Maxima & Minima

Single Correct Answer Type

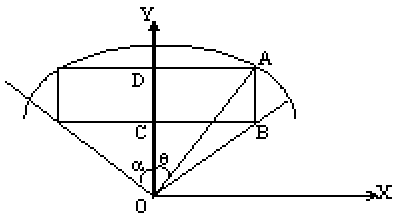
1. A sector subtends an angle 2α at the centre then the greatest area of the rectangle inscribed in the sector is (R is radius of the circle)

A) $R^2 \tan\frac{\alpha}{2}$ B) $\frac{R^2}{2}\tan\frac{\alpha}{2}$ C) $R^2 \tan\alpha$ D) $\frac{R^2}{2}\tan\alpha$

Key. A

Sol. Let A be any point on the arc such that $\angle YOA = \theta$

Where $0 \le \theta \le \alpha$



DA = CB = R sin θ , OD = R cos θ

$\Rightarrow$ CO = CB cot $\alpha = R\sin\theta\cot\alpha$

Now, CD = OD - OC = R cos $\theta - R\sin\theta\cot\alpha$

= R (cos θ – sin θ cot α)

Area of rectangle ABCD , S = 2.CD.CB

= 2R (cosθ - sin θ cot α) R sin θ = $2R^2(\sin\theta\cos\theta - \sin^2\theta\cot\alpha)$

$R^2\left(\sin 2\theta - (1-\cos 2\theta)\cot\alpha\right) = \frac{R^2}{\sin\alpha}\left[\cos(2\theta-\alpha) - \cos\alpha\right]$

$S_{max} = \frac{R^2}{\sin\alpha}(1-\cos\alpha) \quad \left(\text{for } \theta = \alpha/2\right)$

Hence, greatest area of the rectangle = $R^2\tan\frac{\alpha}{2}$

2. Let f(x) = x^2 – bx + c , b is a odd positive integer, f(x) = 0 have two prime numbers as roots and

b + c = 35. Then the global minimum value of f(x) is

A) $-\frac{183}{4}$ B) $\frac{173}{16}$ C) $-\frac{81}{4}$ D) data not sufficient

Key. C

Sol. Let α, β be roots of x^2 – bx + c = 0,

Then $\alpha + \beta$ = b

$\Rightarrow$ one of the roots is ‘2’ (Since α, β are primes and b is odd positive integer)

$\therefore$ f(2) = 0 $\Rightarrow$ 2b - c = 4 and b + c = 35

$\therefore$ b = 13, c = 22

Minimum value = $f\left(\frac{13}{2}\right)=-\frac{81}{4}$.

3. Let $f(x)$ be a positive differentiable function on $[0,a]$ such that $f(0)=1$ and $f(a)=3^{1/4}$ If $f^1(x)\ge(f(x))^3+(f(x))^{-1}$, then, maximum value of a is

a) $\frac{\pi}{12}$ b) $\frac{\pi}{24}$ c) $\frac{\pi}{36}$ d) $\frac{\pi}{48}$

Key. B

Sol. $f^1(x)f(x)\ge(f(x))^4+1$

$\Rightarrow \frac{2f^1(x)f(x)}{\left\{(f(x))^2\right\}^2+1}\ge 2$

$\Rightarrow \int_0^a \frac{2f^1(x)f(x)}{\left\{(f(x))^2\right\}^2+1}\ge 2\int_0^a 1dx$

$\Rightarrow \left|\tan^{-1}(f(x))^2\right|_0^a \ge 2a \Rightarrow \frac{\pi}{3}-\frac{\pi}{4}\ge 2a$

4. The least value of 'a' for which the equation $\frac{4}{\sin x}+\frac{1}{1-\sin x}=a$ for atleast one solution on the interval $\left(0,\frac{\pi}{2}\right)$ is,

a) 1 b) 4 c) 8 d) 9

Key. D

Sol. Q $a=\frac{4}{\sin x}+\frac{1}{1-\sin x}$, where a is least

$\Rightarrow \frac{da}{dx}=\left(\frac{-4}{\sin^2 x}+\frac{1}{(1-\sin x)^2}\right)\cos x=0$

Q $\cos x\ne 0\Rightarrow \sin x=2/3$

$\frac{d^2a}{dx^2}=45>0$ for $\sin x=2/3\Rightarrow \frac{4}{2/3}+\frac{1}{1-2/3}=6+3=9$

5. Let domain and range of $f(x)$ and $g(x)$ are respectively $[0,\infty)$. If $f(x)$ be an increasing function and $g(x)$ be an decreasing function. Also, $h(x)=f(g(x)),h(0)=0$ and $p(x)=h\left(x^3-2x^2+2x\right)-h(4)$ then for every $x\in(0,2]$

a) $p(x)\in(0,-h(4))$ b) $p(x)\in[-h(4),0]$

c) $p(x)\in(-h(4),h(4))$ d) $p(x)\in(h(4),h(4)]$

Key. A

Sol. $h(x)=f(g(x))$

$h^1(x)=f^1(g(x))g^1(x)<0\forall x\in[0,\infty)$

Q $g^1(x)<0\forall x\in[0,\infty)$ and $f^1(g(x))>0\forall x\in[0,\infty)$

Also, $h(0)=0$ and hence, $h(x)<0\forall x\in[0,\infty)$

$p(x)=h(x^3-2x^2+2x)-h(4)$

$p^1(x)=h^1(x^3-2x^2+2x).(3x^2-4x+2)<0\forall x\in(0,2)$

$\because h^1(x^3-2x^2+2x)<0\forall x\in(0,\infty)$ and $3x^2-4x+2>0\forall x\in R$

$\Rightarrow p(x)$ is an decreasing function

$\Rightarrow p(2)<p(x)<p(0)\forall x\in(0,2)$

$\Rightarrow h(4)-h(4)<p(x)<h(0)-h(4)$

$\Rightarrow 0<p(x)<-h(4)$

6. If $f(x)=\begin{cases}3-x^2, x\le 2\\ \sqrt{a+14}-|x-48|, x>2\end{cases}$ and if $f(x)$ has a local maxima at x = 2, then, greatest value of a is

a) 2013 b) 2012 c) 2011 d) 2010

Key. C

Sol. Local maximum at $x=2\Rightarrow$

$$\Rightarrow \lim_{h\to 0} f(2+h)\le f(2)$$

$$\Rightarrow \lim_{h\to 0}\left(\sqrt{a+14}-|2+h-48|\right)\le 3-2^2$$

$$\Rightarrow \sqrt{a+14}\le 45\Rightarrow a\le 2011$$

7. Two runners A and B start at the origin and run along positive x-axis, with B running three times as fast as A. An observer, standing one unit above the origin, keeps A and B in view. Then the maximum angle of sight $'\theta'$ between the observes view of A and B is

a) $\pi/8$ b) $\pi/6$ c) $\pi/3$ d) $\pi/4$

Key. B

Sol. $$\tan\theta=\tan(\theta_2-\theta_1)\Rightarrow \tan\theta=\frac{3x-x}{1+3x.x}=\frac{2x}{1+3x^2}$$

$$\text{let } y=\frac{2x}{1+3x^2}\quad \frac{dy}{dx}=\frac{2(1-3x^2)}{(1+3x^2)^2}$$

$$\frac{dy}{dx}=0\Rightarrow x=\frac{1}{\sqrt{3}} \text{ and } \frac{d^2y}{dx^2}=\frac{-24x}{(1+3x^2)^3}<0 \text{ for } x=1/\sqrt{3}$$

8. If the function $f(x)=ax^3+bx^2+11x-6$ satisfies conditions of Rolle's theorem in [1, 3] and $f'\left(2+\frac{1}{\sqrt{3}}\right)=0$, then value of a and b are respectively

(A) 1, –6 (B) –1, 6 (C) –2, 1 (D) –1, 1/2

Key. A

Sol. $\because$ f (1) = f(3)

$\Rightarrow$ a + b + 11 – 6 = 27a + 9b + 33 – 6

$\Rightarrow$ 13a + 4b = –11

and $f'(x) = 3ax^2 + 2bx + 11$ ... (i)

$\Rightarrow \quad f'\left(2+\frac{1}{\sqrt{3}}\right) = 3a\left(2+\frac{1}{\sqrt{3}}\right)^2 + 2b\left(2+\frac{1}{\sqrt{3}}\right) + 11 = 0$

$\Rightarrow \quad 3a\left(4+\frac{1}{3}+\frac{4}{\sqrt{3}}\right) + 2b\left(2+\frac{1}{\sqrt{3}}\right) + 11 = 0$ ... (ii)

From eqs. (i) and (ii), we get a = 1, b = –6.

9. Let $f(x)$ be a positive differentiable function on $[0,a]$ such that $f(0)=1$ and $f(a)=3^{1/4}$ If $f^1(x) \ge (f(x))^3 + (f(x))^{-1}$, then, maximum value of a is

a) $\frac{\pi}{12}$ b) $\frac{\pi}{36}$ c) $\frac{\pi}{24}$ d) $\frac{\pi}{48}$

Key. C

Sol. $f^1(x)f(x) \ge (f(x))^4 + 1$

$\Rightarrow \frac{2f^1(x)f(x)}{\left\{(f(x))^2\right\}^2 + 1} \ge 2$

$\Rightarrow \int_0^a \frac{2f^1(x)f(x)}{\left\{(f(x))^2\right\}^2 + 1} \ge 2\int_0^a 1dx$

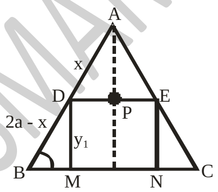
$\Rightarrow \left|\tan^{-1}(f(x))^2\right|_0^a \ge 2a \Rightarrow \frac{\pi}{3} - \frac{\pi}{4} \ge 2a$

Given expansion = $\{x-(1+\cos t)\}^2 + \left\{\frac{K}{x}-(1+\sin t)\right\}^2$

10. A rectangle is inscribed in an equilateral Δ of side length 2a units. Maximum area of this rectangle is

(A) $\sqrt{3}a^2$ (B) $\frac{\sqrt{3}a^2}{4}$ (C) a^2 (D) $\frac{\sqrt{3}a^2}{2}$

Key. D



Sol.

Let AD = x
BD = (2a – x)
In ΔDBM
$\angle B = \frac{\pi}{3}$

Let DM = y_1
DE = $2x_1$
$\sin 60° = \frac{y_1}{2a-x}$
$y_1 = (2a-x) \times \frac{\sqrt{3}}{2}$

In ΔADP

$\angle D = \frac{\pi}{3}$ $\cos 60° = \frac{x_1}{x}$

$x_1 = x \times \frac{1}{2}$

$2x_1 = x$

$\Delta(x)$ = Area of rectangle = $2x_1y$

$\Delta(x) = x \times (2a - x)\frac{\sqrt{3}}{2}$

$\Delta'(x) = \frac{\sqrt{3}}{2}(2a - 2x) = 0 \Rightarrow x = a$

$\Delta''(a) = -ve$

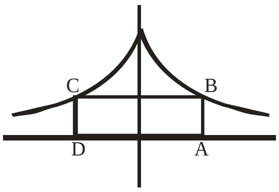
x = a point of maxima

maximum area = $a \times \frac{a\sqrt{3}}{2} = \frac{\sqrt{3}a^2}{2}$

11. The maximum area of a rectangle whose two consecutive vertices lie on the x-axis and another two lie on the curve $y = e^{-|x|}$ is equal to

(A) 2e sq. Units (B) $\frac{2}{e}$ sq. Units (C) e sq. units (D) $\frac{1}{e}$ sq. units

Key. B



Sol.

Let the rectangle is (ABCD)

$A = (t, 0), B = (t, e^{-t}), C = (-t, e^{-t}), D = (-t, 0)$

ABCD = $2te^{-t} = f(t)$

$\frac{df}{dt} = 2(t(-e^{-t}) + e^{-t}) = 2e^{-t}(1 - t)$

$\frac{df}{dt} > 0 \Rightarrow t \in (0, 1)$

$\frac{df}{dt} < 0 \Rightarrow t \in (1, \infty)$

t = 1 is point of maxima

Maximum area = $f(1) = \frac{2}{e}$

12. Let $f : [0, 4] \to R$, be a differentiable function. Then, there exists real numbers $a, b \in (0, 4)$ such that, $(f(4))^2 - (f(0))^2 = Kf^1(a)f(b)$ Where K, is

a) $\frac{1}{4}$ b) 8 c) $\frac{1}{12}$ d) 4

Key. B

Sol. By LMVT, $\exists a \in (0,4) \ni \frac{f(4)-f(0)}{4-0} = f^1(a) \Rightarrow f(4)-f(0) = 4f^1(a)$

Q $\frac{f(4)+f(0)}{2}$ lies between $f(0)$ and $f(4)$, by Intermediate value theorem $\exists b \in (0,4) \ni \frac{f(4)+f(0)}{2} = f(b)$ hence, $\left(f(4)^2\right)-\left(f(0)\right)^2 = 8 \quad f^1(a)f(b)$

13. A window is in the shape of a rectangle surmounted by a semi circle .If the perimeter of the window is of fixed length 'l' then the maximum area of the window is

1) $\frac{l^2}{2\pi+4}$ 2) $\frac{l^2}{\pi+8}$ 3) $\frac{l^2}{2\pi+8}$ 4) $\frac{l^2}{8\pi+4}$

Key. 3

$l = 2x + 2r + \pi r$

Sol. $A = 2rx + \frac{1}{2}\pi r^2$

$\frac{dA}{dV} = 0 \Rightarrow r = \frac{l}{4+\pi}$

14. If the petrol burnt per hour in driving a motor boat varies as the cube of its velocity when going against a current of 'C' kmph , the most economical speed
Is (in kmph)

1) $\frac{C}{2}$ 2) $\frac{3C}{2}$ 3) $\frac{\sqrt{3}C}{2}$ 4) C

Key. 2

Sol. y be the petrol burnt hour $y = kv^3$ 'S' be the distance traveled by boat the petrol burnt = $\frac{S}{V-C} \times kv^3$

$f'(v) = 0 \Rightarrow v = \frac{3c}{2}$

15. A point 'P' is given on the circumference of a Circle of radius 'r' .The chord 'QR' is parallel to the tangent line at 'P' the maximum area of the triangle PQR is

1) $\frac{3\sqrt{2}}{4}r^2$ 2) $\frac{3\sqrt{3}}{4}r^2$ 3) $\frac{3}{8}r^2$ 4) $\frac{3\sqrt{2}}{4}r$

Key. 2

Sol. The area maximum when the triangle is equilateral

16. The minimum value of $f(x) = x^2 + \frac{250}{x}$ is

1) 15 2) 25 3) 45 4) 75

Key. 4

Sol. $f'(x)=0$ *and* $f''(5)>0$ minimum value = $f(5)$

17. The sum of two numbers is '6'.The minimum value of the sum of their reciprocals is

1) $\frac{3}{4}$ 2) $\frac{6}{5}$ 3) $\frac{2}{3}$ 4) $\frac{2}{5}$

Key. 3

Sol. $x=y=\frac{6}{2}=3, \frac{1}{x}+\frac{1}{y}=\frac{2}{3}$

18. Minimum value of $\frac{(6+x)(11+x)}{2+x}$ is

1) 5 2) 15 3) 45 4) 25

Key. 4

Sol. $f'(x)=0$ *when put* $x=4$

19. The maximum area of a rectangle inscribed in a circle of radius 5 cm is

1) $25\ sq.cm$ 2) $50\ sq.cm$ 3) $100\ sq.cm$ 4) $\frac{25}{2} sq.cm$

Key. 2

Sol. $Area = 2r^2 = 50\ sq.cm$

20. The diagonal of the rectangle of maximum area having perimeter 100 cm is

1) $10\sqrt{2}$ 2) 10 3) $25\sqrt{2}$ 4) 15

Key. 3

Sol. The maximum perimeter of the rectangle that can be inscribed in a circle is a square .Here the lengths are $x=\sqrt{2}\ r, y=\sqrt{2}\ r$

21. The maximum value of $x^{-x}, (x>0)$ is

1) e^e 2) $e^{1\backslash e}$ 3) e^{-e} 4) $1\backslash e$

Key. 2

Sol. $f(x)=x^{-x},\ f'(x)=0 \Rightarrow x=e^{-1}$
$f''(e-1)<0$

22. Which fraction exceeds its p^{th} power by the greatest number possible is?

1) p^p 2) $\left(\frac{1}{P}\right)^{P-1}$ 3) $p^{\frac{1}{1-p}}$ 4) $\frac{1}{p^p}$

Key. 3

$y = x - x^{p}$

Sol. $\dfrac{dy}{dx} = 0 \Rightarrow x = \left(\dfrac{1}{p}\right)^{\frac{1}{p-1}}$

23. In $(0, 2\pi)$, $f(x) = x + \sin 2x$ is

1) Minimum at $x = \dfrac{2\pi}{3}$ 2) Maximum at $x = \dfrac{2\pi}{3}$

3) Maximum at $x = \dfrac{\pi}{4}$ 4) Minimum at $x = \dfrac{\pi}{6}$

Key. 1

Sol. $f'(x) = 0 \Rightarrow f''(x) > 0 \ when \ x = \dfrac{2\pi}{3}$

24. The Value of 'a' for which $f(x) = a\sin x + \dfrac{1}{3}\sin 3x$ has an extremum at $x = \dfrac{\pi}{3}$ is

1) 1   2) -1   3) 0   4) 2

Key. 4

Sol. $\dfrac{d^2y}{dx^2} = 0$ then find 'x' and substitute in $\dfrac{dy}{dx}$.

25. A person wishes to lay a straight fence across a triangular field ABC, with $\underline{|A} < \underline{|B} < \underline{|C}$ so as to divide it into two equal areas. The length of the fence with minimum expense, is

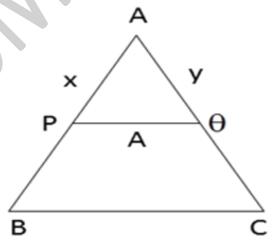
a) $\sqrt{2\Delta \cot \dfrac{B}{2}}$ b) $\sqrt{2\Delta \tan \dfrac{C}{3}}$

c) $\sqrt{\tan \dfrac{A}{2} \tan \dfrac{B}{2} \tan \dfrac{C}{2}}$ d) $\sqrt{2\Delta \tan \dfrac{A}{2}}$

(where 'Δ' represents, area of triangle ABC)

Key. D

Sol.



$$\frac{1}{2}xy\sin A=\frac{1}{2}\left(\frac{1}{2}bc\sin A\right)$$

$$\Rightarrow xy=\frac{1}{2}bc$$

$$z_A^2=(PQ)^2=x^2+y^2-2xy\cos A$$

$$=x^2+\frac{b^2c^2}{4x^2}-bc\cos A$$

$$\Rightarrow 2Z_A\left(\frac{dZ_A}{dx}\right)=2x-\frac{b^2c^2}{2x^3}$$

$$\frac{dZ_A}{dx}=0\Rightarrow x=\sqrt{\frac{bc}{2}},\ \text{and}\ \frac{d^2Z_A}{dx^2}>0$$

Hence Z_A is minimum if $x=\sqrt{\frac{bc}{2}}$ and the minimum value of Z_A,is

$$\sqrt{\frac{bc}{2}+\frac{bc}{2}-bc\cos A}=\sqrt{2\Delta\tan\frac{A}{2}}$$

26. The number of critical point of $f(x)=\frac{|x-1|}{x^2}$ is

1) 1 2) 2 3) 3 4) 0

Key. 2

Sol.
$$f(x)=\left|\frac{x-1}{x^2}\right|,\ f(x)=0\ \ for\ \ x=\pm 2$$

$$f(x)=\pm\left(x-\frac{1}{x}\right)\Rightarrow f'(x)=\pm\left(1+\frac{1}{x^2}\right)\neq 0$$

27. The total cost of producing 'x' pocket radio sets per day is Rs.$\left(\frac{1}{4}x^2+35x+25\right)$ and the price per set at which they may be sold is Rs.$\left(50-x/2\right)$ to obtain maximum profit the daily out put should be--------- radio sets.

1) 10 2) 5 3) 15 4) 20

Key. 1

Sol. If daily out put is x sets and p be the total point then

$$p=x\left(50-\tfrac{1}{2}x\right)-\left(\frac{1}{4}x^2+35x-25\right)$$

$$\frac{dp}{dx}=0\Rightarrow x=10\ and\ \left(\frac{d^2p}{dx^2}\right)_{(x=10)}=-3/2<0$$

28. If $f(x)=a\log|x|+bx^2+x$ has extreme values at $x=-1,\ x=2$ then a = ---- b = --

1) $2,\frac{-1}{2}$ 2) $\frac{-1}{2},2$ 3) $\frac{1}{2},2$ 4) $2,\frac{1}{2}$

Key. 1

Sol. $f'(-1)=0 \Rightarrow -a-2b+1=0$

$f'(2)=0 \Rightarrow -\frac{a}{2}+4b+1=0$

29. A quadratic function in 'x' has the values '10' when x =1 and has minimum value '1' when x = -2 the function is

1) $2x^2+3x+5$ 2) $3x^2+2x+5$ 3) x^2+3x+6 4) x^2+4x+5

Key. 4

Sol. $f(x)=ax^2+bx+c$

$a+b+c=10,\ f'(-2)=0,\ f(-2)=1$

30. The equation of a line passing through the point (3,4) and which forms a triangle of minimum area with the coordinate axes in the first quadrant

1) $4x+3y-24=0$ 2) $3x+4y-12=0$ 3) $2x+3y-12=0$ 4) $3x+2y-24=0$

Key. 1

Sol. (3 ,4) is the mid point of the line segment

31. The maximum of $f(x)=2x^3-9x^2+12x+4$ occurs at x =

1) 1 2) 2 3) -1 4) -2

Key. 1

Sol. $f'(x)=0 \Rightarrow 6x^2-18x+12=0$

$f''(x)=12x-18$

32. $f(x)=4+5x^2+6x^4$ has

1) Only one minimum
2) Neither maximum n or minimum
3) Only one maximum
4) No minimum.

Key. 1

Sol. $f(x)$ is minimum at x = 0

33. At $x=0,\ f(x)=(3-x)e^{2x}-4xe^x-x$

1) Has a minimum
2) Has a maximum
3) Has no extremum
4) Is not defined

Key. 3

Sol.
$At\ x=0,\ f'(x)=0$

$At\ x=0,\ f''(x)=0$

$At\ x=0,\ f'''(x)\neq 0$

$\therefore f(x)$ *is neither* maximum *nor* minimum

34. The number of critical points of $f(x)=\frac{|x-1|}{x^2}$ is

(A) 1 (B) 2 (C) 3 (D) None of these

Key. C

Sol. f(x) is not differentiable at x = 0 and x = 1.
$f'(x)=0$ at $x=2$

35. A differentiable function f(x) has a relative minimum at x =0, then the function y = f(x) + ax + b has a relative minimum at x = 0 for

(A) all a and all b (B) all b > 0 (C) all b, if a = 0 (D) all a > 0

Key. C

Sol. $f'(0)=0$ and $f''(0)>0$
y = f(x) + ax + b has a relative minimum at x = 0.

Then $\frac{dy}{dx}=0$ at x = 0

$f'(x)+a=0 \Rightarrow a=0$

$f''(x)>0 \Rightarrow \quad f''(0)>0$

Hence y has relative minimum at x = 0 if a = 0 and b $\in$ R.

36. Let $f:[0,4]\to R$, be a differentiable function. Then, there exists real numbers $a,b\in(0,4)$ such that, $(f(4))^2-(f(0))^2=Kf^1(a)f(b)$ Where K, is

a) $\frac{1}{4}$ b) 8 c) $\frac{1}{12}$ d) 4

Key. B

Sol. By LMVT, $\exists a\in(0,4) \ni \frac{f(4)-f(0)}{4-0}=f^1(a)\Rightarrow f(4)-f(0)=4f^1(a)$

$\because \frac{f(4)+f(0)}{2}$ lies between $f(0)$ and $f(4)$, by Intermediate value theorem

$\exists b\in(0,4)\ni \frac{f(4)+f(0)}{2}=f(b)$ hence, $\left(f(4)^2\right)-(f(0))^2=8\quad f^1(a)f(b)$

37. If $f(x)=(1-x)^{5/2}$ satisfies the relation, $f(x)=f(o)+xf^1(o)+\frac{x^2}{2}f^{11}(\theta x)$ then, as $x\to 1$, the value of θ, is

a) $\frac{4}{25}$ b) $\frac{25}{4}$ c) $\frac{25}{9}$ d) $\frac{9}{25}$

Key. D

Sol. $f^1(x)=\frac{-5}{2}(1-x)^{3/2}$ and $f^{11}(x)=\frac{15}{4}(1-x)^{1/2}$ and $f(0)=1, f^1(0)=\frac{-5}{2}$,

$f^{11}(\theta x)=\frac{15}{4}(1-\theta x)^{1/2}$

Hence, $(1-x)^{5/2}=\frac{2-5x}{2}+\frac{x^2}{2}(1-\theta x)^{1/2}\times\frac{15}{4}$ as

$x\to 1,\ 0=1-\frac{5}{2}+\frac{15}{8}(1-\theta)^{1/2}\Rightarrow\theta=9/25$

38. $A(1,0),B(e,1)$ are two points on the curve $y=\log_e x$. If P is a point on the curve at which the tangent to the curve is parallel to the chord AB, then, abscissa of P, is

a) $\frac{e-1}{2}$ b) $\frac{e+1}{2}$ c) $e-1$ d) $e+1$

Key. C

Sol. By LMVT, applied to $f(x)=\log_e x$ on $[1,e]$, $\exists$ an $x_0\in(1,e)\ni f^1(x_0)=\frac{f(e)-f(1)}{e-1}$

$\Rightarrow x_0=e-1$

39. Consider the following statements

Statement – I: If f and g are continuous and monotonic on R, then, f + g is also a monotonic function.

Statement- II: If $f(x)$ is a continuous decreasing function $\forall x>0$, and $f(1)$ is positive, then, $f(x)=0$ happens exactly at one value of x. Then,

a) Both I and II are true b) I is true, II is false

c) I is false, II is true d) both I and II are false

Key. D

Sol. I : $f(x)=x$ and $g(x)=-x^2$ on R

II : $f(x)=\frac{1}{x}, x>0$

40. The number of values of x at which the function, $f(x)=(x-1)x^{2/3}$ has extreme values, is

a) 4 b) 3 c) 2 d) 1

Key. C

Sol. $f^1(x)=\dfrac{5x-2}{3x^{1/3}}$

Let $x<0, f^1(x)>0$ and for $x>0, f^1(x)<0$ $\Rightarrow f$ has maximum at x = 0

$x<\dfrac{2}{5}, f^1(x)<0$ and $x>\dfrac{2}{5}, f^1(x)>0 \Rightarrow f$ has minimum at $X=\dfrac{2}{5}$

41. A person wishes to lay a straight fence across a triangular field ABC, with $\underline{|A}<\underline{|B}<\underline{|C}$ so as to divide it into two equal areas. The length of the fence with minimum expense, is

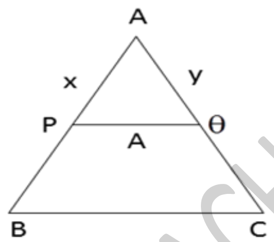
a) $\sqrt{2\Delta\cot\dfrac{B}{2}}$ b) $\sqrt{2\Delta\tan\dfrac{C}{3}}$

c) $\sqrt{\tan\dfrac{A}{2}\tan\dfrac{B}{2}\tan\dfrac{C}{2}}$ d) $\sqrt{2\Delta\tan\dfrac{A}{2}}$

(where 'Δ' represents, area of triangle ABC)

Key. D

Sol.



$\dfrac{1}{2}xy\sin A=\dfrac{1}{2}\left(\dfrac{1}{2}bc\sin A\right)$

$\Rightarrow xy=\dfrac{1}{2}bc$

$z_A^2=(PQ)^2=x^2+y^2-2xy\cos A$

$=x^2+\dfrac{b^2c^2}{4x^2}-bc\cos A$

$\Rightarrow 2Z_A\left(\dfrac{dZ_A}{dx}\right)=2x-\dfrac{b^2c^2}{2x^3}$

$\frac{dZ_A}{dx} = 0 \Rightarrow x = \sqrt{\frac{bc}{2}}$, and $\frac{d^2Z_A}{dx^2} > 0$

Hence Z_A is minimum if $x = \sqrt{\frac{bc}{2}}$ and the minimum value of Z_A,is

$\sqrt{\frac{bc}{2} + \frac{bc}{2} - bc\cos A} = \sqrt{2\Delta \tan\frac{A}{2}}$

42. If the function $f(x) = ax^3 + bx^2 + 11x - 6$ satisfies conditions of Rolle's theorem in [1, 3] and $f'\left(2 + \frac{1}{\sqrt{3}}\right) = 0$, then value of a and b are respectively

(A) 1, –6 (B) –1, 6 (C) –2, 1(D) –1, 1/2

Key. A

Sol. $\because$ f (1) = f(3)

$\Rightarrow$ a + b + 11 – 6 = 27a + 9b + 33 – 6

$\Rightarrow$ 13a + 4b = –11

and $f'(x) = 3ax^2 + 2bx + 11$ ... (i)

$\Rightarrow$ $f'\left(2 + \frac{1}{\sqrt{3}}\right) = 3a\left(2 + \frac{1}{\sqrt{3}}\right)^2 + 2b\left(2 + \frac{1}{\sqrt{3}}\right) + 11 = 0$

$\Rightarrow$ $3a\left(4 + \frac{1}{3} + \frac{4}{\sqrt{3}}\right) + 2b\left(2 + \frac{1}{\sqrt{3}}\right) + 11 = 0$ ... (ii)

From eqs. (i) and (ii), we get a = 1, b = –6.

43. Let $f(x)$ be a positive differentiable function on $[0,a]$ such that $f(0) = 1$ and $f(a) = 3^{1/4}$ If $f^1(x) \geq (f(x))^3 + (f(x))^{-1}$,then, maximum value of a is

a) $\frac{\pi}{12}$ b) $\frac{\pi}{36}$ c) $\frac{\pi}{24}$ d) $\frac{\pi}{48}$

Key. C

Sol. $f^1(x)f(x) \geq (f(x))^4 + 1$

$\Rightarrow \frac{2f^1(x)f(x)}{\left\{(f(x))^2\right\}^2 + 1} \geq 2$

$\Rightarrow \int_0^a \frac{2f^1(x)f(x)}{\left\{(f(x))^2\right\}^2 + 1} \geq 2\int_0^a 1dx$

$\Rightarrow \left|\tan^{-1}(f(x))^2\right|_0^a \geq 2a \Rightarrow \frac{\pi}{3} - \frac{\pi}{4} \geq 2a$

Given expansion = $\{x - (1+\cos t)\}^2 + \left\{\frac{K}{x} - (1+\sin t)\right\}^2$

44. For $x > 0, 0 \le t \le 2\pi, K > \frac{3}{2} + \sqrt{2}$, K being a fixed real number the minimum value of

$x^2 + \frac{K^2}{x^2} - 2\left\{(1+\cos t)x + \frac{K(1+\sin t)}{x}\right\} + 3 + 2\cos t + 2\sin t$ is

a) $\left\{\sqrt{K} - \left(1+\frac{1}{\sqrt{2}}\right)\right\}^2$ b) $\frac{1}{2}\left\{\sqrt{K} - \left(1+\frac{1}{\sqrt{2}}\right)\right\}^2$

c) $3\left\{\sqrt{K} - \left(1+\frac{1}{\sqrt{2}}\right)\right\}^2$ d) $2\left\{\sqrt{K} - \left(1+\frac{1}{\sqrt{2}}\right)\right\}^2$

Key. D

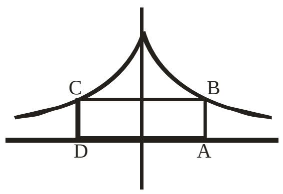
Sol. Given expansion = $\{x-(1+\cos t)\}^2 + \left\{\frac{K}{x} - (1+\sin t)\right\}^2$

45. The maximum area of a rectangle whose two consecutive vertices lie on the x-axis and another two lie on the curve $y = e^{-|x|}$ is equal to

(A) 2e sq. Units (B) $\frac{2}{e}$ sq. Units (C) e sq. units (D) $\frac{1}{e}$ sq. units

Key. B

Sol.



Let the rectangle is (ABCD)

$A = (t,0), B = (t, e^{-t}), C = (-t, e^{-t}), D = (-t, 0)$

ABCD = $2te^{-t} = f(t)$

$\frac{df}{dt} = 2(t(-e^{-t}) + e^{-t}) = 2e^{-t}(1-t)$

$\frac{df}{dt} > 0 \Rightarrow t \in (0,1)$

$\frac{df}{dt} < 0 \Rightarrow t \in (1,\infty)$

t = 1 is point of maxima

Maximum area = $f(1) = \frac{2}{e}$

46. The number of critical points of $f(x) = \frac{|x-1|}{x^2}$ is

(A) 1 (B) 2 (C) 3 (D) None of these

Key. C

Sol. f(x) is not differentiable at x = 0 and x = 1.

$f'(x) = 0$ at $x = 2$

47. A differentiable function f(x) has a relative minimum at x =0, then the function y = f(x) + ax + b has a relative minimum at x = 0 for

(A) all a and all b (B) all b > 0 (C) all b, if a = 0 (D) all a > 0

Key. C

Sol. $f'(0)=0$ and $f''(0)>0$

y = f(x) + ax + b has a relative minimum at x = 0.

Then $\frac{dy}{dx}=0$ at x = 0

$f'(x)+a=0 \Rightarrow a=0$

$f''(x)>0 \Rightarrow f''(0)>0$

Hence y has relative minimum at x = 0 if a = 0 and b $\in$ R.

48. Let A(1, 2), B(3, 4) be two points and C(x, y) be a point such that area of ΔABC is 3 sq.units and $(x-1)(x-3)+(y-2)(y-4)=0$. Then maximum number of positions of C, in the xy plane is

a) 2 b) 4 c)8 d) none of these

Key: D

Hint: (x,y) lies on the circle ,with AB as a diameter . Area

$(\Delta ABC)=3$

$\Rightarrow \left(\frac{1}{2}\right)(AB)(\text{altitude})=3.$

$\Rightarrow \text{altitude } =\frac{3}{\sqrt{2}} \Rightarrow \text{no such "C" exists}$

49. If $y,z>0$ and $y+z=C$, then minimum value of $\sqrt{\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right)}$ is equal to

A) $\frac{C}{2}+1$ B) $\frac{2}{C}+3$ C) $1+\frac{2}{C}$ D) $\frac{C}{2}$

Key: C

Hint: $\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right)=1+\frac{1}{y}+\frac{1}{z}+\frac{1}{yz}$

$=1+\frac{1}{y}+\frac{1}{z}+\frac{1}{yz} \geq 1+\frac{2}{\sqrt{yz}}+\frac{1}{yz}=\left(1+\frac{1}{\sqrt{yz}}\right)^2 = \frac{1}{\sqrt{yz}} \geq \frac{2}{y+z} \geq \frac{2}{C} = \left(1+\frac{1}{\sqrt{yz}}\right)^2 \geq \left(1+\frac{2}{C}\right)^2$

50. Let a, b, c, d, e, f, g, h be distinct elements in the set {–7, –5, –3, –2, 2, 4, 6, 13}. The minimum value of $(a + b + c + d)^2 + (e + f + g + h)^2$ is

a) 30 b) 32 c) 34 d) 40

Key: B

Hint: Note that sum of the elements is 8

Let a + b + c + d = x

$\therefore$ e + f + g + h =8 – x

Again, let y = $x^2 + (8 - x)^2$

$\therefore$ y = $2x^2 - 16x + 64$

= $2[x^2 - 8x + 32]$

= $2(x-4)^2 + 16]$

$\therefore$ $_{min}$ = 32 when x = 4

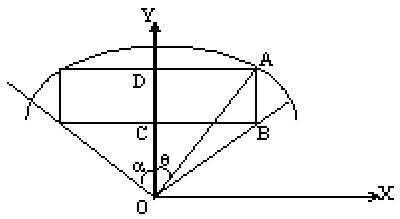
51. A sector subtends an angle 2α at the centre then the greatest area of the rectangle inscribed in the sector is (R is radius of the circle)

a) $R^2 \tan\frac{\alpha}{2}$ b) $\frac{R^2}{2}\tan\frac{\alpha}{2}$ c) $R^2\tan\alpha$ d) $\frac{R^2}{2}\tan\alpha$

Key: A

Hint: Let A be any point on the arc such that $\angle YOA = \theta$

Where $0 \le \theta \le \alpha$



DA = CB = R sin θ, OD = R cos θ

$\Rightarrow$ CO = CB cot $\alpha = R\sin\theta\cot\alpha$

Now, CD = OD - OC = R cos $\theta - R\sin\theta\cot\alpha$

= R (cos θ – sin θ cot α)

Area of rectangle ABCD , S = CD.CB

R = (cos θ - sin θ cot α) R sin θ = R^2(sin θ cos θ - $\sin^2\theta$ cot α)

$\frac{R^2}{2}\left(\sin2\theta - (1-\cos2\theta)\cot\alpha\right)$ $\frac{R^2}{2\sin\alpha}\left[\cos(2\theta-\alpha)\right]$

$S_{man} = \frac{R^2}{\sin\alpha}(1-\cos\alpha)$ $\left(\text{for } \theta = \alpha/2\right)$

Hence, greatest area of the rectangle = $R^2\tan\frac{\alpha}{2}$

52. Let $f:(0,\infty)\to R$ be a (strictly) decreasing function. If $f\left(2a^2+a+1\right) < f\left(3a^2-4a+1\right)$, then the range of $a \in R$ is

(A) $\left(-\infty,\frac{1}{3}\right)\cup(1,\infty)$ (B) (0, 5) (C) $\left(0,\frac{1}{3}\right)\cup(1,5)$ (D) [0, 5]

Key: C

Hint: we have $2a^2+a+1>3a^2-4a+1 \Rightarrow a^2-5a<0 \Rightarrow 0<a<5$ …….(A)

ALSO $3a^2-4a+1>(3a-1)(a-1)>0 \Rightarrow a\in(-\infty,1/3)\cup(1,\infty)$.........(B)

INTERSECTION OF (A) AND (B) YIELDS $a \in (0, 1/3) \cup (1, 5)$

53. The greatest possible value of the expression $\tan\left(x+\frac{2\pi}{3}\right)-\tan\left(x+\frac{\pi}{6}\right)+\cos\left(x+\frac{\pi}{6}\right)$ on the interval $[-5\pi/12, -\pi/3]$ is

(A) $\frac{12}{5}\sqrt{2}$ (B) $\frac{11}{6}\sqrt{2}$ (C) $\frac{12}{5}\sqrt{3}$ (D) $\frac{11}{6}\sqrt{3}$

Key: D

Hint: Let $u = -x - \pi/6$ then $u \in [\pi/6, \pi/4]$ and then $2u \in [\pi/3, \pi/2]$

$\tan(x + 2\pi/3) = -\cot(x + \pi/6) = \cot u$

NOW $\tan(x + 2\pi/3) - \tan(x + \pi/6) + \cos(x + \pi/6)$

$= \cot u + \tan u + \cos u$

$= \frac{2}{\sin 2u} + \cos u$

BOTH $\frac{2}{\sin 2u}$ AND $\cos u$ MONOTONIC DECREASING ON $[\pi/6, \pi/4]$ AND THUS THE GREATEST VALUE OCCURS AT $u = \pi/6$

I.E $\frac{2}{\sin \pi/3} + \cos \pi/6 = \frac{4}{\sqrt{3}} + \frac{\sqrt{3}}{2} = \frac{11}{2\sqrt{3}} = \frac{11\sqrt{3}}{6}$

54. Let the smallest positive value of x for which the function $f(x) = \sin\frac{x}{3} + \sin\frac{x}{11}$, $(x \in R)$ achieves its maximum value be x_0. Express x_0 in degrees i.e, $x_0 = \alpha^0$. Then the sum of the digits in α is

(A) 15 (B) 17 (C) 16 (D) 18

Key: D

Hint The maximum possible values is 2

$\sin(x/3)$ TAKES THE VALUES 1 WHEN

$x/3 = 2n\pi + \pi/2$

I.E $x/3 = 90 + 360m$

$\sin(x/11)$ TAKES THE VALUE 1

WHEN $x/11 = 2n\pi + \pi/2$

I.E $x/11 = 90 + 360n$

WE ARE LOOKING FOR A COMMON SOLUTION

WE HAVE $3m - 11n = 2$. THEN SMALLEST POSITIVE SOLUTION TO THIS IS $m = 8, n = 2$, THUS $x_0 = 8910^0$, GIVING $\alpha = 8910$

55. Let $f(x) = \begin{cases} (x+1)^3 & -2 < x \le -1 \\ x^{2/3} - 1 & -1 < x \le 1 \\ -(x-1)^2 & 1 < x < 2 \end{cases}$

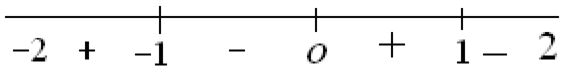
The total number of maxima and minima of $f(x)$ is

(A) 4 (B) 3 (C) 2 (D) 1

KEY : B

HINT : $f'(x)=\begin{cases}3(x+1)^2 & -2<x<-1\\ \frac{2}{3}\times x^{-1/3} & -1<x<1-\{0\}\\ -2(x-1) & 1<x<2\end{cases}$

$f'(x)\,DNE \;\; at \;\; x=-1,0,1$



Sign of $f'(x)$

56. Let f(x) = x^2 – bx + c , b is a odd positive integer, f(x) = 0 have two prime numbers as roots and b + c = 35. Then the global minimum value of f(x) is

(A) $-\frac{183}{4}$ (B) $\frac{173}{16}$

(C) $-\frac{81}{4}$ (D) data not sufficient

KEY : C

SOL : Let α, β be roots of x^2 – bx + c = 0,

Then $\alpha + \beta$ = b

$\Rightarrow$ one of the roots is '2' (Since α, β are primes and b is odd positive integer)

$\therefore$ f(2) = 0 $\Rightarrow$ 2b - c = 4 and b + c = 35

$\therefore$ b = 13, c = 22

Minimum value = $f\left(\frac{13}{2}\right)=-\frac{81}{4}$.

57. Maximum value of $\log_5(3x+4y)$, if $x^2+y^2=25$ is

(A) 2 (B) 3 (C) 4 (D) 5

Key : A

Hint : Since $x^2+y^2=25 \Rightarrow$ x = 5 cos θ and y = 5 sin θ

So, therefore, $\log_5(3x+4y)=\log_5(15\cos\theta+20\sin\theta)$

$\Rightarrow \{\log_5(3x+4y)\}_{max}=2$

58. The greatest area of the rectangular plot which can be laid out within a triangle of base 36 ft. & altitude 12ft equals (Assume that one side of the rectangle lies on the base of the triangle)

(A) 90 (B) 108

(C) 72 (D) 126

Key: B

Hint: Area of rectangle = A = xy ……..(i)

Also $\frac{36}{x}=\frac{12}{12-y}\Rightarrow 3y=\left(36-x\right)$ ……(ii)

$\therefore A=\frac{A}{3}\left(36-x\right)=\frac{1}{3}\left(36x-x^2\right)$

Now $A'\left(x\right)=0\Rightarrow 36-2x=0\Rightarrow x=18$

$A''\left(x\right)=\frac{1}{3}\left(-2\right)<0$

Also $y=\frac{36-x}{3}-\frac{36-18}{3}=6$

$\therefore A_{mas} 18\times 6=108\text{sq.feet}$

59. Let f(x) = $\begin{cases}3x+|a^2-4|, & x<1\\ -x^2+2x+7, & x\ge 1\end{cases}$. Then set of values of a for which f(x) has maximum value at x = 1 is

(A) (3, ∞) (B) [-3, 3]

(C) (-∞, 3) (D) none of these

Key: B

Hint: Since $-x^2 + 2x + 7$ takes maximum value 8 at x = 1, so f(x) take maximum value at x = 1,

if $\lim_{x\to 1} f(x)\le f(1)$

$\Rightarrow |a^2 - 4| \le 5 \Rightarrow a\in [-3, 3]$

60. Let $f\left(x\right)=\left(\sin\theta\right)\left(x^2-2\right)\left(\left(\sin\theta\right)x+\cos\theta\right),\left(\theta\ne m\pi, m\in I\right)$ Then f(x) has

(A) local maxima at certain $x\in R^+$ (B) a local maxima at certain $x\in R^-$

(C) a local minima at certain x = 0 (D) a local minima at certain $x\in R^-$

Key: B

Hint: $f(x) = (\sin^2\theta)x^3 + \frac{1}{2}\sin 2\theta\, x^2 - 2\sin^2\theta.\, x - \sin 2\theta$

$f'(x) = (3\sin^2\theta)x^2 + \sin 2\theta x - 2\sin^2\theta$

Then D > 0 and product of roots < 0

So f(x) has local maxima at some $x\in R^-$

and local minima at some $x\in R^+$

61. Let $g\left(x\right)=\frac{1}{4}f\left(2x^2-1\right)+\frac{1}{2}f\left(1-x^2\right)\forall x\in R$, where $f''\left(x\right)>0\forall x\in R$, g(x) is necessarily increasing in the interval

(A) $\left(-\sqrt{\frac{2}{3}},\sqrt{\frac{2}{3}}\right)$ (B) $\left(-\sqrt{\frac{2}{3}},0\right)\cup\left(\sqrt{\frac{2}{3}},\infty\right)$

(C) $\left(-1,1\right)$ (D) None of these

Key: B

Hint: $f''\left(x\right)>0$

$\Rightarrow f'$ is inc. fn

To find : where g is nec. Inc

g is inc $\Rightarrow g' > 0$

$\Rightarrow \frac{1}{4}.f'\left(2x^2-1\right)(4x)+\frac{1}{2}P\left(1-x^2\right)(-2x)>0$

$\Rightarrow x\left\{f'\left(2x^2-1\right)-f'\left(1-x^2\right)\right\}>0$

Case 1 : $x>0 \to (1) f'\left(2x^2-1\right)>f'\left(1-x^2\right)$

$\Rightarrow 2x^2-1>1-x^2$

$\Rightarrow x \in \left(-\infty, \sqrt{\frac{2}{3}}\right) \cup \left(\sqrt{\frac{2}{3}}, \infty\right) \to (2)$

$(1) \cap (2) \Rightarrow x \in \left(\sqrt{\frac{2}{3}}, \infty\right)$ ....(3)

Case II : $x<0 \to (3)\ f'\left(2x^2-1\right)<f'\left(1-x^2\right)$

$\Rightarrow 2x^2-1<1-x^2$

$\Rightarrow x \in \left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right) \to (4)$

$(3) \cap (4) \Rightarrow \in \left(-\sqrt{\frac{2}{3}}, 0\right) \to (6)$

$\therefore$ g is inc in $x \in (5) \cup (6)$

$\Rightarrow x \in \left(-\sqrt{\frac{2}{3}}, 0\right) \cup \left(\sqrt{\frac{2}{3}}, \infty\right)$

62. A variable line through A(6,8) meets the curve $x^2+y^2=2$ at B and C. P is a point on BC such that AB, AP, AC are in HP. The minimum distance of the origin from the locus of P is

a) 1 b) $1/2$ c) $1/3$ d) $1/5$

Key: D

Hint: Locus of P is the chord of contact of tangent, from A is $3x+4y-1=0$

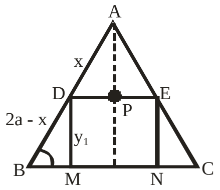
Distance of (0,0) is $1/5$

63. A rectangle is inscribed in an equilateral Δ of side length 2a units. Maximum area of this rectangle is

(A) $\sqrt{3}a^2$ (B) $\frac{\sqrt{3}a^2}{4}$ (C) a^2 (D) $\frac{\sqrt{3}a^2}{2}$

Key. D

Sol.



Let AD = x

BD = (2a − x)

In ΔDBM

$\angle B = \frac{\pi}{3}$

Let DM = y_1

DE = $2x_1$

$\sin 60° = \frac{y_1}{2a - x}$

$y_1 = (2a - x) \times \frac{\sqrt{3}}{2}$

In ΔADP

$\angle D = \frac{\pi}{3}$

$\cos 60° = \frac{x_1}{x}$

$x_1 = x \times \frac{1}{2}$

$2x_1 = x$

Δ(x) = Area of rectangle = $2x_1y$

$\Delta(x) = x \times (2a - x)\frac{\sqrt{3}}{2}$

$\Delta'(x) = \frac{\sqrt{3}}{2}(2a - 2x) = 0 \Rightarrow x = a$

$\Delta''(a) = -ve$

x = a point of maxima

maximum area = $a \times \frac{a\sqrt{3}}{2} = \frac{\sqrt{3}a^2}{2}$

64 If the equation $a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x = 0\,(a_1 \neq 0, n \geq 2)$ has a+ve root $x = \alpha$, then the equation $na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + ... + a_1 = 0$ has a positive root, which is :

1. equal to α 2. $\geq \alpha$ 3. $< \alpha$ 4. $> \alpha$

Key. 3

Sol. $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x = 0$ has a+ve root $x = \alpha$; by observation x = 0 is also a root

$f(\alpha) = f(0) = 0$

f(x) is continuous on $[0, \alpha]$ and differentiable on $(0, \alpha)$ by Rolle's Theorem

$\Rightarrow \exists$ at least one root $c \in (0, \alpha)$

Such that $f'(c) = 0$

$\therefore\ 0 < c < \alpha$

65 The minimum & maximum value of $f(x) = \sin(\cos x) + \cos(\sin x) \forall -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ are respectively.

1. cos 1 and 1+ sin 1
2. sin 1 and 1+ cos 1
3. cos 1 & $\cos\left(\frac{1}{\sqrt{2}}\right)+\sin\left(\frac{1}{\sqrt{2}}\right)$
4. 2

Key. 1

Sol. Given $f(x)=\sin(\cos x)+\cos(\sin x)$

Fact when a function is even & defined in negative as well as positive interval for maxima & minima, we check the maxima/minimum in the positive internal only so it suffices to find the maximum & minimum values of f in $0\le x\le\frac{\pi}{2}$.

Now $x\in[0,\frac{\pi}{2}]$, $\sin(\cos x)\,\&\cos(\sin x)$ are decreasing functions so maximum of f(x) is f(0) & minimum of f(x) is $f(\pi/2)$

$\therefore f(\pi/2)=\sin(\cos\pi/2)+\cos(\sin\pi/2)=\cos 1$

And $f(0)=\sin(\cos 0^0)+\cos(\sin 0^0)=\sin 1+\cos 0^0=1+\sin 1$

66 Let $f(x)=\begin{cases}\frac{\cos(\pi x)}{2} & \forall\, 0\le x<1\\ 3+5x & \forall\, x\ge 1\end{cases}$

1. f(x) has local minimum at x= 1
2. f(x) has local maximum at x= 1
3. f(x) does not have any local maximum or local minimum at x = 1
4. f(x) has a global minimum at x = 1

Key. 1

Sol. $f(x)=\begin{cases}\cos\frac{\pi}{2}x & \forall\, 0\le x<1\\ 5x+3 & \forall\, x\ge 1\end{cases}$

$f'(x)=\begin{cases}-\frac{\pi}{2}\sin\frac{\pi}{2}x & \forall\, 0\le x<1\\ 5 & \forall\, x\ge 1\end{cases}$

$\Rightarrow f'(x)$ changes its sign from –ve to +ve in the immediate neighbourhood of

x= 1

$\Rightarrow f(x)$ changes from decreasing function to increasing function

$\Rightarrow f(x)$ has a local minimum value at x= 1

67 The minimum value of $x^2-x+1+\sin x$ is given by

1. $\frac{1}{4}$
2. $\frac{3}{4}$
3. $-\frac{1}{4}$
4. $-\frac{7}{4}$

Key. 3

Sol. Let $f(x)=x^2-x+1+\sin x$

$= (x-1/2)^2 + (\frac{3}{4} + \sin x)$

$\geq \frac{3}{4} + \sin x$ $\qquad$ $(Q\,(x-\frac{1}{2})^2 \geq 0)$

$\geq \frac{3}{4} - 1 = -1/4$ (Q minimum value of sinx = -1)

68. If $f(x)$ is a differentiable function $\forall x \in R$ so that, $f(2) = 4, f^1(x) \geq 5\ \forall x \in [2,6]$, then, $f(6)$ is

a) ≥ 24 b) ≤ 24 c) ≥ 9 d) ≤ 9

Key. A

Sol. By mean value theorem, $f(6) - f(2) = (6-2)f^1(c)$ where $c \in (2,6)$

$\Rightarrow f(6) = f(2) + 4f^1(c) = 4 + 4f^1(1) > 4 + 4(5)$

$(\therefore f^1(x) \geq 5)$ $f(6) \geq 24$

69. The values of parameter 'a' for which the point of minimum of the function $f(x) = 1 + a^2x - x^3$ satisfies the inequality $\frac{x^2+x+2}{x^2+5x+6} < 0$ are,

a) $(-3\sqrt{3}, -2\sqrt{3}) \cup (2\sqrt{3}, 3\sqrt{3})$ b) $(-5\sqrt{3}, -3\sqrt{3}) \cup (3\sqrt{3}, 5\sqrt{3})$

c) $(-7\sqrt{3}, -5\sqrt{3}) \cup (5\sqrt{3}, 7\sqrt{3})$ d) $(-9\sqrt{3}, -6\sqrt{3}) \cup (6\sqrt{3}, 9\sqrt{3})$

Key. A

Sol. $\frac{x^2+x+2}{x^2+5x+6} < 0 \Rightarrow x \in (-3,-2)$

Let $f(x) = 1 + a^2x - x^3$ for maximum (or) minimum,

$f^1(x) = 0 \Rightarrow a^2 - 3x^2 = 0 \Rightarrow x = \pm\frac{a}{\sqrt{3}}$

And $f^1(x) = -6x$ is positive when x is negative if a > 0 then point of minimum is $\frac{-a}{\sqrt{3}}$

$\Rightarrow -3 < \frac{-a}{\sqrt{3}} < -2$

$\Rightarrow 2\sqrt{3} < a < 3\sqrt{3}$

If $a < 0$, the point of minimum is $a|\sqrt{3}$

$\Rightarrow -3 < \frac{a}{\sqrt{3}} < -2 \Rightarrow -3\sqrt{3} < a < -2\sqrt{3}$

$\Rightarrow a \in (-3\sqrt{3}, -2\sqrt{3}) \cup (2\sqrt{3}, 3\sqrt{3})$

70. Let $\phi(x)=\frac{(x-b)(x-c)}{(a-b)(a-c)}f(a)+\frac{(x-c)(x-a)}{(b-c)(b-a)}f(b)+\frac{(x-a)(x-b)}{(c-a)(c-b)}f(c)-f(x)$ Where $a<c<b$ and $f^{11}(x)$ exists at all points in (a,b). Then, there exists a real number $\mu, a<\mu<b$ such that $\frac{f(a)}{(a-b)(a-c)}+\frac{f(b)}{(b-c)(b-a)}+\frac{f(c)}{(c-a)(c-b)}=$

a) $f^{11}(\mu)$ b) $2f^{11}(\mu)$ c) $\frac{1}{2}f^{11}(\mu)$ d) $\frac{1}{3}f^{111}(\mu)$

Key. C

Sol. Apply RT's, twice

71. If $f(x)$ is a differentiable function $\forall x\in R$ so that, $f(2)=4, f^1(x)\ge 5\ \forall x\in[2,6]$, then, $f(6)$ is

a) ≥ 24 b) ≤ 24 c) ≥ 9 d) ≤ 9

Key. A

Sol. By mean value theorem, $f(6)-f(2)=(6-2)f^1(c)$ where $c\in(2,6)$

$\Rightarrow f(6)=f(2)+4f^1(c)=4+4f^1(1)>4+4(5)$

$(\therefore f^1(x)\ge 5)\ \ f(6)\ge 24$

72. The values of parameter 'a' for which the point of minimum of the function $f(x)=1+a^2x-x^3$ satisfies the inequality $\frac{x^2+x+2}{x^2+5x+6}<0$ are,

a) $(-3\sqrt{3},-2\sqrt{3})\cup(2\sqrt{3},3\sqrt{3})$ b) $(-5\sqrt{3},-3\sqrt{3})\cup(3\sqrt{3},5\sqrt{3})$

c) $(-7\sqrt{3},-5\sqrt{3})\cup(5\sqrt{3},7\sqrt{3})$ d) $(-9\sqrt{3},-6\sqrt{3})\cup(6\sqrt{3},9\sqrt{3})$

Key. A

Sol. $\frac{x^2+x+2}{x^2+5x+6}<0\Rightarrow x\in(-3,-2)$

Let $f(x)=1+a^2x-x^3$ for maximum (or) minimum,

$f^1(x)=0\Rightarrow a^2-3x^2=0\Rightarrow x=\pm\frac{a}{\sqrt{3}}$

And $f^1(x)=-6x$ is positive when x is negative if a > 0 then point of minimum is $\frac{-a}{\sqrt{3}}$

$\Rightarrow -3<\frac{-a}{\sqrt{3}}<-2$

$\Rightarrow 2\sqrt{3}<a<3\sqrt{3}$

If $a<0$, the point of minimum is $a\big|\sqrt{3}$

$\Rightarrow -3<\frac{a}{\sqrt{3}}<-2\Rightarrow -3\sqrt{3}<a<-2\sqrt{3}$

$\Rightarrow a\in(-3\sqrt{3},-2\sqrt{3})\cup(2\sqrt{3},3\sqrt{3})$

73. Let domain and range of $f(x)$ and $g(x)$ are respectively $[0,\infty)$. If $f(x)$ be an increasing function and $g(x)$ be an decreasing function. Also, $h(x)=f(g(x)), h(0)=0$ and $p(x)=h(x^3-2x^2+2x)-h(4)$ then for every $x\in(0,2]$

a) $p(x)\in(0,-h(4))$ b) $p(x)\in[-h(4),0]$

c) $p(x)\in(-h(4),h(4))$ d) $p(x)\in(h(4),h(4)]$

Key. A

Sol. $h(x)=f(g(x))$

$h^1(x)=f^1(g(x))g^1(x)<0\,\forall x\in[0,\infty)$

$\because g^1(x)<0\,\forall x\in[0,\infty)$ and $f^1(g(x))>0\,\forall x\in[0,\infty)$

Also, $h(0)=0$ and hence, $h(x)<0\,\forall x\in[0,\infty)$

$p(x)=h(x^3-2x^2+2x)-h(4)$

$p^1(x)=h^1(x^3-2x^2+2x).(3x^2-4x+2)<0\,\forall x\in(0,2)$

$\because h^1(x^3-2x^2+2x)<0\,\forall x\in(0,\infty)$ and $3x^2-4x+2>0\,\forall x\in R$

$\Rightarrow p(x)$ is an decreasing function

$\Rightarrow p(2)<p(x)<p(0)\,\forall x\in(0,2)$

$\Rightarrow h(4)-h(4)<p(x)<h(0)-h(4)$

$\Rightarrow 0<p(x)<-h(4)$

74. Let $f(x)$ be a positive differentiable function on $[0,a]$ such that $f(0)=1$ and $f(a)=3^{1/4}$ If $f^1(x)\ge(f(x))^3+(f(x))^{-1}$, then, maximum value of a is

a) $\frac{\pi}{12}$ b) $\frac{\pi}{24}$ c) $\frac{\pi}{36}$ d) $\frac{\pi}{48}$

Key. B

Sol. $f^1(x)f(x)\ge(f(x))^4+1$

$\Rightarrow \frac{2f^1(x)f(x)}{\left\{(f(x))^2\right\}^2+1}\ge 2$

$\Rightarrow \int_0^a \frac{2f^1(x)f(x)}{\left\{(f(x))^2\right\}^2+1}\ge 2\int_0^a 1dx$

$\Rightarrow \left|\tan^{-1}(f(x))^2\right|_0^a\ge 2a \Rightarrow \frac{\pi}{3}-\frac{\pi}{4}\ge 2a$

75. The least value of 'a' for which the equation $\frac{4}{\sin x}+\frac{1}{1-\sin x}=a$ for atleast one solution on the interval $\left(0,\frac{\pi}{2}\right)$ is,

a) 1 b) 4 c) 8 d) 9

Key. D

Sol. $\because a = \dfrac{4}{\sin x} + \dfrac{1}{1-\sin x}$, where a is least

$\Rightarrow \dfrac{da}{dx} = \left(\dfrac{-4}{\sin^2 x} + \dfrac{1}{(1-\sin x)^2} \right)\cos x = 0$

$\because \cos x \neq 0 \Rightarrow \sin x = 2/3$

$\dfrac{d^2a}{dx^2} = 45 > 0$ for $\sin x = 2/3 \Rightarrow \dfrac{4}{2/3} + \dfrac{1}{1-2/3} = 6+3 = 9$

76. $f(x) = x^4 - 10x^3 + 35x^2 - 50x + c.$ where c is a constant. the number of real roots of $f'(x) = 0$ and $f''(x) = 0$ are respectively

(1) 1, 0 (2) 3, 2 (3) 1, 2 (4) 3, 0

Key. 2

Sol. $g(x) = (x-1)(x-2)(x-3)(x-4)$

$f(x) = g(x) + c_0 : c_0 = c - 24$

$g(x) = 0$ has 4 roots viz. $x = 1, 2, 3, 4$

$f'(x) = g'(x)$ and $f''(x) = g''(x)$

By Rolle's theorem $g'(x) = 0$ has min. one root in each of the intervals (1, 2); (2, 3); (3, 4)

By Rolle's theorem, between two roots of $f'(x) = 0, f''(x) = 0$ has minimum one root.

77. Let $h(x) = f(x) - (f(x))^2 + (f(x))^3$ for every real number x. Then

(1) h is increasing whenever f is increasing

(2) h is increasing whenever f is decreasing

(3) h is decreasing whenever f is increasing

(4) nothing can be said in general

Key. 1

Sol. $h'(x) = f'(x) - 2f(x)f'(x) + 3(f(x))^2 f'(x)$

$= f'(x)\left[1 - 2f(x) + 3(f(x))^2\right]$

Since, $1 - 2f(x) + 3(f(x))^2 > 0$ for all $f(x)$

$\Rightarrow h'(x) > 0$ if $f'(x) > 0$

$\Rightarrow$ h is increasing when ever f is increasing and $h'(x) < 0$ if $f'(x) < 0$

$\Rightarrow h$ is decreasing when ever f is decreasing.

78. The set of critical points of the function $f(x) = (x-2)^{\frac{2}{3}}.(2x+1)$ is

(1) $\{1, 2\}$ (2) $\left\{-\dfrac{1}{2}, 1\right\}$ (3) $\{-1, 2\}$ (4) $\{1\}$

Key. 1

Sol. $f'(x)=(x-2)^{\frac{2}{3}}.2+(2x+1).\frac{2}{3}\frac{1}{(x-2)^{\frac{1}{3}}}$

$=2\left[\frac{3(x-2)+2x+1}{3(x-2)^{\frac{1}{3}}}\right]$

$=\frac{2}{3}\frac{(5x-5)}{(x-2)^{\frac{1}{3}}}=\frac{10}{3}\frac{(x-1)}{(x-2)^{\frac{1}{3}}}$

Critical points are $x=1$ and $x=2$

79. For $x\in(0,1)$ which of the following is true?

(1) $e^x<1+x$ (2) $\log_e(1+X)<X$ (3) $\sin x>x$ (4) $\log_e x>x$

Key. 2

Sol. Let $f(x)=e^x-1-x, g(x)=\log(1+x)-x$

h(x) = sin x – x, p(x) =log x – x

for g(x) = log(1 + x) – x

$g'(x)=\frac{1}{1+x}-1=\frac{-x}{1+x}<0 \quad \forall x\in(0,1)$

g(x) is decreasing when 0 < x < 1.

g(0) > g(x)_ $\Rightarrow$ log (1 + x) < x

Similarly it can be done for other functions.

80. $f(x)=|x\ln x|:\ x\in(0,1)$, then $f(x)$ has maximum value=

(1) e (2) $\frac{1}{e}$ (3) 1 (4) None of these

Key. 2

Sol. $f(x)=-x\ln x$

$\lim_{x\to 0+} f(x)=0$

$$f'(x)=-(1+\ln x)\begin{cases}>0 & \text{if} \quad 0<x<\frac{1}{e}\\ =0 & \text{if} \quad x=\frac{1}{e}\\ <0 & \text{if} \quad \frac{1}{e}<x<1\end{cases}$$

f has maximum value at $x=\frac{1}{e}$ and $f\left(\frac{1}{e}\right)=\frac{1}{e}$

81. Let $f(x)=\begin{cases}(x+1)^3 & -2<x\le -1\\ x^{2/3}-1 & -1<x\le 1\\ -(x-1)^2 & 1<x>2\end{cases}$

The total number of maxima and minima of $f(x)$ is

(1) 4 (2) 3 (3) 2 (4) 1

Key. 2

Sol.

$$f'(x)=\begin{cases}3(x+1)^2 & -2<x<-1\\ \frac{2}{3}\times x^{-1/3} & -1<x<1-\{0\}\\ -2(x-1) & 1<x<2\end{cases}$$

$f'(x)\, DNE\ \ at\ \ x=-1,0,1$

$-2\quad +\quad -1\quad -\quad 0\quad +\quad 1\quad -\quad 2$

Sign of $f'(x)$

82. Given $f(x)=\begin{cases}x^2e^{2(x-1)} & 0\le x\le 1\\ a\cos(2x-2)+bx^2 & 1<x\le 2\end{cases}$ $f(x)$ is differentiable at $x=1$ provided

(1) $a=-1, b=2$ (2) $a=1, b=-2$

(3) $a=-3, b=4$ (4) $a=3, b=-4$

Key. 1

Sol. $f(1+0)=f(1-0)\Rightarrow a+b=1$

$f^1(1-0)=f^1(1+0)\Rightarrow 4=2b$

$\Rightarrow b=2, a=-1$

83. Define $f:[0,\pi]\to R$ by is continuous at $x=\frac{\pi}{2}$, then k=

(1) $\frac{1}{12}$ (2) $\frac{1}{6}$ (3) $\frac{1}{24}$ (4) $\frac{1}{32}$

Key. 1

Sol. Let $\sin x=t$ and evaluate $\lim\limits_{t\to 1}\frac{t^2}{1-t^2}\left[\sqrt{2t^2+3t+4}-\sqrt{t^2+6t+2}\right]$ by rationalization

84. If $f(x)=\frac{1}{(x-1)(x-2)}$ and $g(x)=\frac{1}{x^2}$, then the number of discontinuities of the composite function $f(g(x))$ is

(1) 2 (2) 3 (3) 4 (4) ≥ 5

Key. 4

Sol. Conceptual

85. Find which function does not obey lagrange's mean value theorem in [0, 1]

(1) $f(x)=\begin{cases}\frac{1}{2}-x: & x<\frac{1}{2}\\ \left(\frac{1}{2}-x\right)^2: & x\geq\frac{1}{2}\end{cases}$ (2) $f(x)=\begin{cases}\frac{\sin x}{x} & : x\neq 0\\ 1 & if \quad x=0\end{cases}$

(3) $f(x)=x|x|$ (4) $f(x)=|x|$

Key. 1

Sol. In (a), $f'\left(\frac{1}{2}-\right)=-1$ while $f'\left(\frac{1}{2}+\right)=0$

f is not differentiable at $x=\frac{1}{2}$.

86. Rolle's theorem holds in [1, 2] for the function $f(x)=x^3+bx^2+cx$ at the point $\frac{4}{3}$. The values of b, c are respectively

(1) 8, –5 (2) –5, 8 (3) 5, –8 (4) –5, –8

Key. 2

Sol. $f(1)=f(2)$ and $f'(4/3)=0$

$3b+c=-7$ and $8b+3c=-16$

$b=-5; c=8$

87. If $f(x)=\begin{cases}x^\alpha \log x, & x>0\\ 0, & x=0\end{cases}$ and Rolle's theorem is applicable to f(x) for $x\in[0, 1]$ then α is equal to

1. -2 2. -1 3. 0 4. 1/2

Key. 4

Sol. for Rolle's theorem in [a, b]

$f(a)=f(b)\Rightarrow f(0)=f(1)=0$

Since the function has to be continuous in [0, 1]

$\underset{x\to 0^+}{Lt} f(x)=f(0)$

$\Rightarrow \underset{x\to 0^+}{Lt} x^\alpha \log x=0$

$\Rightarrow \underset{x\to 0}{Lt} \frac{\log x}{x^{-\alpha}}=0$

Applying L – H rule

$$\underset{x\to 0}{Lt}\frac{1/x}{-\alpha x^{-\alpha-1}}=0$$

$$\Rightarrow \underset{x\to 0}{Lt}\frac{-x^{\alpha}}{\alpha}=0$$

This is true for $\alpha>0$

88. Let $f:(0,\infty)\to R$ be a (strictly) decreasing function.

If $f\left(2a^2+a+1\right)<f\left(3a^2-4a+1\right)$, then the range of $a\in R$ is

a) $\left(-\infty,\frac{1}{3}\right)\cup(1,\infty)$ b) $(0,\ 5)$ c) $\left(0,\frac{1}{3}\right)\cup(1,5)$ d) $[0,\ 5]$

Key. 3

Sol. we have $2a^2+a+1>3a^2-4a+1\Rightarrow a^2-5a<0\ \Rightarrow 0<a<5$ ....(A)

Also $3a^2-4a+1>(3a\ -1)(a-1)>0\ \Rightarrow a\in(-\infty,1/3)\cup(1,\infty)$.........$(B)$

Intersection of (A) and (B) yields $a\in(0,\ 1/3)\cup(1,5)$

89. Suppose $f:[1,2]\to R$ is such that $f(x)=x^3+bx^2+cx$. If f satisfies the hypothesis of Rolle's theorem on $[1,2]$ and the conclusion of Rolle's theorem holds for f on $[1,2]$ at the point $\frac{4}{3}$, then

a) $b=-5$ b) $b=5$ c) $c=-8$ d) $c=9$

Key. 1

Sol. $f(1)=f(2)\Rightarrow 1+b+c=8+4b+2c\Rightarrow 3b+c=-7\to(1)$.

Now, $f'(x)=3x^2+2bx+c$; $\therefore f'\left(\frac{4}{3}\right)=0$ (given) $\Rightarrow 3.\frac{16}{9}+2b.\frac{4}{3}+c=0\ \Rightarrow 8b+3c=-16$ → (2). From $(1),(2)$ we get $b=-5$ and $c=8$.

90. Given a function $f:[0,4]\to R$ is differentiable, then for some $a,b\in(0,4)$ $[f(4)]^2-[f(0)]^2$ =

a) $8f'(b)f(a)$ b) $4f'(b)f(a)$ c) $2f'(b)f(a)$ d) $f'(b)f(a)$

Key. 1

Sol. Since $f(x)$ is differentiable in [0, 4], using Lagrange's Mean Value Theorem.

$f'(b)=\frac{f(4)-f(0)}{4},\ b\in(0,4)$ .... (1)

Now, $\{f(4)\}^2-\{f(0)\}^2=\frac{4\{f(4)-f(0)\}}{4}\{f(4)+f(0)\}$ $=4f'(b)\{f(4)+f(0)\}$ .... (2)

Also, from Intermediate Mean Value Theorem,

$\frac{f(4)+f(0)}{2}=f(a)$ for $a\in(0,4)$

Hence, from (2) $[f(4)]^2-[f(0)]^2=8f'(b)f(a)$

91. Suppose α, β and θ are angles satisfying $0<\alpha<\theta<\beta<\frac{\pi}{2}$, then $\frac{\sin\alpha-\sin\beta}{\cos\beta-\cos\alpha}$ =

a) $\tan\theta$ b) $-\tan\theta$ c) $\cot\theta$ d) $-\cot\theta$

Key. 3

Sol. Let $f(x)=\sin x$ and $g(x)=\cos x$, then f and g are continuous and derivable. Also, $\sin x\neq 0$ for any $x\in\left(0,\frac{\pi}{2}\right)$ so by Cauchy's MVT, $\frac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)}=\frac{f'(\theta)}{g'(\theta)}\Rightarrow\frac{\sin\beta-\sin\alpha}{\cos\beta-\cos\alpha}=\frac{\cos\theta}{-\sin\theta}$

92. If $f''(x)>0$, $\forall\ x\in R$, $f'(3)=0$ and $g(x)=f(\tan^2 x-2\tan x+4)$, $0<x<\frac{\pi}{2}$, then $g(x)$ is increasing in

a) $\left(0,\frac{\pi}{4}\right)$ b) $\left(\frac{\pi}{6},\frac{\pi}{3}\right)$ c) $\left(0,\frac{\pi}{3}\right)$ d) $\left(\frac{\pi}{4},\frac{\pi}{2}\right)$

Key. 4

Sol. $g'(x)=(f'((\tan x-1)^2+3))2(\tan x-1)\sec^2 x$ since $f''(x)>0\Rightarrow f'(x)$ is increasing

So, $f'((\tan x-1)^2+3)>f'(3)=0\ \forall x\in\left(0,\frac{\pi}{4}\right)\cup\left(\frac{\pi}{4},\frac{\pi}{2}\right)$

Also, $(\tan x-1)>0$ for $x\in\left(\frac{\pi}{4},\frac{\pi}{2}\right)$. So, $g(x)$ in increasing in $\left(\frac{\pi}{4},\frac{\pi}{2}\right)$

93. Let $f(x)=2x^3+ax^2+bx-3\cos^2 x$ is an increasing function for all $a,b,x\in R$. Then

a) $a^2-6b-18>0$ b) $a^2-6b+18<0$ c) $a^2-3b-6<0$ d) $a>0$, $b>0$

Key. 2

Sol. $f(x)=2x^3+ax^2+bx-3\cos^2 x$

$\therefore\ f'(x)=6x^2+2ax+b+3\sin 2x$

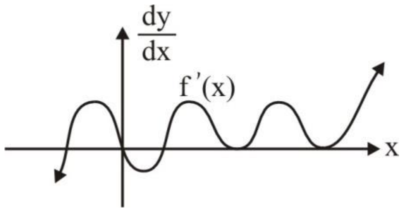
$\therefore\ f(x)$ is increasing for all $x\Rightarrow 6x^2+2ax+b+3\sin 2x>0$

Also, $6x^2+2ax+b+3\sin 2x\geq 6x^2+2ax+b-3$ as $\sin 2x\geq -1$

Hence $6x^2+2ax+b-3>0$

$\therefore\ 4a^2-4\,.\,6(b-3)<0\Rightarrow a^2-6b+18<0$

94. f : R → R be differentiable function. Study following graph of $f'(x)=\frac{dy}{dx}$. Find sum of total no. of points of inflexion and extrema of y = f(x).



Key. 9

Sol. No. of points of inflexion = 6, no. of extrema = 3

95. The minimum value of $(8x^2 + y^2 + z^2)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2$, (x, y, z > 0), is

(A) 8 (B) 27
(C) 64 (D) 125

Key. C

Sol. $$\frac{2(2x)^2+y^2+z^2}{2+1+1} \geq \left(\frac{2(2x)+y+z}{2+1+1}\right)^2 \geq \left(\frac{2+1+1}{\frac{2}{2x}+\frac{1}{y}+\frac{1}{z}}\right)^2 \Rightarrow (8x^2+y^2+z^2)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2 \geq 64.$$

96. Let $f(x) = \begin{cases} (3-\sin(1/x))\,|x|, & x \neq 0 \\ 0 \quad , & x = 0 \end{cases}$. Then at $x = 0$ f has a

(A) maxima (B) minima
(C) neither maxima nor minima (D) point of discontinuity

Key. B

Sol. f is continuous at x = 0
Further f(0 + h) > f(0) and f(0 − h) > f(0), for positive 'h'. Hence f has minimum value at x = 0.

97. A car is to be driven 200kms on a highway at an uniform speed of x km/hrs (speed Rules of the high way require $40 \leq x \leq 70$) . The cost of diesel is Rs 30/litre and is consumed at the rate of $100+\frac{x^2}{60}$ litres per hour. If the wage of the driver is Rs 200 per hour then the most economical speed to drive the car is

a) 55.5 b) 70 c) 40 d) 80

Key. B

Sol. Let cost incurred to travel 200 kms be
$C(x)$.*Then*

$$C(x) = \left(100+\frac{x^2}{60}\right)\frac{200}{x}\times 30 + 200\times\frac{200}{x}$$

$$= \frac{640000}{x} + 100x$$

$$\Rightarrow C'(x) < 0 \;\; for \;\; x \in [40,70]$$

$\Rightarrow C(x)$ is minimum for x = 70 in $x \in [40,70]$.

98. Let a, n $\in$ N such that a $\geq$ n^3 then $\sqrt[3]{a+1}-\sqrt[3]{a}$ is always

(A) less than $\frac{1}{3n^2}$ (B) less than $\frac{1}{2n^3}$

(C) more than $\frac{1}{n^3}$ (D) more than $\frac{1}{4n^2}$

Key. A

Sol. Let $f(x)=x^{1/3} \Rightarrow f'(x)=\frac{1}{3x^{2/3}}$, applying LMVT in [a, a + 1], we get one c$\in$(a, a + 1)

$$f'(c)=\frac{f(a+1)-f(a)}{a+1-a} \Rightarrow \sqrt[3]{a+1}-\sqrt[3]{a}=\frac{1}{3c^{2/3}}<\frac{1}{3a^{2/3}}\leq\frac{1}{3n^2} \Rightarrow \sqrt[3]{a+1}-\sqrt[3]{a}<\frac{1}{3n^2}\ \forall a\geq n^3$$

99. If $x^2 + 9y^2 = 1$, then minimum and maximum value of $3x^2 - 27y^2 + 24xy$ is

(A) 0, 5 (B) – 5, 5

(C) – 5, 10 (D) 0, 10

Key. B

Sol. Put x = cos θ, $y=\frac{1}{3}\sin\theta$

Let $u = 3x^2 - 27y^2 + 24xy$

$u = 3\cos 2\theta + 4\sin 2\theta$

$-5 \leq u \leq 5$.

100. Let the function g : $(-\infty, \infty) \rightarrow \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ be given by $g(u) = 2\tan^{-1}(e^u)-\frac{\pi}{2}$. Then g is

(A) even and is strictly increasing in $(0, \infty)$

(B) odd and is strictly decreasing in $(-\infty, \infty)$

(C) odd and is strictly increasing in $(-\infty, \infty)$

(D) neither even nor odd but is strictly increasing in $(-\infty, \infty)$

Key. C

Sol. $g(-u)=2\tan^{-1}e^{-u}-\frac{\pi}{2}=2\cot^{-1}e^{u}-\frac{\pi}{2}=2\left(\frac{\pi}{2}-\tan^{-1}e^{u}\right)-\frac{\pi}{2}$

$=-\left(2\tan^{-1}e^{u}-\frac{\pi}{2}\right)=-g(u)$

$g'(u)=2.\frac{1}{1+e^{2u}}.e^{u}>0.$

So, g(u) is odd and strictly increasing.

101. Let $f(x)$ be a differentiable function in the interval $(0,2)$, then the value of $\int_0^2 f(x)dx$ is___

a) $f(c)$ where $c\in(0,2)$ b) $2f(c)$ where $c\in(0,2)$

c) $f'(c)$ where $c \in (0,2)$ d) $f''(0)$

Key. B

Sol. Consider $g(t) = \int_0^t f(x)dx$ Applying LMVT in (0,2)

$\dfrac{g(2)-g(0)}{2-0} = g'(c);\ c \in (0,2)$ $\Rightarrow \int_0^2 f(x)dx = 2f(c)$ for $c \in (0,2)$

102. Let $g(x) = \int\limits_{1-x}^{1+x} t\,|f'(t)|\,dt$, where f(x) does not behave like a constant function in any interval (a, b) and the graph of y = f ′(x) is symmetric about the line x = 1. Then

(A) g(x) is increasing $\forall$ x $\in$ R (B) g(x) is increasing only if x < 1

(C) g(x) is increasing if f is increasing (D) g(x) is decreasing $\forall$ x $\in$ R

Key. A

Sol. $g'(x) = (1+x)\,|f'(x+1)| + (1-x)\,|f'(1-x)|$

$= |f'(1+x)|\,(1+x+1-x) > 0 \quad \forall x \in R$

103. The equation $2x^3 - 3x^2 - 12x + 1 = 0$ has in the interval $(-2,1)$

A) no real root B) exactly one real root

C) exactly two real roots D) all three real roots

Key. C

Sol. Let $f(x) = 2x^3 - 3x^2 - 12x + 1$

$f(-2) < 0;\ f(0) > 0;\ f(1) < 0$

$\therefore f(x) = 0$ has atleast two roots in the interval $(-2,1)$.

Suppose all the real roots of $f(x) \in (-2,1)$.

Then by Rolle's theorm, both the roots of the equation $f^1(x) = 0$ should belong to $(-2,1)$

$f^1(x) = 6x^2 - 6x - 12 = 0 \Rightarrow x^2 - x - 2 = 0$

$\Rightarrow (x-2)(x+1) = 0 \Rightarrow x = 2, -1$

104. If f: [1, 5] $\to$ R is defined by $f(x) = (x-1)^{10} + (5-x)^{10}$ then the range of f is

A) $[0, 2^{20}]$ B) $[0, 2^{11}]$ C) $[2^{11}, 2^{20}]$ D) R^+

Key. C

Sol. Conceptual

105. If $3(a+2c) = 4(b+3d) \neq 0$ then the equation $ax^3 + bx^2 + cx + d = 0$ will have

(A) no real solution (B) at least one real root in $(-1,0)$

(C) at least one real root in $(0,1)$ (D) none of these

Key. B

Sol. Consider $f(x) = \dfrac{ax^4}{4} + \dfrac{bx^3}{3} + \dfrac{cx^2}{2} + dx$ and apply Rolle's theorem

106. The function in which Rolle's theorem is verified is

(A) $f(x) = \log\left(\dfrac{x^2+ab}{(a+b)x}\right)$ in $[a,b]$ (where $0 < a < b$) (B) $f(x) = (x-1)(2x-3)$ in $[1,3]$

(C) $f(x)=2+(x-1)^{2/3}$ in $[0\ ,2]$ (D) $f(x)=\cos(1/x)$ in $[-1,1]$

Key. A

Sol. $f(x)=\log\left(\dfrac{x^2+ab}{(a+b)x}\right)$ is continuous in $[a,b]$ and differentiable in (a,b) and $f(a)=f(b)$

107. If $f(x)=x^{\alpha}\log x$ and $f(0)=0$ then the value of α for which Rolle's theorem can be applied in $[0,1]$ is

(A) -2 (B) -1 (C) 0 (D) $\dfrac{1}{2}$

Key. D

Sol. for the function $f(x)=x^{\alpha}\log x$ Rolle's theorem is applicable for $\alpha>0$ in $[0,1]$

108. Let $f(x)=2x^2-ln|x|, x\neq 0$, then $f(x)$ is

a) monotonically increasing in $\left(-\dfrac{1}{2},0\right)\cup\left(\dfrac{1}{2},\infty\right)$

b) monotonically decreasing in $\left(-\dfrac{1}{2},0\right)\cup\left(\dfrac{1}{2},\infty\right)$

c) monotonically increasing in $\left(-\infty,-\dfrac{1}{2}\right)\cup\left(0,\dfrac{1}{2}\right)$

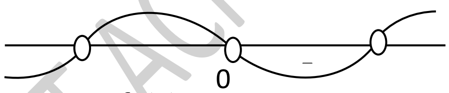
d) monotonically decreasing in $\left(-\infty,-\dfrac{1}{2}\right)\cup\left(0,\dfrac{1}{2}\right)$

Key. A,D

Sol. $\because f(x)=2x^2-\ln|x|$

$\therefore \quad f'(x)=4x-\dfrac{1}{x}$

$=\dfrac{(2x+1)(2x-1)}{x}$



For increasing, $f'(x)>0$

$\therefore \quad x\in\left(-\dfrac{1}{2},0\right)\cup\left(\dfrac{1}{2},\infty\right)$

And for decreasing, $f'(x)<0$

$\therefore \quad x\in\left(-\infty,-\dfrac{1}{2}\right)\cup\left(0,\dfrac{1}{2}\right)$

109. For $x>1, y=\log_e x$ satisfies the inequality

a) $x-1>y$ b) $x^2-1>y$ c) $y>x-1$ d) $\dfrac{x-1}{x}<y$

Key. A,B,D

Sol. Let $f(x)=\log_e x-(x-1)$

$\Rightarrow f'(x)=\dfrac{1}{x}-1=\dfrac{1-x}{x}<0$

$\because f(x)$ is decreasing function ($\because x > 1$)

$$x > 1 \Rightarrow f(x) < f(1)$$

$\Rightarrow \quad \log_e x - (x-1) < 0$

$\Rightarrow \quad (x-1) > \log_e x$

$= y$

Or $\quad (x-1) > y$

Now, let $g(x) = \log_e x - \left(x^2 - 1\right)$.

$\Rightarrow g'(x) = \dfrac{1}{x} - 2x = \left(\dfrac{1-2x^2}{x}\right) < 0$ (for $x > 1$)

$\therefore g(x)$ is decreasing function

$\because \quad x > 1 \Rightarrow g(x) < g(1)$

$\Rightarrow \quad \log_e x - (x^2 - 1) < 0$

$\therefore$

Or $\quad (x^2 - 1) > y$

Again, let $h(x) = \dfrac{x-1}{x} - \log_e x$

$\therefore \quad h'(x) = 0 + \dfrac{1}{x^2} - \dfrac{1}{x} = \dfrac{1-x}{x^2} < 0 \qquad$ (for $x > 1$)

$\therefore h(x)$ is decreasing function

$\because \quad x > 1 \Rightarrow h(x) < h(1)$

$\Rightarrow \quad \dfrac{x-1}{x} - \log_e x < 0$

$\Rightarrow \quad \dfrac{x-1}{x} < y$.

110. Let 'a'(a < 0, a $\notin$ I) be a fixed constant and 't' be a parameter then the set of values of 't' for the function f(x) = $\left(\dfrac{|[t]+1|+a}{|[t]+1|+1-a}\right)$x to be a non increasing function of x,

([·] denotes the greatest integer function) is

a) [[a], [–a + 1]) b) [[a], [–a]) c) [[a + 1], [–a + 1]) d) [[a – 1], [–a + 1])

Key. B

Sol. f′(x) ≤ 0 $\Rightarrow \dfrac{|[t]+1|+a}{|[t]+1|+1-a} \le 0$, but as a < 0, 1 – a > 0.

So |[t] + 1| ≤ -a $\Rightarrow$ a ≤ [t] + 1 ≤ -a $\Rightarrow$ a – 1 ≤ [t] ≤ -a – 1

$\Rightarrow$ [a] ≤ [t] ≤ [-a] – 1 (as a$\notin$I) $\Rightarrow$ [a] ≤ t < [-a]

111. The number of critical values of $f(x) = \dfrac{|x-1|}{x^2}$ is

a) 0 b) 1 c) 2 d) 3

Key. D

Sol. $f'(x)=\dfrac{|x-1|\left\{\dfrac{x^2}{x-1}-2x\right\}}{x^4}$ $\Rightarrow f'(x)=0 \quad at\, x=2$

$\Rightarrow f'(x)$ does not exist at x = 0,1

112. The absolute minimum value of $x^2-4x-10|x-2|+29$ occurs at

a) one value of $x\in R$ b) at two values of $x\in R$ c) *x=7,3* d) no value of $x\in R$

Key. B

Sol. Given function is $(|x-2|-5)^2$ which has global minimum value equal to 0 ,when $|x-2|=5$

113. The function $f(x)=x(x-1)(x-2)(x-3)-----(x-50)$ in (0,50) has *m* local maxima and *n* local minimum then

a) *m=25 , n=26* b) *m=26 , n=25* c) *m=n=26* d) *m=n=25*

Key. D

Sol. From the given conditions, it follows that $f(x)=x^3+1\Rightarrow f^1(2)=3(2)^2=12$

114. The value of *c* in the Lagrange's mean value theorem applied to the function $f(x)=x(x+1)(x+2)$ for $0\le x\le 1$ is

a) $\dfrac{\sqrt{21}}{4}$ b) $\dfrac{\sqrt{21}-3}{3}$ c) $\dfrac{1}{5}$ d) $\dfrac{\sqrt{21}+3}{8}$

Key. B

Sol. $f^1(c)=3c^2+6c+2=\dfrac{f(1)-f(0)}{1}=6\Rightarrow 3c^2+6c-4=0\Rightarrow c=-1+\dfrac{\sqrt{21}}{3}\in(0,1)$

115. A twice differentiable function $f(x)$ on *(a,b)* and continuous on [a,b] is such that $f^{11}(x)<0$ for all $x\in(a,b)$ then for any $c\in(a,b),\dfrac{f(c)-f(a)}{f(b)-f(c)}>$

a) $\dfrac{b-c}{c-a}$ b) $\dfrac{c-a}{b-c}$ c) $(b-c)(c-a)$ d) $\dfrac{1}{(b-c)(c-a)}$

Key. B

Sol. Let $u\in(a,c),v\in(c,b)$ then by LMVT on (a,c),(c,b) it follows

$$f^1(u)=\frac{f(c)-f(a)}{c-a},\ f^1(v)=\frac{f(b)-f(c)}{b-c}.$$

But u<v and $f^{11}(x)<0$ for all $x\in(a,b)\Rightarrow f^1(x)\downarrow\Rightarrow f^1(u)>f^1(v)\Rightarrow\dfrac{f(c)-f(a)}{f(b)-f(c)}>\dfrac{c-a}{b-c}$.

116. The number of roots of $x^5-5x+1=0$ in (-1,1) is

a) 0 b) 1 c) 2 d) 3

Key. B

Sol. Let $f(x)=x^5-5x+1$. Q $f(1)f(-1)<0$ $\exists$ atleast one root say α of $f(x)=0$ in (-1,1). If $\exists$ another root $\beta\ (\alpha<\beta)$ in (-1,1) then by RT applied to $[\alpha,\beta]$, it follows that there exist $\gamma\in(\alpha,\beta)$ such that $f^1(\gamma)=5\gamma^4-5=0\ \ i.e\ \ \gamma=1,-1$ but $\gamma\in(\alpha,\beta)\subset(-1,1)\therefore\gamma\ne1,-1$, *a contradiction*. Hence number of roots of $f(x)=0\ in\ (-1,1)$ is 1.

117. If $\frac{a_0}{5}+\frac{a_1}{4}+\frac{a_2}{3}+\frac{a_3}{2}+a_4=0$ then the equation $a_0x^4+a_1x^3+a_2x^2+a_3x+a_4=0$

A) does not have root between 0 and 1
B) possesses at least one root between 0 and 1
C) has exactly one root between 0 and 1
D) has a root between 1 and 2

Key. B

Sol. Consider the function $f(x)=\frac{a_0x^5}{5}+\frac{a_1x^4}{4}+\frac{a_2x^3}{3}+\frac{a_3x^2}{2}+a_4x$

$f(0)=0$ *and* $f(1)=0$ by hypothesis

$\therefore$ f satisfies all conditions of Rolle's theorem

$\therefore$ $f^1(x)=0$ has at least one root in (0,1)

118. The largest area of the rectangle which has one side on the X-axis and two vertices on the curve $y=e^{-x^2}$ is

A) $\frac{1}{\sqrt{2e}}$
B) $\frac{1}{2e^2}$
C) $\sqrt{\frac{2}{e}}$
D) $\frac{\sqrt{2}}{e^2}$

Key. C

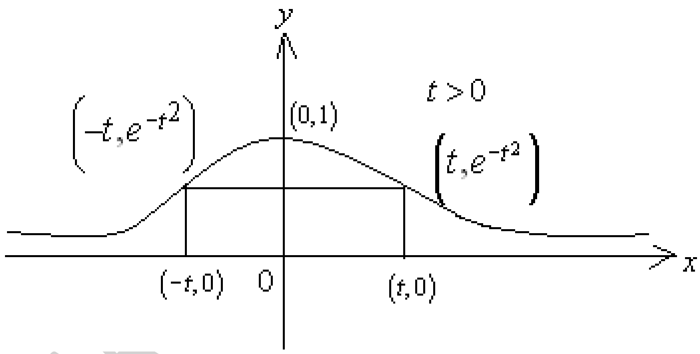
Sol. Let $f(t)=t\ e^{-t^2}$

$f^1(t)=-2t^2\ e^{-t^2}+e^{-t^2}$

$=e^{-t^2}(1-2t^2)$

$f^1(t)=0\Rightarrow t=\frac{1}{\sqrt{2}}$

Max area $=2\times\frac{1}{\sqrt{2}}\times e^{\frac{-1}{2}}=\frac{\sqrt{2}}{\sqrt{e}}$



119. $f(x)=\frac{x}{\sin x}$ and $g(x)=\frac{x}{\tan x}$ where $0<x\le 1$. Then in this interval

(a) f(x) and g(x) both are increasing
(b) f(x) is decreasing and g(x) is increasing
(c) f(x) is increasing and g(x) is decreasing
(d) none of the above

Key. C

Sol. $f'(x)=\frac{\sin x-x\cos x}{\sin^2 x}$

Now $h(x)=\sin x-x\cos x$

$h'(x)=x\sin x>0\quad\forall\, 0<x\le 1$

h(x) is increasing in (0, 1]

h(0) < h(x) $\Rightarrow$ sin x – x cos x > 0 for 0 < x ≤ 1

$\Rightarrow \; f'(x) > 0$

Hence f(x) is increasing. Similarly it can be done for g(x).

120. For $x \in (0,1)$, which of the following is true?

(a) $e^x < 1 + x$ (b) $\log_e(1+x) < x$ (c) $\sin x > x$ (d) $\log_e x > x$

Key. B

Sol. Let $f(x) = e^x - 1 - x$, $g(x) = \log(1+x) - x$

h(x) = sin x – x, p(x) =log x – x

for g(x) = log(1 + x) – x

$$g'(x) = \frac{1}{1+x} - 1 = \frac{-x}{1+x} < 0 \;\; \forall x \in (0, 1)$$

g(x) is decreasing when 0 < x < 1.

g(0) > g(x)_ $\Rightarrow$ log (1 + x) < x

Similarly it can be done for other functions.

121. $f(x) = |x \ln x| : x \in (0,1)$ has maximum value

(A) e (B) $\frac{1}{e}$

(C) 1 (D) None of these

Key. B

Sol. $f(x) = -x \ln x$

$$\lim_{x \to 0+} f(x) = 0$$

$$f'(x) = -(1 + \ln x)\begin{cases} > 0 & \text{if} \quad 0 < x < \frac{1}{e} \\ = 0 & \text{if} \quad x = \frac{1}{e} \\ < 0 & \text{if} \quad \frac{1}{e} < x < 1 \end{cases}$$

f has maximum value at $x = \frac{1}{e}$ and $f\left(\frac{1}{e}\right) = \frac{1}{e}$

122. $f(x) = \begin{cases} x^a \ln x & : \quad x > 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$.

If Lagrange's theorem applies to f on [0, 1] then 'a' can be

(A) –2 (B) –1

(C) 0 (D) $\frac{1}{2}$

Key. D

Sol. f is continuous at $x = 0$

$\therefore 0 = \lim_{x \to 0+} f(x) = \lim_{x \to 0+} x^a \ln x$ forces "$a > 0$" is necessary.

123. Rolle's theorem holds in [1, 2] for the function $f(x) = x^3 + bx^2 + cx$ at the point "$\frac{4}{3}$". The values of *b*, *c* are respectively

(A) 8, –5
(B) –5, 8
(C) 5, –8
(D) –5, –8

Key. B

Sol. $f(1) = f(2)$ and $f'(4/3) = 0$

$3b + c = -7$ and $8b + 3c = -16$

$b = -5; c = 8$

124. Point on the curve $y^2 = 4(x-10)$ which is nearest to the line *x* + *y* = 4 may be

(A) (11, 2)
(B) (10, 0)
(C) (11, –2)
(D) None of these

Key. C

Sol. $P(x_{0,}, y_0) : pt$ on curve nearest to line.

Normal at *P* is perpendicular to the line

Normal at *P* has slope "$-\frac{y_0}{2}$"

$\therefore y_0 = 2$ and $x_0 = 11$; $P(11, -2)$

125. $f(x) = \left(\sin^2 x\right) e^{-2\sin^2 x}$; max $f(x) -$ min $f(x) =$

(A) $\frac{1}{e^2}$
(B) $\frac{1}{2e} - \frac{1}{e^2}$
(C) 1
(D) None of these

Key. D

Sol. Let $t = \sin^2 x; t \in [0,1]$

$f(x) = g(t) = te^{-2t}$

$$g'(t) = (1-2t)\, e^{-2t} \begin{cases} > 0 & if \quad t \in [0, \frac{1}{2}) \\ < 0 & if \quad t \in (\frac{1}{2}, 1] \end{cases}$$

$\max f = \max g = g\left(\frac{1}{2}\right) = \frac{1}{2e}$

$\min f = \min g = \min\{g(0), g(1)\} = 0$

$\max f - \min f = \frac{1}{2e}$.

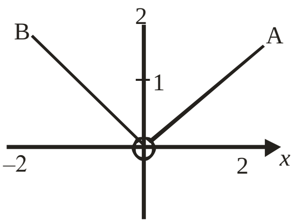
126. $f(x) = \begin{cases} |x| & if \quad 0 < |x| \le 2 \\ 1 & if \quad x = 0 \end{cases}$ HAS AT *X* = 0

(A) LOCAL MAXIMA (B) LOCAL MINIMA

(C) TANGENT (D) NONE OF THESE

KEY. A

SOL.



A(2,0), B(–2, 0)

O(0, 0) is not a point on the graph

127. $f(x) = x^4 - 10x^3 + 35x^2 - 50x + c.$ WHERE C IS A CONSTANT. THE NUMBER OF REAL ROOTS OF $f'(x) = 0$ AND $f''(x) = 0$ ARE RESPECTIVELY

(A) 1, 0 (B) 3, 2 (C) 1, 2 (D) 3, 0

KEY. B

Sol. $g(x) = (x-1)(x-2)(x-3)(x-4)$

$f(x) = g(x) + c_0 : c_0 = c - 24$

$g(x) = 0$ has 4 roots viz. $x = 1, 2, 3, 4$

$f'(x) = g'(x)$ and $f''(x) = g''(x)$

By Rolle's theorem $g'(x) = 0$ has min. one root in each of the intervals (1, 2); (2, 3); (3, 4)

BY ROLLE'S THEOREM, BETWEEN TWO ROOTS OF $f'(x) = 0,\ f''(x) = 0$ HAS MINIMUM ONE ROOT.

128. THE DIFFERENCE BETWEEN THE GREATEST AND LEAST VALUE OF $f(x) = \sin 2x - x : x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(A) $\frac{\sqrt{3}+\sqrt{2}}{2}$ (B) $\frac{\sqrt{3}+\sqrt{2}}{2} + \frac{\pi}{6}$

(C) $\frac{\sqrt{3}}{2} - \frac{\pi}{3}$ (D) NONE OF THESE

KEY. D

Sol. $f'(x) = 2\cos 2x - 1;\ f'(x) = 0$ if $x = -\frac{\pi}{6}, \frac{\pi}{6}$

$f'(x) > 0$ if $x \in \left(-\frac{\pi}{6}, \frac{\pi}{6}\right)$

$f'(x) < 0$ if $x \in [-\frac{\pi}{2}, -\frac{\pi}{6})$ or $x \in (\frac{\pi}{6}, \frac{\pi}{2}]$

Max $f = \max\{f\left(-\frac{\pi}{2}\right), f\left(\frac{\pi}{6}\right)\} = \max\left\{\frac{\pi}{2}, \frac{\sqrt{3}}{2} - \frac{\pi}{6}\right\} = \frac{\pi}{2}$

MIN $f = -\frac{\pi}{2}$ IS F IS AN ODD FUNCTION.

129. $f : R \to R$ IS A FUNCTION SUCH THAT $f(x) = 2x + \sin x$; THEN, F IS

(A) ONE-ONE AND ONTO
(B) ONE-ONE BUT NOT ONTO
(C) ONTO BUT NOT ONE-ONE
(D) NEITHER ONE-ONE NOR ONTO

KEY. A

Sol. $f'(x) = 2 + \cos x > 0$; $\therefore f$ is one-one

f is continuous; $\lim_{x\to\infty} f(x) \equiv \infty$; $\lim_{x\to-\infty} f(x) \equiv -\infty$

$\therefore f$ IS ONE-ONE AND ONTO

130. FIND WHICH FUNCTION DOES NOT OBEY LAGRANGE'S MEAN VALUE THEOREM IN [0, 1]

(A) $f(x) = \begin{cases} \frac{1}{2} - x & : x < \frac{1}{2} \\ \left(\frac{1}{2} - x\right)^2 & : x \ge \frac{1}{2} \end{cases}$
(B) $f(x) = \begin{cases} \frac{\sin x}{x} & : x \ne 0 \\ 1 \quad \text{if} & x = 0 \end{cases}$
(C) $f(x) = x|x|$
(D) $f(x) = |x|$

KEY. A

Sol. In (a), $f'\left(\frac{1}{2}-\right) = -1$ while $f'\left(\frac{1}{2}+\right) = 0$

F IS NOT DIFFERENTIABLE AT $x = \frac{1}{2}$.

131. IF A > 0, B < 0 AND $A = \frac{\pi}{3} + B$ THEN MINIMUM VALUE OF TANA TANB IS

(A) $-\frac{1}{2}$
(B) –1
(C) $-\frac{1}{3}$
(D) NONE OF THESE

KEY. C

Sol. $B_0 = -B > 0$; $A + B_0 = \frac{\pi}{3}$.

By $A.M. - G.M.$, max $\tan A \tan B_0$ happens when

$A = B_0 = \frac{\pi}{6}$

$\therefore$ MIN $\tan A \tan B = -\frac{1}{3}$.

132. The point on the curve $x^2 = 2y$ which is nearest to a (0, 3) may be

(A) (2, 2) (B) $\left(1, \frac{1}{2}\right)$

(C) (0, 0) (D) $\left(-3, \frac{9}{2}\right)$

KEY. A

Sol. Let $P(x_0, y_0)$ be the nearest point

$$PA^2 = (y_0 - 3)^2 + (x_0 - 0)^2$$

$$= y_0^2 - 4y_0 + 9 \text{ as } x_0^2 = 2y_0$$

$$= (y_0 - 2)^2 + 5$$

PA^2 is minimum if $y_0 = 2; x_0 = \pm 2$

$P(\pm 2, 2)$.

Aliter : A lies on normal to curve at *P.*

133. POINT ON THE LINE $x - y = 3$ WHICH IS NEAREST TO THE CURVE $x^2 = 4y$ IS

(A) (0, –3) (B) $(3, 0)$

(C) $(2, -1)$ (D) NONE OF THESE

KEY. B

Sol. $P(x_0, y_0)$ is the nearest point; $y_0 = x_0 - 3$

Line through *P,* perpendicular to $x - y = 3$ is normal to given curve at, say, $Q(x_1, y_1)$

$$\therefore -\frac{2}{x_1} = -1; \;\; x_1 = 2; y_1 = 1.$$

Normal is $y - 1 = -(x - 2)$; This cuts $x - y = 3$ at *P.*

$\therefore P(3, 0)$.

134. $f(x) = \begin{cases} \frac{|x-1|}{x^2} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$ INCREASES IN

(A) (0, 2) (B) [0, 2]

(C) $[0, \infty)$ (D) NONE OF THESE

KEY. D

Sol. $$f(x) = \begin{cases} \frac{x-1}{x^2} & \text{if} \quad x > 1 \\ \frac{1-x}{x^2} & \text{if} \quad x < 1 : x \neq 0 \\ 0 & \text{if} \quad x = 0, 1 \end{cases}$$

$$f'(x) = \begin{cases} \dfrac{2-x}{x^3} & \text{if} \quad x > 1 \\ \dfrac{x-2}{x^3} & \text{if} \quad x \in (0,1) \text{ or } x \in (-\infty, 0) \end{cases}$$

f is not differentiable at *x* = 0, 1

$f'(x) > 0$ IF $x \in (1,2)$ OR $x \in (-\infty, 0)$

Maxima & Minima

Multiple Correct Answer Type

1. Let $f(x) = 2x^2 - ln|x|, x \neq 0$, then f(x) is

A) monotonically increasing in $\left(-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right)$

B) monotonically decreasing in $\left(-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right)$

C) monotonically increasing in $\left(-\infty, -\frac{1}{2}\right) \cup \left(0, \frac{1}{2}\right)$

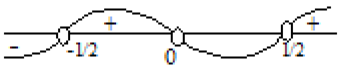
D) monotonically decreasing in $\left(-\infty, -\frac{1}{2}\right) \cup \left(0, \frac{1}{2}\right)$

Key. A,D

Sol. $\because f(x) = 2x^2 - \ln|x|$

$\therefore \quad f'(x) = 4x - \frac{1}{x}$

$= \frac{(2x+1)(2x-1)}{x}$



For increasing, $f'(x) > 0 \quad \therefore \quad x \in \left(-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right)$

And for decreasing, $f'(x) < 0 \quad \therefore \quad x \in \left(-\infty, -\frac{1}{2}\right) \cup \left(0, \frac{1}{2}\right)$

2. If f(x) = $\left(\frac{a^2 - 1}{3}\right)$ $x^3 + (a - 1)x^2 + 2x + 1$ is monotonic increasing for every $x \in R$ then 'a' lies in

A) (1, 2) B) $(1, \infty)$

C) $(-\infty, -3)$ D) $(-10, -7)$

Key. A,B,C,D

Sol. $f(x) = \left(\frac{a^2 - 1}{3}\right)x^3 + (a - 1)x^2 + 2x + 1$

$f'(x) > 0 \Rightarrow \frac{(a^2 - 1)3x^2}{3} + 2x(a - 1) + 2 > 0$

$(a^2 - 1)x^2 + 2x(a - 1) + 2 > 0 \Rightarrow 4(a - 1)^2 - 8(a^2 - 1) < 0 \; \{b^2 - 4ac < 0\}$

$\Rightarrow (a - 1)^2 - 2(a^2 - 1) < 0 \quad \Rightarrow (a + 3)\,(a - 1) > 0$

⇒ (- ¥ - 3)È(1 ¥)

3. If $f^{11}(x) > 0 \forall x \in R, f^{1}(3) = 0$ and $g(x) = f\left(\tan^2 x - 2\tan x + 4\right), 0 < x < \pi/2$, then, $g(x)$ is increasing in

a) $(0, \pi/4)$ b) $(\pi/6, \pi/3)$

c) $(0, \pi/3)$ d) $(\pi/4, \pi/2)$

Key. D

Sol. $g^{1}(x) = f^{1}\left((\tan x - 1)^2 + 3\right).(2\tan x - 2)\sec^2 x$

$\therefore f^{11}(x) > 0 \Rightarrow f^{1}(x)$ is increase

$\Rightarrow f^{1}\left((\tan x - 1)^2 + 3\right) > f^{1}(3) = 0 \forall x \in (0, \pi/4) \cup (\pi/4, \pi/2)$

Also, $(\tan x - 1) > 0 \forall x \in (\pi/4, \pi/2) \therefore g(x)$ is increase in $(\pi/4, \pi/2)$

4. The function $f(x) = \int_0^x \sqrt{1 - t^4}\, dt$ is such that

a) it is defined on the interval $[-1, 1]$ b) it is an increasing function

c) it is an odd function d) the point $(0,0)$ is the point of inflexion

Key. A,B,C,D

Sol. Find $f^{1}(x)$

5. $f(x) = \begin{cases} \dfrac{3 - x^2}{2}, & 0 \le x \le 1 \\ \dfrac{1}{x}, & 1 \le x \le 2 \end{cases}$ Then, the value of 'c' in the LMVT over $[0,2]$, is

a) $\frac{1}{2}$ b) $\frac{1}{3}$ c) $\frac{3}{2}$ d) $\sqrt{2}$

Key. A,D

Sol. By LMVT for $f(x)$ on $[0,1], \exists c \in (0,1) \ni \dfrac{f(1) - f(0)}{1 - 0} = \dfrac{-2c}{2} \Rightarrow c = \dfrac{1}{2}$

By LMVT for f(X) on $[1,2], \exists c \in (1,2) \ni \dfrac{f(2) - f(1)}{2 - 1} = \dfrac{-1}{c^2} \Rightarrow c = \sqrt{2}$

6. Let $f(x)$ be a twice differentiable function such that $f^{11}(x) > 0$ in $[0,1]$. Then,

a) $f(0) + f(1) = 2f(c)$ for some $c \in (0,1)$ b) $f(0) + f(1) = 2f(1/2)$

c) $f(0)+f(1)>2f\left(\frac{1}{2}\right)$ d) $f(0)+f(1)<2f\left(\frac{1}{2}\right)$

Key. A,C

Sol. By IVP, $\frac{f(0)+f(1)}{2}=f(c), 0<c<1$ by

$\text{LMVT}, f(1/2)-f(0)=\frac{1}{2}f^1(c_1), 0<c_1<\frac{1}{2}$

$f(1)-f(1/2)=\frac{1}{2}f^1(c_2), \frac{1}{2}<c_1<1$

Subtracting, we get,

$f(1)+f(0)-2f\left(\frac{1}{2}\right)=\frac{f^1(c_2)-f^1(c_1)}{2}=\frac{c_2-c_1}{2}f^{11}(c)>0$ (UsingLMVT)

$\Rightarrow f(1)+f(0)>2f\left(\frac{1}{2}\right)$

7. The function $f(x)=\int_0^x\sqrt{1-t^4}\,dt$ is such that

a) it is defined on the interval $[-1,1]$

b) it is an increasing function

c) it is an odd function

d) the point $(0,0)$ is the point of inflexion

Key. A,B,C,D

Sol. Find $f^1(x)$

8. Consider the following statements

Statement – I: If $0<\alpha<\beta<\frac{\pi}{2}$, then, $\frac{\tan\beta}{\tan\alpha}>\frac{\alpha}{\beta}$

Statement –II: If $x\geq 0$, then $\frac{x}{1+x}\leq\log(1+x)\leq x$. Then

a) I is true b) I is false c) II is true d) II is false

Key. A,C

Sol. $f(x)=x\tan x\Rightarrow f^1(x)=\tan x+x\sec^2 x>0$

$\frac{f(\beta)-f(\alpha)}{\beta-\alpha}=f^1(c)>0\Rightarrow f(\beta)>f(\alpha)\Rightarrow\frac{\tan\beta}{\tan\alpha}>\frac{\alpha}{\beta}$

$f(x)=\log(1+x)\Rightarrow f^1(x)=\frac{1}{1+x}$

$\frac{f(x)-f(0)}{x-0}=f^1(c)=\frac{1}{1+x}\Rightarrow\log(1+x)=\frac{x}{1+c}$

$0 < c < x \Rightarrow 1 < 1+c < 1+x \Rightarrow \frac{1}{1+x} < \frac{1}{1+c} < 1$

$\Rightarrow \frac{x}{1+x} < \frac{x}{1+c} < x \Rightarrow \frac{x}{1+x} < \log(1+x) < x$

Equality holds good for x = 0

9. $f(x) = \begin{cases} \frac{3-x^2}{2}, & 0 \le x \le 1 \\ \frac{1}{x}, & 1 \le x \le 2 \end{cases}$ Then, the value of 'c' in the LMVT over $[0,2]$, is

a) $\frac{1}{2}$ b) $\frac{1}{3}$ c) $\frac{3}{2}$ d) $\sqrt{2}$

Key. A,D

Sol. By LMVT for $f(x)$ on $[0,1], \exists c \in (0,1) \ni \frac{f(1)-f(0)}{1-0} = \frac{-2c}{2} \Rightarrow c = \frac{1}{2}$

By LMVT for f(X) on $[1,2], \exists c \in (1,2) \ni \frac{f(2)-f(1)}{2-1} = \frac{-1}{c^2} \Rightarrow c = \sqrt{2}$

10. For the function, $f(x) = x\cos\frac{1}{x}, x \ge 1,$

a) For atleast one x in $[1,\infty), f(x+2)-f(x) < 2$

b) $\underset{x\to\infty}{Lt} f^1(x) = 1$

c) $\forall x \ \text{in} [1,\infty), f(x+2)-f(x) > 2$

d) $f^1(x)$ strictly decreases in $[1,\infty)$

Key. B,C,D

Sol. b and d are obvious For 'c' using LMVT for $f(x)$ on

$[x,x+2], x \ge 1, \exists c \in (x,x+2) \ni \frac{f(x+2)-f(x)}{x+2-x} = 2f^1(c) \Rightarrow f(x+2)-f(x) = 2f^1(c)$

Q $\underset{x\to\infty}{Lt} f^1(x) = 1$ and $f^1(x)$ is strictly decreasing $\forall x \ge 1, \Rightarrow f^1(c) > 1$ hence

$f(x+2)-f(x) > 2 \forall x \in (1,\infty)$

11. $f(x)=\begin{cases}\left(\sqrt{2}+\sin\frac{1}{x}\right)e^{\frac{-1}{|x|}} & if \quad x\neq 0\\ 0 & if \quad x=0\end{cases}$

Number of points where $f(x)$ has local extrema when $x\neq 0$ be n_1. n_2 be the value of global minimum of $f(x)$ then $n_1+n_2=$

Key. 0

Sol. Local extremum does not occur at any value of $x\neq 0$. But global minimum $=f(0)=0$

$\therefore n_1=0,\ n_2=0$ then $n_1+n_2=0$

12. Let $f(x)=\frac{e^x}{1+x^2}$ and g(x) = f′(x) then

(A) g(x) has four points of local extremum

(B) g(x) has two points of local extremum

(C) g(x) has a point of local minimum at x = 1

(D) g(x) has a point of local maximum at some $x\in(-1,0)$

Key: B,C,D

Hint f(x) = $\frac{e^x}{1+x^2}$, $g(x)=f'(x)=\frac{(x-1)^2e^x}{(1+x^2)^2}$, $g'(x)=\frac{(x-1)(x^3-3x^2+5x+1)}{(x^2+1)^3}e^x$

Let h(x) = x^3-3x^2+5x+1, h′(x) = $3x^2-6x+5$, D < 0 so h(x) has only one real roots. Also g(–1) g(0) < 0. So the root ∈ (–1, 0). Clearly g(x) has two points of extremum. Maxima at x ∈ (–1, 0) and minima at x = 1.

13. Let $f(x)=x^2.e^{-x^2}$ then

(A) $f(x)$ has local maxima at $x=-1$ and $x=1$

(B) $f(x)$ has local minima at $x=0$

(C) $f(x)$ is strictly decreasing on $x\in R$

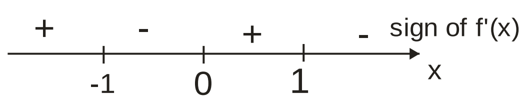
(D) Range of $f(x)$ is $\left[0,\frac{1}{e}\right]$

KEY : A,B,D

HINT $f(x)=x^2.e^{-x^2}$

$f'(x)=2x.e^{-x^2}+x^2.e^{-x^2}(-2x)$

$=2xe^{-x^2}\left[1-x^2\right]$



$f(x)$ has local maxima at x = -1 and 1

$f(x)$ has local minima at x = 0

Now ; $f(0)=0$

$f(1)=\frac{1}{e}$ and as $x\to\infty, f(x)\to 0$

So, Range of $f(x)$ is $\left[0,\frac{1}{e}\right]$

14. Let $f(x)=\begin{cases} 2x-4 & x\le 2 \\ -x^2+\frac{k^3(k-1)^2}{k^2-k-2}+4, & x>2 \end{cases}$. $f(x)$ attains local maximum at x = 2 if k lies in

(A) $(0,1)$ (B) $(3,\infty)$ (C) $(-\infty,-1)$ (D) $(1,2)$

KEY : A, C, D

HINT : When $f(x)$ is continuous at x = 2 $f'(x)\,DNE\ \ at\ \ x=2$

and $f'(x)$ changes sign from + to –

$\Rightarrow f(x)$ attains max. At x = 2 if $\frac{k^3(k-1)^2}{k^2-k-2}=0 \Rightarrow k=0,1$

When $f(x)$ is discontinuous at x = 2, $f'(x)$ changes its sign from + to $-$. $f(x)$ will attain maximum if $\underset{x\to 2^+}{Lt} f(x)<f(2)$ as

$\underset{x\to 2^-}{Lt} f(x)=f(2)\ \ i.e\ \ if\ \ k\in(-\infty,-1)\cup(0,1)\cup(1,2)$

$\Rightarrow k\in(-\infty,-1)\cup[0,2)$

15. Which of the following functions will not have absolute minimum value?

A) $\cot(\sin x)$ B) $\tan(\log x)$

C) $x^{2005}-x^{1947}+1$ D) $x^{2006}+x^{1947}+1$

KEY : A,B,C

SOL : Even degree polynomial with leading coefficient +ve will have absolute minimum.

16. More than one option.

$$f(x)=\begin{matrix} |x+1|; & -2<x<0 \\ \sqrt[3]{1-x}; & 0<x<1 \\ 2; & x=0 \\ \sqrt{x+1}; & x\ge 1 \end{matrix}$$

Then $f(x)$

A) has neither maximum nor minimum at $x=0$

B) has maximum at $x=0$

C) has neither maximum nor minimum at $x=1$

D) no global maximum

KEY : B,C,D

17. Which of the following are true for $\forall x \in (0,\infty)$

a) $\ln(1+x) > x-\frac{x^2}{2}$ b) $\ln(1+x) > \frac{x}{x+1}$

c) $4\cos x + x > 0$ d) $2\tan^{-1} x < x+1$

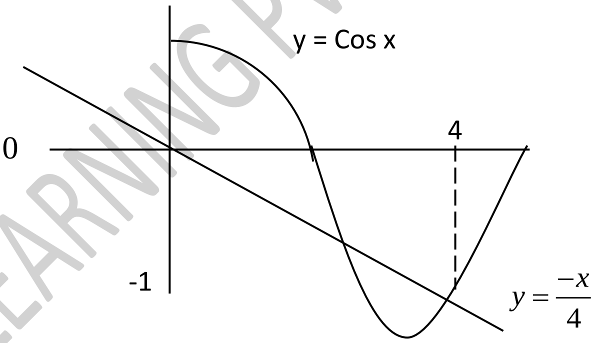
Key: A, B, C

Hint:

$(a)\ f(x)=\ln(1+x)-x+\frac{x^2}{2}$

$f^1(x)=\frac{1}{1+x}-1+x=\frac{x^2}{1+x}$

$f(0)=0,\ f^1(x)>0\forall x\in(0,\infty)\Rightarrow f(x)>0$



$(b)\ f(x)=\ln(1+x)-\frac{x}{1+x}$

$f^1(x)=\frac{1}{1+x}-\frac{1}{(x+1)^2}=\frac{x}{(1+x)^2}$

$f(0)=0,\ f^1(x)>0\forall x\in(0,\infty)\Rightarrow f(x)>0$

$(c)\ y=\cos x,\ y=-\frac{x}{4}$

by graph it is clear that

$\cos x > \frac{-x}{4}$ *is not true* $\forall x\in(0,\infty)$

$(d)\ f(x)=x+1-2\tan^{-1}x$

$f(0)=1$

$f^1(x)=\frac{x^2-1}{1+x^2}=\frac{(x-1)(x+1)}{x^2+1}$

at $x=1$ $f(x)$ min *ma*

$f(1)=2\left(1-\frac{\pi}{4}\right)>0\Rightarrow f(x)>0\forall x\in(0,\infty)$

18. If $\log_2\left(\log_{\frac{1}{2}}\left(\log_2(\text{x})\right)\right)=\log_3\left(\log_{\frac{1}{3}}\left(\log_3(\text{y})\right)\right)=\log_5\left(\log_{\frac{1}{5}}\left(\log_5(\text{z})\right)\right)=0$ for positive x,y and z, then which of the following is/ are NOT true?

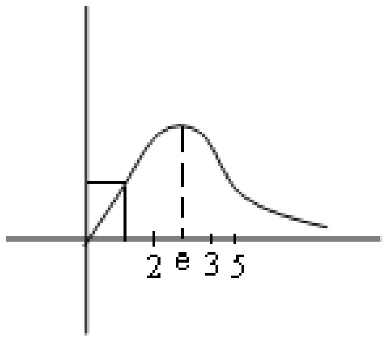
(A) z < x < y (B) x < y < z

(C) y < z < x (D) z < y < x

Key: B, C, D

Hint: Solving we get $x = 2^{1/2}, y = 3^{1/3}, z = 5^{1/5}$

Using graph of $x^{1/x}$



$\Rightarrow 3^{1/3} > 5^{1/3}$

Also $2^{1/2} < 3^{1/3}$ as $2^3 < 3^2 \Big| 2^{1/2} > 5^{1/5}$ as $2^5 > 5^2$

$\Rightarrow y > x > z$

Hence (b), (c), & (d) are NOT true.

19. For the function $f(x) = x\cos\dfrac{1}{x},\ x \ge 1$

a) for at least one x in the interval $[1,\infty),\ f(x+2) - f(x) < 2$

b) $\lim\limits_{x\to\infty} f^1(x) = 1$

c) for all x in the interval $[1,\infty),\ f(x+2) - f(x) > 2$

d) $f^1(x)$ is strictly decreasing in the interval $[1,\infty)$

Key. B,C,D

Sol. $f(x) = x\cos\dfrac{1}{x}, x^3\ 1$

\ $f'(x) = \cos\dfrac{1}{x} + \dfrac{1}{x}\sin\dfrac{1}{x}$

20. $f(x) = \dfrac{1}{1+|x|} + \dfrac{1}{1+|x-1|}$. Let x_1, x_2 are points where $f(x)$ attains local minimum and global maximum respectively. Let $k = f(x_1) + f(x_2)$ then $6k - 9$

Key. 8

Sol. Local minimum $= f\left(\dfrac{1}{2}\right) = \dfrac{4}{3}$

Global maximum $= f(0) = f(1) = \dfrac{3}{2}$ $\qquad k = \dfrac{4}{3} + \dfrac{3}{2} = \dfrac{17}{6}$

21. Let $g(x)$ and $h(x)$ be twice differentiable functions on R and $f(2) = 8, g(2) = 0, f(4) = 10, g(4) = 8$, then

a) $g^1(x) > 4f^1(x) \forall x \in (2,4)$

b) $g(x) > f(x) \forall x \in (2,4)$

c) $3g^1(x) = 4f^1(x)$ for at least one $x \in (2,4)$ d) $g^1(x) = 4f^1(x)$ for at least one $x \in (2,4)$

Key. D

Sol. Let $h(x) = g(x) - 4f(x)$. Verify RT in $(2,4)$

$h(2) = g(2) - 4f(2) = -32$

$h(4) = g(4) - 4f(4) = -32$

$\therefore h^1(x) = 0$ for at least one $x \in (2,4) \Rightarrow g^1(x) = 4f^1(x)$ for at least one $x \in (2,4)$

22. If $f^{11}(x) > 0 \forall x \in R, f^1(3) = 0$ and $g(x) = f\left(\tan^2 x - 2\tan x + 4\right), 0 < x < \pi/2$, then, $g(x)$ is increasing in

a) $(0, \pi/4)$ b) $(\pi/6, \pi/3)$

c) $(0, \pi/3)$ d) $(\pi/4, \pi/2)$

Key. D

Sol. $g^1(x) = f^1\left((\tan x - 1)^2 + 3\right).(2\tan x - 2)\sec^2 x$

$\therefore f^{11}(x) > 0 \Rightarrow f^1(x)$ is increase

$\Rightarrow f^1\left((\tan x - 1)^2 + 3\right) > f^1(3) = 0 \forall x \in (0, \pi/4) \cup (\pi/4, \pi/2)$

Also, $(\tan x - 1) > 0 \forall x \in (\pi/4, \pi/2) \therefore g(x)$ is increase in $(\pi/4, \pi/2)$

23. If $f(x) = 2x^3 - 15x^2 + 24x$ and $g(a) = \int_0^a f(x)dx + \int_0^{5-a} f(x)dx$ $(0 < a < 5)$, then

a) g is increasing in $\left(0, \frac{5}{2}\right)$ b) g is decreasing in $\left(\frac{5}{2}, 5\right)$

c) g is increasing in (0, 5) d) None of these

Key. A,B

Sol. $f'(x) = 6x^2 - 30x + 24 = 6(x-4)(x-1)$

By graph

$g'(a) = f(a) - f(5-a)$

If $a < 5 - a \Rightarrow a < \frac{5}{2}$

$g'(a) > 0 \Rightarrow g$ is increasing in $\left(0, \frac{5}{2}\right)$

And if $a > \frac{5}{2}, g'(a) < 0 \Rightarrow g$ is decreasing in $\left(\frac{5}{2}, 5\right)$.

24. Let $f(x)=\frac{e^x}{1+x^2}$ and g(x) = f′(x) then

(A) g(x) has four points of local extremum

(B) g(x) has two points of local extremum

(C) g(x) has a point of local minimum at x = 1

(D) g(x) has a point of local maximum at some $x \in (-1, 0)$

Key. B,C,D

Sol. f(x) = $\frac{e^x}{1+x^2}$, $g(x)=f'(x)=\frac{(x-1)^2e^x}{(1+x^2)^2}$, $g'(x)=\frac{(x-1)(x^3-3x^2+5x+1)}{(x^2+1)^3}e^x$

Let $h(x) = x^3 - 3x^2 + 5x + 1$, $h'(x) = 3x^2 - 6x + 5$, D < 0 so h(x) has only one real roots. Also g(–1) g(0) < 0. So the root $\in (-1, 0)$. Clearly g(x) has two points of extremum. Maxima at $x \in (-1, 0)$ and minima at x = 1.

25. The function $f(x) = a(x^2 - 1)(ax + b)$ $(a \neq 0)$ has

(A) a local maxima at certain $x \in R^+$

(B) a local minima at certain $x \in R^+$

(C) a local maxima at certain $x \in R^-$

(D) a local minima at certain $x \in R^-$

Key. B,C

Sol. $f(x) = a^2x^3 + abx^2 - a^2x - ab$. Coefficient of x^3 is +ve.

$f'(x) = 3a^2x^2 + 2abx - a^2$. Product of roots is –ve

So, f(x) has two points of maxima/minima and maxima at R^- and minima at R^+.

26. If the function $f(x)=x^3-6x^2+ax+b$ defined in $[1,3]$ satisfies the rolle's theorem for $C=\frac{2\sqrt{3}+1}{\sqrt{3}}$ then

(A) a = 11 (B) b = 6 (C) $a\in R$ (D) $b\in R$

Key. A,D

Sol. Conceptual

27. If $f(x)=(x-4)(x-5)(x-6)(x-7)$ then $f'(x)=0$ has roots in

(A) $(4,5)$ (B) $(5,6)$ (C) $(6,7)$ (D) $(3,4)$

Key. A,B,C

Sol. Conceptual

28. If f(x), g(x) (where x > 1) are non–negative and non–positive functions respectively, such that $f'(x) \le \alpha f(x)$, $g'(x) \ge \beta g(x)$ for some $\alpha, \beta > 0$ and f(1) = 0, g(1) = 0, then

a) $\frac{f(e)+f(\pi)}{f^2(e)+f^2(\pi)-4} = 0$

b) $\frac{f(e)+f(\pi)}{f^2(e)+f^2(\pi)-4} = -4$

c) $\frac{g(\sqrt{e})+g(\sqrt{\pi})}{g(\sqrt[3]{e})+g(\sqrt[3]{\pi})-3} = 0$

d) $\frac{g(\sqrt{e})+g(\sqrt{\pi})}{g(\sqrt[3]{e})+g(\sqrt[3]{\pi})-3} = -3$

Key. A,C

Sol. $f(x) \ge 0$

$f'(x) - \alpha f(x) \le 0$

$\Rightarrow \frac{d}{dx}[e^{-\alpha x} f(x)] \le 0 \quad \forall x > 1$

$\Rightarrow$ f(x) $\le$ 0 $\forall$ x > 1 $\because$ f(1) = 0

$\Rightarrow$ f(x) = 0 $\forall$ x > 1

Similarly, g(x) = 0 $\forall$x > 1

Now $\frac{f(e)+f(\pi)}{f^2(e)+f^2(\pi)-4}$ = 0 and $\frac{g(\sqrt{e})+g(\sqrt{\pi})}{g(\sqrt[3]{e})+g(\sqrt[3]{\pi})-3}$ = 0

29. Let f(x)= $\int_0^x (u-1)(u-2)^2\, du$, then for the function f(x)

a) (1, –3) is a point of minimum
b) (2, –4/3) is a point of inflexion
c) $\left(1, -\frac{17}{12}\right)$ is a point of minimum
d) $\left(\frac{4}{3}, -\frac{112}{81}\right)$ is a point of inflexion.

Key. B,C,D

Sol. f^1 (x) = (x – 1) (x – 2)$^2 \Rightarrow$ x = 1, 2

Around x = 1, f′(x) changes sign from –ve to +ve f(x) is minimum at x = 1, f(1) = $-\frac{17}{12}$

f″(x) changes sign around x = 2, $\frac{4}{3}$,

x = 2, x = $\frac{4}{3}$ are the point of inflexion f(2) = $-\frac{4}{3}$, $f\left(\frac{4}{3}\right) = -\frac{112}{81}$.

30. $f(x) = \left||x+2| - 2|x-2|\right| + |x| \ (x \in R)$ can never be

a) $\frac{1}{2}$ b) $\frac{1}{4}$ c) 1 d) 2

Key. A,B

Sol. Global minimum of $f(x)$ *is* $\frac{2}{3}$

31. The function $f(x) = \frac{x+a}{\sqrt{x^2+a^2}}, a > 0$ can never be

a) $\sqrt{2}$ b) 2 c) 3 d) 4

Key. B,C,D

Sol. Range of f(x) = $\left(-1, \sqrt{2}\right]$

29. Consider $f(x) = x^{\frac{1}{\log_e x}} + \sin \pi[x],\ 2 \le x \le 6$, where [x] denotes integer part of x. Then the true statement among the following is/are

A) Rolle’s theorem can be applied to f

B) Rolle’s theorem cannot be applied to f

C) Lagrange’s mean value theorem can be applied to f

D) Lagrange's mean value theorem cannot be applied to f

Key. A,C

Sol. $f(x)=e, 2\le x\le 6$

30. Consider the function $f:R-\{0\}\to R$ defined by $f(x)=x+\frac{1}{x}$. Then the true statements among the following are

A) f is one-one but not onto

B) f possesses local extrema but f does not have absolute extrema

C) If $g:(0,1]\to[2,\infty)$ is defined by $g(x)=x+\frac{1}{x}$, then $g^{-1}(x)=\frac{x-\sqrt{x^2-4}}{2}$

D) If $h:(-\infty,-1]\to(-\infty,-2]$ is defined by $h(x)=x+\frac{1}{x}$, then $h^{-1}(x)=\frac{x-\sqrt{x^2-4}}{2}$

Key. B,C,D

Sol. $y=f(x)=x+\frac{1}{x}, x\ne 0\ |y|\ge 2 \Rightarrow x^2-yx+1=0 \Rightarrow x=\frac{y\pm\sqrt{y^2-4}}{2}$

g: $(0,1]\to[2,\infty), g(x)=x+\frac{1}{x}=y$

$0<x\le 1$ and $y\ge 2\Rightarrow \frac{y}{2}\ge 1$

$x=\frac{y+\sqrt{y^2-4}}{2}\ge 1$ but $x\le 1$

$\therefore x=g^{-1}(y)=\frac{y-\sqrt{y^2-4}}{2}\Rightarrow g^{-1}(x)=\frac{x-\sqrt{x^2-4}}{2}$

$h:(-\infty,-1]\to(-\infty,-2], h(x)=x+\frac{1}{x}=y$

$x\le -1$ and $y\le -2\Rightarrow \frac{y}{2}\le -1$

$\frac{y-\sqrt{y^2-4}}{2}$ is clearly –ve when $y\le -2$

Also $\frac{y-\sqrt{y^2-4}}{2}\le -1\Rightarrow h^{-1}(y)=\frac{y-\sqrt{y^2-4}}{2}$

31. $f(x)=\begin{cases}3x^2+12x-1 & ,-1\le x\le 2\\ 37-x & ,2<x\le 3\end{cases}$, then

A) f is increasing on $[-1,2]$

B) f is differentiable at $x=2$

C) f does not attain absolute minimum in $[-1,2]$

D) Absolute maximum value of f is 35

Key. A,D

Sol. Conceptual

32. Consider the curves $y=f(x)=1+\frac{x^2}{a^3}, a\ne 0$ and $y=g(x)=4\sqrt{x}$. Then the true statements among the following are:

A) If $a<0$ then the curves have a unique point in common.

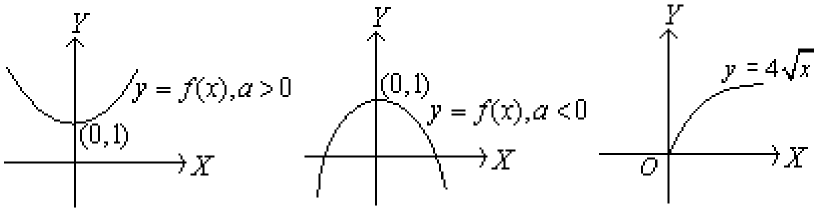
B) If $a = \frac{1}{3}$ the curves touch each other

C) If $0 < a < \frac{1}{3}$ then curves intersect at two distinct points

D) If $a > \frac{1}{3}$ then the curves do not meet

Key. A,B

Sol.



Consider $y = f(x)$ and $y = g(x)$ when $a > 0$

Suppose they touch at (x_1, y_1)

Then $y_1 = f(x_1) = g(x_1)$ and $f^1(x_1) = g^1(x_1)$

33. Let $f(x) = \frac{x-1}{x^2}$ then which of the following is correct?

(a) f(x) has minima but no maxima

(b) f(x) increases in the interval (0, 2) and decreases in the interval $(-\infty, 0) \cup (2, \infty)$

(c) f(x) is concave down in $(-\infty, 0) \cup (0, 3)$

(d) x = 3 is the point of inflection

Key. B,C,D

Sol. $f'(x) = \frac{-(x-2)}{x^3}$ anf $f''(x) = \frac{2(x-3)}{x^4}$

$f'(x)$: $-$ | 0 | $+$ | 2 | $-$

$f''(x)$: $-$ | 3 | $+$

So f increases (0, 2) and 3 is point of inflection.

34. Let $g'(x) > 0$ and $f'(x) < 0$ $\forall x \in R$, then

(a) $g(f(x+1)) > g(f(x-1))$ (b) $g(f(x+1)) < g(f(x-1))$

(c) $f(g(x-1)) > f(g(x+1))$ (d) $f(g(x-1)) < f(g(x+1))$

Key. B,C

Sol. x + 1 > x – 1

Now g(x) is increasing and f is decreasing.

g(x + 1) > g(x – 1) and f(x + 1) < f(x – 1)

so f(g(x + 1)) < f(g(x – 1)) (Greater input gives smaller output)

and g(f(x + 1)) < g(f(x – 1)) (Greater input gives greater output)

35. The interval in which $y = x^3$ increases more rapidly than $y = 6x^2 + 15x + 5$ is

(A) $(-\infty, -1)$ (B) $(5, \infty)$

(C) $(-1, 5)$ (D) $(0, \infty)$

Key. B

Sol. $f(x) = x^3; f'(x) = 3x^2$

$g(x) = 6x^2 + 15x + 5; g'(x) = 12x + 15$

$f'(x) > g'(x);$

$\therefore x^2 - 4x - 5 > 0$

$x < -1$ or $x > 5$

36. The inequality $1 + \ln x \le x$ is true in the regions

(A) $(1, \infty)$ (B) (0, 1)

(C) (0, *e*) (D) None of these

Key. A,B,C

Sol. $f(x) = 1 + \ln x - x : x > 0$

$$f'(x) = \frac{1}{x} - 1 \begin{cases} > 0 & if \quad 0 < x < 1 \\ < 0 & if \quad x > 1 \end{cases}$$

$f(1) = 0. \ \therefore \ f(x) \le 0 \quad \forall x > 0$

37. $f(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ decreases in the region

(A) $(-1, 0)$ (B) (0, 1)

(C) $(-\infty, -1)$ (D) $(-\infty, 1)$

Key. A,C

Sol. $h(x) = \frac{1-x^2}{1+x^2} = \frac{2}{1+x^2} - 1$

$\cos^{-1}$ is a decreasing function

f decreases when h increases

i.e., when $x \in (-\infty, 0)$.

Maxima & Minima

Assertion Reasoning Type

a) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1

b) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1

c) Statement-1 is True, Statement-2 is False

d) Statement-1 is False, Statement-2 is True

1. Statement 1:-Let f be a continuous function on the internal $[0,1]\to R$, such that $f(0)=f(1)$.Then, there exists a point C in $[0,1/2]$ such that $f(C)=f(C+1/2)$

Statement 2:- Let f be a real valued function defined on $[a,b]$ such that, i) f is continuous on $[a,b]$, ii) f is differentiable in $[a,b]$, iii) $f(a)=f(b)$, then, there is at least one value C of x in (a,b) for which $f^1(c)=0$

Key. B

Sol. Consider the function $g(x)=f(x)-f\left(x+\frac{1}{2}\right)$.

$g(0)=f(0)-f\left(\frac{1}{2}\right)$

$g\left(\frac{1}{2}\right)=f\left(\frac{1}{2}\right)-f(1)=f\left(\frac{1}{2}\right)-f(0)$

$\Rightarrow g(0)$ and $g\left(\frac{1}{2}\right)$ are of opposite signs

Hence, there exists at least are $c\in\left(0,\frac{1}{2}\right)$, such that

$$g(c)=0(or)f(c)-f\left(c+\frac{1}{2}\right)=0\Rightarrow f(c)=f\left(1+\frac{1}{2}\right)$$

2. Statement 1:- let f is a continuous function on $[a,b]$ and differentiable on (a,b) and satisfies $f^2(a)-f^2(b)=a^2-b^2$. Then, the equation $f(x)f^1(x)=x$ has at least one solution in (a,b)

Statement 2:- If f is continuous on $[a,b]$ and $f(a)\neq f(b)$, then, for any value $C\in(f(a),f(b))$,there is at least one number x_0 in (a,b) for which $f(x_0)=C$

Key. B

Sol. Let $g(x)=\frac{1}{2}\left((f(x))^2-x^2\right)$

$g(a)=\frac{1}{2}\left[f^2(a)-a^2\right]$ and $g(b)=\frac{1}{2}\left[f^2(b)-b^2\right]$ by $g(a)=g(b)$, so there exists atleast one $x\in(a,b)\Rightarrow g^1(x)=0$ (By RT) i.e., $\frac{1}{2}\left[2f(x)f^1(x)-2x\right]=0$

$\Rightarrow f^1(x)f^1(x)=x$

3. Statement 1:- The minimum distance of the fixed point $(0,\alpha)$, where, $0 \le \alpha \le \frac{1}{2}$ from the curve $y = x^2$ is α

Statement 2:- Maxima and minima of a function is always a root of the equation $f^1(x) = 0$

Key. C

Sol. Let point be (t,t^2) then, $d^2 = t^2 + (t^2 - \alpha)^2$

$= t^4 + (1-2\alpha)t^2 + \alpha^2$

$= z^2 + (1-2\alpha)Z + \alpha^2, z \ge 0$

Its vertex is at $x = \alpha - \frac{1}{2} < 0$

$\therefore$ minimum value of d^2 is at z = 0 i.e., $t^2 = 0$

$\Rightarrow d = \alpha$

$\therefore$ Statement I is true

Statement II is false because extremum can occur at a point where $f^1(x)$ One

4. STATEMENT-1 : If the point (x, y) lies on the curve $2x^2 + y^2 - 24y + 80 = 0$ then the maximum value of $x^2 + y^2$ is 400.

STATEMENT-2 : The point (x, y) is at a distance of $\sqrt{x^2 + y^2}$ from origin.

Key: A

Hint: The given equation represents ellipse $\frac{x^2}{32} + \frac{(y-12)^2}{64} = 1$. The maximum value of $\sqrt{x^2 + y^2}$ is the distance between (0, 0), (0, 20).

5. Statement-1 : The least value of $x + \frac{9}{x}, x > 0$ is 6.

Statement-2 : The absolute value of the sum of a non-zero real number and its reciprocal is never less than 2. Also $f(x)$ and $kf(x)$ *with* $k > 0$ attain their least value simultaneously.

Key: A

Hint Rewrite $x + \frac{9}{x} = 3\left(\frac{x}{3} + \frac{3}{x}\right)$

6. Statement-1 : The function $f(x) = x^3 - 3x^2 + 12x$ is increasing on R.

Statement-2 : If a differentiable function g(x) is increasing implies $g'(x) > 0$.

Key: C

Hint Statement 2 is false. Take for counter example $f(x) = x^3$, $f'(x) = 3x^2$ and so $f'(0) = 0$.

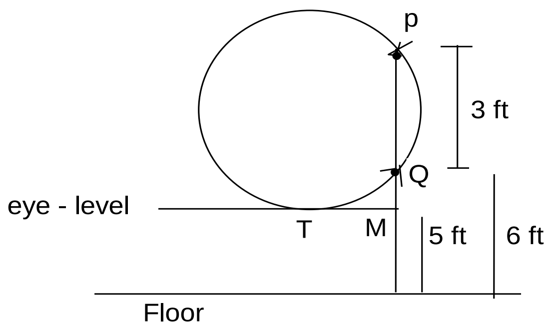
7. A painting of height 3 m hangs on the wall of a museum with the bottom of the painting 6 feet above the floor. The eyes of an observer are 5 feet above the floor.

Statement-1 : In order to maximize his angle of vision the observer must stand at a distance of 2 m from the base of the wall.

Statement-2 : Given a line L, and two points A and B ,(not on the line) on the same side of it, the point on the line at which AB subtends the maximum angle is the point at which the circle with AB as a chord touches the line L.

KEY : D

Sol.



.

PQ is the painting. Now $TM^2 = MQ.MP$

$\Rightarrow TM^2 = 1\times 4 = 4$

$\therefore TM = 2$

8. STATEMENT 1: If $f(x) = \dfrac{\cos x}{\left[\dfrac{x}{2\pi}\right] + \dfrac{1}{2}}, x \in (-2\pi, 2\pi)$,then f(x) is an odd function.

Where [.] denotes greatest integer function.

STATEMENT 2: Odd functions are symmetric about y-axis

KEY : C

HINT

$f(x) = -2\cos x \quad -2\pi < x < 0$

$= 2\cos x \quad 0 \le x < 2\pi$

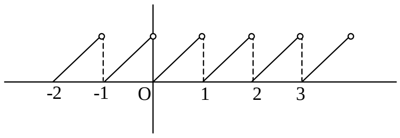
f is symmetric about the origin

9. STATEMENT–1: For f(x) = {x}, every integral point is a point of neither maxima nor minima.

STATEMENT–2: If $f(n) < f(n^+)$ and $f(n) < f(n^-)$ then every integer n is a point of minima.

KEY : D

SOL :



10. STATEMENT – 1: Between any two roots of $e^x \cos x = 1$ there exists a root of $e^x \sin x = 1$

STATEMENT – 2: For a differentiable function f(x) between any two roots of f(x) = 0 there exists atleast one root of f′(x) = 0.

KEY : A

HINT: 2 is according to the Rolle's theorem, So is correct and it correctly explains 1

Let f(x) = cos x – e^{-x} . f(x) = 0

$\Rightarrow$ cos x = $e^{-x} \Rightarrow e^x$ cos x = 1

f′(x) = –sin x + e^{-x} = 0 $\Rightarrow e^{-x}$ = sin x

$\Rightarrow e^x$ sin x = 1

11. Let $f(x)$ satisfy the requirement of Lagrange's mean value theorem in $[0,2]$. If $f(0)=0$ and $|f'(x)| \le \frac{1}{2}$ for all $x \in [0,2]$, then which of the following is not true ?

(A) $f(x) \le 2$ (B) $|f(x)| \le 2x$

(C) $f(x) = 3$, for at least one $x \in [0,2]$ (D) $|f(x)| \le 1$

KEY : C

HINT : Let $x \in [0, 2]$. Since $f(x)$ satisfies the requirements of LMVT in $[0,2]$, So it also satisfies in $[0,x]$. Consequently, there exists at least one c in $(0,x)$ such that

$$f'(c) = \frac{f(x)-f(0)}{x-0} \Rightarrow f'(c) = \frac{f(x)}{x} \Rightarrow \left|\frac{f(x)}{x}\right| \le \frac{1}{2} \Rightarrow |f(x)| \le \frac{|x|}{2} \Rightarrow |f(x)| \le 1 \; (\because |x| \le 2)$$

12. Let $f(x)$ be a twice differentiable function in $[a,e]$ such that

$f(a)=2,\ f(b)=-2,\ f(c)=3,\ f(d)=-4,\ f(e)=4$ where $a<b<c<d<e$ then

STATEMENT-1: $f''(x)=0$ has minimum two zeroes in (a,e)

STATEMENT-2: Between any two roots of $f'(x)=0$ there lies atleast one root of $f''(x)=0$

KEY : A

HINT : $f(x)=0$ has atleast four roots in (a,e)

$f'(x)=0$ has atleast three roots in (a,e)

$f''(x)=0$ has atleast two roots in (a,e)

13. Let f be a function such that $f(x).\, f'(x) < 0,\ \forall\ x \in R$, then

STATEMENT–1: $|f(x)|$ is decreasing $\forall$ x $\in$ R.

STATEMENT–2: f(x) is not continuous if it is not always of same sign.

Key: B

Hint: If f(x) is positive, f(x) is decreasing and if f(x) is negative, then f(x) is increasing so $|f(x)|$ is decreasing $\forall$ x $\in$ R.

As $f(x) \neq 0$

$\therefore$ If f(x) changes its sign then it has been to be discontinuous at some x.

14. Statement 1:-Let f be a continuous function on the internal $[0,1]\to R$, such that $f(0)=f(1)$.Then, there exists a point C in $[0,1/2]$ such that $f(C)=f(C+1/2)$

Statement 2:- Let f be a real valued function defined on $[a,b]$ such that, i) f is continuous on $[a,b]$, ii) f is differentiable in $[a,b]$, iii) $f(a)=f(b)$, then, there is at least one value C of x in (a,b) for which $f^1(c)=0$

Key. B

Sol. Consider the function $g(x)=f(x)-f\left(x+\frac{1}{2}\right)$.

$g(0)=f(0)-f\left(\frac{1}{2}\right)$

$g\left(\frac{1}{2}\right)=f\left(\frac{1}{2}\right)-f(1)=f\left(\frac{1}{2}\right)-f(0)$

$\Rightarrow g(0)$ and $g\left(\frac{1}{2}\right)$ are of opposite signs

Hence, there exists at least are $c\in\left(0,\frac{1}{2}\right)$, such that

$$g(c)=0(or)f(c)-f\left(c+\frac{1}{2}\right)=0\Rightarrow f(c)=f\left(1+\frac{1}{2}\right)$$

15. Statement 1:- let f is a continuous function on $[a,b]$ and differentiable on (a,b) and satisfies $f^2(a)-f^2(b)=a^2-b^2$. Then, the equation $f(x)f^1(x)=x$ has at least one solution in (a,b)

Statement 2:- If f is continuous on $[a,b]$ and $f(a)\neq f(b)$, then, for any value $C\in(f(a),f(b))$,there is at least one number x_0 in (a,b) for which $f(x_0)=C$

Key. B

Sol. Let $g(x)=\frac{1}{2}\left((f(x))^2-x^2\right)$

$g(a)=\frac{1}{2}\left[f^2(a)-a^2\right]$ and $g(b)=\frac{1}{2}\left[f^2(b)-b^2\right]$ by $g(a)=g(b)$, so there exists atleast one $x\in(a,b)\Rightarrow g^1(x)=0$ (By RT) i.e., $\frac{1}{2}\left[2f(x)f^1(x)-2x\right]=0$

$\Rightarrow f^1(x)f^1(x)=x$

16. Statement 1:- The minimum distance of the fixed point $(0,\alpha)$, where, $0\le\alpha\le\frac{1}{2}$ from the curve $y=x^2$ is α

Statement 2:- Maxima and minima of a function is always a root of the equation $f^1(x)=0$

Key. C

Sol. Let point be (t,t^2) then, $d^2=t^2+(t^2-\alpha)^2$

$=t^4+(1-2\alpha)t^2+\alpha^2$

$=z^2+(1-2\alpha)Z+\alpha^2, z\ge 0$

Its vertex is at $x = \alpha - \frac{1}{2} < 0$

$\therefore$ minimum value of d^2 is at z = 0 i.e., $t^2 = 0$

$\Rightarrow d = \alpha$

$\therefore$ Statement I is true

Statement II is false because extremum can occur at a point where $f^1(x)$

One

17. Statement 1:- If f is an increasing function with concave upwards then, $f^{-1}(x)$ is also concave upwards

Statement 2:- If f is decreasing function with concave upwards then $f^{-1}(x)$ is also concave upwards

Key. D

Sol. Conceptual

18. Assertion (A) : The equation $3x^5 + 15x - 8 = 0$ has one and only one real root in (-1,1) .

Reason (R) : If a function $y = f(x)$ is continuous in [a,b] and derivable in (a,b) and $f(a) = f(b)$ then there exists one and only one $c \in (a,b)$ such that $\underset{x \to c}{Lt} \frac{f(x) - f(c)}{x - c} = 0$

Key. C

Sol. Using Rolle's theorem for $f(x) = 3x^5 + 15x - 8$ we have $f(x_1) = f(x_2) = 0$ if there are two roots $x_1, x_2 \in (-1,1)$. But $f'(x) = 15(1 + x^4) \neq 0$

$\therefore$ There is no c value in $(-1,1)$ satisfying Rolle's theorem. Hence our assumption of two distinct roots x_1, x_2 is wrong. So there is a unique root for f(x) = 0 in (–1, 1) as f(1) f(–1) < 0. $\therefore$ Statement – I is correct Statement – II is statement of Rolle's theorem. But there exists at least one $c \in (a,b)$ such that $f'(c) = 0$. We can't declare that there is one and only one root c such that $f'(c) = 0$. $\therefore$ Statement II is wrong

19. Assertion(A): $f(x) = 1 + 4x - x^2 \ \forall x \in R$ & $g(x) = \max.$

$$\{f(t); x \le t \le x + 1;\ 0 \le x < 3\}$$

$$= \min.\{x + 3 : 3 \le x \le 5\}$$ Then g(x) is continuous at x=3

Reason (R) : If f(x) is strictly increasing in [a,b] then max. $\{f(x); a \le x \le b\} = f(b)$

Key. D

Sol. $g(x) = 4 + 2x - x^2 : [0,1)$

$= 5 : \quad [1,2]$

$= 1 + 4x - x^2 : (2,3)$

$= 6 : \quad [1,2]$

is clearly discontinuous at x = 3 $\therefore$ Statement I is false while statement II is true

20. STATEMENT-1 : If $f(x) = \cos x + 4x^2 - x^3 - 7x + 15$ then $f(|\sin x|) \geq f(\sin^2 x)$.
because
STATEMENT-2 : If a continuous function $f(x)$ is decreasing $\forall x \in R$ then $f(a) < f(b)$ when $a > b$.

Key. D

Sol. $f(x) = \cos x + 4x^2 - x^3 - 7x + 15$
$f'(x) = -\sin x + 8x - 3x^2 - 7$.
$= -\sin x - 1 - (3x^2 - 8x + 6), -\sin x - 1 \leq 0 \ \& \ (3x^2 - 8x + 6) < 0 \ \forall x \in r$
$\Rightarrow$ f(x) is decreasing $\forall x \in r$ as $|\sin x| \geq \sin^2 x$.
$\Rightarrow$ $f(|\sin x|) \leq f(\sin^2 x)$

21. Statement–1 : The minimum value of $\left(x_2 - x_1\right)^2 + \left(\sqrt{1 - x_1^2} - \sqrt{4 - x_2^2}\right)^2 = 1$

Statement–2 : The expression attains minimum value if one of the perfect squares vanishes.

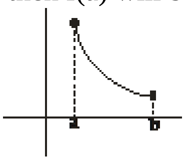
Key. C

Sol. Let $y_1 = \sqrt{1 - x_1^2} \Rightarrow x_1^2 + y_1^2 = 1$
$y_2 = \sqrt{4 - x_2^2} \Rightarrow x_2^2 + y_2^2 = 4$
The minimum distance between (x_1, y_1) and (x_2, y_2) is 1.

22. STATEMENT-1 : If $f'(x) > 0 \ \forall x \in (a, b)$ then maximum value of f(x) in [a, b] is f(a).
because
STATEMENT-2 : If f(x) is continuous $\forall x \in [a, b]$ and $f'(x) > 0 \ \forall x \in (a, b)$ then f(a) is least and f(b) is greatest value of f(x) in [a, b].

Key. D

Sol. $f'(x) < 0 \ \forall x \in (a, b)$
$\Rightarrow$ f(x) is decreasing $\forall \in (a, b)$
if $f(a) \geq \lim_{x \to a^-} f(x)$
then f(a) will be max. value of f(x) in [a, b].



23. Statement – 1 : Minimum distance between $y^2 - 4x - 8y + 40 = 0$ and $x^2 - 8x - 4y + 40 = 0$ is $\sqrt{2}$
Because
Statement – 2 : Minimum distance between two curves lie along their common normal.

Key. A

Sol. Parabolas are symmetric about y = x
$$2y\frac{dy}{dx} - 4 - 8\frac{dy}{dx} = 0$$
$$\Rightarrow \quad \frac{dy}{dx} = \frac{2}{y-4} = 1$$
y = 6, x = 7
distance from (7, 6) to the line x – y = 0
$$= \frac{1}{\sqrt{2}}$$

Shortest distance $= \frac{2}{\sqrt{2}} = \sqrt{2}$

24. STATEMENTS–1:

Consider the function f(x), $\frac{f(x_1+x_2)}{2} < \frac{f(x_1)+f(x_2)}{2}$

because

STATEMENTS–2:

$f'(x) > 0$, $f''(x) > 0$ where $x_1 < x_2$

Key. A

Sol. $f(x_1) = f\begin{pmatrix} \frac{x_1+x_2}{2} + \frac{x_1-x_2}{2} \\ a \quad\quad b \end{pmatrix}$

$= f\frac{(x_1+x_2)}{2} + \frac{x_1-x_2}{2} f'\frac{(x_1-x_2)}{2} + \frac{(x_1+x_2)^2}{2!} f''(x_1)$

$\therefore$ $f(x_1) + f(x_2) > 2f\left(\frac{x_1+x_2}{2}\right)$

$\Rightarrow f\left(\frac{x_1+x_2}{2}\right) < \frac{1}{2}(f(x_1)+f(x_2))$

25. Statement – 1: The function $f(x) = \begin{cases} |x|, & 0<|x|\le 2 \\ 1, & x=0 \end{cases}$ has no local extremum at x = 0

Statement – 2: If $g^1(a) = 0$ and $g^{11}(a) \ne 0$ then the function *g* has a local extremum at x = a

Key. D

Sol. Conceptual

26. Assertion (A) : If the function $f:[0,4] \to R$ is differentiable then $f^2(4) - f^2(0) = 8f'(a)f(b)$ for a, $b \in (0,4)$

Reason (R) : For any continuous function $f(x)$ defined on I , $\frac{f(x_1)+f(x_2)}{2} = f(b)$ for $b \in (x_1, x_2)$ where $x_1, x_2 \in I$

Key. B

Sol. By Legrange's mean value theorem

$f'(a) = \frac{f(4)-f(0)}{4}$

$\Rightarrow 4f'(a) = f(4) - f(0) \to (1)$

Also $2f(b) = f(4) + f(0) \to (2)$

From (1) & (2) it follows

27. **STATEMENT–1:** $9^8 > 8^9$

STATEMENT–2: The function $f(x) = x^{1/x}$ (x > e) is a decreasing function of x.

Key. D

Sol. $f(x) = x^{1/x}$

f′(x) = $x^{1/x}\dfrac{1-\log x}{x^2}$ < 0 for x > e

So, f(x) is decreasing function of x $\forall$x > e

So, f(9) < f(8)

$9^{1/9} < 8^{1/8} \Rightarrow 9^8 < 8^9$

28. Let f(x) = $x^2 . e^{-x^2}$

Statement 1: f(x) has local maxima at x= -1and x=1 and f(x) has local minima at x = 0

Statement 2: y = f' (x) changes its sign at x = 1, - 1, 0

Key. B

Sol. Statement -1 is true as $f'(x) < 0$ if $-1 < x < 0,\ x > 1$ and $f'(x) > 0$ if $0 < x < 1,\ x < -1$.

Statement- 2 explains statement-1.

29. Consider the function $f(x) = \dfrac{1}{2\{-x\}} - \{x\}$, where $\{x\}$ is the fractional part of x (x not being integer)

Statement – 1: Least value of f(x) is $\sqrt{2}-1$

Statement – 2: If the product of two positive numbers is a constant, then the least value of their sum is twice the square root of their product.

Key. B

Sol. Since the function is periodic of period one it is enough to consider the function in (0,1)

$\{-x\} = 1-\{x\}$ for x$\in$ (0,1).

$\therefore f(x) = \dfrac{1}{2(1-x)} - x$ for all x $\in$ (0,1), $f'(x) = \dfrac{1}{2(1-x)^2} - 1 = 0 \Rightarrow x = \left(1-\dfrac{1}{\sqrt{2}}\right)$,

$f'(x) < 0$, for x < $1-\dfrac{1}{\sqrt{2}}$, and $f'(x) > 0$, for x > $1-\dfrac{1}{\sqrt{2}}$.

$\therefore f(x)$ is minimum at x = $1-\dfrac{1}{\sqrt{2}}$ $\therefore$ minimum value of f(x) is $\sqrt{2}-1$.

$\therefore$ statement - 1 is true, statement – 2 is true, but statement-2 does not explain statement-1

30. **Statement-1:** $f:[-1,1] \to$ R given by $f(x) = -|x|$ has maximum at x = 0.

Statement-2: If $f(x)$ increases in (a, c) and decreases in (c, b) then f has local maximum at $x = c$.

Key. C

Sol. Statement 1 is true

Statement 2 is false; e.g., consider the function

$$f(x) = \begin{cases} 1-|x| & if \quad x \neq 0 \\ o & if \quad x = 0 \end{cases}$$

31. **Statement-1:** $0.4 < \tan^{-1} 2 < 2$.

Statement-2: $f(x) = \dfrac{1}{1+x^2} : x \in [0, 2]$ has range [0.2, 1].

Key. A

Sol. Statement 2 is true

$x:0\to 2$

$1+x^2:1\to 5$

$\therefore f(x):1\to 0.2$

Statement 1 is true

By Lagrange's theorem applied to $g(x)=\tan^{-1}x : x\in[0,2],$ we have

$\frac{g(2)-g(0)}{2-0}=g'(c)$ for some $c\in]0,2[$

$\frac{\tan^{-1}2}{2}=\frac{1}{1+c^2}\in(0.2,1)$ as $f(x)=\frac{1}{1+x^2}$ decreases.

$\tan^{-1}\in(0.4,2)$.

The reason for statement 1 is Lagrange's theorem and statement 2. Statement 2 is not a complete explanation of statement 1.

32. **Statement-1:** $303^{202}<202^{303}$.

Statement-2: $f(x)=\frac{\ln x}{x}$ is a decreasing function in (e,∞).

Key. A

Sol. $g(x)=x^{1/x}; f(x)=\ln g(x): x>0$

$g(x)=e^{f(x)}$ and e^x is an increasing function.

$$f'(x)=\frac{1-\ln x}{x^2};\ f'(x)\begin{cases}<0 & if \quad x>e\\ >0 & if \quad 0<x<e\\ 0 & if \quad x=e\end{cases}$$

$\therefore$ statement 2 is true

$\therefore$ g is a decreasing function

$\therefore\ (202)^{\frac{1}{202}}>(303)^{\frac{1}{303}}$

$\therefore$ statement 1 is true

33. **Statement-1:** $f(x)=x^{2/3}$ has vertical tangent at x = 0.

Stetement-2: A function to have local extrema at a point need not be differentiable at that point.

Key.D

Sol. Statement 1 is false as at x = 0

$f'(0-)\equiv-\infty$ and $f'(0+)\equiv\infty$

Statement 2 is true; e.g., $f(x)=|x|$ at x = 0

34. STATEMENT–1 : $y=x^{1/3}$ has a vertical tangent at x = 0

and

STATEMENT–2 : If $y = f(x)$ has left hand and right hand derivative both ∞ or both $-\infty$ at *x* = *a* then *f* is continuous at *x* = *a*.

Key. C

Sol. Statement 1 is true for $g(x) = x^{1/3}$ as $g'(0-) = g'(0+) \equiv \infty.$
Statement 2 is false; e.g., $y = \text{sgn}(x)$ at $x = 0$.

35. STATEMENT–1 : $y = \frac{1}{x}$ decreases in $R \sim \{0\}$

and

STATEMENT–2 : Derivative of $y = \frac{1}{x}$ is negative throughout its domain.

Key. D

Sol. Statement 1 is false:

$x_1 = -1, x_2 = 1; x_1 < x_2$

$f(x_1) = -1, f(x_2) = 1; f(x_1) < f(x_2)$
Statement 2 is true

36. ***STATEMENT–1 :*** $y = x^3$ ***has a tangent at* x = *0.***

and

STATEMENT–2 : Tangent at a point *x* = *a* is a line having only the point $'x = a'$ common with the curve.

Key. C

Sol. Statement 1 is true.
Statement 2 is false; e.g., $y = \sin x$ has line $y = 1$ as tangent at infinite points.

37. STATEMENT –1 : $f:(0,1) \to R$ such that $f(x) = x$ has no local extrema.

and

STATEMENT –2 :A function *f* can have local extrema only if its domain is a closedinterva

Key. C

Sol. Statement 1 is true
Statement 2 is false; e.g., $y = \sin x : x \in (0, \pi)$

Maxima & Minima

Comprehension Type

Paragraph – 1

Let $f(x)=\frac{(x-1)(x-2)}{(x-a)(x-b)}$

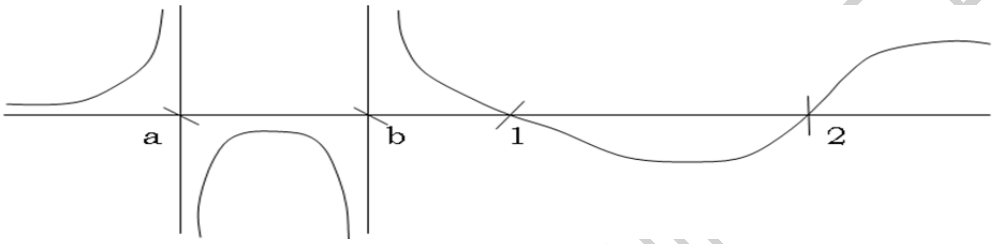
1. If $a<b<1$, then, $f(x)$ has

a) Neither a maximum nor a minimum b) a maximum

c) a minimum d) a maximum and a minimum

Key. D

Sol.



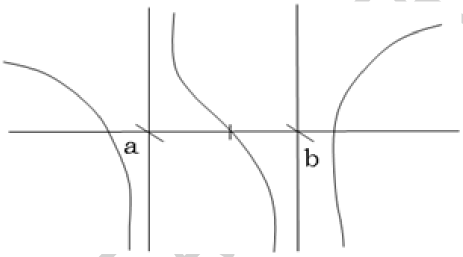
2. If $1<a<b<2$, then, $f(x)$ has

a) Neither a maximum nor a minimum b) a maximum

c) a minimum d) a maximum and a minimum

Key. C

Sol.



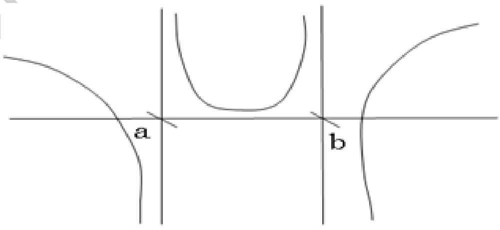
3. If $1<a<2<b$, then, $f(x)$ has

a) Neither maximum nor minimum b) a maximum

c) a minimum d) a maximum and a minimum

Key. A

Sol.



Paragraph – 2

Let α,β,γ be the positive roots of $x^3+ax^2+bx+c=0$, then,

4. If $c = \frac{-1}{64}$, then the minimum value of $\alpha+\beta+\gamma$, is

a) $\frac{4}{3}$ b) $\frac{3}{4}$ c) $\frac{5}{6}$ d) $\frac{6}{5}$

Key. B

Sol. Conceptual

5. If $a=-1,$ then the maximum value of $\alpha\beta^2\gamma^3$, is

a) $\frac{1}{234}$ b) $\frac{1}{324}$ c) $\frac{1}{456}$ d) $\frac{1}{432}$

Key. D

Sol. $\frac{\alpha+\beta+\gamma}{3} \ge (\alpha\beta\gamma)^{1/3} \Rightarrow \alpha+\beta+\gamma \ge \frac{3}{4}$

$$\frac{\alpha+\frac{\beta}{2}+\frac{\beta}{2}+\frac{\gamma}{3}+\frac{\gamma}{3}+\frac{\gamma}{3}}{6} \ge \left(\alpha.\frac{\beta^2}{2^2}.\frac{\gamma^3}{3^3}\right)^{1/6}$$

Paragraph – 3

Let $f(x) = \frac{(x-1)(x-2)}{(x-a)(x-b)}$

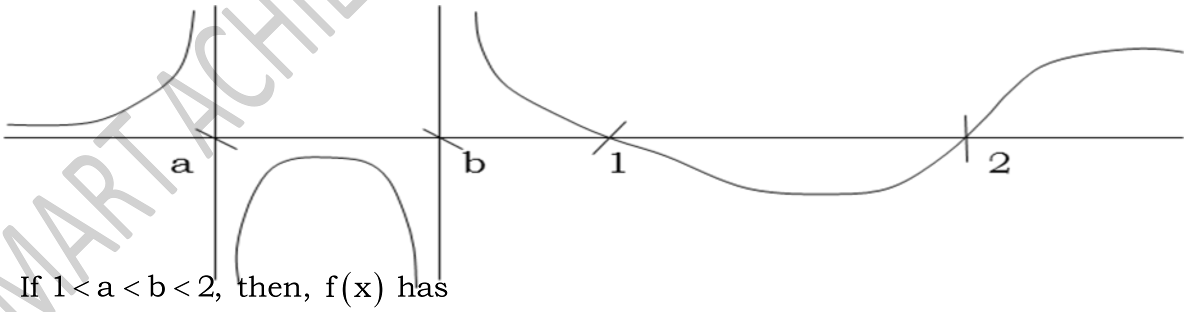
6. If $a<b<1,$ then, $f(x)$ has

b) Neither a maximum nor a minimum b) a maximum

c) a minimum d) a maximum and a minimum

Key. D

Sol.



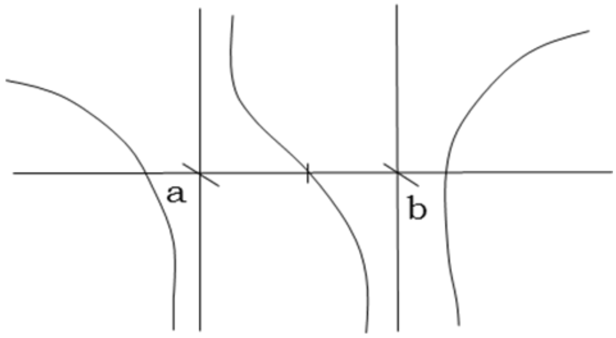
7. If $1<a<b<2,$ then, $f(x)$ has

b) Neither a maximum nor a minimum b) a maximum

c) a minimum d) a maximum and a minimum

Key. C

Sol.



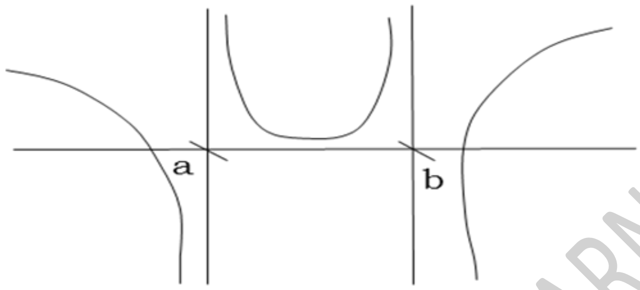
8. If $1 < a < 2 < b$, then, $f(x)$ has

b) Neither maximum nor minimum b) a maximum

c) a minimum d) a maximum and a minimum

Key. A

Sol.



Paragraph – 4

Let α,β,γ be the positive roots of $x^3 + ax^2 + bx + c = 0$, then,

9. If $c = \frac{-1}{64}$, then the minimum value of $\alpha+\beta+\gamma$, is

a) $\frac{4}{3}$ b) $\frac{3}{4}$ c) $\frac{5}{6}$ d) $\frac{6}{5}$

Key. B

10. If $a = -1$, then the maximum value of $\alpha\beta^2\gamma^3$, is

a) $\frac{1}{234}$ b) $\frac{1}{324}$ c) $\frac{1}{456}$ d) $\frac{1}{432}$

Key. D

Sol. $\frac{\alpha+\beta+\gamma}{3} \ge (\alpha\beta\gamma)^{1/3} \Rightarrow \alpha+\beta+\gamma \ge \frac{3}{4}$

$$\frac{\alpha+\frac{\beta}{2}+\frac{\beta}{2}+\frac{\gamma}{3}+\frac{\gamma}{3}+\frac{\gamma}{3}}{6} \ge \left(\alpha.\frac{\beta^2}{2^2}\cdot\frac{\gamma^3}{3^3}\right)^{1/6}$$

Paragraph – 5

For a polynomial function y = f(x)

Points of extrema are obtained at points where

$f'(x) = 0$

$f''(x_1) > 0 \Rightarrow x_1$ is a point of minima

$f''(x_1) < 0 \Rightarrow x_1$ is a point of maxima

Let $f(x) = x^3 - 3(7 - a)x^2 - 3(9 - a^2)x + 2$

11. The values of parameter 'a' if f(x) has a negative point of local minimum are

(A) ϕ (B) $\left(-\infty, \frac{58}{14}\right)$ (C) $(-3, 3)$ (D) none of these

Key. A

12. The values of parameter 'a' if f(x) has a positive point of local maxima are

(A) ϕ (B) $(-\infty, -3) \cup \left(\frac{58}{14}, \infty\right)$ (C) $\left(-\infty, \frac{58}{14}\right)$ (D) none of these

Key. B

13. The values of parameter 'a' if f(x) has points of extrema which are opposite in sign are

(A) ϕ (B) $(-3, 3)$ (C) $\left(-\infty, \frac{58}{14}\right)$ (D) none of these

Key. B

Sol. 11 – 13

$f(x) = x^3 - 3(7-a)x^2 - 3(9-a^2)x + 2$

$f'(x) = 3x^2 - 6(7-a)x - 3(9-a^2)$

For real root $D \geq 0$

$a \leq \frac{58}{14}$ ...(i)

Let $f'(x)$ has two roots x_1 & x_2 ($x_2 > x_2$)

Minima at $x = x_2$

11. Both roots – ve $\Rightarrow 2(7 - a) < 0 \Rightarrow a > 7$

Not possible

12. Both roots are +ve

Sum of roots > 0 $\Rightarrow a < 7$ ...(ii)

Product of roots > 0 $\Rightarrow a \in (-\infty, -3) \cup (3, \infty)$ ...(iii)

From 1, 2, 3, we get $(-\infty, -3) \cup \left(\frac{58}{14}, \infty\right)$

13. For points of opposite sign,

Product of roots < 0

$a \in (-3, 3)$

Paragraph – 6

Consider the equation $\log_2 x^2 - 4\log_2 x - m^2 - 2m - 13 = 0$ in the variable x, 'm' being a parameter $(m \in R)$. Let the real roots of the equation be x_1, x_2 where $x_1 < x_2$.

14. The set of all values of 'm' for which the equation has real roots is

A) $(-\infty, 0)$ B) $(0, \infty)$ C) $[1, \infty)$ D) R

15. The maximum value of x_1 is

A) $1/2$ B) $-1/4$ C) $1/4$ D) $-1/2$

16. The minimum value of x_2 is

A) 32 B) -28 C) 64 D) 48

Key: D-C-C

Hint: Question nos: 14 – 16

Put $\log_2 x = t$ or $x = 2^t, x > 0$

$t^2 - 4t - (m^2 + 2m + 13) = 0$. Disc = $4(m^2 + 2m + 17) > 0 \forall m \in R$

Let t_1, t_2 be roots where $t_1 < t_2$

$\therefore x_1 = 2^{t_1}$ and $x_2 = 2^{t_2}$.

Paragraph – 7

Let $a \in \boldsymbol{R}$ *and* $f(x) = 2x^3 - 3(a-3)x^2 + 6ax + (a+2)$

17. The set of values of 'a' for which f has no point of extrema is

a) (1,9) b) [1,9] c) $(-\infty, 0)$ d) ϕ

18. The set of values of 'a' for which f has exactly one point of local maxima and one point of local minima is

a) $(-\infty, 1) \cup (9, \infty)$ b) $(-\infty, 1] \cup [9, \infty)$ c) [1,9] d) (1,9)

19. The set of values of 'a' for which f has a local maximum at some negative real number and a local minimum at some positive real number is

a) (1, 9) b) [1,9] c) $(-\infty, 1)$ d) $(-\infty, 0)$

Key: B-A-D

Hint:

17. f has no point of extrema $\Rightarrow f'(x) = 0$ has no real root or $f'(x) = 0$ has a double root

$\Rightarrow (a-1)(a-9) \le 0 \Rightarrow a \in [1, 9]$

18. $f'(x)=6\left(x^2-(a-3)x+a\right)$. *so* $f'(x)=0$ has a pair of real and distinct roots α *and* β $(\alpha<\beta)$ if and only if $(a-3)^2>4a$ *i.e* $(a-1)(a-9)>0(or)$ *iff* $a<1$ *or* $a>9$

19. The roots of $f'(x)=0$ are given by

$$\alpha=\frac{(a-3)-\sqrt{(a-1)(a-9)}}{2} \text{ and } \beta=\frac{(a-3)+\sqrt{(a-1)(a-9)}}{2}$$

Since $\alpha<\beta$, f has a local maximum at $\alpha(<0)$ and a local minimum at $\beta(>0)$ only if

$$\alpha\beta=\frac{(a-3)^2-(a-1)(a-9)}{4}<0 \text{ i.e only if } a<0$$

Paragraph – 8

Let $f(x)=ax^2+bx+c; a,b,c\in R$

It is given that $|f(x)|\le 1, \forall |x|\le 1$

20. The possible value of $|a+b|$ if $4a^2+3b^2$ is maximum is

(A) 1 (B) 0

(C) 2 (D) 3

Key: C

21. The possible values of $|a+b|$ if $\frac{8}{3}a^2+2b^2$ is maximum is given by

(A) 1 (B) 0

(C) 2 (D) 3

Key: C

22. The possible maximum value of $\frac{8}{3}a^2+2b^2$ is given by

(A) 32 (B) $\frac{32}{3}$

(C) $\frac{2}{3}$ (D) $\frac{16}{3}$

Key: B

Hint: Now $|f(1)-f(0)|\le 2 \Rightarrow |a+b|\le 2 \Rightarrow (a+b)^2\le 4$

$|f(-1)-f(0)|\le 2 \Rightarrow |a-b|\le 2 \Rightarrow (a-b)^2\le 4$

Now, $4a^2+3b^2=2(a+b)^2+2(a-b)^2-b^2\le 16$

$\left(4a^2+3b^2\right)_{max}=16$ When b = 0

$\Rightarrow |a+b|=|a-b|=|a|=2$

Also the possible ordered triplet (a,b,c) are (2,0, -1) or (-2,0,1)

Also $\frac{8}{3}a^2+2b^2=\frac{2}{3}\left(4a^2+3b^2\right)\le\frac{2}{3}\times 16.\le\frac{32}{3}$

Paragraph – 9

Let $f(x)=0$ be a polynomial equation with real coefficients. Then between any two distinct real roots of $f(x)=0$, there exists at least one real root of the equation $f'(x)=0$.

This result is a consequence of the celebrated Rolle's theorem applied to polynomials. Much information can be extracted about the roots of $f(x)=0$ from the roots of $f'(x)=0$.

23. The range of values of k for which the equation $x^4-14x^2+24x-k=0$ has four unequal real roots is
    (A) $8<k<11$ (B) $4<k<8$ (C) $8<k<15$ (D) $4<k<13$

24. If the roots of $x^3-12x+k=0$ lie in (-4, -3), (0, 1) and (2, 3), then the range of values of k is
    (A) $4<k<11$ (B) $9<k<11$ (C) $8<k<13$ (D) $4<k<13$

25. The range of values of k for which the equation $x^4+4x^3-8x^2+k=0$ has four real and unequal roots is
    (A) $0<k<3$ (B) $0<k<8$ (C) $3<k<8$ (D) $3<k<13$

KEY : A-B-A

HINT: (23-25)

23. $f(x)=x^4-14x^2+24x-k$

$f'(x)=4x^3-28x+24=4(x+3)(x-1)(x-2)$

$f'(x)=0$ has two roots -3, 1, 2

$f(-\infty)>0$

$f(\infty)>0$

$f(-3)=-117-k$

$f(1)=11-k$

$f(2)=8-k$

For the equation $f(x)=0$ to have four real and unequal roots we require

$f(-3)<0, f(1)>0 \quad and\ f(2)<0$

We get 8 < k < 11

24. $f(x)=x^3-12x+k$

Note that $k\neq 0$ otherwise '0' is root of $f(x)=0$ which goes against the hypothesis.

If k < 0, then $f(0)=k\ and\ f(1)=-11+k$ have the same sign which can't happen because f(x) has a root between 0 and 1.

Thus k > 0, now we can complete the solution

$f(-4) = -16 + k < 0$

$f(-3) = 9 + k > 0$, $f(0) = k > 0$

$f(1) = -11 + k < 0$, $f(2) = -16 + k < 0$

$f(3) = -9 + k > 0$

The range is found by the intersection i.e, $9 < k < 11$

25. $f(x) = x^4 + 4x^3 - 8x^2 + k$

$f'(x) = 4x^3 + 12x^2 - 16x$

$= 4x(x+4)(x-1)$

Then each of the intervals $(-\infty, -4), (-4, 0)(0,1)$ *and* $(1, \infty)$ contain a root of f(x)=0

$f(-4) = -128 + k < 0$

$f(0) = k > 0$

$f(1) = k - 3 < 0$

The range is found by intersection i.e. 0 < k < 3

Paragraph – 10

Let f be a function satisfying $f\left(\frac{x_1 + x_2}{2}\right) \le \frac{f(x_1) + f(x_2)}{2}$ for all x_1, x_2 in its domain. If we draw the graph of a continuous function satisfying this inequality, we will notice that the chord joining any two points on the curve will always be above the portion of the curve between those two points. We can also prove that if the given inequality is true, then the similar result follows for four values x_1, x_2, x_3, x_4. Indeed

$$f\left(\frac{x_1 + x_2 + x_3 + x_4}{4}\right) = f\left(\frac{\frac{x_1 + x_2}{2} + \frac{x_3 + x_4}{2}}{2}\right) \le \frac{f\left(\frac{x_1 + x_2}{2}\right) + f\left(\frac{x_3 + x_4}{2}\right)}{2}$$

$$\le \frac{\frac{f(x_1) + f(x_2)}{2} + \frac{f(x_3) + f(x_4)}{2}}{2} \le \frac{f(x_1) + f(x_2) + f(x_3) + f(x_4)}{4}$$

26. If , $f\left(\frac{x_1 + x_2}{2}\right) \le \frac{f(x_1) + f(x_2)}{2}$ then $f\left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right) \le \frac{f(x_1) + f(x_2) + \ldots + f(x_n)}{n}$ for

a) only when n is of type 2^k b) only when n is even positive integer

c) for all d) Nothing can be said

27. The result $f\left(\frac{x_1+x_2+x_3}{3}\right) \le \frac{f(x_1)+f(x_2)+f(x_3)}{3}$ follows from

$f\left(\frac{x_1+x_2+x_3+x_4}{4}\right) \le \frac{f(x_1)+f(x_2)+f(x_3)+f(x_4)}{4}$ in the later x_4 is replaced by

a) $\frac{x_1+x_2+x_3}{2}$ b) $\frac{x_1+x_2+x_3}{4}$ c) $\frac{x_1+x_2+x_3}{3}$ d) $x_1 + x_2 + x_3$

28. Which of the following functions satisfy $f\left(\frac{x_1+x_2}{2}\right) \le \frac{f(x_1)+f(x_2)}{2}$ for all x_1, x_2 in a domain D

A) $\sin x, 0 < x < \frac{\pi}{2}$ B) $\log x, 0 < x < \infty$

C) $\tan x, 0 < x < \frac{\pi}{2}$ D) All of the above

KEY : C,B,C

HINT :

26. If $f\left(\frac{x_1+x_2}{2}\right) \le \frac{f(x_1)+f(x_2)}{2}$, then the function will be convex and the convexity will remain true for any number of variables, Algebraically it can be shown in the following manner :

(i) Show that whenever it is true for k it is also true for 2k.

(ii) Whenever it is true for k, then it is also true for k – 1.

27. We can test one-by-one and determine that $\frac{x_1+x_2+x_3}{3}$ is the correct replacement.

28. Only tan x is convex (chord will be above the curve.)

Paragraph – 11

If f be a twice differentiable function such that $f''(x) > 0\ \forall x \in R$. Let h(x) is defined by $h(x) = f(\sin^2 x) + f(\cos^2 x)\ where |x| < \frac{\pi}{2}$

29. The number of critical points of h(x) are

a) 1 b) 2 c) 3 d) more than 3

Key: C

Hint: Passage $f^{11}(x) > 0 \Rightarrow f^1(x)$ is an increasing function

$f^1(x_1) > f^1(x_2) \Rightarrow x_1 > x_2$, $f^1(x_1) = f^1(x_2) \Rightarrow x_1 = x_2$

$h^1(x) = \sin 2x\left(f^1(\sin^2 x) - f^1(\cos^2 x)\right)$

$$f^1(x)=0 \Rightarrow \sin 2x=0 \Rightarrow x=0$$

$$or\ f^1(\sin^2 x)=f^1(\cos^2 x) \Rightarrow \sin^2 x=\cos^2 x \Rightarrow \tan^2 x=1 \Rightarrow x=\pm\frac{\pi}{4}$$

30. $f'(\sin^2 x)<f'(\cos^2 x)$ *for* $x\in$

a) $\left(-\frac{\pi}{4},\frac{\pi}{4}\right)$ b) $\left(-\frac{\pi}{2},-\frac{\pi}{4}\right)\cup\left(\frac{\pi}{4},\frac{\pi}{2}\right)$

c) $\left(-\frac{\pi}{4},0\right)\cup\left(\frac{\pi}{4},\frac{\pi}{2}\right)$ d) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

Key: A

Hint: $f^1(\sin^2 x)<f^1(\cos^2 x) \Rightarrow \sin^2 x<\cos^2 x \Rightarrow \tan^2 x<1 \Rightarrow x\in\left(-\frac{\pi}{4},\frac{\pi}{4}\right)$

31. h (x) is increasing for $x\in$

a) $\left(-\frac{\pi}{4},\frac{\pi}{4}\right)$ b) $\left(-\frac{\pi}{2},-\frac{\pi}{4}\right)\cup\left(\frac{\pi}{4},\frac{\pi}{2}\right)$

c) $\left(-\frac{\pi}{4},0\right)\cup\left(\frac{\pi}{4},\frac{\pi}{2}\right)$ d) $\left(-\frac{\pi}{2},-\frac{\pi}{4}\right)\cup\left(0,\frac{\pi}{4}\right)$

Key: B

Hint: $h(x)$ is increasing $\Rightarrow h^1(x)>0$

$$\left.\begin{array}{l} Case\ I\ \ (i)\sin 2x>0 \Rightarrow x\in\left(0,\frac{\pi}{2}\right) \\ (ii)\ f^1(\sin^2 x)>f^1(\cos^2 x) \Rightarrow \tan^2 x>1 \end{array}\right\} \Rightarrow x\in\left(\frac{\pi}{4},\frac{\pi}{2}\right)$$

$$\left.\begin{array}{l} Case\ II\ (i)\sin 2x<0 \Rightarrow x\in\left(-\frac{\pi}{2},0\right) \\ (ii)\ f^1(\sin^2 x)<f^1(\cos^2 x) \Rightarrow \tan^2 x<1 \end{array}\right\} \Rightarrow x\in\left(\frac{-\pi}{2},-\frac{\pi}{4}\right)$$

Paragraph – 12

If the function f(x) is continuous and has continuous derivatives through order n-1 on the interval $[a,b]$ and has a finite derivative of the n^{th} order at every interior point of the interval. Then,
at $x\in[a,b]$,

$$f(x)=f(a)+f^1(a)f(x-a)+f^{11}(a)\frac{(x-a)^2}{2!}+....+f^{(n-1)}(a)\frac{(x-a)^{n-1}}{(n-1)!}+f^n(\in)\frac{(x-a)^n}{n!}$$

where $\in=a+\theta(x-a),0<\theta<1$ is called TAYLOR'S FORMULA of the function f (x).
Put a = 0, we obtain MACLAURIN'S FORMULA!.
The last term in both TAYLOR AND MACLAURINE formulae are called REMAINDER ANSWER THE FOLLOWING

32. The polynomial $p(x)=x^5-2x^4+x^3-x^2+2x-1$ can be expressed in powers of $(x-1)$ as,

a) $3(x-1)^3+3(x-1)^4+(x-1)^5$ b) $5(x-1)^3+5(x-1)^4+(x-1)^5$

c) $6(x-1)^3+6(x-1)^4+(x-1)^5$ d) $2(x-1)^3+2(x-1)^4+(x-1)^5$

Key. A

Sol. Put a = 1 in Taylor series

33. In the expansion of $f(x)=\log(1+x)$, on $[0,1]$ the remainder $R_{10}(x)$ in maclaurin's form, $0<\in<x$, is given by

a) $\frac{-x^{10}}{10(1+\in)^{10}}$ b) $\frac{x^{10}}{10(1+\in)^{10}}$ c) $\frac{x^{10}}{10(1-\in)^{10}}$ d) $\frac{-x^{10}}{10(1-\in)^{10}}$

Key. A

Sol. Conceptual

Paragraph – 13

Let f(x) = $\frac{e^x}{1+x^2}$ and g(x) = f ′(x)

34. The number of point(s) of local maxima/minima of g(x) is (are)

(A) zero (B) one

(C) two (D) four

Key. C

35. The point of local extrema of g(x) for x > 0 is (are)

(A) x = 1/2 (B) x = 1

(C) both (A) and (B) (D) no such point exists

Key. B

36. The tangent line of the curve y = f(x) at x = 2

(A) lies above the curve in the neighborhood of x = 2

(B) lies below the curve in the neighborhood of x = 2

(C) crosses the curve at x = 2

(D) can not say

Key. B

Sol. 34. f(x) = $\frac{e^x}{1+x^2}$, $g(x)$ = f′(x)

g′(x) = f″(x) = $\frac{e^x(x-1)}{1+x^2}$ (x^3-3x^2+5x+1)

Let $h(x)=x^3-3x^2+5x+1$, $h'(x)=3x^2-6x+5$, D < 0

So, h(x) is strictly increasing will exactly have one real root. Product of roots < 0, so the real root is negative. So, g′(x) will change sign only at two points.

35. For x > 0 it is x = 1.

36. f″ (2) = g′(2) > 0, so f(x) is concave upward at x = 2.

Paragraph – 14

If $f(x)=\begin{vmatrix}\sin^2 x & \cos^2 x & l\\ \cos^2 x & l & \sin^2 x\\ l & \sin^2 x & \cos^2 x\end{vmatrix}$, where *l* is a root of $16l^2-16l+7=0$ then there exists two real numbers $(a,b)\in\left(0,\frac{\pi}{2}\right)$ and a < b for which f (x) = 0 . Does there exist $c\in(a,b)$ for which $f'(c)=0$. Then

37. The value of 'a' is

a) $\frac{\pi}{6}$ b) $\frac{\pi}{4}$ c) $\frac{\pi}{3}$ d) $\frac{\pi}{2}$

Key. A

38. The value of 'b' is

a) $\frac{\pi}{6}$ b) $\frac{\pi}{4}$ c) $\frac{\pi}{3}$ d) $\frac{\pi}{2}$

Key. C

39. The value of 'c' is

a) $\frac{\pi}{6}$ b) $\frac{\pi}{4}$ c) $\frac{\pi}{3}$ d) $\frac{\pi}{2}$

Key. B

Sol. 37. 38. 39. : obliviously f(x) is a circulant matrix

$\therefore -f(x)=\left(\sin^2 x+\cos^2 x+l\right)\left(\sin^4 x+\cos^4 x+l^2-\sin^2 x\cos^2 x-l\cos^2 x-l\sin^2 x\right)$

= $(l+1)\left(l^2+1-3\sin^2 x\cos^2 x-l\right)$

Now f(x) = 0 $\Rightarrow l^2-l+1=3\sin^2 x\cos^2 x=0\ as\ l=-1$

$\Rightarrow l^2-l+1=\frac{3}{4}\sin 2x \Rightarrow \frac{7}{16}+1=\frac{3}{4}\sin^2 2x\ \ as\ l$ is a root of $16l^2-16l+7=0$

$\Rightarrow \sin^2 2x=\frac{3}{4}\Rightarrow \sin 2x=\frac{\sqrt{3}}{2}$ as we have to find roots of . $f(x)=0 in\left(0,\frac{\pi}{2}\right)$.

$\Rightarrow x=\frac{\pi}{6},\frac{\pi}{3}$. Thus $f\left(\frac{\pi}{6}\right)=f\left(\frac{\pi}{3}\right)=0$ further f is a continuous function and differentiable $\forall\ x\in R$. Rolle's Theorem assert that $f'(c)=0$ for some $c\in\left(\frac{\pi}{6},\frac{\pi}{3}\right)$.

$\therefore\ f'(x)=\frac{3}{2}(l+1)\sin 4x=0\ \Rightarrow x=\frac{\pi}{4}$

So finally $a=\frac{\pi}{6},c=\frac{\pi}{4},b=\frac{\pi}{3}$.

Paragraph – 15

Consider the function f : R $\rightarrow$ R defined by $f(x)=\frac{x^2-ax+1}{x^2+ax+1}$, $a\in(0,2)$

40. Which of the following is true?
(A) $(2 + a)^2 . f''(1) + (2 - a)^2 . f''(-1) = 0$ (B) $(2 - a)^2 f''(1) - (2 + a)^2 f''(-1) = 0$
(C) $f'(1) f'(-1) = (2 - a)^2$ (D) $f'(1) f'(-1) = -(2 + a)^2$

Key. A

41. Which of the following is true?
(A) f(x) is decreasing on (–1, 1) and has a local minimum at x = 1
(B) f(x) is increasing on (–1, 1) and has a local maximum at x = 1
(C) f(x) is increasing on (–1, 1) but has neither a local maximum nor a local minimum at x = 1
(D) f(x) is decreasing on (–1,1) but has neither a local maximum nor a local minimum at x = 1

Key. A

42. Let $g(x) = \int_0^{e^x} \frac{f'(t)}{1+t^2} dt$, which of the following is true?
(A) g′(x) is positive on (–∞, 0) and negative on (0, ∞)
(B) g′(x) is negative on (–∞, 0) and positive on (0, ∞)
(C) g′(x) changes sign on both (–∞, 0) and (0, ∞)
(D) g′(x) does not change sign on (–∞, ∞)

Key. B

Sol. 40. $f(x) = 1 - \frac{2ax}{x^2 + ax + 1}$

$\Rightarrow \quad f'(x) = \frac{2a(x^2 + ax + 1) - 2ax(2x + a)}{(x^2 + ax + 1)^2} = \frac{2a(x^2 - 1)}{(x^2 + ax + 1)}$

$\therefore \quad f''(x) = \frac{4a(x^2 + ax + 1) - 2ax(x^2 - 1).2(x^2 + ax + 1).(2x + a}{(x^2 + ax + 1)^4}$

$= \frac{4ax(x^2 + ax + 1) - 4a(x^2 - 1)(2x + a)}{(x^2 + ax + 1)^3}$

$\therefore \quad f''(1) = \frac{4a(2 + a)}{(2 + a)^3} = \frac{4a}{(2 + a)^2}; \; f''(-1) = \frac{-4a(2 - a)}{(2 - a)^3} = \frac{-4a}{(2 - a)^2}$

$\therefore \quad (2 + a)^2.f''(1) + (2 - a)^2.f''(-1) = 4a - 4a = 0$

41. In (–1, 1), $f'(x) = \frac{2a(x^2 - 1)}{(x^2 + ax + 1)^2} \Rightarrow f'(x) = \frac{(+)(-)}{(+)} < 0$

$\therefore$ f(x) is decreasing
$f'(x) = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = 1, -1$

$\therefore \quad f''(1) = \frac{4a}{(2 + a)^2} > 0$

$\therefore$ f(x) has local minimum at x = 1.

42. Here, $g'(x) = \frac{dg(x)}{d(e^x)} . \frac{d(e^x)}{dx} = \frac{f'(e^x)}{1 + e^{2x}}.e^x$

$= \frac{e^x}{1 + e^{2x}} . \frac{2a(e^{2x} - 1)}{(e^{2x} + ae^x + 1)^2} > 0$ when $x \in (0, \infty)$ < 0, when $x \in (-\infty, 0)$

Paragraph – 16

If $f(x)$ is defined in $[a,b]$ and (i) $f(x)$ is continuous on $[a,b]$ (ii) $f(x)$ is derivable on (a,b) (iii) $f(a) = f(b)$. Then f has atleast one point $c \in (a,b)$ such that

$f'(c)=0$. This is known as Rolle's theorem. This means $f(x)$ is a polynomial then between any two roots of $f(x)=0$ there is always lies a root of $f'(x)=0$

43. A twice differentiable function f such that $f(a)=f(b)=0 \, and \, f(c)>0$ and for $a<c<b$. Then there is atleast one value 'ε' between a and b for which

(A) $f''(\varepsilon)=0$ (B) $f''(\varepsilon)>0$

(C) $f''(\varepsilon)<0$ (D) $f''(\varepsilon)\geq 0$

Key. C

44. If a function f is such that its derivative f' is continuous on $[a,b]$ and derivable on (a,b) then there exists a number 'c' between 'a' and 'b' such that

$f(b)=f(a)+(b-a)f'(a)+(b-a)^2K$ then K =

(A) $f''(c)$ (B) $2f''(c)$ (C) $\frac{1}{2}f''(c)$ (D) $-f''(c)$

Key. C

45. If $f(x)$ is continuous in $[a,b]$ and differentiable in (a,b) then $\frac{f'(c)}{3c^2}=\frac{f(b)-f(a)}{b^3-a^3}$ holds for

(A) at least one $c\in(a,b)$ (B) atmost one $c\in(a,b)$

(C) exactly one value of c (D) none of these

Key. A

Sol. 21. Use Legrange's Mean Value Theorem

23. Let $\phi(x)=f(x)-f(a)+A(x^3-a^3)$

Where A is constant to be obtained from $\phi(b)=\phi(a)$

Since $\phi(b)=0$

$\Rightarrow A=\frac{-f(b)-f(a)}{b^3-a^3}$

Apply rolle's theorem

Paragraph – 17

If $f(x)$ is differentiable function wherever it is continuous and $f'(c_1)=f'(c_2)=0$, $f''(c_1).f''(c_2)<0$, $f(c_1)=5$, $f(c_2)=0$ and $(c_1<c_2)$

46. If $f(x)$ is continuous in $[c_1,c_2]$ and $f''(c_1)-f''(c_2)>0$ then the minimum number of roots of $f'(x)=0$ in $[c_1-1,c_2+1]$ is

(A) 2 (B) 3 (C) 4 (D) 5

Key. C

47. If $f(x)$ is continuous in $[c_1,c_2]$ and $f''(c_1)-f''(c_2)<0$ then the minimum number of roots of $f'(x)=0$ in $[c_1-1,c_2+1]$ is

(A) 1 (B) 2 (C) 3 (D) 4

Key. B

48. If $f(x)$ is continuous in $[c_1,c_2]$ and $f''(c_1)-f''(c_2)>0$ then minimum number of roots of $f(x)=0$ in $[c_1-1,c_2+1]$ is

(A) 2 (B) 3 (C) 4 (D) 5

Key. A

Sol. 46. Since $f''(c_1)-f''(c_2)>0$ and $f''(c_1)\times f''(c_2)<0$

$\Rightarrow f''(c_1)>0\ f''(c_2)<0$

It follows that $f'(x)=0$ has at least four roots in the given interval

47. $f''(c_1)<0\quad f''(c_2)>0$

It follows that $f'(x)=0$ has at least two roots in the given interval

48. $f''(c_1)>0\quad f''(c_2)<0$

$\Rightarrow f(x)=0$ has at least two roots

Paragraph – 18

If $f''(x)<0(>0)$ on an interval (a,b) then the curve $y=f(x)$ on this interval is convex (concave) i.e. it is situated below (above) any of its tangent lines. If $f''(x_0)=0$ or does not exist but $f'(x_0)$ does exist and the second derivative changes sign when passing through the point x_0 then the point $(x_0, f(x_0))$ is the point of inflexion of the curve $y=f(x)$

49. If $y=x^4+x^3-18x^2+24x-12$ then

a) $(-2,-24)$ is a point of inflexion
b) $\left(3/2,-8\frac{1}{16}\right)$ is a point of inflexion
c) $(-2,3/2)$ is a point of inflexion
d) y is convex on $(3/2,\infty)$

Key. B

50. If $y=x\sin(\log x)$ then

a) y has only two points of inflexion
b) y has only 4 points of inflexion
c) $\pi/4$ is only point of inflexion
d) y has infinite no. of points of inflexion

Key. D

51. $y=x^4+ax^3+\frac{3}{2}x^2+1$ is concave along the entire number scale then

a) $|a|\ge 1$
b) $|a|\le 1$
c) $|a|\le 2$
d) $|a|>2$

Key. C

Sol. 49. $y'=4x^3+3x^2-36x+24$

$y''=12\left(x^2+\frac{x}{2}-3\right)$

$y''>0$ on $(-\infty,-2)\cup(3/2,\infty)$ $y''<0$ on $(-2,3/2)$

$y''0$ at $x_2=-2\ x_2=3/2$

$\therefore(-2,-124)$ and $\left(\frac{3}{2},-8\frac{1}{16}\right)$ are points of inflection

50. $y'=\sin(\log x)+\cos(\log x)$

$y''=\frac{\sqrt{2}\sin(\pi/4-\log x)}{x}$

$y''=0$ at $x_k=e^{\pi/4+K\pi}$ $K=0,\pm1,\pm2,....$

y'' changes sign when passing through each point x_k

$\therefore f(x)$ has infinite no. of points of inflection

51. $y''=12x^2+6ax+3$

$y'' \geq 0 \ \forall \in R$

$\therefore y$ is concave on entire number scale

$4x^2 + 2ax + 1 \geq 0 \quad \forall x \in R$

$\therefore 4a^2 - 16 \leq 0$

$\Rightarrow |a| \leq 2$

Paragraph – 19

Maximum and minimum values of functions are not always found by calculus.
At times algebraic and trigonometric methods become very elegant. Some of the results which are frequently used are:

i) Arithemtic mean of positive numbers $\geq$ geometric mean with equality being attained when all numbers are equal

ii) $a\cos x + b\sin x \in \left[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}\right]$ for all real 'x'

iii) $\sqrt{a^2\sec^2 x + b^2\operatorname{cosec}^2 x} \ (a, b > 0)$ will have a minimum value (a + b)

52. If a, b > 0, a + b = 1 then minimum value of $a^4 + b^4$ must be

a) $1/2$ b) $1/4$ c) $1/8$ d) $1/16$

Key. C

53. The minimum value of $\left(\sin^{-1} x\right)^3 + \left(\cos^{-1} x\right)^3, \ \left(|x| \leq 1\right)$ must be

a) $\frac{7\pi^3}{8}$ b) $\frac{\pi^3}{4}$ c) $\frac{\pi^3}{8}$ d) $\frac{\pi^3}{32}$

Key. D

54. If x, y > 0 then maximum value of product $xy(72 - 3x - 4y)$ is

a) 1155 b) 1152 c) 1122 d) 1144

Key. B

Sol. 52. $AM \geq GM \Rightarrow ab \leq 1/4$

$\Rightarrow a^2 + b^2 = 1 - 2ab \geq \frac{1}{2}$

$\Rightarrow a^4 + b^4 = \left(a^2 + b^2\right)^2 - 2a^2b^2 \geq \frac{1}{8} \Rightarrow a^4 + b^4 \geq 1/8$

53. Let $\cos^{-1} x = t$

$f(x) = \frac{3\pi}{2}\left(t - \frac{\pi}{4}\right)^2 + \frac{\pi^3}{32}$

$\because t \in [0, \pi], \left(t - \frac{\pi}{4}\right)^2$ is capable of becoming zero

$\Rightarrow \text{minimum} \, f(x) = \frac{\pi^3}{32}$

54. $xy(72 - 3x - 4y) = \frac{1}{12}(3x)(4y)(72 - 3x - 4y)$

$3x, 4y, 72 - 3x - 4y$ have a constant sum 72

thus we seek values of x and y satisfying

$3x = 4y = 72 - 3x - 4y$

$\therefore x = 8\ y = 6$

$\therefore \max xy(72-3x-4y) = 8\times 6\times 24 = 1152$

Paragraph – 20

Let f be a function satisfying $f\left(\frac{x_1+x_2}{2}\right) \le \frac{f(x_1)+f(x_2)}{2}$, for all x_1, x_2 in its domain.

If we draw the graph of a continuous function satisfying this inequality, we will notice that the chord joining any two points on the curve always be above the portion of the curve between two points. We can also prove that, if the given inequality is true then the similar result follows for four values x_1, x_2, x_3, x_4. Indeed

$$f\left(\frac{x_1+x_2+x_3+x_4}{4}\right) = f\left(\frac{\frac{x_1+x_2}{2}+\frac{x_3+x_4}{2}}{2}\right) \le \frac{f\left(\frac{x_1+x_2}{2}\right)+f\left(\frac{x_3+x_4}{2}\right)}{2}$$

$$\le \frac{\frac{f(x_1)+f(x_2)}{2}+\frac{f(x_3)+f(x_4)}{2}}{2} \le \frac{f(x_1)+f(x_2)+f(x_3)+f(x_4)}{4}$$

55. If $f\left(\frac{x_1+x_2}{2}\right) \le \frac{f(x_1)+f(x_2)}{2}$ then the result of

$f\left(\frac{x_1+x_2+.......+x_n}{n}\right) \le \frac{f(x_1)+f(x_2)+....+f(x_n)}{n}$ is true.

a) for all $n \in N$
b) Only when n is even positive integer
c) Only when n is of the type 2^k
d) Only when n is odd positive integer

Key. A

56. The result $f\left(\frac{x_1+x_2+x_3}{3}\right) \le \frac{f(x_1)+f(x_2)+f(x_3)}{3}$ follows from

$f\left(\frac{x_1+x_2+x_3+x_4}{4}\right) \le \frac{f(x_1)+f(x_2)+f(x_3)+f(x_4)}{4}$, if x_4 in the latter is replaced by

a) $x_1+x_2+x_3$
b) $\frac{x_1+x_2+x_3}{2}$
c) $\frac{x_1+x_2+x_3}{3}$
d) $\frac{x_1+x_2+x_3}{4}$

Key. C

57. Which of the following functions satisfy $f\left(\frac{x_1+x_2}{2}\right) \le \frac{f(x_1)+f(x_2)}{2}$

a) $\sin x, 0 < x < \frac{\pi}{2}$
b) $\tan x, 0 < x < \frac{\pi}{2}$
c) $-x^2, 0 < x < \infty$
d) $\log x, 0 < x < \infty$

Key. B

Sol. 55. If $f\left(\frac{x_1+x_2}{2}\right) \le \frac{f(x_1)+f(x_2)}{2}$, then the function will be convex and the convexity will remain true for any number of variables. Algebraically it can be shown in the following manner:

(i) Whenever it is true for k it is also true for 2k.
(ii) Whenever it is true for k then it is also true for k-1.

56. We can test one-by-one and determine that $\frac{x_1 + x_2 + x_3}{3}$ is the correct replacement.

57. Only $\tan x$ is convex chord will be above the curve

Paragraph – 21

$f(x) = \sin 2\pi x + \{x\} : x \in [0, 10]$

58. Number of points where $f(x)$ achieves local maximum is
(A) 20 (B) 10
(C) 11 (D) None of these
Key. B

59. Number of roots of $f(x) = 0$ in (0, 10) is
(A) 20 (B) 30
(C) 31 (D) None of these
Key. D

60. Number of points where $f(x)$ achieves local minima is
(A) 10 (B) 15
(C) 11 (D) 19
Key. D

Sol.58. $f(x)$ is periodic of period 1

Consider $x \in [0,1)$

$f'(x) = 2\pi \cos 2\pi x + 1$

$f'(x) = 0$ if $x = \alpha, \beta$ where $\frac{1}{4} < \alpha < \frac{1}{3} < \frac{1}{2} < \beta < \frac{3}{4}$

$f'(x) > 0$ if $x \in [0, \alpha)$ or $(\beta, 1)$

$f'(x) < 0$ if $x \in (\alpha, \beta)$

$f(0) = f(1) = 0$; $f(\alpha) > 0$

f has local maxima at α.

59. Use graphical solution.

60. *f* has one point of local minima in [0, 1) at $x = \beta$
Number of points of local minima is 10.

Paragraph – 22

$f(x) = \left(\tan^{-1} x\right)^2 + \frac{2}{\sqrt{1+x^2}}$

61. *f* increases in the region
(A) $(0, \infty)$ (B) ***R***
(C) $(-\infty, 0)$ (D) None of these
Key. A

62. Maximum value of *f*
(A) is $\pi^2 + 1$ (B) is $\frac{\pi^2}{4}$

(C) is 1 (D) does not exist

Key. D

Sol.

63. Number of points of local extrema of *f* is

(A) 0 (B) 1

(C) 2 (D) None of these

Key. B

Sol.61. f has domain R and f is even.

$$f'(x) = \frac{2\tan^{-1}x}{1+x^2} - \frac{2x}{(1+x^2)^{3/2}}$$

$$= \frac{2}{1+x^2}g(x) \text{ where}$$

$$g(x) = \tan^{-1}x - \frac{x}{\sqrt{1+x^2}}; \quad x = \tan\theta : \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) = \theta - \sin\theta$$

$$f'(x) > 0 \text{ if } g(x) > 0; \; i.e., \text{ if } \theta \in \left(0, \frac{\pi}{2}\right); x > 0.$$

f is even $\therefore f'(x) < 0$ if $x < 0$.

62. As $x \to \infty$, $f(x) \to \frac{\pi^2}{4} + 0 = \frac{\pi^2}{4}$ which is not achieved.

63. $x = 0$ is the only point of local extrema.

Paragraph – 23

$$f(x) = \ln\left(\sqrt{1-x^2} - x\right)$$

64. $f(x)$ increases in the region

(A) $[0, \frac{1}{\sqrt{2}})$ (B) $\left(-1, -\frac{1}{\sqrt{2}}\right)$

(C) $(-1, 0)$ (D) None of these

Key. B

65. *f* has local maximum at x =

(A) 1 (B) 0

(C) $\frac{1}{\sqrt{2}}$ (D) $-\frac{1}{\sqrt{2}}$

Key. D

66. Least value of *f* is

(A) 0 (B) 1

(C) 2 (D) Does not exist

Key. D

Sol.64. $\ln$ is an increasing function

$\sqrt{1-x^2}-x$ is defined for $x\in[-1,1]$

If $x\in[-1,0], \sqrt{1-x^2}-x>0$

If $x>0,$ then $\sqrt{1-x^2}>x\Rightarrow 1-x^2>x^2$

$i.e.,\ x\in\left(0,\dfrac{1}{\sqrt{2}}\right)$

Domain of $f=[-1,\dfrac{1}{\sqrt{2}})$

$g(x)=\sqrt{1-x^2}-x=\cos\theta-\sin\theta:\theta\in[-\dfrac{\pi}{2},\dfrac{\pi}{4});x=\sin\theta$

$g(x)=\sqrt{2}\cos\left(\theta+\dfrac{\pi}{4}\right):\theta+\dfrac{\pi}{4}\in[-\dfrac{\pi}{4},\dfrac{\pi}{2})$

g increases if $\theta+\dfrac{\pi}{4}\in[-\dfrac{\pi}{4},0)$

if $\theta\in[-\dfrac{\pi}{2},-\dfrac{\pi}{4})$

if $x\in[-1,\dfrac{-1}{\sqrt{2}})$

f increases in $[-1,-\dfrac{1}{\sqrt{2}})\supseteq\left(-1,\dfrac{-1}{\sqrt{2}}\right)$

65. *f* decreases in $\left(-\dfrac{1}{\sqrt{2}},\dfrac{-1}{\sqrt{2}}\right)$

f has local max at $x=-\dfrac{1}{\sqrt{2}}$

66. Least value of *f* = min (Range of *f*).

$f(-1)=0;\ \lim\limits_{x\to\frac{1}{\sqrt{2}}-} f(x)\equiv-\infty$ as

$g(x)\to 0+$ as $x\to\dfrac{1}{\sqrt{2}}-$

f has no minima.

Paragraph – 24

$$f(x)=\frac{x^3}{(4+x+x^2)^3}$$

67. $f(x)$ has local minimum at x =

(a) $-\frac{1}{2}$ (b) –2

(c) 0 (d) none of these

Key. B

68. $f(x)$ has local maximum at x =

(a) 1 (b) –1

(c) 2 (d) none of these

Key. C

69. Range of f is

(a) $(-\infty, \infty)$ (b) $\left[-\frac{1}{3}, \frac{1}{125}\right]$

(c) $\left[-\frac{1}{27}, \frac{1}{125}\right]$ (d) none of these

Key. C

Sol. 67. $h(x) = x^3$ increases over ***R.***

$$f(x) = hog(x) : g(x) = \frac{x}{x^2 + x + 4} = \begin{cases} 0 & \text{if} \quad x = 0 \\ \dfrac{1}{2\left(\alpha + \dfrac{1}{\alpha} + \dfrac{1}{2}\right)} & \text{if} \quad x \neq 0; \dfrac{x}{2} = \alpha \end{cases}$$

$\alpha : -\infty \to -1 \to 0-$	$0+ \to 1 \to \infty$
$\alpha + \frac{1}{\alpha} : -\infty \to -2 \to -\infty$	$\infty \to 2 \to \infty$
$g(x) : 0- \to -\frac{1}{3} \to 0-$	$0+ \to \frac{1}{5} \to 0+$

$g(x)$ has local minimum at $\alpha = -1; x = -2$

f has local minimum at $x = -2$

68. $g(x)$ has local maximum at $\alpha = 1; x = 2$

$\therefore f$ has local maximum at x = 2.

69. $g_{\min} = g(-2) = -\frac{1}{3}; \ f_{\min} = -\frac{1}{27}$

$g_{\max} = g(2) = \frac{1}{5}; \ f_{\max} = \frac{1}{125}$

Range of $f \equiv \left[-\frac{1}{27}, \frac{1}{125}\right]$

Paragraph – 25

$f(x)=\dfrac{1}{\sin^{-1}2x\sqrt{1-x^2}}$.

70. f increases in the region

(a) $\left(-1,-\dfrac{1}{\sqrt{2}}\right)$ (b) $\left(\dfrac{-1}{\sqrt{2}},0\right)$

(c) (0, 1) (d) none of these

Key. A

71. Number of critical points of f is

(a) 0 (b) 1

(c) 2 (d) 3

Key. C

72. f has local maximum value

(a) $\dfrac{\pi}{2}$ (b) $-\dfrac{2}{\pi}$

(c) $\dfrac{4}{\pi}$ (d) none of these

Key. B

Sol. 70. $f(x)=\dfrac{1}{g(x)}$ has domain $(-1,1)\sim(0): g(x)=\sin^{-1}2x\sqrt{1-x^2}$

$g(x)=\sin^{-1}h(x): h(x)=2x\sqrt{1-x^2}$

$\sin^{-1}$ is an increasing function

$h(x)=\sin 2\theta : x=\sin\theta ; \theta\in\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)\sim\{0\}$

$2\theta : -\pi+\rightarrow-\dfrac{\pi}{2}\rightarrow 0-$	$0+\rightarrow\dfrac{\pi}{2}\rightarrow\pi-$
$h(x): 0-\rightarrow -1\rightarrow 0-$	$0+\rightarrow 1\rightarrow 0+$
$g(x): 0-\rightarrow-\dfrac{\pi}{2}\rightarrow 0-$	$0+\rightarrow\dfrac{\pi}{2}\rightarrow 0+$
$f(x): -\infty\rightarrow-\dfrac{2}{\pi}\rightarrow-\infty$	$\infty\rightarrow\dfrac{2}{\pi}\rightarrow\infty$

f increases when $x\in\left(-1,-\dfrac{1}{\sqrt{2}}\right)$ or $x\in\left(\dfrac{1}{2},1\right)$

71. $f'(x)=\dfrac{-g'(x)}{(g(x))^2}$ where

$$g(x)=\begin{cases} 2\sin^{-1}x & \text{if} \quad x\in\left[-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right]\sim\{0\} \\ \pi-2\sin^{-1}x & \text{if} \quad x\in[\frac{1}{\sqrt{2}},1) \\ -\pi-2\sin^{-1}x & \text{if} \quad x\in(-1,-\frac{1}{\sqrt{2}}] \end{cases}$$

g is not differentiable at $x=\pm\frac{1}{\sqrt{2}}$

$g'(x)$ is never zero in $(-1,1)\sim(0)$.

$\therefore f$ has two critical points *viz.*, $x=\pm\frac{1}{\sqrt{2}}$.

72. f has local maximum value $'-\frac{2}{\pi}'$ at $x=-\frac{1}{\sqrt{2}}$.

Maxima & Minima

Integer Answer Type

1. From a point perpendicular tangents are drawn to ellipse $x^2 + 2y^2 = 2$. The chord of contact touches a circle which is concentric with given ellipse. Then find the ratio of maximum and minimum area of circle.

Key. 4

Sol. The director circle of ellipse $\frac{x^2}{2}+\frac{y^2}{1}=1$ is $x^2 + y^2 = 3$

Let a point $P(\sqrt{3}\cos\theta, \sqrt{3}\sin\theta)$

Equation of chord of contact is

$x.\sqrt{3}\cos\theta + 2y\sqrt{3}\sin\theta - 2 = 0$

It touches $x^2 + y^2 = r^2$

$r = \frac{2}{\sqrt{3\cos^2\theta+12\sin^2\theta}}=\frac{2}{\sqrt{3+9\sin^2\theta}}$

$r_{max} = \frac{2}{\sqrt{3}}$ & $r_{min} = \frac{2}{\sqrt{12}} \Rightarrow \frac{A_{max}}{A_{min}} = 4$.

2. The maximum value of the function $f(x)=2x^3-15x^2+36x-48$ on the set $A=\{x \mid x^2+20\le 9x\}$ is

Key. 7

Sol. The given function is $f(x)=2x^3-15x^2+36x-48$ and $A=\{x \mid x^2+20\le 9x\}$

$\Rightarrow \quad A=\{x \mid x^2-9x+20\le 0\}$

$\Rightarrow \quad A=\{x \mid (x-4)(x-5)\le 0\}$

$\Rightarrow \quad A=[4,5]$

Also

$f'(x)=6x^2-30x+36=6(x^2-5x+6)=6(x-2)(x-3)$

Clearly $\forall x\in A,\ f'(x)>0$

$\therefore$ f is strictly increasing function on

$\therefore$ Maximum value of f on A

$= f(5)=2\times 5^3-15\times 5^2+36\times 5-48=250-375+180-48=7$

3. If $a,b,c\in N$, and if $\frac{ax^4-bx^3+cx^2-bx+a}{\left(x^2+1\right)^2}$ attains minimum value at x = 2 or 1/2 then the A.M of the least possible values of a, b and c is __________

Key. 4

Sol. Put $x+\frac{1}{x}=t$ $a=1, b=4, c=7, \Rightarrow$ AM is $\frac{1+4+7}{3}=4$

4. If the greatest value of $\left(3-\sqrt{4-x^2}\right)^2+\left(1+\sqrt{4-x^2}\right)^3$ is α, then the numerical value of $\left(\frac{\alpha}{7}\right)$, is

Key. 4

Sol. Let $t=\sqrt{4-x^2}, 0\le t\le 2$

$\therefore F(t)=(3-t)^2+(1+t)^3$ and maximum of $f(x)$ is 10

5. If the graph of $f(x)=2x^3+ax^2+bx, a,b\in N$ cuts the x-axis at three real and distinct points, then the minimum value of $\left(a^2+b^2-4\right)$, is

Key. 6

Sol. $f^1(x)=6x^2+2ax+b \Rightarrow 4a^2-24b\ge 0$

$\Rightarrow a^2\ge 6b$

$\Rightarrow a\ge 3, b\ge 1, \Rightarrow a=3, b=1$

6. If $a,b,c\in N$, and if $\frac{ax^4-bx^3+cx^2-bx+a}{\left(x^2+1\right)^2}$ attains minimum value at x = 2 or 1/2 then the A.M of the least possible values of a, b and c is __________

Key. 4

Sol. Put $x+\frac{1}{x}=t$ $a=1, b=4, c=7, \Rightarrow$ AM is $\frac{1+4+7}{3}=4$

7. The maximum value of the function $f(x)=2x^3-15x^2+36x-48$ on the set $A=\left\{x\middle|x^2+20\le 9x\right\}$ is

Key. 7

Sol. The given function is $f(x) □ 2x^3 □ 15x^2 □ 36x □ 48$ and $A □ \{x \mid x^2 □ 20 □ 9x\}$

□ $A □ \{x \mid x^2 □ 9x □ 20 □ 0\}$ □ $A □ \{x \mid (x □ 4)(x □ 5) □ 0\}$ □ $A □ [4, 5]$

Also $f'(x) □ 6x^2 □ 30x □ 36 □ 6(x^2 □ 5x □ 6)$ □ $6(x □ 2)(x □ 3)$

Clearly □$x □ A$, $f'(x) □ 0$

□ f is strictly increasing function on A.

□ Maximum value of f on A

□ $f(5) □ 2□5^3 □ 15□5^2 □ 36□5 □ 48$ □ $250 □ 375 □ 180 □ 48 □ 7$

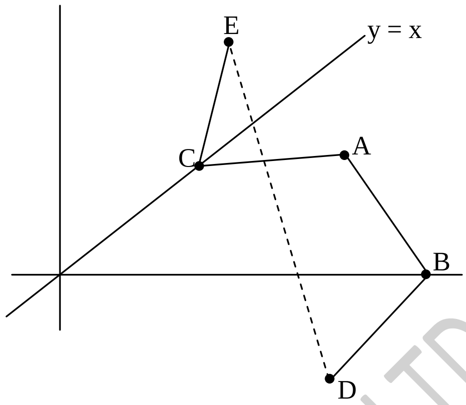
8. Given a point (2, 1). If the minimum perimeter of a triangle with one vertex at (2, 1), one on the x-axis, and one on the line y = x, is k, then [k] is equal to (where [ ] denotes the greatest integer function)

Key. 3

Sol.

Let, $D=(2,-1)$ be the reflection of A in x-axis, and let $E=(1,2)$ be the reflection in the line y = x. Then AB = BD and AC = CE, so the perimeter of ABC is

DB + BC + CE ≥ DE = $\sqrt{1+9}=\sqrt{10}$



9. The minimum value of, $\frac{\sec^4\alpha}{\tan^2\beta}+\frac{\sec^4\beta}{\tan^2\alpha},\alpha,\beta\neq\frac{K\pi}{2},K\in I,$ is

Key. 8

Sol.

$$\frac{(a+1)^2}{b}+\frac{(b+1)^2}{a}=\frac{a^2}{b}+\frac{1}{b}+\frac{b^2}{a}+\frac{1}{a}+2\left(\frac{a}{b}+\frac{b}{a}\right)\geq 4\left[\frac{a^2}{b}\cdot\frac{1}{b}\cdot\frac{b^2}{a}\cdot\frac{1}{a}\right]^{\frac{1}{4}}+4\left(\frac{a}{b}\cdot\frac{b}{a}\right)^{\frac{1}{2}}\geq 8$$

Where a = $\tan^2\alpha, b=\tan^2\beta$

10. If one root of $x^2-4ax+a+f(a)=0$ is three times the other and if minimum value of $f(a)$ is α, then $|12\alpha|$ has a value

Key. 1

Sol. θ and $3\theta \Rightarrow 4\theta=4a\Rightarrow\theta=a$ and $a-4a^2+f(a)=0$

$\Rightarrow f(a)=3a^2-a\Rightarrow f_{min}$ is $\frac{-1}{12}$

11. For a twice differentiable function $f(x)$, a function $g(x)$ is defined as $g(x)=(f^1(x))^2+f(x)f^{11}(x)$ on $[a,e]$. If $a<b<c<d<e$ and $f(a)=0, f(b)=2, f(c)=-1, f(d)=2, f(e)=0,$ then, the minimum number of roots of the equation $g(x)=0$, is/are

Key. 6

Sol. $\because f(b)f(c)<0$ and $f(c)f(d)<0$

$\Rightarrow f(x)=0$ has at least four roots,

$a,c_1,c_2,e,$ Where $c_1\in(b,c)$ and $c_2\in(c,d)$. Then, by RT, $f^1(x)=0$ has at least three roots in, $(a,c_1),(c_1,c_2),(c_2,e)$

$\therefore f(x)f^1(x)=0$ has at least 7 roots, by RT and hence,

$g(x) = \frac{d}{dx}\{f(x)f^1(x)\} = 0$ has at least 6 roots

12. Let $P(x)$ be a polynomial of degree 4 having extremum at $x = 1,2$ and $\underset{x\to 0}{\text{Let}}\left(1+\frac{P(x)}{x^2}\right) = 2$, then, the value of $P(2)$, is

Key. 0

Sol. Let $P(x) = a_0x^4 + ... + a_4$ by hypothesis, $P^1(1) = 0$ and $P^1(2) = 0$

$\Rightarrow 4a_0 + 3a_1 + 2a_2 + a_3 = 0$ and $32a_0 + 12a_1 + 4a_2 + a_3 = 0$

Also, $\underset{x\to 0}{\text{Lt}} \frac{P(x)}{x^2} = 1 \Rightarrow a_4 = 0$ and $a_3 = 0$ hence $\underset{x\to 0}{\text{Lt}}(a_0x^3 + a_1x + a_2) = 1 \Rightarrow a_2 = 1$

Solving, we get, $a_0 = \frac{1}{4}, a_1 = -1, a_2 = 1, a_3 = 0, a_4 = 0$

$\therefore P(x) = \frac{1}{4}x^4 - x^3 + x^2 \Rightarrow P(2) = 0$

13. In the coordinate plane, the region M consists of all points (x, y) satisfying the inequalities $y \geq 0, y \leq x,$ and $y \leq 2 - x$ simultaneously. The region N which varies with parameter t, consists of all the points (x, y) satisfying the inequalities $t \leq x \leq t+1$ and $0 \leq t \leq 1$ simultaneously. If the area of the region $M \cap N$ is a function of t, i,e., $M \cap N = f(t)$ and if α is the value of t for which this area is maximum, then the numerical value of 2α is

Key. 1

Sol. $M \cap N = f(t) = -t^2 + t + 1/2$

$= \frac{3}{4} - \left(t - \frac{1}{2}\right)^2$ f(t) is maximum for t = 1/2 i.e. $\alpha = \frac{1}{2} \Rightarrow 2\alpha = 1$

14. Let $M(-1, 2)$ and $N(1, 4)$ be two points in a plane rectangular coordinate system XOY. P is a moving point on the x-axis. When $\angle$MPN takes its maximum value, the x-coordinate of point P is

Key. 1

Sol. The centre of a circle passing through points M and N lies on the perpendicular bisector y = 3 – x of MN. Denote the centre by C(a, 3 – a), the equation of the circle is

$(x - a)^2 + (y - 3 + a)^2 = 2(1 + a^2)$

Since for a chord with a fixed length the angle at the circumference subtended by the corresponding arc will become larger as the radius of the circle becomes smaller. When $\angle$MPN reaches its maximum value the circle through the three points M, N and P will be tangent to the x-axis at P, which means

$2(1 + a^2) = (a - 3)^2 \Rightarrow a = 1$ or $a = -7$

Thus the point of contact are P(1, 0) or $P'(-7, 0)$ respectively.

But the radius of circle through the points M, N and P' is larger than that of circle through points M, N and P.

Therefore, $\angle MPN > \angle MP'N$. Thus P = (1, 0)

$\therefore$ x-coordinate of P = 1.

15. Put numbers 1, 2, 3, 4, 5, 6, 7, 8 at the vertices of a cube, such that the sum of any three numbers on any face is not less than 10. The minimum sum of the four number on a face is k, then k/2 is equal to

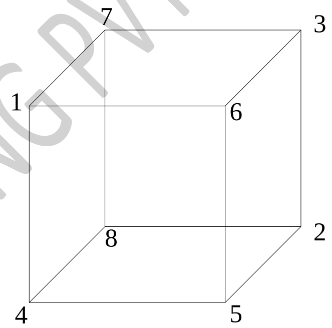
Key. 8

Sol. Suppose that the four numbers on face of the cube is a_1, a_2, a_3, a_4 such that their sum reaches the minimum and $a_1 < a_2 < a_3 < a_4$.

Since the maximum sum of any three numbers less than 5 is 9, we have $a_4 \geq 6$ and $a_1 + a_2 + a_3 + a_4 \geq 16$.

As seen in figure, we have

2 + 3 + 5 + 6 = 16



and that means minimum sum of four numbers on a face is 16.

16. Rolle's theorem holds for the function $f(x) = x^3 + mx^2 + nx$ on the interval $[1, 2]$ and the value of c is $\frac{4}{3}$. Then $m + n =$

Key. 3

Sol. $f(1) = f(2) \Rightarrow 1 + m + n = 8 + 4m + 2n \Rightarrow 3m + n + 7 = 0.$

$$f^1(C) = 0 \Rightarrow 3C^2 + 2mC + n = 0 \Rightarrow \frac{16}{3} + \frac{8m}{3} + n = 0 (C = \frac{4}{3})$$

$\Rightarrow 8m + 3n + 16 = 0$ on solving we get $m = -5, n = 8$ Hence m + n = 3

17. If the greatest value of $\left(3-\sqrt{4-x^2}\right)^2 + \left(1+\sqrt{4-x^2}\right)^3$ is α, then the numerical value of $\left(\frac{\alpha}{7}\right)$, is

Key. 4

Sol. Let $t = \sqrt{4-x^2}, 0 \leq t \leq 2$

$\therefore F(t) = (3-t)^2 + (1+t)^3$ and maximum of $f(x)$ is 10

18. If the graph of $f(x)=2x^3+ax^2+bx, a,b\in N$ cuts the x-axis at three real and distinct points, then the minimum value of $\left(a^2+b^2-4\right)$, is

Key. 6

Sol. $f^1(x)=6x^2+2ax+b\Rightarrow 4a^2-24b\ge 0$

$$\Rightarrow a^2\ge 6b$$

$\Rightarrow a\ge 3, b\ge 1, \Rightarrow a=3, b=1$

19. The minimum value of, $\frac{\sec^4\alpha}{\tan^2\beta}+\frac{\sec^4\beta}{\tan^2\alpha}, \alpha,\beta\ne\frac{K\pi}{2}, K\in I,$ is

Key. 8

Sol.

$$\frac{(a+1)^2}{b}+\frac{(b+1)^2}{a}=\frac{a^2}{b}+\frac{1}{b}+\frac{b^2}{a}+\frac{1}{a}+2\left(\frac{a}{b}+\frac{b}{a}\right)\ge 4\left[\frac{a^2}{b}\cdot\frac{1}{b}\cdot\frac{b^2}{a}\cdot\frac{1}{a}\right]^{\frac{1}{4}}+4\left(\frac{a}{b}\cdot\frac{b}{a}\right)^{\frac{1}{2}}\ge 8$$

Where $a=\tan^2\alpha, b=\tan^2\beta$

20. If one root of $x^2-4ax+a+f(a)=0$ is three times the other and if minimum value of $f(a)$ is α , then $|12\alpha|$ has a value

Key. 1

Sol. θ and $3\theta\Rightarrow 4\theta=4a\Rightarrow\theta=a$ and $a-4a^2+f(a)=0$

$\Rightarrow f(a)=3a^2-a\Rightarrow f_{min}$ is $\frac{-1}{12}$

21. The sum of greatest and least values of $f(x)=\left|x^2-5x+6\right|$ in $\left[0,\frac{5}{2}\right]$, is

Key. 6

Sol. Sketch its graph

22. If $A=(0,2), B=(5,10)$ are two points. If P is a Point on x-axis, then, the sum of the digits in the minimum value of AP+PB ,is

Key. 4

Sol. If $P=(x,0)$, then $AP+PB=f(x)=\sqrt{x^2+2^2}+\sqrt{(x-5)^2+10^2}$

$\Rightarrow x = \frac{5}{6}$ is a point of minima

$\therefore$ minimum value of $f(x) = \sqrt{\frac{169}{36}} + \sqrt{\frac{625+3600}{36}} = \frac{13}{6} + \frac{65}{6} = \frac{78}{6} = 13$

23. For a twice differentiable function $f(x)$, a function $g(x)$ is defined as $g(x) = \left(f^1(x)\right)^2 + f(x)f^{11}(x)$ on $[a,e]$. If $a < b < c < d < e$ and $f(a) = 0, f(b) = 2, f(c) = -1, f(d) = 2, f(e) = 0,$ then, the minimum number of roots of the equation $g(x) = 0$, is/are

Key. 6

Sol. $\because f(b)f(c) < 0$ and $f(c)f(d) < 0$

$\Rightarrow f(x) = 0$ has at least four roots,

$a, c_1, c_2, e,$ Where $c_1 \in (b,c)$ and $c_2 \in (c,d)$. Then, by RT, $f^1(x) = 0$ has at least three roots in, $(a,c_1),(c_1,c_2),(c_2,e)$

$\therefore f(x)f^1(x) = 0$ has at least 7 roots, by RT and hence,

$g(x) = \frac{d}{dx}\{f(x)f^1(x)\} = 0$ has at least 6 roots

24. Let $f(x) = 0$ be an equation of degree six, having integer coefficients and whose one root is $2\cos\frac{\pi}{18}$. Then, the sum of all the roots of $f^1(x) = 0$, is

Key. 0

Sol. Let $\theta = \frac{\pi}{18} \Rightarrow 6\theta = \frac{\pi}{3} \Rightarrow \cos 6\theta = \frac{1}{2}$

$\Rightarrow 4\cos^3 2\theta - 3\cos 2\theta = \frac{1}{2} \Rightarrow 8\left(2\cos^2\theta - 1\right)^3 - 6\left(2\cos^2\theta - 1\right) = 1$ let $2\cos\theta = x$

$\Rightarrow 8\left(2.\frac{x^2}{4} - 1\right)^3 - 6\left(2.\frac{x^2}{4} - 1\right) = 1$

$\Rightarrow \left(x^2 - 2\right)^3 - 3\left(x^2 - 2\right) = 1$

$\Rightarrow x^6 - 6x^4 + 9x^2 - 3 = 0$

$f^1(x) = 6x\left(x^4 - 4x^2 + 3\right)$

$f^1(x) = 0 \Rightarrow x = 0, \pm 1, \pm\sqrt{3}$

25. Let α and β respectively be the number of solutions of $e^x = x^2$ and $e^x = x^3$. Then, the numerical value of $2\alpha + 3\beta$, is

Key. 8

Sol. Sketch the graphs

26. Let $P(x)$ be a polynomial of degree 4 having extremum at $x = 1,2$ and $\underset{x\to 0}{Lt}\left(1+\frac{P(x)}{x^2}\right) = 2$, then, the value of $P(2)$, is

Key. 0

Sol. Let $P(x) = a_0x^4 + ... + a_4$ by hypothesis, $P^1(1) = 0$ and $P^1(2) = 0$

$\Rightarrow 4a_0 + 3a_1 + 2a_2 + a_3 = 0$ and $32a_0 + 12a_1 + 4a_2 + a_3 = 0$

Also, $\underset{x\to 0}{Lt}\frac{P(x)}{x^2} = 1 \Rightarrow a_4 = 0$ and $a_3 = 0$ hence $\underset{x\to 0}{Lt}\left(a_0x^3 + a_1x + a_2\right) = 1 \Rightarrow a_2 = 1$

Solving, we get, $a_0 = \frac{1}{4}, a_1 = -1, a_2 = 1, a_3 = 0, a_4 = 0$

$\therefore P(x) = \frac{1}{4}x^4 - x^3 + x^2 \Rightarrow P(2) = 0$

27. Let $f(x) = \begin{cases} \left|x^2 - 3x\right| + a, & 0 \le x < \frac{3}{2} \\ -2x + 3, & x \ge \frac{3}{2} \end{cases}$. If f(x) has a local maxima at x = $\frac{3}{2}$, and the greatest value of 'a' is k, then $|4k|$ is...........

Key. 9

Sol. $f\left(\frac{3}{2}\right) = 0 \Rightarrow \lim_{x\to\frac{3}{2}}\left|x^2 - 3x\right| + a \le 0$

$a \le -\frac{9}{4}$

Hence, $|4k| = 9$

28. If $a,b,c \in N$, and if $\frac{ax^4 - bx^3 + cx^2 - bx + a}{\left(x^2+1\right)^2}$ attains minimum value at x = 2 or $1/2$ then the A.M of the least possible values of a, b and c is __________

Key. 4

Sol. Put $x + \frac{1}{x} = t$ $a = 1, b = 4, c = 7, \Rightarrow$ AM is $\frac{1+4+7}{3} = 4$

29. The maximum value of the function $f(x) = 2x^3 - 15x^2 + 36x - 48$ on the set $A = \left\{x \middle| x^2 + 20 \le 9x\right\}$ is

Key. 7

Sol. The given function is $f(x) = 2x^3 - 15x^2 + 36x - 48$ and $A = \{x \mid x^2 + 20 \le 9x\}$

$\Rightarrow \quad A = \{x \mid x^2 - 9x + 20 \le 0\} \Rightarrow \quad A = \{x \mid (x-4)(x-5) \le 0\} \Rightarrow \quad A = [4, 5]$

Also $f'(x) = 6x^2 - 30x + 36 = 6(x^2 - 5x + 6)$ $= 6(x-2)(x-3)$

Clearly $\forall x \in A,\ f'(x) > 0$

$\Rightarrow$ f is strictly increasing function on A.

$\therefore$ Maximum value of f on A

$= f(5) = 2 \cdot 5^3 - 15 \cdot 5^2 + 36 \cdot 5 - 48$ $= 250 - 375 + 180 - 48 = 7$

30. In the coordinate plane, the region M consists of all points (x, y) satisfying the inequalities $y \ge 0, y \le x,$ and $y \le 2 - x$ simultaneously. The region N which varies with parameter t, consists of all the points (x, y) satisfying the inequalities $t \le x \le t + 1$ and $0 \le t \le 1$ simultaneously. If the area of the region $M \cap N$ is a function of t, i,e., $M \cap N = f(t)$ and if α is the value of t for which this area is maximum, then the numerical value of 2α is

Key. 1

Sol. $M \cap N = f(t) = -t^2 + t + 1/2$

$= \dfrac{3}{4} - \left(t - \dfrac{1}{2}\right)^2$ f(t) is maximum for t = 1/2 i.e. $\alpha = \dfrac{1}{2} \Rightarrow 2\alpha = 1$

31. Let $M(-1, 2)$ and $N(1, 4)$ be two points in a plane rectangular coordinate system XOY. P is a moving point on the x-axis. When $\angle$MPN takes its maximum value, the x-coordinate of point P is

Key. 1

Sol. The centre of a circle passing through points M and N lies on the perpendicular bisector y = 3 – x of MN. Denote the centre by C(a, 3 – a), the equation of the circle is

$(x - a)^2 + (y - 3 + a)^2 = 2(1 + a^2)$

Since for a chord with a fixed length the angle at the circumference subtended by the corresponding arc will become larger as the radius of the circle becomes smaller. When $\angle$MPN reaches its maximum value the circle through the three points M, N and P will be tangent to the x-axis at P, which means

$2(1 + a^2) = (a - 3)^2 \Rightarrow$ a = 1 or a = –7

Thus the point of contact are P(1, 0) or $P'(-7, 0)$ respectively.

But the radius of circle through the points M, N and P' is larger than that of circle through points M, N and P.

Therefore, $\angle$MPN > $\angle \mathrm{MP'N}$. Thus P = (1, 0)

$\therefore$ x-coordinate of P = 1.

32. $f(x)=\frac{1}{1+|x|}+\frac{1}{1+|x-1|}$. Let x_1, x_2 are points where $f(x)$ attains local minimum and global maximum respectively. Let $k=f(x_1)+f(x_2)$ then $6k-9$

Key. 8

Sol. Local minimum $=f\left(\frac{1}{2}\right)=\frac{4}{3}$

Global maximum $=f(0)=f(1)=\frac{3}{2}$ $k=\frac{4}{3}+\frac{3}{2}=\frac{17}{6}$

33. $f(x)=\begin{cases}\left(\sqrt{2}+\sin\frac{1}{x}\right)e^{\frac{-1}{|x|}} & if \quad x\neq 0\\ 0 & if \quad x=0\end{cases}$

Number of points where $f(x)$ has local extrema when $x\neq 0$ be n_1. n_2 be the value of global minimum of $f(x)$ then $n_1+n_2=$

Key. 0

Sol. Local extremum does not occur at any value of $x\neq 0$. But global minimum $=f(0)=0$

$\therefore$ $n_1=0, n_2=0$ then $n_1+n_2=0$

34. A = (- 3,0) and B = (3 ,0) are the extremities of the base AB of triangle PAB. If the vertex P varies such that the internal bisector of angle APB of the triangle divides the opposite side AB into two segments AD and BD such that AD : BD = 2 : 1 , then the maximum value of the length of the altitude of the triangle drawn through the vertex P is

Ans: 4

Hint: The point E dividing $\overline{AB}$ externally in the ratio 2 : 1 is (9 , 0) . Since P lies on the circle described on $\overline{DE}$ as diameter, coordinates of P are of the form $(5+4\cos\theta, 4\sin\theta)$

$\therefore$ maximum length of the altitude drawn from P to the base $AB=|4\sin\theta|_{\max}=4$

35. Find the maximum value of ($\log_{2^{1/5}}$ a).($\log_{2^{1/2}}$ b). It is given that coefficient of 2nd, 3rd and 4th term in expansion of $(a + b)^n$ are in A.P and the value of 3rd term is equal to 84 (a, b > 1).

Key: 1

Hint: In expansion of $(a + b)^n$ the coefficient of 2nd, 3rd and 4th term are in A.P. which gives n = 7 also ${}^7C_2\, a^5 b^2 = 84 \Rightarrow a^5b^2 = 4$

Now $\frac{\log_2 a^5 + \log_2 b^2}{2} \geq (\log_2 a^5 . \log_2 b^2)^{1/2} \Rightarrow k \leq \left(\frac{\log_2 a^5 b^2}{2}\right)^2$

$k \leq 1 \Rightarrow$ maximum value of k is 1.

36. From a point perpendicular tangents are drawn to ellipse $x^2 + 2y^2 = 2$. The chord of contact touches a circle which is concentric with given ellipse. Then find the ratio of maximum and minimum area of circle.

Ans: 4

Hint: The director circle of ellipse $\frac{x^2}{2} + \frac{y^2}{1} = 1$ is $x^2 + y^2 = 3$

Let a point P($\sqrt{3}\cos\theta$, $\sqrt{3}\sin\theta$)

Equation of chord of contact is

$x.\sqrt{3}\cos\theta + 2y\sqrt{3}\sin\theta - 2 = 0$

It touches $x^2 + y^2 = r^2$

$r = \frac{2}{\sqrt{3\cos^2\theta + 12\sin^2\theta}} = \frac{2}{\sqrt{3 + 9\sin^2\theta}}$

$r_{max} = \frac{2}{\sqrt{3}}$

$r_{min} = \frac{2}{\sqrt{12}} \Rightarrow \frac{A_{max}}{A_{min}} = 4.$

37. Let $f(x) = 30 - 2x - x^3$, then find the number of positive integral values of x which satisfies $f(f(f(x))) > f(f(-x))$

Key: 2

Hint: $f(x) = 30 - 2x - x^3$

$f(x) = -2 - 3x^2 < 0 \Rightarrow f(x)$ is decreasing function

Hence $f(f(f(x))) > f(f(-x)) \Rightarrow f(f(x)) < f(-x)$

$\Rightarrow f(x) > -x$

$\Rightarrow 30 - 2x - x^3 > -x \Rightarrow x^3 + x - 30 < 0 \Rightarrow (x-3)(x^2 + 3x + 10) < 0$

$\Rightarrow x < 3$

38. The sum of greatest and least values of $f(x) = |x^2 - 5x + 6|$ in $\left[0, \frac{5}{2}\right]$, is

Key. 6

Sol. Sketch its graph

39. If $A = (0,2), B = (5,10)$ are two points. If P is a Point on x-axis, then, the sum of the digits in the minimum value of AP+PB ,is

Key. 4

Sol. If $P=(x,0)$, then $AP+PB=f(x)=\sqrt{x^2+2^2}+\sqrt{(x-5)^2+10^2}$

$\Rightarrow x=\frac{5}{6}$ is a point of minima

$\therefore$ minimum value of $f(x)=\sqrt{\frac{169}{36}}+\sqrt{\frac{625+3600}{36}}=\frac{13}{6}+\frac{65}{6}=\frac{78}{6}=13$

40. If $a,b,c\in N$, and if $\frac{ax^4-bx^3+cx^2-bx+a}{(x^2+1)^2}$ attains minimum value at x = 2 or $1/2$ then the A.M of the least possible values of a, b and c is __________

Key. 4

Sol. Put $x+\frac{1}{x}=t$ $a=1,b=4,c=7,\Rightarrow$ AM is $\frac{1+4+7}{3}=4$

41. In the coordinate plane, the region M consists of all points (x,y) satisfying the inequalities $y\ge 0, y\le x,$ and $y\le 2-x$ simultaneously. The region N which varies with parameter t, consists of all the points (x,y) satisfying the inequalities $t\le x\le t+1$ and $0\le t\le 1$ simultaneously. If the area of the region $M\cap N$ is a function of t, i,e., $M\cap N=f(t)$ and if α is the value of t for which this area is maximum, then the numerical value of 2α is

Key. 1

Sol. $M\cap N=f(t)=-t^2+t+1/2$

$=\frac{3}{4}-\left(t-\frac{1}{2}\right)^2$ f(t) is maximum for t = 1/2 i.e. $\alpha=\frac{1}{2}\Rightarrow 2\alpha=1$

42. Let $P=x^3-\frac{1}{x^3}$, $Q=x-\frac{1}{x}$ and a is the minimum value of P/Q^2. Then the value of [a] is where [x] = the greatest integer $\le x$.

Key. 3

Sol. $Q^3 = P - 3Q$

$\Rightarrow \frac{P}{Q^2}=Q+\frac{3}{Q}$

$f(Q)=Q+\frac{3}{Q}$

$f'(Q)=1-\frac{3}{Q^2}\Rightarrow Q=\pm\sqrt{3}$

f(Q)f will be minimum at Q = $\sqrt{3}$

So minimum value of f(Q) is $2\sqrt{3}$

i.e. minimum of $\left[\frac{P}{Q^2}\right]=\left[2\sqrt{3}\right]=3$

43. Let f(x) = (x – a)(x – b)(x – c)(x – d); a < b < c < d. Then minimum number of roots of the equation $f''(x)=0$ is

Key. 2

Sol. f(a) = f(b) = f(c) = f(d) = 0

f(x) = 0 (4 times). Graph of f(x) will intersect 4 times the x-axis. So there will be minimum three turnings.

and $f'(x)=0$ minimum (3 times). So $f''(x)=0$ will be minimum (2 times).

44. If $f(x)=x^3+x^2f'(1)+xf''(2)+f'''(3)\ \forall x\in R$. Then the value of $f'(1)+f''(2)+f'''(3)$ is

Key. 3

Sol. Let $f'(1)=a,\ f''(2)=b,\ f'''(3)=c$

so $f'(x)=3x^2+2ax+b,\quad f''(x)=6x+2a$

a = 3 + 2a + b

b = 12 + 2a and c = 6.

$\Rightarrow$ a = –5, b = 2 and c = 6. so a + b + c = 3

45. Let f be twice differentiable such that $f''(x)=-f(x)$ and $f'(x)=g(x)$. If h(x) = $(f(x))^2+(g(x))^2$, where h(5) = 9. Then the value of h(10) is

Key. 9

Sol. $h'(x)=2f(x)f'(x)+2g(x)g'(x)$

$f'(x)=g(x)\Rightarrow\ f''(x)=g'(x)$

$\Rightarrow\quad g'(x)=-f(x)$

$\therefore\quad h'(x)=0$ h(x) = constant

h(5) = 9 $\Rightarrow$ h(10) is also 9.

Maxima & Minima

Matrix-Match Type

1. Match the following Column – 1 with Column – 2

Column – 1		Column – 2	
(A)	$f(x) = x^2 \log_e x$	(p)	f (x) has one point of minima
(B)	$f(x) = x \log_e x$	(q)	f (x) has one point of maxima
(C)	$f(x) = \frac{\log_e x}{x}$	(r)	f (x) increases in 0, e)
(D)	$f(x) = x^{-x}$	(s)	f (x) decreases in (0, 1/e)

Key. (A→ p, s), (B→ p, s), (C→ q, r), (D→ q)

Sol. (A) $f(x) = x^2 \log x$

For $f'(x) = x(2\log x + 1) = 0, \ \Rightarrow x = \frac{1}{\sqrt{e}}$

Which is the point of minima as derivative changes sign from negative to positive

Also, the function decreases in $\left(0, \frac{1}{\sqrt{e}}\right)$

(B) $y = x \log x$

$\Rightarrow \frac{dy}{dx} = x \times \frac{1}{x} + \log x \times 1 = 1 + \log x$ and $\frac{d^2y}{dx^2} = \frac{1}{x}$

For $\frac{dy}{dx} = 0 \Rightarrow \log x = -1 \Rightarrow x = \frac{1}{e}$

$\frac{d^2y}{dx^2} = \frac{1}{1/e} = e > 0$ at $x = \frac{1}{e}$

$\Rightarrow y$ is min for $x = \frac{1}{e}$

(C) $f(x) = \frac{\log x}{x}$

For $f'(x) = \frac{1 - \log x}{x^2} = 0, \ x = e$. Also, derivative changes sign from positive to negative at x = e, hence it is the point of maxima.

(D) $f(x) = x^{-x}$

$f'(x) = -x^{-x}(1 + \log x) = 0 \Rightarrow x = \frac{1}{e},$

Which is clearly point of maxima.

2. Match the following

	Column I		Column II
(A)	$f(x) = (x-1)^3(x-2)^5$	(p)	Has points of maxima
(B)	$f(x) = 3\sin x + 4\cos x - 5x$	(q)	Has points of minima
(C)	$f(x) = \sin\left(\frac{\pi x}{2}\right), 0 < x \le 1$ $= x^2 - 4x + 4, 1 < x < 2$	(r)	Has points of inflection
(D)	$f(x) = (x-1)^{3/5}$	(s)	Has no points of extrema

Key. a – qr ; b – rs ; c – pr ; d – rs

Sol. (A) f(x) = $(x-1)^2(x-2)^5$

$f'(x) = (x-1)^2(x+2)^4(8x+1)$

1 point of minima at $x = -\frac{1}{8}$

$f''(x) = 0$ for x = 1, –2

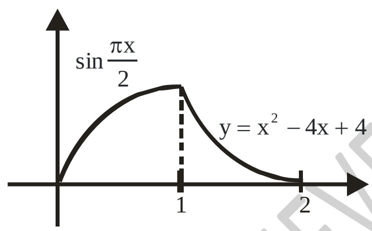
Two points of inflection

(B) $f(x) = 3\sin x + 4\cos x - 5x$

$f'(x) = 3\cos x - 4\sin x - 5 \le 0$

$f''(x) = -3\sin x - 4\cos x = 0$ for infinite value of x

(C)



x = 1 point of maxima as well as point of inflection

(D) $f'(x) = \frac{3}{5}(x-1)^{-2/5} \ge 0 \, \forall x \in R$

$f''(x) = \frac{-3}{5} \times \frac{2}{5}(x-1)^{-7/5}$

which changes sign at x = 1

x = 1 point of inflection

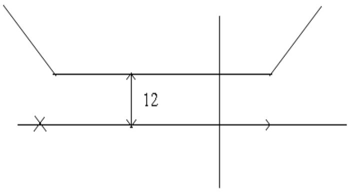
3. Consider the equation in real numbers $|x-3| + |x+9| = a,\ a \in R$.

Match the statements/expressions in column I with those in column II.

	Column – I		Column – II
(A)	As a ranges over R, the maximum possible number of integral solutions is	(P)	2
(B)	For $a > \log_{27} 81$, the number of solutions cannot be	(Q)	0
(C)	For a = 13, the number of solutions is	(R)	13
(D)	For at least one solution 'a' cannot take the value(s)	(S)	1

KEY : A-R; B-R, S; C-P; D-P, Q, S

Sol. The line $y = k,\ k < 12$ doesn't meet the above curve



k= 2 meets the curve at infinite points belonging to $[-9,-3]$

k > 12 meets the curve at two points

For k = 12 , the line y = k meets the curve

$y = |x-3| + |x+9|$ at 13 integral points. Note that $\log_{27} 81 = 4/3$

A) The maximum possible number of integral solutions is 13 which happens when a = 12

B) For 4/3 < a<12 the number of solutions = 0

a=12, the number of solutions is infinite

a>12, the number of solutions is two.

4.

Column – 1		Column – 2	
(A)	The maximum value of $\sec^{-1}\left(\dfrac{7-5(x^2+3)}{2(x^2+2)}\right)$ is	(p)	$\dfrac{\pi}{6}$
(B)	The minimum value of $\operatorname{cosec}^{-1}\left[3x^2+\dfrac{5}{4}\right]+\sec^{-1}\left[3x^2+\dfrac{1}{4}\right]$	(q)	$\dfrac{\pi}{4}$
(C)	Points of non-differentiability of the function $f(x) = \min\left\{\tan\left(x+\dfrac{\pi}{12}\right), \cot\left(x+\dfrac{\pi}{12}\right)\right\}$ $\forall x \in \left(0, \dfrac{3\pi}{2}\right)$ is/are	(r)	$\dfrac{2\pi}{3}$
(D)	Tangent is drawn to hyperbola $\dfrac{x^2}{8}-\dfrac{y^2}{1}=1$ at $(2\sqrt{2}\sec\theta, \tan\theta)$; $\theta \in \left(0, \dfrac{\pi}{2}\right)$. The value of θ such that sum of intercepts on axes made by this tangent is maximum is	(s)	$\dfrac{7\pi}{6}$

Key : (A-r), (B -p), (C –p, r, s), (D -q)

Sol : (A) $\sec^{-1}\left(\dfrac{7-5(x^2+3)}{2(x^2+2)}\right) = \sec^{-1}\left(\dfrac{1}{x^2+2}-\dfrac{5}{2}\right)$

Q $\frac{1}{x^2+2} \le \frac{1}{2}$

$\frac{1}{x^2+2} - \frac{5}{2} \le -2$

(B) minimum value = $\text{cosec}^{-1}2 + \sec^{-1}1 = \frac{\pi}{6}$

when $\left[3x^2 + \frac{1}{4}\right] = 1$

(C) f(x) is non-differentiable at $x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}$.

(D) Equation of tangent is $\frac{x\sec\theta}{2\sqrt{2}} - \frac{y\tan\theta}{1} = 1$

If it cuts the coordinate axes at A and B, then

$A \equiv \left(2\sqrt{2}\cos\theta, 0\right)$

$B \equiv \left(0, -\cot\theta\right)$

$S = 2\sqrt{2}\cos\theta - \cot\theta$

$\frac{dS}{d\theta} = -2\sqrt{2}\sin\theta + \text{cosec}^2\theta = 0 = \sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$

$\frac{d^2S}{d\theta^2} < 0$

$\therefore$ S is maximum

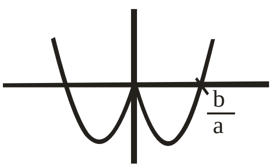
5. For the function f(x) = $ax^2 - b|x|$

	Column – I		Column – II
(A)	f(x) has local max. at x = 0	(p)	When a > 0, b > 0
(B)	f(x) has local min at x = 0	(q)	When a > 0, b < 0
(C)	f(x) has local extremum at $x = \frac{b}{2a}$	(r)	When a < 0, b < 0
(D)	f(x) is not diff. at x = 0	(s)	When a < 0, b > 0

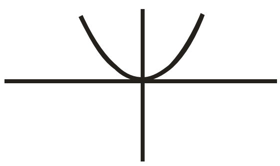
Key : A – P,S, B – Q,R, C – P,R, D – PQRS

Sol.

When a > 0, b > 0

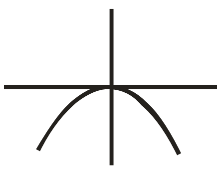


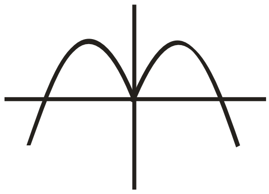
When a > 0, b < 0



When a < 0, b < 0

When a < 0, b > 0





6. Match the following: -

	Column – I	*Column – II*	
(A)	If a, b > 0 a + b = 1, then minimum value of $\left(a^2+\frac{1}{a^2}\right)^2+\left(b^2+\frac{1}{b^2}\right)^2$ is	(p)	$\frac{3}{2}$
(B)	The perpendicular distance of the image of the point (3, 4 –12) in the xy-plane from the z-axis is	(q)	5
(C)	The area of the quadrilateral whose vertices are 1, i, ω, iω is (ω is the cube root of unity)	(r)	8
(D)	The minimum value of $(\sin^2x + \cos^2x + \text{cosec}^2 2x)^3$ is	(s)	$\frac{289}{8}$

KEY : (A) → Q,
(B) →Q,
(C) → R,
(D) → (S)

7. Match the following inequalities with intervals given such that inequalities are valid

	Column I		Column II
(A)	$\frac{x}{1+x}<\ln(1+x)$	(P)	$(0,\infty)$
(B)	$x-\frac{x^2}{2}<\ln(1+x)$	(Q)	$(-1,0)$
(C)	ln(1+x)<x	(R)	$(1,\infty)$
(D)	$\frac{1}{\ln(1+x)}-\frac{1}{x}<1$	(S)	$(-1,0)\cup(0,1)$

Key: (A) → (PQRS)
(B) → (PR)
(C) → (PQRS)
(D) → (PQRS)

Sol. Conceptual

8. Let f(x) = $(2^x - 1)(2^x - 2)$ and g(x) = 2 sin x + cos 2x

Column I		Column II	
(A)	f increases on	(P)	(π, ∞)
(B)	f decreases on	(Q)	$(-\infty, 0)$
(C)	g decreases on	(R)	$\left(\frac{\pi}{2}, \frac{5\pi}{6}\right)$
(D)	g increases on	(S)	$\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$

Key: A-R, B-Q, C-S, D-R

Hint: $f'(x) = 2^x\left(2^{x+1} - 3\right)\log 2$

Since $2^x > 0$ and log 2 > 0 so $f'(x) > 0$. If $2^{x+1} - 3 > 0 \Rightarrow x > \log_2 3 - 1$

The period of g is 2π so it is enough to consider g on $[0, 2\pi]$.

$g'(x) = 2\cos x - 2\sin 2x = 2\cos x(1 - 2\sin x)$

$g'(x) > 0 \Rightarrow \cos x > 0$ and $1 - 2\sin x > 0$ or $\cos x < 0$ and $1 - 2\sin x < 0$

$\Rightarrow x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$, $\sin x < \frac{1}{2}$ or $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ and $\sin x > \frac{1}{2}$

$\Rightarrow x \in \left(0, \frac{\pi}{6}\right)$ or $x \in \left(\frac{\pi}{2}, \frac{5\pi}{6}\right)$

9.

COLUM – I	COLUMN – II
a) $f(x) = \begin{cases} 4x - x^3 + \log_e\left(a^2 - 3a + 3\right), \text{if } 0 \le x \le 3 \\ x - 18, \text{if } x \ge 3 \end{cases}$ Then, the complete set of values of 'a' such that f(x) has a local maxima at x = 3 lie in the interval	p) $(-\infty, -2) \cup (2, \infty)$
b) The equation $x + \cos x = a$ has exactly a positive root, then, complete set of values of a lie in the interval	q) $[1, 2]$
c) If $f(x) = \begin{cases} 3 + \lvert x - k \rvert, & x \le k \\ a^2 - 2 + \frac{\sin(x - k)}{x - k}, & x > k \end{cases}$ has minimum at x = k, then, the complete set of values of 'a' lie in the interval	r) $(-20, -16)$
d) If $f(x) = x^3 - 9x^2 + 24x + a$ has three real and distinct roots, then, possible values of	s) $(1, \infty)$

a lie in the interval Look at, $f(x)=x^4+32x+K, K\in R$

Key. $A\to Q; B\to S;$
$C\to P; D\to R$

Sol. A. F is decreasing for x < 3 and increasing for x > 3

$\therefore$ for $x=3$ to be point of local maximum, $f(3)\ge f(3-0)$

$\Rightarrow -15\ge 12-27+\log_e (a^2-3a+3)$

$\Rightarrow 0<a^2-3a+3\le 1$

B. $f(x)=x+\cos x-a\Rightarrow f^1(x)=1-\sin x\ge 0\forall x\in R$

$\Rightarrow f$ is increasing on R

For positive root, $f(0)=1-a<0$

D. $\frac{dy}{dx}=3x^2-18x+24=3(x^2-6x+8)$

$=3(x-2)(x-4)\Rightarrow f(2).f(4)<0$

10. COLUMN – I COLUMN – II

COLUMN – I	COLUMN – II
a) The function $f(x)=(x-1)^2+(x-2)^2+(x-3)^2$ has a minimum for x =	p) 0
b) The least value of the function $f(x)=2.3^{3X}-3^{2X}.4+2.3^X$ in $[-1,1]$ is	q) 2
c) Let $f(x)=\frac{4}{3}x^3-4x, 0\le x\le 2$, then the global minimum value of the function is	r) $\frac{8}{27}$
d) Let $f(x)=6-12x+9x^2-2x^3, x\in[1,4]$, then absolute maximum value of f (x) is	s) $\frac{-8}{3}$

Key. $A\to Q; B\to P;$
$C\to S; D\to R$

Sol.

A) $f(x)=3x^2-12x+14$

$f^1(x)=0\Rightarrow 6x-12=0\Rightarrow x=2$

B) Put $3^x=t\Rightarrow f(t)=2t^3-4t^2+2t, t\in[1/3,3]$

$f^1(t)<0$ in $(1/3,1)$ and $f^1(t)>0$ in $(1,3)$ $f(t)$ minimum = $f(1)=0$

C) $f^1(x)=4(x^2-1)\Rightarrow f$ is decreasing for 0 < x < 1 and f is increasing for 1<x ≤ 2

$\therefore$ f minimum is $f(1)=4/3-4=-8/3$

D) $f^1(x)=-12x+18x-6x^2=-6(x-1)(x-2)$

F is increasing in $(1,2)$ and decreasing in $(2,4)$

$\therefore$ absolute maximum = maximum $\{f(1),f(2)\}$ = $\max\text{imum}\ \{1,2\}=2$

11. Match the following: -

Column – 1		Column – 2	
(A)	The maximum value of $\sec^{-1}\left(\dfrac{7-5\left(x^2+3\right)}{2\left(x^2+2\right)}\right)$ is	(p)	$\dfrac{\pi}{6}$
(B)	The minimum value of $\operatorname{cosec}^{-1}\left[3x^2+\dfrac{5}{4}\right]+\sec^{-1}\left[3x^2+\dfrac{1}{4}\right]$	(q)	$\dfrac{\pi}{4}$
(C)	Points of non-differentiable of the function $f(x)=\min\left(\tan\left(x+\dfrac{\pi}{12}\right),\cot\left(x+\dfrac{\pi}{12}\right)\right)$ $\forall x\in\left(0,\dfrac{3\pi}{2}\right)$ is/are	(r)	$\dfrac{2\pi}{3}$
(D)	Tangent is drawn to hyperbola $\dfrac{x^2}{8}-\dfrac{y^2}{1}=1$ at $(2\sec\theta,\tan\theta)$; $\theta\in\left(0,\dfrac{\pi}{2}\right)$. The value of θ such that sum of intercepts on axes made by this tangent is maximum is	(s)	$\dfrac{7\pi}{6}$

Key. A - (r); B - (p); C - (p, r, s); D - (q)

Sol. (A) $\sec^{-1}\left(\dfrac{7-5\left(x^2+3\right)}{2\left(x^2+2\right)}\right)=\sec^{-1}\left(\dfrac{1}{x^2+2}-\dfrac{5}{2}\right)$

$$\because \frac{1}{x^2+2}\le\frac{1}{2}$$

$$\frac{1}{x^2+2}-\frac{5}{2}\le -2$$

(B) minimum value = $\operatorname{cosec}^{-1}2+\sec^{-1}1=\dfrac{\pi}{6}$

when $\left[3x^2+\dfrac{1}{4}\right]=1$

(C) f(x) is non-differentiable at $x=\dfrac{\pi}{6},\dfrac{2\pi}{3},\dfrac{7\pi}{6}$.

(D) Equation of tangent is $\dfrac{x\sec\theta}{2\sqrt{2}}-\dfrac{y\tan\theta}{1}=1$

If it cuts the coordinate axes at A and B, then

$$A\equiv\left(2\sqrt{2}\cos\theta,0\right)$$

$$B\equiv\left(0,-\cot\theta\right)$$

$$S=2\sqrt{2}\cos\theta-\cot\theta$$

$$\frac{dS}{d\theta}=-2\sqrt{2}\sin\theta+\operatorname{cosec}^2\theta=0=\sin\theta=\frac{1}{\sqrt{2}}\Rightarrow\theta=\frac{\pi}{4}$$

$$\frac{d^2S}{d\theta^2}<0$$

$\therefore$ S is maximum

22. Given that $(x-2)^2+(y-2)^2=1$

Column – I		Column – II	
a)	Maximum value of $x+y$ is	p)	$4+\sqrt{2}$
b)	Maximum value of $x-y$ is	q)	$\frac{9+4\sqrt{2}}{2}$
c)	Maximum value of xy is	r)	$\frac{4+\sqrt{7}}{3}$
d)	Maximum value of $\frac{x}{y}$ is	s)	$\sqrt{2}$

Key. a) p; b) s; c) q; d) r

Sol. Let $x=2+\cos\theta$ and $y=2+\sin\theta$

$x+y=4+\sin\theta+\cos\theta \Rightarrow$ maximum of (*x* + *y*) is $4+\sqrt{2}$

$x-y=\cos\theta-\sin\theta \Rightarrow$ maximum of $x-y$ is $\sqrt{2}$

$f(\theta)=xy=4+2(\sin\theta+\cos\theta)+\sin\theta\cos\theta$

$f'(\theta)=2(\cos\theta-\sin\theta)+\cos 2\theta=0 \Rightarrow \tan\theta=1$ or $2+\sin\theta+\cos\theta=0$ rejected

$f\left(\frac{\pi}{4}\right)=4+2\sqrt{2}+\frac{1}{2}$

$g(\theta)=\frac{x}{y}=\frac{2+\cos\theta}{2+\sin\theta} \Rightarrow g'(\theta)=\frac{(2+\sin\theta)(-\sin\theta)-(2+\cos\theta)\times\cos\theta}{(2+\sin\theta)^2}$

$g'(\theta)=0 \Rightarrow 2\sin\theta+\sin^2\theta+2\cos\theta+\cos^2\theta=0 \Rightarrow \sin\theta+\cos\theta=-\frac{1}{2}$

$\theta=\frac{\pi}{4}-\cos^{-1}\left(\frac{-1}{2\sqrt{2}}\right)$. Let $\alpha=\cos^{-1}\left(\frac{-1}{2\sqrt{2}}\right)$

$\cos\left(\frac{\pi}{4}-\alpha\right)=\frac{\sqrt{7}-1}{4}$ and $\sin\left(\frac{\pi}{4}-\alpha\right)=\frac{-\sqrt{7}-1}{4}$

Maximum value of $\frac{x}{y}$ is $\frac{7+\sqrt{7}}{7-\sqrt{7}}=\frac{4+\sqrt{7}}{3}$

20. Match the following List

Column I (curve)		Column II (Number of critical points)	
(A)	$f(x)=\text{sgn}(x)$	(p)	2
(B)	$f(x)=\sin\frac{1}{x}$	(q)	1
(C)	$f(x)=\frac{\lvert x-2\rvert}{x^2}$	(r)	0
(D)	$f(x)=\frac{1}{x^2-x}$	(s)	infinite

Key. A-s; B-s; C-p; D-q

Sol. (a) $f'(x)=0\ \ \forall x\neq 0$

f has vertical tangent at *x* = 0

Infinite number of critical points.

(b) $f'(x)=-\dfrac{1}{x^2}\cos\dfrac{1}{x};\ \ f'(x)=0$ when $\dfrac{1}{x}=(2n+1)\dfrac{\pi}{2};n\in I$

$f'(0)$ does not exist

Infinite number of critical points.

(c) $f'(x)=\begin{cases}\dfrac{4-x}{x^3} & \text{if} \quad x>2\\ \dfrac{x-4}{x^3} & \text{if} \quad x\in(-\infty,2)\sim(0)\end{cases}$

$f'(x)=0$ at $x=4;\ f'(2)$ does not exist

$x=0$ is not a point in domain.

Two critical points.

(d) $f(x)=\dfrac{1}{x-1}-\dfrac{1}{x}$

$f'(x)=\dfrac{1}{x^2}-\dfrac{1}{(x-1)^2}=\dfrac{1-2x}{(x^2-x)^2}$

$f'(x)=0$ if $x=\dfrac{1}{2}$

One critical point.

21. Match the following List

Column I Equation of tangent		Column II	
(A)	to $y=xe^{-\|x\|}$ at the point where the curve achieves local maxima	(p)	$y=x+2$
(B)	Common to $y^2=8x$ and $xy=-1.$	(q)	$2y=2-\sqrt{5}$
(C)	to $y=x^2$ at the point where its slope is abscissa of the point	(r)	$y=\dfrac{1}{e}$
(D)	to $(1+x^2)y=2-x:x>0,$ which tangent is parallel to x-axis	(s)	$y=0$

Key. A-r; B-p; C-s; D-q

Sol. (a) $f(x)=\begin{cases}xe^{-x} & \text{if} \quad x\geq 0\\ xe^{x} & \text{if} \quad x\leq 0\end{cases}$

$$f'(x) = \begin{cases} (1-x)e^{-x} & \text{if} \quad x > 0 \\ (1+x)e^{x} & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x = 0 \end{cases}$$

$f'(x) > 0$ if $x \in (0,1)$ or $x \in (-1,0)$

< 0 if $x > 1$ or $x < -1$

f has local maxima at *x* = 1

$f'(1) = 0;\ f(1) = \frac{1}{e}$

Tangent : $y = \frac{1}{e}.$

(b) $y = mx + \frac{2}{m}$ is tangent to $y = -\frac{1}{x}$

Solving the 2 equations

$x(mx + \frac{2}{m}) = -1$ has unique sol. for *x*

$m^2x^2 + 2x + m = 0$ has discriminant = 0

$m = 1$

Tangent : $y = x + 2.$

(c) $P(x_0, x_0^2)$ is the corresponding point

$2x_0 = x_0;\ P(0,0)$

Tangent : $y = 0$

(d) $P(x_0, y_0)$ is the point of tangency

$(1 + x^2)y = 2 - x$

$2xy + (1 + x^2)\frac{dy}{dx} = -1$

$\left.\frac{dy}{dx}\right|_P = 0 \Rightarrow 2x_0y_0 + 1 = 0$

$y_0 = -\frac{1}{2x_0}$ & $(1 + x_0^2)y_0 = 2 - x_0$

$x_0^2 - 4x_0 - 1 = 0;\ x_0 = 2 + \sqrt{5};\ y_0 = 1 - \frac{\sqrt{5}}{2}$

Tangent : $y = 1 - \frac{\sqrt{5}}{2}$

22. Match the following List

Column I (curve)		Column II (Angle of inter section)	
(A)	$y^2 = x \,\&\, x^2 = y : x > 0$	(p)	$\tan^{-1} 3$
(B)	$y^2 = 4x \,\&\, xy = 16$	(q)	$\tan^{-1} 2\sqrt{2}$
(C)	$y = \sin x \,\&\, y = \cos x$	(r)	$\pi/2$
(D)	$x^3 - 3xy^2 + 2 = 0 \,\&\, 3x^2y - y^3 = 2$	(s)	$\tan^{-1}\frac{3}{4}$

Key. A-s; B-p; C-q; D-r

Sol. (a) $P(1,1)$: point of intersection

Slopes of tangents $\frac{1}{2}, 2$

$$\tan\theta = \frac{2-\frac{1}{2}}{1+2.\frac{1}{2}}$$

$$\theta = \tan^{-1}\frac{3}{4}$$

(b) Point of intersection : P(4, 4)

Slopes of tangents : $\frac{1}{2}, -1$

$$\tan\theta = \frac{\frac{1}{2}+1}{1+\frac{1}{2}(-1)}$$

$$\theta = \tan^{-1} 3$$

(c) Sufficient to consider $x \in [0, 2\pi)$

Points of intersection : $P_1\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right); P_2\left(\frac{5\pi}{4}, \frac{-1}{\sqrt{2}}\right)$

Slopes of tangents at $P_1 : \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$

Slopes of tangents at $P_2 : -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

$\tan\theta = 2\sqrt{2}$ in each case

$\theta = \tan^{-1} 2\sqrt{2}$

(d) For the curve $x^3 - 3xy^2 + 2 = 0, \frac{dy}{dx} = \frac{x^2 - y^2}{2xy}$

For the curve $3x^2y - y^3 = 2, \frac{dy}{dx} = -\frac{2xy}{x^2 - y^2}$

Product of slopes = –1 at any point of intersection. $P(1,1)$ is one point of intersection

Angle of intersection $= \frac{\pi}{2}$