC, $2a = p\left(\frac{1}{r_2} + \frac{1}{r_3}\right) + q\left(\frac{1}{r} - \frac{1}{r_1}\right)$, then p+q=

1) Δ 2) 2Δ 3) 3Δ 4) 4Δ

Key. 2

Sol. $r_1 r_2 = r_3 = \frac{\sqrt{3}}{2}, r = \frac{1}{2\sqrt{3}}$

28. In a triangle, if $= r_1 = 2r_2 = 3r_3$, then $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} =$

1) $\frac{75}{100}$ 2) $\frac{155}{60}$ 3) $\frac{176}{60}$ 4) $\frac{191}{60}$

Key. 4

Sol. $\frac{\Delta}{s-a} = 2, \frac{\Delta}{s-b} = 3, \frac{\Delta}{s-c} = k$

$\Rightarrow a = \frac{5}{k}, b = \frac{4}{k}, c = \frac{3}{k}$

29. In a triangle ABC, medians AD and CE are drawn . If AD=5, $\angle DAC = \frac{\pi}{8}$ and $\angle ACE = \frac{\pi}{4}$ then the area of triangle ABC is equal to

1) $\frac{25}{9}$ 2) $\frac{25}{3}$ 3) $\frac{25}{18}$ 4) $\frac{10}{3}$

Key. 2

Sol. $AG = \frac{2}{3}, AD = \frac{10}{3}$

$$\frac{GC}{\sin\frac{\pi}{8}} = \frac{AG}{\sin\frac{\pi}{4}} \Rightarrow GC = \frac{10}{3} \times \frac{\sin\frac{\pi}{8}}{\sin\frac{\pi}{4}}$$

$\therefore$ Area fo $\Delta ABC = 3$ Area of ΔAGC

$$3\left(\frac{1}{2}\frac{10}{3} \times \left(\frac{10}{3} \times \frac{\sin\frac{\pi}{8}}{\sin\frac{\pi}{4}}\right)\right) \times \sin\left(\frac{\pi}{2} + \frac{\pi}{8}\right) = \frac{25}{3}$$

30. In a triangle ABC, $r = 1, R = 4, \Delta = 8$ then the value of $ab + bc + ca =$

1) 18 2) 81 3) 72 4) 27

Key. 2

Sol. $r_1 + r_2 + r_3 - r = 4R$

$r(r_1 + r_2 + r_3) = ab + bc + ca - S^2$

31. If in a triangle ABC $r_1=3$, $r_2=10$, $r_3=15$ then the value of R equals

1) $\frac{15}{2}$ 2) $\frac{11}{2}$ 3) $\frac{9}{2}$ 4) $\frac{13}{2}$

Key. 4

Sol. $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$

$r_1 + r_2 + r_3 - r = 4R$

32. In a triangle ABC, the maximum value of $\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$ is

1) $\frac{s}{2R}$ 2) $\frac{R}{2s}$ 3) $\frac{s}{2r}$ 4) $\frac{r}{2s}$

Key. 2

Sol. $\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} = \frac{\Delta}{s(s-a)} \cdot \frac{\Delta}{s(s-b)} \cdot \frac{\Delta}{s(s-c)}$

$= \frac{\Delta}{s^2} = \frac{r}{s} \le \quad (\because 2r \le R)$

33. In triangle ABC, $\frac{r_1 + r_3}{1 + \cos B} =$

1) $\frac{abc}{4\Delta}$ 2) $\frac{abc}{2\Delta}$ 3) $\frac{2ab}{c\Delta}$ 4) $\frac{2(a+b)}{c\Delta}$

Key. 2

Sol. $\frac{4R\cos^2\frac{B}{2}}{2\cos^2\frac{B}{2}} = \frac{abc}{2\Delta}$

34. If in a triangle ABC, $r_1 = 8,\ r_2 = 12,\ r_3 = 24$ then C =

1) $\frac{\pi}{4}$ 2) $\frac{\pi}{6}$ 3) $\frac{\pi}{3}$ 4) $\frac{\pi}{2}$

Key. 4

Sol. $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$, $\tan^2\frac{C}{2} = \frac{rr_3}{r_1 r_2}$

35. If H is the orthocenter of a acuteangled triangle ABC whose circumcircle is $x^2 + y^2 = 16$ then curcumdiametre of the triangle HBC is

1) 1 2) 2 3) 4 4) 8

Key. 4

Sol. since $\angle HBC = 90 - C$

$$\frac{HC}{\sin(90 - c)} = 2R^1$$

$$\therefore 2R^1 = \frac{2R\cos c}{\cos c} = 2R$$

36. In triangle ABC , I is the incentre of the triangle . Then IA.IB.IC =

1) $4r^2R$ 2) $4R^2r$ 3) r^2R 4) R^2r

Key. 1

Sol. $I_A.I_B.I_C$ = r cosec A/2. r cosec B/2. r cosec C/2

$$\frac{r^3}{\sin A/2 \sin B/2 \sin C/2} \cdot \frac{4R}{4R} = \frac{4Rr^3}{r} = 4Rr^2$$

37. In a right angled triangle ABC with $A = \frac{\pi}{2}$, a circle is drawn touching the side AB,AC and incircle of the triangle. It's radius is equal to

1) $\left(2-\sqrt{2}\right)r$ 2) $\left(3-\sqrt{2}\right)r$ 3) $\left(3+\sqrt{2}\right)r$ 4) $\left(3-2\sqrt{2}\right)r$

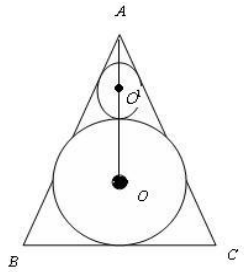
Key. 4

Sol. let r_1 be radius of required circle

$A0^1 = r_1 \operatorname{cosec}\frac{A}{2} = \sqrt{2}r_1$ $A0 = r \operatorname{cosec}\frac{A}{2} = \sqrt{2}r$

$00^1 = \sqrt{2}r\left(r-r_1\right)$ But $00^1 = r_1 + r$

$\because r_1 + r = \sqrt{2}\left(r-r_1\right) \Rightarrow r_1 \frac{\left(\sqrt{2}-1\right)}{\left(\sqrt{2+1}\right)} r = \left(3-2\sqrt{2}\right)r$

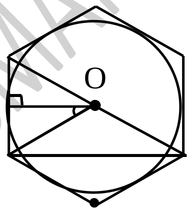


38. Let S_1 and S_2 be the areas of inscribed and circumscribed polygons of n sides respectively and S_3 is the area of regular polygon of 2n sides inscribed in a circle, then

A) $2S_3 = S_1 + S_2$ B) $S_3^2 = S_1S_2$

C) $\frac{1}{S_3} = \frac{1}{S_1} + \frac{1}{S_2}$ D) $\frac{2}{S_3} = \frac{1}{S_1} + \frac{1}{S_2}$

Key. B



Sol.

$$\tan\frac{\pi}{n} = \frac{x}{r}$$

$$x = r\tan\frac{\pi}{n}$$

$$S_1 = n \times \frac{1}{2} \times r^2 \times \sin\frac{2\pi}{n}$$

$$S_2 = n.r^2 \tan\frac{\pi}{n}$$

$$S_3 = \frac{2n}{2} \times r^2 \sin\frac{\pi}{n} \qquad S_3^2 = n^2 r^4 \sin^2\frac{\pi}{n}$$

$$S_1 S_2 = n^2 r^4 \frac{1}{2} \times 2\sin\frac{\pi}{n}\cos\frac{\pi}{n} . \frac{\sin\frac{\pi}{n}}{\cos\frac{\pi}{n}}$$

$$= n^2 r^4 \sin^2\frac{\pi}{n} = S_3^2$$

39. In ΔABC if $\frac{\sin A}{\sin B} + \frac{\sin B}{c} + \frac{\sin C}{b} = \frac{c}{ab} + \frac{b}{ac} + \frac{a}{bc}$ then angle A is

A) 120^0 B) 90^0 C) 60^0 D) 30^0

Key. B

Sol. $\frac{a}{bc} + \frac{b}{2Rc} + \frac{c}{2Rb} = \frac{c}{ab} + \frac{b}{ac} + \frac{a}{bc}$

$\Rightarrow 2R = a \Rightarrow A = 90^0$

40. In ΔABC, $A = \frac{2\pi}{3}, b - c = 3\sqrt{3}$ cm and area of ΔABC = $\frac{9\sqrt{3}}{2} cm^2$, then BC =

A) $6\sqrt{3}$ cm B) 9cm C) 18cm D) 27cm

Key. B

Sol. $\frac{1}{2} bc \sin\frac{2\pi}{3} = \frac{9\sqrt{3}}{2} \Rightarrow bc = 18 \Rightarrow b^2 + c^2 - 36 = 27 \Rightarrow b^2 + c^2 = 63$

$a^2 = 63 - 2 \times 18 \times \frac{-1}{2} = 81 \Rightarrow a = 9$

41. In ΔABC, if $\cot A = \sqrt{ac}, \cot B = \sqrt{\frac{c}{a}}, \cot C = \sqrt{\frac{a^3}{c}}$ then which of the following can be true?

A) $a + a^2 = 1 - c$ B) $a + a^2 = 1 + c$

C) $a+a^2=2-c$ D) $a+a^2=2+c$

Key. A

Sol. $\cot A\cot B=C,\ \cot B\cot C=a,\ \cot C\cot A=a^2$

But $\sum\cot A\cot B=1\Rightarrow c+a+a^2=1\Rightarrow a+a^2=1-c$

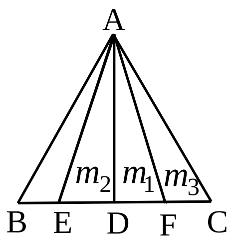
42. Let AD be a median of ΔABC. If AE and AF are medians of ΔABD and ΔADC respectively and AD = m_1, AE = m_2, AF = m_3, BC = a, then $\frac{a^2}{8}=$

A) $m_2^2+m_3^2-2m_1^2$ B) $m_1^2+m_2^2-2m_3^2$

C) $m_1^2+m_3^2-2m_2^2$ D) $m_1^2+m_2^2+m_3^2$

Key. A

Sol. $m_2^2+m_3^2=2\left(m_1^2+ED^2\right)\Rightarrow m_2^2+m_3^2-2m_1^2=\frac{a^2}{8}$

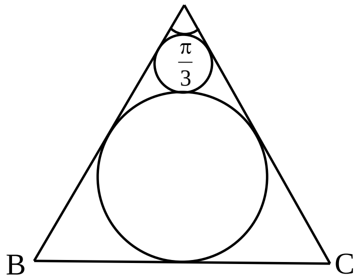


43. In ΔABC, $\angle A=\frac{\pi}{3}$ and its inradius is 6 units. The radius of the circle touching the sides AB, AC internally and the incircle of ΔABC externally is

A) 3units B) 3/2 units C) 2 units D) 4 units

Key. C

Sol. Angle between the direct common tangents is $\frac{\pi}{3}$



$\therefore\ 2\sin^{-1}\left(\frac{6-r}{6+r}\right)=\frac{\pi}{3}\Rightarrow\frac{6-r}{6+r}=\frac{1}{2}$

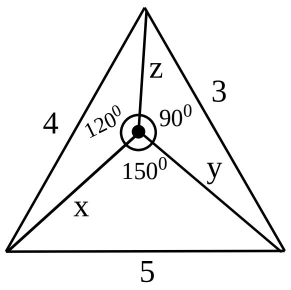
$\Rightarrow 12-2r=6+r\Rightarrow 6=3r\Rightarrow r=2$.

44. Three positive real numbers x, y, z satisfy the equations $x^2+\sqrt{3}xy+y^2=25$, $y^2+z^2=9$ and $x^2+xz+z^2=16$ then the value of $xy+2yz+\sqrt{3}xz$ is

A) 18 B) 24 C) 30 D) 36

Key. B

Sol.



Area of triangle = $\frac{1}{2}\times 3\times 4=\frac{1}{2}xz\frac{\sqrt{3}}{2}+\frac{1}{2}xy\times\frac{1}{2}+\frac{1}{2}yz$

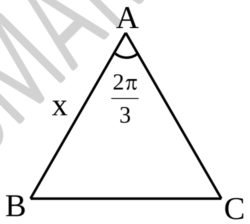
$$\Rightarrow 24=\sqrt{3}xz+xy+2yz$$

45. Let ABC be a triangle with $\angle BAC=\frac{2\pi}{3}$ and AB = x such that AB.AC=1. If x varies then the largest possible length of internal angular bisector AD is

A) 1 B) 2 C) $\frac{1}{2}$ D) $\frac{1}{4}$

Key. C

Sol.



Angular bisector $AD=\frac{2bc}{b+c}\cos\frac{A}{2}$

$$=\frac{2\times x\times\frac{1}{x}}{x+\frac{1}{x}}\times\frac{1}{2}$$

46. The sides of a triangle inscribed in a given circle subtend angles α, β, γ at the centre. Then, the minimum value of the A.M of $\cos\left(\alpha+\frac{\pi}{2}\right), \cos\left(\beta+\frac{\pi}{2}\right), \cos\left(\gamma+\frac{\pi}{2}\right)$ is

(A) $-\frac{\sqrt{3}}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) $\frac{1}{\sqrt{2}}$ (D) none of these

Key. A

Sol. Clearly, $\angle A = \frac{\alpha}{2}, \angle B = \frac{\beta}{2}, \angle C = \frac{\gamma}{2}$

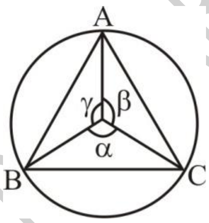
$\therefore \alpha+\beta+\gamma = 2\pi$

$$\text{A.M.} = \frac{1}{3}\left[\cos\left(\alpha+\frac{\pi}{2}\right)+\cos\left(\beta+\frac{\pi}{2}\right)+\cos\left(\frac{\gamma}{2}\right)\right]$$

$$= -\frac{1}{3}[\sin\alpha+\sin\beta+\sin\gamma]$$

$$= -\frac{4}{3}\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right)$$

$$= -\frac{4}{3}\sin A \sin B \sin C$$



A.M. will be least if $\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right)$ is greatest i.e. sinA sinB sinC is greatest, we know that in a ΔABC, sin A sin B sin C is greatest if $A = B = C = \frac{\pi}{3}$

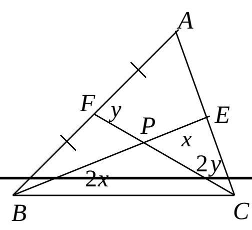
$$\therefore \text{Least A.M.} = -\frac{4}{3}\left(\frac{\sqrt{3}}{2}\right)^3 = -\frac{\sqrt{3}}{2}$$

47. In the triangle ABC the medians from B and C are perpendicular. The value of $\cot B + \cot C$ cannot be

A) $\frac{1}{3}$ B) $\frac{2}{3}$ C) $\frac{4}{3}$ D) $\frac{5}{3}$

Key : A

Sol. $\tan B = \dfrac{\frac{y}{2x}+\frac{y}{x}}{1-\frac{y^2}{2x^2}} = \dfrac{3xy}{2x^2-y^2}$



$\tan\lfloor PBF = \frac{y}{2x}$

$\tan\lfloor PBC = \frac{y}{x}$

$$\cot B = \frac{2x^2 - y^2}{3xy},\ \cot C = \frac{2y^2 - x^2}{3xy}$$

$$\cot B + \cot C = \frac{x^2 + y^2}{3xy} \geq \frac{2}{3}$$

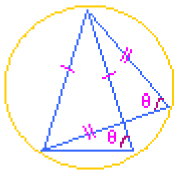
48. T_1 is an isosceles triangle with circumcircle K. Let T_2 be another isosceles triangle inscribed in K whose base is one of the equal side of T_1 and which overlaps the interior of T_1. Similarly create isosceles triangles T_3 from T_2, T_4 from T_3 and so on to the triangle T_n. Then the base angle of the triangle T_n as $n \to \infty$ is

a) 30^0 b) 60^0 c) 90^0 d) 120^0

Key : B

Sol : T_1 is an isosceles triangle with circumcricle K. Let T_2 be another isosceles triangle inscribed in K whose base is one of the equal sides of T_1 and which overlaps the interior of T_1. Similarly create isosceles triangles T_3 from T_2, T_4 from T_3 and so on, do the triangles T_n approach an equilateral triangle as $n \to \infty$?

Note that the base angle of T_n is equal to the angle opposite the base of T_{n+1} (as the figure indicates). Therefore, if θ is the base angle for T_n, then the base of angle for the next triangle (T_{n+1}) is $\frac{180^0 - \theta}{2} = 90^0 - \frac{\theta}{2}$.



Suppose, now that θ is the base angle for T_1, then the base angle for T_n is

$$90 - \frac{90}{2} + \frac{90}{4} - \frac{90}{8} + ... + (-1)^{n-2} + \frac{90}{2^{n-2}} + (-1)^{n-1} \frac{\theta}{2^{n-1}}.$$

Note that the limit as $n \to \infty$ of the above is $\frac{90}{1+1/2} = 60^0$ by formula for the sum of an infinite

49. R is the circum radius of $\Delta le\ ABC$ whose circum centre is 'S'. is the circum radius of $\Delta le\ SBC$. Then the ratio $R : R'$ is

a) 1 b) depends upon side BC

c) independent of d) depends on

c) A is true , R is false

d) A is false, R is true

KEY : D

HINT. $R = \frac{a}{\text{Sin}A}, R' = \frac{a}{\text{Sin}2A}$

$\therefore \dfrac{R}{R'} = 2\text{CosA}$

50. In a triangle ABC, $A - B = 120^0$ and $R = 8r$ then the value of $\cos C$ is

(A) $\dfrac{1}{4}$ (B) $\dfrac{\sqrt{15}}{4}$

(C) $\dfrac{7}{8}$ (D) $\dfrac{\sqrt{3}}{2}$

KEY : C

HINT : $r = 4R\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$

$$\Rightarrow 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{1}{16}$$

$$\Rightarrow \left[\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right]\sin\frac{C}{2} = \frac{1}{16}$$

$$\Rightarrow \left(\frac{1}{2} - \sin\frac{C}{2}\right)\sin\frac{C}{2} = \frac{1}{16}$$

$$\Rightarrow \left(\frac{1}{4} - \sin\frac{C}{2}\right)^2 = 0 \Rightarrow \sin\frac{C}{2} = \frac{1}{4}$$

Hence $\cos C = 1 - 2\sin^2\dfrac{C}{2} = 1 - 2\times\dfrac{1}{16} = \dfrac{7}{8}$

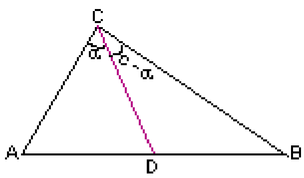
51. In a scalene DABC, D is a point on the side AB such that $CD^2 = AD \cdot DB$, if $\text{SinA}\cdot\text{SinB} = \text{Sin}^2\dfrac{C}{2}$ then CD is

a) Median through C b) Internal bisector of

c) Altitude through C d) Divides AB in the ratio 1 : 2

Key : B

Sol : Let $\angle ACD = \alpha \Rightarrow \angle DCB = (C - \alpha)$



Applying the sine rule in ΔACD and in ΔDCB respectively, we get

$\dfrac{AD}{\sin\alpha} = \dfrac{CD}{\sin A}$ and $\dfrac{BD}{\sin(C-\alpha)} = \dfrac{CD}{\sin B}$

$$\Rightarrow \quad \frac{AD\cdot BD}{\sin\alpha\cdot\sin(C-\alpha)}=\frac{CD^2}{\sin A\cdot\sin B}$$

$$\Rightarrow \quad \frac{1}{2}\left[\cos(2\alpha-C)-\cos C\right]=\frac{1}{2}\left[\cos(2\alpha-c)-1+2\sin^2\frac{C}{2}\right]=\sin^2\frac{C}{2}-\frac{1}{2}\left(1-\cos(2\alpha-C)\right)$$

since, $1-\cos(2\alpha-C)\geq 0$

$$\Rightarrow \sin A\cdot\sin B\leq\sin^2\frac{C}{2}$$

and equality sign holds, if $1-\cos(2\alpha-C)=0$

$$\Rightarrow \alpha=\frac{C}{2}$$

That means equality sign holds, if CD is the internal angle bisector of angle C.

52. The perimeter of a triangle ABC is 6 times the arithmetic mean of the sines of its angles. If the side a is 1,then $\underline{|A}$ is

a) $\frac{\pi}{6}$ b) $\frac{\pi}{3}$

c) $\frac{\pi}{2}$ d) $\frac{2\pi}{3}$

Key: A

Hint $2s=6\left(\frac{\sin A+\sin B+\sin C}{3}\right)$

53. The radii of the escribed circles of ΔABC are r_a, r_b and r_c respectively. If $r_a + r_b = 3R$ and $r_b + r_c = 2R$, then the smallest angle of triangle is

a) $\tan^{-1}(\sqrt{2}-1)$ b) $\frac{1}{2}\tan^{-1}(\sqrt{3})$

c) $\frac{1}{2}\tan^{-1}(\sqrt{2}+1)$ d) $\tan^{-1}(2-\sqrt{3})$

sol : We have $r_a + r_b = 3R \Rightarrow \frac{\Delta}{s-a}+\frac{\Delta}{s-b}=3r=\frac{3abc}{4\Delta}\left(R=\frac{abc}{4\Delta}\right)$

$$\Rightarrow \frac{\Delta(s-b+s-a)}{(s-a)(s-b)}=\frac{3abc}{4\Delta}\Rightarrow\frac{c\Delta}{(s-a)(s-b)}=\frac{3abc}{4\Delta}\Rightarrow\frac{\Delta^2}{(s-a)(s-b)}=\frac{3ab}{4}$$

$\Rightarrow 4s(s-c)=3ab \Rightarrow (a+b+c)(a+b-c)=3ab$

$\Rightarrow (a+b)^2-c^2=3ab$ $\Rightarrow a^2+b^2-c^2=ab$

$\Rightarrow c^2=a^2+b^2-ab$

$\Rightarrow$ $a^2 + b^2 - 2ab \cos C = a^2 + b^2 - ab$ (As $c^2 = a^2 + b^2 - 2ab \cos C$)

$\Rightarrow \cos C = \frac{1}{2} \Rightarrow \angle C = 60^o$ …………….. (1)

Clearly from $r_b + r_c = 2R$

$\Rightarrow \frac{\Delta}{s-b} + \frac{\Delta}{s-c} = 2R \quad \Rightarrow \frac{\Delta(2s-b-c)}{(s-b)(s-c)} = \frac{2abc}{4\Delta} \Rightarrow \frac{2\Delta^2}{(s-b)(s-c)} = bc$

$\Rightarrow 2s(s-a) = bc \Rightarrow (b+c+a)(b+c-a) = 2bc \Rightarrow (b+c)^2 - a^2 = 2bc$

Note : Angles A, C, B are in AP can be converted into more than one

54. With usual notations, in a triangle ABC, a cos(B – C) + b cos(C – A) + c cos(A – B) is equal to

(A) $\frac{abc}{R^2}$ (B) $\frac{abc}{4R^2}$ (C) $\frac{4abc}{R^2}$ (D) $\frac{abc}{2R^2}$

Key. A

Sol. Here a(cosB cosC + sinB sinC) + ...

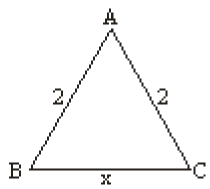
using $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

a (cosB cosC + $\frac{bc}{4R^2}$) + ...

$= \frac{3abc}{4R^2}$ + a cosB cosC + b cosC cosA + c cosA cosB = $\frac{3abc}{4R^2}$ + c cosC + c cosA cosB

$= \frac{3abc}{4R^2}$ + c [cosA cosB – cos(A + B)] = $\frac{3abc}{4R^2}$ + c sinA sinB = $\frac{3abc}{4R^2} + \frac{abc}{4R^2} = \frac{abc}{R^2}$

55. An isosceles triangle has sides of length 2, 2, and x. The value of x for which the area of the triangle is maximum, is



(A) 1 (B) $\sqrt{2}$

(C) 2 (D) $2\sqrt{2}$

Key. D

Sol. $\frac{1}{2} \times 2 \times 2 \sin A$ which is maximum if A = 90° $\Rightarrow$ x = $2\sqrt{2}$]

56, In a Δ ABC if b + c = 3a then $\cot \frac{B}{2} \cdot \cot \frac{C}{2}$ has the value equal to :

(A) 4 (B) 3 (C) 2 (D) 1

Key. C

Sol. $\cot \frac{B}{2} \cdot \cot \frac{C}{2} = \frac{s(s-b)}{\Delta} \cdot \frac{s(s-c)}{\Delta} \cdot \frac{(s-a)}{s-a} = \frac{s}{s-a} = \frac{2s}{2s-2a}$

but given that a + b + c = 4a $\Rightarrow$ 2s = 4a Hence $\cot\frac{B}{2}\cdot\cot\frac{C}{2} = \frac{4a}{2a} = 2$

57. Let f, g, h be the lengths of the perpendiculars from the circumcentre of the Δ ABC on the sides a, b and c respectively . If $\frac{a}{f}+\frac{b}{g}+\frac{c}{h} = \lambda\frac{a\,b\,c}{f\,g\,h}$ then the value of λ is :

(A) 1/4 (B) 1/2 (C) 1 (D) 2

Key. A

Sol. $\tan A = \frac{a}{2f} \Rightarrow \frac{1}{2}\Sigma\tan A = \frac{1}{2}\Pi\tan A$

$= \frac{1}{4}\left(\frac{a}{f}.\frac{b}{g}.\frac{c}{h}\right) \Rightarrow$ A]

58. In a triangle ABC, $R(b + c) = a\sqrt{bc}$ where R is the circumradius of the triangle. Then the triangle is

(A) Isosceles but not right (B) right but not isosceles

(C) right isosceles (D) equilateral

Key. C

Sol. $R(b + c) = a\sqrt{bc}$

$R(b + c) = 2R\sin A\,\sqrt{bc}$

$\therefore$ $\sin A = \frac{b+c}{2\sqrt{bc}}$

now applying AM $\geq$ GM for b and c

$\frac{b+c}{2bc} \geq \sqrt{bc}$; $\therefore$ $\frac{b+c}{2bc} \geq 1$

hence $\sin A \geq 1$ which is not possible.

hence $\sin A = 1$ $\Rightarrow$ $A = 90°$

$\therefore$ $A = 90°$ and $b = c$ $\Rightarrow$ (C)

59. A triangle with integral sides has perimeter 8 cm. Then the area of the triangle, is

(A) $2\sqrt{2}\,cm^2$ (B) $\frac{16}{9}\sqrt{3}\,cm^2$ (C) $2\sqrt{3}\,cm^2$ (D) $4\sqrt{2}\,cm^2$

Key. A

Sol. Only possibility for the sides can be 3, 3, 2 (think !)

$A = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{4\times1\times1\times2} = 2\sqrt{2}\,cm^2$

60. In triangle ABC, $a^2 + c^2 = 2002b^2$ then $\frac{\cot A + \cot C}{\cot B} =$

A) $\frac{1}{2001}$ B) $\frac{2}{2001}$ C) $\frac{3}{2001}$ D) $\frac{4}{2001}$

Key. B

Sol. $\frac{\cot A+\cot C}{\cot B} = \frac{\sin(A+C)\sin B}{\sin A\sin C\sin B} = \frac{\sin^2 B}{\sin A\cos B\sin C}$

$$= \frac{4R^2b^2}{4R^2ac\cos B} = \frac{2b^2}{2ac\cos B} = \frac{2b^2}{a^2+c^2-b^2}$$

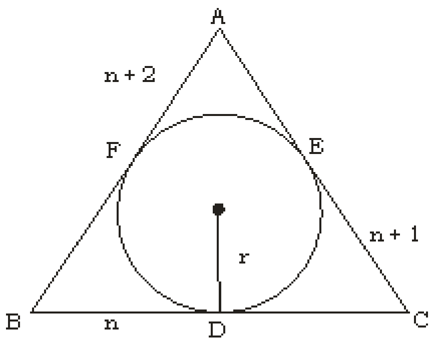
$$= \frac{2b^2}{2002b^2-b^2} = \frac{2}{2001}$$

61. The circle touches the sides BC, CA and AB of respectively at D, E and F. If the lengths BD, CE and AF are consecutive integers then the largest side of the triangle is equal to

a) 13
b) 14
c) 15
d) cannot be determined

Sol: Let BD = n, CE = n + 1, AF = n + 2.

Then BD = BF = n, CE = CD = n + 1, AF = AE = n + 2



$\therefore$ a = BC = 2n +1, b =2n +3, c =2n+2, s = 3n+3

$$r = \frac{\Delta}{s} = \frac{\sqrt{(3n+3)(n+2)n(n+1)}}{3n+3}$$

$$\therefore 4 = \sqrt{\frac{(n+2)n}{3}} \Rightarrow n(n+2) = 48 \Rightarrow n = 6$$

$\therefore$ the largest side of the triangle is 2n + 3 = 15.

62. In a ΔABC, medians AD and BE are drawn. If AD = 4, $\angle DAB = \frac{\pi}{6}$ and $\angle ABE = \frac{\pi}{3}$ then the area of ΔABC is

(A) $\frac{64}{3}$
(B) $\frac{8}{3\sqrt{3}}$
(C) $\frac{16}{3}$
(D) $\frac{32}{3\sqrt{3}}$

Key. D

Sol. The medians intersect at centroid G with AG = $\frac{8}{3}$ ($\because$ AG : GD = 2 : 1)

$$\angle AGB = \frac{\pi}{2} \Rightarrow BG = \frac{8}{3}\cot\frac{\pi}{3} = \frac{8}{3\sqrt{3}}$$

Area of $\Delta AGB = \frac{1}{2} \times \frac{8}{3\sqrt{3}} \times \frac{8}{3} = \frac{32}{9\sqrt{3}}$ $\therefore$ Area of $\Delta ABC = \frac{32}{3\sqrt{3}}$

63. In a triangle ABC, $A - B = 120^0$ and $R = 8r$ then the value of $\cos C$ is

(A) $\frac{1}{4}$ (B) $\frac{\sqrt{15}}{4}$

(C) $\frac{7}{8}$ (D) $\frac{\sqrt{3}}{2}$

Key. (c)

Sol. $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$\Rightarrow 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{1}{16}$

$\Rightarrow \left[\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right]\sin\frac{C}{2} = \frac{1}{16}$

$\Rightarrow \left(\frac{1}{2} - \sin\frac{C}{2}\right)\sin\frac{C}{2} = \frac{1}{16}$

$\Rightarrow \left(\frac{1}{4} - \sin\frac{C}{2}\right)^2 = 0 \Rightarrow \sin\frac{C}{2} = \frac{1}{4}$

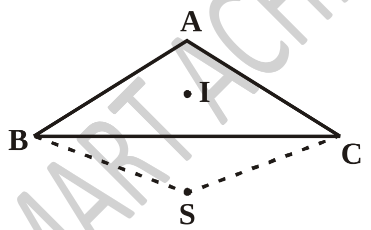
Hence $\cos C = 1 - 2\sin^2\frac{C}{2} = 1 - 2\times\frac{1}{16} = \frac{7}{8}$

64. In a Δ ABC the incentre and circumcentre are *reflections* of each other in side BC. Hence the
measure of ∠ BAC (in degrees) is

(a) 120 (b) 108 (c) 135 (d) 105

Key. (b)

Sol.



I : the incentre

S : the circumcentre

$\angle BIC = 90^0 + \frac{A}{2}$ (standard result)

and reflex $\angle BSC = 2A \Rightarrow \angle BSC = 360^0 - 2A$

Hence $90^0 + \frac{A}{2} = 360^0 - 2A$

65. ABC is a triangle. Put $x = a\cos A$, $y = b\cos B$, $z = c\cos C$.

x , y, z are the side lengths of a triangle

(a) only if Δ ABC is equilateral (b) only if Δ ABC is obtuse

(c) only if Δ ABC is a right triangle (d) for any acute Δ ABC

Key. (d)

Sol. For any acute triangle ABC , x , y and z are the side lengths of the triangle formed by the feet of the altitudes of Δ ABC.

66. If ABC is a triangle in which $\frac{\pi}{2} < C < \pi$, then the quantity $\frac{a^2+b^2}{c^2}$ lies in the interval

(a) $(0, \frac{1}{2})$ (b) $(1, \frac{3}{2})$ (c) $(\frac{3}{2}, 2)$ (d) $(\frac{1}{2}, 1)$

Key. (d)

Sol. $\frac{\pi}{2} < C < \pi \Rightarrow \frac{a^2+b^2-c^2}{2ab} = \cos C < 0$

$\Rightarrow a^2 + b^2 < c^2$

$\Rightarrow \frac{a^2+b^2}{c^2} < 1.$

Further $\frac{a^2+b^2}{2} \geq \left(\frac{a+b}{2}\right)^2 > \left(\frac{c}{2}\right)^2 \Rightarrow \frac{a^2+b^2}{c^2} > \frac{1}{2}$

67. If cos A + cosB + 2cosC = 2 then the sides of the Δ ABC are in

(A) A.P. (B) G.P (C) H.P. (D) none

Key. A

Sol. $\cos A + \cos B = 2(1-\cos C) = 4\sin^2\frac{C}{2}$ or $2\cos\frac{A+B}{2}\cos\frac{A-B}{2} = 4\sin^2\frac{C}{2}$

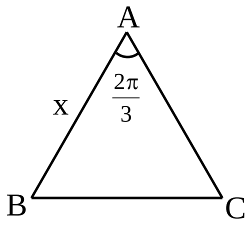
or $\cos\frac{A-B}{2} = 2\sin\frac{C}{2}$ or $2\cos\frac{C}{2}\cos\frac{A-B}{2} = 4\sin\frac{C}{2}\cos\frac{C}{2} = 2\sin C$

$2\sin\frac{A+B}{2}\cos\frac{A-B}{2} = 2\sin C$ or $\sin A + \sin B = 2\sin C \Rightarrow$ a, c, b are in A.P.

68. Let ABC be a triangle with $\angle BAC = \frac{2\pi}{3}$ and AB = x such that AB.AC=1. If x varies then the largest possible length of internal angular bisector AD is

A) 1 B) 2 C) $\frac{1}{2}$ D) $\frac{1}{4}$

Key. C



Sol.

Angular bisector $AD = \frac{2bc}{b+c}\cos\frac{A}{2}$

$$= \frac{2 \times x \times \frac{1}{x}}{x + \frac{1}{x}} \times \frac{1}{2}$$

69. Let I be the incentre of the triangle ABC, where $\frac{\overrightarrow{BC}}{|\overrightarrow{BC}|} + \frac{\overrightarrow{BA}}{|\overrightarrow{BA}|} = \frac{\overrightarrow{BI}}{k}$ then the diameter of the circumcircle of the triangle is

(A) k(cos A/2 + cos C/2) (B) k(sin A/2 + sin C/2)
(C) k(cot A/2 + cot C/2) (D) k (tan A/2 + tan C/2)

Key. C

Sol. Taking modulus both sides

$2\cos B/2 = \frac{1}{k}|\overrightarrow{BI}| = \frac{1}{k}\frac{r}{\sin B/2} = \frac{kR\sin A/2\sin C/2}{k}$

$\Rightarrow 2R = \frac{k\sin\left(\frac{A+C}{2}\right)}{\sin A/2\ \sin C/2} = k\ (\cot A/2 + \cot C/2)$

70. Let in a triangle ABC, $\frac{\overrightarrow{BC}}{|\overrightarrow{BC}|} + \frac{\overrightarrow{BA}}{|\overrightarrow{BA}|} = \frac{1}{k}\overrightarrow{BI}$ then the diameter of the circumcircle of the ΔABC is

(A) k (cosA/2 + cosC/2) (B) k (sinA/2 + sinC/2)
(C) k (cot A/2 + cot C/2) (D) k (tan A/2 + tan C/2)

Key. C

Sol. Taking modulus both sides

$2\cos B/2 = \frac{1}{k}|\overrightarrow{BI}| = \frac{1}{k}.\frac{r}{\sin B/2} = \frac{4R\sin\frac{A}{2}\sin\frac{C}{2}}{k}$

$\Rightarrow 2R = \frac{k\sin\left(\frac{A+C}{2}\right)}{\sin\frac{A}{2}\sin\frac{C}{2}}$ = k (cot A/2 + cot C/2)

71. In ΔABC, $A=\frac{2\pi}{3}, b-c=3\sqrt{3}$ cm and area of ΔABC = $\frac{9\sqrt{3}}{2}cm^2$, then BC =

A) $6\sqrt{3}$ cm B) 9cm C) 18cm D) 27cm

Key. B

Sol. $\frac{1}{2}bc\sin\frac{2\pi}{3}=\frac{9\sqrt{3}}{2}\Rightarrow bc=18\Rightarrow b^2+c^2-36=27\Rightarrow b^2+c^2=63$

$a^2=63-2\times 18\times\frac{-1}{2}=81\Rightarrow a=9$

72. In ΔABC, if $\cot A=\sqrt{ac}, \cot B=\sqrt{\frac{c}{a}}, \cot C=\sqrt{\frac{a^3}{c}}$ then which of the following can be true?

A) $a+a^2=1-c$ B) $a+a^2=1+c$

C) $a+a^2=2-c$ D) $a+a^2=2+c$

Key. A

Sol. $\cot A\cot B=C,\ \cot B\cot C=a,\ \cot C\cot A=a^2$

But $\sum\cot A\cot B=1\Rightarrow c+a+a^2=1\Rightarrow a+a^2=1-c$

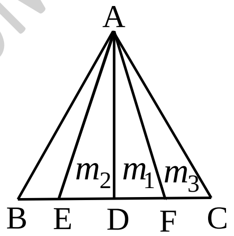
73. Let AD be a median of ΔABC. If AE and AF are medians of ΔABD and ΔADC respectively and AD = m_1, AE = m_2, AF = m_3, BC = a, then $\frac{a^2}{8}=$

A) $m_2^2+m_3^2-2m_1^2$ B) $m_1^2+m_2^2-2m_3^2$

C) $m_1^2+m_3^2-2m_2^2$ D) $m_1^2+m_2^2+m_3^2$

Key. A

Sol. $m_2^2+m_3^2=2\left(m_1^2+ED^2\right)\Rightarrow m_2^2+m_3^2-2m_1^2=\frac{a^2}{8}$

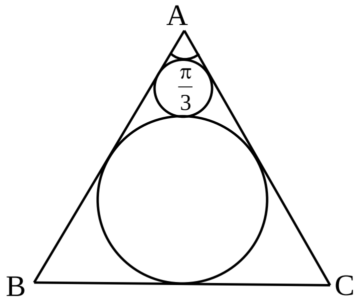


74. In ΔABC, $\angle A=\frac{\pi}{3}$ and its inradius is 6 units. The radius of the circle touching the sides AB, AC internally and the incircle of ΔABC externally is

A) 3units B) 3/2 units C) 2 units D) 4 units

Key. C

Sol. Angle between the direct common tangents is $\frac{\pi}{3}$



$\therefore\ 2\sin^{-1}\left(\frac{6-r}{6+r}\right)=\frac{\pi}{3}\Rightarrow\frac{6-r}{6+r}=\frac{1}{2}$

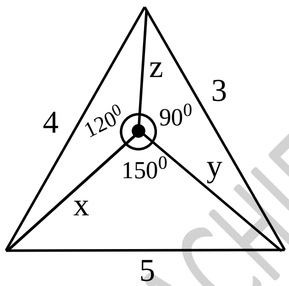
$\Rightarrow 12-2r=6+r\Rightarrow 6=3r\Rightarrow r=2$.

75. Three positive real numbers x, y, z satisfy the equations $x^2+\sqrt{3}xy+y^2=25$, $y^2+z^2=9$ and $x^2+xz+z^2=16$ then the value of $xy+2yz+\sqrt{3}xz$ is

A) 18 B) 24 C) 30 D) 36

Key. B

Sol.



Area of triangle = $\frac{1}{2}\times3\times4=\frac{1}{2}xz\frac{\sqrt{3}}{2}+\frac{1}{2}xy\times\frac{1}{2}+\frac{1}{2}yz$

$$\Rightarrow 24=\sqrt{3}xz+xy+2yz$$

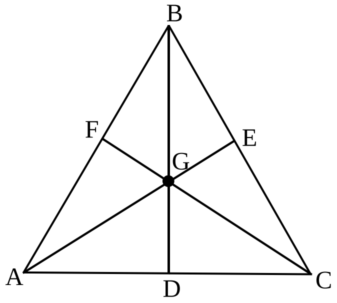
76. If m_a, m_b, m_c are lengths of medians through the vertices A, B, C of triangle ABC respectively, then length of side c =

A) $\frac{1}{3}\sqrt{2m_a^2+2m_c^2-m_b^2}$ B) $\frac{2}{3}\sqrt{2m_a^2+2m_c^2-m_b^2}$

C) $\frac{1}{3}\sqrt{2m_a^2+2m_b^2-m_c^2}$ D) $\frac{2}{3}\sqrt{2m_a^2+2m_b^2-m_c^2}$

Key. D

Sol. $AG=\frac{2}{3}ma,\ CG=\frac{2}{3}mc$



$$c^2+\frac{4}{9}mc^2=2\left(\frac{4}{9}ma^2+\frac{4}{9}mb^2\right)$$

77. If the bisector of angle 'A' of triangle ABC makes an angle 'θ' with $\overline{BC}$, then $\sin\theta$ is equal to

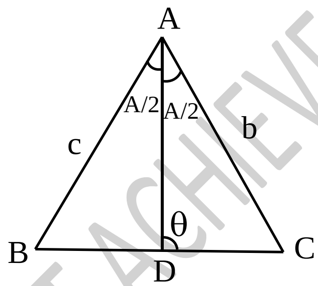
A) $\cos\left(\frac{B-C}{2}\right)$ B) $\sin\left(\frac{B-C}{2}\right)$

C) $\sin\left(B-\frac{A}{2}\right)$ D) $\sin\left(C-\frac{A}{2}\right)$

Key. A

Sol. $\theta=B+\frac{A}{2}=B+\frac{180^0-(B+C)}{2}$

$$=90^0+\left(\frac{B-C}{2}\right)$$

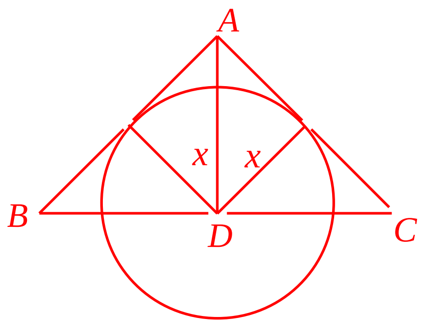


$$\sin\theta=\cos\left(\frac{B-C}{2}\right)$$

78. A circle of diameter '$2x$' is drawn on the side BC of triangle ABC such that it touches the sides, AB and AC. Then x =

A) $\frac{\Delta}{2(b+c)}$ B) $\frac{2\Delta}{b+c}$ C) $\frac{bc}{2\Delta}$ D) $\frac{b+c}{2\Delta}$

Key. B



Sol.

$$\Delta = \frac{1}{2}x(AB+AC) \Rightarrow x = \frac{2\Delta}{b+c}$$

79. If in a triangle ABC, $b\cos^2\frac{A}{2}+a\cos^2\frac{B}{2}=\frac{3c}{2}$ then minimum value of $\frac{a+c}{2c-a}+\frac{b+c}{2c-b}$ is equal to

A) 2 B) 4 C) 6 D) 8

Key. B

Sol. L.H.S. = $\frac{1}{2}\left(b+b\cos A+a+a\cos B\right)$

$$\Rightarrow \frac{1}{2}(a+b+c)=\frac{3}{2}c \Rightarrow 2c=a+b$$

$$\frac{a+c}{2c-a}+\frac{b+c}{2c-b}=\frac{a+c}{b}+\frac{b+c}{a}=\frac{a}{b}+\frac{b}{a}+\frac{c}{a}+\frac{c}{b}$$

$$\geq 4\left(\frac{c^2}{ab}\right)^{1/4} \geq 4$$

80. A right angled triangle ABC of maximum area is inscribed in a circle of radius R, then (Here Δ is area and s is semi perimeter, r_1, r_2, r_3 exradi of Δ ABC)

A) $\Delta = 2R^2$

B) $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{\sqrt{2}+1}{R}$

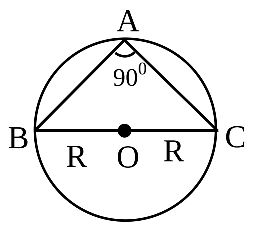
C) $r=\left(\sqrt{2}-1\right)R$

D) $s=\left(2+\sqrt{2}\right)R$

Key. B

Sol. In Δ ABC, $AB=AC=\sqrt{2}R$

$$S=R\left(\sqrt{2}+1\right),\ \Delta = R^2$$



$$r=\frac{\Delta}{s}=\frac{R}{\sqrt{2}+1}\Rightarrow\frac{1}{r}=\frac{\sqrt{2}+1}{R}$$

81 If an acute angled triangle ABC, if H is the orthocenter $AH = x$, $BH = y$, $CH = z$ then $x^2+y^2+z^2=$

A. $16R^2-(a^2+b^2+c^2)$ B. $12R^2-(a^2+b^2+c^2)$

C. $9R^2-(a^2+b^2+c^2)$ D. $8R^2-(a^2+b^2+c^2)$

KEY. B

SOL. $AH=2R\cos A, BH=2R\cos B, CH=2R\cos C$

$x^2+y^2+z^2=4R^2(\cos^2 A+cos^2 B+\cos^2 C)$

$=4R^2\{3-\sin^2 A-\sin^2 B-\sin^2 C\}$

$=12R^2-(a^2+b^2+c^2)$

82 Let ABC be a triangle such that $\angle ACB=\frac{\pi}{6}$ and let a,b,c denote the length of the sides opposite to A,B and C respectively . The value of x for which $a=x^2+x+1, b=x^2-1, c=2x+1$ is

A. $2+\sqrt{3}$ B. $2-\sqrt{3}$ C. $1+\sqrt{3}$ D. $4\sqrt{3}$

KEY. C

SOL. $A=120^0, C=30^0, B=30^0$

$b=c\Rightarrow x^2-1=2x+1$

$x^2-2x-2=0$

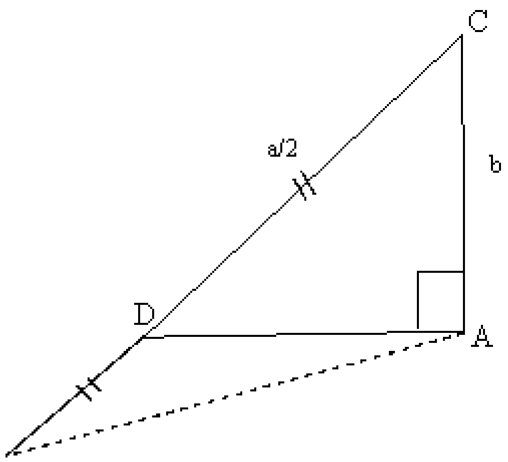
$x=\frac{2\pm\sqrt{4-4(1)(-2)}}{2.1}=\frac{2\pm\sqrt{12}}{2}=\frac{2\pm2\sqrt{3}}{2}=1\pm\sqrt{3}$

$\Rightarrow x=1+\sqrt{3}$

83 In ΔABC , D is the midpoint of BC. If AD is perpendicular to AC. Then cosA.cosC=

A. $\frac{c^2-a^2}{3ac}$ B. $\frac{3(c^2-a^2)}{2ac}$ C. $\frac{2(c^2-a^2)}{3ac}$ D. $\frac{2(a^2-c^2)}{3ac}$

KEY. C



SOL.

$$\cos C = \frac{b}{\left(\frac{a}{2}\right)} = \frac{2b}{a}$$

$$\frac{a^2+b^2-c^2}{2ab} = \frac{2b}{a} \Rightarrow a^2+b^2+c^2 = 4b^2$$

$$a^2 - c^2 = 3b^2$$

$$\cos A.\cos c = \frac{b^2+c^2-a^2}{2bc}\left(\frac{2b}{a}\right) = \frac{2(c^2-a^2)}{3ac}$$

84 In ΔABC if r =1, R =5 , $\Delta = 10$ then ab + bc + ca =

A. 81 B. 121 C. 141 D. 111

KEY. B

SOL. $r(r_1+r_2+r_3)+s^2 = ab+bc+ca$

$$1(r+4r)+s^2 = ab+bc+ca$$

$$1(1+4.5)+10^2 = ab+bc+ca \Rightarrow 100+21=121$$

$$r = \frac{\Delta}{s}$$

$$1 = \frac{10}{s}$$

$$s = 10$$

85. If in an equilateral triangle, inradius is a rational number then which of the following is not true?

(A) circum-radius is always rational (B) area is always irrational

(C) ex-radii are always rational (D) perimeter is always rational

Key. D

Sol. Clearly $r = \frac{R}{2} \Rightarrow R \in Q$, now $r_1 = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = 4R\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)^2 \in Q$.

Similarly $r_2, r_3 \in Q$. Now $\Delta = \frac{abc}{4R} = 2R^2 \sin A \sin B \sin C = 2R^2 \left(\frac{\sqrt{3}}{2}\right)^3 \notin Q$

Also s = a + b + c = 2R(sinA + sinB + sinC) = $= 3\sqrt{3}R \notin Q$

86. If in equilateral triangle, in-radius is a rational number then which of the following is not true?

(A) circum-radius is always rational (B) area is always irrational

(C) ex-radii are always rational (D) perimeter is always rational

Key. D

Sol. Clearly $r = \frac{R}{2} \Rightarrow R \in Q$, now $r_1 = 4R \sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = 4R\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)^2 \in Q$.

Similarly $r_2, r_3 \in Q$. Now $\Delta = \frac{abc}{4R} = 2R^2 \sin A \sin B \sin C = 2R^2 \left(\frac{\sqrt{3}}{2}\right)^3 \notin Q$

Also s = a + b + c = 2R(sinA + sinB + sinC) = $= 3\sqrt{3}R \notin Q$

87. In an isosceles triangle ABC, AB=AC. If vertical angle A is 20^0, then a^3+b^3 is equal to

a) $3a^2b$ b) $3b^2c$ c) $3c^2a$ d) abc

Key. C

Sol. Q $\angle A = 20^o$

$\therefore \angle B = \angle C = 80^o$

Then, $b = c$

$\therefore \frac{a}{\sin 20^o} = \frac{b}{\sin 80^o} = \frac{c}{\sin 80^o}$

Or $\frac{a}{\sin 20^o} = \frac{b}{\cos 10^o}$

$\Rightarrow a = 2b \sin 10^o$

$\therefore a^3 + b^3 = 8b^3 \sin^3 10^o + b^3 = b^3\left\{2\left(4\sin^3 10^o\right)+1\right\} = b^3\left\{6\sin 10^o\right\} = 3ac^2$

88. Which of the following pieces of data does not uniquely determine acute angled ΔABC (R = circum radius)

a) a, sin A ,sinB b) a,b,c c) a, sinB, R d) a, sinA , R

Key. D

Sol. Q In a ΔABC, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin\{\pi - (A+B)\}} = 2R$

$$\Rightarrow \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin(A+B)} = 2R$$

Alternate. (a) : If we know a, sinA, sinB then we can find b, c, A, B and C.

Alternate. (b) : We can find A, B, C by using cosine rule.

Alternate. (c) : Q a, sinB, R are given then we can find sinA, b and hence.

$$\sin C = \sin\{\pi - (A+B)\} = \sin C$$

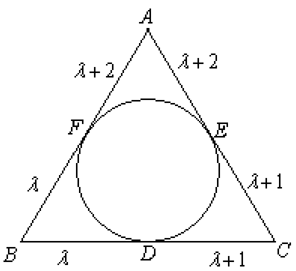
Alternate. (d) : a, sinA, R are given then we know only the ratio $\frac{b}{\sin B}$ or $\frac{c}{\sin(A+B)}$; we cannot determine the values of b, c, sinB, sinC separately.

$\therefore$ Triangle ABC cannot be determined in this case.

89. The incircle of a ΔABC touches the sides BC, CA, AB at the points D, E, F respectively. If the lengths of BD, CE, AF respectively are consecutive positive integers and the inradius of the triangle is 4 units, then the perimeter of the triangle is

A) 42 B) 35 C) 84 D) 57

Key. A

Sol. 

Now applying $\Delta = rs$, we get λ

90. Tangents at P,Q,R on a circle of radius r form a triangle whose sides are 3r, 4r, 5r then $PR^2 + RQ^2 + QP^2 =$

A) $\frac{84}{5}r^2$ B) $\frac{184}{5}r^2$ C) $\frac{176}{5}r^2$ D) None of these

Key. C

Sol. In ΔAIQ Q $AI = \frac{r}{\sin A/2}$

AQ = r cot A/2 In ΔARQ

$$RQ = \sqrt{(AR)^2 + (AQ)^2} - 2(AR)(AQ)A$$

= 2(AR) sin A/2

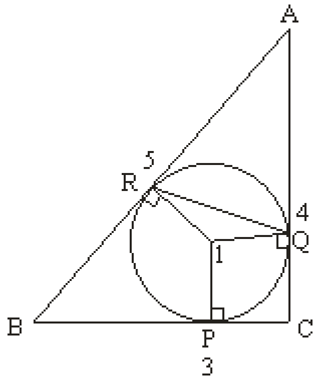
RQ = 2r cos A/2

RP = 4rcos $\left(\frac{B}{2}\right)$, $PQ = 4r\cos\left(\frac{C}{2}\right)$

$$PR^2 + RQ^2 + QP^2 = 16r^2\left[\cos^2\left(\frac{A}{2}\right) + \cos^2\left(\frac{B}{2}\right) + \cos^2\left(\frac{C}{2}\right)\right]$$

$$= 16r^2\left[\frac{1+\cos A}{2}+\frac{1+\cos B}{2}+\frac{1}{2}\right]$$

$$= 8r^2\left[3+\frac{3}{5}+\frac{4}{5}\right] = 8r^2\left[\frac{15+7}{5}\right] = \frac{176r^2}{5}$$



91. In a triangle ABC, if a : b : c = 7 : 8 : 9 then cos A : cos B =

A) $\frac{11}{63}$ B) $\frac{22}{63}$ C) $\frac{2}{9}$ D) none of these

Key. D

Sol. $\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{64+81-49}{2.\,8.\,9} = \frac{145-49}{144} = \frac{96}{144}$

$\cos B = \frac{a^2+c^2-b^2}{2bc} = \frac{49+81-64}{2.\,7.\,9} = \frac{66}{126} = \frac{11}{21}$

92. In a triangle ABC, if cos A + cos B + cos C = $\frac{7}{4}$ then $\frac{R}{r}$ is equal to

A) $\frac{3}{4}$ B) $\frac{4}{3}$ C) $\frac{2}{3}$ D) $\frac{3}{2}$

Key. A

Sol. $\cos A+\cos B+\cos C = \frac{7}{4}$

$1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{7}{4}$

$4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{3}{4}$ (Q R = 4r sin A/2 sin B/2 sin C/2)

$\frac{R}{r} = \frac{3}{4}$

93. In $\Delta ABC \cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}$ is equal to

A) $\frac{\Delta}{r^2}$ B) $\frac{(a+b+c)^2}{abc}$, 2R C) $\frac{\Delta}{r}$ D) $\frac{\Delta}{Rr}$

Key. A

Sol. $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2} = \frac{s(s-a)}{\Delta} = \frac{s(s-b)}{\Delta}+\frac{s(s-c)}{\Delta}$

$= \frac{s}{\Delta}\left[3s-(a+b+c)\right]$

$=\frac{s[3s-2s]}{\Delta}=\frac{s^2}{\Delta}$

$=\left(\frac{a+b+c}{2}\right)^2\times\frac{4R}{abc}=\frac{(a+b+c)^2 R}{abc}$ $\left[\because\ \Delta=\frac{abc}{4R}\right]$

also $\frac{s^2}{\Delta}=\frac{\Delta^2}{r^2\Delta}$ $=\frac{\Delta}{r^2}$

94. In acute angled triangle ABC, $r+r_1=r_2+r_3$ and $\angle B>\frac{\pi}{3}$ then

A) b + 2c < 2a < 2b + 2c
B) b + 4c < 4a < 2b + 4c
C) b + 4c < 4a < 4b + 4c
D) b + 3c < 3a < 3b + 3c

Key. D

Sol. $r-r_2=r_3-r_1$

$\frac{\Delta}{s}-\frac{\Delta}{s-b}=\frac{\Delta}{s-c}-\frac{\Delta}{s-a}$

$\frac{-b}{s(s-b)}=\frac{-a+c}{(s-a)(s-c)}$

$\frac{(s-a)(s-c)}{s(s-b)}=\frac{a-c}{b}$

$\tan^2(B/2)=\frac{a-c}{b}$

But $\frac{B}{2}\in\left(\frac{\pi}{6},\frac{\pi}{4}\right)$ $\Rightarrow$ $\tan^2\frac{B}{2}\in\left(\frac{1}{3},1\right)$

$\Rightarrow$ $\frac{1}{3}<\frac{a-c}{b}<1$

b < 3a – 3c < 3b

b + 3c < 3a < 3b + 3c

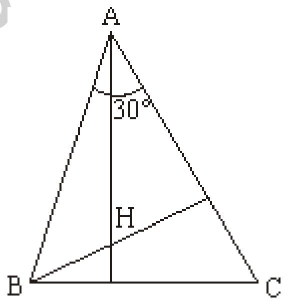
95. In a triangle ABC, $\angle A=30°$, $BC=2+\sqrt{5}$, then the distance of the vertex A from the orthocenter of the triangle is

A) 1 B) $\left(2+\sqrt{5}\right)\sqrt{3}$ C) $\frac{\sqrt{3}+1}{2\sqrt{2}}$ D) $\frac{1}{2}$

Key. B

Sol. $R=\frac{a}{2\sin A}=\frac{2+\sqrt{5}}{2\sin 30°}=\frac{2+\sqrt{5}}{2\times\frac{1}{2}}=\left(2+\sqrt{5}\right)$

Now, AH = 2R cos A = 2 $\left(2+\sqrt{5}\right)$ cos 30° = $\left(2+\sqrt{5}\right)\sqrt{3}$



96. If $c^2=a^2+b^2$, $2s=a+b+c$, then 4s(s – a) (s – b)(s – c) =

A) s^4 B) b^2c^2 C) c^2a^2 D) a^2b^2

Key. D

Sol. $c^2 = a^2 + b^2 \quad \Rightarrow \quad \angle C = \frac{\pi}{2}$

$\therefore \quad \Delta = \frac{1}{2}\,ab\sin C = \frac{1}{2}ab \Rightarrow \sqrt{s(s-a)(s-b)(s-c)} = \frac{1}{2}ab$

$\Rightarrow \quad 4s(s-a)(s-b)(s-c) = a^2b^2$.

97. If $\cot\frac{A}{2} = \frac{b+c}{a}$, then the ΔABC is

A) isosceles B) equilateral C) right angled D) none of these

Key. C

Sol. $\cot\frac{A}{2} = \frac{b+c}{a} \Rightarrow \frac{\cos A/2}{\sin A/2} = \frac{\sin B + \sin C}{\sin A}$

$\Rightarrow \quad \frac{\cos A/2}{\sin A/2} = \frac{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}{2\sin\frac{A}{2}\cos\frac{A}{2}}$

$\Rightarrow \quad \cos\frac{A}{2} = \cos\left(\frac{B-C}{2}\right) \Rightarrow \frac{A}{2} = \frac{B-C}{2}$

$\Rightarrow \quad A = B - C \quad \Rightarrow \quad A + C = B$

But A + B + C = π. Therefore, $B = \frac{\pi}{2}$

98. In a triangle ABC, (a + b + c) (b + c – a) = λ bc if

A) $\lambda < 0$ B) $\lambda > 6$ C) $0 < \lambda < 4$ D) $\lambda > 4$

Key. C

Sol. 2s(2s – 2a) = λbc

i.e., $4\frac{s(s-a)}{bc} = \lambda$ i.e., $\sin^2\frac{A}{2} = \frac{\lambda}{4}$

$\therefore \quad 0 < \frac{\lambda}{4} < 1$ i.e. $0 < \lambda < 4$

Alternative solution

$(b+c)^2 - a^2 = \lambda\, bc$

$b^2 + c^2 - a^2 = (\lambda - 2)bc$

$\frac{b^2+c^2-a^2}{2bc} = \frac{\lambda-2}{2}$ i.e. $\cos A = \frac{\lambda-2}{2}$

$\therefore \quad -1 < \frac{\lambda-2}{2} < 1$ i.e. $-2 < \lambda - 2 < 2$

i.e. $0 < \lambda < 4$

99. If ‘a’, ‘b’, ‘c’ are the sides of a triangle than the minimum value of $\frac{2a}{b+c-a} + \frac{2b}{c+a-b} + \frac{2c}{a+b-c}$ is

A) 3 B) 9 C) 6 D) 1

Key. C

Sol. Let a + b + c = 2S

Than we have to find minimum value of

$$\frac{a}{S-a}+\frac{b}{S-b}+\frac{c}{S-c}=-3+\frac{S}{S-a}+\frac{S}{S-b}+\frac{S}{S-c}$$

Also, $\dfrac{\dfrac{S}{S-a}+\dfrac{S}{S-b}+\dfrac{S}{S-c}}{3}\ge\dfrac{3}{\dfrac{S-a}{S}+\dfrac{S-b}{S}+\dfrac{S-c}{S}}\because\dfrac{S-a}{S}+\dfrac{S-b}{S}+\dfrac{S-c}{S}=1$

$\Rightarrow\quad \dfrac{S}{S-a}+\dfrac{S}{S-b}+\dfrac{S}{S-c}\ge 9$.

Thus minimum value of the expression is 6.

100. In triangle ABC, medians AD and BE are mutually perpendicular, then such a triangle would exist if

A) $\frac{1}{4}<\frac{a}{b}<\frac{1}{2}$ B) $\frac{1}{4}<\frac{b}{a}<\frac{3}{4}$ C) $\frac{1}{4}<\frac{a}{b}<\frac{3}{4}$ D) $\frac{1}{2}<\frac{b}{a}<2$

Key. D

Sol. AD and BE are perpendicular thus $b^2+a^2=5c^2$

Since $|a-b|<c \quad\Rightarrow\quad a^2+b^2>5(a-b)^2$

$\Rightarrow\quad 4a^2-10ab+4b^2<0 \quad\Rightarrow\quad \frac{1}{2}<\frac{a}{b}<2$

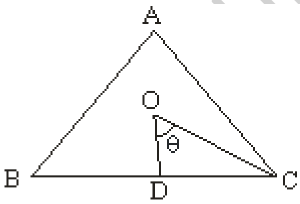
101. Consider a given acute angled triangle ABC having O as its circumcentre. Let D be a variable interior point of the side BC. The limiting value of the circumradius of the ΔOCD as point D approaches towards vertex C is equal to

A) $\frac{R}{2\cos A}$ B) $\frac{R}{\cos A}$ C) $\frac{R}{\sin A}$ D) $\frac{R}{2\sin A}$

Key. B

Sol. In the adjacent figure we have $\angle OCB=\frac{\pi}{2}-A$

Let $\angle ODC=\pi-\left(\frac{\pi}{2}-A+\theta\right)=\frac{\pi}{2}+(A-\theta)$



If R_1 be the circumradius of ΔOCD then

$\dfrac{OC}{\sin\left(\frac{\pi}{2}+(A-\theta)\right)}=2R_1 \quad\Rightarrow\quad 2R_1=\dfrac{R}{\cos(A-\theta)}$

As $D\to C\ \ \theta\to 0 \quad\Rightarrow\quad 2R_1\to\dfrac{R}{\cos A}$

102. If circumradius and inradius of a triangle be 8 and 3, then value of $\frac{a}{\tan A}+\frac{b}{\tan B}+\frac{c}{\tan C}$ equals

A) 11 B) 33 C) 44 D) 55

Key. D

Sol. $\dfrac{a}{\tan A}+\dfrac{b}{\tan B}+\dfrac{c}{\tan C}=a\cot A+b\cot B+c\cot C$

= 2 (R + r) = 2 (8 + 3) = 22 Ans.

103. ABCD is a quadrilateral circumscribed about a circle of unit radius then

A) $AB\sin\frac{C}{2}.\sin\frac{A}{2} = CD\sin\frac{B}{2}\sin\frac{D}{2}$ B) $AB\sin\frac{A}{2}.\sin\frac{B}{2} = CD\sin\frac{C}{2}\sin\frac{D}{2}$

C) $AB\sin\frac{A}{2}.\sin\frac{A}{2} = CD\sin\frac{C}{2}\sin\frac{B}{2}$ D) $AB\sin\frac{A}{2}.\cos\frac{B}{2} = CD\sin\frac{C}{2}\cos\frac{D}{2}$

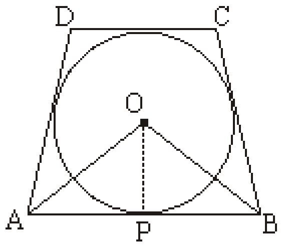
Key. B

Sol. Let 'O' be the centre of circle and 'P' be its point of contact with side AB. We have

$AP = OP.\cot\frac{A}{2} = \cot\frac{A}{2}$ and

$PB = OP.\cot\frac{B}{2} = \cot\frac{B}{2}$

$\Rightarrow \quad AP + PB = \cot\frac{A}{2} + \cot\frac{B}{2}$



$= \dfrac{\sin\left(\frac{A+B}{2}\right)}{\sin\frac{A}{2}.\sin\frac{B}{2}} = AB$

Since $A+B+C = 2\pi \quad \Rightarrow \frac{A+B}{2} = \pi - \frac{C+D}{2}$

$\Rightarrow \quad \sin\left(\frac{A+B}{2}\right) = \sin\left(\frac{C+D}{2}\right)$

$\Rightarrow \quad AB.\sin\frac{A}{2}.\sin\frac{B}{2} = \sin\frac{C}{2}.\sin\frac{D}{2}.CD$

104. In triangle ABC, a : b : c = (1 + x): 1: (1 – x) where $x \in (0,1)$. If $\angle A = \frac{\pi}{2} + \angle C$, then x is equal to

A) $\frac{1}{\sqrt{6}}$ B) $\frac{1}{2\sqrt{6}}$ C) $\frac{1}{\sqrt{7}}$ D) $\frac{1}{2\sqrt{7}}$

Key. C

Sol. $a = (1+x)h,\ b = h,\ c = (1-x)h, \frac{A}{2} - \frac{C}{2} = \frac{\pi}{4}$

$\Rightarrow \quad \cos\frac{A}{2}.\cos\frac{C}{2} + \sin\frac{A}{2}\sin\frac{C}{2} = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \sqrt{\frac{S^2(S-a)(S-c)}{bc.ab}} + \sqrt{\frac{(S-b)(S-c)(S-a)(S-b)}{bc.ab}} = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \frac{S}{b}\sqrt{\frac{(S-a)(S-c)}{ac}} + \frac{(S-b)}{b}\sqrt{\frac{(S-a)(S-c)}{ac}} = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \left(\frac{2S-b}{b}\right)\sqrt{\frac{(S-a)(S-b)}{ac}}=\frac{1}{\sqrt{2}} \Rightarrow \quad \frac{a+c}{b}\sqrt{\frac{(S-a)(S-b)}{ac}}=\frac{1}{\sqrt{2}}$

$\Rightarrow \quad 2\left(\frac{a+c}{b}\right)^2=\frac{ac}{(s-a)(s-c)}$

Now a + c = 2h, b = h

$\Rightarrow \quad \frac{a+c}{b}=2,\ s=\frac{a+b+c}{2}=\frac{3h}{2}$

$\Rightarrow \quad S-a=\frac{(1-2x)h}{2},(S-c)=\frac{(1-2x)h}{2}$

$\Rightarrow \quad 8=\frac{(1+x^2)4}{(1-4x^2)} \quad \Rightarrow \quad x=\frac{1}{\sqrt{7}}$

Properties of Triangles

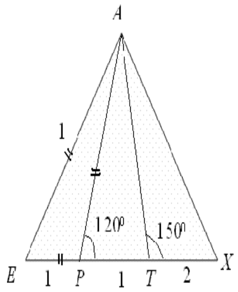
Multiple Correct Answer Type

1. In a ΔAEX, T is the mid point of XE, and P is the mid point of ET. If the ΔAPE is equilateral of side length equal to unity then which of the following alternative(s) is/are correct?

(A) AX = $\sqrt{13}$ (B) $\angle$EAT = 90°

(C) cos $\angle XAE$ = $\frac{-1}{\sqrt{13}}$ (D) AT = $\frac{1}{\sqrt{3}}$

Key. A,B,C



Sol.

A) $Ax^2 = AE^2 + EX^2 - 2AE.EX \cos\angle AEX$

$= 1 + 6 - 2.1.4.\frac{1}{2} = 13$

$\therefore AX = \sqrt{13}$

B) since $\angle APT$ isosceles $\angle ATP = \angle PAT = 30^0$ then $\angle EAT = 90^0$

And also $\frac{AT}{\sin 120^0} = \frac{AP}{\sin 30^0} \Rightarrow AT = \frac{\sqrt{3}}{2}.\frac{1}{1/2} = \sqrt{3}$

C) since $EX^2 = AE^2 + AX^2 - 2AE.AX \cos\angle XAE$

$16 = 1 + 13 - 2.1\sqrt{13}.\cos\angle XAE$

$\cos\angle XAE = \frac{-1}{\sqrt{13}}$

3. In $\Delta ABC, \angle C = 2\angle A \, and \, AC = 2BC,$ then which of the following is/are True

A) Angles A,B,C are in arithmetic progression.

B) Angles A,C,B are in arithmetic progression

C) ΔABC is a right angled isosceles triangle

D) $BC^2 + CA^2 + AB^2 = 8R^2$ where R is the circum-radius of of ΔABC

Key. B,D

Sol. $\angle C = 2\angle A$ and $b = 2a \Rightarrow \sin B = 2\sin A$

$\Rightarrow \sin(A + C) = 2\sin A$

$\sin 3A = 2\sin A$

$3 - 4\sin^2 A = 2$

$\Rightarrow \sin A = \frac{1}{2} \Rightarrow A = 30^0$

$C = 60^0, \; B = 90^0$

$$a^2+b^2+c^2=8R^2 \Leftrightarrow \sin^2 A+\sin^2 B+\sin^2 C=2$$
$$\Leftrightarrow 3-(\cos 2A+\cos 2B+\cos 2C)=4$$
$$\Leftrightarrow \cos A\cos B\cos C=0$$

3. In a Δ ABC, BC = 2, CA = $\sqrt{3}+1$ and $\angle$C = 60°. Feet of the perpendicular from A, B and C on the opposite sides BC, CA and AB are D, E and F respectively and are concurrent at P. then which of the following is / are true

(A) Radius of the circle circumscribing the Δ DEF is $\frac{1}{\sqrt{2}}$

(B) Area of the Δ DEF is $\frac{\sqrt{3}}{4}$

(C) Radius of the circle inscribed in the Δ DEF is $\frac{\sqrt{6}-\sqrt{2}}{4}$

D) $\sin D+\sin E+\sin F=\frac{3+\sqrt{3}}{2}$

Key. A,B,C,D

Sol. $a=2$; $b=\sqrt{3}+1$ $\angle C=\frac{\pi}{3}$

DEF is pedal triangle of ABC

For triangle DEF, circumradius = $\frac{R}{2}$

In radius = 2R cos A cos B cos C and area = $\frac{1}{2}R^2\sin 2A\sin 2B\sin 2C$

$\angle A=45^0$ $\angle B=75^0$ $\angle C=60^0$ $R=\sqrt{2}$

$\angle D=90^0$ $\angle E=30^0$ $\angle F=60^0$

4. If in a triangle ABC, $BC=5, CA=4, AB=3$ and D,E are points on BC such that

$BD=DE=EC, \underline{|CAE}=\theta$

A) $AE^2=73/3$ B) $AE^2=73/9$

C) $\tan\theta=3/8$ D) $\cos\theta=\frac{3}{\sqrt{73}}$

Key. B,C

Sol. take A=(0,0)
B=(3,0)
C=(0,4)

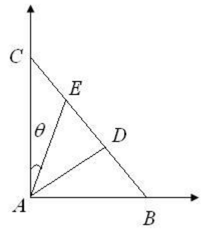
D,E are point of trisection of BC

$\therefore D=\left(2,\frac{4}{3}\right)$ $\therefore E=(1,8/3)$

$\therefore AE=\sqrt{1+\frac{64}{9}}=\sqrt{\frac{73}{9}}$

Slop of $\overline{AB}=\tan(90-\theta)=\frac{8}{3}\Rightarrow \tan\theta=\frac{3}{8}$

And $\cos\theta = \frac{8}{\sqrt{73}}$



5. In triangle ABC if $2a^2b^2 + 2b^2c^2 = a^4 + b^4 + c^4$ then angle B is equal to

A) 45^0 B) 135^0 C) 120^0 D) 60^0

Key. A,B

Sol. $a^4 + b^4 + c^4 - a^2b^2 - 2b^2c^2 = 0$

$$\Rightarrow \left(a^2 - b^2 + c^2\right)^2 = 2c^2a^2 \Rightarrow \left(\frac{a^2 - b^2 + c^2}{2ac}\right)^2 = \frac{1}{2}$$

$$\Rightarrow \cos^2 B = \frac{1}{2} \Rightarrow \cos B = \frac{1}{\sqrt{2}}$$

$$\therefore\ B = 45^0 \text{ or } 135^0$$

6. Let ABC be an isosceles triangle with base BC. If 'r' is the radius of the circle inscribed in Δ ABC and r_1 be the radius of the circle described opposite to the angle A, then r_1r =

A) $R^2\sin^2 A$ B) $R^2\sin^2 2B$ C) $\frac{a^2}{2}$ D) $\frac{a^2}{4}$

Key. A,B,D

Sol. $rr_1 = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

$= 4R^2\sin^2\frac{A}{2}\sin B\sin C$ but $2B = 180^0 - A \Rightarrow B = C = 90^0 - \frac{A}{2}$

$$= 4R^2\sin^2\frac{A}{2}\cos^2\frac{A}{2} = R^2\sin^2 A$$

$$= R^2\sin^2 2B = \frac{a^2}{4}$$

7. If A is the area and 2S the sum of sides of a triangle, then

A) $A < \frac{S^2}{4}$ B) $A \le \frac{S^2}{3\sqrt{3}}$

C) $A > \frac{S^2}{\sqrt{3}}$ D) $A \le \frac{S^2}{\sqrt{3}}$

Key. A,B

Sol. $\dfrac{S+S-a+S-b+S-c}{4} \geq \sqrt[4]{S(S-a)(S-b)(S-c)}$

$\dfrac{S}{2} \geq \Delta^{1/2} \Rightarrow \dfrac{S^2}{4} \geq \Delta \Rightarrow \Delta \leq \dfrac{S^2}{4}$

$\dfrac{S-a+S-b+S-c}{3} \geq \sqrt[3]{(S-a)(S-b)(S-c)}$

$\dfrac{S^3}{27} \geq \dfrac{(S-a)(S-b)(S-c)S}{S}$

$\Rightarrow \Delta^2 \leq \dfrac{S^4}{27} \Rightarrow \Delta \leq \dfrac{S^2}{3\sqrt{3}}$

8. In ΔABC which of the following statements are true?

A) Maximum value of $\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$ is $\dfrac{1}{4}$

B) $R \geq 2r$ where R is circum radius and r is in radius

C) $R^2 \geq \dfrac{abc}{a+b+c}$

D) ΔABC is right angled if r + 2R = s where 's' is semi perimeter

Key. B,C

Sol. $\dfrac{r_1+r_2+r_3}{3} \geq \dfrac{3}{\dfrac{1}{r_1}+\dfrac{1}{r_2}+\dfrac{1}{r_3}}$

$4R+r \geq 9r \Rightarrow 4R \geq 8r \Rightarrow R \geq 2r$

$R^2 \geq 4r^2 \geq \dfrac{4\Delta^2}{S^2} \geq \dfrac{4a^2b^2c^2}{16R^2S^2} \Rightarrow R^4 \geq \dfrac{a^2b^2c^2}{4S^2}$

$\Rightarrow R^2 \geq \dfrac{abc}{a+b+c}$

9. There exists a triangle ABC satisfying the following conditions.

A) b sinA = a, $A < \dfrac{\pi}{2}$

B) b sinA > a, $A > \dfrac{\pi}{2}$

C) b sinA > a, $A < \dfrac{\pi}{2}$

D) b sinA < a, $A < \dfrac{\pi}{2}, b > a$

Key. A,D

Sol. $b \times \dfrac{a}{2R} = a \Rightarrow a = 2R \Rightarrow B = 90^0,\ A < \dfrac{\pi}{2}$

$\therefore$ A triangle is possible.

$b\sin A < a \Rightarrow b < 2R,\ A < \dfrac{\pi}{2}$ but b > a a triangle is possible.

$\Rightarrow b < \dfrac{\pi}{2}$

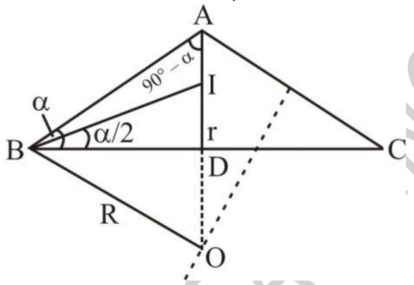
10. Given an isosceles triangle with equal side of length b, base angle $\alpha < \pi/4$. R, r the radii and O, I the centres of the circumcircle and incircle, respectively. Then

(A) $R = \frac{1}{2} b \operatorname{cosec} \alpha$ (B) $R = \frac{2}{3} b \cos \alpha$

(C) $r = \frac{b \sin 2\alpha}{2(1+\cos\alpha)}$ (D) $OI = \left|\frac{b\cos(3\alpha/2)}{2\sin\alpha\cos(\alpha/2)}\right|$

Key. A,C,D

Sol. Let ABC be the isosceles triangle with AB = AC = b and $\angle B = \angle C = \alpha$. Let AD be the perpendicular bisector of the side BC. Since ΔABC is isosceles, AD is also the bisector of angle A, So that O and I both lie on AD. We have OB = R and ID = r. Also, since O is the circumcentre, we get OA = OB = R. Therefore, from isosceles triangle OAB



$$\frac{OB}{\sin(90^o - \alpha)} = \frac{AB}{\sin 2\alpha}$$

$$\Rightarrow R = \frac{b\cos\alpha}{2\sin\alpha\cos\alpha}$$

$$= \frac{1}{2} b \operatorname{cosec}\alpha$$

(a) is correct. Again

$$\Delta = BD.AD = b\cos\alpha . b\sin\alpha$$

$$= \frac{1}{2} b^2 \sin 2\alpha$$

$$r = \frac{\Delta}{s} = \frac{\frac{1}{2}b^2\sin 2\alpha}{\frac{1}{2}(b+b+2b\cos\alpha)} = \frac{b\sin 2\alpha}{2(1+\cos\alpha)}$$

(d) $OI^2 = R^2 - 2Rr$

11. The in-circle of ΔABC touches side BC at D. Then difference between BD and CD (R is circum-radius of ΔABC)

A) $\left|4R\sin\frac{A}{2}\sin\frac{B-C}{2}\right|$ B) $\left|4R\cos\frac{A}{2}\sin\frac{B-C}{2}\right|$ C) $|b-c|$ D) $\left|\frac{b-c}{2}\right|$

Key. A,C

Sol.

$$|BD-CD|=\left|r\left(\cot\frac{B}{2}-\cot\frac{C}{2}\right)\right|=r\left|\frac{\sin\left(\frac{B-C}{2}\right)}{\sin\frac{B}{2}\sin\frac{C}{2}}\right|=$$

$$\left|4R\sin\frac{A}{2}\sin\frac{B-C}{2}\right|=\left|4R\cos\frac{B+C}{2}.\sin\frac{B-C}{2}\right|=\left|2R(\sin B-\sin C)\right|=|b-c|$$

12. If the orthocenter of an isosceles triangle lies on the incircle of the triangle then

A) the base angle of the triangle is $\cos^{-1}\frac{2}{3}$ B) the triangle is acute

C) the base angle of the triangle is $\tan^{-1}\frac{\sqrt{5}}{2}$

D) If S, I are the circumcentre and incentre and R is circumradius then $\frac{SI}{R}=\frac{1}{3}$

KEY : A,B,C,D

HINT. Let ABC be the triangle in which AB = AC. Let I, P respectively be the incentre and the orthocenter of the triangle.

$$AI=r\operatorname{cosec}\frac{A}{2}, AP=2R\cos A$$

$$r\operatorname{cosec}\frac{A}{2}=2R\cos A+r$$

13. If in a triangle ABC, cos A cosB + sin A sin B Sin^3 C = 1, then, with usual notation in ΔABC,

a) the triangle is isosceles b) the triangle is right angled

c) $R:r=\left(\sqrt{2}+1\right):1$ d) $r_1:r_2:r_3=1:1:\left(\sqrt{2}+1\right)$

KEY : A,B,C,D

HINT. The given relation implies cos (A-B) =1 and so, A=B and C = 90°

14. A circle of radius 4 cm is inscribed in $\Delta ABC,$ which touches the side BC at D. if $BD=6cm$: $DC=8cm$ then

(A) The triangle is necessarily acute angled Δ

(B) $\tan\frac{A}{2}=\frac{4}{7}$;

(C) perimeter of the triangle ABC is 42 cm

(D) Area of ΔABC is 84 cm^2

KEY : A,B,C,D

HINT : $\tan\frac{B}{2}=\frac{2}{3}.\tan\frac{C}{2}=\frac{1}{2}\Rightarrow\tan\frac{A}{2}=\frac{4}{7}$

$$\tan\frac{B}{2}.\tan\frac{C}{2}=\frac{s-a}{s}\Rightarrow 2s=3a=42$$

$\therefore$ Perimeter=42

$\therefore \Delta = r.s. = 84cm^2$

$\therefore \tan\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2}$ all are less than 1. All angles are acute.

15. If a right angled ΔABC of maximum area is inscribed within a circle of radius R, then (Δ represents area of triangle ABC and r, r_1, r_2, r_3 represent inradius and exradii, and s is the semi perimeter of ΔABC then

A) $\Delta = R^2$

B) $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3} = \frac{\sqrt{2}+1}{R}$

C) $r = \left(\sqrt{2}-1\right)R$

D) $s = \left(1+\sqrt{2}\right)R$

KEY : A,B,C,D

SOL : For a right angled triangle inscribed in a circle of radius R the length of the hypotenuse is 2R
Then area is maximum when its is isosceles triangle
With each side $= \sqrt{2}R$

$$\therefore S = \frac{1}{2}\left(2\sqrt{2}+2\right)R = \left(\sqrt{2}+1\right)R$$

$$\Delta = \frac{1}{2}\sqrt{2}R.\sqrt{2}R = R^2$$

$$r = \frac{\Delta}{S} = \frac{R^2}{\left(\sqrt{2}+1\right)R} \Rightarrow r = \left(\sqrt{2}-1\right)R$$

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3} = \frac{1}{r} = \frac{1}{\left(\sqrt{2}-1\right)R} = \frac{\sqrt{2}+1}{R}$$

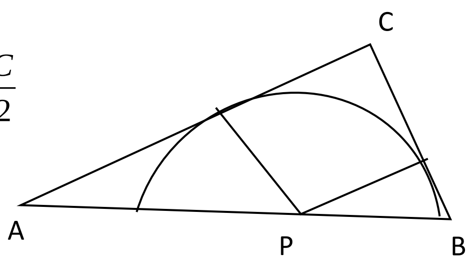
16. In a triangle ABC with sides a, b and c, a semicircle touching the sides AC and CB is inscribed whose diameter lies on AB, then the radius of the semicircle is

a) $\frac{a}{2}$

b) $\frac{2abc}{s(a+b)}\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}$

c) $\frac{2\Delta}{a+b}$

d) $\frac{2abc}{s(a+b)}.\cos\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}$



Key: c,d

Hint:

$$\Delta_{ABC} = \Delta_{BPC} + \Delta_{APC}$$

$$\Delta = \frac{1}{2}ar + \frac{1}{2}br$$

$$r = \frac{2\Delta}{a+b}$$

$$\cos\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2} = \sqrt{\frac{s(s-a)}{bc}}.\sqrt{\frac{s(s-b)}{ac}}.\sqrt{\frac{s(s-c)}{ab}} = \frac{s\Delta}{abc}$$

17. In a ΔABC sides b, c $\angle C$ are given, which of the following cannot determine a unique ΔABC

(A) $c > b\sin C, \angle C < \pi/2, \ c > b$ (B) $c > b\sin C, \angle C < \pi/2, \ c < b$

(C) $c > b\sin C, \angle C > \pi/2, c < b$ (D) $c > b\sin C, \angle C < \pi/2, \ b = c$

KEY : B, C

HINT : $\cos C = \dfrac{a^2+b^2-c^2}{2ab} \Rightarrow a = b\cos C \pm \sqrt{c - b\sin C}$

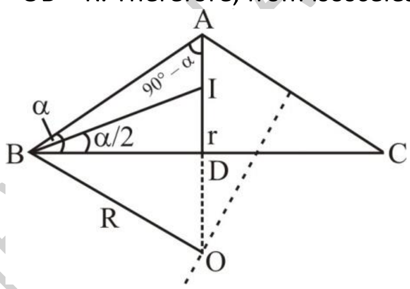
18. Given an isosceles triangle with equal side of length b, base angle $\alpha < \pi/4$. R, r the radii and O, I the centres of the circumcircle and incircle, respectively. Then

(A) $R = \frac{1}{2} b \operatorname{cosec} \alpha$ (B) $R = \frac{2}{3} b\cos\alpha$

(C) $r = \dfrac{b\sin 2\alpha}{2(1+\cos\alpha)}$ (D) $OI = \left|\dfrac{b\cos(3\alpha/2)}{2\sin\alpha\cos(\alpha/2)}\right|$

Key. A,C,D

Sol. Let ABC be the isosceles triangle with AB = AC = b and $\angle B = \angle C = \alpha$. Let AD be the perpendicular bisector of the side BC. Since ΔABC is isosceles, AD is also the bisector of angle A, So that O and I both lie on AD. We have OB = R and ID = r. Also, since O is the circumcentre, we get OA = OB = R. Therefore, from isosceles triangle OAB



$$\frac{OB}{\sin(90^o - \alpha)} = \frac{AB}{\sin 2\alpha}$$

$$\Rightarrow R = \frac{b\cos\alpha}{2\sin\alpha\cos\alpha}$$

$$= \frac{1}{2} b \operatorname{cosec}\alpha$$

(a) is correct. Again

$$\Delta = BD.AD = b\cos\alpha . b\sin\alpha$$

$$= \frac{1}{2} b^2 \sin 2\alpha$$

$$r = \frac{\Delta}{s} = \frac{\frac{1}{2} b^2 \sin 2\alpha}{\frac{1}{2}(b + b + 2b\cos\alpha)} = \frac{b\sin 2\alpha}{2(1+\cos\alpha)}$$

(c) correct. Further

OI = |OD + DI| = |OD + r|

Because $\alpha < \frac{\pi}{4}, A > \frac{\pi}{2}$ and O lies on AD produced. Now, from right-angled triangle ODB, we get

$OD^2 = OB^2 - BD^2 = R^2 - (b\cos\alpha)^2$

$= \frac{1}{4}\frac{b^2}{\sin^2\alpha} - b^2\cos^2\alpha$

$= \frac{b^2(1-4\sin^2\alpha\cos^2\alpha)}{4\sin^2\alpha}$ [from (a)]

$= \frac{b^2(\cos^2\alpha - \sin^2\alpha)^2}{4\sin^2\alpha} = \frac{b^2\cos^2 2\alpha}{(2\sin\alpha)^2}$

$\Rightarrow OD = \frac{b\cos 2\alpha}{2\sin\alpha}$

$\therefore OI = \left|\frac{b\sin 2\alpha}{2(1+\cos\alpha)} + \frac{b\cos 2\alpha}{2\sin\alpha}\right|$

$= \left|\frac{b\sin 2\alpha}{4\cos^2(\alpha/2)} + \frac{b\cos 2\alpha}{4\sin(\alpha/2)\cos(\alpha/2)}\right|$

$= \left|\frac{b}{4\cos(\alpha/2)}.\left(\frac{\sin 2\alpha\sin(\alpha/2) + \cos 2\alpha\cos(\alpha/2)}{\sin(\alpha/2)\cos(\alpha/2)}\right)\right|$

$= \frac{b\cos(3\alpha/2)}{2\sin\alpha\cos(\alpha/2)}$

(d) is also correct

19. In a triangle, the lengths of the two larger sides are 10 and 9 respectively . If the angles are in A.P., then the length of the third side can be :

(A) $5-\sqrt{6}$ (B) $3\sqrt{3}$

(C) 5 (D) $6 \pm \sqrt{5}$

Key. A,D

Sol. *a > b > c ; B = 60º ; use cos B as cosine rule to get two values of c*

20. In a ΔABC, a , c , A are given and b_1, b_2 are two values of the third side b is such that $b_2 = 2b_1$ then

(A) $\sin A = \frac{\sqrt{9a^2 - c^2}}{8a^2}$ (B) $a > \frac{c}{3}$

(C) $\frac{\sqrt{9a^2 - c^2}}{8c^2}$ (D) $\sin A = \frac{\sqrt{9a^2 - c^2}}{8c^2}$

Key. B,D

Sol. $b^2 - 2bc\cos A + c^2 - a^2 = 0 \Rightarrow b_1 + b_2 = 2c\cos A, b_1 b_2 = c^2 - a^2$

$b_2 = 2b_1;\ b_1 = \frac{2c}{3}\cos A,\ b_2 = \frac{\sqrt{c^2 - a^2}}{2} \Rightarrow \frac{2c}{3}\cos A = \frac{c^2 - a^2}{2}$

$$\Rightarrow \sin A = \frac{\sqrt{9a^2 - c^2}}{8c^2}$$

$a = 20 - 6\pi;$ $\quad b = 10\pi - 30$ $\quad c = 10 - 3\pi$

$\therefore a + b + c = \pi$

21. There exists a triangle ABC satisfying the conditions

(A) $b\sin A = a, A < \frac{p}{2}$ (B) $b\sin A > a, A > \frac{p}{2}$

(C) $b\sin A > a, A < \frac{p}{2}$ (D) $b\sin A < a, A < \frac{p}{2}$

Key. A,D

Sol. The sine formula is $\frac{a}{\sin A} = \frac{b}{\sin B}$ Þ

a sin B b sin A cohice (A)

b sin A = a Þ a sin B = a Þ B = p/2

\ $ÐA < \frac{p}{2}$

\ the triangle is possible

Choice (b) and (c)

b sin A > a a sin B > a sin B > 1

\ DABC is not possible.

choice (D)

b sin A < a Þ a sin B < a Þ sin B < 1

Þ $ÐB$ does exist

Now, if b > a Þ B > A \ A < p/2

\ the triangle is possible

22. The angles of a Δ ABC satisfy. $\frac{\cos A + 2\cos C}{\cos A + 2\cos b} = \frac{\sin B}{\sin C}$ Δ ABC

a) may be isosceles b) must be equilateral

c) may be a right triangle d) must be acute angled

Key. (a), (c).

Sol. Cross-multiplying and de-factorising , we get

sin C cos A + sin2C = sin B cos A + sin 2B

⇔ (sin C – sinB) cos A = sin 2B – sin 2C = 2 cos (B + C) sin (B – C) = –2 cos A sin (B – C)

⇔ cos A = 0 or sin C – sin B = – 2 sin (B – C).

⇕

B = C

23. In a triangle ABC, length of the bisector of an angle is

(A) $\dfrac{2bc\sin\left(\frac{A}{2}\right)}{b+c}$ (B) $\dfrac{2bc\cos\left(\frac{A}{2}\right)}{b+c}$

(C) $\dfrac{abc}{2R(b+c)}\ \text{cosec}\,\dfrac{A}{2}$ (D) $\dfrac{2\Delta}{(b+c)}\ \text{cosec}\,\dfrac{A}{2}$

Key. B,C,D

24. If the sides a , b , c of a ΔABC satisfy the relation , $a^4 + b^4 + c^4 = 2c^2(a^2 + b^2)$ then the possible values of the angle C can be

(A) $\dfrac{\pi}{4}$ (B) $\dfrac{\pi}{3}$ (C) $\dfrac{5\pi}{6}$ (D) $\dfrac{3\pi}{4}$

Key. A,D

25. The triangle ABC has $\angle A = 60°, \angle B = 45°$. The bisector of $\angle A$ meets the side BC at T where AT = 24. Then which of the following is correct?

(A) length of BC = $18\sqrt{2}$ (B) length of AC < AT

(C) radius of the circumcircle is $12\sqrt{2}$ (D) area of the triangle is $72\left(3+\sqrt{3}\right)$ sq. units.

Key. C,D

26. If 'l' is the median from the vertex A to the side BC of a ΔABC, then

(a) $4l^2 = 2b^2 + 2c^2 - a^2$ (b) $4l^2 = b^2 + c^2 + 2bc\cos A$

(c) $4l^2 = a^2 + 4bc\cos A$ (d) $4l^2 = (2s - a)^2 - 4bc\sin^2\left(\dfrac{A}{2}\right)$

Key. A,B,C,D

27. If AD and AD′ are the internal and external bisectors of $\angle A$ of ΔABC (b < c). The points B, D, C, D′ are collinear, then

(A) BD = $\dfrac{ac}{c+b}$ (B) BD′ = $\dfrac{ac}{c-b}$.(C) DD′ = $\dfrac{2abc}{c^2-b^2}$ (D) a AD > b BD

Key. A,B,C

28. Triangle ABC is right – angled at point A, radii of described circles are r_1, r_2, r_3, given that r_2 = 2, r_3 = 3 then which of the following hold

(a) a = 5 (b) r_1 = 6 (c) r = 1 (d) None of these

Key. A,B,C

29. In ΔABC if $\angle C$ = 120º and cos A + cos B = 1 then

(a) cos (A – B) = $\dfrac{1}{\sqrt{3}}$ (b) |cos A – cos B| = $\dfrac{\sqrt{6}}{3}$

(c) cos (A – B) = - $\dfrac{1}{3}$ (d) $\cos^2 A + \cos^2 B - \cos^2 C = \dfrac{1}{4}$.

Key. B,C

30. In triangle ABC if $2a^2b^2 + 2b^2c^2 = a^4 + b^4 + c^4$ then angle B is equal to

A) 45^0 B) 135^0 C) 120^0 D) 60^0

Key. A,B

Sol. $a^4 + b^4 + c^4 - a^2b^2 - 2b^2c^2 = 0$

$\Rightarrow \left(a^2 - b^2 + c^2\right)^2 = 2c^2a^2 \Rightarrow \left(\dfrac{a^2 - b^2 + c^2}{2ac}\right)^2 = \dfrac{1}{2}$

$\Rightarrow \cos^2 B = \dfrac{1}{2} \Rightarrow \cos B = \dfrac{1}{\sqrt{2}}$

$\therefore\ B = 45^0 \text{ or } 135^0$

31. Let ABC be an isosceles triangle with base BC. If 'r' is the radius of the circle inscribed in Δ ABC and r_1 be the radius of the circle described opposite to the angle A, then r_1r =

A) $R^2 \sin^2 A$ B) $R^2 \sin^2 2B$ C) $\dfrac{a^2}{2}$ D) $\dfrac{a^2}{4}$

Key. A,B,D

Sol. $rr_1 = 4R\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}.4R\sin\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$

$= 4R^2\sin^2\dfrac{A}{2}\sin B\sin C$ but $2B = 180^0 - A \Rightarrow B = C = 90^0 - \dfrac{A}{2}$

$= 4R^2\sin^2\dfrac{A}{2}\cos^2\dfrac{A}{2} = R^2\sin^2 A$

$= R^2\sin^2 2B = \dfrac{a^2}{4}$

32. If A is the area and 2S the sum of sides of a triangle, then

A) $A < \dfrac{S^2}{4}$ B) $A \le \dfrac{S^2}{3\sqrt{3}}$

C) $A > \dfrac{S^2}{\sqrt{3}}$ D) $A \le \dfrac{S^2}{\sqrt{3}}$

Key. A,B

Sol. $\dfrac{S + S - a + S - b + S - c}{4} \ge \sqrt[4]{S(S-a)(S-b)(S-c)}$

$\dfrac{S}{2} \ge \Delta^{1/2} \Rightarrow \dfrac{S^2}{4} \ge \Delta \Rightarrow \Delta \le \dfrac{S^2}{4}$

$\dfrac{S - a + S - b + S - c}{3} \ge \sqrt[3]{(S-a)(S-b)(S-c)}$

$\dfrac{S^3}{27} \ge \dfrac{(S-a)(S-b)(S-c)S}{S}$

$\Rightarrow \Delta^2 \le \dfrac{S^4}{27} \Rightarrow \Delta \le \dfrac{S^2}{3\sqrt{3}}$

33. In ΔABC which of the following statements are true?

A) Maximum value of $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ is $\frac{1}{4}$

B) $R \geq 2r$ where R is circum radius and r is in radius

C) $R^2 \geq \frac{abc}{a+b+c}$

D) Δ ABC is right angled if r + 2R = s where 's' is semi perimeter

Key. B,C

Sol. $\frac{r_1+r_2+r_3}{3} \geq \frac{3}{\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}}$

$4R + r \geq 9r \Rightarrow 4R \geq 8r \Rightarrow R \geq 2r$

$R^2 \geq 4r^2 \geq \frac{4\Delta^2}{S^2} \geq \frac{4a^2b^2c^2}{16R^2S^2} \Rightarrow R^4 \geq \frac{a^2b^2c^2}{4S^2}$

$$\Rightarrow R^2 \geq \frac{abc}{a+b+c}$$

34. There exists a triangle ABC satisfying the following conditions.

A) b sinA = a, $A < \frac{\pi}{2}$

B) b sinA > a, $A > \frac{\pi}{2}$

C) b sinA > a, $A < \frac{\pi}{2}$

D) b sinA < a, $A < \frac{\pi}{2}, b > a$

Key. A,D

Sol. $b \times \frac{a}{2R} = a \Rightarrow a = 2R \Rightarrow B = 90^0,\ A < \frac{\pi}{2}$

$\therefore$ A triangle is possible.

$b\sin A < a \Rightarrow b < 2R,\ A < \frac{\pi}{2}$ but b > a a triangle is possible.

$$\Rightarrow b < \frac{\pi}{2}$$

35. In ΔABC, internal angle bisector of $\angle A$ meets side BC in D . $DE \perp AD$ meets AC in E and AB in F. Then

A. AE is HM of b&c

B. $AD = \frac{2bc}{b+c}\cos\frac{A}{2}$

C. $EF = \frac{4bc}{b+c}\sin\frac{A}{2}$

D. ΔAEF is isosceles

KEY. A,B,C,D

SOL. $AD = \frac{2bc}{b+c}\cos\frac{A}{2}; AD = AE\cos\frac{A}{2}$

$AE = \frac{2bc}{b+c} = HM\ of\ b\ and\ c$

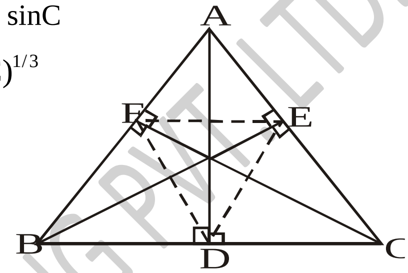
$EF = 2DE = 2AD\tan\frac{A}{2} = \frac{4bc}{b+c}\sin\frac{A}{2}$

$\Rightarrow a, b, c, d$

36. In a triangle ABC, perpendiculars drawn from vertices A, B, C meet the opposite sides BC, CA, AB at D, E, F respectively, triangle DEF is completed. The perimeter of triangle DEF is greater than or equal to $3\sqrt{3}\,r$, where r is in-radius of triangle ABC. Also $r=\sqrt{3}$ and perimeter of triangle ABC is 18. Then
(A) triangle ABC is right angled
(B) triangle ABC is equilateral
(C) area of triangle ABC is $9\sqrt{3}$
(D) ratio of area of triangle ABC to triangle DEF is 4 : 1

Key. B,C,D

Sol. Here EF = a cosA, FD = b cos B, ED = c cos C
Or EF+FD+ED = R(sin2A+sin2B+sin2C)=4RsinA sinB sinC



For a ΔABC, $\frac{\sin A+\sin B+\sin C}{3} \ge (\sin A \sin B \sin C)^{1/3}$

$\Rightarrow \quad \sin A \sin B \sin C \le \left(\frac{\sqrt{3}}{2}\right)^3 = \frac{3\sqrt{3}}{8}$

$\therefore$ EF + FD + ED = 4R sinA sinB sinC $\le \frac{3\sqrt{3}}{2}R = 3\sqrt{3}r$

But given EF + FD + ED ≥ $3\sqrt{3}r$, which is possible only when triangle is equilateral.

Also $r=(9-a)\tan\frac{A}{2} \Rightarrow \sqrt{3}=(9-a)\frac{1}{\sqrt{3}}$

$\Rightarrow$ a = b = c = 6 (since triangle is equilateral)

$$\frac{\text{Ar.of } \Delta ABC}{\text{Ar.of } \Delta DEF} = \frac{\frac{1}{2}bc\sin A}{\frac{1}{2}b\cos B\ c\cos C\sin(\pi-2A)} = \frac{1}{2\cos A\cos B\cos C} = \frac{4}{1}$$

Also area of ΔABC = $\frac{1}{2}$ bc sinA $=\frac{36}{2}\frac{\sqrt{3}}{2} = 9\sqrt{3}$ sq. units.

37. If the sines of the angles A and B of a triangle ABC satisfy the equation

$c^2x^2 - c(a+b)x + ab = 0$, then the triangle

a) is acute angled b) is right angled c) is obtuse angled d) satisfy sin A +cosA= $\frac{(a+b)}{c}$

Key. B,D

Sol. ∵ sinA and sinB are the roots of $c^2x^2 - c(a+b)x + ab = 0$

Then $\sin A + \sin B = \frac{a+b}{c}$

And $\sin A \sin B = \frac{ab}{c^2}$

$\Rightarrow \frac{a}{2R}+\frac{b}{2R}=\frac{a+b}{c}$

And $\frac{a}{2R}\times\frac{b}{2R}=\frac{ab}{c^2}$

$\therefore c=2R \Rightarrow 2R\sin C=2R$

$\therefore \sin C=1$

$\therefore \angle C=90^o$

$A+B=90^o$

$B=90^o-A$

$\because \sin A+\sin B=\frac{a+b}{c}$

$\Rightarrow \sin A+\sin\left(90^o-A\right)=\frac{a+b}{c} \Rightarrow \sin A+\cos A=\frac{a+b}{c}$

38. If $\cos(\theta-\alpha)$, $\cos\theta$, $\cos(\theta+\alpha)$ a are in HP, then $\cos\theta\sec\frac{\alpha}{2}$ is equal to

a) -1 b) $-\sqrt{2}$ c) $\sqrt{2}$ d) 2

Key. B,C

Sol. Since, $\cos(\theta-\alpha),\cos\theta,\cos(\theta+\alpha)$ are in HP

$\therefore \frac{1}{\cos(\theta-\alpha)},\frac{1}{\cos\theta},\frac{1}{\cos(\theta+\alpha)}$ are in AP

$\Rightarrow \frac{2}{\cos\theta}=\frac{1}{\cos(\theta-\alpha)}+\frac{1}{\cos(\theta+\alpha)}=\frac{2\cos\theta\cos\alpha}{\cos(\theta-\alpha)\cos(\theta+\alpha)}$

$\Rightarrow \cos^2\theta\cos\alpha=\cos^2\theta-\sin^2\alpha$

$\Rightarrow \cos^2\theta(1-\cos\alpha)=\sin^2\alpha$

$\Rightarrow \cos^2\theta(1-\cos\alpha)=(1+\cos\alpha)(1-\cos\alpha)$

$\Rightarrow \cos^2\theta=(1+\cos\alpha)$

$\Rightarrow \cos^2\theta=2\cos^2\frac{\alpha}{2}$

$\Rightarrow \cos^2\theta\sec^2\frac{\alpha}{2}=2$

$\therefore \cos\theta\sec\frac{\alpha}{2}=\pm\sqrt{2}$

39. In a triangle ABC, if the radii of ex-circles r_1, r_2 and r_3 are given by r_1=8, r_2=12 and r_3 = 24, then

a) in radius r=8 b) side a=12 c) side b=16 d) side c=20

Key. B,C,D

Sol. $a=\frac{r_1(r_2+r_3)}{\sqrt{r_1r_2+r_2r_3+r_3r_1}}=\frac{8\times 36}{24}=12$

$b=\frac{r_2(r_3+r_1)}{\sqrt{r_1r_2+r_2r_3+r_3r_1}}=\frac{12\times 32}{24}=16$

$c=\frac{r_3(r_1+r_2)}{\sqrt{r_1r_2+r_2r_3+r_3r_1}}=\frac{24\times 20}{24}=20$

Also, $r_1r_2r_3=rs^2\Rightarrow\frac{8\times 12\times 24}{(24)^2}=4$

40. $\cos(\sin x)=\frac{1}{\sqrt{2}}$, then x must lie in the intrerval

a) $\left(\frac{\pi}{4},\frac{\pi}{2}\right)$ b) $\left(-\frac{\pi}{4},\frac{\pi}{2}\right)$ c) $\left(\pi,\frac{3\pi}{2}\right)$ d) $\left(\frac{\pi}{2},\pi\right)$

Key. A,B,C,D

Sol. $\cos(\sin x)=\frac{1}{\sqrt{2}}=\cos\frac{\pi}{4}$

$\sin x=2n\pi\pm\frac{\pi}{4},n\in I$

$y=\sin x=\pm\frac{\pi}{4}$

$y=\sin x$ and $y=\pm\frac{\pi}{4}$

41. If inside a big circle exactly 24 small circles each of radius 2 can be drawn in such a way that each small circle touches the big circle and also touch both its adjacent small circles. Then the radius of the big circle is ____

a) $2\left(1+\operatorname{cosec}\frac{\pi}{24}\right)$ b) $\left(\frac{1+\tan\frac{\pi}{24}}{\cos\frac{\pi}{24}}\right)$

c) $2\left(1+\operatorname{cosec}\frac{\pi}{12}\right)$ d) $\frac{2\left(\sin\frac{\pi}{48}+\cos\frac{\pi}{48}\right)^2}{\sin\frac{\pi}{24}}$

Key. A,D

Sol. $\sin\frac{\pi}{24}=\frac{2}{R-2}\Rightarrow R=2\left(1+\operatorname{cosec}\frac{\pi}{24}\right)$

42. In a $\Delta ABC, 2\cos\frac{(A-C)}{2} = \frac{a+c}{\sqrt{a^2+c^2-ac}}$ then

A) $B = \frac{\pi}{3}$ B) $B = C$ C) A, B, C are in A.P D) $B + C = A$

Key. A,C

Sol. $2\cos\frac{A-C}{2} = \frac{2\sin\frac{A+C}{2}\cos\frac{A-C}{2}}{\sin\frac{A+C}{2}} = \frac{\sin A + \sin C}{\sin\frac{A+C}{2}}$

$\cos B = \frac{a^2+c^2-b^2}{2ac}$ if $B = 60^0$

$= \frac{a+c}{\sin 60} = \frac{a+c}{b} = \frac{a+c}{\sqrt{a^2+c^2-ac}}$

43. Let ABC be an isosceles triangle with base BC. If 'r' is the radius of the circle inscribed in the Δ ABC and ρ be the radius of the circle escribed opposite to the angle A, then the product $\rho\, r$ can be equal to:

A) $R^2 \sin^2 A$ B) $R^2 \sin^2 2B$ C) $\frac{1}{2}a^2$ D) $\frac{a^2}{4}$

Where R is the radius of the circumcircle of the ΔABC

Key. A

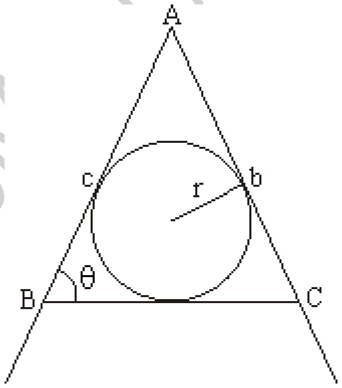
Sol. $r = \frac{\Delta}{s}$; $p = \frac{\Delta}{s-a}$

$pr = \frac{\Delta^2}{s(s-a)} = \frac{s(s-a)(s-b)(s-c)}{s(s-a)}$

$= (s-b)(s-c) \;\; = (s-b)^2 \quad (\therefore b = c)$

$= \frac{(2s-2b)^2}{4} = \frac{(a+b+c-2b)^2}{4} \quad (\therefore b = c)$

$= \frac{a^2}{4} = \frac{4R^2 \sin A}{4} = R^2 \sin^2 A$



Also if $\angle B = \theta \Rightarrow \angle A = \pi - 2\theta$

$pr = R^2 \sin^2(\pi - 2\theta) = R^2 \sin^2 2\theta = R^2 \sin^2 2B$

44. The sides of a Δ ABC satisfy the equation, $2a^2 + 4b^2 + c^2 = 4ab + 2ac$. Then

A) the triangle is isosceles B) the triangle is obtuse

C) $B = \cos^{-1}\frac{7}{8}$ D) $A = \cos^{-1}\frac{1}{4}$

Key. A,C,D

Sol. given expression $(a-c)^2 + (a+2b)^2 = 0 \Rightarrow a = 2b$ and c = a. Sides are 2b, b, 2b

$\Rightarrow$ isosceles and $\cos B = \frac{7}{8}$ and $\cos A = \frac{1}{4}$

$$\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{7b^2}{8b^2} = \frac{7}{8}$$

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{1}{4}$$

45. If in a triangle ABC p,q,r are the altitudes from the vertices A,B,C to the opposite sides, then which of the following hold(s) good.

A) $(\Sigma\, p)\left(\Sigma \frac{1}{p}\right) = (\Sigma\, a)\left(\Sigma\, \frac{1}{a}\right)$ B) $(\Sigma\, p)\,(\Sigma\, a) = \left(\Sigma \frac{1}{p}\right)\left(\Sigma\, \frac{1}{a}\right)$

C) $(\Sigma\, p)\,(\Sigma\, pq)(\Pi\, a) = (\Sigma\, a)(\Sigma\, ab)\,(\Pi\, p)$ D) $\left(\Sigma\, \frac{1}{p}\right)\Pi\left(\frac{1}{p}+\frac{1}{q}-\frac{1}{r}\right)\Pi\ a^2 = 16\,R^2$

where R is the circum-radius of ΔABC

Key. A,C,D

Sol. $p = \frac{2\Delta}{a}, q = \frac{2\Delta}{b}, r = \frac{2\Delta}{c}$

A) $(\Sigma p)\left(\Sigma\frac{1}{p}\right) = 2\Delta\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{a+b+c}{2\Delta}\right)$

$= (\Sigma a)\left(\Sigma\frac{1}{a}\right)$

C) $(\Sigma p)(\Sigma pq)(\Pi a)$

$= \left(\frac{2\Delta}{a}+\frac{2\Delta}{b}+\frac{2\Delta}{c}\right)\left(\frac{4\Delta^2}{ab}+\frac{4\Delta^2}{bc}+\frac{4\Delta^2}{ca}\right)abc$

$= \frac{2\Delta(ab+bc+ca)}{abc}.\ 4\Delta^2\frac{(a+b+c)}{abc}.\ abc$

$= (\Sigma a)(\Sigma ab)(\Pi p)$

D) $\left(\Sigma\frac{1}{p}\right)\Pi\left(\frac{1}{P}+\frac{1}{q}-\frac{1}{r}\right)\Pi a^2$

$= \left(\frac{a+b+c}{2\Delta}\right)\left(\frac{a+b-c}{2\Delta}\right)\left(\frac{a-b+c}{2\Delta}\right)\left(\frac{b+c-a}{2\Delta}\right).\ a^2b^2c^2$

$= \frac{S}{\Delta}.\frac{(S-c)}{\Delta}.\frac{(S-b)}{\Delta}.\frac{(S-a)}{\Delta}(abc)^2$

$= \left(\frac{abc}{\Delta}\right)^2 = (4R)^2 = 16R^2.$

46. If a,b,c are in H.P, then which of the following is true.

A) $\frac{a}{1-2a}, \frac{b}{1-2b}, \frac{c}{1-2c}$ are in H.P. B) $ln\left(a-\frac{b}{2}\right),\ ln\frac{b}{2}, ln\left(c-\frac{b}{2}\right)$ are in H.P.

C) $c-\frac{b}{2},\frac{b}{2},a-\frac{b}{2}$ are in G.P. D) $e^{1/a}, e^{1/b}, e^{1/c}$ are in G.P.

Key. A,C,D

Sol. A) $\frac{a}{1-2a},\frac{b}{1-2b},\frac{c}{1-2c}$ are in H.P. $\Leftrightarrow$ $\frac{1-2a}{a},\frac{1-2b}{b},\frac{1-2c}{c}$ are in A.P.

$\Leftrightarrow$ $\frac{1}{a},\frac{1}{b},\frac{1}{c}$ are in A.P. $\Leftrightarrow$ a,b,c are in H.P.

B) $a-\frac{b}{2},\frac{b}{2},c-\frac{b}{2}$ are in G.P. $\Rightarrow$ $l n\left(a-\frac{b}{2}\right), l n\frac{b}{2}, l n\left(c-\frac{b}{2}\right)$ are in A.P.

C) $c-\frac{b}{2},\frac{b}{2}, a-\frac{b}{2}$ are in G.P.

D) $\frac{1}{a},\frac{1}{b}\frac{1}{c}$ are in A.P. $\Rightarrow$ $e^{1/a},e^{1/b},e^{1/c}$ are in G.P.

47. In a triangle ABC, with usual notations the length of the bisector of angle A is

A) $\frac{2bc\cos\frac{A}{2}}{b+c}$ B) $\frac{2bc\sin\frac{A}{2}}{b+c}$ C) $\frac{abc\operatorname{cosec}\frac{A}{2}}{2R(b+c)}$ D) none

Key. A,C

Sol. $\Delta=\Delta_1+\Delta_2$

$$\frac{1}{2}bc\sin A=\frac{1}{2}cx\sin\frac{A}{2}+\frac{1}{2}xb\sin\frac{A}{2}$$

Progressions of Triangles

Assertion Reasoning Type

(A) Statement-1 is true, statement-2 is true; statement-2 is correct explanation for statement-1.

(B) Statement-1 is true, statement-2 is true; statement-2 is NOT the correct explanation for statement-1.

(C) Statement-1 is true, statement-2 is false.

(D) Statement-1 is false, statement-2 is true

1. ABC is a triangle. h_a is the length of the altitude from A. h_b has a similar meaning.

Given $a \le h_a$ and $b \le h_b$.

(A) : $a : b : c = 1 : 1 : \sqrt{2}$

(R) : In any triangle the length of an altitude does *not exceed* that of the corresponding median.

(a) Both A and R are true, and R is a correct explanation of A

(b) Both A and R are true, but R is not a correct explanation of A

(c) A is true and R is false

(d) A is false and R is true.

Key. (b) $a \le h_a = b \sin C$ and $b \le h_b = a \sin C$

$\Rightarrow \quad ab \le ab \sin^2 C$

$\Rightarrow \quad 1 \le \sin^2 C$

Hence sin C = 1 (i.e. $C = \frac{\pi}{2}$)

Further, equality must occur in the given statements. Hence (A) is true.

Clearly , (R) is true but has no relevance with (A).

2. Consider the following statments

STATEMENT 1: In a right angled triangle ABC, $\sin^2A + \sin^2B + \sin^2C = 2$

Because

STATEMENT 2 : In any triangle ABC, $\sin^2A + \sin^2B + \sin^2C = 2 - 2 \cos A \cos B \cos C$.

Key. (c)

Sol. Let A = 90° then C = 90° - B

$\sin^2C = \cos^2B$

statement 1 is true

$$\because \sum \sin^2 A = 2 + 2\pi \cos A$$

statement 2 is false

3. Let a, b, c be 3 positive real numbers , such that $\sqrt[3]{\frac{a^3+b^3+c^3+3abc}{2}}$ > max (a,b,c), then

Statement - 1 : It is impossible to form a triangle whose sides have length p, q and r.

Statement - 2 : p > max (a , b, c) p > a , p > b and p > c .

Key. (d)

Sol.

$\frac{a^3 + b^3 + c^3 + 3abc}{2} > a^3 \quad \Rightarrow \quad b^3 + c^3 + (-a)^3 - 3(-a)(b)(c) > 0$

$\Rightarrow \quad \frac{1}{2}(b + c - a)[(b - c)^2 + (c + a)^2 + (b + a)^2] > 0$

$\Rightarrow \quad b + c - a > 0 \quad \Rightarrow \quad b + c > a$

Siumilarly $c + a > b$ and $a + b > c$

$\Rightarrow$ a, b, c will form the sides of a triangle.

4. STATEMENT -1 : In a ΔABC, if cos A + 2 cos B + cos C = 2, then a, b, c are in A.P.
and
STATEMENT-2 : In a ΔABC, we have $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
and $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$, $\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$, $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$, where
$2s = a + b + c$

Key. A

5. Statement : In ΔABC, tan A, tan B, tan C are distinct values in A.P.

Conclusion I : $\frac{\pi}{3} < B < \frac{\pi}{2}$

Conclusion II : One angle less than $\frac{\pi}{3}$ and other two angles lie between $\frac{\pi}{3}$ *and* $\frac{\pi}{2}$

Key. C

6. ABC is a triangle. h_a is the length of the altitude from A. h_b has a similar meaning.
Given $a \le h_a$ and $b \le h_b$.

(A) : $a : b : c = 1 : 1 : \sqrt{2}$

(R) : In any triangle the length of an altitude does not exceed that of the corresponding median.

(a) Both A and R are true, and R is a correct explanation of A
(b) Both A and R are true, but R is not a correct explanation of A
(c) A is true and R is false
(d) A is false and R is true.

Key. (b) $a \le h_a = b\sin C$ and $b \le h_b = a\sin C$

$\Rightarrow \quad ab \le ab\sin^2 C$

$\Rightarrow \quad 1 \le \sin^2 C$

Hence sin C = 1 (i.e. $C = \frac{\pi}{2}$)

Further, equality must occur in the given statements. Hence (A) is true.
Clearly , (R) is true but has no relevance with (A).

7. STATEMENT – 1

In triangle $ABC, \frac{a^2 + b^2 + c^2}{\Delta} \ge 4\sqrt{3}$

STATEMENT – 2 If $a_i > 0, i = 1, 2, 3, ..., n$ which are not identical, then

$$\frac{a_1^m + a_2^m + ... + a_n^m}{n} > \left(\frac{a_1 + a_2 + ... + a_n}{n}\right)^m$$, if m < 0 or m > 1.

Key. A

Sol. We know that in any triangle

$$\frac{s^2}{3\sqrt{3}} \ge \Delta$$

Now, $$\frac{a^2 + b^2 + c^2}{3} \ge \left(\frac{a+b+c}{3}\right)^2$$

$$= \left(\frac{2s}{3}\right)^2$$

$$= \frac{4}{9}s^2 \ge \frac{4}{9} \times 3\sqrt{3}\Delta$$ [from Eq. (i)]

8. The angles of a right angled triangle ABC are in A.P.

STATEMENT–1

r/R = $\frac{\sqrt{3}-1}{2}$

because

STATEMENT–2

$$\frac{r}{s} = \frac{2-\sqrt{3}}{\sqrt{3}}$$

Key. B

Sol. $(\alpha - \beta) + \alpha + (\alpha + \beta) = \pi$ & $\alpha + \beta = \pi/2$ $\Rightarrow$ angle are $\pi/6, \pi/3, \pi/2$ $\Rightarrow$ a = R, b = $\sqrt{3}$ R, c = 2R

$$\therefore \text{ r} = \frac{\Delta}{s} = \frac{\sqrt{3}}{3+\sqrt{3}}R \Rightarrow \frac{r}{R} = \frac{\sqrt{3}}{3+\sqrt{3}} = \frac{\sqrt{3}(3-\sqrt{3})}{6} = \frac{\sqrt{3}-1}{2}$$

Also $$\frac{r}{2s} = \frac{\sqrt{3}R}{3+\sqrt{3}} \times \frac{1}{(3+\sqrt{3})R} = \frac{1}{(3+\sqrt{3})(\sqrt{3}+1)} = \frac{1}{6+4\sqrt{3}} = \frac{2-\sqrt{3}}{2\sqrt{3}}$$

Hence (B) is the correct option.

9. Statement – 1: In a ΔABC, if a < b < c and r is inredius and r_1, r_2, r_3 are the exradii opposite to angle A,B,C respectively then $r < r_1 < r_2 < r_3$

Statement – 2: For, ΔABC $r_1r_2 + r_2r_3 + r_3r_1 = \frac{r_1 r_2 r_3}{r}$

Key. B

Sol. Statement – 1:

a < b < c

s – a > s – b > s – c

s > s – a > s – b > s – c

$$\frac{\Delta}{s} < \frac{\Delta}{s-a} < \frac{\Delta}{s-b} < \frac{\Delta}{s-c}$$

$r < r_1 < r_2 < r_3$

Statement – 2:

$r_1 = \frac{\Delta}{s-a}, r_2 = \frac{\Delta}{s-b}, r_3 = \frac{\Delta}{s-c}, r = \frac{\Delta}{s}$

$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{s}{\Delta} = \frac{1}{r}$

10. Statement – 1: If the sides of a triangle are 13, 14, 15 then the radius of incircle = 4

Statement – 2: In a ΔABC, $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{a+b+c}{2}$ and $r = \frac{\Delta}{s}$

Key. A

Sol. s = 21

$\Delta = \sqrt{21.8.7.6} = \sqrt{3.7.2^4.7.3} = 3.7.4 = 84$

$r = \frac{\Delta}{s} = \frac{84}{21} = 4$

11. Statement – 1: In a ΔABC, $\sum \frac{\cos^2 \frac{A}{2}}{a}$ has the value equation to $\frac{s^2}{abc}$

Statement – 2: In a

Δ ABC, $\cos\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$, $\cos\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}}$, $\cos\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

Key. C

Sol. $\frac{\cos^2 \frac{A}{2}}{a} = \frac{s(s-a)}{abc}$

$\therefore \quad \sum \frac{\cos^2 \frac{A}{2}}{a} \frac{s^2}{abc}$

12. Statement 1: If I is incentre of ΔABC and I_1 excentre opposite to A and P is the intersection of II_1 and BC then $IP.\ I_1P = BP.\ PC$

Statement – 2: In a ΔABC, I is incentre and I_1 is excentre opposite to A then IBI_1C must be square

Key. C

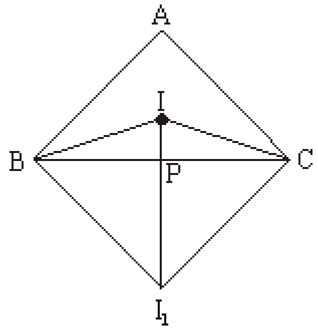
Sol. $ICI_1 = \frac{\pi}{2}$

$IBI_1 = \frac{\pi}{2}$

$\therefore$ $BICI_1$, is cyclic

Quadrilateral

BP. PC = IP. I_1P



13. All the notations used in statement – 1 and statement – 2 are usual

Statement – 1: In triangle ABC, if $\frac{\cos A}{a}=\frac{\cos B}{b}=\frac{\cos C}{c}$, then value of $\frac{r_1+r_2+r_3}{r}$ is equal to 9.

Statement – 2: In ΔABC; $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R$, where R is circumradius.

Key. A

Sol. Statement – 2 is true.

Statement – 1 tan A + tan B = tan C

A = B = C i.e., a = b = c

$r_1=r_2=r_3$

$$\therefore \quad \frac{r_1+r_2+r_3}{r}=3.\frac{r_1}{r}=3.\frac{\frac{\Delta}{s-a}}{\frac{\Delta}{s}}=3\frac{a+b+c}{b+c-a}=9$$

Progressions of Triangles

Comprehension Type

Paragraph – 1

In triangle ABC, BC = a, CA = b, AB = c. R is the circum radius and r is in radius and s is the semi perimeter and it is given that $\left(\cot\frac{A}{2}\right)^2+\left(2\cot\frac{B}{2}\right)^2+\left(3\cot\frac{C}{2}\right)^2=\left(\frac{6s}{7r}\right)^2$ then answer the following.

1. The ratio of the sides of triangle ABC, a:b:c is

A) 1 : 1 : 1 B) 13 : 40 : 45 C) 1 : 2 : 3 D) 7 : 15 : 45

Key. B

2. The greatest angle of the triangle is

A) $\pi-\cos^{-1}\left(\frac{16}{65}\right)$ B) $\sin^{-1}\left(\frac{16}{65}\right)$ C) $\cos^{-1}\left(\frac{63}{65}\right)$ D) None

Key. A

Sol. 1. $\left(\frac{S(S-a)}{\Delta}\right)^2+4\left(\frac{S(S-b)}{\Delta}\right)^2+9\left(\frac{S(S-c)}{\Delta}\right)^2=\frac{36s^2}{49r^2}$

$\Rightarrow\frac{S^2}{\Delta^2}\left((S-a)^2+4(S-b)^2+9(S-c)^2\right)=\frac{36}{49}\frac{s^2.S^2}{\Delta^2}$

$\Rightarrow\left(\frac{S-a}{6}\right)^2+\left(\frac{S-b}{3}\right)^2+\left(\frac{S-c}{2}\right)^2=\frac{s^2}{49}$

Let $\frac{S-a}{6}=l,\ \frac{S-b}{3}=m,\ \frac{S-c}{2}=n\Rightarrow S=6l+3m+2n$

$\Rightarrow 49\left(l^2+m^2+n^2\right)=(6l+3m+2n)^2$

$\Rightarrow\frac{l}{6}=\frac{m}{3}=\frac{n}{2}=k$ (from canchy's inequality)

$\therefore$ S – a = 36k, S – b = 9k, S – c = 4k

$\Rightarrow S=49k,\ a=13k,\ b=40k,\ c=45k$

$\therefore a:b:c=13:4045$

2. $\cos c=\frac{13^2+40^2-45^2}{2\times13\times40}=\frac{-256}{2\times13\times40}=\frac{-16}{65}$

$c=\cos^{-1}\left(\frac{-16}{65}\right)=\pi-\cos^{-1}\left(\frac{16}{65}\right)$

Paragraph – 2

Let x and y represent the sum and product of two sides of a triangle such that $x^2=y+z^2$ where z is the third side, then answer the following.

3. The triangle is

A) Equilateral triangle B) Right angled triangle

C) Acute angled triangle D) Obtuse angled triangle

Key. D

4. In radius of the triangle is

A) $\frac{y}{2(z+x)}$ B) $\frac{z}{2(x+y)}$ C) $\frac{\sqrt{3}y}{2(x+z)}$ D) $\frac{\sqrt{3}z}{x+y}$

Key. C

5. Area of the triangle is

A) $\frac{\sqrt{3}y}{4}$ B) $\frac{\sqrt{3}x}{4}$ C) $\frac{\sqrt{3}z}{4}$ D) None

Key. A

Sol. 3 to 5.

Let x = a+b, y = ab $\Rightarrow (a+b)^2 = ab + c^2$

z = c $\Rightarrow a^2 + b^2 + ab = c^2 \Rightarrow c = 120^0$

$\therefore$ The triangle is obtuse angled.

$$r = \frac{\Delta}{S} = \frac{\frac{1}{2}ab \times \frac{\sqrt{3}}{2}}{\frac{x+z}{2}} = \frac{\sqrt{3}y}{2(x+z)}$$

Area = $\frac{1}{2} y \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}y}{4}$

Paragraph – 3

The sides of a triangle ABC are 7,8,6 the smallest angle being 'C'

6. The length of the altitude from vertex 'C' is

a) $5\sqrt{3}$ b) $\frac{\sqrt{35}}{4}$ c) $\frac{7}{3}\sqrt{15}$ d) $\frac{7}{4}\sqrt{15}$

7. The length of the median from vertex 'C' is

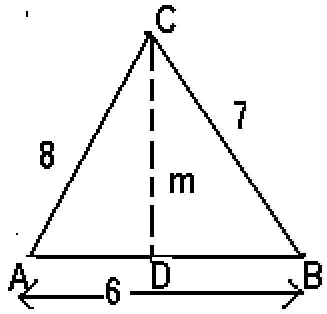
a) $\frac{\sqrt{95}}{4}$ b) $\frac{\sqrt{95}}{2}$ c) $\sqrt{\frac{95}{2}}$ d) $\frac{\sqrt{95}}{3}$,

8. The length of the internal bisector of angle 'C' is

a) $\sqrt{30}$ b) $\frac{14}{5}\sqrt{6}$ c) $\frac{14}{5}$ d) $2\sqrt{6}$

Sol. 6 (D) Altitude from 'C' = $\frac{2\Delta}{c}$

7 (C) $2(9+m^2) = 7^2 + 8^2$ where 'm' is the length of the median from vertex 'C'



8 (B)Length of the internal bisector of angle 'C' = $\frac{2\sqrt{abs(s-c)}}{a+b}$

Paragraph – 4

At times the methods of coordinates becomes effective in solving problems of properties of triangles. We may choose one vertex of the triangle as origin and one side passing through this vertex as x-axis. Thus without loss of generality, we can assume that every triangle ABC has a vertex situated at (0 ,0) another at (x, 0) and third one at (h, k).

9. If in $\Delta ABC, AC = 3, BC = 4$ medians AD and BE are perpendicular then area of ΔABC ______ sq.units.

a) $\sqrt{7}$

b) $\sqrt{11}$

c) $2\sqrt{2}$

d) $2\sqrt{11}$

10. Suppose the bisector AD of the interior angle A of ΔABC divides side BC into segments BD=4; DC=2 then

a) $b > c$ and c<4

b) $2 < b < 6$ and c<1

c) $2 < b < 6$ and 4<c<12

d) $b < c$ and c>4

11. If in the above question (34), altitude $AE > \sqrt{10}$ and suppose lengths of AB and AC are integers, then b will be

a) 3

b) 6

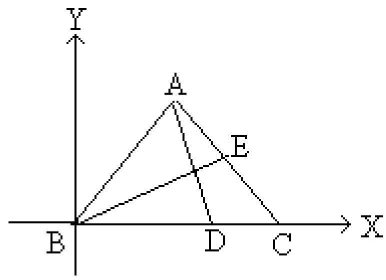
c) 4 or 5

d) 3 or 6

KEY : 9 - B 10 - C or D 11 - C

HINT

9. Take B as origin, BC as x-axis and take A as (h,k) C (4,0).



Area of $\Delta ABC = \frac{1}{2} 4 \times k = 2k$ -----(1)

D = (2,0) and $E\left(\frac{h+4}{2}, \frac{k}{2}\right)$

$\because AD \perp BE$ slope of AD × slope of BE = -1

$\Rightarrow k^2 + (h+4)(h-2) = 0 \rightarrow (2)$

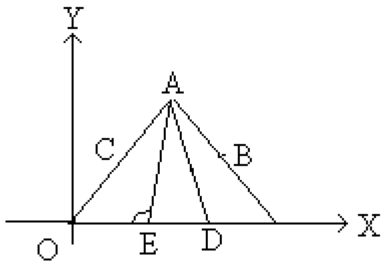
Also AC = 3 $\Rightarrow (h-4)^2 + k^2 = 9 \rightarrow (3)$

(2) - (3) $\Rightarrow h = \frac{3}{2}$ and $k^2 = \frac{11}{4}$

$k = \frac{\sqrt{11}}{2}$

From (1): Area of $\Delta ABC = \sqrt{11}$

10. Now AD is the bisector



$$\frac{AB}{AC} = \frac{BD}{DC} \Rightarrow c = 2b$$

$$b + c > a \Rightarrow b + c > 6$$

$$\therefore b > 2$$

Again $\frac{b^2 + 4b^2 - 3b}{4b^2} < 1$

$$\Rightarrow b < 6$$

$\therefore 2 < b < b$ and consequently $4 < c < 12$

11. Now $c^2 = h^2 + k^2$ and $b^2 = (h - b)^2 + k^2$

$$c^2 - b^2 = 12h - 3b \Rightarrow h = \frac{b^2 + 12}{4}$$

Given that $k^2 > 10 \Rightarrow c^2 - h^2 > 10$

$$\Rightarrow 4b^2 - \left(\frac{b^2 + 12}{4}\right)^2 > 10$$

$$\Rightarrow b^2 + \left(20 - \sqrt{96}, 20 + \sqrt{96}\right)$$

B is either 4 or 5

Paragraph – 5

Let ABC be any triangle and P be a point inside it such that $\angle PAB = \frac{\pi}{18}, \angle PBA = \frac{\pi}{9}$, $\angle PCA = \frac{\pi}{6}, \angle PAC = \frac{2\pi}{9}$. Let $\angle PCB = x$

12. x =

a) $\frac{\pi}{9}$ b) $\frac{2\pi}{9}$ c) $\frac{\pi}{3}$ d) none of these

Key. A

13. ΔABC is

a) Equilateral b) Isosceless c) Scalene d) Right angled

Key. B

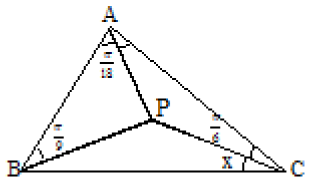
14. Which of the following is true

a) BC > AC b) AC < AB c) AC > AB d) BC = AC

Key. C

Sol. 12. $\frac{PA}{\sin 20} = \frac{PB}{\sin 10} \Rightarrow \frac{PA}{PB} = \frac{\sin 20}{\sin 10}$

Similarly $\frac{PB}{PC} = \frac{\sin x}{\sin(80 - x)}, \frac{PC}{PA} = \frac{\sin 40}{\sin 30}$



$$\frac{PA}{PB}\times\frac{PB}{PC}\times\frac{PC}{PA}=1$$

$$\Rightarrow \frac{\sin 20}{\sin 10}\times\frac{\sin x}{\sin(80-x)}\times\frac{\sin 40}{\sin 30}=1$$

sin(80 - x) = 4 cos 10 sin 40 sin x = 2

sin 50 sin x + sin x

$\Rightarrow$ sin (80 - x) - sin x = 2 sin 50 sin x

$\Rightarrow$ 2 sin (40 - x) cos 40 = 2 sin 50 sin x

$\Rightarrow$ sin(40 - x) = sin x

$\Rightarrow$ x = 20^{o}

13. $\angle A = \angle C = 50$

14. $\angle ABC = 80 \Rightarrow AC$ is longest side

Paragraph – 6

The sides of a triangle ABC are 7,8,6 the smallest angle being ‘C’

15. The length of the altitude from vertex ‘C’ is

a) $5\sqrt{3}$ b) $\frac{\sqrt{35}}{4}$

c) $\frac{7}{3}\sqrt{15}$ d) $\frac{7}{4}\sqrt{15}$

16. The length of the median from vertex ‘C’ is

a) $\frac{\sqrt{95}}{4}$ b) $\frac{\sqrt{95}}{2}$ c) $\sqrt{\frac{95}{2}}$ d) $\frac{\sqrt{95}}{3}$,

17. The length of the internal bisector of angle ‘C’ is

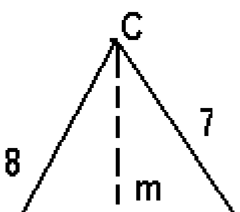
a) $\sqrt{30}$ b) $\frac{14}{5}\sqrt{6}$ c) $\frac{14}{5}$ d) $2\sqrt{6}$

15 (D)

Altitude from ’C’ = $\frac{2\Delta}{c}$

16 (C)

$2\left(9+m^2\right)=7^2+8^2$ where ‘m’ is the length of the median from vertex ‘C’

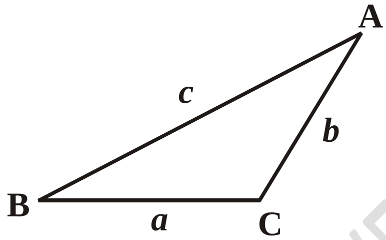


17. (B)

Length of the internal bisector of angle 'C' = $\dfrac{2\sqrt{abs(s-c)}}{a+b}$

Paragraph – 7

ABC is a triangle such that C > 90^0 , A = 2B and the side lengths are integers. We wish to find the *minimum* perimeter.



18. A relation among a , b and c is

(a) $a^2 - 2b^2 = bc$ (b) $a^2 - b^2 = 2bc$

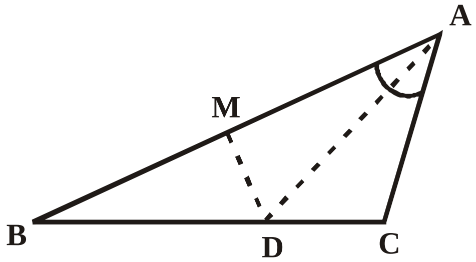
(c) $a^2 - b^2 = bc$ (d) $a^2 - 3b^2 = bc$

19. To seek the minimum perimeter, we must assume that gcd (a , b , c) = 1. There exist positive integers m and n such that $b = m^2$, $b + c = n^2$. This leads to cos B =

(a) $\dfrac{2n}{m}$ (b) $\dfrac{n}{2m}$ (c) $\dfrac{n}{m}$ (d) $\dfrac{n}{3m}$

20. Using the range of B, we finally get the minimum perimeter =

(a) 77 (b) 83 (c) 87 (d) 93



Sol. 18. (C)

Draw the bisector AD of $\angle$ BAC. Put BD = x $\Rightarrow$ AD = x

From $\frac{b}{c} = \frac{a-x}{x}$ we get $x = \frac{ca}{b+c}$.

Draw DM ⊥ AB. cos B = $\Rightarrow a^2 - b^2 = bc$

19. (b) We have $a^2 = b\,(b + c) = m^2\,n^2 \Rightarrow a = mn$

Use cosB = to yield cos B =

20. (a) Note that C > 90^0 & A = 2B $\left\{ \begin{array}{l} \frac{c^2+a^2-b^2}{2ca} \\ \frac{c}{2x} = \frac{b+c}{2a} \end{array} \right\} \Rightarrow 0 < B < 30^0$

$$\Rightarrow \frac{b+c}{2a} < \cos B < 1$$

$$\Rightarrow \sqrt{3} < \frac{n}{m} < 2$$

The *least* values of (n , m) , satisfying this, are (7, 4).

Paragraph – 8

The internal bisectors of angles of ΔABC meet opposite sides at D, E, F respectively.

21. $\frac{\Delta CDE}{\Delta ABC}$ equals

(A) $\frac{ab}{(a+c)(b+c)}$ (B) $\frac{bc}{(b+a)(c+a)}$ (C) $\frac{ca}{(c+b)(a+b)}$ (D) none of these

Key. A

22. $\frac{\Delta DEF}{\Delta ABC}$ equals

(A) $\frac{abc}{(a+b)(b+c)(c+a)}$ (B) $\frac{2abc}{(a+b)(b+c)(c+a)}$

(C) $\frac{abc}{2(a+b)(b+c)(c+a)}$ (D) none of these

Key. B

Paragraph – 9

Let the sides of a triangle are three consecutive integers and the largest angle is twice the smallest. Then

23. Smallest angle of the triangle is

(A) $\tan^{-1}\left(\frac{4}{3}\right)$ (B) $\cos^{-1}\left(\frac{3}{4}\right)$ (C) $\text{cosec}^{-1}\left(\frac{4}{3}\right)$ (D) $\sin^{-1}\left(\frac{3}{5}\right)$

Key. B

24. Perimeter of the triangle is

(A) 9 (B) 12 (C) 15 (D) 18

Key. C

25. Radius of the circle inscribed in the triangle is

(A) $\frac{\sqrt{7}}{2}$ (B) $\frac{\sqrt{7}}{4}$ (C) $\sqrt{7}$ (D) $\sqrt{\frac{7}{2}}$

Key. A

Paragraph – 10

In triangle ABC, BC = a, CA = b, AB = c. R is the circum radius and r is in radius and s is the semi perimeter and it is given that

$$\left(\cot\frac{A}{2}\right)^2+\left(2\cot\frac{B}{2}\right)^2+\left(3\cot\frac{C}{2}\right)^2=\left(\frac{6s}{7r}\right)^2$$ then answer the following.

26. The ratio of the sides of triangle ABC, a:b:c is

A) 1 : 1 : 1 B) 13 : 40 : 45 C) 1 : 2 : 3 D) 7 : 15 : 45

Key. B

27. The greatest angle of the triangle is

A) $\pi-\cos^{-1}\left(\frac{16}{65}\right)$ B) $\sin^{-1}\left(\frac{16}{65}\right)$ C) $\cos^{-1}\left(\frac{63}{65}\right)$ D) None

Key. A

Sol. 26. $\left(\frac{S(S-a)}{\Delta}\right)^2+4\left(\frac{S(S-b)}{\Delta}\right)^2+9\left(\frac{S(S-c)}{\Delta}\right)^2=\frac{36s^2}{49r^2}$

$$\Rightarrow\frac{S^2}{\Delta^2}\left((S-a)^2+4(S-b)^2+9(S-c)^2\right)=\frac{36}{49}\frac{s^2.S^2}{\Delta^2}$$

$$\Rightarrow\left(\frac{S-a}{6}\right)^2+\left(\frac{S-b}{3}\right)^2+\left(\frac{S-c}{2}\right)^2=\frac{s^2}{49}$$

Let $\frac{S-a}{6}=l,\ \frac{S-b}{3}=m,\ \frac{S-c}{2}=n\Rightarrow S=6l+3m+2n$

$$\Rightarrow 49\left(l^2+m^2+n^2\right)=(6l+3m+2n)^2$$

$\Rightarrow\frac{l}{6}=\frac{m}{3}=\frac{n}{2}=k$ (from canchy's inequality)

$\therefore$ S – a = 36k, S – b = 9k, S – c = 4k

$$\Rightarrow S=49k,\ a=13k,\ b=40k,\ c=45k$$

$$\therefore a:b:c=13:4045$$

27. $\cos c=\frac{13^2+40^2-45^2}{2\times13\times40}=\frac{-256}{2\times13\times40}=\frac{-16}{65}$

$$c=\cos^{-1}\left(\frac{-16}{65}\right)=\pi-\cos^{-1}\left(\frac{16}{65}\right)$$

Paragraph – 11

Let x and y represent the sum and product of two sides of a triangle such that $x^2=y+z^2$ where z is the third side, then answer the following.

28. The triangle is

A) Equilateral triangle B) Right angled triangle

C) Acute angled triangle D) Obtuse angled triangle

Key. D

29. In radius of the triangle is

A) $\dfrac{y}{2(z+x)}$ B) $\dfrac{z}{2(x+y)}$ C) $\dfrac{\sqrt{3}y}{2(x+z)}$ D) $\dfrac{\sqrt{3}z}{x+y}$

Key. C

30. Area of the triangle is

A) $\dfrac{\sqrt{3}y}{4}$ B) $\dfrac{\sqrt{3}x}{4}$ C) $\dfrac{\sqrt{3}z}{4}$ D) None

Key. A

Sol. 28 to 30.

Let x = a+b, y = ab $\Rightarrow (a+b)^2 = ab + c^2$

z = c $\Rightarrow a^2 + b^2 + ab = c^2 \Rightarrow c = 120^0$

$\therefore$ The triangle is obtuse angled.

$$r = \frac{\Delta}{S} = \frac{\frac{1}{2}ab \times \frac{\sqrt{3}}{2}}{\frac{x+z}{2}} = \frac{\sqrt{3}y}{2(x+z)}$$

Area = $\dfrac{1}{2} y \times \dfrac{\sqrt{3}}{2} = \dfrac{\sqrt{3}y}{4}$

Paragraph – 12

In triangle ABC, AB = AC. AD is drawn perpendicular to BC. BC is produced to a point E such that AE = 10. Let $\angle$BAD = DAC = β and $\angle$EAD = α. Let tan($\alpha - \beta$), tan α, tan($\alpha + \beta$) be in G.P. and cot β, cot ($\alpha - \beta$) and cot α be in A.P. Then answer the following questions.

31. α must be equal to

(A) 30º (B) 45º

(C) 60º (D) 75^0

Key. B

32. The value of tanβ must be equal to

(A) $\dfrac{1}{4}$ (B) $\dfrac{1}{5}$

(C) $\dfrac{1}{3}$ (D) $\dfrac{2}{3}$

Key. C

33. The area of triangle ABC must be

(A) $\dfrac{40}{3}$ sq. units (B) $\dfrac{50}{3}$ sq. units

(C) 20 sq. units (D) $\dfrac{25}{3}$ sq. units

Key. B

Sol. 31. $\tan\alpha^2 = \tan(\alpha - \beta).\tan(\alpha + \beta)$

$$= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}.\frac{\tan\alpha + \tan\beta}{1 - \tan\beta.\tan\beta}$$

$$= \frac{\tan^2\alpha - \tan^2\beta}{1 - \tan^2\alpha.\tan^2\beta}$$

$\Rightarrow$ $\tan^2\alpha - \tan^4\alpha.\tan^2\beta = \tan^2\alpha - \tan^2\beta$

$\Rightarrow \quad \tan^2 \alpha - 1 = 0$

$\Rightarrow \quad \alpha = 45^\circ$ (Q $0 < \alpha, \beta < 90^\circ, \tan\alpha > 0$)

(b) is correct

32. Q $\cot \alpha = 1$ and $\cot \alpha$, $\cot(\alpha - \beta)$ and $\cot \beta$ are in A.P.

$\Rightarrow \cot \alpha + \cot \beta = 2\cot(\alpha - \beta)$

$\Rightarrow 1 + \cot \beta = 2 \cot (45^\circ - \beta)$

Put, $45^\circ - \beta = \theta$

$\Rightarrow 1 + \cot \beta = 2 \cot \theta$

But $45^\circ = \theta + \beta$

$$\Rightarrow 1 = \cot(\theta + \beta) = \frac{\cot \beta . \cot \theta - 1}{\cot \beta + \cot \theta}$$

$\Rightarrow \cot \beta + \cot \theta = \cot \theta \cot \theta - 1$

Solving this we get,

$(\cot \beta + 1)(\cot \beta - 1) = 2(\cot \beta + 1)$

$\Rightarrow \cot \beta = 3$

$$\Rightarrow \tan \beta = \frac{1}{3}$$

(C) is correct

33. ΔADE is right-angled, isosceles triangle with

$AD = DE = 5\sqrt{2}$ (Q AE = 10 given)

Also in right-triangle, ACD, $\tan \beta = \frac{CD}{AD}$

$\Rightarrow A(\Delta ABC) = AD.CD = (AD)^2.\tan\beta = 50.\tan\beta$

$$= \frac{50}{3} \qquad \left(Q \tan \beta = \frac{1}{3} \right)$$

$\Rightarrow$ (B) is correct.

Paragraph – 13

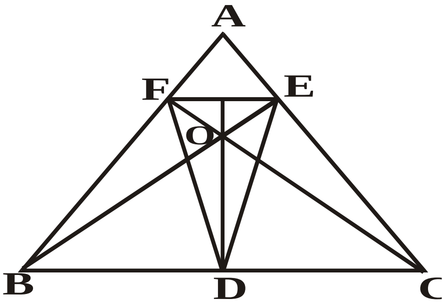
Let ABC be a triangle, from vertices A, B and C altitudes AD, BE and CF are drawn to opposite sides BC, CA and AB respectively which meet at a point O. Now triangle DEF is completed, also length OA, OB and OC respectively are p, q and r.

34. Ratio of area of ΔDEF to ΔABC is

(A) 2 cosA cosB cosC
(B) 2 sinA sinB sinC
(C) 2 cosA cosB sinC
(D) 2 sinA cosB cosC

Key. A

Sol. $\Delta DEF = \frac{1}{2} a \cos A b \cos B \sin(\pi - 2c)$



$= ab \cos A \cos B \sin C \cos C$

$$\frac{\Delta DEF}{\Delta ABC} = 2\cos A\cos B\cos C$$

35. Radius of incircle of ΔDEF is

(A) a cotA cosB cosC (B) $\frac{a}{2}$ cotA cosB cosC

(C) $\frac{abc\cos A\cos B\cos C}{\text{area of }\ \Delta ABC}$ (D) 2RsinA sin B sin C

Key. A

Sol. In radius of $\Delta DEF = \frac{\Delta_{DEF}}{S_{DEF}}$

$$= \frac{(2\cos A\cos B\cos C)\frac{1}{2}ab\sin C}{\left(\frac{a\cos A + b\cos B + c\cos C}{2}\right)}$$

= 2RcosAcosBcosC= acotA cosB cosC.

36. Value of $\frac{a}{p}+\frac{b}{q}+\frac{c}{r}$ is equal to (where a, b, c are length of sides BC, CA and AB)

(A) $\frac{abc}{pqr}$ (B) $\frac{p+q+r}{a+b+c}$

(C) $\frac{p+q+r}{a+b-c}$ (D) $\frac{a+b+c}{p+q+r}$

Key. A

Sol. $\frac{a}{p}+\frac{b}{q}+\frac{c}{r} = \frac{2R\sin A}{2R\cos A}+\frac{2R\sin B}{2R\cos B}+\frac{2R\sin C}{2R\cos C}$

=tanA + tanB + tanC = tanA tanB tanC = $\frac{abc}{pqr}$

Paragraph – 14

Let ABC be a triangle, from vertices A, B and C altitudes AD, BE and CF are drawn to opposite sides BC, CA and AB respectively which meet at a point O. Now triangle DEF is completed, also length OA, OB and OC respectively are p, q and r.

37. Ratio of area of ΔDEF to ΔABC is

(A) 2 cosA cosB cosC (B) 2 sinA sinB sinC

(C) 2 cosA cosB sinC (D) 2 sinA cosB cosC

Key. A

38. Radius of In circle of ΔDEF is

(A) a cotA cosB cosC (B) $\frac{a}{2}$ cotA cosB cosC

(C) $\frac{abc\cos A\cos B\cos C}{\Delta ABC}$ (D) none of these

Key. A

39. Value of $\frac{a}{p}+\frac{b}{q}+\frac{c}{r}$ is equal to (where a, b, c are length of sides BC, CA and AB)

(A) $\frac{abc}{pqr}$ (B) $\frac{p+q+r}{a+b+c}$

(C) $\frac{p+q+r}{a+b-c}$ (D) $\frac{a+b+c}{p+q+r}$

Key. A

Sol. 37. $\Delta DEF=\frac{1}{2}a\cos Ab\cos B\sin(\pi-2c)$

$=ab\cos A\cos B\sin C\cos C$

$\frac{\Delta DEF}{\Delta ABC}=2\cos A\cos B\cos C$

38. In radius of $\Delta DEF = \frac{\Delta_{DEF}}{S_{DEF}}$

$$=\frac{(2\cos A\cos B\cos C)\frac{1}{2}ab\sin C}{\left(\frac{a\cos A+b\cos B+c\cos C}{2}\right)}$$

= 2RcosAcosBcosC= acotA cosB cosC.

39. $\frac{a}{p}+\frac{b}{q}+\frac{c}{r}=\frac{2R\sin A}{2R\cos A}+\frac{2R\sin B}{2R\cos B}+\frac{2R\sin C}{2R\cos C}$

=tanA + tanB + tanC = tanA tanB tanC = $\frac{abc}{pqr}$

Paragraph – 15

Let a,b,c are the sides opposite to angle A,B,C respectively in a ΔABC

$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}$ and $\frac{a}{\sin A}=\frac{b}{\sin B}$. If a = 6, b = 3 and $\cos(A-B)=\frac{4}{5}$

40. Angle C is equal to

A) $\frac{\pi}{4}$ B) $\frac{\pi}{2}$ C) $\frac{3\pi}{4}$ D) $\frac{2\pi}{3}$

Key. B

41. Area of the triangle is equal to

A) 8 B) 9 C) 10 D) 11

Key. B

42. Value of sin A is equal to

A) $\frac{1}{\sqrt{5}}$ B) $\frac{2}{\sqrt{5}}$ C) $\frac{1}{2\sqrt{5}}$ D) $\frac{1}{\sqrt{3}}$

Key. B

Sol. (40-42)

$\cos(A-B)=\frac{4}{5} \Rightarrow \frac{1-\tan^2\frac{A-B}{2}}{1+\tan^2\frac{A-B}{2}}=\frac{4}{5} \Rightarrow \frac{2\tan^2\frac{A-B}{2}}{2}=\frac{1}{9}$

$\Rightarrow \tan\frac{A-B}{2}=\frac{1}{3} \Rightarrow \frac{1}{3}=\frac{6-3}{6+3}\cot\frac{C}{2} \Rightarrow \cot\frac{C}{2}=1 \Rightarrow C=90^\circ$

Area of triangle $=\frac{1}{2}ab\sin C \Rightarrow$ Area $=\frac{1}{2}\times 6\times 3\times 1=9$

$\frac{a}{\sin A}=\frac{\sqrt{a^2+b^2}}{1} \Rightarrow \frac{6}{\sin A}=\sqrt{45} \Rightarrow \sin A=\frac{2}{\sqrt{5}}$

Paragraph – 16

Consider a triangle ABC, where x,y,z are the length of perpendicular drawn from the vertices of the triangle to the opposite sides a,b,c respectively let the letters R, r, S, Δ denote the circumradius, inradius semiperimeter and area of the triangle respectively

43. If $\frac{bx}{c}+\frac{cy}{a}+\frac{az}{b}=\frac{a^2+b^2+c^2}{k}$, then the value of k is

A) R B) S C) 2R D) $\frac{3}{2}R$

Key. C

Sol. $\frac{bc}{c}+\frac{cy}{a}+\frac{az}{b}=b\sin B+c\sin C+a\sin A=\frac{b^2+c^2+a^2}{2R}$

$\therefore \quad k=2R$

44. If $\cot A+\cot B+\cot C=k\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)$, then the value of k is

A) R^2 B) rR C) Δ D) $a^2+b^2+c^2$

Key. C

Sol. $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{a^2}{4\Delta^2}+\frac{b^2}{4\Delta^2}=\frac{c^2}{4\Delta^2}=\frac{a^2+b^2+c^2}{4\Delta^2}$

$\cot A+\cot B+\cot C=\frac{R}{abc}\left(b^2+c^2-a^2+c^2+a^2-b^2+a^2+b^2-c^2\right)$

$=\frac{R}{abc}\left(b^2+c^2+a^2\right)=\frac{R}{abc}\left(\frac{4\Delta^2}{x^2}+\frac{4\Delta^2}{y^2}+\frac{4\Delta^2}{z^2}\right)$

$=\frac{4\Delta^2 R}{abc}\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)$

$=\frac{4\Delta^2 R}{abc}.\,\Delta\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)=\Delta\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)$

$\therefore \quad k=\Delta$

45. The value of $\frac{c\sin B+b\sin C}{x}+\frac{a\sin C+c\sin A}{y}+\frac{b\sin A+a\sin B}{z}$ is equal to

A) $\frac{R}{r}$ B) $\frac{S}{R}$ C) 2 D) 6

Key. D

Sol. $=\sum\frac{c\sin B+b\sin C}{x},\ \sum\frac{x+x}{x}=6$

Progressions of Triangles

Integer Answer Type

1. If a, b, c are sides of a triangle satisfying $a^2+b^2+c^2=6$, then the AM of all the integral values which lie in the interval of ab + bc + ca is

Key. 5

Sol. $\frac{1}{2}\left\{(a-b)^2+(b-c)^2+(c-a)^2\right\}\ge 0 \Rightarrow a^2+b^2+c^2\ge ab+bc+ca \Rightarrow a^2+b^2+c^2\ge 6$

$a^2=b^2+c^2-2b\cos A>b^2+c^2-2bc\,Q\cos A<1$

$b^2>a^2+c^2-2ac$ and $c^2>a^2+b^2-2ab$

$\therefore a^2+b^2+c^2>2\left(a^2+b^2+c^2-ab-bc-ca\right)$

$\Rightarrow ab+bc+ca>\frac{1}{2}\left(a^2+b^2+c^2\right)=3.$ hence $ab+bc+ca\in(3,6]$

2. Let the lengths of the altitudes drawn from the veritices of a triangle ABC to the opposite sides are 2,2 and 3. if the area of ΔABC is Δ, then find the value of $2\sqrt{2}\Delta$

Key. 9

Sol. $\frac{2\Delta}{a}=2, \frac{2\Delta}{b}=2, \frac{2\Delta}{C}=3$

$\Rightarrow a=\Delta,\ b=\Delta,\ c=\frac{2\Delta}{3}$

$\therefore \Delta^2=\left(\frac{4\Delta}{3}\right)\left(\frac{\Delta}{3}\right)\left(\frac{\Delta}{3}\right)\left(\frac{2\Delta}{3}\right)$

$\Rightarrow 8\Delta^2=81$

$\Rightarrow 2\sqrt{2}\Delta=9$

3. If r and R are respectively the radii of the inscribed and circumscribed circles of a regular polygon of n sides such that $\frac{R}{r}=\sqrt{5}-1$, then n is equal to

Key. 5

Sol. $r=\frac{a}{2}\cot\frac{\pi}{n};\ R=\frac{a}{2}\operatorname{cosec}\frac{\pi}{n}$

$\therefore \frac{R}{r}=\sec\frac{\pi}{n}=\sqrt{5}-1=\sec 36^0$

$\therefore \frac{\pi}{n}=\frac{\pi}{5}\Rightarrow n=5$

4. In a $\Delta ABC, a=5, b=4$ and $\tan\frac{C}{2}=\sqrt{\frac{7}{9}}$, then measure of side 'c' is

Key. 6

Sol. $\cos C=\dfrac{1-\frac{7}{9}}{1+\frac{7}{9}}=\dfrac{1}{8}$

$\therefore c^2=25+16-2.5.4.\frac{1}{8}=36$

5. In triangle ABC $a=\sqrt{5};\ b=2;\angle A=\pi/6$ and $c_1 and\, c_2$ are the two possible values of third side then $|c_1-c_2|$ is

Key. 4

Sol. $a^2=b^2+c^2-2ba\cos A$

$c^2-2.2\frac{\sqrt{3}}{2}.c+4-5=0$

$c^2-2\sqrt{3}.C+-1=0$

$c_1+c_2=2\sqrt{3};c_1c_2=-1$

$(c_1-c_2)^2=16$

6. The ratios of the lengths of the sides BC and AC of a triangle ABC to the radius of a circumscribed circle are equal to 2 and 3/2 respectively. If the ratio of the lengths of the bisectors of the interior angles B and C is $\alpha\left(\frac{\sqrt{\alpha}-1}{9\sqrt{\beta}}\right)$ $then\ \alpha+\beta\ is$

Key. 9

Sol. Given $\frac{a}{R}=2$; $\frac{b}{R}=\frac{3}{2}$

$\frac{a}{2}=\frac{b}{3/2}=R$

$\frac{BE}{CF}=\dfrac{\frac{2ac}{a+c}\cos\frac{B}{2}}{\frac{2ab}{a+b}\cos\frac{c}{2}}$

$=\frac{a+b}{a+c}.\dfrac{\cos\frac{B}{2}}{\cos\frac{C}{2}}.\frac{c}{b}$ ---------------------(1)

$\because a=2R$ use here

$a^2=b^2+c^2\Rightarrow c=\frac{\sqrt{7}R}{2}$

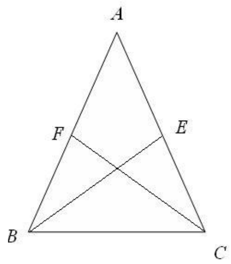
$\sin B=\frac{3}{4};\sin=\frac{\sqrt{7}}{4}$

$\cos^2\frac{B}{2}=\frac{4+\sqrt{7}}{2}$;

$$\cos^2\frac{B}{2}=\frac{4+\sqrt{7}}{2};\cos^2\frac{c}{2}=\frac{7}{8}$$

Now from -----------------------------(1)

$$\frac{BE}{CF}=\frac{7\left(\sqrt{7}-1\right)}{9\sqrt{2}}$$



7. In $\Delta ABC.\ 3a=b+c$ then cot B/2 cot C/2 is

Key. 2

Sol.
$$3a=b+c$$
$$4a=2s$$
$$s=2a$$
$$\cot\frac{B}{2}\cot\frac{C}{2}=\frac{s}{s-a}=\frac{2a}{2a-a}=2$$

8. In Δ ABC, if $R(a+b)=c\sqrt{ab}$ and a = 2+$\sqrt{2}$, then in radius r is

Key. 1

Sol.
$$\frac{a+b}{\sqrt{ab}}=\frac{C}{R}\Rightarrow\frac{a+b}{2\sqrt{ab}}=\frac{C}{2R}=\sin C$$
$$\cos^2 C=1-\frac{(a+b)^2}{4ab}=\frac{-(a-b)^2}{4ab}$$

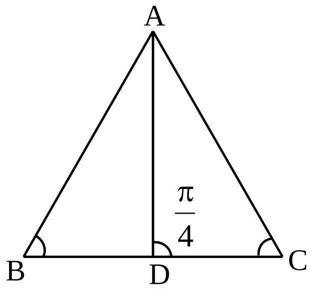
But $\cos^2 C\ge 0\Rightarrow a=b\Rightarrow c=\frac{\pi}{2}$

$$r=\frac{\Delta}{S}=\frac{\frac{1}{2}ab}{\frac{a+b+c}{2}}=\frac{a^2}{2a+\sqrt{2}a}=\frac{a}{2+\sqrt{2}}=\frac{2+\sqrt{2}}{2+\sqrt{2}}=1$$

9. If the median AD of triangle ABC makes an angle $\frac{\pi}{4}$ with the side BC, then the value of $\left|\cot B-\cot C\right|$ is

Key. 2

Sol.



From m – n theorem

$$2\cot\frac{\pi}{4} = |\cot C|\cot B \Rightarrow |\cot B - \cot C| = 2$$

10. If $A = 30^0, a = 7$ and b = 8 in ΔABC, then the number of triangles that can be constructed is

Key. 2

Sol. $\frac{7}{1/2} = \frac{8}{\sin B} \Rightarrow \sin B = \frac{8}{14} = \frac{4}{7}$

$\Rightarrow$ B can take two different values.

$\therefore$ No.of triangles = 2

11. In ΔABC if cosA + 2 cosB + cosC = 2, then the value of $\frac{2s}{b}$ (where 's' is the semi peri meter) is

Key. 3

Sol. $\cos A + \cos C = 2(1 - \cos B)$

$$\Rightarrow 2\cos\frac{A+C}{2}\cos\frac{A-C}{2} = 2\times 2\sin^2\frac{B}{2}$$

$$\Rightarrow \frac{\cos\left(\frac{A-C}{2}\right)}{\sin\frac{B}{2}} = 2 \Rightarrow \frac{a+c}{b} = 2 \Rightarrow a + c = 2b$$

$$\therefore \frac{2S}{b} = \frac{3b}{b} = 3$$

12. In ΔABC, if $r_1 = 6$, R = 5, r = 2, then the value of 3tanA is

Key. 4

Sol. $r_1 - r = 4R\sin^2\frac{A}{2} \Rightarrow 4 = 20\sin^2\frac{A}{2} \Rightarrow \sin\frac{A}{2} = \frac{1}{\sqrt{5}}$

$$\tan A = \frac{2\times\frac{1}{2}}{1-\frac{1}{4}} = \frac{4}{3} \Rightarrow 3\tan A = 4$$

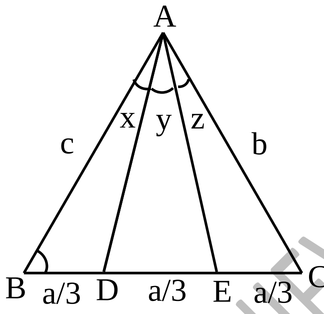
13. In ΔABC if $\angle C = 90^0$, then the value of $\frac{a+b}{r+R}$ is

Key. 2

Sol. $\angle c = 90^0 \Rightarrow 2(r+R) = a+b \Rightarrow \frac{a+b}{r+R} = 2$

14. Points D, E are taken on the side BC of an acute angled ΔABC, such that BD=DE=EC. If $\angle BAD = x,\ \angle DAE = y,\ \angle EAC = z$, then the value of $\frac{\sin(x+y).\sin(y+z)}{\sin x.\sin z}$ is

Key. 4



Sol.

$$\frac{a}{3\sin x} = \frac{AD}{\sin B} \quad \text{--------- (1)}$$

$$\frac{2a}{3\sin(x+y)} = \frac{AE}{\sin B} \quad \text{--------- (2)}$$

$$\frac{(1)}{(2)} \Rightarrow \frac{\sin(x+y)}{2\sin x} = \frac{AD}{AE} \quad \text{--------------- (3)}$$

$$\frac{2a}{3\sin(y+z)} = \frac{AD}{\sin C} \quad \text{----------- (4)}$$

$$\frac{a}{3\sin z} = \frac{AE}{\sin C} \quad \text{----------- (5)}$$

$$\frac{(5)}{(4)} \Rightarrow \frac{\sin(y+z)}{2\sin z} = \frac{AE}{AD} \quad \text{--------------- (6)}$$

$$\therefore\quad (3)\times(6) \Rightarrow \frac{\sin(x+y)\sin(y+z)}{\sin x.\sin z} = 4$$

15. If the circumcentre of triangle ABC lies on its incircle, then $\left[4\left(\cos A+\cos B+\cos C\right)\right]$

(Where $\left[x\right]$ is greatest integer less than or equal to x is)

Key. 5

Sol. $SI = r \Rightarrow R^2 - 2Rr - r^2 = 0$

$\Rightarrow \left(\frac{R}{r}\right)^2 - 2\left(\frac{R}{r}\right)^2 - 1 = 0$

$\Rightarrow \frac{R}{r} = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$

$\therefore \frac{R}{r} = \sqrt{2} + 1$

$\therefore \cos A + \cos B + \cos C = 1 + \frac{r}{R} = 1 + \sqrt{2} - 1 = \sqrt{2}$

$\therefore \left[4\left(\cos A + \cos B + \cos C\right)\right] = \left[4\sqrt{2}\right] = 5$

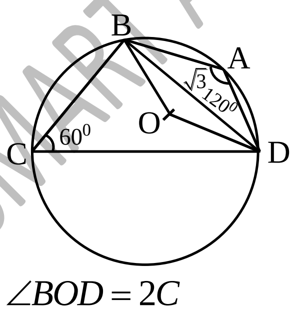
16. The area of a cyclic Quadrilateral ABCD is $\frac{3\sqrt{3}}{4}$. The radius of the circle circumscribing the Quadrilateral is 1. If AB = 1, BD = $\sqrt{3}$ then the value of 3.BC.CD is

Key. 6

Sol. $3 = 1 + AD^2 - 2 \times 1 \times AD \times \frac{-1}{2}$

$AD^2 + AD - 2 = 0$

$\left(AD + 2\right)\left(AD - 1\right) = 0 \Rightarrow AD = 1$



$\angle BOD = 2C$

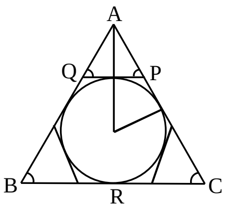
$\cos 2C = \frac{-1}{2} \Rightarrow C = 60^0,\ A = 120^0$

$\therefore \frac{1}{2} \times 1 \times 1 \times \frac{\sqrt{3}}{2} + \frac{1}{2} \times BC.CD \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2} B$

$BC.CD = 2$

17. The lengths of the tangents drawn from the vertices A, B, C to the incircle of ΔABC are 5, 3, 2 respectively. If the lengths of the parts of tangents with in the triangle which are drawn parallel to the sides BC, CA, AB of the triangle to the incircle be α, β, γ respectively, then $[\alpha+\beta+\gamma]$ where ($[g]$ is G.I.F. is)

Key. 6

Sol. 

$$r = (S-a)\tan\frac{A}{2}$$

$$\frac{r}{S-a} = \frac{r}{AP} \Rightarrow S-a=5,\ S-b=3,\ S-c=2$$

$$\Rightarrow S=10 \Rightarrow a=5,\ b=3,\ c=8$$

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \Rightarrow \frac{\alpha}{5}+\frac{\beta}{7}+\frac{\gamma}{8}=1$$

$$\alpha = \frac{ra}{S\tan\frac{A}{2}},\ \beta = \frac{rb}{S\tan\frac{B}{2}},\ \gamma = \frac{rc}{S\tan\frac{C}{2}}$$

$$\alpha = \frac{(S-a)a}{S},\ \beta = \frac{(S-b)b}{S},\ \gamma = \frac{(S-c)c}{S}$$

$$\alpha = \frac{5\times5}{10},\ \beta = \frac{3\times7}{10},\ \gamma = \frac{2\times8}{10}$$

$$\alpha+\beta+\gamma = \frac{25+21+16}{10} = \frac{62}{10} \Rightarrow [\alpha+\beta+\gamma]=6$$

18. In ΔABC, $\frac{r}{r_1}=\frac{1}{2}$, then the value of $4\tan\left(\frac{A}{2}\right)\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)$ must be

Key. 2

Sol. $\frac{r}{r_1} = \tan\frac{B}{2}\tan\frac{C}{2} = \frac{1}{2}$

$$\tan\frac{A}{2}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right) = 1-\tan\frac{B}{2}\tan\frac{C}{2} = \frac{1}{2}$$

$$\therefore\ 4\tan\frac{A}{2}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right) = 2$$

19. If a, b, c are sides of a triangle satisfying $a^2+b^2+c^2=6$, then the AM of all the integral values which lie in the interval of ab + bc + ca is

Key. 5

Sol.

$$\frac{1}{2}\left\{(a-b)^2+(b-c)^2+(c-a)^2\right\}\ge 0 \Rightarrow a^2+b^2+c^2 \ge ab+bc+ca \Rightarrow a^2+b^2+c^2 \ge 6$$

$$a^2=b^2+c^2-2b\cos A > b^2+c^2-2bc \because \cos A<1$$

$$b^2>a^2+c^2-2ac \text{ and } c^2>a^2+b^2-2ab$$

$$\therefore a^2+b^2+c^2>2\left(a^2+b^2+c^2-ab-bc-ca\right)$$

$$\Rightarrow ab+bc+ca>\frac{1}{2}\left(a^2+b^2+c^2\right)=3. \text{ hence } ab+bc+ca\in(3,6]$$

20. In ΔABC, $\frac{r}{r_1}=\frac{1}{2}$, then the value of $4\tan\left(\frac{A}{2}\right)\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)$ must be

Key. 2

Sol. $\frac{r}{r_1}=\tan\frac{B}{2}\tan\frac{C}{2}=\frac{1}{2}$

$$\tan\frac{A}{2}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)=1-\tan\frac{B}{2}\tan\frac{C}{2}=\frac{1}{2}$$

$$\therefore\ 4\tan\frac{A}{2}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)=2$$

21. With usual notation in triangle ABC, the numerical value of $\left(\frac{a+b+c}{r_1+r_2+r_3}\right)\left(\frac{a}{r_1}+\frac{b}{r_2}+\frac{c}{r_3}\right)$ is

ANS : 4

Sol. $$\sum\frac{a}{r_1}=2R\sum\frac{2\sin A/2\cos A/2}{4R\sin A/2\cos B/2\cos C/2}=\sum\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)$$

$$=2\sum\tan\frac{A}{2}=2\sum\frac{r_1}{s}=4\left(\frac{r_1+r_2+r_3}{a+b+c}\right)$$

22. In $\Delta ABC\,\frac{(r_1+r_2)(r_2+r_3)(r_3+r_1)}{Rs^2}=$______(where r_1,r_2,r_3 are exradii & R is circum radius and

s is Semiperimeter of triangle ABC)

KEY : 4

23. If in a triangle ABC, $b\cos^2\frac{A}{2}+a\cos^2\frac{B}{2}=\frac{3c}{2}$ *then* minimum value of $\frac{a+c}{2c-a}+\frac{b+c}{2c-b}$ is equal to …………

Key: 4

Hint:

$$LHS=\frac{1}{2}\left[b+b\cos A+a+a\cos B\right]=\frac{1}{2}\left(a+b+c\right)=\frac{3c}{2}\Rightarrow 2c=a+b$$

$$\frac{a+c}{2c-a}+\frac{b+c}{2c-b}=\frac{a+c}{b}+\frac{b+c}{a}=\frac{a}{b}+\frac{b}{a}+\frac{c}{b}+\frac{c}{a}\geq 4\left(\frac{c^2}{ab}\right)^{1/4}\geq 4$$

$$\left(\because \frac{a+b}{2}\geq\sqrt{ab}\Rightarrow c^2\geq ab\Rightarrow\frac{c^2}{ab}\geq 1\right)$$

24. If length of the side BC of a ΔABC is 4cm and $\angle BAC=120^0$, then the distance between incentre & excentre of the circle touching the side BC internally is

Key: 8

Hint: $II_1=AI_1-AI$

$$=\left(r_1-r\right)\cos cA/2$$

$$=a\tan A/2\cos cA/2$$

$$=\frac{a}{\cos A/2}=8$$

25. In ΔABC, $\frac{r}{r_1}=\frac{1}{2}$, then the value of $4\tan\left(\frac{A}{2}\right)\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)$ must be

Key. 2

Sol. $\frac{r}{r_1}=\tan\frac{B}{2}\tan\frac{C}{2}=\frac{1}{2}$

$$\tan\frac{A}{2}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)=1-\tan\frac{B}{2}\tan\frac{C}{2}=\frac{1}{2}$$

$$\therefore\ 4\tan\frac{A}{2}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)=2$$

26. Given a parallelogram whose acute angle is , if the squares of length of the diagonals are in ratio 1 : 3 tnen $\frac{a}{b}$ is (where a,b are the length of the sides)

____

Sol. 1

Applying cosines rule in ΔABD and ΔBCD, we get $2a^2+2b^2=d_1^2+d_2^2$, where d_1,d_2 are length of diagonals $\left(d_1<d_2\right)$ given $d_2^2=3d_1^2$

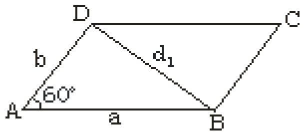
$\Rightarrow a^2+b^2=2d_1^2$

also ab = d_1^2

$\Rightarrow a^2+b^2+2ab=3ab+d_1^2=4d_1^2$ ...(1)

$\Rightarrow\left(a+b\right)^2=4d_1^2\Rightarrow a+b=2d_1$ ...(2)

using (1) and (2), we get a = b = d_1 i.e. $\frac{a}{b}=1$.



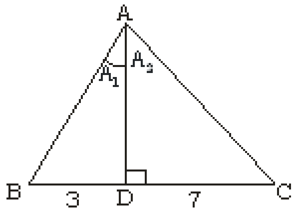
27. In a triangle ABC, the foot of the perpendicular from A divides the opposite side into parts of lengths 3 and 17 and $\tan A = \frac{22}{7}$. Let a ΔPQR is a right angle triangle (right angle at Q) such that $\angle A = \angle P$ and PQ = 7 units, the $\left[\frac{\text{Area }(ABC)}{\text{Area }(PQR)}\right]$ is __________ (where [.] denotes the greatest integer function)

Sol. 1

tan A = tan(A_1 + A_2)

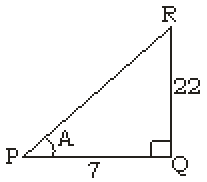
$\Rightarrow AD = 11$

hence the area of $\Delta ABC = \frac{1}{2} \times 11 \times 20 = 110$



$\Rightarrow$ Area of PQR $= \frac{1}{2} \times 22 \times 7 = 77$.

$\left[\frac{\text{Area }(ABC)}{\text{Area }(PQR)}\right] = 1$.



28. ABC is an acute angle triangle, $\angle A = 30°$.H is the orthocentre and M is the midpoint of BC. On the line HM a point T is taken such that HM = MT. If BC = 4cm, then the length of AT is ____________

Key. 8

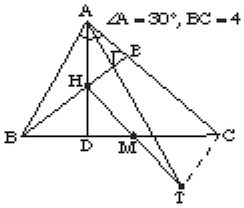
Sol. ΔBMH and ΔCMT are similar, $\angle MBH = \angle MCT$,

$\Rightarrow$ BH CT, CT $\perp$ AC and CT = BH

Now, $AT^2 = TC^2 + AC^2 = BH^2 + AC^2$

$= BD^2 \operatorname{cosec}^2 C + AD^2 \operatorname{cosec}^2 C$

$= \frac{AB^2}{\sin^2 C} = \frac{a^2}{\sin^2 A} = 16 \times 4 = 64 \Rightarrow AT = 8$



29. If I be the incentre of triangle ABC and R_1, R_2, R_3 be the circum radius of the triangle BIC, CIA, AIB respectively, then maximum value of $\frac{a^2}{R_1^2}+\frac{b^2}{R_2^2}+\frac{c^2}{R_3^2}$ is .....

Key. 9

30. The radii r_1,r_2,r_3 of inscribed circles of the triangle ABC are in H.P. If its area is 24 sq. cm and its perimeter is 24 cm, then the length of its largest side is

Key. 10

31. If in ΔABC, circle with altitude AD as diameter intersect AB at P and AC at Q such that $PQ = \lambda \frac{\Delta}{R}$, when Δ, R are area and circumradius of triangle ABC respectively, then λ is equal to

Key. 1

32. In a triangle ABC, CH and CM are the lengths of the altitude and median to the base AB. If a = 10, b = 26, c = 32 then length (HM) ?

Key. 9

33. In a ΔABC, perpendiculars are drawn from the angles A,B,C of an acute angled triangle on the opposite sides and produced to meet the circumscribing circle at D, E and F respectively. If these produced parts be α, β, γ respectively.

Then the value of $\frac{\left(\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}\right)}{\tan A+\tan B+\tan C}$ is ....

Key. 2

34. In ΔABC, if $R(a+b)=c\sqrt{ab}$ and a = 2+$\sqrt{2}$, then in radius r is

Key. 1

Sol. $\frac{a+b}{\sqrt{ab}}=\frac{C}{R}\Rightarrow\frac{a+b}{2\sqrt{ab}}=\frac{C}{2R}=\sin C$

$$\cos^2 C = 1-\frac{(a+b)^2}{4ab}=\frac{-(a-b)^2}{4ab}$$

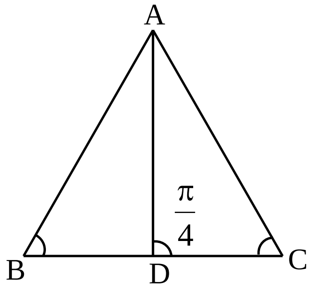
But $\cos^2 C \geq 0 \Rightarrow a=b \Rightarrow c=\frac{\pi}{2}$

$$r=\frac{\Delta}{S}=\frac{\frac{1}{2}ab}{\frac{a+b+c}{2}}=\frac{a^2}{2a+\sqrt{2}a}=\frac{a}{2+\sqrt{2}}=\frac{2+\sqrt{2}}{2+\sqrt{2}}=1$$

35. If the median AD of triangle ABC makes an angle $\frac{\pi}{4}$ with the side BC, then the value of $\left|\cot B - \cot C\right|$ is

Key. 2

Sol.



From m – n theorem

$$2\cot\frac{\pi}{4} = \left|\cot C\right|\cot B \Rightarrow \left|\cot B - \cot C\right| = 2$$

36. If $A = 30^0, a = 7$ and b = 8 in ΔABC, then the number of triangles that can be constructed is

Key. 2

Sol. $\frac{7}{1/2} = \frac{8}{\sin B} \Rightarrow \sin B = \frac{8}{14} = \frac{4}{7}$

$\Rightarrow$ B can take two different values.

$\therefore$ No.of triangles = 2

37. In ΔABC if cosA + 2 cosB + cosC = 2, then the value of $\frac{2s}{b}$ (where ‘s’ is the semi peri meter) is

Key. 3

Sol. $\cos A + \cos C = 2(1 - \cos B)$

$$\Rightarrow 2\cos\frac{A+C}{2}\cos\frac{A-C}{2} = 2\times 2\sin^2\frac{B}{2}$$

$$\Rightarrow \frac{\cos\left(\frac{A-C}{2}\right)}{\sin\frac{B}{2}} = 2 \Rightarrow \frac{a+c}{b} = 2 \Rightarrow a + c = 2b$$

$$\therefore \frac{2S}{b} = \frac{3b}{b} = 3$$

38. In ΔABC, if r_1 = 6, R = 5, r = 2, then the value of 3tanA is

Key. 4

Sol. $r_1 - r = 4R\sin^2\frac{A}{2} \Rightarrow 4 = 20\sin^2\frac{A}{2} \Rightarrow \sin\frac{A}{2} = \frac{1}{\sqrt{5}}$

$$\tan A = \frac{2\times\frac{1}{2}}{1-\frac{1}{4}} = \frac{4}{3} \Rightarrow 3\tan A = 4$$

39. In ΔABC if $\angle C = 90^0$, then the value of $\frac{a+b}{r+R}$ is

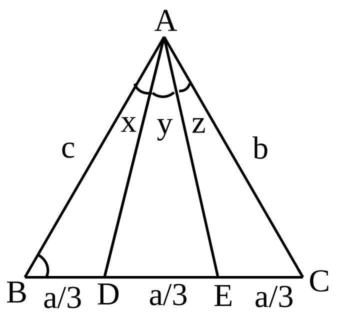
Key. 2

Sol. $\angle c = 90^0 \Rightarrow 2(r+R) = a+b \Rightarrow \frac{a+b}{r+R} = 2$

40. Points D, E are taken on the side BC of an acute angled ΔABC, such that BD=DE=EC. If $\angle BAD = x,\ \angle DAE = y,\ \angle EAC = z$, then the value of $\frac{\sin(x+y).\sin(y+z)}{\sin x.\sin z}$ is

Key. 4

Sol.



$\frac{a}{3\sin x} = \frac{AD}{\sin B}$ --------- (1)

$\frac{2a}{3\sin(x+y)} = \frac{AE}{\sin B}$ --------- (2)

$\frac{(1)}{(2)} \Rightarrow \frac{\sin(x+y)}{2\sin x} = \frac{AD}{AE}$ --------------- (3)

$\frac{2a}{3\sin(y+z)} = \frac{AD}{\sin C}$ ----------- (4)

$\frac{a}{3\sin z} = \frac{AE}{\sin C}$ ----------- (5)

$\frac{(5)}{(4)} \Rightarrow \frac{\sin(y+z)}{2\sin z} = \frac{AE}{AD}$ --------------- (6)

$\therefore\ (3)\times(6) \Rightarrow \frac{\sin(x+y)\sin(y+z)}{\sin x.\sin z} = 4$

41. If the circumcentre of triangle ABC lies on its incircle, then $\left[4(\cos A + \cos B + \cos C)\right]$

(Where $[x]$ is greatest integer less than or equal to x is)

Key. 5

Sol. $SI = r \Rightarrow R^2 - 2Rr - r^2 = 0$

$$\Rightarrow\left(\frac{R}{r}\right)^2-2\left(\frac{R}{r}\right)^2-1=0$$

$$\Rightarrow\frac{R}{r}=\frac{2\pm2\sqrt{2}}{2}=1\pm\sqrt{2}$$

$$\therefore\frac{R}{r}=\sqrt{2}+1$$

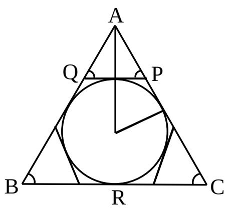
$$\therefore\cos A+\cos B+\cos C=1+\frac{r}{R}=1+\sqrt{2}-1=\sqrt{2}$$

$$\therefore\left[4\left(\cos A+\cos B+\cos C\right)\right]=\left[4\sqrt{2}\right]=5$$

42. The lengths of the tangents drawn from the vertices A, B, C to the incircle of ΔABC are 5, 3, 2 respectively. If the lengths of the parts of tangents with in the triangle which are drawn parallel to the sides BC, CA, AB of the triangle to the incircle be α,β,γ respectively, then $[\alpha+\beta+\gamma]$ where ($[g]$ is G.I.F. is)

Key. 6

Sol.



$$r=(S-a)\tan\frac{A}{2}$$

$$\frac{r}{S-a}=\frac{r}{AP}\Rightarrow S-a=5,\ S-b=3,\ S-c=2$$

$$\Rightarrow S=10\Rightarrow a=5,\ b=3,\ c=8$$

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1\Rightarrow\frac{\alpha}{5}+\frac{\beta}{7}+\frac{\gamma}{8}=1$$

$$\alpha=\frac{ra}{S\tan\frac{A}{2}},\ \beta=\frac{rb}{S\tan\frac{B}{2}},\ \gamma=\frac{rc}{S\tan\frac{C}{2}}$$

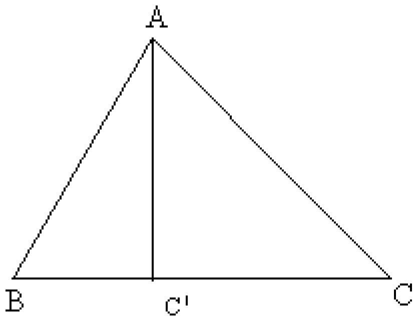
$$\alpha=\frac{(S-a)a}{S},\ \beta=\frac{(S-b)b}{S},\ \gamma=\frac{(S-c)c}{S}$$

$$\alpha=\frac{5\times5}{10},\ \beta=\frac{3\times7}{10},\ \gamma=\frac{2\times8}{10}$$

$$\alpha+\beta+\gamma=\frac{25+21+16}{10}=\frac{62}{10}\Rightarrow[\alpha+\beta+\gamma]=6$$

43. Let ABC and ABC^1 be two non congruent triangles with sides $AB = 4, AC = AC^1 = 2\sqrt{2}$ and angle $B = 30^0$, the absolute value of the difference between the area of these triangles is

KEY. 4



SOL.

$$AB = 4, AC = AC^1 = 2\sqrt{2}, B = 30^0$$

$$C = 4, b = b^1 = 2\sqrt{2}, B = 30^0$$

$$\frac{b}{\sin 30} = \frac{c}{\sin C} \Rightarrow C = 45^0$$

$$\underline{|CAC^1} = 90^0$$

$$Area\ of\ \Delta ABC - \Delta ABC^1 = \Delta ACC^1$$

$$= \frac{1}{2} AC.AC^1$$

$$= \frac{1}{2} 2\sqrt{2}.2\sqrt{2} = 4$$

44. Consider a triangle ABC and let a,b and c denote the lengths of the sides opposite to the vertices A,B and C respectively .Suppose a = 6, b = 10 and the area of the triangle is $15\sqrt{3}$. If $\angle ACB$ is obtuse and if 'r' denote the radius of the incirlce of the triangle then $r^2 =$

KEY. 3

SOL. $\Delta = \frac{1}{2} ab \sin C \Rightarrow \sin C = \frac{\sqrt{3}}{2} \Rightarrow C = 120^0$

$$c^2 = a^2 + b^2 - 2ab \cos C \Rightarrow c = 14$$

$$r = \frac{\Delta}{s} = \sqrt{3} \Rightarrow r^2 = 3$$

45. If $B=60^0$, $C=45^0$ and D divides BC internally in the ratio 1 : 3 and $\frac{\sin\angle CAD}{\sin\angle BAD}=\lambda$, then $\lambda^3+\lambda^2-6\lambda+3=$

KEY. 9

SOL. By sine Rule $\frac{\sin\angle CAD}{\sin\angle BAD}=\sqrt{6}=\lambda$

$$\lambda^3+\lambda^2-6\lambda+3=6\sqrt{6}+6-6\sqrt{6}+3=9$$

46. If p_1, p_2 and p_3 are the altitudes of a triangle from vertices A,B and C respectively, and Δ is the area of the triangle, prove that $\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_3}=\frac{2ab}{(a+b+c)\Delta}\cos^2\frac{C}{2}$

Ans. $\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_3}=\frac{a+b-c}{2\Delta}=\frac{2s-2c}{2\Delta}=\frac{s-c}{\Delta}$

Sol. We have $\cos^2\frac{C}{2}=\frac{s(s-c)}{ab}$

so that $\frac{2ab}{(a+b+c)\Delta}\cos^2\frac{C}{2}=\frac{2ab}{2s\Delta}.\frac{s(s-c)}{ab}=\frac{s-c}{\Delta}$

Now, the area of triangle ABC is $\Delta=\frac{1}{2}ap_1$, i.e., $p_1=2\Delta/a$. Similarly, $p_2=2\Delta/b$ and $p_3=2\Delta/c$.

$$\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_3}=\frac{a+b-c}{2\Delta}=\frac{2s-2c}{2\Delta}=\frac{s-c}{\Delta}$$

47. In a ΔABC, the angles A,B,C are in A.P. Show that $2\cos\frac{A-C}{2}=\frac{a+c}{\sqrt{a^2-ac+c^2}}$

Ans. $2\cos\frac{A-C}{2}$

Sol.

$$\frac{a+c}{\sqrt{a^2-ac+c^2}}=\frac{\sin A+\sin C}{\sqrt{\sin^2 A-\sin A\sin C+\sin^2 C}}=\frac{2\sin\frac{A+C}{2}\cos\frac{A-C}{2}}{\sqrt{\frac{1-\cos 2A}{2}-\frac{\cos(A-C)-\cos(A+C)}{2}+\frac{1-\cos 2C}{2}}}$$

$$=\frac{2\sqrt{2},\frac{\sqrt{3}}{2}\cos\frac{A-C}{2}}{\sqrt{2-(\cos 2A+\cos 2C)-\cos(A-C)+\cos(A+C)}}$$

$$=\frac{\sqrt{6}\cos\frac{A-C}{2}}{\sqrt{\frac{3}{2}-2\cos(A+C)\cos(A-C)-\cos(A-C)}}=\frac{\sqrt{6}\cos\frac{A-C}{2}}{\sqrt{\frac{3}{2}+\cos(A-C)-\cos(A-C)}}$$

$$=2\cos\frac{A-C}{2}$$

48. Let AD, BE, CF be the length of internal bisectors of angles A,B,C of triangle ABC. Show that the harmonic mean of AD $\sec\frac{A}{2}$. BE $\sec\frac{B}{2}$, $CF\sec\frac{C}{2}$ is the harmonic mean of the sides of the triangle

Key. $\Sigma\frac{1}{AD\sec\frac{A}{2}}=\Sigma\frac{1}{a}$

Sol. $AD=\frac{2bc}{b+c}\cos\frac{A}{2}$ $\Rightarrow$ $\frac{1}{AD\sec\frac{A}{2}}=\frac{1}{2}\left(\frac{1}{b}+\frac{1}{c}\right)$

$\therefore$ $\Sigma\frac{1}{AD\sec\frac{A}{2}}=\Sigma\frac{1}{a}$

49. Let ABC be a triangle with altitudes h_1,h_2,h_3 and inradius r. Prove that $\frac{h_1+r}{h_1-r}+\frac{h_2+r}{h_2-r}+\frac{h_3+r}{h_3-r}\ge 6$

Ans.

Sol. $\frac{h_1+r}{h_1-r}+\frac{h_2+r}{h_2-r}+\frac{h_3+r}{h_3-r}\ge 6$

$\Delta=\frac{1}{2}ah$ $\Rightarrow$ $h_1=\frac{2\Delta}{a}$

Similarly $h_1=\frac{2\Delta}{b}$, $h_3=\frac{2\Delta}{c}$

So $\frac{h_1+r}{h_1-r}+\frac{h_2+r}{h_2-r}=\frac{h_3+r}{h_3-r}=\frac{2\Delta/a+\Delta/s}{2\Delta/a-\Delta/s}+\frac{2\Delta/b+\Delta/s}{2\Delta/c-\Delta/s}+\frac{2\Delta/c+\Delta/s}{2\Delta/c-\Delta/s}$

$=\frac{2s+a}{2s-a}=\frac{2s+b}{2s-b}+\frac{2s+c}{2s-c}=\frac{4s}{2s-a}+\frac{4s}{2s-b}+\frac{4s}{2s-c}-3$

$=3\left[\frac{1}{3}\left\{\frac{4s}{2s-a}+\frac{4s}{2s-b}+\frac{4s}{2s-c}\right\}\right]-3\ge 3\left(\frac{3}{\frac{2s-a}{4s}+\frac{2s-b}{4s}+\frac{2s-c}{4s}}\right]=3$ Since $(AM\ge HM)\ge 6$

50. Find the point inside a Δ from which the sum of the squares of distances to the three sides is minimum. Also find the minimum value of the sum of squares of distances.

Ans. $\frac{4(s-a)(s-b)(s-c)s}{a^2+b^2+c^2}$

Sol. If a,b, c are the lengths of the sides of the Δ and x,y,z are the length of perpendicular from the point on the sides BC, CA, AB respectively we have to minimize $x^2+y^2+z^2=t$

we have

$\frac{1}{2}ax+\frac{1}{2}by+\frac{1}{2}cz=\Delta$

$\Rightarrow$ ax + by + cz = 2Δ

Where Δ is the area of the ΔABC we have the identity;

$\Rightarrow$ $(x^2+y^2+z^2)(a^2+b^2+c^2)-(ax+bx+cz)^2=(ax-by)^2+(by-cz)^2+(cz+ax)^2$

$\Rightarrow$ $(x^2+y^2+z^2)(a^2+b^2+c^2)\ge(ax+by+cz)^2$

$\Rightarrow$ $(x^2+y^2+z^2)(a^2+b^2+c^2)\ge 4\Delta^2$

$\Rightarrow \qquad x^2 + y^2 + z^2 \geq \frac{4\Delta^2}{a^2 + b^2 + c^2}$ and equality only when

$$\frac{x}{a} - \frac{y}{b} - \frac{z}{c} = \frac{ax + by + cz}{a^2 + b^2 + c^2} = \frac{2\Delta}{a^2 + b^2 + c^2}$$

The minimum value of t is $\frac{4\Delta^2}{a^2 + b^2 + c^2}$

$$t \min = \frac{4(s-a)(s-b)(s-c)s}{a^2 + b^2 + c^2}$$ Ans.

Properties of Triangles

Matrix-Match Type

1. In ΔABC Column-I Column-II

a)	$If\ \frac{s-a}{11}=\frac{s-b}{12}=\frac{s-c}{13}$, then 33 $\tan^2 A/2$ is	p)	6
b)	$If\ a=5, b=4\ and\ \cos(A-B)=\frac{31}{32}, then\ \ c=$	q)	52
c)	In a ΔABC, line joining the circum centre orthocenter is parallel to side BC then $4\tan^2 B\tan^2 C=$	r)	26
d)	If the radius of the circum circle of a triangle is 12 and that of the in circle is 4 then the sum of the radii of the escribed circles is	s)	13
		t)	36

Key. A-P;B-P;C-S;D-Q

Sol. A). $s-a=11k$

$s-b=12k$

$s-c=12k$

$s=36k$

$\tan^2\frac{A}{2}=\frac{(s-b)(s-c)}{s(s-a)}=\frac{12\times 13}{36\times 11}=\frac{13}{33}$

B) $\frac{a-b}{a+b}\cot\frac{c}{2}=\tan\left(\frac{A-B}{2}\right)$

$\left(\frac{5-4}{5+4}\cot\frac{C}{2}\right)^2=\frac{1-\cos(A-B)}{1+\cos(A-B)}$

$\Rightarrow \frac{1}{81}\cot^2\frac{c}{2}=\frac{1}{63}$

$\Rightarrow \tan^2\frac{c}{2}=\frac{7}{9}$

$\therefore \cos c=\frac{1}{8}$

$c^2=a^2+b^2-2ab\cos c$

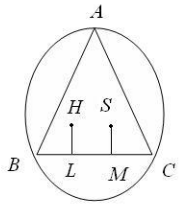
$=25+16+2.5.4.\frac{1}{8}=36$

C) $\tan B\tan C=3$

$(Q\ 2R\cos B\cos C=R\cos A$

$\Rightarrow 2\cos B\cos C=-\cos(B+C)$

$\Rightarrow \sin B\sin c=3\cos B\cos C$



D) $r_1 + r_2 + r_3 = r + 4R$

2. In ΔABC

Column-I Column-II

a)	If in a triangle ABC, $\frac{r}{r_1} = \frac{1}{4}$, Then the value of $\tan\frac{A}{2}\left(\tan\frac{B}{2} + \tan\frac{C}{2}\right)$ is equal to	p)	$\frac{2}{3}$
b)	In a triangle the least value of $\frac{r_1 r_2 r_3}{r^3}$ is	q)	$\frac{3}{4}$
c)	In any triangle ABC, $\frac{a^2+b^2+c^2}{3R^2}$ has the maximum value	r)	1
d)	Let P be an interior point of the triangle ABC and the line AP,BP and CP when Produced meet the opposite sides in D,E and F respectively then $\frac{PD}{AD} + \frac{PE}{BE} + \frac{PF}{CF}$ is equal to	s)	3
		t)	27

Key. A-Q;B-T;C-QR;D-R

Sol. A) $\frac{r}{r_1} = \tan\frac{B}{2}\tan\frac{C}{2} = \frac{1}{4}$

In a ΔABC $\sum \tan\frac{A}{2}\tan\frac{B}{2} = 1$

$\therefore \tan\frac{A}{2}\left(\tan\frac{B}{2} + \tan\frac{C}{2}\right) = 1 - \frac{1}{4} = \frac{3}{4}$

B) $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}$ apply $AM \geq GM$

C) $\frac{a^2+b^2+c^2}{3R^2} = \frac{4\left(\sin^2 A + \sin^2 + \sin^2 c\right)}{3} \leq \frac{4}{3}.\frac{9}{4} = 3$

D) $\frac{PD}{AD} + \frac{PE}{BE} + \frac{PF}{CF} = 1$

3. Match the following:

If in $\Delta ABC,\ r=2,\ r_1=4,\ s=12$ and $a<b<c,$ then

Column -I		Column -II	
(A)	Area of ΔABC is	(p)	22
(B)	$4+4R$ is	(q)	24
(C)	$\angle A$ of triangle is	(r)	$\sin^{-1}\frac{4}{5}$
(D)	$\angle B$ of triangle is	(s)	$\sin^{-1}\frac{3}{5}$

Key. (A) $\to$(q); (B) $\to$(q); (C) $\to$(s); (D) $\to$(r)

Sol. $r_1r_2+r_2r_3+r_3r_1=\frac{r_1r_2r_3}{r}=s^2=144$

$4(r_1+r_3)+r_2r_3=2r_2r_3=144 \Rightarrow r_2r_3=72,\ r_2+r_3=18 \Rightarrow r_2=6,\ r_3=12$

$\frac{r}{r_1}=\frac{s-a}{s}=\frac{1}{2} \Rightarrow a=6$

Similarly, $b=8, c=10 \Rightarrow ABC$ is right angle triangle.

Smallest angle is $\sin^{-1}\frac{3}{4}$, $\Delta=\frac{1}{2}\times 6\times 8=24$ sq.units $R=5$; $|r_2-r_3|=6$

4. Match the following:

Column -I | Column -II

(A) If $p^2-2p\cos x=673$ and $\tan\frac{x}{2}=7,$ the integral value of $'p'$ is — (p) 8

(B) If $\sin\theta+\cos\theta=m$ then the maximum value of m^2 is — (q) 9

(C) r_1,r_2,r_3 are the radii of the circles drawn on the altitudes PD, PE and PF of $\Delta PBC,\ \Delta PCA,\ \Delta PAB$ respectively as diameter where $'P'$ is the circum-centre of the acute angled $\Delta ABC.$ The minimum value of $\frac{1}{18}\left[\frac{a^2}{r_1^2}+\frac{b^2}{r_2^2}+\frac{c^2}{r_3^2}\right]$ is (a,b,c are sides of ΔABC) — (r) 2

(D) In a $\Delta ABC,\ a=6, b=3$ and $\cos(A-B)=\frac{4}{5}$ then the area of the ΔABC is — (s) 25

Key. (A) $\rightarrow$(s); (B) $\rightarrow$(r); (C) $\rightarrow$(p); (D) $\rightarrow$(q)

Sol. A) $\tan\frac{x}{2}=7 \Rightarrow \sin\frac{x}{2}=\pm\frac{7}{\sqrt{50}}$

$p^2-2p\cos x=673$

$(p-1)^2+2p(1-\cos x)=674$

$\Rightarrow (p-1)^2+4p\sin^2\frac{x}{2}=674$

$\Rightarrow (p-1)^2+\frac{98}{25}p=674$

Put $p=25$

$(24)^2+98=674$ which is true

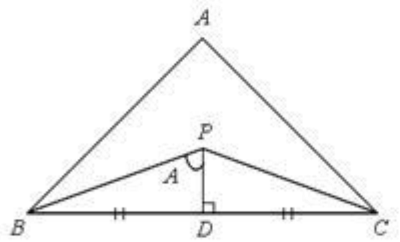
B) $\sin\theta+\cos\theta=m$

$\sqrt{2}\le\sin\theta+\cos\theta\le\sqrt{2}$, so the maximum value of m^2 is 2.

C) $\tan A=\frac{BD}{PD}=\frac{\frac{a}{2}}{2r_1}=\frac{a}{4r_1}$

Similarly, $\tan B=\frac{b}{4r_2}$

$\tan C=\frac{c}{4r_3}$



So, $\frac{a^2}{r_1^2}+\frac{b^2}{r_2^2}+\frac{c^2}{r_3^2}=16\left(\tan^2 A+\tan^2 B+\tan^2 C\right)$

But $\tan^2 A+\tan^2 B+\tan^2 C\ge 3\left[\tan A\tan B\tan C\right]^{\frac{2}{3}}$ (using $A.M.\ge G.M.$)

Also, in an acute angled ΔABC

$\tan A+\tan B+\tan C\ge 3\left(\tan A.\tan B.\tan C\right)^{\frac{1}{3}}$ (using $A.M.\ge G.M.$)

But $\tan A+\tan B+\tan C=\tan A\tan B\tan C$

So, $\left(\tan A\tan B\tan C\right)^{\frac{2}{3}}\ge 3$

Therefore, $\tan^2 A + \tan^2 B + \tan^2 C \geq 9$

So, $\frac{a^2}{r_1^2} + \frac{b^2}{r_2^2} + \frac{c^2}{r_3^2} \geq 144$

The minimum value of $\frac{1}{18}\left[\frac{a^2}{r_1^2} + \frac{b^2}{r_2^2} + \frac{c^2}{r_3^2}\right]$ is 8.

D) $\cos(A-B) = \frac{4}{5}$

$$\Rightarrow \frac{1-\tan^2\left(\frac{A-B}{2}\right)}{1+\tan^2\left(\frac{A-B}{2}\right)} = \frac{4}{5}$$

$\tan\left(\frac{A-B}{2}\right) = \pm\frac{1}{3}$

But $a > b \Rightarrow A > B$

So, $\tan\left(\frac{A-B}{2}\right) = \frac{1}{3}$

Again, $\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}.\cot\frac{C}{2}$

$$\Rightarrow \frac{1}{3} = \frac{3}{9}.\cot\frac{C}{2} \Rightarrow \cot\frac{C}{2} = 1 \Rightarrow \frac{C}{2} = 45^0 \Rightarrow C = 90^0$$

So, the area of $\Delta ABC = \frac{1}{2}ab\sin C = \frac{1}{2}\times 6\times 3\times 1$

5. Match the Following:

Column I		Column II	
(A)	Number of triangle to which an acute angle triangle ABC can act as a pedal triangle is	(p)	3
(B)	DEF is a pedal triangle of ABC, If R_1 *and* R_2 are circumradius of ΔDEF *and* ΔABC respectively then $\frac{R_2}{R_1}$ is	(q)	2
(C)	Three points D,E,F are taken on side BC, CA and AB such that AD, BE and CF are concurrent then $\frac{BD.CE.AF}{DC.AE.FB}$ is equal to	(r)	4
(D)	$\frac{3}{2}\left(1-\frac{r_1}{r_2}\right)\left(1-\frac{r_1}{r_3}\right)$ is (where r_1, r_2, r_3 are radii of	(s)	1

	excircles of ΔABC which is right angled at A)		
		(t)	5

KEY: (A-r), (B –q), (C -s), (D –p)

Hint: (A) ABC is pedal triangle of acute angle $I_1I_2I_3$ and of obtuse angled triangle II_1I_2, II_2I_3 *and* II_3I_1.

(B) $\frac{R_2}{R_1} = 2$

(C) In ΔABC If AD, BE and CF are concurred and if $\frac{BD}{DC} = \frac{\mu}{\beta}$ *and* $\frac{CE}{AB} = \frac{\beta}{\gamma}$

Then $\frac{AF}{FB} = \frac{\gamma}{\alpha}$

$$\Rightarrow BD.CE.AF = DC.AE.FB$$

(D) $\frac{3}{2}\left(1-\frac{s-b}{s-a}\right)\left(1-\frac{s-c}{s-a}\right)$

6. If in ΔABC, sin(A-B) = $\frac{3}{5}$, and $\frac{a}{b} = 2$

Column I		Column II	
(A)	$\tan\left(\frac{A-B}{2}\right)$	(p)	0
(B)	$\cot\frac{C}{2}$	(q)	1/ 3
(C)	Area of triangle ABC	(r)	1
(D)	r + c - s (r-In radius, s - semi perimeter)	(s)	b^2

Key: (A-q), (B-r), (C-s), (D-p)

Hint: $\sin\theta = \frac{2\tan\theta/2}{1+\tan^2\theta/2}$ $\Rightarrow \tan\left(\frac{A-B}{2}\right)$ = 3, $\frac{1}{3}$

by Napeir's rule

$\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\frac{C}{2}$

$\Rightarrow \cot\frac{C}{2} = 1$, 9

If $\cot\frac{C}{2} = 1 \Rightarrow$ area of Δ = b^2

r = (s - c) $\tan\frac{c}{2}$ $\Rightarrow$ r + c - s = 0

7. Match the following

Column – I	Column – II
a) If in a triangle ABC, $\sin^2 A + \sin^2 B = \sin(A+B)$ then the triangle must be	p) right angled
b) If in a triangle ABC, $\frac{bc}{2\cos A} = b^2 + c^2 - 2bc \ \cos A$ then the triangle must be	q) equilateral
c) If in a triangle ABC, $\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} = \sqrt{3}$ then the triangle must be	r) isosceles
d) If in a triangle the sides and the attitudes are in AP then the triangle must be	s) obtuse angled

Key. $a:p, b:r, c:q, d:q$

Sol. a) $\sin^2 A + \sin^2 B = \sin(A+B) \quad \Rightarrow 1 - \frac{\cos 2A + \cos 2B}{2} = \sin C$

$1 - \cos(A+B)\cos(A-B) = \sin C$

$\cos^2 C \cos^2(A-B) + 2\cos C \cos(A-B) = \cos^2 C$

$\Rightarrow \cos C = 0 \Rightarrow$ triangle is right angled.

b) $\frac{bc}{2\cos A} = b^2 + c^2 - 2bc\cos A \quad \Rightarrow \cos A = \frac{bc}{2a^2} \Rightarrow b^2c^2 = a^2(b^2 + c^2 - a^2)$

$(a^2 - b^2)(a^2 - c^2) = 0 \Rightarrow a = b \ or \ a = c$ Hence triangle is isosceles.

c) $\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} = \sqrt{3}$

$\left(\tan\frac{A}{2} - \tan\frac{B}{2}\right)^2 + \left(\tan\frac{B}{2} - \tan\frac{C}{2}\right)^2 + \left(\tan\frac{C}{2} - \tan\frac{A}{2}\right)^2 = 0$

$\Rightarrow$ triangle is equilateral.

d) If a, b, c are in AP and h_a, h_b, h_c are in AP, where h_a, h_b, h_c are the altitudes, then $a = b = c \Rightarrow$ the triangle is equilateral.

8.

Column I	Column II
a) The lengths of the altitudes of a triangle are in H.P. Hence the sines of the corresponding angles are	p) in A.P.
b) The radii of the escribed circles of a triangle are in H.P. Hence the corresponding sides are	q) in H.P.
c) Let H be the orthocentre of Δ ABC. If HA^2, HB^2, HC^2 are in A. P. , then a^2, b^2, c^2 are	r) in G.P.
d) The angles A, B, C of a triangle are in A.P. . a^2, $\frac{b^2}{2}$, c^2 are	s) not in A.P.

Key. (a) – (p) ; (b) – (p) ; (c) – (p) ; (d) – (s)

Sol. (a) h_a , h_b , h_c are in H.P.

$\Rightarrow$ $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in H.P.

$\Rightarrow$ a , b , c are in A.P.

$\Rightarrow$ sinA, sinB , sinC are in A.P.

(b) r_1 , r_2 , r_3 are in H.P.

$\Rightarrow$ $\frac{1}{s-a}, \frac{1}{s-b}, \frac{1}{s-c}$ are in H.P.

$\Rightarrow$ $s - a$, $s - b$, $s - c$ are in A.P.

$\Rightarrow$ a , b , c are in A.P.

(c) The distance of the orthocentre H from the vertex A is 2R cosA (?)

HA^2 , HB^2 , HC^2 are in A.P.

$\Rightarrow$ $\cos^2 A$, $\cos^2 B$, $\cos^2 C$ are in A.P.

$\Rightarrow$ $\sin^2 A$, $\sin^2 B$, $\sin^2 C$ are in A.P.

$\Rightarrow$ a^2, b^2 , c^2 are in A.P.

(d) A, B, C are in A.P.

$\Rightarrow$ $B = \frac{\pi}{3}$ $\Rightarrow$ $c^2 + a^2 - b^2 = bc$

9. Column – I Column – II

Column – I	Column – II
(P) In any ΔABC, $b^2 \sin 2C + c^2 \sin 2B$ =	(A) $\frac{1}{\sqrt{6}}$
(Q) In a ΔABC, $\angle B = \pi/3$ & $\angle C = \pi/4$ Let D divide BC internally in the ratio 1 : 3, then $\frac{\sin\angle BAD}{\sin\angle CAD} =$	(B) 1
(R) In a ΔABC, if C = 90º, then $\frac{a^2+b^2}{a^2-b^2}\sin(A-B) =$	(C) 2
(S) In a ΔABC, $\angle$C = 45º, then (1 + cot A) (1 + cot B) =	(D) 4Δ

Key. (P) $\to$ (D), (Q) $\to$ (A), (R) $\to$ (B), (S) $\to$ (C)

10. In a triangle ABC, AD is perpendicular to BC and DE is perpendicular to AB.

Column – I	Column – II
(a) Area of ΔADB	(p) $(b^2/4)\sin 2C$
(b) Area of ΔADC	(q) $(c^2/4)\cos^2 B \sin 2B$
(c) Area of ΔADE	(r) $(c^2/4)\sin 2B$
(d) Area of ΔBDE	(s) $(c^2/4)\sin^2 B \sin 2B$

Key. (a) $\to$ r, (b) $\to$ p, (c) $\to$ s, (d) $\to$ q

11. Match the following:

If in $\Delta ABC,\ r=2,\ r_1=4,\ s=12$ and $a<b<c,$ then

Column -I		Column -II	
(A)	Area of ΔABC is	(p)	22
(B)	$4+4R$ is	(q)	24
(C)	$\angle A$ of triangle is	(r)	$\sin^{-1}\frac{4}{5}$
(D)	$\angle B$ of triangle is	(s)	$\sin^{-1}\frac{3}{5}$

Key. (A) $\rightarrow$(q); (B) $\rightarrow$(q); (C) $\rightarrow$(s); (D) $\rightarrow$(r)

Sol. $r_1r_2+r_2r_3+r_3r_1=\frac{r_1r_2r_3}{r}=s^2=144$

$4(r_1+r_3)+r_2r_3=2r_2r_3=144 \Rightarrow r_2r_3=72,\ r_2+r_3=18 \Rightarrow r_2=6,\ r_3=12$

$\frac{r}{r_1}=\frac{s-a}{s}=\frac{1}{2} \Rightarrow a=6$

Similarly, $b=8, c=10 \Rightarrow ABC$ is right angle triangle.

Smallest angle is $\sin^{-1}\frac{3}{4}$, $\Delta=\frac{1}{2}\times 6\times 8=24$ sq.units $R=5$; $|r_2-r_3|=6$

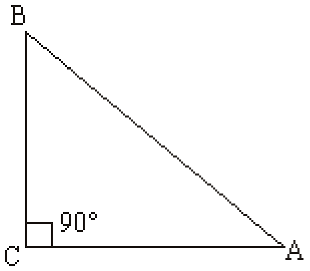
12. Match the following: -

Column – I		Column – II	
(A)	In a ΔABC, let $\angle C=\frac{\pi}{2}$, r = in radius R = circum radius then 2(r + R)	(p)	a + b + c
(B)	It l, m, n are pendicular drawn from the vertices of triangle having sides a,b and c then $\sqrt{2R\left(\frac{bl}{c}+\frac{cm}{a}+\frac{an}{b}\right)+2ab+2bc+2ca}$	(q)	a – b
(C)	In a ΔABC, $R\left(b^2\sin 2C+c^2\sin 2B\right)$ equals	(r)	a + b
(D)	In a right angle triangle ABC $\angle C=\frac{\pi}{2}$, then $4R\sin\frac{A+B}{2}.\sin\frac{(A-B)}{2}$	(s)	abc

Key. A $\rightarrow$ r, B $\rightarrow$ p, C $\rightarrow$ s, D $\rightarrow$ q

Sol. A) $2(r+R)=2\left((s-c)\tan\frac{C}{2}+\frac{C}{2}\right)$

$2\left(s-\frac{C}{2}\right)=a+b$



B) $\sin C = \dfrac{l}{b}$ …(i)

$\sin B = \dfrac{n}{a}$ …(ii)

$\sin A = \dfrac{m}{c}$ …(iii)

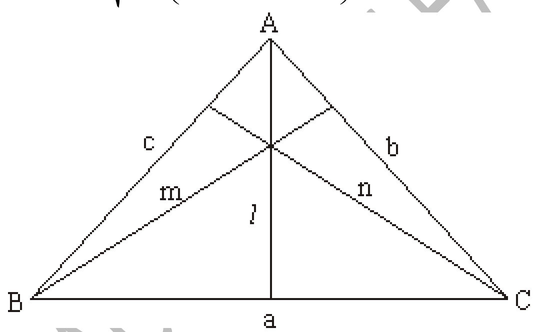
$2R\left(\dfrac{bl}{c}\right) + 2R\left(\dfrac{cm}{a}\right) + 2R\left(\dfrac{an}{b}\right)$

$\dfrac{c}{\sin c}\dfrac{bl}{c} + \dfrac{a}{\sin A}\dfrac{cm}{a} + \dfrac{b}{\sin B}\dfrac{an}{b}$

From (i), (ii), (iii)

$2R\left(\dfrac{bl}{c} + \dfrac{cm}{a} + \dfrac{an}{b}\right) = a^2 + b^2 + c^2$

$\therefore \quad \sqrt{2R\left(\dfrac{bl}{c} + \dfrac{cm}{a} + \dfrac{an}{b}\right) + 2ab + 2bc + 2ac} = a + b + c$



C) $R = \dfrac{b}{2\sin B} = \dfrac{c}{2\sin C}$

$Rb^2 \sin 2C + Rc^2 \sin 2B$

$b^2 c\cos C + c^2 b\cos B$

bc (b cos C + c cos B)

abc

D) 2R (cos A + cos B)

$2R\left(-\dfrac{b}{\sqrt{a^2+b^2}} + \dfrac{a}{\sqrt{a^2+b^2}}\right)$

Q $\quad R = \sqrt{a^2+b^2}/2$

$2R\left(-\dfrac{b}{\sqrt{a^2+b^2}} + \dfrac{a}{\sqrt{a^2+b^2}}\right) = a - b$

13. Match the following: -

	Column – I		Column – II
(A)	In a ΔABC, (a + b + c) (b + c – a) $=\lambda bc$, where $\lambda \in I$, then greatest value of λ is	(p)	3
(B)	In a ΔABC, tanA + tanB + tan C = 9 If $\tan^2 A+\tan^2 B+\tan^2 C=k$, then least value of k satisfying is	(q)	$9(3)^{1/3}$
(C)	In a triangle ABC, then line joining the circumcenter to the incentre is parallel to BC, then value of cosB + cosC is	(r)	1
(D)	If in a ΔABC, a = 5, b = 4 and cos (A – B) $=\frac{31}{32}$, then the third side c is equal to	(s)	6

Key. A $\to$ p; B $\to$ q; C $\to$ r; D $\to$ s

Sol. A) $(b+c)^2-a^2=\lambda bc$

or $b^2+c^2-a^2=(\lambda-2)bc$

$$\frac{b^2+c^2-a^2}{2bc}=\frac{\lambda-2}{2}$$

$$\cos A=\frac{\lambda-2}{2}<1$$

or $\lambda-2<2$

or $\lambda=3$

B) tan A + tan B + tan C = 9

In any triangle tan A + tan B + tan C = tan A tan B tan C

$$\frac{\tan^2 A+\tan^2 B+\tan^2 C}{3}\ge(\tan A\tan B\tan C)^{2/3}$$

$$k\ge 3(9)^{2/3}$$

$$k\ge 9.3^{1/3}$$

C) Since the line joining the circumcenter to the incentre is parallel to BC

$\therefore$ $r=R\cos A$

$\therefore$ $2R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=R\cos A$

$\therefore$ - 1 + cos A cos B + cos C = cos A

$\therefore$ cos B + cos C = 1

D) a = 5, b = 4

$$\cos(A-B)=\frac{31}{32}$$

$$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}=\frac{1}{9}\cot\frac{C}{2}$$

$$\cos(A-B)=\frac{1-\tan^2\frac{A-B}{2}}{1+\tan^2\frac{A-B}{2}}$$

$$\frac{31}{32}=\frac{1-\frac{1}{81}\cot^2\frac{C}{2}}{1+\frac{1}{81}\cot^2\frac{C}{2}}$$

$$31+\frac{31}{81}\cot^2\frac{C}{2}=32-\frac{32}{31}\cot^2\frac{C}{2}$$

$$\frac{7}{9}\cot^2\frac{C}{2}=1$$

$$\cot^2\frac{C}{2}=\frac{9}{7}$$

$$\cos C=\frac{1-\tan^2\frac{C}{2}}{1+\tan^2\frac{C}{2}}=\frac{1-\frac{7}{9}}{1+\frac{7}{9}}=\frac{2}{16}=\frac{1}{8}$$

$$\cos C=\frac{a^2+b^2-c^2}{2ab} \quad \Rightarrow \quad \frac{25+16-c^2}{2\times 20}=\frac{1}{8}$$

$25 + 16 - c^2 = 5$

$c^2 = 36$

$c = 6$

14. Match the following: -

	Column – I		Column – II
(A)	If $\cos A=\frac{\sin B}{2\sin C}$, then ΔABC is	(p)	isosceles
(B)	If $\frac{\cos A+2\cos C}{\cos A+2\cos B}=\frac{\sin B}{\sin C}$, then ΔABC may b	(q)	obtuse angle
(C)	If $\frac{2\cos A}{a}+\frac{\cos B}{b}+\frac{2\cos C}{c}=\frac{a}{bc}+\frac{b}{ca}$, then ΔABC is	(r)	right angle
(D)	If $\frac{a^2-b^2}{a^2+b^2}=\frac{\sin(A-B)}{\sin(A+B)}$, then ΔABC may be	(s)	acute angle
		(t)	equilateral

Key. A $\rightarrow$ p; B $\rightarrow$ p,r; C $\rightarrow$ r; D $\rightarrow$ p,r

Sol. A) Since $\cos A=\frac{\sin B}{2\sin C}$, we have $\frac{a^2+c^2-a^2}{2bc}=\frac{b}{2c}$

or $b^2+c^2-a^2=b^2$ or $c^2=a^2$

Hence c = a and so the ΔABC is isosceles

B) cos A(sin B – sin C) + (sin 2B – sin 2C) = 0

Or cosA(sinB – sin C) + 2 sin (B – C) cos (B + C) = 0

Or cosA(sinB – sin C) – 2 cos A sin (B – C) = 0

$\therefore$ either cos A = 0 $\Rightarrow$ A = 90°

Or (sinB – sin C) – 2(sin B cos C – cosB sin C) = 0

$\therefore$ $(b-C)-2\left[b.\frac{a^2+b^2+c^2}{2ab}-c.\frac{c^2+a^2-b^2}{2ca}\right]=0$

Or $a(b-c)-2(b^2-c^2)=0$

$\Rightarrow$ (b – c) [a – 2(b + c)] = 0

$\therefore$ b – c = 0 or b = c

$\therefore$ isosceles

C) Combine first and thiral and put the value of cos B

$\therefore$ $\frac{2}{ac}.\quad (b)+\frac{1}{b}\frac{c^2+a^2-b^2}{2ca}=\frac{a^2+b^2}{abc}$

or $4b^2+c^2+a^2-b^2=2a^2+2b^2$

$\therefore$ $b^2+c^2=a^2$

$\therefore \quad \angle A = 90°$

D) $\dfrac{\sin(A-B)}{\sin(A+B)} = \dfrac{k^2(\sin^2 A - \sin^2 B)}{k^2(\sin^2 A + \sin^2 B)}$ by sine formula

or $\dfrac{\sin(A-B)}{\sin C} = \dfrac{\sin(A-B)\sin(A+B)}{\sin^2 A + \sin^2 B}$

or $\sin(A-B)\left[\dfrac{1}{\sin C} - \dfrac{\sin C}{\sin^2 A + \sin^2 B}\right] = 0$

$\therefore$ either sin (A – B) = 0

$\therefore$ A = B i.e., Δ is isosceles

or $\sin^2 A + \sin^2 B = \sin^2 C$ or $a^2 + b^2 = c^2$

$\therefore$ Δ is right angled