Continuity & Differentiability

Single Correct Answer Type

1. A function $f(x)$ is defined by ,

$$f(x)=\begin{cases}\frac{[x^2]-1}{x^2-1}, & for\ x^2\neq 1\\ 0 & , for\ x^2=1\end{cases} \qquad Where[.] denotes GIF$$

A) Continuous at $x=-1$ B) Discontinuous at $x=1$

C) Differentiable at $x=1$ D) None of these

Key. B

Sol.

$$f(x)=\begin{cases}\frac{[x^2]-1}{x^2-1} & , for\ x^2\neq 1\\ 0 & , for\ x^2=1\end{cases}$$

$$=\begin{cases}\frac{-1}{x^2-1} & , for\ 0<x^2<1\\ 0 & , for\ x^2=1\\ 0 & , for\ 1<x^2<2\end{cases}$$

$\therefore$ RHL at x = 1 is 0

Also LHL at x = 1 is ∞

2. If $f(x)=\text{sgn}(x)$ and $g(x)=x(1-x^2)$ then $(fog)(x)$ is discontinuous at

(A) exactly one point (B)exactly two points

(C) exactly three points (D) no point.

Key. C

Sol. Given $f(x)=Sgnx=\begin{cases}-1\ if\ x<0\\ 0\ if\ x=0\\ 1\ if\ x>0\end{cases}$

And $g(x)=x(1-x^2)$

Now $fog(x)=-1$ if $x(1-x^2)<0$ solving

$= 0$ if $x(1-x^2)=0$, $x(1-x^2)<0$

$= 1$ if $x(1-x^2)>0$ we have $x\in(-1,0)\cup(1,\infty)$

$\therefore\ fog(x) = -1$ if $x \in (-1,0) \cup (1,\infty)$

$= 0$ if $x \in \{-1,0,1\}$

$= 1$ if $x \in (-\infty,-1) \cup (0,1)$

$\therefore\ fog(x)$ is discontinuous at $x = -1,0,1$

3. If $f(x)$ is a polynomial satisfying the relation $f(x) + f(2x) = 5x^2 - 18$ then $f^1(1)$ is equal to

(A) 1

(B) 3

(C) cannot be found since degree of $f(x)$ is not given

(D) 2

Key. D

Sol. Let $f(x) = ax^2 + bx + c$ (By hypothesis)

$f(x) + f(2x) = 5x^2 - 8$

$\Rightarrow f(x) = x^2 - 9 \therefore f^1(1) = 2.$

4. Let 'f' be a real valued function defined on the interval $(-1,1)$ such that $e^{-x}.f(x) = 2 + \int_0^x \sqrt{t^4+1}\, dt \quad \forall x \in (-1,1)$ and let 'g' be the inverse function of 'f'.

Then $g^1(2)$ = _____

(A) 3 (B) $1/2$ (C) $1/3$ (D) 2

Key. C

Sol. Differentiating given equation we get

$e^{-x}.f^1(x) - e^{-x}.f(x) = \sqrt{1+x^4}$

Since $(g\,of)(x) = x.as$ '$\underline{g}$' is inverse of f.

$\Rightarrow g[f(x)] = x$

$\Rightarrow g^1[f(x)].f^1(x) = 1$

$\Rightarrow g^1[f(0)] = \frac{1}{f^1(0)}$

$\Rightarrow g^1(2) = \frac{1}{f^1(0)}$

(Here $f(0) = 2$ observe from hypothesis)

Put $x = 0$ in (1) we get $f^1(0) = 3$.

5. If $y = f(x)$ represents a straight line passing through origin and not passing through any of the points with integral Co-ordinates in the co-ordinate plane. Then the number of such continuous functions on 'R' is _____(it is known that straight line represents a function)

(A) 0 (B) finite (C) infinite (D) at most one

Key. C

Sol. $\exists$ infinitely many continuous functions of the form $f(x) = mx$. When m is Irrational, and when slope is irrational the line obviously will not pass through any of the pts in the Co-ordinate plane with integral Co-ordinates. We know a straight line is always continuous.

6. If a function $y = \phi(x)$ is defined on $[a,b]$ and $\phi(a)\phi(b) < 0$ then

(A) $\exists$ no $c \in (a,b)$ such that $\phi(c) = 0$ if and only if $'\phi'$ is continuous

(B) $\exists$ a function $\phi(x)$ differentiable on $R - \{0\}$ satisfying the given hypothesis

(C) If $\phi(c) = 0$ satisfying the given hypothesis then $\phi(x)$ must be discontinuous

(D) None of these

Key. B

Sol. Consider the function $\phi(x) = \begin{cases} \frac{1}{x}, & if \quad x \neq 0 \\ 1, & if \quad x = 0 \end{cases}$ defined on $[-1,1]$,clearly $\phi(-1) \times \phi(1) < 0$, and $\phi(x)$ is differentiable on $R1\{0\}$

But there is no point $c \in [-1,1] \ni \phi(c) = 0$.

7. Let $f : R \to R$ be a differentiable function satisfying $f(y)f(x-y) = f(x) \; \forall \, x, y \in R$ and $f^1(0) = p, f^1(5) = q$ then $f(5)$ is

A. p^2/q B. p/q C. q/p D. q

Key. C

Sol. $y = 0 \Rightarrow f(0) = 1$ and $x = 0 \Rightarrow f(-y) = \dfrac{1}{f(y)}$.

Hence $f(x+y) = f(x)f(y)$ $f^1(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = f(x) \lim_{h\to 0} \dfrac{f(x) - 1}{h} = f(x).f^1(0) = pf(x)$ put $x = 5$ $f(5) = \dfrac{q}{p}$

8. If both f(x) and g(x) are differentiable functions at x = x_0, then the function defined as h(x) = maximum {f(x), g(x)} :

(A) is always differentiable at x = x_0

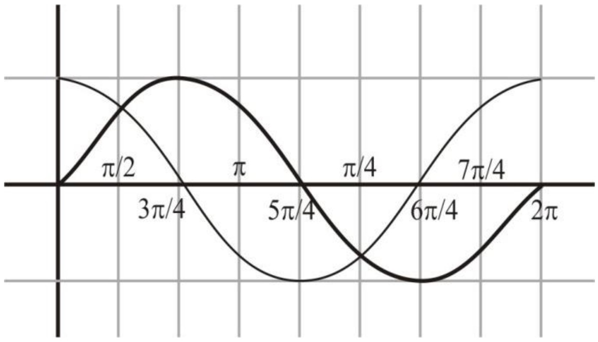
(B) is never differentiable at x = x_0

(C) is differentiable at x = x_0 provided $f(x_0) \neq g(x_0)$

(D) cannot be differentiable at x = x_0 if $f(x_0) \neq g(x_0)$

Key. C

Sol. Consider the graph of f(x) = max(sinx, cosx), which is non-differentiable at x = $\pi/4$, hence statement (A) is false. From the graph y = f(x) is differentiable at x = $\pi/2$, hence statement (B) is false.
Statement (C) is false
Statement (D) is false as consider g(x) = max (x, x^2) at x = 0, for which x = x^2 at x = 0, but f(x) is differentiable at x = 0.



9. $$\left.\begin{aligned} f(x) &= \left(\tan\left(\frac{\pi}{4}+x\right)\right)^{\frac{1}{x}} && if \quad x \neq 0 \\ &= \lambda && if \quad x = 0 \end{aligned}\right\}$$ is continuous at x = 0 then value of λ is

1) 1 2) e 3) e^2 4) 0

Key. 3

Sol. $$\lambda = \lim_{x\to 0}\left(\frac{1+\tan x}{1-\tan x}\right)^{\frac{1}{x}} = \frac{e}{e^{-1}} = e^2$$

10. $f(x) = \frac{1}{q}$ If $x = \frac{p}{q}$ where p and q are integer and $q \neq 0$, G.C.D of (p,q) = 1 and $f(x) = 0$
If x is irrational then set of continuous points of $f(x)$ is

1) all real numbers 2) all rational numbers 3) all irrational number 4) all integers

Key. 3

Sol. Let $x = \frac{p}{q}$

$f(x) = \frac{1}{q}$

When $x \to \frac{p}{q}$ $f(x) = 0$ for every irrational number $\in nbd(p/q)$

$= \frac{1}{n}$ $if \quad n = \frac{m}{n} \in nbd(p/q)$

$\frac{1}{n} \to 0 \ as \ n \to \infty$ since

There ∞ - number of rational $\in nbd(p/q)$

$\therefore \lim_{x\to \frac{p}{q}} f(x) = 0$ but $f\left(\frac{p}{q}\right) = \frac{1}{q} \neq 0$

Discontinuous at every rational

If $x = \alpha$ is irrational $\Rightarrow f(\alpha) = 0$

Now $\lim\limits_{x \to \alpha} f(x)$ is also O

$\therefore$ continuous for every irrational α

11. $f(x) = \max\{3 - x, 3 + x, 6\}$ is differentiable at

A) All points B) No point

C) All points except two D) All points expect at one point

Key. C

Sol.

$$f(x) = \begin{cases} 3 - x & x < -3 \\ 6 & -3 \le x \le 3 \\ 3 + x & x > 3 \end{cases}$$

Since these expressions are linear function in x or a constant
It is clearly differentiable at all points except at the border points at -3 and 3

At $x = -3,\ LHD = -1, RHD = 0$

At $x = 3,\ LHD = 0, RHD = 1$

$\therefore$ At $x = -3$ and x = 3 it is not differentiable

12. If ([.] denotes the greatest integer function) then $f(x)$ is

A) continuous and non-differentiable at $x = -1$ and $x = 1$

B) continuous and differentiable at $x = 0$

C) discontinuous at $x = 1/2$

D) continuous but not differentiable at $x = 2$

Key. C

Sol.

$$f(x)=\begin{cases}-1 & ,\frac{1}{2}<x<1\\ 0 & ,0<x\le\frac{1}{2}\\ 1 & ,x=0\\ 0 & ,-\frac{1}{2}\le x<0\\ -1 & ,-\frac{3}{2}<x<-\frac{1}{2}\\ 2-x & ,1\le x<2\end{cases}$$ clearly discontinuous at $x=\frac{1}{2}$

13. A function $f(x)$ is defined by ,

$$f(x)=\begin{cases}\frac{[x^2]-1}{x^2-1}, & \text{for } x^2\ne 1\\ 0 & , \text{for } x^2=1\end{cases}$$,Where [.] denotes G.I.F

A) Continuous at $x=-1$
B) Discontinuous at $x=1$
C) Differentiable at $x=1$
D) None of these

Key. B

Sol.

$$f(x)=\begin{cases}\frac{[x^2]-1}{x^2-1}, & \text{for } x^2\ne 1\\ 0 & , \text{for } x^2=1\end{cases}$$

$$=\begin{cases}\frac{-1}{x^2-1} & , \text{for } 0<x^2<1\\ 0 & , \text{for } x^2=1\\ 0 & , \text{for } 1<x^2<2\end{cases}$$

$\therefore$ RHL at x = 1 is 0

Also LHL at x = 1 is ∞

14. $f(x)=\frac{\sin 2\pi[\pi^2 x]}{5+[x^2]}$. Where $[.]$ denotes the greatest integer function then $f(x)$ is

A) Continuous
B) Discontinuous

C) $f'(x)$ exist but $f''(x)$ does not exist

D) $f'(x)$ is not differentiable

Key. A

Sol. $2\pi[\pi^2 x]$ is integral multiple of π ,there fore f(x)=0 $\forall$ x

$\Rightarrow$f(x) is constant function

$\Rightarrow$f(x) is continuous and differentiable any number of times

15. The no. of points of discontinuous of $g(x) = f(f(x))$ where $f(x)$ is defined as, $f(x) = \begin{cases} 1+x, 0 \le x \le 2 \\ 3-x, 2 < x \le 3 \end{cases}$

A) 0

B) 1

C) 2

D) >2

Key. C

Sol.

$$g(x) = \begin{cases} 2+x, 0 \le x \le 1 \\ 2-x, 1 < x \le 2 \\ 4-x, 2 < x \le 3 \end{cases}$$

16. $$Let\ f(x) = \begin{cases} x^n \sin\left(\frac{1}{x}\right), & x \ne 0 \\ 0, & x = 0 \end{cases}$$

then f (x) is continuous but not differentiable at $x = 0$, if

A) $n \in (0,1]$

B) $n \in [1, \infty)$

C) $n \in (-\infty, 0)$

D) $n = 0$

Key. A

Sol.

$R.H.L = \lim_{x\to 0^+} f(x)$

$= \lim_{h\to 0} f(0+h)$

$= \lim_{h\to 0} h^n . \sin\left(\frac{1}{h}\right)$

$= 0^n . \sin(\infty)$

$= 0^n . \{-1 \text{ to } 1\}$

$\because V.F = f(0) = 0$

$\therefore n > 0..........(1)$

$Rf^I(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$

$\lim_{h\to 0} \frac{h^n \sin\left(\frac{1}{h}\right) - 0}{h}$

$\lim_{h\to 0} h^{n-1} \sin\left(\frac{1}{h}\right) = 0^{n-1}(-1 \text{to } 1)$

For not differentiable

$n - 1 \le 0$

$n \le 1............(2)$

From equation 1 and 2

$0 < n \le 1$

$n \in (0,1]$

17. The function f(x) is defined as

$f(x) = \begin{cases} \frac{1}{|x|}, |x| > 2 \\ a + bx^2, |x| \le 2 \end{cases}$ where a and b are constants. Then which one of the following is true?

A) f is differentiable at x = - 2 if and only if a =3/4, b = -1/16

B) f is differentiable at x = - 2 whatever be the values of a and b

C) f is differentiable at x = - 2 if $b = -\frac{1}{16}$, whatever be the values of a

D) f is differentiable x = - 2 if $b = \frac{1}{16}$, whatever be the values of a.

Key. A

Sol. Conceptual

18. Total number of points belonging to $(0, 2\pi)$ where $f(x) = \min\{\sin x, \cos x, 1 - \sin x\}$ is not differentiable

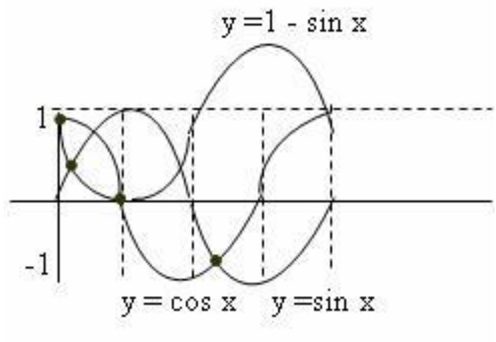
A) 2 B) 3 C) 4 D) 5

Key. B

Sol. By figure it is clear

$x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{4}$ *are*

The points where f(x) is not differentiable



19. If $f(x) = \begin{cases} \alpha + \frac{\sin[x]}{x} & x > 0 \\ 2 & x = 0 \\ \beta + \left[\frac{\sin x - x}{x^3}\right] & x < 0 \end{cases}$

Where [.] is G.I.F. If f(x) is continuous at x = 0 then $\beta - \alpha$ equal to

A) 1 B) -1 C) 2 D) -2

Key. A

Sol. Conceptual

$RHL(x = 0) = \alpha + 0 = \alpha$

$$\frac{\sin x - x}{x^3} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - x}{x^3} = \frac{-1}{3!} + \frac{x^2}{5!} - \ldots\ldots$$

$$\underset{x \to 0}{lt} \frac{\sin x - x}{x^3} = \frac{-1}{6}$$

$LHL = \beta - 1$

20. Given $f(x)=\begin{cases} x^2e^{2(x-1)} & 0\le x\le 1 \\ a\cos(2x-2)+bx^2 & 1<x\le 2 \end{cases}$

$f(x)$ is differentiable at $x=1$ provided

A) $a=-1, b=2$ B) $a=1, b=-2$ C) $a=-3, b=4$ D) $a=3, b=-4$

Key. A

Sol. $f(1+0)=f(1-0)\Rightarrow a+b=1$

$$f^1(x)=\begin{cases} 2x^2e^{2(x-1)}+e^{2(x-1)}.2x & 0<x<1 \\ -2a\sin(2x-2)+2bx & 1<x<2 \end{cases}$$

$f^1(1-0)=f^1(1+0)\Rightarrow 4=2b$

$\Rightarrow b=2,\ a=-1$

21. The function $f(x)=\dfrac{x}{1+|x|}$ is differentiable in

A) R B) $R-\{0\}$ C) $[0,\infty)$ D) $(0,\infty)$

Key. A

Sol. The function f(x) is an odd function with Range $(-1,1)\Rightarrow$ it is differentiable every where

$$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{1}{1+|x|}=1$$

22. The domain of the derivative of the function $f(x)=\begin{cases} \tan^{-1}x & \text{if } |x|\le 1 \\ \frac{1}{2}(|x|-1) & \text{if } |x|>1 \end{cases}$ is

A) $\mathbb{R}-\{0\}$ B) $\mathbb{R}-\{1\}$ C) $\mathbb{R}-\{-1\}$ D) $\mathbb{R}-\{-1,1\}$

Key. D

Sol. The given function is $f(x)=\begin{cases}\tan^{-1}x & \text{if } |x|\le 1\\ \frac{1}{2}(|x|-1) & \text{if } |x|>1\end{cases}$

$$\Rightarrow f(x)=\begin{cases}\frac{1}{2}(-x-1) & if \quad x<-1\\ \tan^{-1}x & if \quad -1\le x\le 1\\ \frac{1}{2}(x-1) & if \quad x>1\end{cases}$$

Clearly L.H.L at $(x=-1)=\lim_{h\to 0} f(-1-h)$

R.H.L at $(x=-1)=\lim_{h\to 0} f(-1+h)=\lim_{h\to 0}\tan^{-1}(-1+h)=-\pi/4$

$\therefore$ L.H.L $\ne$ R.H.L at $x=-1$

$\therefore$ f(x) is discontinuous at $x=-1$

Also we can prove in the same way, that f(x) is discontinuous at x = 1

$\therefore$ f(x) can not be found for $x=\pm 1$ or domain of $f'(x)=R-\{-1,1\}$

23. $If \ f(x)=\frac{[x]}{|x|}, x\ne 0$ where [.] denotes the G.I.F then $f'(1)$ *is*

A) -1 B) 1 C) ∞ D) Does not exist

Key. D

Sol. $f(x)=\frac{[x]}{|x|}=\begin{cases}0 \ , \ 0<x<1\\ 1 \ , \ 1\le x<2\end{cases}$

Clearly $\lim_{x\to 1^-} f(x)=0, \ \lim_{x\to 1^+} f(x)=1$

$\therefore$ f(x) is not continuous at x = 1

f(x) is not differentiable at x = 1

$\therefore f'(1)$ does not exist

24. If f(x) = $\sin\left\{\frac{\pi}{3}[x]-x^2\right\}$ for 2 < x < 3 and ([x] denotes the G.I.F) then $f'\left(\sqrt{\frac{\pi}{3}}\right)$ is

A) $\sqrt{\frac{\pi}{3}}$ B) $-\sqrt{\frac{\pi}{3}}$ C) $-\sqrt{\pi}$ D) $\sqrt{\pi}$

Key. B

Sol. For 2 < x < 3 , we have [x] = 2

$\therefore f(x) = \sin\left(\frac{2\pi}{3} - x^2\right)$

$f^1(x) = -2x\cos\left(\frac{2\pi}{3} - x^2\right)$

$f^1\left(\sqrt{\frac{\pi}{3}}\right) = -2\sqrt{\frac{\pi}{3}}\cos\left(\frac{2\pi}{3} - \frac{\pi}{3}\right)$

$= -\sqrt{\frac{\pi}{3}}$

25. The derivation of $f(\tan x)$ with respect to $g(\sec x)$ at $x = \frac{\pi}{4}$. *If* $f'(1) = 2, g'(\sqrt{2}) = 4$

A) $\frac{1}{\sqrt{2}}$ B) $\sqrt{2}$ C) $\frac{1}{2}$ D) 1

Key. A

Sol. *Let* $u = f(\tan x)$

$\frac{du}{dx} = f'(\tan x).\sec^2 x$

$v = g(\sec x)$

$\frac{dv}{dx} = g'(\sec x).\sec x \tan x$

Now $\left(\frac{du}{dv}\right) = \frac{f'(\tan x).\sec^2 x}{g'(\sec x).\sec x \tan x} = \frac{f'(1)2}{g'(\sqrt{2}).\sqrt{2}} = \frac{2.2}{4.\sqrt{2}} = \frac{1}{\sqrt{2}}$

26. If $y = \tan^{-1}\frac{1}{x^2+x+1} + \tan^{-1}\frac{1}{x^2+3x+3} + \tan^{-1}\frac{1}{x^2+5x+7} + \ldots\text{nterms}$ then $\frac{dy}{dx} =$

A) $\frac{1}{1+(x+n)^2} - \frac{1}{1+x^2}$ B) $\frac{1}{1+(x+n)^2} + \frac{1}{1+x^2}$ C) $\frac{1}{1-(x+n)^2} - \frac{1}{1+x^2}$ D) $\frac{1}{1-(x+n)^2} + \frac{1}{1+x^2}$

Key. A

Sol. $y = \tan^{-1}\frac{1}{x^2+x+1} + \tan^{-1}\frac{1}{x^2+3x+3} + \tan^{-1}\frac{1}{x^2+5x+7} + \ldots\text{nterms}$

$$y=\tan^{-1}\left(\frac{(x+1)-x}{1+x(x+1)}\right)+\tan^{-1}\left(\frac{(x+2)-(x+1)}{1+(x+1)(x+2)}\right)+\tan^{-1}\left(\frac{(x+3)-(x+2)}{1+(x+2)(x+3)}\right)+\ldots+\tan^{-1}\left(\frac{(x+n)-(x+n-1)}{1+(x+n)(x+n-1)}\right)$$

$$y=\tan^{-1}(x+1)-\tan^{-1}x+\tan^{-1}(x+2)-\tan^{-1}(x+1)+\tan^{-1}(x+3)-\tan^{-1}(x+2)+\ldots.+\tan^{-1}(x+n)-\tan^{-1}(x+n-1)$$

$$y=\tan^{-1}(x+n)-\tan^{-1}x\Rightarrow\frac{dy}{dx}=\frac{1}{1+(x+n)^2}-\frac{1}{1+x^2}$$

27. Let f (x) = x[x] , (where [.] denotes the G.I.F) . If x is not an integer, then $f'(x)$ is

A) 2x B) x C) [x] D) 3x

Key. C

Sol. f(x)=x[x]

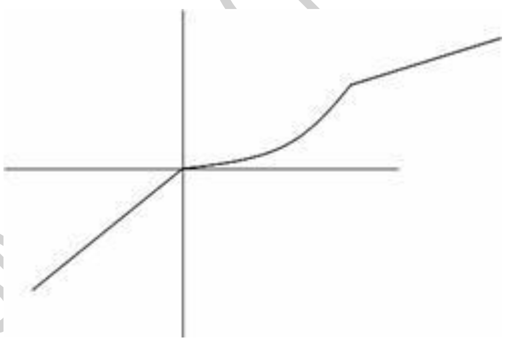
$f'(x)=[x]$

28. Number of points at which the function $f(x)=\begin{cases}\min.(x,x^2)\text{ if }-\infty<x<1\\ \min.(2x-1,x^2)\text{ if }x\ge 1\end{cases}$ is not derivable is

A) 0 B) 1 C) 2 D) 3

Key. C

Sol.



29. Given $f(x)=\begin{cases}x^2e^{2(x-1)} & 0\le x\le 1\\ a\cos(2x-2)+bx^2 & 1<x\le 2\end{cases}$

$f(x)$ is differentiable at $x=1$ provided

A) $a=-1, b=2$ B) $a=1, b=-2$ C) $a=-3, b=4$ D) $a=3, b=-4$

Key. A

Sol. $f(1+0)=f(1-0)\Rightarrow a+b=1$

$$f^1(x)=\begin{cases}2x^2e^{2(x-1)}+e^{2(x-1)}.2x & 0<x<1\\ -2a\sin(2x-2)+2bx & 1<x<2\end{cases}$$

$$f^1(1-0)=f^1(1+0)\Rightarrow 4=2b$$

$$\Rightarrow b=2,\ a=-1$$

30. The function $f(x)=\dfrac{x}{1+|x|}$ is differentiable in

A) R B) $R-\{0\}$ C) $[0,\infty)$ D) $(0,\infty)$

Key. A

Sol. The function f(x) is an odd function with Range $(-1,1)\Rightarrow$ it is differentiable every where

$$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{1}{1+|x|}=1$$

31. The value of $\lim_{x\to\infty}\left(\dfrac{a_1^{1/x}+a_2^{1/x}+\ldots\ldots a_n^{1/x}}{n}\right)^{nx}$ is

A) $a_1+a_2+\ldots\ldots+a_n$ B) $e^{a_1+a_2+\ldots\ldots+a_n}$ C) $\dfrac{a_1+a_2+\ldots..+a_n}{n}$ D) $a_1 a_2\ldots..a_n$

Key. D

Sol. Let $x=\dfrac{1}{y}$. Then, $x\to\infty, y\to 0$

$$=\lim_{x\to\infty}\left(\frac{a_1^{1/x}+a_2^{1/x}+\ldots.+a_n^{1/x}}{n}\right)^{nx}$$

$$=\lim_{y\to 0}\left(\frac{a_1^y+a_2^y+\ldots.+a_n^y}{n}\right)^{n/y}=1^{\infty}$$

$$= e^{\lim\limits_{y\to 0}\left(\frac{1+a_1^y+a_2^y+....+a_n^y-n}{n}\right)^{n/y}}$$

$$= e^{\lim\limits_{y\to 0}\frac{n}{y}\left(\frac{a_1^y+a_2^y+......+a_n^y}{n}-1\right)}$$

$$= e^{\lim\limits_{y\to 0}\left(\frac{a_1^y-1}{y}+\frac{a_2^y-1}{y}+......+\frac{a_n^y-1}{y}\right)}$$

$$= e^{\log a_1+\log a_2+\log a_3........+\log a_n}$$

$$= e^{\log(a_1.a_2.a_3..........a_n)} = (a_1 a_2 a_3 ......a_n)$$

32. If $f(x)=\frac{\sin(e^{x-2}-1)}{\log(x-1)}$, then $\lim\limits_{x\to 2} f(x)$ is given by

A) -2 B) -1 C) 0 D) 1

Key. D

Sol. $\lim\limits_{x\to 2} f(x) = \lim\limits_{x\to 2}\frac{\sin(e^{x-2}-1)}{\log(x-1)}$

$$\lim_{x\to 2}\left[\frac{\sin(e^{x-2}-1)}{e^{x-2}-1}.\frac{e^{x-2}-1}{1}.\frac{x-2}{\log(1+(x-2))}\right]$$

$= 1.1.1 = 1$

33. The value of $\lim\limits_{x\to\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right)$ is

A) 0 B) $\frac{1}{2}$ C) $\frac{1}{4}$ D) 1

Key. B

Sol. $\lim\limits_{x\to\infty}\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}$

$$= \lim_{x\to\infty} \frac{\sqrt{x+\sqrt{x}}}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{1+\frac{1}{\sqrt{x}}}}{\sqrt{1+\sqrt{\frac{1}{x}+\frac{1}{\sqrt{x^3}}}}+1} = \frac{\sqrt{1+0}}{\sqrt{1+0+0}+1} = \frac{1}{2}$$

34. Let $f(x,y)$ be a periodic function satisfying the condition $f(x,y)=f(2x+2y,\,2y-2x)$ for all $x,y\in R$ and let $g(x)=f(2^x,0)$. Then the period of $g(x)$ is

A) 2 B) 6 C) 12 D) 24

Key. C

Sol. $f(x,y)=f(2x+2y,\,2y-2x)$ ....(1)

$= f\big(2(2x+2y)+2(2y-2x),\ 2(2y-2x)-2(2x+2y)\big)$

$= f(8y,-8x)$ ....(2)

$f(8y,-8x)=f(-64x,-64y)$ ....(3)

$f(-64x,-64y)=f(2^{12}x,2^{12}y)$

Replace x by 2^x

$f(x,0)=f(2^{12}x,0)=f(2^{x+12},0)$

$g(x)=g(x+12)$

35. The fundamental period of the function $f(x)=\left|\sin\frac{x}{2}\right|+\left|\cos|x|\right|$ is

A) 2π B) π C) 4π D) $\frac{\pi}{2}$

Key. A

Sol. The fundamental period of $\left|\sin\frac{x}{2}\right|$ is 2π and that of $\left|\cos|x|\right|$ is π. L.C.M of π and 2π is 2π

So fundamental period of $f(x)$ is 2π

36. If $\cos x = \tan y$, $\cos y = \tan z$, $\cos z = \tan x$ then the value of $\sin x$ is

A) $\sin 36^0$ B) $\cos 36^0$ C) $2\sin 18^0$ D) $2\cos 18^0$

Key. C

Sol. $\cos x = \tan y \Rightarrow \cos^2 x = \tan^2 y$

$= \sec^2 y - 1 = \cot^2 z - 1 = \operatorname{cosec}^2 z - 2 = \dfrac{1}{1-\cos^2 z} - 2 = \dfrac{1}{1-\tan^2 x} - 2$

$= \dfrac{2\tan^2 x - 1}{1-\tan^2 x}$

$\Rightarrow \cos^2 x = \dfrac{2\sin^2 x - \cos^2 x}{\cos^2 x - \sin^2 x} \Rightarrow 1 - \sin^2 x = \dfrac{3\sin^2 x - 1}{1 - 2\sin^2 x}$

$\Rightarrow 1 - 2\sin^2 x - \sin^2 x + 2\sin^4 x = 3\sin^2 x - 1$

$\Rightarrow 2\sin^4 x - 6\sin^2 x + 2 = 0$

$\Rightarrow \sin^4 x - 3\sin^2 x + 1 = 0$

$\sin x = \dfrac{\sqrt{5}-1}{2} = 2\sin 18^0$

37. Define $f:[0,\pi] \to R$ by

$$f(x) = \begin{cases} \tan^2 x\left[\sqrt{2\sin^2 x + 3\sin x + 4} - \sqrt{\sin^2 x + 6\sin x + 2}\right] & , x \neq \pi/2 \\ k & , x = \pi/2 \end{cases}$$ is continuous at

$x = \dfrac{\pi}{2}$,then k =

A) $\dfrac{1}{12}$ B) $\dfrac{1}{6}$ C) $\dfrac{1}{24}$ D) $\dfrac{1}{32}$

Key. A

Sol. Let $\sin x = t$ and evaluate $\lim\limits_{t\to 1} \dfrac{t^2}{1-t^2}\left[\sqrt{2t^2+3t+4} - \sqrt{t^2+6t+2}\right]$ by rationalization

38. Let $|a_1 \sin x + a_2 \sin 2x + ... + a_8 \sin 8x| \leq |\sin x|$ for $x \in R$

Define $P = a_1 + 2a_2 + 3a_3 + ... + 8a_8$ .Then P satisfies

A) $|P| \leq 1$ B) $|P| < 1$ C) $|P| > 1$ D) $|P| \geq 1$

Key. A

Sol. f(x) = a_1 sinx + a_2 sin2x + ...+ a_8 sin 8x

$|a_1 + 2a_2 + ... + 8a_8| = |f'(0)| = \lim\limits_{x\to 0}\left|\dfrac{f(x)-0}{x}\right|$

$= \lim_{x \to 0} \left|\frac{f(x)}{\sin x}\right| \left|\frac{\sin x}{x}\right|$

$= \lim_{x\to 0}\left|\frac{f(x)}{\sin x}\right| \le 1$

$|p| \le 1$

39. If $f(x) = \begin{cases} a + \dfrac{\sin[x]}{x}, & x > 0 \\ 2, & x = 0 \\ b + \left[\dfrac{\sin x - x}{x^3}\right], & x < 0 \end{cases}$ (where [.] denotes the greatest integer function). If $f(x)$ is continuous at $x = 0$, then b is equal to

A. $a-1$ B. $a+1$ C. $a+2$ D. $a-2$

Key. B

Sol. $f(0+) = \lim_{x\to 0} a + \frac{\sin[x]}{x} = a$

since $\lim_{x\to 0} \frac{\sin x - x}{x^3} = \frac{-1}{6}$; we get $f(0-) = b-1$

Hence $b = a+1$

40. If $f(x)$ is a continuous function $\forall x \in R$ and the range of $f(x) = (2, \sqrt{26})$ and $g(x) = \left[\frac{f(x)}{a}\right]$ is continuous $\forall x \in R$ (where [.] denotes the greatest integral function). Then the least positive integral value of a is

A. 2 B. 3 C. 6 D. 5

Key. C

Sol. $g(x)$ is continuous only when $\frac{f(x)}{a}$ lies between two consecutive integers Hence $(\frac{2}{a}, \frac{\sqrt{26}}{a})$ should not contain any integer. The least integral value of a is $6(\sin ce \frac{\sqrt{26}}{a} < 1)$

41. $f(x) = [x^2] - [x]^2$, then (where [.] denotes greatest integer function)

A. f is not continuous $x=0$ *and* $x=1$

B. f is continuous at $x=0$ but not at $x=1$

C. f is not continuous at $x=0$ but continuous at $x=1$

D. f is continuous at $x=0$ and $x=1$

Key. C

Sol. $f(0-)=0-(-1)^2=-1 and\ f(0)=0$. Hence f is not continuous at $x=0$ (1) $f(1-)=0-0=0$, $f(1+)=1-1=0$ $f(1)=0$ and Thus f is continuous at $x=1$

42. Let $f(x)=\sec^{-1}([1+\sin^2 x]);$ where [.] denotes greatest integer function. Then the set of points where $f(x)$ is not continuous is

A. $\left\{\frac{n\pi}{2}, n\in I\right\}$ B. $\left\{(2n-1)\frac{\pi}{2}, n\in I\right\}$ C. $\left\{(n-1)\frac{\pi}{2}, n\in I\right\}$ D. $\{n\pi / n\in I\}$

Key. B

Sol. $f(n\pi+)=\sec^{-1}1=0$ and $f(n\pi-)=\sec^{-1}1=0$ and $f(n\pi)=0$

$\therefore f$ is continuous at $x=n\pi$

$f((2n-1)\frac{\pi}{2}+)=\sec^{-1}1=0$ but $f((2n-1)\frac{\pi}{2})=\sec^{-1}2=\frac{\pi}{3}$

$\therefore f$ is discontinuous at $x=(2n-1)\frac{\pi}{2}$ for all $n\in I$

43. The number of points at which the function $f(x)=\max.\{a-x, a+x, b\}, -\infty<x<\infty, 0<a<b$ cannot be differentiable is,

A. 2 B. 3 C. 1 D. 0

Key. A

Sol. $f(x)=\begin{cases} a-x & if \quad x<a-b \\ b & if \quad a-b\le x\le b-a \\ a+x & if \quad x>b-a \end{cases}$

Hence f is not differentiable at $x=a-b, b-a$

44. $\lim\limits_{x\to -1-}\left[x\sin\pi x\right]=$ [.] $\to$ denotes greatest integer function

1) -1 2) 1 3) 0 4) does not exist

Key. 1

Sol. $x<-1\Rightarrow \pi x<-\pi \Rightarrow \pi x\in$ 2nd quadrant

$\Rightarrow \sin\pi x>0$

$x<0$

$\Rightarrow x\sin\pi x<0$

$\left[x\sin\pi x\right]=-1$

45. The function $f(x)=(x^2-1)\left|x^2-3x+2\right|+\cos(|x|)$ is not differentiable at

A) – 1 B) 0 C) 1 D) 2

Key. D

Sol. Here cos (|x|) = cos (± x) cos x

$f(x) = -(x^2 - 1)(x^2 - 3x + 2) + \cos x, 1 \le x \le 2$

$= (x^2 - 1)(x^2 - 3x + 2) + \cos x,\ x \le 1$ or $x \ge 2$

Clearly f(1) = cos 1, $\underset{x \to 1}{\text{Lt}} f(x) = \cos 1$

f(2) = cos 2, $\underset{x \to 2}{\text{Lt}} f(x) = \cos 2$

Hence f(x) is continuous at x = 1, 2

Now $f'(x) = -2x(x^2 - 3x + 2) - (x^2 - 1)(2x - 3) - \sin x, 1 \le x < 2$

$= 2x(x^2 - 3x + 2) + (x^2 - 1)(2x - 3) - \sin x, x < 1$ or x > 2

$f'(1-0) = -\sin 1, f'(1+0) = -\sin 1$

$f'(2-0) = -3 - \sin 2$,

$f'(2+0) + 3 - \sin 2$

Hence f(x) is not differentiable at x = 2.

46. If f(x) is a function such that f(0) = a, $f'(0) = ab$, $f''(0) = ab^2$, $f'''(0) = ab^3$, and so on and b > 0, where dash denotes the derivatives, then $\underset{x \to -\infty}{\text{Lt}} f(x) =$

A) ∞ B) $-\infty$ C) 0 D) none of these

Key. C

Sol. Given f(0) = a, $f'(0) = ab$, $f''(0) = ab^2$

$f'''(0) = ab^3$ and so on.

$\therefore$ $f(x) = ae^{bx}$

$\therefore$ $\underset{x \to -\infty}{\text{Lt}} f(x) = \underset{x \to -\infty}{\text{Lt}} ae^{bx} = 0\ [\because b > 0]$

47. If $f(x) = p|\sin x| + qe^{|x|} + r|x|^3$ and f(x) is differentiable at x = 0, then

A) p = q = r = 0 B) p = 0, q = 0, r = any real number

C) q = 0, r = 0, p is any real number D) r = 0, p = 0, q is any real number

Key. B

Sol. At x = 0,

L. H. derivative of p | sin x | = - p

R.H. derivative of p | sin x | = p

$\therefore$ for p | sin x| to be differentiable at x = 0, p = - p or p = 0

at x = 0, L.H. derivative of $qe^{|x|} = -q$

R.H. derivative of $qe^{|x|} = q$

For $qe^{|x|}$ to be differentiable at x = 0,

- q = q or q = 0

d.e. of r $|x|^3$ at x = 0 is 0

$\therefore$ for f (x) to be differentiable at x = 0

P = 0, q = 0 and r may be any real number.

Second Method:

$f'(0-0) = \underset{h\to 0-0}{Lt} \frac{f(h)-f(0)}{h}$

$\underset{h\to 0-0}{Lt} \frac{p|\sinh| + qe^{|h|} + r|h|^3 - q}{h}$

$\underset{h\to 0-0}{Lt} \frac{-p\sinh + qe^{-h} - rh^3 - q}{h}$

$= \underset{h\to 0-0}{Lt} \left\{-p\frac{\sinh}{h} - \frac{q(e^{-h}-1)}{-h} - rh^2\right\}$

= - p – q

Similarly, $f'(0+0) = p+q$

Since f(x) is differentiable at x = 0

$\therefore \quad f'(0-0) = f'(0+0) \Rightarrow -p-q = p+q$

$\Rightarrow \quad p+q=0$

Here r may be any real number.

$\therefore$ Correct choice is (b)

48. The number of points in (1, 3), where $f(x) = a^{[x^2]}$, $a > 1$, is not differentiable where [x] denotes the integral part of x is

A) 0 B) 3 C) 5 D) 7

Key. D

Sol. Here 1 < x < 3 and in this interval x^2 is an increasing function.

$\therefore \quad 1 < x^2 < 9$

$[x^2] = 1, 1 \le x < \sqrt{2}$

$= 2, \sqrt{2} \le x < \sqrt{3}$

$= 3, \sqrt{3} \le x < 2$

$= 4, 2 \le x < \sqrt{5}$

$= 5, \sqrt{5} \le x < \sqrt{6}$

$= 6, \sqrt{6} \le x < \sqrt{7}$

$= 7, \sqrt{7} \le x < \sqrt{8}$

$= 8, \sqrt{8} \le x < 3$

Clearly $[x^2]$ and also $a^{[x^2]}$ is discontinuous and not differentiable at only 7 points $x = \sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}$

49. Let f(x) be defined in [- 2, 2] by $f(x) = \max\left(\sqrt{4-x^2}, \sqrt{1+x^2}\right), -2 \le x \le 0$

$= \min\left(\sqrt{4-x^2}, \sqrt{1+x^2}\right), 0 < x \le 2$, then f(x)

A) is continuous at all points B) has a point of discontinuity

C) is not differentiable only at one point D) is not differentiable at more than one point

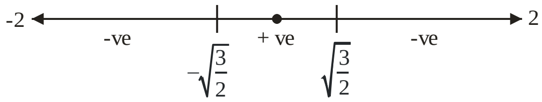
Key. B,D

Sol. $\sqrt{4-x^2} - \sqrt{1+x^2}$

$$= \frac{3-2x^2}{\sqrt{4-x^2}+\sqrt{1+x^2}}$$

$\therefore$ Sign scheme for $\left(\sqrt{4-x^2} - \sqrt{1+x^2}\right)$ is same as that of $3-2x^2$

Sign scheme for $3-2x^2$ is



$\therefore \quad f(x) = \sqrt{1+x^2}, -2 \le x \le -\sqrt{\frac{3}{2}}$

$= \sqrt{4-x^2}, -\sqrt{\frac{3}{2}} \le x \le 0$

$= \sqrt{1+x^2}, 0 < x \le \sqrt{\frac{3}{2}}$

$= \sqrt{4-x^2}, \sqrt{\frac{3}{2}} \le x \le 2$

Clearly f(x) is continuous at $x = -\sqrt{\frac{3}{2}}$ and $x = \sqrt{\frac{3}{2}}$ but it is discontinuous at x = 0

Also $f'(x) = \frac{x}{\sqrt{1+x^2}} - 2 \le x < -\sqrt{\frac{3}{2}}$

$= -\frac{x}{\sqrt{4-x^2}} - \sqrt{\frac{3}{2}} < x < 0$

$= \frac{x}{\sqrt{1+x^2}}, 0 < x < \sqrt{\frac{3}{2}}$

$= -\frac{x}{\sqrt{4-x^2}} \sqrt{\frac{3}{2}} < x \le 2$

F(x) is not differentiable at $x = \pm\sqrt{\frac{3}{2}}$ and also at x = 0 as it is discontinuous at x = 0.

50. If $f(x) = a\left|\sin^7 x\right| + be^{|x|} + c|x|^5$ and if f(x) is differentiable at x = 0, then which of the following in necessarily true

A) a = b = c = 0 B) a = 0, b = 0, $c \in R$

C) b = c = 0, $c \in R$ D) b 0 and a and $c \in R$

Key. D

Sol. $\therefore\ a\left|\sin^7 x\right|$ is differentiable at x = 0 and its d.e. is 0 for all $a \in R$ and $c|x|^5$ is differentiable at x = 0 and its d.e. is 0 for all $c \in R$.

But at x = 0, L.H. derivative of $be^{|x|} = -b$ and R.H. derivative = b

$\therefore$ for $be^{|x|}$ to be differentiable at x = 0, b = - b

$\Rightarrow \quad b = 0$

51. If [x] denotes the integral part of x and

$$f(x) = [x]\left\{\frac{\sin\frac{\pi}{[x+1]} + \sin\pi[x+1]}{1+[x]}\right\}$$; then

A) f(x) is continuous in R

B) f(x) is continuous but not differentiable in R

C) $f''(x)$ exists for all x in R

D) f(x) is discontinuous at all integral points in R

Key. D

Sol. $\sin\pi[x+1] = 0$.

Also [x + 1] = [x] + 1

$\therefore \quad f(x) = \frac{[x]}{1+[x]}\sin\frac{\pi}{[x]+1}$

at x = n, $n \in I,\ f(x) = \frac{n}{1+n}\sin\frac{\pi}{n+1}$

For n < x < n + 1, $n \in I$,

$$f(x) = \frac{n}{1+n}\sin\frac{\pi}{n+1}$$

For n – 1 < x < n, [x] = n – 1

$\therefore \quad f(x) = \frac{n-1}{n}\sin\frac{\pi}{n}$

Hence $\underset{x\to n=0}{Lt} f(x) = \frac{n-1}{n}\sin\frac{\pi}{4}$,

$f(n) = \frac{n}{1+n}\sin\frac{\pi}{n+1}$

$\therefore$ f(x) is discontinuous at all $n \in I$

52. In $x \in \left[0, \frac{\pi}{2}\right]$, let $f(x)\ \underset{n\to\infty}{Lt}\frac{2^x - x^n \sin x}{1+x^n}$, then

A) f(x) is a constant function

B) f(x) is continuous at x = 1

C) f(x) is discontinuous at x = 1

D) none of these

Key. C

Sol. $f(x) = \underset{n\to\infty}{Lt}\frac{2^x - x^n\sin x}{1+x^n}$

$$=\begin{cases} 2^x, & 0 \le x < 1 \\ \dfrac{2^x - \sin x}{2}, & x = 1 \\ -\sin x & x > 1 \end{cases}$$

Now $f(1) = \dfrac{2 - \sin 1}{2}$

$\underset{x \to 1-0}{\text{Lt}} f(x) = \underset{x \to 1-0}{\text{Lt}} 2^x = 2$

Hence f(x) is discontinuous at x = 1

53. Let f(x) = [cos x + sin x], 0 < x < 2 π, where [x] denotes the integral part of x, then the number of points of discontinuity of f (x) is

A) 3 B) 4 C) 5 D) 6

Key. C

Sol. $f(x) = \left[\sqrt{2}\cos\left(x - \dfrac{\pi}{4}\right)\right]$

But [x] is discontinuous only at integral points.

Also $-\sqrt{2} \le \sqrt{2}\cos\left(x - \dfrac{\pi}{4}\right) \le \sqrt{2}$

Integral values of $\sqrt{2}\cos\left(x - \dfrac{\pi}{4}\right)$ when

0 < x < 2π are

- 1, at $x = \pi, \dfrac{3\pi}{2}$

0, at $x = \dfrac{3\pi}{4}, \dfrac{7\pi}{4}$

1, at $x = \dfrac{\pi}{2}$

$\therefore$ In $(0, 2\pi), f(x)$ is discontinuous at $x = \dfrac{\pi}{2}, \dfrac{3\pi}{4}, \pi, \dfrac{3\pi}{2}, \dfrac{7\pi}{4}$.

54. If [x] denotes the integral part of x and in $(0, \pi)$, we define

$f(x) = \left[\dfrac{2(\sin x - \sin^n x) + |\sin x - \sin^n x|}{2(\sin x - \sin^n x) - |\sin x - \sin^n x|}\right]$. Then for n > 1.

A) f(x) is continuous but not differentiable at $x = \dfrac{\pi}{2}$

B) both continuous and differentiable at $x = \dfrac{\pi}{2}$

C) neither continuous nor differentiable at $x = \dfrac{\pi}{2}$

D) $\underset{x \to \frac{\pi}{2}}{\text{Lt}} f(x)$ exists but $\underset{x \to \frac{\pi}{2}}{\text{Lt}} f(x) \ne f\left(\dfrac{\pi}{2}\right)$

Key. B

Sol. For $0 < x < \frac{\pi}{2}$ or $\frac{\pi}{2} < x < \pi$,

$0 < \sin x < 1$

$\therefore$ for n > 1, $\sin x > \sin^4 x$

$\therefore$ $f(x) = \left[\frac{3(\sin x - \sin^n x)}{\sin x - \sin^n x}\right] = 3, x \neq \frac{\pi}{2}$

$= 3, x = \frac{\pi}{2}$

Thus in $(0, \pi), f(x) = 3$.

Hence f(x) is continuous and differentiable at $x = \frac{\pi}{2}$.

55. If [x] denotes the integral part of x and f(x) = [n + p sin x], 0 < x < π, $n \in I$ and p is a prime number, then the number of points where f(x) is not differentiable is

A) p – 1 B) p C) 2p – 1 D) 2p + 1

Key. C

Sol. [x] is not differentiable at integral points.

Also [n + p sin x] = n + [p sin x]

$\therefore$ [p sin x] is not differentiable, where

P sin x is an integer. But p is prime and 0 < sin $x \leq 1 [\because 0 < x < \pi]$

$\therefore$ p sin x is an integer only when

$\sin x = \frac{r}{p}$, where 0 < r $\leq$ p and $r \in N$

For r = p, sin x = 1 $\Rightarrow x = \frac{\pi}{2}$ in $(0, \pi)$

For 0 < r < p, sin $x = \frac{r}{p}$

$\therefore$ $x = \sin^{-1}\frac{r}{p}$ or $\pi - \sin^{-1}\frac{r}{p}$

Number of such values of

x = p – 1 + p – 1 = 2p – 2

$\therefore$ Total number of points where f(x) is not differentiable = 1 + 2p – 2 = 2p – 1

56. Let $f(x)$ and $g(x)$ be two differentiable functions, defined as

$f(x) = x^2 + x g'(1) + g''(2)$ and $g(x) = f(1)x^2 + x f'(x) + f''(x)$.

The value of $f(1) + g(-1)$ is

A) 0 B) 1 C) 2 D) 3

Key. C

Sol. $f(x) = x^2 + xg'(1) + g''(2)$

$f'(x) = 2x + g'(1)$

$f''(x) = 2$

$f'''(x) = 0$

and $g(x) = f(1)x^2 + xf'(x) + f''(x)$

$g(x) = f(1)x^2 + x\{2x + g'(1)\} + 2$

$= f(1)x^2 + 2x^2 + xg'(1) + 2 \;= x^2\{2 + f(1)\} + xg'(1) + 2$

$g'(x) = 2x\{2 + f(1)\} + g'(1)$

$g''(x) = 2\{2 + f(1)\}$

$\therefore f(1) + g(-1)$

$= 1 + g'(1) + g''(2) + f(1).(-1)^2 + f'(-1)(-1) + f''(-1)$

$[\because g'(2) = 4 + 2f(1)$

$f''(-1) = 2$

$f'(-1) = 1 - g'(1) + g''(2)]$

$= 1 + g'(1) + 4 + 2f(1) + f(1) - \{1 - g'(1) + g''(2)\} + 2$

$= 6 + 2g'(1) + 3f(1) - g''(2)$

$= 6 + 2g'(1) + 3f(1) - \{4 + 2f(1)\} \;= 2 + f(1) + 2g'(1)$

$f(x) = x^2 + xg'(1) + g''(2)$

$f'(x) = 2x + g'(1)$

$f''(x) = 2$

$f'''(x) = 0$

$f^{iv}(x) = 0$

$g(x) = f(1)x^2 + x.f'(x) + f''(x)$

$g'(x) = 2f(1)x + x.f''(x) + f'(x).1 + f'''(x)$

$g''(x) = 2f(1) + x.f'''(x) + f''(x).1 + f''(x) + f^{iv}(x)$

$\therefore g'(x) = 2f(1)x + 2x + 2x + g'(x) + 0$

$g'(x) = \{2f(1) + 4\}x + g'(x)$

$g''(x) = 2f(1) + 0 + 2 + 2 + 0$

$g''(x) = 4 + 2f(1)$

$\therefore f(1)+g(-1)$

$= 1+g'(1)+g''(2)+1+(-1)g'(-1)+g''(2)$

$= 2+2g''(2)+g'(1)-g'(-1)$

$= 2+2\{4+2f(1)\}+0 \quad [\because g'(1)=g'(-1)]$

$= 2+2\{0\}+(0)=2$

57. Let f(x) be a real function not identically zero, such that
$f\left(x+y^{2n+1}\right)=f(x)+\{f(y)\}^{2n+1}; n\in N$ and x,y are real numbers and $f'(0)\ge 0$. Find the values of f(5) and $f'(10)$.

Sol. As in the preceding example, $f'(x)=0$ or $\{f(x)\}^{2n}=x^{2n}\Rightarrow f(x)=f(0)=0$ or f(x) = x.
But f(x) is given to be not identically zero.
$\therefore$ f(x) = 0 is inadmissible. Hence f(x) = x.
$\therefore$ f(x) = 5 and $f'(10)=1$.

58. If $f(x)+f(y)=f\left(\frac{x+y}{1-xy}\right)$ for all $x,y\in R$ and $xy\ne 1$ and $\underset{x\to 0}{Lt}\frac{f(x)}{x}=2$, find $f\left(\sqrt{3}\right)$ and $f'(-2)$.

Sol. Given that $f(x)+f(y)=f\left(\frac{x+y}{1-xy}\right)$.
Putting x = 0, y = 0, we have f(0) = 0.
Differentiating both sides with respect to x, treating y as constant, we get

$$f(x)+0=f'\left(\frac{x+y}{1-xy}\right)\left\{\frac{(1-xy).1-(x+y).(-y)}{(1-xy)^2}\right\}$$

$$=f'\left(\frac{x+y}{1-xy}\right)\left\{\frac{1-xy+xy+y^2}{(1-xy)^2}\right\}=f'\left(\frac{x+y}{1-xy}\right)\left\{\frac{1+y^2}{(1-xy)^2}\right\} \quad ...(1)$$

Similarly differentiating both sides with respect to y, keeping x as constant, we get

$$f'(y)=f'\left(\frac{x+y}{1-xy}\right)\left\{\frac{1+x^2}{(1-xy)^2}\right\} \quad ...(2)$$

From (1) and (2), we get

$$\frac{f'(x)}{f'(y)}=\frac{1+y^2}{1+x^2}\Rightarrow\left(1+x^2\right)f'(x)=\left(1+y^2\right)f'(y)=k(say)\{=f'(0)\}$$

$$\Rightarrow \quad f'(x)=\frac{k}{1+x^2}\Rightarrow f(x)=k\int\frac{1}{1+x^2}dx=k\tan^{-1}x+\alpha.$$

Putting x = 0, we have $f(0)=k\times 0+\alpha\Rightarrow\alpha=0, \because f(0)=0$.

Thus $f(x)=k\tan^{-1}x$.

Again $\frac{f(x)}{x}=k\frac{\tan^{-1}x}{x}\Rightarrow\underset{x\to 0}{Lt}\frac{f(x)}{x}=k\ Lt\frac{\tan^{-1}}{x}\Rightarrow 2=k\times 1\Rightarrow k=2$.

Hence f(x) = 2 $\tan^{-1} x$.

$\therefore\ f\left(\sqrt{3}\right) = 2\tan^{-1}\left(\sqrt{3}\right) = 2 \times \frac{\pi}{3} = \frac{2\pi}{3}$ and $f'(-2) = \frac{2}{1+(-2)^2} = \frac{2}{5}$.

59. If $2f(x) = f(xy) + f\left(\frac{x}{y}\right)$ for all $x, y \in R^+$, $f(1) = 0$ and $f'(1) = 1$, find f(e) and $f'(e)$.

Sol. Given $2f(x) = f(xy) + f\left(\frac{x}{y}\right)$.

Differentiating partially with respect to x (keeping y as constant), we get

$$2f'(x) = f'(xy).\ y + f'\left(\frac{x}{y}\right).\frac{1}{y} \qquad ...(1)$$

Again, differentiating partially with respect to y (keeping x as constant), we get

$$0 = f'(xy).x + f'\left(\frac{x}{y}\right).x\left(-\frac{1}{y^2}\right) \qquad ...(2)$$

(2) $\Rightarrow$ $\frac{x}{y^2} f'\left(\frac{x}{y}\right) = xf'(xy) \Rightarrow f'\left(\frac{x}{y}\right) = y^2 f'(x)$.

Hence from (1), $2f'(x) = yf'(xy) = 2f'(xy) \Rightarrow f'(x) = yf'(xy)$.

Now, putting x = 1, we have $yf'(y) = f'(1) = 1$.

$\Rightarrow$ $f'(y) = \frac{1}{y} \Rightarrow \int f'(y)dy = \int \frac{1}{y}dy \Rightarrow f(y) = \log y + c$.

Putting y = 1, we have f(1) = 0 + c $\Rightarrow$ 0 = c; Q f (1) = 0

$\therefore$ c = 0.

Hence f(y) = log y i.e. f(x) = log x (x > 0).

Hence f(e) = log e = 1 and $f'(e) = \frac{1}{e}$

60. A function y = f(x) is defined for all $x \in [0,1]$ and f(x) + f(y) $= f\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$.

And $f(0) = \frac{\pi}{2}, f\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$ Find the function y = f(x)

Sol. Given f(x) + f(y) $= f\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$ ...(1)

Differentiating partially with respect to x (treating y as constant), we get

$$f'(x) + 0 = f'\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right) \times \left\{y - \sqrt{1-y^2}, \frac{-2x}{2\sqrt{1-x^2}}\right\}$$

$$\Rightarrow \quad f'(x) = f'\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right) \times \left\{\frac{y\sqrt{1-x^2} + x\sqrt{1-y^2}}{\sqrt{1-x^2}}\right\} \qquad ...(2)$$

Similarly, differentiating (2) partially with respect to y (treating x as constant), we get

$$f'(y)f'\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right)\times\left\{\frac{x\sqrt{1-y^2}+y\sqrt{1+x^2}}{\sqrt{1-y^2}}\right\} \quad ...(3)$$

Now, dividing (2) by (3), we get

$$\frac{f'(x)}{f'(y)}=\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}\Rightarrow\sqrt{1-x^2}f'(x)=\sqrt{1-y^2}f'(y)=k \text{ (say)}$$

Thus, $\sqrt{1-x^2}\,f'(x)=k\Rightarrow f'(x)=\frac{k}{1-x^2}$

$\Rightarrow \quad \int f'(x)dx=k\int\frac{1}{\sqrt{1-x^2}}\,dx\Rightarrow f(x)=k\sin^{-1}x+\alpha$ ...(4)

Now, x = 0 $\Rightarrow$ f(0) = k. $0+\alpha\Rightarrow\frac{\pi}{2}=\alpha$.

Again $x=\frac{1}{\sqrt{2}}\Rightarrow f\left(\frac{1}{\sqrt{2}}\right)=k\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)+\alpha$

$\Rightarrow \quad \frac{\pi}{4}=k\frac{\pi}{4}=\alpha\Rightarrow\frac{\pi}{4}=k\frac{\pi}{4}+\frac{\pi}{2},\ Q\ \alpha=\frac{\pi}{2}$

$\Rightarrow \quad k\frac{\pi}{4}=\frac{\pi}{4}-\frac{\pi}{2}=-\frac{\pi}{4}\Rightarrow k=-1.$

Hence putting k = - 1 and $\alpha=\frac{\pi}{2}$ in (4), we get $f(x)=-\sin^{-1}x+\frac{\pi}{2}=\cos^{-1}x$.

61. Let $f(x)=\underset{n\to\infty}{\text{Lt}}\sum_{r=0}^{n-1}\frac{x}{(rx+1)\{(r+1)x+1\}}$, then

A) f(x) is continuous but not differentiable at x = 0

B) f(x) is both continuous and differentiable at x = 0

C) f(x) is neither continuous not differentiable at x = 0

D) f(x) is a periodic function

Key. C

Sol. $t_{r+1}=\frac{x}{(rx+1)\{(r+1)x+1\}}$

$=\frac{(r+1)x+1-(rx+1)}{(rx+1)[(r+1)x+1]}$

$=\frac{1}{(rx+1)}-\frac{1}{(r+1)x+1}$

$\therefore\ S_n=\sum_{r=0}^{n-1}t_{r+1}\frac{1}{nx+1}$

$=1,\ x\neq 0$

= 0, x = 0

$\therefore \quad \underset{n\to\infty}{\text{Lt}}S_n=\underset{n\to\infty}{\text{Lt}}\left(1-\frac{1}{nx+1}\right)$

Thus, f(x) is neither continuous nor differentiable at x = 0.

Clearly f(x) is not a periodic function.

62. If f(x) is a polynomial function which satisfy the relation

$(f(x))^2 f'''(x) = (f''(x))^3 f'(x)$, $f'(0) = f'(1) = f'(-1) = 0$, $f(0) = 4$, $f(\pm 1) = 3$, then $f''(i)$ (where $i = \sqrt{-1}$) is equal to

(A) 10 (B) 15

(C) –16 (D) –15

Key. C

Sol. Solving the equation

We will get $f(x) = x^4 - 2x^2 + 4$

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(A) 10 (B) 15

(C) –16 (D) –15

Key. C

Sol. Solving the equation

We will get $f(x) = x^4 - 2x^2 + 4$

65. Let a function f(x) be such that $f''(x) = f'(x) + e^x$ and $f(0) = 0$, $f'(0) = 1$, then $\ln\left(\frac{(f(2))^2}{4}\right)$ equal to

(A) $\frac{1}{2}$ (B) 1

(C) 2 (D) 4

Key. D

Sol. $f''(x) - f'(x) = e^x$

put $f'(x) = v$

$$\frac{dv}{dx} + v(-1) = e^x$$

$$\Rightarrow \quad ve^{-x} = \int e^x . e^{-x} dx$$

$ve^{-x} = x + C_1$, $f'(0) = 1 \Rightarrow C_1 = 1$

$f'(x) = xe^x + e^x$

$f(x) = xe^x + C_2$

$\Rightarrow$ f(0) = 0 $\Rightarrow C_2 = 0$

$\Rightarrow f(x) = xe^x \Rightarrow f(2) = 2e^2$

$\ln\left(\frac{(f(2))^2}{4}\right) = 4.$

66. If $\int_{\sin x}^{1} t^2.f(t)dt = 1 - \sin x, \forall x \in \left(0, \frac{\pi}{2}\right)$ then the value of $f\left(\frac{1}{\sqrt{3}}\right)$ is

(A) $\frac{1}{\sqrt{3}}$ (B) $\sqrt{3}$

(C) $\frac{1}{3}$ (D) 3

Key. D

Sol. $\int_{\sin x}^{1} t^2.f(t)dt = 1 - \sin x$

Differentiating both sides with respect to 'x'

$0 - \sin^2 x.f(\sin x).\cos x = -\cos x \Rightarrow \cos x\left[1 - \sin^2 x.f(\sin x)\right] = 0$

But $\cos x \neq 0$

So, $f(\sin x) = \frac{1}{\sin^2 x}$

$f\left(\frac{1}{\sqrt{3}}\right) = 3$

67. Let $f:(0,\infty) \to R$ and $F(x) = \int_1^x f(t)dt$. If $F(x^2) = x^2(1 + x)$ then f(4) equals

(A) 5/4 (B) 7 (C) 4 (D) 2

Key. C

Sol. $F'(x) = f(x)$

$F(x) = x(1 + \sqrt{x}) = x + x^{3/2}$

$\therefore F'(x) = f(x) = 1 + \frac{3}{2}\sqrt{x}$

$\therefore$ f(4) = 4

68. If $f(x) = \int_0^x (1 + t^3)^{-1/2} dt$ and g (x) is the inverse of f, then the value of $\frac{g''(x)}{g^2(x)}$ is

(A) 3/2 (B) 2/3 (C) 1/3 (D) 1/2

Key. A

Sol. $f(x)=\int_0^x \left(1+t^3\right)^{-1/2} dt$

i.e. $f\left[g(x)\right]=\int_0^{g(x)} \left(1+t^3\right)^{-1/2} dt$

i.e. $x=\int_0^{g(x)} \left(1+t^3\right)^{-1/2} dt$ [∵ g is inverse of f $\Rightarrow$ f[g(x)] = x]

Differentiating with respect to x, we have

$1 = (1 + g^3)^{-1/2} . g'$

i.e. $(g')^2 = 1 + g^3$

Differentiating again with respect to x, we have

$2g'g''=3g^2g'$

gives $\frac{g''}{g^2}=\frac{3}{2}$

69. If $f(x)$ be positive, continuous and differentiable on the interval (a,b). If $\lim\limits_{x\to a^+} f(x)=1$ and $\lim\limits_{x\to b^-} f(x)=3^{1/4}$ also $f'(x)>\left(f(x)\right)^3+\frac{1}{f(x)}$ then

a) $b-a>\frac{\pi}{24}$ b) $b-a<\frac{\pi}{24}$ c) $b-a=\frac{\pi}{12}$ d) $b-a=\frac{\pi}{24}$

Key. B

Sol. $\frac{f'(x)f(x)}{f(x)^4+1}>1$

Integrating both sides with respect to "x" from a to b

$\Rightarrow \frac{1}{2}\left[\tan^{-1}\left(\left(f(x)\right)^2\right)\right]_a^b>(b-a)$

$\Rightarrow \frac{1}{2}\left\{\pi/3-\pi/4\right\}>(b-a)$

$\Rightarrow b-a<\frac{\pi}{24}$

70. $f(x)=\left(\tan\left(\frac{\pi}{4}+x\right)\right)^{\frac{1}{x}}$ if $x\neq 0$; $=\lambda$ if $x=0$ is continuous at x = 0 then value of λ is

1) 1 2) e 3) e^2 4) 0

Key. 3

Sol. $\lambda=\lim\limits_{x\to 0}\left(\frac{1+\tan x}{1-\tan x}\right)^{\frac{1}{x}}=\frac{e}{e^{-1}}=e^2$

71. $f(x)=\frac{1}{q}$ If $x=\frac{p}{q}$ where p and q are integer and $q\neq 0$, G.C.D of (p,q) = 1 and $f(x)=0$

If x is irrational then set of continuous points of $f(x)$ is

1) all real numbers 2) all rational numbers 3) all irrational number 4) all integers

Key. 3

Sol. Let $x=\frac{p}{q}$

$f(x)=\frac{1}{q}$

When $x\to\frac{p}{q}$ $f(x)=0$ for every irrational number $\in nbd(p/q)$

$=\frac{1}{n}\ if\ n=\frac{m}{n}\in nbd(p/q)$

$\frac{1}{n}\to 0\ as\ n\to\infty$ since

There ∞ - number of rational $\in nbd(p/q)$

$\therefore \lim\limits_{x\to\frac{p}{q}} f(x)=0$ but $f\left(\frac{p}{q}\right)=\frac{1}{q}\neq 0$

Discontinuous at every rational

If $x=\alpha$ is irrational $\Rightarrow f(\alpha)=0$

Now $\lim\limits_{x\to\alpha} f(x)$ is also O

$\therefore$ continuous for every irrational α

72. If a function $f:[-2a,2a]\to R$ is an odd function such that $f(x)=f(2a-x)$ for $x\in[a,2a]$ and the left hand derivative at x=a is zero then left hand derivative at $x=-a$ is____

a) a b) 0 c) –a d) 1

Key. B

Sol. LHD at x = –a is $\lim\limits_{h\to 0}\frac{f(-a)-f(-a-h)}{h}=-\lim\limits_{h\to 0}\frac{f(a)-f(2a-a+h)}{h}$

$=-\lim\limits_{h\to 0}\frac{f(a)-f(a-h)}{h}=0$ by hypothesis

73. Let $f(x)=\begin{cases}x^n\sin\left(\frac{1}{x}\right), & x\neq 0\\ 0; & x=0\end{cases}$,then f(x) is continuous but not differentiable at x = 0 if

a) $n\in(0,1]$ b) $n\in[1,\infty)$ c) $n\in(-\infty,0)$ d) n = 0

Key. A

Sol. $\lim\limits_{x\to 0}x^n\sin\frac{1}{x}=0$ for n > 0 $\therefore$ continuous for n > 0 Similarly f(x) is non-differentiable for $n\leq 1$

$\therefore n \in (0,1]$ for f(x) to be continuous and non-differentiable at x = 0.

74. If $f(x)$ is continuous on [-2,5] and differentiable over (-2,5) and -4 $\le f'(x) \le 3$ for all x in (-2,5) then the greatest possible value of $f(5) - f(-2)$ is

a) 7 b) 9 c) 15 d) 21

Key. D

Sol. Using LMVT in [–2, 5]

$$\frac{f(5)-f(-2)}{5-(-2)} = f^1(c); c \in (-2,5)$$

$\therefore f(5)-f(-2) = 7f^1(c) \le 21$ Since $-4 \le f^1(x) \le 3$

$\therefore \max\{f(5)-f(-2)\} = 21$

75. If [.] denotes the integral part of x and $f(x) = [x]\left\{\frac{\sin\frac{\pi}{[x+1]} + \sin\pi[x+1]}{1+[x]}\right\}$, then

(A) f(x) is continuous in R

(B) f(x) is continuous but not differentiable in R

(C) f'(x) exists $\forall x \in R$

(D) $f(x)$ is discontinuous at all integral points in R

Key: D

Hint: At x = n, $f(n) = \frac{n}{n+1}\sin\left(\frac{\pi}{n+1}\right) = f(n^+)$

$f(n) = \frac{n-1}{n}\sin\frac{\pi}{n}$

$\Rightarrow$ f(x) is discontinuous at all $n \in 1$

76. If $f(x) = \begin{cases} x, & x \le 1 \\ x^2 + bx + c, & x > 1 \end{cases}$ and $f(x)$ is differentiable for all $x \in R$, then

a) $b = -1, c \in R$ b) $c = 1, b \in R$ c) $b = 1, c = -1$ d) $b = -1, c = 1$

Key. 4

Sol. $Lf'(1) = 1,\ Rf'(1) = 2 + b$ $\Rightarrow b = -1$

$f(1-) = 1$ AND $f(1+) = 1 + b + c$ $\Rightarrow c = 1$

77. If $f(x) = \begin{cases} x^m \sin\frac{1}{x} & x \ne 0 \\ 0 & x = 0 \end{cases}$ then the interval in which m lies so that $f(x)$ is both continuous and differentiable at $x = 0$ is

a) ¡ b) $(0,\infty)$ c) $(0,1]$ d) $(1,\infty)$

Key. 4

Sol. $\underset{x\to 0}{Lt}\, f(x)=\underset{x\to 0}{Lt}\, x^m \sin\frac{1}{x}$ exists if m > 0 I.E., $m\in[0,\infty)$

$$f'(0)=\underset{x\to 0}{Lt}\frac{f(x)-f(0)}{x-0}=\underset{x\to 0}{Lt}\, x^{m-1}\sin\frac{1}{x}$$ EXISTS IF M – 1 > 0 IF M > 1 OR $m\in(1,\infty)$

78. $f(x)=Max\ \{x,\ x^3\}$, then at x = 0

a) $f(x)$ is both continuous and differentiable

b) $f(x)$ is neither continuous nor differentiable

c) $f(x)$ is continuous but not differentiable

d) $f(x)$ is differentiable but not continuous

Key. 3

Sol. $\mathrm{f}(\mathrm{x})=\begin{cases}\mathrm{x} & 0\le \mathrm{x}\le 1\\ \mathrm{x}^3 & -1\le \mathrm{x}\le 0\end{cases}$ $\mathrm{f}(0+)=0$ $\mathrm{f}(0-)=0=\mathrm{f}(0)$ $\mathrm{Lf}'(0)=0$ $\mathrm{Rf}'(0)=1$

79. $f(x)=\begin{cases}\left(\dfrac{e^{\frac{1}{x}}-e^{-\frac{1}{x}}}{e^{\frac{1}{x}}+e^{-\frac{1}{x}}}\right) & x\ne 0\\ 0 & x=0\end{cases}$ then at x = 0

a) $f(x)$ is both continuous and differentiable

b) $f(x)$ is neither continuous nor differentiable

c) $f(x)$ is continuous but not differentiable

d) $f(x)$ is differentiable but not continuous

Key. 2

Sol. $\underset{x\to 0+}{Lt}\, e^{-\frac{1}{x}}=0$, $\underset{x\to 0-}{Lt}\, e^{\frac{1}{x}}=0$ $f(0-)=\underset{x\to 0-}{Lt}\left(\dfrac{e^{\frac{2}{x}}-1}{e^{\frac{2}{x}+1}}\right)=\underset{x\to 0-}{Lt}\left(\dfrac{0-1}{0+1}\right)=-1$

$f(0+)=\underset{x\to 0+}{Lt}\left(\dfrac{1-e^{-\frac{1}{x}}}{1+e^{-\frac{1}{x}}}\right)=1$ $\underset{\mathrm{x}\to 0}{\mathrm{Lt}}\,\mathrm{f}(\mathrm{x})$ DOES NOT EXIST

80. If $f\left(\dfrac{x+2y}{3}\right)=\dfrac{f(x)+2f(y)}{3}\ \forall x,y\in R$ and $f^{'}(0)=1$; then $f(x)$ is

a). a second degree polynomial in x b). Discontinuous $\forall x \in R$

c). not differentiable $\forall x \in R$ d). a linear function in x

Key. 4

Sol. We have $f\left(\frac{x+2y}{3}\right)=\frac{f(x)+2f(y)}{3} \forall x, y \in R$ → (1) replacing x by $3x$ and putting $y=0$ in (1),

we get $f(x)=\frac{f(3x)+2f(0)}{3}$. $\Rightarrow f(3x)=3f(x)-2f(0) \to (2)$

. Now, $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ = $\lim_{h\to 0}\frac{f\left(\frac{3x+2.\frac{3h}{2}}{3}\right)-f(x)}{h}$

= $\lim_{h\to 0}\frac{\frac{f(3x)+2.f\left(\frac{3h}{2}\right)}{3}-f(x)}{h}$ (from (1))

= $\lim_{h\to 0}\frac{f(3x)+2f\left(\frac{3h}{2}\right)-3f(x)}{3h}$ = $\lim_{h\to 0}\frac{2f\left(\frac{3h}{2}\right)-2f(0)}{3h}$ (from(2))

= $\lim_{h\to 0}\frac{f\left(\frac{3h}{2}\right)-f(0)}{\frac{3h}{2}}=f'(0)=1$ (given) $\Rightarrow f'(x)=1 \Rightarrow f(x)=x+c$. $\therefore f(x)$ is a linear function in x, continuous $\forall x \in R$ and differentiable $\forall x \in R$. $\therefore$ Only 4 is correct option

81. Let f be a function defined by $f(x)=2^{|\log_2 x|}$, then at $x = 1$

(A) f is continuous as well as differentiable (B) continuous but not differentiable

(C) differentiable but not continuous (D) neither continuous nor differentiable

Key. B

Sol. $f(x)=\begin{cases}1/x, & 0<x<1\\ x, & x\ge 1\end{cases}$, f is continuous

$f'(x)=\begin{cases}-1/x^2, & 0<x<1\\ 1, & x>1\end{cases}$, f is not differentiable at x = 1.

82. If the function $f(x)=\left[\frac{(x-2)^3}{a}\right]\sin(x-2)+a\cos(x-2)$ [.] GIF, is continuous and differentiable in (4, 6), then a belongs

A) [8, 64] B) (0, 8] C) $(64,\infty)$ D) (0, 64)

Key. C

Sol. $a > (x-2)^3$

$8 \le (x-2)^3 \le 64 \Rightarrow a > 64$

83. The equation $x^7 + 3x^3 + 4x - 9 = 0$ has

A) no real root B) all its roots real

C) a unique rational root D) a unique irrational root

Key. D

Sol. Let $f(x) = x^7 + 3x^3 + 4x - 9$

$f^1(x) = 7x^6 + 9x^2 + 4 > 0 \;\; \forall x \in R$

$\therefore f$ is strictly increasing.

$\therefore f(x) = 0$ has a unique real root.

$f(1)f(2) < 0$

$\therefore$ The real root belongs to the interval (1, 2). If $f(x) = 0$ has rational roots, they must be integers. But there are no integers between 1 and 2.

84. A function $f : R \to R$ is such that $f(0) = 4, f^1(x) = 1$ in $-1 < x < 1$ and $f^1(x) = 3$ in $1 < x < 3$. Also f is continuous every where. Then $f(2)$ is

A) 5 B) 7 C) 8 D) Can not be determined

Key. C

Sol. If $-1 < x < 1$ then $f(x) = x + 4$

If $1 < x < 3$ then $f(x) = 3x + c$

But f is continuous at $x = 1$

$\therefore f(1) = 1 + 4 = 3 + c \Rightarrow c = 2$ and $f(1) = 5$

$\therefore f(2) = 8$

85. $f(x) = a|\sin x| + be^{|x|} + c|x|^3$. If $f(x)$ is differentiable at x = 0, then

a) $a + b + c = 0$ b) $a + b = 0$ and c can be any real number

c) $b = c = 0$ and a can be any real number d) $c = a = 0$ and b can be any real number.

Key. B

Sol. $f(x) = -a\sin x + be^{-x} - cx^3, x \le 0$

$= a\sin x + be^{x} + cx^3, x \ge 0$

Clearly continuous at 0, for differentiability $-a - b = a + b$

86. Let $f : [0,1] \to [0,1]$ be a continuous function. The equation $f(x) = x$

a) will have at least one solution. b) will have exactly two solutions.

c) will have no solution d) None of these

Key. A

Sol. $g(x) = f(x) - x$

$g(0)g(1) = f(0)(f(1) - 1) \le 0$

87. The value of $f(0)$, so that the function $f(x) = \dfrac{1 - \cos(1 - \cos x)}{x^4}$ is continuous everywhere, is

a) 1/8 b) 1/2 c) 1/4 d) 1/16

Key. A

Sol. $f\left(0\right)=\lim\limits_{h\to 0}\frac{1-\cos(1-\cos h)}{h^4}\times\frac{1+\cos(1-\cos h)}{1+\cos(1-\cos h)}$

$$=\lim_{h\to 0}\frac{\sin^2(1-\cos h)}{h^4.(1+\cos(1-\cos h)}.\frac{(1-\cos h)^2}{(1-\cos h)^2}$$

$$=\lim_{h\to 0}\left[\frac{\sin\,(1-\cos h)}{(1-\cos h)}\right]^2\times\lim_{h\to 0}\left(\frac{1-\cos h}{h^2}\right)^2\times\lim_{h\to 0}\frac{1}{1+\cos(1-\cos h)}$$

$$=\left(1\right)^2\times\frac{1}{4}\times\frac{1}{2}=\frac{1}{8}.$$

88. Let $f(x+y)=f(x)f(y)$ for all *x* and *y*. Suppose that *f* (3) = 3 and $f'(0)=11$ then $f'(3)$ is given by

a) 22 b) 44 c) 28 d) 33

Key. D

Sol. $\because f'\left(x\right)=\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to 0}\frac{f(x)f(h)-f(x)}{h}$$

$$=f(x)\lim_{h\to 0}\frac{f(h)-1}{h}$$

$=f(x)f'(0)$ since $1=f\left(0\right)$ [By putting $x=3, y=0$, we can show that $f\left(0\right)=1$]

$$f'(3)=f(3)f'(0)$$

$$=3\times 11=33.$$

89. Let $f(x)=\left[\cos x+\sin x\right], 0<x<2\pi,$ where $\left[x\right]$ denotes the greatest integer less than or equal to x.

The number of points of discontinuity of $f(x)$ is

a) 6 b) 5 c) 4 d)3

Key. B

Sol. $\left[\cos x+\sin x\right]=\left[\sqrt{2}\cos(x-\pi/4\right]$

We know that $\left[x\right]$ is discontinuous at integral values of *x*,

Now, $\sqrt{2}\cos(x-\pi/4)$ is an integer.

at $x=\pi/2,\ 3\pi/4,\ \pi, 3\pi/2, 7\pi/4$

90. The function f defined by $f(x)=\begin{cases}\frac{1}{2} if\ x\ is\ rational\\ \frac{1}{3} if\ x\ is\ Irrational\end{cases}$

(a) Discontinuous for all x

(b) Continuous at x = 2

(c) Continuous at $x=\frac{1}{2}$

(d) Continuous at x = 3

Key. A

Sol. If x is Rational any interval there lie many rationals as well as infinitely many Irrationals

$\therefore \forall n\in N\ \exists$ an Irrational number x_n such that $x-\frac{1}{n}<x_n<x+\frac{1}{n}\Rightarrow\left|x_n-x\right|<\frac{1}{n}, \forall n$

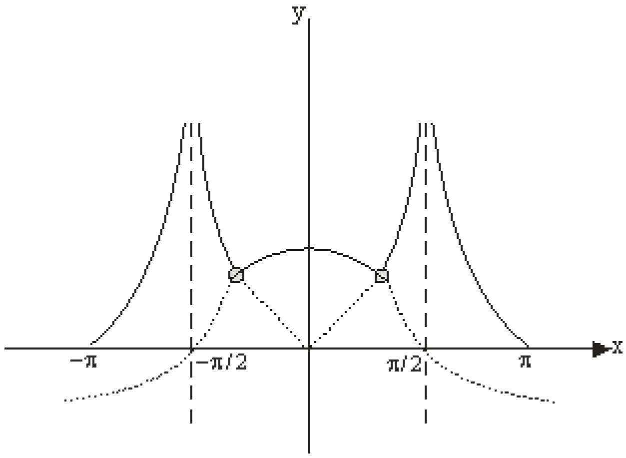
$\Rightarrow \underset{n\to\infty}{Lt}\ f(x_n)=\frac{1}{3}$, Similarly in case of Irrational

91. Number of points where the function f(x) = max (|tan x|, cos |x|) is non differentiable in the interval $(-\pi, \pi)$ is

A) 4 B) 6 C) 3 D) 2

Key. A

Sol. The function is not differentiable and continuous at two points between $x = -\pi/2 \& x = \pi/2$ also function is not continuous at $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$ hence at four points function is not differentiable

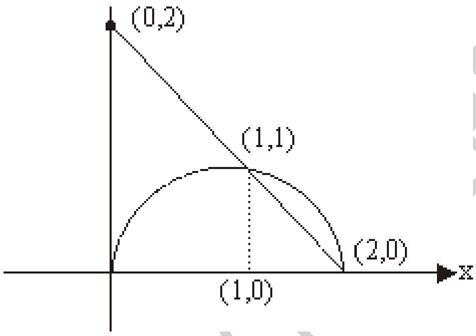


92. The function f(x) = maximum $\left\{\sqrt{x(2-x)}, 2-x\right\}$ is non-differentiable at x equal to

A) 1 B) 0.2 C) 0, 1 D) 1,2

Key. D

Sol.

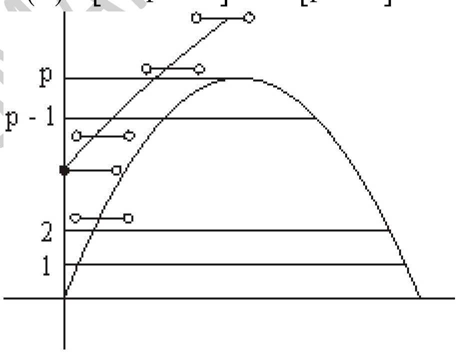


93. Let $f(x) = [n + p\sin x], x \in (0, \pi), n \in Z, p$ is a prime number and [x] is greatest integer less than or equal to x. The number of points at which f(x) is not differentiable is

A) p B) p – 1 C) 2p + 1 D) 2p – 1

Key. D

Sol. $f(x) = [n + p\sin x] = n + [p\sin x]$



$$[p\sin x]=\begin{cases}0 & 0\le \sin x<\frac{1}{p}\\ 1 & \frac{1}{p}\le \sin x<\frac{2}{p}\\ 2 & \frac{2}{p}\le \sin x<\frac{3}{p}\\ p-1 & \frac{p-1}{p}\le \sin x<1\\ p & \sin x=1\end{cases}$$

$\therefore$ Number of points of discontinuituy are 2 (p – 1) + 1 = 2p – 1 else where it is differentiable and the value = 0

94. Let $f:R\to R$ be any function and $g(x)=\frac{1}{f(x)}$. Then g is

A) onto if f is onto
B) one-one if f is one-one
C) continuous if f is continuous
D) differentiable if f is differentiable

Key. B

Sol. $f:R\to R,\quad g(x)=\frac{1}{f(x)}$

$g'(x)=-\frac{1}{f(x)^2}.f'(x)$

$\Rightarrow$ g is one – one if f is one – one

95. If f(x) = [x] (sin kx)^p is continuous for real x, then

A) $k\in\{n\pi,\ n\in I\},p>0$
B) $k\in\{2n\pi,\ n\in I\},p>0$
C) $k\in\{n\pi,n\in I\},p\in R-\{0\}$
D) $k\in\{n\pi,\ n\in I,n\ne 0\},\ p\in R-\{0\}$

Key. A

Sol. f(x) = [x] (sin kx)^p

$(\sin kx)^p$ is continuous and differentiable function $\forall\ x\in R,\ k\in R$ and p > 0.

[X] is discoutinuous at $x\in I$

For k = n π, n $\in$ I

$f(x)=[x](\sin(n\pi x))^p$

$\lim_{x\to a} f(x)=0,\ a\in I$

and f(a) = 0

So. f(x) becomes coutinuous for all $x\in R$

96. $f(x)=\begin{cases}x+2 & x<0\\ -x^2-2 & 0\le x<1\\ x & x\ge 1\end{cases}$

Then the number of points of discontinuity of |f(x)| is

A) 1
B) 2
C) 3
D) none of these

Key. A

Sol. $f(x)=\begin{cases}x+2 & x<0\\ -x^2-2 & 0\le x<1\\ x & x\ge 1\end{cases}$

$$\therefore \qquad |f(x)| = \begin{cases} -x-2 & x<-2 \\ x+2 & -2 \le x < 0 \\ x^2+2 & 0 \le x < 1 \\ x & x \ge 1 \end{cases}$$

Discontinuous at x = 1

$\therefore$ number of points of discount. 1

97. $f(x) = \begin{cases} \dfrac{e^{e/x} - e^{-e/x}}{e^{1/x} + e^{-1/x}} & , \ x \ne 0 \\ x & , \ x = 0 \end{cases}$

A) f is continuous at x, when k = 0

B) f is not continuous at x = 0 for any real k.

C) $\lim\limits_{x \to 0} f(x)$ exist infinitely

D) None of these

Key. B

Sol. $\lim\limits_{x \to 0^+} \dfrac{e^{e/x} - e^{-e/x}}{e^{1/x} + e^{-1/x}} = \lim\limits_{x \to 0^+} \dfrac{e^{\frac{e-1}{ex}}\left(1 - e^{-2e/x}\right)}{\left(1 + e^{-2/x}\right)} = +\infty$

$\lim\limits_{x \to 0^-} \dfrac{e^{e/x} - e^{-e/x}}{e^{1/x} + e^{-1/x}} = \lim\limits_{x \to 0^-} \dfrac{e^{-e/x}\left(e^{2e/x} - 1\right)}{e^{-e/x}\left(e^{+2/x} + 1\right)} = \lim\limits_{x \to 0^-} e^{-\left(\frac{e-1}{x}\right)}\left(\dfrac{e^{2e/x} - 1}{e^{2/x} + 1}\right) = -\infty$

Limit doesn't exist So f(x) is discoutinous

98. The correct statement for the function $f(x) = \begin{cases} x & , \quad x \in Q \\ -x & , \quad x \in R \sim Q \end{cases}$ IS

A) continuous every where

B) f(x) is a periodic function

C) discontinuous everywhere except at x = 0

D) f(x) is an even function

Key. C

Sol. $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} x = a, \ x \in Q$

$\lim\limits_{x \to a} f(x) \lim\limits_{x \to a}(-x) = -a, \ x \in R \sim Q$

The limit exists $\Leftrightarrow$ a = 0

99. If f(x) = sgn(x) and g(x) = x(1 – x^2), then the number of points of discontinuity of function f(g(x)) is

A) exact two

B) exact three

C) finite and more than 3

D) infinitely many

Key. B

Sol. $f(g(x)) = \begin{cases} 1 & , \quad x < -1 \\ 0 & , \quad x = -1 \\ -1 & , \quad -1 < x < 0 \\ 0 & , \quad x = 0 \\ 1 & , \quad 0 < x < 1 \\ 0 & , \quad x = 1 \\ -1 & , \quad x > 1 \end{cases}$

100. The value of Argz + Arg $\bar{z} = 0, \ z = x + iy, \ \forall \ x, y \in R$ is (Arg z stands for principal argument of z)

A) 0

B) Non-zero real number

C) Any real number

D) Can't say

Key. D

Sol. Let z = - 2 + 0i, then $\bar{z} = -2 - 0i$

$\therefore \quad Arg(z) + Arg\left(\bar{z}\right) = 2\pi \neq 0$

If z = 2 + 3i

Arg (2 – 3i) is $\tan^{-1}\left(-\frac{3}{2}\right)$

Arg(2 + 3i) + Arg (2 – 3i) = 0

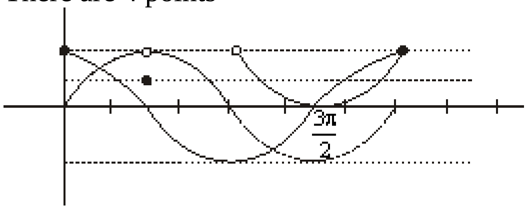
101. If f(x) = maximum $\left(\cos x, \frac{1}{2}, \{\sin x\}\right), 0 \le x \le 2\pi$, where { . } represents fractional part function, then number of points at which f(x) is continuous but not differentiable, is

A) 1 B) 2 C) 3 D) 4

Key. D

Sol. See figure

There are 4 points



102. Function $\begin{cases} 2x\tan x - \dfrac{\pi}{\cos x} & , \quad x \neq \dfrac{\pi}{2} \\ k & , \quad x = \dfrac{\pi}{2} \end{cases}$ is continuous at $x = \frac{\pi}{2}$ if k =

A) – 2 B) 2 C) $\frac{1}{2}$ D) no such values of k exists

Key. A

Sol. $\lim_{x \to \frac{\pi}{2}}\left(2x\tan x - \frac{\pi}{\cos x}\right)$

$= \lim_{x \to \frac{\pi}{2}}\left(\frac{2x\sin x - \pi}{\cos x}\right) = \lim_{h \to 0}\left(\frac{2\left(\frac{\pi}{2} + h\right)\cos h - \pi}{-\sin h}\right)$

$= \lim_{x \to 0} - \frac{2h\cos h}{\sin h} = -2 \quad \therefore \quad k = -2$

103. If $f(x) = \begin{cases} x^2\{e^{1/x}\} & x \neq 0 \\ k & x = 0 \end{cases}$ is continuous at x = 0, then

({ } denotes fractional part function)

A) It is differentiable at x = 0 B) k = 1

C) continuous but not differentiable at x = 0 D) continuous everywhere in its domain

Key. A

Sol. $\lim_{x \to 0} f(x) = 0$ { Q $\lim_{x \to 0} x^2 = 0$ and $\{e^{1/x}\}$ is a bounded function}

$\lim_{x \to 0}\frac{f(0 + x) - f(0)}{x} = \lim_{x \to 0} x\{e^{1/x}\} = 0$

$\therefore \quad f'(0) = 0$

not continuous at $x = \log_2 e, \log_3 e, ...$ etc.

104. Let $f(x) = a|\sin x| + be^{|x|} + c|x|^3$. If $f(x)$ is differentiable at x = 0 then

A) c = a = 0 and b can be any real number B) a + b =0 and c can be any real number

C) b = c = 0 and a can be any real number D) a = b = c = 0

Key. B

Sol. we have f(x)= $\begin{Bmatrix} -a\sin x + be^{-x} - cx^3 \ if \ x < 0 \\ a\sin x + be^{x} + cx^3 \ if \ x \geq 0 \end{Bmatrix}$

f(x) is obviously continuous at zero.

L.H.D = R.H.D

$$(-a\cos x - be^{-x} - 2cx^2)_{x=0} = (a\cos x + be^{x} + 2cx^2)_{x=0}$$

$$\Rightarrow -a - b = a + b$$

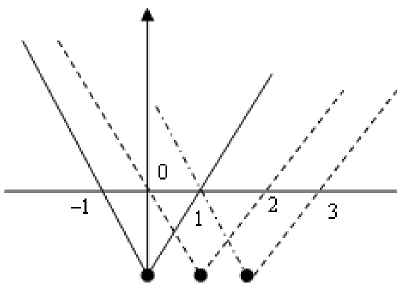
$\Rightarrow$ a+b=0 , and c can be any real number

105. The function $f(x) = \min\{|x|-1, |x-2|-1, |x-1|-1\}$ is not differentiable at

A) 2 points B) 5 points C) 4 points D) 3 points

Key. B

Sol. From the graph , it is clear that function is non-differentiable at 0, ½, 1,3/2, 2.



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Continuity & Differentiability

Multiple Correct Answer Type

1. Which is discontinuous at $x=1$

(A) $g(x)=\lim_{n\to\infty}\frac{1}{1+n\sin^2(\pi x)}$ (B) $f(x)=\frac{1}{1+2^{\tan x}}$

(C) $h(x)=2^{-2^{\left(\frac{1}{1-x}\right)}}$, $x\neq 1$ and $h(1)=1$ (D) $\phi(x)=\frac{x-1}{|x-1|+2(x-1)^2}$, $x\neq 1$ and $\phi(1)=1$

Key. A,C,D

Sol. a) $f(x)$ is count at n =1

b) $g(1^+)$ = 0 , $g(1^+)=1\Rightarrow g(x)$ is discontinuous at n =1

c) $\left.\begin{array}{l}h(1^+)=1\\ h(1^-)=0\end{array}\right\}\Rightarrow h(x)$ is discontinuous at n = 1

d) $L.L\neq RL\Rightarrow \phi(x)$ is discontinuous at n = 1

2. Let a function $f:R\to R$ satisfies the equation $f(x+y)=f(x)+f(y)$, $\forall x,y\in R$ then

(A) f is continuous for all $x\in R$ if it is continuous at x = 0

(B) $f(x)=x.f(1)\forall x\in R$, if 'f' is continuous

(c) $f(x)=(f(1))^x\ \forall x\in R$, if 'f' is continuous

(D) $f(x)$ is differentiable for all $x\in R$

Key. A,B

Sol. (i)Since $f(x)$ is continuous at x = 0, $\lt_{x\to a} f(x)=f(0)$ - (1)

Let $a\in R$ then $\lt_{x\to a} f(x)=\lt_{h\to 0} f(a+h)$

$= \lt_{h\to 0} f(a)+f(h)$

$= f(a)+\lt_{x\to 0} f(h)$

$= f(a)+f(0)=f(a+0)$

$= f(a)$

$\Rightarrow$ 'f' is continuous $\forall x\in R$, as 'a' is arbitrary

(ii) $\because f(xe\,y) = f(x) + f(y) \Rightarrow f(0) = 0.f(1)$ - (1)

For any +ve inteteger 'n'

$f(x) = f(1+1+........=1) = n.f(1)$ - (2)

For any –ve integer m we have

$0 = f(0) = f[m+(-m)] = f(m) + f(-m)$

$\Rightarrow f(m) = -f(-m) = -(-m).f(1)$

$= m.f(1)$ - (3)

(iii) let p/q be any rational number where 'q' is $a+ve$ integer and p is any $+ve$ integer, $+ve, -ve$ or zero.

Then $f\left(q.\frac{p}{q}\right) = f\left(\frac{p}{q}+\frac{p}{q}+.............q\ times\right)$

$= f\left(\frac{p}{q}\right) + f\left(\frac{p}{q}\right) + ...q\,times$

$= q.f\left(\frac{p}{q}\right)$

$\Rightarrow f(p) = q.f(p/q)$ - (4)

But $f(p) = p.f(1)$ from previous cases.

$\therefore\ p.f(1) = q.f(p/q)$

$\Rightarrow\ f(p/q) = \frac{p}{q}.f(1)$ - (5)

3. Let a function $f: R \to R$ satisfies the equation $f(x+y) = f(x) + f(y)$, $\forall x, y \in R$ then

(A) f is continuous for all $x \in R$ if it is continuous at x = 0

(B) $f(x) = x.f(1) \forall x \in R$, if 'f' is continuous

(C) $f(x)$ is not a periodic function

(D) $f(x)$ is differentiable for all $x \in R$

Key. A,B,C,D

Sol. (i) Since $f(x)$ is continuous at x = 0, $\lim\limits_{x\to a} f(x) = f(0)$ - (1)

Let $a \in R$ then $\lim\limits_{x\to a} f(x) = \lim\limits_{h\to 0} f(a+h)$

$= \lim\limits_{h\to 0} f(a) + f(h)$

$$= f(a) + \underset{x\to 0}{lt} f(h)$$

$$= f(a) + f(0) = f(a+0)$$

$$= f(a)$$

$\Rightarrow 'f'$ is continuous $\forall x \in R$, as 'a' is arbitrary

(ii) $\because f(xe\,y) = f(x) + f(y) \Rightarrow f(0) = 0.f(1)$ - (1)

For any +ve inteteger 'n'

$f(x) = f(1+1+........=1) = n.f(1)$ - (2)

For any –ve integer m we have

$0 = f(0) = f[m+(-m)] = f(m) + f(-m)$

$\Rightarrow f(m) = -f(-m) = -(-m).f(1)$

$$= m.f(1) \text{ - (3)}$$

(iii) let p/q be any rational number where 'q' is $a + ve$ integer and p is any $+ve$ integer, $+ve, -ve$ or zero.

Then $f\left(q.\frac{p}{q}\right) = f\left(\frac{p}{q} + \frac{p}{q} + ...q\ times\right)$

$$= f\left(\frac{p}{q}\right) + f\left(\frac{p}{q}\right) + ...q\,times$$

$$= q.f\left(\frac{p}{q}\right)$$

$\Rightarrow f(p) = q.f(p/q)$ - (4)

But $f(p) = p.f(1)$ from previous cases.

$\therefore\ p.f(1) = q.f(p/q)$

$\Rightarrow f(p/q) = \frac{p}{q}.f(1)$ - (5)

iv. Let 'x' be a real number, since 'f' is continuous $x_n \to x \Rightarrow f(x_n) \to f(x)$ where $< x_n >$ represents sequence of rational numbers representing 'x' as x_n is a rational number.

$f(x_n) = x_n.f(1)$

$\underset{n\to\infty}{lt} f(x_n) = \underset{n\to\infty}{lt}[x_n.f(1)]$

$\Rightarrow \underset{n\to\infty}{lt} f(x_n) = f(1)\left[\underset{n\to\infty}{lt} x_n\right]$

$$= x.f(1) \text{ - (6)}$$

$\therefore f(x) = x.f(1)\ \forall x \in R$

From all the above cases, we have $f(x) = kx, \forall x$ taking $f(1) = k$, where $'k'$ is a constant.

(iii), (iv) are obvious from $f(x) = kx$

4. A function $f: R \to R$ satisfies the equation $f(x+y) = f(x).f(y)$ for all x, y in R and $f(x) \neq 0$ for any $x \in R$. Let the function be differentiable at x = 0 and $f'(0) = 2$ then.

(A) $f'(x) = 2f(x) \forall x \in R$ (B) $f(x) = e^{2x}$

(C) $f(x)$ is every where continuous (D) $f\left(\frac{1}{2}\right)$ is an Irrational number

Key. A,B,C,D

Sol. Clearly for $x = y = 0;\ f(0) = 1$

$$f'(x) = \underset{h \to 0}{lt} \frac{f(x+h) - f(x)}{h} = 2.f(x)$$

Integrating $f(x) = e^{2x}$ from this all the remaining follows.

5. Let $\varnothing\left(\frac{x+2y}{3}\right) = \frac{\varnothing(x) + 2\varnothing(y)}{3} \forall x, y \in R$ and $\varnothing'(0) = 1$ and $\varnothing(0) = 2$ then

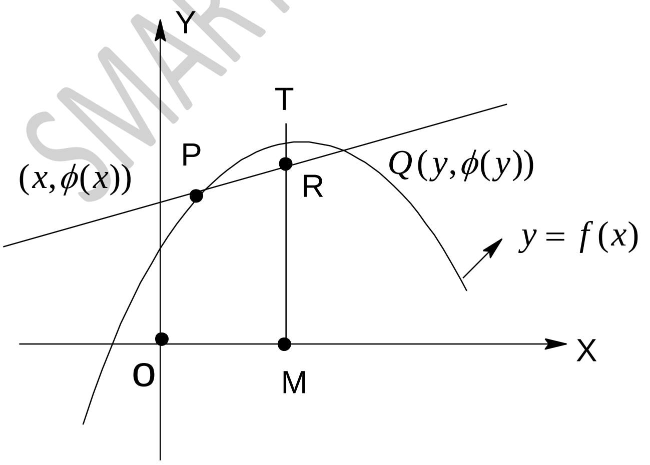
(A) $\varnothing(x)$ is continuous $\forall x \in R$ (B) $\varnothing(x)$ is differentiable $\forall x \in R$

(C) $\varnothing(x)$ is both continuous and differentiable

(D) $\varnothing(x)$ is discontinuous at x = 0

Key. A,B,C

Sol.



Take $p = (x, \varnothing(x)); Q = (y, \varnothing(y))$ be any two points the curve $y = \varnothing(x)$

Let 'R' divides the line segment $\overline{PQ}$ in the ratio 2:1 then $R = \left(\frac{x+2y}{3}, \frac{\varnothing(x)+2\varnothing(y)}{3}\right)$

Clearly TM > RM

$$\Rightarrow \varnothing\left(\frac{x+2y}{3}\right) > \frac{\varnothing(x)+2\varnothing(y)}{3}$$

Equality holds iff $\varnothing(x)$ is a linear function.

$\therefore \varnothing(x) = ax + b$

Q $\varnothing'(0) = 1 \Rightarrow a = 1$ Q$\varnothing(0) = 2$

b = 2

Q$\varnothing(x) = x + 2$

6. Consider the function $'f'$ defined in $[0,1]$ as

$$\phi(x) = \begin{cases} x^2.\sin\left(\frac{1}{x}\right) & ; \ if \quad x \neq 0 \\ 0 & ; \ if \quad x = 0 \end{cases}$$. Then

(A) $\phi(x)$ has right derivate at $x = 0$ (B) $\phi^1(x)$ is discontinuous at $x = 0$

(C) $\phi^1(x)$ is continuous at $x = 0$ (D) $\phi^1(x)$ is differentiable at $x = 0$

Key. A,B

Sol. Clearly $\phi^1(x) = 2x.\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) if \ x \neq 0$

= 0 if x = 0.

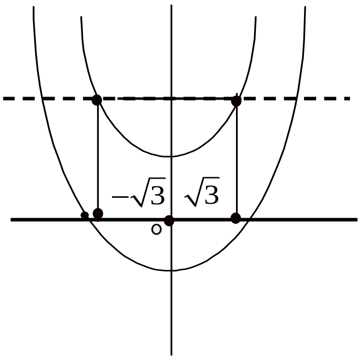
$\Rightarrow \phi^1(x)$ is distinuous at $x = 0$, as $\cos\left(\frac{1}{x}\right)$ is oscillating in the neighbour hood of $'\underline{0}'$

7. If $f(x) =$ maximum $\{4, 1+x^2, x^2-1\} \ \forall x \in R$. Then the total number of points where $f(x)$ is not-differentiable at

(A) $\sqrt{3}$ (B) $-\sqrt{3}$ (C) Two irrational points (D) none

Key. A,B,C

Sol. Draw graph, clearly at $x = \pm\sqrt{3}$, $f(x)$ is not differentiable.



8. The in-circle of ΔABC touches side BC at D. Then difference between BD and CD (R is circum-radius of ΔABC)

A) $\left|4R\sin\frac{A}{2}\sin\frac{B-C}{2}\right|$ B) $\left|4R\cos\frac{A}{2}\sin\frac{B-C}{2}\right|$ C) $|b-c|$ D) $\left|\frac{b-c}{2}\right|$

Key. A,C

Sol.

$$|BD-CD| = \left|r\left(\cot\frac{B}{2}-\cot\frac{C}{2}\right)\right| = r\left|\frac{\sin\left(\frac{B-C}{2}\right)}{\sin\frac{B}{2}\sin\frac{C}{2}}\right| =$$

$$\left|4R\sin\frac{A}{2}\sin\frac{B-C}{2}\right| = \left|4R\cos\frac{B+C}{2}.\sin\frac{B-C}{2}\right| = \left|2R(\sin B-\sin C)\right| = |b-c|$$

9. Which of the following functions are not differentiable at $x=0$

A) $\cos|x|+|x|$ B) $\cos|x|-|x|$ C) $\sin|x|+|x|$ D) $\sin|x|-|x|$

Key. A,B,C

Sol. $|x|$ is not differentiable at $x=0$

$\cos|x| = \cos x$ is differentiable for all 'x'

however $\sin|x|-|x|$ has both right and left derivatives are zero at $x=0$

$\therefore \sin|x|-|x|$ is differentiable at $x=0$

10. If $8f(x)+6f\left(\frac{1}{x}\right)=x+5$; *where* $x\neq 0$ and $y=x^2f(x)$ then

A) $f(x)=\frac{1}{28}\left[8x-\frac{6}{x}+10\right]$ B) $\left(\frac{dy}{dx}\right)=\frac{1}{28}\left[24x^2-6+20x\right]$

C) $\left(\frac{dy}{dx}\right)_{(x=-1)}=-\frac{1}{14}$ D) $\left(\frac{dy}{dx}\right)_{(x=1)}=\frac{19}{14}$

Key. A,B,C,D

Sol. $8f(x)+6f\left(\frac{1}{x}\right)=x+5$ .....(1)

Replacing x by $\frac{1}{x}$, we set

$6f(x)+8f\left(\frac{1}{x}\right)=\frac{1}{x}+5$ ....(2)

From (1) & (2)

$f(x)=\frac{1}{28}\left(8x-\frac{6}{x}+10\right)$ ....(3)

$\therefore y=x^2f(x)=\frac{1}{28}\left[8x^3-6x+10x^2\right]$

$\frac{dy}{dx}=\frac{1}{28}\left[24x^2-6+20x\right]$

$\left(\frac{dy}{dx}\right)_{(x=-1)}=\frac{1}{28}(24-6-20)=-\frac{1}{14}$

11. If p(x) is a polynomial such that

$p\left(x^2+1\right)=\{p(x)\}^2+1$ and $p(0)=0$ then

A) $p(x)=x$ B) $p'(0)=1$ C) $p'(1)=1$ D) $p'(1)=0$

Key. A,B,C

Sol. $p\left(x^2+1\right)=(p(x))^2+1$

$\therefore p(x)=x$ $(\because p(x)$ is an identity function$)$

$p'(x)=1$

12. If $f(x)=\begin{cases}\frac{x}{1+e^{1/x}}, & x\neq 0\\ 0, & x=0\end{cases}$ then

A) $f'\left(0^{+}\right)=1$ B) $f'\left(0^{+}\right)=0$ C) $f'\left(0^{-}\right)=1$ D) $f'\left(0^{-}\right)=0$

Key. B,C

Sol. $f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x}=\lim_{x\to 0}\frac{1}{1+e^{1/x}}$

$f'\left(0^{+}\right)=\lim_{x\to 0^{+}}\frac{1}{1+e^{1/x}}=0$

$f'\left(0^{-}\right)=\lim_{x\to 0^{-}}\frac{1}{1+e^{1/x}}=1$

13. Consider the function f defined by

$$f(x)=\begin{cases}0 & , x=0 \quad or\ x\ is\ irrational\\ \frac{1}{n} & , where\ x\ is\ a\ non-zero\ rational\ \text{number}\ \frac{m}{n},\ n>0\ and\ \frac{m}{n}\ is\ in\ lowest\ term\end{cases}$$

Which of the following statements is true?

A) Any irrational number is a point of discontinuity of f

B) Any irrational number is a point of continuity of f

C) The points of discontinuities of f are rational numbers

D) The points of discontinuities are non-zero rational numbers.

Key. B,D

Sol. Case : I

Let c be rational .We show that the function is continuous only at c= 0. At all other points its discontinuous. Let f be continuous at c. As there are irrational numbers arbitrarily close to ‘c’ so by continuity, f(c) = 0 and then c = 0.

Also f (x) is continuous at x = 0, since as rational numbers $\frac{m}{n}$ approach 0, their denominators approach ∞, and so f (m/n) = $\frac{1}{n}$ approach 0, which is f(0)

Case: II

c is irrational : then f(c) = 0

But as rational numbers m/n approach c, their denominators n approach ∞, and so the values f(m/n) =1/n approach 0 = f(c).Thus any irrational number is a point of continuity

Summary : f is continuous at x = 0 and any irrational number .f is discontinuous at all non-zero rationals

14. Suppose that $f:R\to R$ is continuous and satisfying the equation f(x) .

f (f(x)) = 1, for all real x.

Let f (1000) = 999, then which of the following is true ?

A) $f(500)=\frac{1}{500}$ B) $f(199)=\frac{1}{199}$

C) $f\left(x\right)=\frac{1}{x}\forall\ x\in R-\left\{0\right\}$ D) $f\left(1999\right)=\frac{1}{1999}$

Key. A,B

Sol. f (1000) f (f (1000)) = 1

$\Rightarrow$ f (1000) f (999) = 1

$\Rightarrow$ 999 f (999) = 1

$\therefore$ f (999) = $\frac{1}{999}$

The numbers 999 and $\frac{1}{999}$ are in the range of f. Hence by intermediate value property (IVP) of continuous function,function takes all values between 999 and $\frac{1}{999}$, then there exists

$\alpha\in\left(\frac{1}{999},999\right)$ such that f $\left(\alpha\right)$ = 500

Than $f\left(\alpha\right)f\left(f\left(\alpha\right)\right)=1\Rightarrow$ f (500) = $\frac{1}{500}$

Similarly 199 $\in\left(\frac{1}{199},999\right)$, thus f (199) = $\frac{1}{199}$

But there is nothing to show that 1999 lies in the range of f

Thus (D) is not correct and so 'C' also

15. Let f be a function with two continuous derivatives and f(0)=0, $f'\left(0\right)=0$. Define a function g by

$$g(x)=\begin{cases}\frac{f(x)}{x}, x\neq 0\\ 0\ , x=0\end{cases}$$

Then which of the following statements is correct?

A) g has a continuous first derivative

B) g has a first derivative

C) g is continuous but g fails to have a derivative

D) g has a first derivative but the first derivative is not continuous

Key. A,B

Sol. One can easily establish that $g'\left(0\right)=\frac{1}{2}f''\left(o\right)$ using definition continuity of g' at '0' is also easy to check.

16. The function

$$\left.\begin{aligned} f\left(x\right)&=\frac{x^2}{a}\ ,0\le x<1\\ &=a\quad ,1\le x<\sqrt{2}\\ &=\frac{2b^2-4b}{x^2}\quad ,\sqrt{2}\le x<\infty\end{aligned}\right\}$$

is continuous for $0 \le x < \infty$. Then which of the following statements is correct?

A) The number of all possible ordered pairs (a, b) is 3

B) The number of all possible order pairs (a, b) is 4

C) The product of all possible values of b is – 1

D) The product of all possible values of b is 1.

Key. A,C

Sol. We get (a,b) = (-1,1) , $\left(1,1+\sqrt{2}\right),\left(1,1-\sqrt{2}\right)$

17. Which of the following statements are true?

A) If f is differentiable at x = c, then $\lim_{h\to 0}\frac{f(c+h)-f(c-h)}{2h}$ exists and equals $f'(c)$.

B) Given a function f and a point c in the domain of f, if the $\lim_{h\to 0}\frac{f(c+h)-f(c-h)}{h}$ exists, then the function is differentiable at x = c

C) Let g(x)= $\begin{cases} x^2\sin\frac{1}{x^2}, x\neq 0 \\ 0, x=0 \end{cases}$, then g' exists

D) Let g(x)= $\begin{cases} x^2\sin\frac{1}{x^2}, x\neq 0 \\ 0, x=0 \end{cases}$, then g' exists and is continuous.

Key. A,C

Sol. (A) is true

$$\lim_{h\to 0}\frac{f(c+h)-f(c)+f(c)-f(c-h)}{h}$$

$$=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}+\lim_{h\to 0}\frac{f(c-h)-f(c)}{-h}$$

$= f'(c)+f'(c)$

= 2f '(c) (f is differentiable)

(B) is false. Existence of limit is no guarantee for differentiability

(C) is true

(D) is false

18. $f(x)=\begin{cases} x^2\left|\cos\frac{\pi}{x}\right| & when \quad x\neq 0 \\ 0 & when \quad x=0 \end{cases}$

A) $f(x)$ is not differentiable at $x=0$

B) $f(x)$ is differentiable at $x=0$

C) $f(x)$ is not differentiable at $x=\frac{2}{2n+1}:n\in Z$

D) $f(x)$ is differentiable at $x\neq\frac{2}{2n+1}:n\in Z$

Key. B,C,D

Sol. $f(x)$ is obviously differentiable at $x=0$ & for $x_n=\frac{2}{2n+1}$ where $n=0,2,4,\ldots$

We get $f'(x_n^+)=\pi$ & $f'(x_n^-)=-\pi$ & for $x_n=\frac{2}{2n+1}$ where $n=1,3,5,\ldots$

We get $f'(x_n^+)=\pi$ & $f'(x_n^-)=-\pi$ and

$\because$ $f(x)$ is even function $f(x)$ is not differentiable at $x_n=\frac{2}{2n+1}:n\in Z$

19. $f(x)=[|x|]+\sqrt{\{|x|\}}$ here [.] is integral part of $'x'$ and {.} fractional part of $'x'$ functions then $f(x)$ is

A) Continuous in $(-2,2)$

B) Non differentiable at 3 points in $(-2,2)$

C) Monotonically increasing in $(-2,2)$

D) Discontinuous at 2 points in $(-2,2)$

Key. A,B

Sol. Conceptual

20. Consider the function $y=f(x)=\sqrt{1-\sqrt{1-x^2}}$. Then the true statements among the following is/are

A) f is continuous in its domain

B) f is differentiable in (-1, 1)

C) $Rf'(0)=\frac{1}{\sqrt{2}}$ and $Lf'(0)=-\frac{1}{\sqrt{2}}$

D) If $\pi<\theta<\frac{3\pi}{2}$ then $f'(\sin\theta)=\frac{\sin\frac{\theta}{2}}{\sqrt{2}\cos\theta}$

Key: A,C,D

Hint f is continuous in its domain $[-1,1]$

$$f^1(x)=\frac{x}{2\sqrt{1-\sqrt{1-x^2}}\,\sqrt{1-x^2}},\ x\neq 0, x\neq\pm 1$$

21. Let $f(x)=\int_0^x e^t\sin(x-t)dt$ and $g(x)=f(x)+f''(x)$ for all real x. Which of the following statements is / are correct ?

a) $g(x)>0$ for all $x\in\mathbf{R}$

b) g (1) = e

c) $g'(x) = g(x)\ for\ all\ x \in \mathbf{R}$ d) range of g is $[0, \infty)$

Key; A, B, C

Hint $f(x) = \frac{1}{2}(e^x - \sin x - \cos x) and\ g(x) = e^x$

22. If $f(x) = |x-a|\phi(x)$, where $\phi(x)$ is a continuous function, then

A. $f^1(a+) = \phi(a)$ B. $f^1(a-) = -\phi(a)$ C. $f^1(a+) = f^1(a-)$ D. $f^1(a)$ does not exist

Key. A,B,D

Sol. $f(x) = \begin{cases} (x-a)\varphi(x) & if \quad x \geq a \\ (a-x)\varphi(x) & if \quad x < a \end{cases}$

$\therefore f^1(a+) = \underset{x\to a}{Lt}\ (x-a)\varphi^1(x) + \varphi(x) = \varphi(a)$

$f^1(a-) = \underset{x\to a}{Lt}\ (a-x)\varphi^1(x) - \varphi(x) = -\varphi(a)$

23. $f(x) = \begin{cases} \left[|x|\left[\frac{1}{|x|}\right]\right], & |x| \neq \frac{1}{n}, n \in N, \\ 0, & |x| = \frac{1}{n} \end{cases}$ then, (where [.] denotes greatest integer function)

A. f is differentiable everywhere B. f is continuous everywhere

C. f is periodic D. f is not an odd function

Key. A,B,C

Sol. If $|x| < 1$ and $|x| \neq \frac{1}{n}$, then $\frac{1}{|x|} - 1 < [\frac{1}{|x|}] < \frac{1}{|x|}$

$\Rightarrow 1 - |x| < |x|[\frac{1}{|x|}] < 1$

$\Rightarrow f(x) = 0$

If $|x| >|$, then $0 < \frac{1}{|x|} < 1$ and hence $(\frac{1}{|\lambda|}) = 0$. Then $f(x) = 0$

Hence $f(x) = 0$ for all $x \in R$

24. If $f(x) = 2 + |\sin^{-1} x|$, it is :

A. continuous no where B. continuous everywhere in its domain

C. differentiable no where in its domain D. not differentiable at $x = 0$

Key. B,D

Sol. $f(x)=2-\sin^{-1}x$ if $-1\le x\le 0$

$2+\sin^{-1}x$ if $0<x\le 1$

Hence f is continuous everywhere on the domain

$f'(x)=\dfrac{-1}{\sqrt{1-x^2}}$ if $-1<x<0$

$\dfrac{1}{\sqrt{1-x^2}}$ if $0<x<1$

$\therefore f$ is not differentiable at $x=0$

25. $f(x)=\cos\pi(|x|+[x])$, then f is (where [.] denotes greatest integer function)

A. continuous at $x=1/2$

B. continuous at $x=0$

C. differentiable in $(-1,0)$

D. differentiable in $(0,1)$

Key. A,C,D

Sol. $f(x)=-\cos\pi x$ if $-1<x<0$

$1 \quad if \quad x=0$

$\leftarrow\cos\pi x \quad if \quad 0<x<1$

$\therefore f$ is not continuous at $x=0$

26. If $\sin^{-1}x+|y|=2y$ then y as a function of x is

A. defined for $-1\le x\le 1$

B. continuous at $x=0$

C. differentiable for all x

D. such that $\dfrac{dy}{dx}=\dfrac{1}{3\sqrt{1-x^2}}$ for $x<0$

Key. A,B,D

Sol. If $y<0$ then $3y=\sin^{-1}x$

if $y\ge 0$ then $y=\sin^{-1}x$

Thus $y=\begin{cases}\dfrac{\sin^{-1}x}{3} & if \quad -1\le\lambda<0\\ \sin^{-1}x & if \quad 0\le\lambda\le 1\end{cases}$

y is not differentiable at $x=0$

27. Which is discontinuous at $x=1$

(A) $g(x)=\lim\limits_{n\to\infty}\dfrac{1}{1+n\sin^2(\pi x)}$

(B) $f(x)=\dfrac{1}{1+2^{\tan x}}$

(C) $h(x)=2^{-2^{\left(\frac{1}{1-x}\right)}}$, $x \neq 1$ and $h(1)=1$ (D) $\phi(x)=\dfrac{x-1}{|x-1|+2(x-1)^2}$, $x \neq 1$ and $\phi(1)=1$

Key. A,C,D

Sol. a) $f(x)$ is count at n =1

b) $g(1^+)$ = 0 , $g(1^+)=1 \Rightarrow g(x)$ is discontinuous at n =1

c) $\left.\begin{array}{l} h(1^+)=1 \\ h(1^-)=0 \end{array}\right\} \Rightarrow h(x)$ is discontinuous at n = 1

d) $L.L \neq RL \Rightarrow \phi(x)$ is discontinuous at n = 1

28. Let a function $f : R \to R$ satisfies the equation $f(x+y)=f(x)+f(y)$, $\forall x, y \in R$ then

(A) f is continuous for all $x \in R$ if it is continuous at x = 0

(B) $f(x)=x.f(1) \forall x \in R$, if 'f' is continuous

(c) $f(x)=(f(1))^x \forall x \in R$, if 'f' is continuous

(D) $f(x)$ is differentiable for all $x \in R$

Key. A,B

Sol. (i)Since $f(x)$ is continuous at x = 0, $\underset{x \to a}{lt} f(x)=f(0)$ - (1)

Let $a \in R$ then $\underset{x \to a}{lt} f(x) = \underset{h \to 0}{lt} f(a+h)$

$= \underset{h \to 0}{lt} f(a)+f(h)$

$= f(a)+\underset{x \to 0}{lt} f(h)$

$= f(a)+f(0)=f(a+0)$

$= f(a)$

$\Rightarrow$ 'f' is continuous $\forall x \in R$, as 'a' is arbitrary

(ii) $\because f(xe\,y)=f(x)+f(y) \Rightarrow f(0)=0.f(1)$ - (1)

For any +ve inteteger 'n'

$f(x)=f(1+1+........=1)=n.f(1)$ - (2)

For any –ve integer m we have

0 = $f(0) = f[m+(-m)] = f(m) + f(-m)$

$\Rightarrow f(m) = -f(-m) = -(-m).f(1)$

= $m.f(1)$ - (3)

(iii) let p/q be any rational number where 'q' is $a+ve$ integer and p is any $+ve$ integer, $+ve, -ve$ or zero.

Then $f\left(q.\frac{p}{q}\right) = f\left(\frac{p}{q}+\frac{p}{q}+.............q\ times\right)$

= $f\left(\frac{p}{q}\right)+f\left(\frac{p}{q}\right)+........q\,times$

= $q.f\left(\frac{p}{q}\right)$

$\Rightarrow f(p) = q.f(p/q)$ - (4)

But $f(p) = p.f(1)$ from previous cases.

$\therefore\ p.f(1) = q.f(p/q)$

$\Rightarrow\ f(p/q) = \frac{p}{q}.f(1)$ - (5)

29. $Let\ f(x) = \begin{cases} \int_0^x (1+|1-t|)\,dt, if\ \ x>2 \\ 5x+1\ \ if\ \ \ x\le 2 \end{cases}\quad then$

(A) $f(x)$ is discontinuous at x = 2

(B) $f(x)$ is continuous but not differentiable at x =2

(C) $f(x)$ is differentiable every where

(D) The right derivative of $f(x)$ at x =2 does not exist

Key. A,D

Sol. $f(x) = \int_0^1 (2-t)\,dt + \int_1^2 t\,dt + \int_2^x t\,dt$

$= \frac{x^2}{2}+1$

$\therefore\ f(x) = \begin{cases} \frac{x^2}{2}+1, if\ \ x>2 \\ 5x+1, if\ \ x\le 2 \end{cases}$

Clearly (1) , (4) are true.

30. If $f(x)=\left|[x]x\right|$ in $-1\le x\le 2$ here [.] denotes the greatest integer $\le x$ then $f(x)$ is.

(A) discontinuous at x = 0 (B)continuous at x = 0

(C) discontinuous at x = 1(D) not differentiable at x = 0

Key. B,C,D

Sol. $f(x)=\begin{cases}-x \text{ if } -1\le x<0\\ 0 \text{ if } 0\le x<1\\ x \text{ if } 1\le x<2\\ 2x \text{ if } x=2\end{cases}$

Now verify the statements

31. Let f(x) be a non negative differentiable function such that $f^1(x)\le f(x)\forall x\ge 0$ and $f(0)=0$ then

a) $f(1)+f(2)=0$ b) $f(2)=f\left(\frac{1}{2}\right)$

c) $f(1)-f(2)=0$ d) $f(1)+f(2)=3$

Key. A,B,C

Sol. $f^1(x)\le f(x)$, let $f^1(x)-f(x)=K\Rightarrow f(x)=-K+Ke^x = K\left(e^z-1\right)\ge 0\forall x\ge 0 \Rightarrow K=0$

$\therefore$ f is a constant function but f(0) = 0

$\therefore f\equiv 0$

32. Consider a function $f(x)=\sqrt{1-e^{-x^2}}$ then

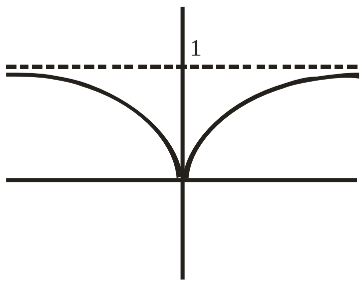
(A) f(x) is continuous at x = 0 (B) f(x) is discontinuous at x = 0

(C) f(x) is differentiable at x = 0 (D) f(x) is not differentiable at x = 0

Key. A,D

SOL. $\lim_{h\to 0} f(a+h)=\lim_{h\to 0}\sqrt{1-e^{-}(a+h)^2}$ WHEN A = 0

= 0 = F(0) $\Rightarrow$ CONTINUOUS AT X = 0



CLEARLY AT X = 0

TANGENTS IS Y-AXIS

$\Rightarrow$ f(x) is not differentiable at x = 0

33. $f(x)=\begin{cases}3x^2+12x-1 & ,-1\le x\le 2\\ 37-x & ,2<x\le 3\end{cases}$, then

A) f is increasing on $[-1,2]$

B) f is differentiable at $x=2$

C) f does not attain absolute minimum in $[-1,2]$

D) Absolute maximum value of f is 35

Key. A,D

Sol. Conceptual

34. Let $f(x)=[x]^2+[x+1]-3$ where $[x]$ = the greatest integer $\le x$. Where $f:R\to R$,Then

a) $f(x)$ is a many-one and into function

b) $f(x)=0$ for infinite number of values of x

c) $f(x)=0$ for only two real values

d) none of these

Key. A,B

Sol. $f(x)=[x]^2+[x]+1-3=\{[x]+2\}\{[x]-1\}$

So, $x=1,1.1,1.2,...\Rightarrow f(x)=0$

Only integral values will be attained.

35. Let $h(x)=\min\{x,x^2\}$ for every real number x. Then

a) *h* is continuous for all x

b) *h* is differentiable for all x

c) $h'(x)=1$ for all $x>1$

d) *h* is not differentiable at two values of *x*

Key. A,C,D

Sol. If $x<x^2$

Then, $h(x)=x, x(x-1)>0$

$\therefore$ $x>1$ *or* $x<0$

$\therefore$ $h'(x)=1$.

and if $x^2<x, x(x-1)<0$

$\Rightarrow$ $0<x<1$

Then, $h(x)=x^2$.

36. $f(x)=|(x-a)|g(x)$ where $g(x)$ is a continuous function then

(a) $R\,f'(a)=g(a)$

(b) $L\,f'(a)=-g(a)$

(c) f is derivable at a

(d) None of these

Key. A,B

Sol. L.H.D = $\underset{x\to a^-}{Lt}\dfrac{f(x)-f(a)}{x-a}=\underset{h\to o^-}{Lt}\dfrac{f(a-h)-f(a)}{a-h-a}=\underset{h\to o^-}{lt}\dfrac{|-h|g(a-h)}{h}=-g(a)$

R.H.D=g(a)

37. Which of the following function(s) has/have removable discontinuity at x = 1.

A) $f(x)=\frac{1}{ln|x|}$ B) $f(x)=\frac{x^2-1}{x^3-1}$ C) $f(x)=2^{-2^{\frac{1}{1-x}}}$ D) $f(x)=\frac{\sqrt{x+1}-\sqrt{2x}}{x^2-x}$

Key. B,D

Sol. (A) $\lim_{x\to1} f(x)$ does not exist

(B) $\lim_{x\to1} f(x)=\frac{2}{3}$ $\therefore$ f(x) has removable discontinuity at x = 1

(C) $\lim_{x\to1} f(x)$ does not exist

(D) $\lim_{x\to1} f(x)=\frac{-1}{2\sqrt{2}}$ $\therefore$ f(x) has removable discontinuity at x = 1

38. A function f(x) satisfies the relation f(x + y) = f(x) + f(y) + xy (x + y) $\forall\ x, y \in R$. If $f'(0)=-1$, then

A) f(x) is a polynomial function

B) f(x) is an exponential function

C) f(x) is twice differentiable for all $x \in R$

D) $f'(3)=8$

Key. A,C,D

Sol. f(x + y) = f(x) + f(y) + xy (x + y)

f(0) = 0 $\therefore$ $\lim_{h\to0}\frac{f(h)}{h}=-1$

$\therefore$ $\lim_{h\to0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to0}\frac{f(x)+f(h)+xh(x+h)-f(x)}{h}=\lim_{h\to0}\frac{f(h)}{h}+\lim_{h\to0}x(x+h)=-1+x^2$

$\therefore$ $f'(x)=-1+x^2$

$\therefore$ $f(x)=\frac{x^3}{3}-x+c$

$\therefore$ f(x) is a polynomial function, f(x) is twice differentiable for all $x \in R$ and $f'(3)=3^2-1=a$

39. Let $f(x)=\int_{-2}^{x}|t+1|dt$, then

A) f(x) is continuous in [- 1, 1]

B) f(x) is differentiable in [- 1, 1]

C) $f'(x)$ is continuous in [- 1, 1]

D) $f'(x)$ is differentiable in [- 1, 1]

Key. A,B,C,D

Sol. $f(x)=\int_{-2}^{x}|t+1|dt$

$=-\int_{-2}^{-1}(t+1)dt+\int_{-1}^{x}(t+1)dt$

$=\frac{1}{2}+\left(\frac{t^2}{2}+t\right)_{-1}^{x}=\frac{x^2}{2}+x+1$ for $x\ge-1$

f(x) is a quadratic polynomial

$\therefore$ f(x) is continuous as well as differentiable in [- 1, 1]

Also $f'(x)$ is continuous as well as differentiable in [- 1, 1]

40. $f(x)=\frac{[x]+1}{\{x\}+1}$ for $f:\left[0,\frac{5}{2}\right)\to\left(\frac{1}{2},3\right]$, where [.] represents greatest integer function and { . } represents fractional part of x, then which of the following is true.

A) f(x) is injective discontinuous function

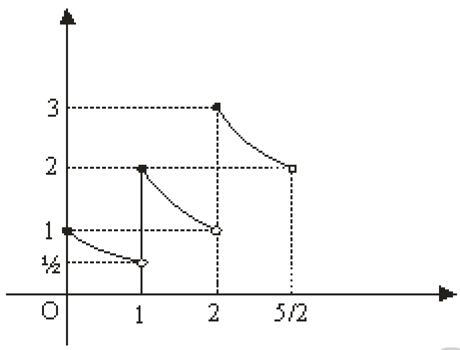
B) f(x) is surjective non differentiable function

C) $\min\left(\min_{x\to1-} f(x), \lim_{x\to1+} f(x)\right)=f(1)$

D) max(x values of point of disconutinuity) = f(1)

Key. A,B,D

Sol. $f(x)\begin{cases}\frac{1}{x+1} & , \quad 0\le x<1\\ \frac{2}{x} & , \quad 1\le x<2\\ \frac{3}{x-1} & , \quad 2\le x<\frac{5}{2}\end{cases}$



Clearly f(x) is discontinuous and bijective function

$\lim_{x\to1^-} f(x)=\frac{1}{2}$

$\lim_{x\to1^+} f(x)=2$

$\min\left(\lim_{x\to1} f(x) \lim_{x\to1^-} f(x)\right)=\frac{1}{2}\ne f(1)$

max(1, 2,) = 2 = f(1)

41. If f(x) = 0 for x < 0 and f(x) is differentiable at x = 0, then for x > 0, f(x) may be

A) x^2 B) x C) sin x D) $-x^{3/2}$

Key. A

Sol. both $x^2, -x^{3/2}$ have their RHL = 0 and RHD = 0

Continuity & Differentiability

Assertion Reasoning Type

A) Statement 1 is true, statement 2 is true and statement 2 is correct explanation for statement 1

B) Statement 1 is true, statement 2 is true and statement 2 is NOT the correct explanation for statement 1

C) Statement 1 is true, statement 2 is false

D) Statement 1 is false, statement 2 is true

1. Statement-I:- Let $f(x) = \cos x \, and \, g(x) = \sin x$, then $f(x) = g(x)$ for atleast one point in $(0, \pi/2)$

   Statement-II:- If f and g are continuous an $[a,b]$ and

   $f(a) \geq g(a) \, and \, f(b) \leq g(b)$, then $f(x_0) = g(x_0)$ for atleast one $x_0 \in [a,b]$

Key. A

Sol. Statement I is true, because $f(\pi/4) = g(\pi/4)$

Statement II is also true and it is correct explanation of I.

If either $f(a) = g(a) \, or \, f(b) = g(b)$ we are through.

If $f(a) > g(a) \, and \, f(b) < g(b)$.

Define $Q(x) = f(x) - g(x) \; for \, x \in [a,b]$

Clearly $Q(x)$ is continuous.

$Q(a)Q(b) < 0, \therefore$ By Intermediate property

$Q(x) = 0$ for some $x_0 \in (a,b)$, hence the result.

2. Statement-I:- The function $f(x) = \begin{cases} x \, if \, 'x' \, is \, rational \\ -x \, if \, x \, is \, Irrational \end{cases}$ is discontinuous at only one point and continuous at all other points on R.

   Statement-II:- The above function can be continuous at only one point in its domain and discontinuous every where else.

Key. D

Sol. It is self explanatory.

3. Statement-I:- Given the function $f(x) = \frac{1}{1-x}$ the number of points of discontinuity of the composite function $y = f^{3n}(x)$, where $f^n(x) = fofof \ldots\ldots\ldots of$ (n times) are 2

   Statement-II:- If $f(x) = \frac{1}{1-x}$, $x \neq 0, 1$, then $fofof(x) = x$

Key. A

Sol. Conceptual

4. Let $f(x) = \cos\left(x\cos\frac{1}{x}\right)$

Statement-I: f(x) is discontinuous at x = 0.

Because

Statement-II: $\lim_{x\to 0^+} f(x)$ does not exist.

Key. C

Sol. $\lim_{x\to 0^+} f(x)$ exists and is equal to 1 but f(0) is not defined $\Rightarrow$ S-I is correct but S-II is incorrect

5. Statement-1: Given the function $f(x) = \frac{1}{1-x}$ the number of points of discontinuity of the composite function $y = f^{3n}(x)$, where $f^n(x) = fofo.....of\ (n\ times)$ is 2

Statement-2: If $f(x) = \frac{1}{1-x}, x \neq 0, 1,$ then $fofof(x) = x$

Key. A

Sol. Use for $f(x) = f(f(x))$

6. STATEMENT1 $\lim_{x\to 0}\frac{2^{1/x}}{1+2^{1/x}} = 1$.

STATEMENT2 $\lim_{x\to 0}\frac{\cos^{-1}(1-x)}{\sqrt{x}} = \sqrt{2}$.

Key. D

Sol. $\lim_{x\to 0}\frac{2^{1/x}}{1+2^{1/x}} = \lim_{x\to 0}\frac{1}{1+2^{-1/x}} = 1$

$\lim_{x\to 0}\frac{\cos^{-1}(1-x)}{\sqrt{x}} = \lim_{x\to 0^+}\frac{\theta}{\sqrt{1-\cos\theta}}$ $\left(let,\ \cos^{-1}(1-x) = \theta \Rightarrow x = \cos\theta\right)$

$\lim_{x\to 0^+}\frac{\theta}{\sqrt{2}\sin\left(\frac{\theta}{2}\right)} = \sqrt{2}$

7. Suppose f : $A \to B$ and $g : B \to C$ are such that gof is onto and g is one- one. Then,

STATEMENT-1: f is onto

STATEMENT-2: g is a bijection

Key. B

Sol. Since gof is onto, g is onto. There fore S-2 is correct. To see that S-1 is correct, we observe that $b \in B \Rightarrow g(b) = c \in C \Rightarrow g(b) = (gof)(a)$ for some $a \in A$

$\Rightarrow b = f(a)$ $(since\ g\ is\ one-one)$

8. Statement -1 : Let $|x| \le 1,$ then the value of

$$\sin^{-1}\left[\cos\left(\sin^{-1}x\right)\right] + \cos^{-1}\left[\sin\left(\cos^{-1}x\right)\right] is\ \pi/2$$

Statement -2 : For $|x| \le 1,$ the values of $\cos(\sin^{-1}x)$ and $\sin(\cos^{-1}x)$ are equal

Key. A

Sol. For $|x| \le 1, \cos(\sin^{-1}x) = \sqrt{1-x^2}$ and $\sin(\cos^{-1}x) = \sqrt{1-x^2}$

$\therefore$ Both I and II are true and II is the correct explanation of I

9. Statement - 1: If f is twice differentiable function and $f(a) = 0, f(b) = 1, f(c) = -1, f(d) = 0$ where $a < b < c < d$ then the minimum number of roots of the equation $[f'(x)]^2 + f(x)f''(x) = 0$ in $[a, d]$ is 4.

Statement - 2: If f is continuous in $[\alpha, \beta]$ and $f(\alpha)f(\beta) < 0$ then $\exists r \in (\alpha, \beta)$ such that $f(r) = 0$ and if further function f is differentiable in (α, β) and $f(\alpha) = f(\beta)$ then $\exists \delta \in (\alpha, \beta)$ such that $f'(\delta) = 0$.

KEY : A

HINT: Conceptual Question

10. Consider the function $f(x) = (|x| - |x-1|)^2$

STATEMENT - 1 $f(x)$ is not differentiable at $x = 0$ and 1

STATEMENT - 2 $f'(0^-) = 0, f'(0^+) = -4, f'(1^-) = 4, f'(1^+) = 0$

Key. A

Sol.

$f(x) = (|x| - |x-1|)^2$

$$\therefore f(x) = \begin{cases} (-x+x-1)^2 = 1 & x < 0 \\ (x+x-1)^2 = (2x-1)^2, & 0 \le x \le 1 \\ (x-x+1)^2 = 1, & x > 1 \end{cases}$$]

$\therefore f'(x) = 0 \Rightarrow f'(0^{-1}) = 0$

$f'(x) = 2(2x-1).2 \Rightarrow f'(0^+) = -4$

$f'(x) = 4(2x-1) \Rightarrow f'(1^-) = 4$

$f'(x) = 1 \Rightarrow f'(1^+) = 0$

11. Statement (1) : If f is continuous and differentiable in $(a-\delta, a+\delta)$, where $a, \delta \in R$ and $\delta > 0$, then $f'(x)$ is continuous at $x = a$

and

Statement (2) : Every differentiable function at $x = a$ is continuous at $x = a$

Key. D

Sol. Statement 1 : $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$

Since $\lim\limits_{x\to 0} \dfrac{x^2 \sin\frac{1}{x} - 0}{x} = 0$

$f'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$, which is clearly not continuous at $x = 0$

$\therefore$ Statement (1): is false

Statement (2): is true (standard result)

12. STATEMENT - 1 $f(x) = |x[x]|$ is discontinuous at all Integers , where [.] denotes G.I.F

STATEMENT - 2 If a function is non-differentiable at a point then it may be continuous at that point

Key. D

Sol. $f(x) = |x[x]|$

$f(x)$ is continuous at $x = 0$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^-} f(x) = f(0) = 0$

13. Assertion (A) : There exists a function $g : R ® R$ continuous at $x = a$ satisfying $f(x) - f(a) = (x - a)g(x) " x Î R$

Reason (R) : $f : R ® R$ is a differentiable function at $x = a$. Then f(x) is also continuous at $x = a$

Key. A

Sol. $\lim\limits_{x\to\alpha} g(x) = \lim\limits_{x\to\alpha} \dfrac{f(x) - f(\alpha)}{x - \alpha} = f'(\alpha)$

Hence we can define $g(x)=\dfrac{f(x)-f(\alpha)}{x-\alpha};x\neq\alpha$

$= f'(a);\quad x=a$

Such that g(x) is continuous at $x=\alpha$

$\therefore$ Statement II is correct explanation for Statement – I.

14. Assertion (A) : $f:R\to R$ is a continuous function and $f\left(\dfrac{x+y}{3}\right)=\dfrac{f(x)+f(y)+f(0)}{3}\ \forall x,y\in R$

then $f(x)$ is differentiable for all $x\in R$

Reason (R) : If $f(x)$ is differentiable at x = 0, then $f'(0)=l$ (finite)

Key. D

Sol. Using first principle

$$f^1(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \qquad =\lim_{h\to 0}\frac{f\left(\dfrac{3x+3h}{3}\right)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(3x)+f(3h)+f(0)-3f(x)}{3h} \qquad =\lim_{h\to 0}\frac{f(3h)+f(0)}{3h-0}=f^1(0)=\lambda$$

Since we have by letting 3x for x and y = 0 in given equation $3f(x)=f(3x)+2f(0)$

$\therefore$ Statement I is supported by II

15. STATEMENT -1: Consider $f(x)=\dfrac{sgn\{x\}}{[x]}$ where [.] and {} denotes integral and fractional part respectively and $sgn(x)=\begin{cases}-1 & x<0\\ 0 & x=0\\ 1 & x>0\end{cases}$ then f(x) is discontinuous at x = n ($n\in I^+$).

because

STATEMENT-2: f(x) is said to be continuous at x = a if $\lim\limits_{x\to a^-}f(x)=\lim\limits_{x\to a^+}f(x)=f(a)$.

Key. A

Sol. at x = n, $n\in I^+$

$$\text{L.H.L}=\lim_{h\to 0}\frac{sgn\{n-h\}}{[n-k]}=\text{not defined when n = 1}$$

$$=\frac{1}{n-1},\ n\neq 1$$

$$\text{R.H.L}=\lim_{h\to 0}\frac{sgn\{n+h\}}{[n+h]}=\frac{1}{n}$$

f(n) = 0

so f(x) is discontinuous at x = n ($n\in I^+$)

16. STATEMENT-1. : Consider $f(x)=\begin{cases}\dfrac{x(2e^{1/x}-e^{-1/x})}{3e^{1/x}-4e^{-1/x}} & x\neq 0\\ 0 & x=0\end{cases}$

then f(x) is continuous at x = 0
because
STATEMENT-2. : If a function f(x) is so defined that $f'(a^+)$ and $f'(a^-)$ are finite and $f'(a^+)\neq f'(a^-)$ then f(x) must be continuous at x = a.

Key. A

Sol. Clearly at x = 0

$\left.\begin{array}{l}\text{LHL}=0\\ \text{RHL}=0\\ f(0)=0\end{array}\right\}\Rightarrow$ f(x) is continuous at x = 0.

17. Statement – 1 : $f(x)=[x]\cos\left(\left[\frac{2x-1}{2}\right]\pi\right)$ where [g] denotes the greatest integer function, is

discontinuous at $x=\frac{n}{2}, n\in I-\{1\}$

Because

Statement – 2 : If the domain of f (x) is $x\in R-(-1,1)$ then the domain of the function

$f\left([\sin x]\cos\frac{x}{[x-1]}\right)$ $\left(where[\ ]denotes\ \ the\ G.I.F\right)$ is $x\in\phi$.

Key. B

Sol. Statement – 1 : Case (i) $f(x)=x\cos\left[\frac{2x-1}{2}\right]\pi$ for $x\in N$,

$f(n)=n\ \cos(n-1)\pi$

$\lim\limits_{x\to n+} f(x)=n\cos(n-1)\pi$

$\lim\limits_{x\to n^-} f(x)=(n-1)\cos(n-1)\pi$

$\therefore$ Limit exists if $\cos(n-1)\pi=0$ which is not possible

$\therefore$ $f(x)$ is discontinuous at all $x\in I$

Case – II : when x is not an integer

$Let\frac{2x-1}{2}=m,$ m is integer then $x=\frac{2m+1}{2}=m+\frac{1}{2}$

$\lim\limits_{x\to\left(m+\frac{1}{2}\right)^+} f(x)=\ m\cos(m-1)\pi,\quad \lim\limits_{x\to\left(m+\frac{1}{2}\right)^-} f(x)=m\ \cos\ m\pi$

$\therefore$ limit exists only when m = 0 i.e. $x=\frac{1}{2}$ $Hence f(x)$ is discontinuous at

$x=\frac{n}{2}, n\in I-\{1\}$

Statement – 2 : Verify domain of given function is $x \in Q$

18. Statement – 1 : f(x) = [x] + [–x], where [.] greatest integer function is not continuous at an integral point n.
Because
Statement – 2 : $\lim_{x\to n^-} f(x) \neq \lim_{x\to n^+} f(x)$

Key. C

Sol. LHL = [x – h] + [–n + h] = n – 1 – 1
RHL = [x + h] + [–n – h] = n – (n + 1) = –1

19. Statement - 1: If $f(x)$ is discontinuous at x = e and $\underset{x\to a}{\text{lt}}\, g(x) = e;$ then $\underset{x\to a}{\text{lt}}\, f(g(x))$ can't be equal to $f\left(\underset{x\to a}{\text{lt}}\, g(x)\right)$

Statement - 2: If $f(x)$ is continuous at x = e and $\underset{x\to a}{\text{lt}}\, g(x) = e$ then $\underset{x\to a}{\text{lt}}\, f(g(x)) = f\left(\underset{x\to a}{\text{lt}}\, g(x)\right)$

Key. D

Sol. Statement 1 is incorrect because if $\underset{x\to a^-}{\text{lt}}\, g(x)$ and $\underset{x\to a^+}{\text{lt}}\, g(x)$ approach 'e' from the same side of e (say from right side). And $\underset{x\to e^+}{\text{lt}}\, f(x) = f(e) \neq \underset{x\to e^-}{\text{lt}}\, f(x)$, then

$$\underset{x\to a}{\text{lt}}\, f(g(x)) = f(e^+) = f(e)$$

20. Assertion (A) : $f(x) = \begin{cases} -1, x < 0 \\ 0, x = 0 \\ 1, x > 0 \end{cases}$ and $g(x) = x(1-x^2)$, then $g(f(x))$ is continuous at $x = 0$

Reason (R) : If $f(x)$ is discontinuous at $x = a, g(f(x))$ is also discontinuous at that point

Key. C

Sol. Conceptual

21. Assertion (A) : $\cos|x| + (x-3)^5 |x^2 - 4x + 3|$ is non-differentiable at x = 3

Reason (R) : $|x^2 - 4x + 3|$ is a non differentiable function.

Key. D

Sol. Conceptual

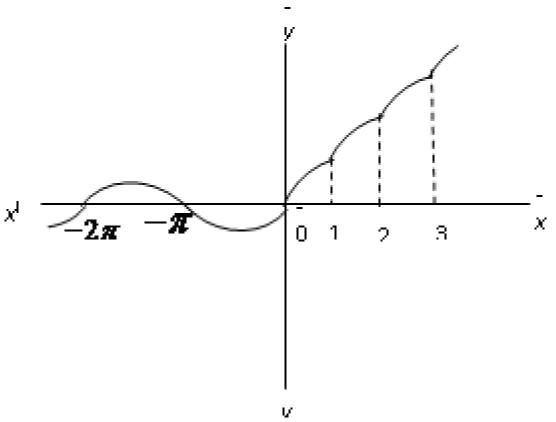
22. Let $f(x)=\begin{cases}[x]+\sqrt{x-[x]} & , \quad x\ge 0\\ \sin x & , \quad x<0\end{cases}$, where [.] denotes the greatest integer function.

Statement – 1 : $f(x)$ is continuous everywhere.

Statement – 2 : $f(x)$ is a periodic function.

Key. C

Sol. Hence, $f(x)$ is continuous everywhere but non periodic function.

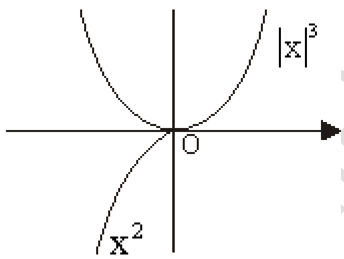


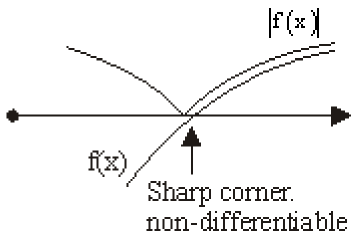
23. Statement – 1: $\left|x^3\right|$ is differentiable at x = 0

Statement – 2: |f(x) is differentiable at x = a then |f(x)| is also differentiable at x = a.

Key. C

Sol.





24. Statement – 1: f(x) = sin x + [x] is discontinuous at x = 0.

Statement – 2: If g(x) is continuous & h(x) is discontinuous at x = a, then g(x) + h(x) will necessarily be discontinuous at x = a

Key. A

Sol. $\lim\limits_{x\to 0^+}\left(\sin x+[x]\right)=0$

$\lim\limits_{x\to 0^-}\left(\sin x+[x]\right)=-1$

Limit doesn't exist

$\lim\limits_{x\to a}\left(f(x)+h(x)\right)=\lim\limits_{x\to a}f(x)+\lim\limits_{x\to a}h(x)$

$\neq f(a)+h(a)$

$\therefore$ f(x) + h(x) is discontinuous function

25. Statement – 1: f(x) = |x|, sin x is differentiable at x = 0

Statement – 2: If f(x) is not differentiable and g(x) is differentiable at x = a, then f(x). g(x) can still be differentiable at x = a

Key. A

Sol. f(x) = |x| sin x

$$\text{L.H.D} = \lim_{h\to 0}\frac{|0-h|\sin(0-h)-0}{h}$$

$$= \lim_{h\to 0}\frac{-h\sinh}{h} = 0$$

$$\text{R.H.D} = \lim_{h\to 0}\frac{|0+h|\sin(0+h)-0}{h}$$

f(x) is differentiable at x = 0

26. Statement – 1 f(x) = |[x] x| in ∈ [-1, 2], where [.] represents greatest integer function, is non differentiable at x = 2

Statement – 2: Discontinuous function is always non differentiable

Key. A

Sol. f(2) = 4

$$f(2^-) = \lim_{x\to 2^-}|[x]x| = 2$$

Discontinuous ⇒ Non. Differentiable

27. Statement – 1: Sum of left hand derivative and right hand devivative of $f(x) = |x^2 - 5x + 6|$ at x = 2 is equal to zero

Statement – 2: Sum of left hand derivative and right hand derivative of f(x) = |(x – a)(x – b)| at x = a (a < b) is equal to zero, (where a, b ∈ R)

Key. A

Sol. Statement – 1 $f(x) = \begin{cases} x^2 - 5x + 6 & , \quad x \le 2 \\ -x^2 + 5x = -6 & , \quad 2 \le x \le 3 \\ x^2 - 5x + 6 & , \quad x \ge 3 \end{cases}$

$$f'(x) = \begin{cases} 2x - 5 & , \quad x < 2 \\ -2x + 5 & , \quad 2 < x < 3 \\ 2x - 5 & , \quad x > 3 \end{cases}$$

$$f'(2^-) + f'(2^+) = -1 + 1 = 0$$

Statement – 2 $f(x) = \begin{cases} (x-a)(x-b) & , \quad x < a \\ -(x-a)(x-a) & , \quad a \le x \le b \\ (x-a)(x-b) & , \quad x > b \end{cases}$

$$f'(x) = \begin{cases} 2x - a - b & , \quad x < a \\ -2x + a + b & , \quad a < x < b \\ 2x - a - b & , \quad x > b \end{cases}$$

∴ $f'(a-) = a - b,\ f'(a^+) = -a + b$

∴ $f'(a-) + f'(a^+) = 0$

Statement – 2 explains statement 1.

28. Statement – 1: If f: R $\to$ R is a continuous function such that f(x) = f(3x) $\forall x \in R$, then f is constant function.

Statement – 2: If f is continuous at $x = \lim_{x\to a} g(x)$, then $\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$

Key. A

Sol. Statement – 2

$$f\left(\lim_{x\to a} g(a)\right) = f(b) = \lim_{x-b} f(x) = \lim_{g(x)\to b} f(g(x)) = \lim_{x\to a} f(g(x))$$

$\therefore \quad \lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$

$\therefore$ Statement is true

Statement – 1:

Since f is continuous on R

and $\quad f(x) = f\left(\frac{x}{3}\right) = f\left(\frac{x}{3^2}\right) ...... = f\left(\frac{x}{3^n}\right)$

and $\quad \lim_{h\to\infty} \frac{x}{3^n} = 0$

$\therefore \quad \lim_{n\to\infty} f(x) = \lim_{n\to\infty} f\left(\frac{x}{3^n}\right) = f\left(\lim_{x\to 0} \frac{x}{3^n}\right) = f(0)$

$\therefore$ f is a constant function

$\therefore$ Statement is true

29. Statement – 1: If f is continuous and differentiable in (a - δ, a + δ), where a, $\delta \in R$ and $\delta > 0$, then $f'(x)$ is continuous at x = a

Statement – 2: Every differentiable function at x = a is continuous at x = a

Key. D

Sol. Statement – 1: $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$,

Since $\lim_{x\to 0} f(x) = 0$, therefore, f(x) is continuous

$$f'(0) = \lim_{x\to 0} \frac{x^2 \sin\frac{1}{x} - 0}{x} = 0$$

$f'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x} & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$, which is clearly not continuous at x = 0.

$\therefore$ statement is false

Statement – 2 is true (standard result)

30. Statement I : The function $y = \sin^{-1}(\cos x)$ is not differentiable at $x = n\pi, n \in Z$, is particular at $x = \pi$

Statement II : $\frac{dy}{dx} = \frac{-\sin x}{|\sin x|}$ so the function is not differentiable at the points where $\sin x = 0$

Key. A

Sol. Reason is the solution of assertion.

31. Statement I : The function $x\tan\frac{1}{x}$ is discontinuous at x = 0..

Statement II : The function $x \tan\frac{1}{x}$ is not differentiable at x = 0.

Key. B

Sol. R.H.L = $\lim_{h\to 0^+} h\tan\left(\frac{1}{h}\right) = \lim it\ not\ exist$

A is true

$g(x)=\tan\left(\frac{1}{x}\right) \Rightarrow g'(x)=\frac{-\sec^2\left(\frac{1}{x}\right)}{x^2}$

$g(x)$ is discontinuous at $x=0$ thus $g'(x)$ may not be $-ve\forall x\in(-1,1)$

Continuity & Differentiability

Comprehension Type

Paragraph – 1

Let $p(x)$ be a polynomial with positive leading coefficient and $p(0)=0$; and

$p(p(x)) = x.\int_0^x p(t)\,dt, \forall\, x \in R.$ Then

1. Degree of the polynomial $p(x)$ is_____

(A) 4 (B) 3 (C) 5 (D) 2

Key. D

2. $\dfrac{p^1(x)}{|x|}$ is discontinuous at x =

(A) 0 (B) 1 (C) -1 (D) none of these

Key. A

3. If $p(1)=3; p(-1)=5$ and $g(x)$ is inverse of $p^1(x)\, then\, g^1(0)$ _______

(A) is equal to $\dfrac{1}{4}$ (B) is equal to $\dfrac{1}{8}$ (C) is equal to 8 (D) does not exist.

Key. B

Sol. (1) Degree of $p(x)$ is 2

(2) $\dfrac{p^1(x)}{|x|} = \dfrac{2ax+b}{|x|}$ is discontinuous at $x=0$

Q $\dfrac{2ax}{|x|}$ is discontinuous. We know if f is continuous and $'\underline{g}'$ is discontinuous then $f+g$ is discontinuous.

(3) $p(1)=3, p(-1)=5 \Rightarrow a=4, b=-1, \therefore\ p(x)=4x^2-x.$

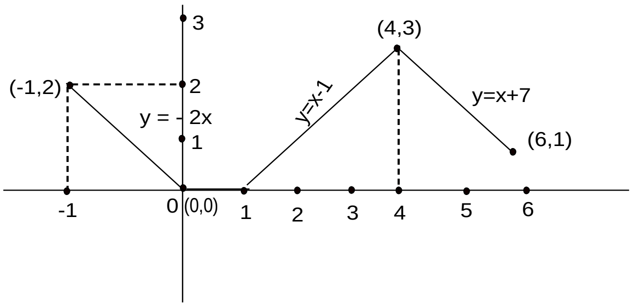
$p^1(x)=8x-1,$Q $'\underline{g}'$ is inverse of $p^1(x)$

$g(x)=\dfrac{x+1}{8}; g^1(0)=\dfrac{1}{8}$

Passage-II

From graph, $f^1(x)=\begin{cases} -2 & if \quad -1\le x<0 \\ 0 & if \quad 0\le x<1 \\ 1 & if \quad 1\le x<4 \\ -1 & if \quad 4\le x<6 \end{cases}$

$\therefore$ the graph of $f(x)$ will be as follows.



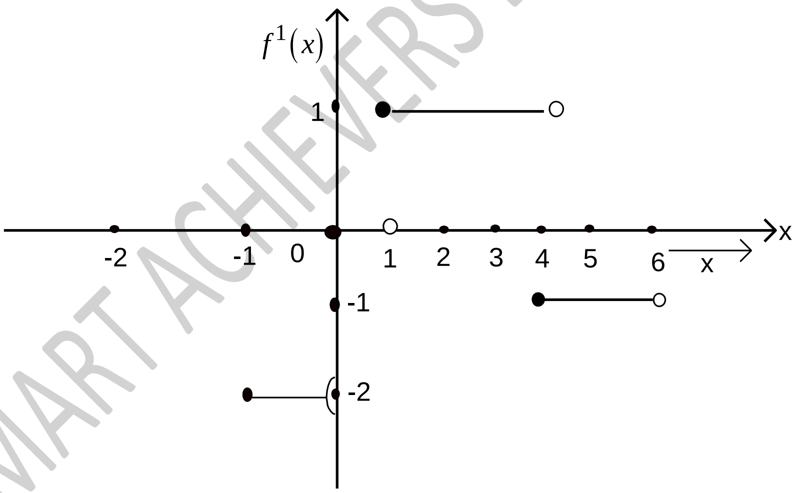
Paragraph – 2

Let $y = f(x)$ be a continuous function for $-1 \le x \le 6$ such that $f(0) = 0$ and

(1) The graph of $f^1(x)$ is made of line segments joined such that left end point is included and right end point is excluded in each sub interval

(2) The graph starts at the point $(-1,-2)$.

(3) The derivatives of $f(x)$, where defined, agrees with the step pattern as shown here.



Using the above information answer the following.

4. The range of $f(x)$ is _____

(A) $[-1,3]$ (B) $(0,3)$ (C) $[0,3]$ (4) $(0,3]$

Key. C

5. Number of integral roots of the equation $f(x)=1$ is

(A) exactly 3 (B) exactly 4 (C) exactly 2 (D) none

Key. C

6. The set of points of discontinuities of $f^{1}(x)$ in its domain are

(A) $\{0,1,4\}$ (B) $\{1,4,6\}$ (C) $\{0,1,4,6\}$ (D) none

Key. A

Sol. (4) Clearly range = $[0,3]$

(5) $y=1$ intersect the graph at '$\underline{3}$' points. Hence 3 – solutions.

(6) Points of discontinuities are $\{0,1,4\}$

Paragraph – 3

For $x>0;$ let $f(x)=\lim\limits_{n\to\infty}\dfrac{\log(2+x)+x^{2^n}\sin x}{1+x^{2^n}}$

Answer the following questions.

7. $\lim\limits_{x\to0^+} f(x)$ is equal to

(A) 0 (B) $\log_e 3$ (C) $\log_e 2$ (D) does not exist

Key. C

8. at $x=1,'\underline{f}'$_______

(A) continuous
(B) discontinuous
(C) both continuous and differentiable
(D) continuous but not differentiable.

Key. B

9. In $[0,\pi/2]$, the number of points at which '$\underline{f}$' vanishes is

(A) 0 (B) 1 (C) 2 (D) 3

Key. A

Sol. (7) Case(i):- Let $0<x<1$ then $f(x)=\log(2+x)\left(\because \lim\limits_{n\to\infty} x^{2^n}=0\right)$

(8) Case(ii):- Let $x=1,\ f(x)=\frac{1}{2}(\log3+\sin1)$

(9) Case(iii):- If $x>1,$ then

$$\lim_{n\to\infty}\frac{\log(2+x)+x^{2^n}\sin x}{1+x^{2^n}}=\lim_{n\to\infty}\frac{\left[\dfrac{\log(2+x)}{x^{2^n}}+\sin x\right]}{\dfrac{1}{x^{2^n}}+1}$$

$$\therefore f(x) = \begin{cases} \log(2+x) & if \quad 0 < x < 1 \\ \frac{1}{2}(\log 3 + \sin 1) & if \quad x = 1 \\ \sin x & if \quad x > 1 \end{cases} \qquad = \sin x.\left(\because \lim_{n\to\infty} \frac{1}{x^{2^n}} = 0\right)$$

All the 3-results obviously follows.

Paragraph – 4

$$f(x) = x^2 + xg'(1) + g''(2) \;\; and \;\; g(x) = f(1)x^2 + xf'(x) + f''(x)$$

10. The value of $f(3)$ is

A) 1 B) 0 C) -1 D) -2

11. The value of g(0) is

A) 0 B) -3 C) 2 D) 1

12. The domain of function $\sqrt{\frac{f(x)}{g(x)}}$ is

A) $(-\infty, 1] \cup (2, 3]$ B) $(-2, 0] \cup (1, \infty)$

C) $(-\infty, 0] \cup (\frac{2}{3}, 3]$ D) $(-\infty, \infty)$

Sol. 10. (B) Here put $g'(1) = a, g''(2) = b$....................(1)

Then $f(x) = x^2 + ax + b, f(1) = 1 + a + b \Rightarrow f'(x) = 2x + a$

$f''(x) = 2$

$\therefore g(x) = (1+a+b)x^2 + (2x+a)x + 2 = x^2(3+a+b) + ax + 2$

$\Rightarrow g'(x) = 2x(3+a+b) + a$ and $g''(x) = 2(3+a+b)$

Hence , $g'(1) = 2(3+a+b) + a$............(2)

$g''(2) = 2(3+a+b)$........................(3)

From (1) ,(2) and (3), we have

$a = (3+a+b) + a$ and $b = 2(3+a+b)$

$\Rightarrow 3 + a + b = 0$ and $b + 2a + 6 = 0$

Hence $b = 0$ and $a = -3$. So $, f(x) = x^2 - 3x$

$\Rightarrow f(3)=0$

11. (C)

$g(x)=-3x+2$

$\Rightarrow g(0)=2$

12. (C)

$\sqrt{\frac{f(x)}{g(x)}}=\sqrt{\frac{x^2-3x}{-3x+2}}$ is defined if $\frac{x^2-3x}{-3x+2}\geq 0$

$\Rightarrow \frac{x(x-3)}{\left(x-\frac{2}{3}\right)}\leq 0$

$\Rightarrow x\in(-\infty,0]\cup(\frac{2}{3},3]$

Paragraph –5

A function $f(x)$ is said to be continuous at $x=a$ if $\lim_{x\to a^-} f(x)=\lim_{x\to a^+} f(x)=f(a)$. i.e., $\lim_{x\to a} f(x)=f(a)$. When $f(x)$ is not continuous at $x=a$ we say that $f(x)$ is iscontinuous at $x=a$.

13. If $f(x)=\lim_{m\to\infty}\sin^{2m}x$, then number of point(s) where $f(x)$ is discontinuous is

A) 0 B) 1 C) 2 D) infinitely many

14. $\lim_{x\to 1}\frac{\sqrt{1-\cos 2(x-1)}}{x-1}$

A) exists and it equals $\sqrt{2}$ B) exists and it equals $-\sqrt{2}$

C) does not exists because $L.H.L\neq R.H.L$ D) exists and it equals 1/2

15. In order that the function $f(x)=(x+1)^{\cot x}$ is continuous at $x=0$, $f(0)$ must be defined as

A) 0 B) e C) 1/e D) 1

Sol. 13. (D)

$f(x)=\lim_{x\to\infty}\left(\sin^2 x\right)^m=1,\ x=(2n+1)\frac{\pi}{2},\ n\in I$

$=0, x\neq(2n+1)\frac{\pi}{2}, n\in I$

$\therefore f(x)$ is discontinuous at $x=(2n+1)\frac{\pi}{2}, n\in I$

14. (C) $L.H.L \neq R.H.L$, as $\lim_{x\to 1^-} -\sqrt{2}\left(\frac{\sin(x-1)}{x-1}\right) = -\sqrt{2}$ and $\lim_{x\to 1^+} \sqrt{2}\left(\frac{\sin(x-1)}{x-1}\right) = \sqrt{2}$

15. (B) $f(0) = \lim_{x\to 0}(1+x)^{\cot x} = e^{\lim_{x\to 0}\cot(1+x-1)} = e^{\lim_{x\to 0}\frac{x}{\tan x}} = e$

Paragraph – 6

Let α, β be the roots of $ax^2+bx+c=0\,(a\neq 0)$ and $ax^2+bx+c = a(x-\alpha)(x-\beta)$ then evaluate the following limits:

16. $\lim_{x\to\beta}\frac{1-\cos(ax^2+bx+c)}{(x-\beta)^2}$ is

A) $\frac{a^2(\beta-\alpha)^2}{2}$ B) $b^2\alpha^2\beta^2$ C) $\frac{c^2(\alpha+\beta)^2}{2}$ D) $\frac{a^2(\beta+\alpha)^2}{2}$

17. $\lim_{x\to 1/\alpha}\frac{1-\cos(cx^2+bx+a)}{(1-\alpha x)^2}$ is

A) $\frac{a^2}{2\beta^2}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)^2$ B) $\frac{c^2}{2\alpha^2}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)^2$ C) $\frac{b^2}{2\alpha^2}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)^2$ D) $\frac{a^2}{2\alpha^2}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)^2$

18. If α, β are the roots of $x^2+ax+b=0$, then $\lim_{x\to\alpha}\frac{1-\cos(x^2+ax+b)}{(x-\alpha)^2}$

A) $\frac{a^2(\alpha-\beta)^2}{2}$ B) $\frac{b^2(\alpha-\beta)^2}{2}$ C) $\frac{(\alpha-\beta)^2}{2}$ D) $\frac{(\alpha-\beta)^2}{4}$

Sol. 16. (A)

$$\lim_{x\to\beta}\frac{2\sin^2\frac{a(x-\alpha)(x-\beta)}{2}}{a^2(x-\beta)^2\frac{(x-\alpha)^2}{4}}\times\frac{a^2(x-\alpha)^2}{4} = \frac{a^2(\beta-\alpha)^2}{2}$$

17. (B)

$$\lim_{x\to 1/\alpha}\frac{2\sin^2\frac{c\left(x-\frac{1}{\alpha}\right)\left(x-\frac{1}{\beta}\right)}{2}}{\alpha^2.\frac{\left(x-\frac{1}{\alpha}\right)^2\left(x-\frac{1}{\beta}\right)^2}{4}}\times\frac{c^2\left(x-\frac{1}{\beta}\right)^2}{4} = \frac{c^2}{2\alpha^2}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)^2$$

18. (C)

$$\lim_{x\to\alpha}\frac{2\sin^2\frac{(x-\alpha)(x-\beta)}{2}}{\frac{(x-\alpha)^2(x-\beta)^2}{4}}\times\frac{(x-\beta)^2}{4}=\frac{(\alpha-\beta)^2}{2}$$

Paragraph – 7

A real function f has the intermediate value property on an interval I containing [a, b] if f(a) < v < f(b) or f(b)< v < f(a) ; that is, if v is between f(a) and f(b), there is between a and b some c such that f(c) = v.

19. Which of the following statements is false?

A) Any continuous function defined on a closed and bounded interval [a, b] possesses intermediate value property on that interval.

B) If a function is discontinuous on [a, b] then it doesn't possess intermediate property on that interval.

C) If f has a derivative at every point of the closed interval [a, b], then f takes on every value between $f(a)$ and $f(b)$.

D) If f has a derivative at every point of the closed interval [a, b], then f' takes on every value between $f'(a)$ and $f'(b)$.

Key. B

Sol. A) is well known to be true.

C) is true because then f become continuous

D) is known as Darboux's theorem although derivatives are not continuous they still enjoy intermediate value property

B) is false .There are discontinuous functions enjoying intermediate value property .Consider f on the interval $\left[-\frac{2}{\pi},\frac{2}{\pi}\right]$

$f(x)=\sin\frac{1}{x}, x\neq 0$

$= 0, x=0$

On the interval $\left[-\frac{2}{\pi},\frac{2}{\pi}\right]$, this function takes on all values between $f(-2/\pi)$ and $f(2/\pi)$ that is between -1, and 1 an infinite number of times as x varies from $-2/\pi$ to $-2/\pi$ but f is not continuous at this interval being discontinuous of x = 0

20. Consider the statements P and Q

P: If $f:(a,b)\to R$ is continuous, then given x_1,x_2,x_3,x_4 in (a, b), there exist $x_0\in(a,b)$ such that $f(x_0)=\frac{1}{4}(f(x_1)+f(x_2)+f(x_3)+f(x_4))$.

Q: If f and g have the intermediate value property on [a, b], then so has f+g on that interval. Which of the following is correct?

A) P is false but Q is true B) P is true but Q is false

C) Both P and Q are false D) Both P and Q are true

Key. B

Sol. Put $m = \min \{f(x_1), f(x_2)........f(x_n)\}$

$M = \max \{f(x_1),.....f(x_n)\}$

Then $m \le \frac{1}{n}\left(f(x_1)+f(x_2)+........+f(x_n)\right) \le M$

Then $\exists\ x_0 \in (a, b)$ such that

$f(x_0) = \frac{1}{x}\ \left(f(x_1)+f(x_2)+......+f(x_n)\right)$

So P is true, But Q is false so the counter example.

Define $f(x) = \sin\frac{1}{x-a}, a < x \le b$

0 , x = a

And $g(x) = -\sin\frac{1}{x-a},\ a < x \le b$

= 1 , x = a

f and g have intermediate value property from [a,b] but f + g doesn't have.

21. Consider the statements P and Q

P: For a non zero polynomial p, the equation $|p(x)| = e^x$ has at least one solution.

Q: There exists a continuous function $f: R \to R$ which attains each of its values exactly two times.

A) P is false but Q is true B) P is true but Q is false

C) Both P and Q are false D) Both P and Q are true

Key. B

Sol. Let $f(x) = e^{-x}|p(x)|$

$\lim_{x\to\infty} e^{-x}|p(x)| = 0$ and $\lim_{x\to\infty} e^{-x}|p(x)| = \infty$

Then f $\exists\ x_0 \in$ such that $f(x_0) = 1$

$\Rightarrow e^{-x_0}|p(x_0)| = 1, \therefore |P(x)| = e^{x_0}$

Then p is true

Q is false (proof by contradiction)

Suppose that f is a continuous function that attains each of its values exactly twice.Let x_1, x_2 be such that $f(x_1) = f(x_2) = b$,then $f(x) \neq b$ for $x \neq x_1, x_2$.

On (x_1,x_2) assume $f(x) > b$, (similar analysis will hold for $f(x) < b$), Let x_0 be the point which f attains its maximum on $[x_1, x_2]$.There can be exactly one such x_0. For it there are more, say 2 points at which the function attained its maximum value on $[x_1, x_2]$, then f should assume some values more than twice in $[x_1, x_2]$, But the function is for bidden to do so

Again, outside $[x_1, x_2]$, there is exactly one point x_0 such that $c = f(x_0) = f(x'_0) > b$

The intermediate value property implies that every value in (b,c) is attained at least three times. A contradiction.

Paragraph – 8

L' Hopital's rule has many versions. One of them is this.

Suppose f, g: (a, b) $\to \mathbb{R}$ are differentiable on (a, b). Suppose further that

(i) $g'(x) \neq 0$ for $x \in (a, b)$ (ii) $\lim_{x\to a^+} g(x) \to \infty \ (\text{or} -\infty)$

(iii) $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$ Then $\lim_{x\to a^+} \frac{f(x)}{g(x)} = L$

(This rule can be extended to cover the case when a or b tends to infinity or L tends to infinity)

22. Let f be a differentiable function on $(0, \infty)$

If $\lim_{x\to\infty}\left(\sin\left(\frac{\pi}{10}\right).f(x) + f'(x)\right) = \sec\left(\frac{\pi}{5}\right)$, then $\lim_{x\to\infty} f(x)$ equals

A) $\frac{1}{4}$ B) 4 C) $3-\sqrt{5}$ D) $\frac{3+\sqrt{5}}{4}$

Key. B

Sol. If $\lim_{x\to\infty}\left(a f(x) + f'(x)\right) = l$ then $\lim_{x\to\infty} f(x) = l/a$, a > 0

We have $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{e^{ax} f(x)}{e^{ax}} = \lim_{x\to\infty} \frac{e^{ax}\left(a f(x) + f'(x)\right)}{ae^{ax}} = l/a$

23. Let f be a differentiable function on $(0, \infty)$.

If $\lim_{x\to\infty}\left(\tan\left(\frac{\pi}{8}\right).f(x) + 2\sqrt{x}f'(x)\right) = \cot\frac{\pi}{12}$, then $\lim_{x\to\infty} f(x)$ equals

A) $\sqrt{8}-\sqrt{6}+\sqrt{4}-\sqrt{3}$ B) $\sqrt{8}+\sqrt{6}-\sqrt{4}-\sqrt{3}$

C) $\sqrt{3}+\sqrt{4}+\sqrt{6}+\sqrt{8}$ D) $\sqrt{8}-\sqrt{6}-\sqrt{4}+\sqrt{3}$

Key. C

Sol. $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{e^{a\sqrt{x}} f(x)}{e^{a\sqrt{x}}} = \lim_{x\to\infty} \frac{e^{a\sqrt{x}}\{f'(x) + \frac{a}{2\sqrt{x}} f(x)}{\frac{a}{2\sqrt{x}} e^{a\sqrt{x}}}$

$$= \lim_{x\to\infty} \frac{1}{a}\left(af(x) + 2\sqrt{x}f'(x)\right) = \frac{l}{a}$$

24. Let f be three times differentiable on $(0,\infty)$ and such that $f(x)>0, f'(x)>0, f''(x)>0$ for $x>0$

If $\lim\limits_{x\to\infty}\dfrac{f'(x)f'''(x)}{(f''(x))^2}=\tan\dfrac{\pi}{12}$, then $\lim\limits_{x\to\infty}\dfrac{xf''(x)}{f'(x)}$ equals

A) $2+\sqrt{3}$ B) $2-\sqrt{3}$ C) $\dfrac{\sqrt{3}-1}{2}$ D) $\dfrac{\sqrt{3}+1}{2}$

Key. D

Sol. $\lim\limits_{x\to\infty}\left(1-\dfrac{f'(x)}{xf''(x)}\right)=\lim\limits_{x\to\infty}\dfrac{\left(x-\dfrac{f'(x)}{f''(x)}\right)'}{x}$

$=\lim\limits_{x\to\infty}\dfrac{f'(x)f'''(x)}{\left(f''(x)\right)^2}=c$

Then $\lim\limits_{x\to\infty}\dfrac{f'(x)}{xf''(x)}=1-c$

Paragraph – 9

At $x=c,$ a function f is said to have

(i) Removable discontinuity if $\lim\limits_{x\to c} f(x)$ exists but not equals to $f(c)$

(ii) Jump discontinuity if $f(c+), f(c-)$ exist but not equal

(iii) Infinite discontinuity if $f(c-)$ or $f(c+)$ or both fail to exist

Answer the following

25. $f(x)=\begin{cases} x^2+5 & if \quad x<2 \\ 10 & if \quad x=2 \\ 1+x^3 & if \quad x>2 \end{cases}$. Then $x=2$ is

A. a point of continuity **B.** a removable discontinuity
C. a jump discontinuity **D.** an infinite discontinuity

Key. B

Sol. $f(2-)=9$ $f(2+)=9$ $f(2)=10$

26. $g(x)=\begin{cases} x+7 & if \quad x<-3 \\ |x-2| & if \quad -3\le x<-1 \\ x^2-2x & if \quad -1\le x<3 \\ 2x-3 & if \quad x\ge 3 \end{cases}$ then g has

A. jump discontinuity at $x=-1$
B. infinite discontinuity at $x=3$
C. jump discontinuity at $x=-3$
D. Removable discontinuity at $x=-1$

Key. C

Sol. $f((-3)-)=4,\ f((-3+)=1 \quad f(-3)=1$

$\therefore f$ has jump discontinuity at $x=-3$

Paragraph – 10

$f'(a^-)$ denotes left hand derivative at x = a and $f'(a^+)$ denotes right hand derivative at x = a. If $f'(a^-)=f'(a^+)$, then f is derivable at x = a. otherwise f is not derivable at x=a

27. $f(x)=\begin{cases} 1 \quad for \quad x<0 \\ 1+\sin x \quad for \quad 0\le x\le \frac{\pi}{2} \\ 2+\left(x-\frac{\pi}{2}\right)^2 \quad for \quad x\ge \frac{\pi}{2} \end{cases}$ then f is derivable at x =

A) 0 only
B) $\frac{\pi}{2}$ only
C) Both 0 and $\frac{\pi}{2}$
D) ϕ

28. $f(x)=\begin{cases} 1-x \quad , \quad x<1 \\ (1-x)(2-x), \quad 1\le x\le 2 \\ 3-x \quad , \quad x>2 \end{cases}$ then $f(x)$ at $x=2$ is

A) Continuous but not differentiable

B) Differentiable but not continuous

C) Both differentiable and continuous

D) Not differentiable and not continuous

29. $f(x)=\begin{cases} 1 & x>0 \\ -1 & x\le 0 \end{cases}$ then f(x) is

A) Differentiable at x=0

B) Continuous at x=0

C) Both differentiable and continuous at x=0

D) Neither differentiable nor continuous at x=0

Sol. 27. (B) $f^1(0-)=0, f^1(0+)=1$ and $f^1\left(\frac{\pi}{2}-\right)=0, f^1\left(\frac{\pi}{2}+\right)=0$

28. (D) L.H.L ≠ R.H.L at x=2

Not continuous

∴ not differentiable

29. (D) Graph of f(x) broken at x=0

Paragraph – 11

Let a real valued function f be defined by the setting

$$f(x)=\begin{cases} x^{\alpha}\sin\frac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$$ where α is a non-zero integer.

30. The set of all values of α for which f is continuous at the origin is

A) $\alpha>0$ B) $\alpha\geq 2$ C) $\alpha>1$ D) $\alpha\geq 3$

31. The set of all values of α for which f is differentiable at the origin is

A) $\alpha\geq 2$ B) $\alpha\geq 1$ C) $\alpha>3$ D) $\alpha>4$

32. The set of all values of α for which f' is continuous at the origin is

A) $\alpha>1$ B) $\alpha\geq 4$ C) $\alpha>2$ D) $\alpha>4$

Sol. 30. (A)

$$\lim_{x\to 0} f(x)=\lim_{x\to 0} x^{\alpha}\sin\frac{1}{x}$$

As $\left|\sin\frac{1}{x}\right|\leq 1$, the above limit tends to zero when $\alpha>0$.

31. (A)

$$f'(0)=\lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

$$=\lim_{h\to 0}\frac{h^{\alpha}\sin\frac{1}{h}}{h}$$

$$=\lim_{h\to 0} h^{\alpha-1}\sin\frac{1}{h}$$

From the reasoning similar to that in the previous question $\alpha-1>0\Rightarrow\alpha>1$.

32. (C)

It easily follows on the lines of above two questions that for f' to be continuous at origin $\alpha > 2$.

Paragraph –12

A real valued function f satisfies the following conditions

(i) $f(x-y) = f(x)f(y) - f(a-x)f(a+y)$ for all $x, y \in \mathbf{R}$ (where a is a constant)

(ii) $f(0) \neq 0$ (iii) $f'(0) = 0$ (iv) $f'(a) = 1$

33. The value of $f(2a)$ equals

A) -1 B) 0 C) 1 D) a

34. $f'(x)$ equals

A) $f(a-x)+f(a+x)$ B) $f(a+x)-f(a-x)$

C) $f(a-x)$ D) $f(a+x)$

35. $g(x) = e^x f(x) \Rightarrow g'(0) =$

A) -1 B) 1 C) 0 D) e

Sol. 33. (A) Put $x = a$ and $y = 0$ in (i).

Then $f(a) = f(a)f(0) - f(0)f(a) = 0$

Again choosing $x = y = 0$ in (i),we get

$f(0) = [f(0)]^2 - [f(a)]^2 = [f(0)]^2 \quad and\ so, f(0) = 1$

Put $x = a$ and $y = -a$ in (i) to get $f(2a) = -1$

34. (C) $\underset{h\to 0}{Lt} \dfrac{f(x+h)-f(x)}{h} = \underset{h\to 0}{Lt} \dfrac{f(x)f(-h)-f(a-x)f(a-h)-f(x)}{h}$

$= \underset{h\to 0}{Lt}\left\{f(x)\left[\dfrac{f(-h)-f(0)}{h}\right]+f(a-x)\left[\dfrac{f(a-h)-f(a)}{-h}\right]\right\}$

$= f(a-x)f'(a) - f(x)f'(0) = f(a-x)$

$\therefore f'(x) = f(a-x) \ \ for\ \ all\ x \in R$

35. (B)

$g(x) = e^x f(x) \Rightarrow g'(x) = e^x(f(x)+f'(x) = e^x\left[f(a-x)+f(x)\right] \Rightarrow g'(0) = f(0) = 1$

Paragraph –13

It can be shown that if $f(x)$ is continuous at 0 then $xf(x)$ is differentiable at x = 0. by changing origin, we can say that if $f(x)$ is continuous at a then $(x-a)f(x-a)$ is differentiable at $x = a$

36. The largest set over which $\frac{x\sin|x|}{1-|x|^2}$ is differentiable is

a) $R-\{0,1,-1\}$ b) R c) $R-\{-1,1\}$ d) None

Key. C

37. The number of points where the function $(x-3)|x^2-7x+12|+\cos|x-3|$ is not differentiable is

a) one b) two c) three d) infinite

Key. A

38. Let $f(x)=|x|$, $g(x)=\sin x$, $h(x)=g(x).f(g(x))$, then

a) $h(x)$ is continuous but not differentiable at x = 0

b) $h(x)$ is continuous and differentiable everywhere.

c) $h(x)$ is continuous everywhere and differentiable only at x = 0

d) None of these

Key. B

Sol. 36. At $x=1,-1$ its not differentiable

37. At $x=4$ its not differentiable.

38. $h(x)=\sin x|\sin x|$

Check only at $x=n\pi$

LHD = RHD = 0

Paragraph –14

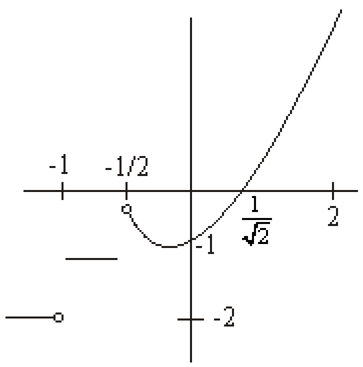
Let a function of defined as $f(x)=\begin{cases}[x] & , -2\le x\le -\frac{1}{2}\\ 2x^2-1 & , -\frac{1}{2}<x\le 2\end{cases}$, where [.] denotes greatest integer function. Answer the following question by using the above information.

39. The number of points of discontinuity of f(x) is

A) 1 B) 2 C) 3 D) N. O. T

Key. B

Sol.



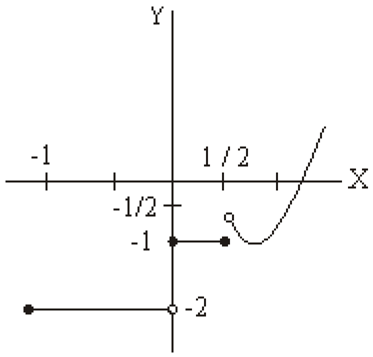
Two pt of discount. – 1, -1/2

40. The function f(x – 1) is discontinuous at the points

A) $-1,-\frac{1}{2}$ B) $-\frac{1}{2},1$ C) $0,\frac{1}{2}$ D) 0, 1

Key. C

Sol.

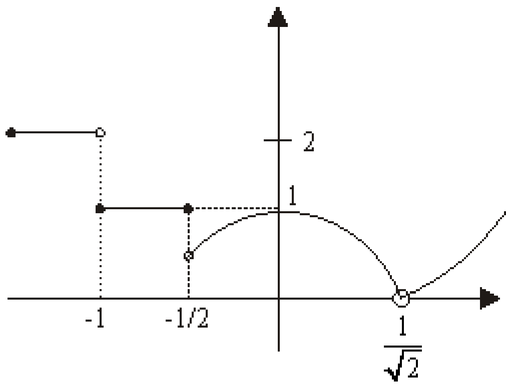


41. Number of points where |f(x)| is not differentiable is

A) 1 B) 2 C) 3 D) 4

Key. C

Sol.



At – 1, - ½, $1/\sqrt{2}$ the function is not differentiable

Paragraph –15

Consider two function y = f(x) and y = g(x) defined as $f(x)=\begin{cases} ax^2+b & , \quad 0\le x\le 1 \\ 2bx+2b & , \quad 1<x\le 3 \\ (a-1)x+2a-3 & , \quad 3<x\le 4 \end{cases}$

and $g(x)\begin{cases} cx^2+d & , \quad 0\le x\le 2 \\ dx+3-c & , \quad 2<x<3 \\ x^2+b+1 & , \quad 3\le x\le 4 \end{cases}$

42. f(x) is continuous at x = 1 but not differentiable at x = 1, if

A) a = 1, b = 0 B) a = 1, b = 2 C) a = 3, b = 1 D) a and b are integers

Key. C

Sol. $\lim_{h\to 1^-} f(x)=a+b$

$\lim\limits_{h\to 1^+} f(x) = 4b$

For continuity a + b = ab i.e. a = 3b …(i)

$$f'(1^+) = \lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+}\frac{2b(1+h)+2b-(a+b)}{h} = \lim_{h\to 0^+}\frac{3b-a+2bh}{h} = 2a$$

$$f'(1^-) = \lim_{h\to 0^+}\frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0^+}\frac{a(1-h)^2+b-a-b}{-h} = \lim_{h\to 0^+}\frac{a(-2h+h^2)}{-h} = 2a$$

2a + 2b, $a \neq b$

43. g(x) is continuous at x = 2, if

A) c = 1, d = 2 B) c = 2, d = 3 C) c = 1, d = - 1 D) c = 1, d = 4

Key. A

Sol. $\lim\limits_{x\to 2^-} g(x) = \lim\limits_{x\to 2^-} cx^2 + d = 4c + d$

$\lim\limits_{x\to 2^+} g(x) = \lim\limits_{x\to 2^+} (dx+3-c) = 2d+3-c$

g(2) = 4c + d

∴ 4c + d = 2d + 3 – c

∴ d = 5c – 3

44. If f is continuous and differentiable at x = 3, then

A) $a = -\frac{1}{3}, b = \frac{2}{3}$ B) $a = \frac{2}{3}, b = -\frac{1}{3}$ C) $a = \frac{1}{3}, b = -\frac{2}{3}$ D) $a = 2, b = \frac{1}{2}$

Key. D

Sol. $\lim\limits_{x\to 3^-} f(x) = 8b,\ \lim\limits_{x\to 3^+} f(x) = 3(a-1)+2a-3 = 5a-6$

Since f(x) is continuous at x = 3

∴ 8b = 5a – 6

$$f'(3^-) = \lim_{h\to 0^+}\frac{f(3+h)-f(3)}{h} = \lim_{h\to 0^+}\frac{(a-1)(3+h)+2a-3-8b}{h}$$

Since f is differentiable at x = 3

∴ $\lim\limits_{h\to 0^+} (a-1)(3+h)+2a-3-8b = 0$ i.e. 5a - 8b – 6 = 0

∴ $f'(3^+) = a-1$

thus a – 1 = 2b …(ii)

from (i) and (ii), we get a = 2, $b = \frac{1}{2}$

Paragraph –16

Let hand derivative and Right hand derivative of a function f(x) at a point x = a are defined as

$$f'(a^-) = \lim_{h\to 0^+}\frac{f(a)-f(a-h)}{h} = \lim_{h\to 0^-}\frac{f(a+h)-f(a)}{h} \text{ and}$$

$$f'(a^+) = \lim_{h\to 0^+}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0^-}\frac{f(a)-f(a-h)}{h} = \lim_{x\to a^+}\frac{f(a)-f(x)}{a-x} \text{ respectively.}$$

Let f be a twice differentiable function.

45. If f is odd, which of the following is Left hand derivative of f at x = - a

A) $\lim\limits_{h\to 0^-}\dfrac{f(a-h)-f(a)}{-h}$ B) $\lim\limits_{h\to 0^-}\dfrac{f(h-a)-f(a)}{h}$

C) $\lim\limits_{h\to 0^+}\dfrac{f(a)+f(a-h)}{-h}$ D) $\lim\limits_{h\to 0^-}\dfrac{f(-a)-f(-a-h)}{-h}$

Key. A

Sol. $L.H.D=\lim\limits_{h\to 0^+}\dfrac{f(-a+h)-f(-a)}{h}=\lim\limits_{h\to 0^-}\dfrac{-f(a-h)+f(a)}{h}=\lim\limits_{h\to 0^-}\dfrac{f(a-h)-f(a)}{-h}$

46. If f is even which of the following is Right hand derivative of f' at x = a.

A) $\lim\limits_{h\to 0^-}\dfrac{f'(a)+f'(-a+h)}{h}$ B) $\lim\limits_{h\to 0^+}\dfrac{f'(a)+f'(-a-h)}{h}$

C) $\lim\limits_{h\to 0^-}\dfrac{-f'(-a)+f'(-a-h)}{-h}$ D) $\lim\limits_{h\to 0^+}\dfrac{f'(a)+f'(-a+h)}{-h}$

Key. A

Sol. If f is even, then $f'(-x)=-f'(x)$

$\therefore$ $f'(a^+)=\lim\limits_{h\to 0^-}\dfrac{f'(a-h)-f'(a)}{-h}=\lim\limits_{h\to 0^-}\dfrac{f'(a)-f'(a-h)}{h}=\lim\limits_{h\to 0^-}\dfrac{f'(a)+f'(h-a)}{h}$

47. The statement $\lim\limits_{h\to 0}\dfrac{f(-x)-f(-x-h)}{h}=\lim\limits_{h\to 0}\dfrac{f(x)-f(x-h)}{-h}$ implies that

A) f is odd B) f is even

C) f is neither odd nor even D) nothing can be concluded

Key. B

Sol. $\lim\limits_{h\to 0}\dfrac{f(-x)-f(-x-h)}{h}=f'(-x)$ and $\lim\limits_{h\to 0}\dfrac{f(x)-f(x-h)}{-h}=-f'(x)$

$\Rightarrow$ $f'(-x)=-f'(x)$

$\therefore$ $f'(x)$ is an odd function

$\therefore$ f is an even function

Paragraph –17

There are two systems S_1 and S_2 of definitions of limit and continuity. In system S_1 the definition are as usual in system S_2 the definition of limit is as usual but the continuity is defined as follows:

A function f(x) is defined to be continuous at x = a if

(i) $\left|\lim\limits_{x\to a^-}f(x)-\lim\limits_{x\to a^+}f(x)\right|\le 1$ and

(ii) f(a) lies between the values of $\lim\limits_{x\to a^-}f(x)$ and $\lim\limits_{x\to a^+}f(x)$ if $\lim\limits_{x\to a^-}f(x)\ne\lim\limits_{x\to a^+}f(x)$ else $f(a)=\lim\limits_{x\to a^-}f(x)=\lim\limits_{x\to a^+}f(x)$

Read the above passage carefully and answer the following

48. If $f(x)=\begin{cases} x+2.7 & , \quad x<0 \\ 2.9 & , \quad x=0 \\ 2x+3 & , \quad x>0 \end{cases}$ and $g(x)=\begin{cases} 3x+3 & , \quad x<0 \\ 2.8 & , \quad x=0 \\ -x^2+2.7 & , \quad x>0 \end{cases}$, then consider statements

(i) f(x) is discontinuous under the system S_1

(ii) f(x) is continuous under the system S_2

(iii) g(x) is continuous under the system S_2

Which of the following option is correct

A) Only (i) is true B) only (i) and (ii) are true

C) only (ii) and (iii) are true D) all (i), (ii), (iii) are true

Key. D

Sol. If $f(x)=\begin{cases} x+2.7 & , \quad x=0 \\ 2.9 & , \quad x=0 \\ 2x+3 & , \quad x>0 \end{cases}$ and $g(x)=\begin{cases} 3x+3 & , \quad x<0 \\ 2.8 & , \quad x=0 \\ -x^2+2.7 & , \quad x>0 \end{cases}$

Then $\lim\limits_{x\to 0^-} f(x)=2.7,\ \lim\limits_{x\to 0^+} f(x)=3$

∴ |3 – 2.7| 0.3 < 1 and f(0) = 2.9 lies in (2, 7, 3)

∴ f(x) is continuous under the system S_2

g(x) is also continuous under the system S_2

under system S_1, since $\lim\limits_{x\to 0} f(x)$ does not exist

∴ f(x) is not continuous

∴ (i), (ii) and (iii) all are true

49. If each of f(x) and g(x) is continuous at x = a in S_2, then in S_2 which of the following is continuous

A) f + g B) f – g

C) f . g D) None of these

Key. D

Sol. Let $f(x)=\begin{bmatrix} x+2.7 & , \quad x<0 \\ 2.9 & , \quad x=0 \\ 2x+3 & , \quad x>0 \end{bmatrix}$ and $g(x)=\begin{bmatrix} 3x+3 & , \quad x<0 \\ 2.9 & , \quad x=0 \\ -x^2+2.75 & , \quad x>0 \end{bmatrix}$

∴ $(f+g)(x)=\begin{bmatrix} 4x+5.7 & , \quad x<0 \\ 5.8 & , \quad x=0 \\ 2x-x^2+5.75 & , \quad x>0 \end{bmatrix}$

∴ $\lim\limits_{x\to 0^-}(f+g)(x)=5.7$ and $\lim\limits_{x\to 0^+}(f+g)(x)=5.75$

∴ $\left|\lim\limits_{x\to 0^-}(f+g)-\lim\limits_{x\to 0^+}(f+g)\right|=0.5<1$ is satisfied

∴ (f + g) (0) = 5.8 which do not lie in (5. 7, 5.75)

∴ f + g is not continuous

Similarly we can show that f – g and f.g are not continuous under S_2

50. Which of the following is incorrect

A) a continuous function under the definition in S_1 must also be continuous under the definition in S_2

B) A continuous function under the definition in S_2 must also be continuous under the definition in S_1

C) A discontinuous function under the definition in S_1 must also be discontinuous under the definition in S_2

D) A discontinuous function under the definition in S_1 must be continuous under the definition in S_2

Key. B

Sol. A function continuous under system S_2 may not be continuous under system S_1

Paragraph –18

Let a function $f(x)$ be defined by $f(x)=\begin{cases} b\sin^{-1}\dfrac{x+c}{2},\ if -\dfrac{1}{2}<x<0 \\ \dfrac{1}{2} \quad ,if\ x=0 \\ \dfrac{e^{\frac{ax}{2}}-1}{x} \quad ,if\ 0<x<\dfrac{1}{2} \end{cases}$, it is given that $|c|<\dfrac{1}{2}$.

Answer the following

51. $f(x)$ may be continuous for

A) $a=1$ B) $a=-\dfrac{1}{2}$ C) $a=\dfrac{1}{2}$ D) $a=2$

Key. A

52. If $f(x)$ is differentiable at x = 0, if

A) $16b^2=4-c^2$ B) $64b^2=4-c^2$ C) $4b^2=4-c^2$ D) $16b^2=c^2+4$

Key. C

53. If $f(x)$ is differentiable at zero, then in addition to the conditions imposed on a,b,c described in a,b,c described in (58) and (59) above, we must have

A) $c=\dfrac{1}{2}\sin^{-1}\dfrac{1}{2b}$ B) $c=2\sin\dfrac{1}{2b}$ C) $c=\sin^{-1}\dfrac{1}{2b}$ D) $c=\sin^{-1}(2b)$

Key. B

Sol. 51. At x=0,

L.H.L $=b\sin^{-1}\dfrac{c}{2}$ R.H.L= $\dfrac{a}{2}$

for f(x) to be continuous at x=0 and f(0)= $\frac{1}{2}$

L.H.L $\big|_{X=0} = R.H.L\big|_{X=0} = f(0)$

$$\frac{a}{2} = \frac{1}{2} = b\sin^{-1}\left(\frac{c}{2}\right)$$

a=1

52. f(x) need to be continuous first $\Rightarrow$ a=1

now let f(x) be differentiable also at x=0,then

$$L.H.D\Big|_{x=0} = \frac{1}{2}\frac{b}{\sqrt{1-\left(\frac{x+c}{2}\right)^2}}\Bigg|_{x=0} = \frac{b}{2\sqrt{1-\frac{c^2}{4}}} ------(1)$$

$$R.H.D\Big|_{x=0} = \lim_{h\to 0+}\left|\frac{e^{\frac{h}{2}}-1-\frac{1}{2}}{h}\right| = \frac{1}{2} ----------(2)$$

$$(1) = (2) \Rightarrow \frac{b}{2\sqrt{1-\frac{c^2}{4}}} = \frac{1}{2} \Rightarrow 4b^2 = 4 - c^2.$$

53. If f(x) is differentiable at x=0,then

$$\frac{1}{2} = b\sin^{-1}\left(\frac{c}{2}\right) \Rightarrow 2\sin\left(\frac{1}{2b}\right) = c \quad \text{constant}$$

Paragraph –19

It can be shown that if f(x) is differentiable at 0 then f(x) is continuous at 0. By changing origin, we can say that if f(x) is continuous at a then (x-a) f(x-a) is differentiable at a.

54. The largest set over which $\frac{x\sin|x|}{1-|x|^2}$ is differentiable is

A) $R-\{0,1,-1\}$ B) R C) $R-\{1,-1\}$ D) $R-\{1,2\}$

Key. C

55. The number of points where the function $(x-3)\left|x^2-7x+12\right|+\cos|x-3|$ is not differentiable is

A) one B) two C) three D) infinite

Key. A

56. Let f(x) = $|x|$, g(x) = sin x and h(x) = g(x) f(g(x)), then

A) h(x) is continuous but not differentiable at 0.
B) h(x) is continuous and differentiable everywhere.
C) h(x) is continuous everywhere and differentiable only at x = 0.
D)None of these.

Key. B

Sol. 54. By given fact $\frac{x\sin|x|}{1-|x|^2}$ is differentiable at zero.but it is certainly not continuous at x=1 and x= -1 .

$\Rightarrow$ Not differentiable at x=1,x=-1

55. $f(x)=(x-3)|(x-3)(x-4)|+\cos(x-3)$

$[\because \cos|x-3|=\cos(x-3)]$

It is evident that f(x) is not differentiable at x = 4

56. It is clear that h(x) = $\sin x|\sin x|$

Whose differentiability is doubtful only at $n\pi$ .At any $n\pi$, h(x)= $-\sin^2 x \;\; or \sin^2 x$

$\Rightarrow R.H.D., L.H.D.$ vanish at $x=n\pi$

$\Rightarrow R.H.D = L.H.D$

$\Rightarrow$ h(x) is differentiable everywhere

Continuity & Differentiability

Integer Answer Type

1. The function $f(x)=\left|x^2-3x+2\right|+\cos|x|$ is not differentiable at how many values of x.

Key. 2

Sol. $\because f(x)=\left|x^2-3x+2\right|+\cos|x|$

$$=|(x-1)||(x-2)|+\cos|x|$$

$$f(x)=\begin{cases} x^2-3x+2+\cos x, x<0 \\ x^2-3x+2+\cos x,\ 0\le x<1 \\ -x^2-3x-2+\cos x, 1\le x<2 \\ x^2-3x+2+\cos x, x>2 \end{cases}$$

$$\therefore f'(x)=\begin{cases} 2x-3-\sin x, x<0 \\ 2x-3-\sin x, 0\le x<1 \\ -2x+3-\sin x, 1\le x<2 \\ 2x-3-\sin x, x>2 \end{cases}$$

it is clear f(x) is not differentiable at x = 1.

$\therefore\ f'(1^-)=-1-\sin 1$

and $f'(1^+)=1-\sin 1$.

2. Let $f(x)=[x]+\left[x+\frac{1}{4}\right]+\left[x+\frac{1}{2}\right]+\left[x+\frac{3}{4}\right]$. Then no. of points of discontinuity of f(x) in [0,1] is [[.] denotes G.I.F]

Key. 4

Sol. $[x]+\left[x+\frac{1}{4}\right]+\left[x+\frac{2}{4}\right]+\left[x+\frac{3}{4}\right]=[4x]$

$\therefore$ f(x) = [4x] which will become discontinuous at $x=\frac{1}{4},\frac{2}{4},\frac{3}{4},\frac{4}{4}$

3. The number of two digits numbers 'a' whose sum of digits is 9 such that

$f(x)=\left[\left(\frac{x-2}{a}\right)^3\right]\sin(x-2)+a\cos(x-2)$ is continuous in $[4,6]$ is.

Here $[.]$ denotes the greatest integer function

Key. 9

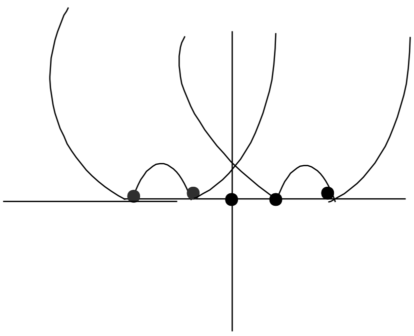
Sol. Clearly $\left(\frac{(x-2)^3}{a}\right)=0,\quad x\in[4,6]$

$(x-2)^3 \in (8,64)$ $\Rightarrow a > 64 \Rightarrow a = 72, 81, 90$

No of values

4. If a Î (- ¥ , - 1)È(- 1, 0) then the number of points where the function $f(x) = \left|x^2 + (\alpha - 1)|x| - \alpha\right|$ is not differentiable is.

Key. 5



Sol. given $f(x) = \left|x^2 + (\alpha - 1)|x| - \alpha\right|$

Take $g(x) = x^2 + (\alpha - 1)x - \alpha$

$\Rightarrow f(x) = (|x| - 1)(|x| + \alpha)$

From graph it is clear that $f(x)$ is not differentiable at '5' points.

5. If the function f defined by $f(x) = \dfrac{x(1 + a\cos x) - b\sin x}{x^3}$ if $x \neq 0$ and $f(0) = 1$ is continuous at $x = 0$ then $2a - 8b =$

Key. 7

Sol. $1 = f(0) = \lim_{x \to 0} f(x) = \lim_{x \to 0} \dfrac{x(1 + a(1 - \frac{x^2}{\lfloor 2} + ...) - b(x - \frac{x^3}{\lfloor 3} + ...}{x^3}$

$= \lim_{x \to 0} \dfrac{x(1 + a - b) + x^3(\frac{-a}{2} + \frac{b}{6}) + x^5(\lambda) + ...}{x^3}$

$\Rightarrow 1 + a - b = 0$ and $\dfrac{-a}{2} + \dfrac{b}{6} = 1 \Rightarrow a = \dfrac{-5}{2}, b = \dfrac{-3}{2}$ and $2a - 8b = 7$

6. If $f\left(\dfrac{x+y}{2}\right) = \dfrac{f(x) + f(y)}{2}$ for all $x, y \in R$, $f^1(0)$ exists and equals to -1 and $f(0) = 1$ then $5 - f(2) =$

Key. 6

Sol. $f(x+y) = \dfrac{f(2x) + f(2y)}{2}$ and $f(2x) = 2f(x) - 1$ $(put\ \ y = 0)$

Now $f^1(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0} \dfrac{f(2x)+f(2h)-2f(x)}{2h} = \lim_{h\to 0} \dfrac{f(2h)-1}{2h}$

$= f^1(0) = -1$

/home/mod_jklog/mod_jk.log since $f(0)=1$

$\therefore f(x) = 1-x$ and $5-f(2) = 5-(-1) = 6$

7. The number of two digits numbers 'a' whose sum of digits is 9 such that

$$f(x) = \left[\left(\frac{x-2}{a}\right)^3\right]\sin(x-2) + a\cos(x-2) \text{ is continuous in } [4,6] \text{ is.}$$

Here $[.]$ denotes the greatest integer function

Key. 9

Sol. Clearly $\left(\dfrac{(x-2)^3}{a}\right) = 0,\quad x\in[4,6]$

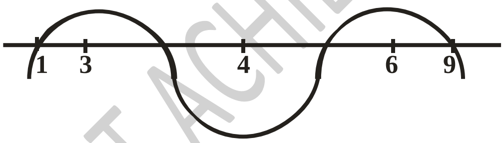
$(x-2)^3 \in (8,64) \qquad \Rightarrow a > 64 \Rightarrow a = 72, 81, 90$

No of values

8. If f(x) is twice differentiable function such that f(1) = 0, f(3) = 2, f(4) = –5, f(6) = 2, f(9) = 0 then the minimum number of zero's of $g'(x) = x^2 f''(x) + 2xf'(x) + f''(x)$ in the interval (1,9) is

Key. (2)

Sol. $f'(x) = 0$ has minimum three solution between (1,9)



$f''(x) = 0$ has minimum two solution between (1,9)

Given equations $\dfrac{d}{dx}\left\{(x^2+1)f'(x)\right\} = 0$

9. In ΔABC, $\dfrac{r}{r_1} = \dfrac{1}{2}$, then the value of $4\tan\left(\dfrac{A}{2}\right)\left(\tan\dfrac{B}{2}+\tan\dfrac{C}{2}\right)$ must be

Key. 2

Sol. $\dfrac{r}{r_1} = \tan\dfrac{B}{2}\tan\dfrac{C}{2} = \dfrac{1}{2}$

$\tan\dfrac{A}{2}\left(\tan\dfrac{B}{2}+\tan\dfrac{C}{2}\right) = 1-\tan\dfrac{B}{2}\tan\dfrac{C}{2} = \dfrac{1}{2}$

$\therefore\ 4\tan\frac{A}{2}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)=2$

10. Let $f(x)=\begin{cases} x^2\sum\limits_{r=0}^{\left[\frac{1}{|x|}\right]} r & ; x\neq 0 \\ \frac{k}{2}; & \text{otherwise}\end{cases}$ ([.]denotes the greatest integer function)

The value of k such that f become continuous at x=0 is

Key. 1

Sol. In the vicinity of x=0,we have $x^2\sum\limits_{r=0}^{\left[\frac{1}{|x|}\right]} r = x^2\left(1+2+3+\ldots\left[\frac{1}{|x|}\right]\right)$

Use sandwich theorem

$$P=\left(1+2+3+\left[\frac{1}{|x|}\right]\right)=\frac{x^2\left(1+\left[\frac{1}{|x|}\right]\right)}{2}\left[\frac{1}{|x|}\right]$$

So $\frac{1}{2}(1-|x|)<P\le\frac{1}{2}(1+|x|)$

Then the limit is $\frac{1}{2}$

11. Let $f:(-\infty,\infty)\to[0,\infty)$ be a continuous function

such that $f(x+y)=f(x)+f(y)+f(x)f(y), \forall x,y\in R$. Also $f'(0)=1$.

Then $\left[\frac{f(4)}{f(2)}\right]$ equals ([g] represents greatest integer function)

Key. 8

Sol. Rewrite the equation as

$1+f(x+y)=(1+f(x))(1+f(y))$

Put g(x) = 1 + f (x) to get

g(x+y) = g(x) g(y)

As g(x) ≥ 1, the function $\ln$ g(x) is defined.

Also continuous of f implies continuity of g

Let h(x) = $\ln$ g(x), we get

h(x+y) = h(x) + h(y)

The only continuous solution of this is h(x) = kx

$\therefore f(x)=e^{kx}-1, f'(0)=1$ gives $k=1$

12. Let f(x) = $[x^2]\sin\pi x, x \in R$, the number of points in the interval $\left(0,3\right]$ at which the function is discontinuous is____

Key. 6

Sol. f(x) = 0 $0 < x < 1$

= $\sin\pi x$ $1 \le x < \sqrt{2}$

= $2\sin\pi x$ $\sqrt{2} \le x < \sqrt{3}$

= $3\sin\pi x$ $\sqrt{3} \le x < 2$

= $4\sin\pi x$ $2 \le x < \sqrt{5}$ etc.

The function is discontinuous at $x = \sqrt{2}, \sqrt{3}, \sqrt{5}, ...\sqrt{K}$ where K is not a perfect square.

Points of discontinuity (desired) = $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}$

13. The number of integral solution for the equation x + 2y = 2xy is

Key. 2

Sol. $2y = \frac{x}{x-1}$

Since y is an integer 2y is even such that x and x – 1 are consecutive integers and hence the only values of x that satisfy are 2 and 0.

14. The function $f\left(x\right) = \left|x^2 - 3x + 2\right| + \cos|x|$ is not differentiable at how many values of x.

Key : 2

Sol : $\because f\left(x\right) = \left|x^2 - 3x + 2\right| + \cos|x|$

$= \left|(x-1)\right|\left|(x-2)\right| + \cos|x|$

$$f\left(x\right) = \begin{cases} x^2 - 3x + 2 + \cos x, x < 0 \\ x^2 - 3x + 2 + \cos x,\ 0 \le x < 1 \\ -x^2 - 3x - 2 + \cos x, 1 \le x < 2 \\ x^2 - 3x + 2 + \cos x, x > 2 \end{cases}$$

$$\therefore f'\left(x\right) = \begin{cases} 2x - 3 - \sin x, x < 0 \\ 2x - 3 - \sin x, 0 \le x < 1 \\ -2x + 3 - \sin x, 1 \le x < 2 \\ 2x - 3 - \sin x, x > 2 \end{cases}$$

it is clear f(x) is not differentiable at x = 1.

$\therefore\ f'\left(1^-\right) = -1 - \sin 1$

and $f'\left(1^+\right) = 1 - \sin 1.$

15. If the function f defined by $f(x)=\dfrac{\log(1+x)^{1+x}}{x^2}-\dfrac{1}{x}$ if $x\neq 0$ is continuous at $x=0$, then $6(f(0))=$

Key. 3

Sol. $f(0)=\lim\limits_{x\to 0}\dfrac{\ln(1+x)^{1+x}-x}{x^2}=\lim\limits_{x\to 0}\dfrac{(1+x)\ln(1+x)-x}{x^2}$

$=\lim\limits_{x\to 0}\dfrac{1+\ln(1+x)-1}{2x}=\dfrac{1}{2}\ \therefore 6f(0)=3$

16. A function $f:R\to R$ where R is a set of real numbers satisfies the equation $f\left(\dfrac{x+y}{3}\right)=\dfrac{f(x)+f(y)+f(0)}{3}$ for all $x,y\in R$. If the function is differentiable at x = 0 then show that it is differentiable for all x in R

Sol. $f\left(\dfrac{x+y}{3}\right)-\dfrac{f(x)+f(y)+f(0)}{3}$

$\lim\limits_{h\to 0}\dfrac{f(0+h)-f(0)}{h}=\text{exist}$.

$\lim\limits_{h\to 0}\dfrac{f(x-h)-f(x)}{h}=\lim\limits_{h\to 0}\dfrac{f\left(\dfrac{3x+3h}{3}\right)-f\left(\dfrac{3x+0}{3}\right)}{h}$

$\lim\limits_{h\to 0}\dfrac{1}{h}\left[\dfrac{f(3x)+f(3h)+f(0)}{3}-\dfrac{f(3x)+f(0)+f(0)}{3}\right]=\lim\limits_{h\to 0}\dfrac{1}{h}\left[\dfrac{f(3h)-f(0)}{3}\right]$

$=\lim\limits_{h\to 0}\dfrac{f(3h)-f(0)}{3h}=f'(0)$

17. If $f(x)=\begin{cases}\dfrac{\tan\left[x^2\right]\pi}{ax^2}+ax^3+b & ,\ 0\le x\le 1\\ 2\cos\pi x+\tan^{-1}x & ,1<x\le 2\end{cases}$ is differentiable in [0, 2], then $b=\dfrac{\pi}{4}-\dfrac{26}{k_2}$. Find $k_1^2+k_2^2$ {where [ ] denotes greatest integer function}.

Ans. 180

Sol. $f(x)=\begin{cases}ax^3+b & ,\ 0\le x\le 1\\ 2\cos\pi x+\tan^{-1} & ,\ 1<x\le 2\end{cases}$

$f'(x)=\begin{cases}3ax^2 & ,\ 0<x<1\\ -2\pi\sin\pi x+\dfrac{1}{1+x^2} & ,\ 1<x<2\end{cases}$

As the function is differentiable in [0, 2] $\Rightarrow$ function is differentiable at x = 1

$\therefore\quad f'(1^-)=f'(1^+)$

$\Rightarrow\quad 3a=\dfrac{1}{2}\ \Rightarrow\quad a=\dfrac{1}{6}$

Function will also be continuous at x = 1

$\therefore \quad \lim\limits_{x\to1^-} f(x) = \lim\limits_{x\to1^-} f(x)$

$\Rightarrow \quad a+b=-2+\dfrac{\pi}{4}$

$\therefore \quad b=-2-\dfrac{1}{6}+\dfrac{\pi}{4}=\dfrac{\pi}{4}-\dfrac{13}{6} \Rightarrow k_1=6 \;\&\; k_2=12 \qquad \Rightarrow \qquad k_1^2+k_2^2=180$ Ans.

18. Let $f(x)=\begin{cases} |x|^p \sin\dfrac{1}{x}+|\tan x|^q & , x\neq 0 \\ 0 & , x=0 \end{cases}$ be differentiable at x = 0, then find the least possible value of [p + q], (where[.] represents greatest integer function)

Ans. 1

Sol. $\lim\limits_{x\to0^+} \dfrac{|x|^p \sin\dfrac{1}{x}+x|\tan x|^q-0}{x}$

$=\lim\limits_{x\to0^+}\left(x^{p-1}\sin\dfrac{1}{x}+|\tan x|^q\right)=0$ if p – 1 > 0 and q > 0 ...(i)

$\lim\limits_{x\to0^-}\left((-1)^p x^{p-1}\sin\dfrac{1}{x}+|\tan x|^q\right)=0$ if p – 1 > 0 and q > 0 ...(ii)

19. (i) If $f(x)=\sin^{-1}2x\sqrt{1-x^2}$, then find the values of $f'(1/2)$ and $f'(-1/2)$.

(ii) If $f(x)=\cos^{-1}(1-2x^2)$, then find the values of $f'(1/2)$ and $f'(-1/2)$.

Ans. $\dfrac{-4}{\sqrt{3}}$

Sol. (i) $f(x)=\sin^{-1}\left(2x\sqrt{1-x^2}\right)=\begin{cases} -\pi-2\sin^{-1}x & , -1\le x<-\dfrac{1}{\sqrt{2}} \\ \dfrac{2}{\sqrt{1-x^2}} & , -\dfrac{1}{\sqrt{2}}<x<\dfrac{1}{\sqrt{2}} \\ \dfrac{-2}{\sqrt{1-x^2}} & , \dfrac{1}{\sqrt{2}}<x<1 \end{cases}$

$f'(1/2)=\dfrac{4}{\sqrt{3}}, \quad f'(-1/2)=\dfrac{4}{\sqrt{3}}$

(ii) $f(x)=\pi-\cos^{-1}(2x^2-1)=\pi-\cos^{-1}(\cos 2\theta)$, where $x=\cos\theta, \quad 0\le\theta\le\pi$

$=\begin{cases} \pi-2\theta & , 0\le\theta\le\dfrac{\pi}{2} \\ \pi-(2\pi-2\theta) & , \dfrac{\pi}{2}<\theta\le\pi \end{cases}=\begin{cases} \pi-2\cos^{-1}x & , 0\le x\le 1 \\ 2\cos^{-1}x-\pi & , -1\le x<0 \end{cases}$

$\therefore \quad f'(x)=\begin{cases} \dfrac{2}{\sqrt{1-x^2}} & , 0<x<1 \\ \dfrac{-2}{\sqrt{1-x^2}} & , -1<x<0 \end{cases} \qquad \therefore \quad f'\left(\dfrac{1}{2}\right)=\dfrac{4}{\sqrt{3}}, f'\left(-\dfrac{1}{2}\right)=\dfrac{-4}{\sqrt{3}}$

Continuity & Differentiability
Matrix-Match Type

1. Column – I Column – II

(A) The function $f(x)=\begin{cases} x^2+3x+a; x\le 1 \\ bx+2; \ x>1 \end{cases}$ (P) $|a|=3$

Is differentiable $\forall x \in R$ then

(B) The function $f(x)=\begin{cases} \dfrac{1}{|x|}; \ |x|\ge 1 \\ ax^2+b, |x|<1 \end{cases}$ (Q) b = 5

Is differentiable every where

(C) The function $f(x)=\begin{cases} ax^2-bx+2 \, if \ x<3 \\ bx^2-3; if \ x\ge 3 \end{cases}$ (R) $a=\dfrac{35}{9}$

Is differentiable every where then

(D) If $f(x)=\begin{cases} \dfrac{a+3\cos x}{x^2}, if \ x<0 \\ -\sqrt{3}b.\tan\left(\dfrac{\pi}{[x+3]}\right) if \ x\ge 0 \end{cases}$ (S) b = 3/2

is continuous at x = 0 then ([.] denotes the greatest integer $\le x$ (T) $a=-1/2$

Key. A – p,q; B – s,t; C – r; D – p

Sol. DO yourself

2. $let \ f(x)=\begin{cases} \dfrac{5e^{1/x}+2}{3-e^{1/x}}; x\ne 0 \\ 0 \ ; \ x=0 \end{cases}$ Now match column – I to column – II

Column – I	Column – II
(a) $y=f(x)$ is	(p) continuous at x =0
(b) $y=xf(x)$ is	(q) discontinuous at x =0

(c) $y = x^2 f(x)$ is (r) differentiable at $x = 0$

(d) $y = x^{-1}.f(x)$ is (s) non – differentiable at $x = 0$

(t) discontinuous and not differentiable at x=0

Key. A – q,s,t; B – p,s; C – p,r; D – q,s,t

Sol. Apply Def for all bits.

3. Match the following:

Column I		Column II	
(A)	$f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ Is not differentiable at	(p)	$x = 1$
(B)	$f(x) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ Is not differentiable at	(q)	$x = -1$
(C)	$f(x) = \cos^{-1}\left(4x^3 - 3x\right)$ Is not differentiable at	(r)	$x = \frac{1}{2}$
(D)	$f(x) = \sin^{-1}\left(3x - 4x^3\right)$ Is not differentiable at	(s)	$x = \frac{-1}{2}$

Key. (A) $\rightarrow$(p, q); (B) $\rightarrow$(p, q); (C) $\rightarrow$(r, s, p); (D) $\rightarrow$(r, s, p)

Sol.

$$A)\ y = \begin{cases} \pi - 2\tan^{-1} x, & x > 1 \\ 2\tan^{-1} x, & -1 \le x \le 1 \\ -\pi - 2\tan^{-1} x, & x < -1 \end{cases}$$

$$\therefore \frac{dy}{dx} = \begin{cases} \frac{-2}{1+x^2}, & |x| > 1 \\ \frac{2}{1+x^2}, & |x| < 1 \\ \text{does not exist}, & \text{at } |x| = 1 \end{cases}$$

f(x) is not differentiable at $x = -1, 1$

$$B)\ \ y=\begin{cases}\pi+2\tan^{-1}x & ,\ x<-1\\ 2\tan^{-1}x & ,\ -1<x<1\\ -\pi+2\tan^{-1}x & ,\ x>1\end{cases}$$

$$\therefore \frac{dy}{dx}=\begin{cases}\dfrac{-2}{1+x^2} & ,\ x\in R-\{-1,1\}\\ does\ not\ exist & ,\ atx=\{-1,1\}\end{cases}$$

f(x) is not differentiable at $x=-1,1$

$$C)\ \ f(x)=\begin{cases}-2\pi+3\cos^{-1}x & ,\ -1\le x\le -\dfrac{1}{2}\\ 2\pi-3\cos^{-1}x & ,\ -\dfrac{1}{2}\le x\le \dfrac{1}{2}\\ 3\cos^{-1}x & ,\ \dfrac{1}{2}\le x\le 1\end{cases}$$

$$\therefore \frac{dy}{dx}=f'(x)=\begin{cases}\dfrac{-3}{\sqrt{1-x^2}} & ,\ \dfrac{1}{2}<|1|<1\\ does\ not\ exixt & ,\ |x|=\dfrac{1}{2}\\ \dfrac{3}{\sqrt{1-x^2}} & ,\ |x|<\dfrac{1}{2}\end{cases}$$

f(x) is not differentiable at $x=\frac{-1}{2},\frac{1}{2}$

$$D)\ \ f(x)=\begin{cases}\pi-3\sin^{-1}x & ,\ \dfrac{1}{2}\le x\le 1\\ 3\sin^{-1}x & ,\ -\dfrac{1}{2}\le x\le \dfrac{1}{2}\\ -\pi-3\sin^{-1}x & ,\ -1\le x\le -\dfrac{1}{2}\end{cases}$$

$$f'(x)=\begin{cases}\dfrac{-3}{\sqrt{1-x^2}} & ,\ \dfrac{1}{2}<|x|<1\\ does\ not\ exist & ,\ |x|=\dfrac{1}{2}\\ \dfrac{3}{\sqrt{1-x^2}} & ,\ |x|<\dfrac{1}{2}\end{cases}$$

f(x) is not differentiable at $x=\frac{-1}{2},\frac{1}{2}$

4. Match the Following:

Column I		Column II	
(A)	$f(x)=\|x\|$	(p)	Continuous at $x=0$
(B)	$f(x)=x^n\|x\|, n\in N$	(q)	Discontinuous at $x=0$
(C)	$f(x)=\begin{cases} x\ln\|\sin x\|, x\neq 0 \\ 0 \quad , x=0\end{cases}$	(r)	Differentiable at $x=0$
(D)	$f(x)=\begin{cases} x e^{1/x} \quad , x\neq 0 \\ 0 \quad , x=0\end{cases}$	(s)	Non- differentiable at $x=0$

Key. (A) $\rightarrow$(p, s); (B)$\rightarrow$(p, r); (C)$\rightarrow$(p, s); (D)$\rightarrow$(q, s)

Sol.

(A) $f(x)=\begin{cases} -x, x<0 \\ x, x\geq 0\end{cases}$

continuous but not differentiable at $x=0$

(B) $f(x)=x^n|x|$

$\Rightarrow LHD = RHD = 0$ at $x=0$

(C) $LHL = RHL = f(0)=0$ but LHD and RHD are not finite

(D) $LHL = 0, RHL = \underset{x\to 0}{Lt}\frac{e^{1/x}}{1/x}$

$=\underset{x\to 0}{Lt}\frac{e^{1/x}(-1/x^2)}{(-1/x^2)}=\underset{x\to 0}{Lt}\, e^{1/x}=\infty$

5. Match the Following:

Column I		Column II	
(A)	If $f(x)\begin{cases} \frac{a+3\cos x}{x^2}, & x<0 \\ b\tan\left(\frac{\pi}{[x+3]}\right), & x\geq 0\end{cases}$ is continuous at $x=0$, then (where [.] denotes the greatest integer function)	(p)	$[a-2b]=-2$ (where [.] denotes G.I.F)

(B)	If $f(x)=\begin{cases}-2\sin x, -\pi \le x \le -\frac{\pi}{2} \\ a\sin x + b, -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \cos x, \frac{\pi}{2} \le x \le \pi\end{cases}$ is continuous in $[-\pi, \pi]$, then	(q)	$\lvert a-b\rvert = 2$
(C)	If $f(x)=\begin{cases}\left(\frac{3}{2}\right)^{(\cos 3x)/(\cot 2x)} & 0 < x < \frac{\pi}{2} \\ b+3, & x=\frac{\pi}{2} \\ (1+\lvert\cos x\rvert)^{\left(\frac{a\lvert\tan x\rvert}{b}\right)}, & \frac{\pi}{2} < x < \pi\end{cases}$ is continuous at $x=\frac{\pi}{2}$, then	(r)	$\lvert a+2b\rvert = 1$
(D)	If $f(x)=\frac{a\sin x + b}{x}, f(0)=1$ and f(x) is continuous then	(s)	$\lvert a+2b\rvert = 4$

Key. (A) $\rightarrow$(p); (B)$\rightarrow$(q, r); (C)$\rightarrow$(q, s); (D)$\rightarrow$(r)

Sol. Conceptual

6. Match the Following:

Column I		Column II	
(A)	$f(x)=\sin(\pi[x])$ (where [.] denote G.I.F)	(p)	Differentiable every where
(B)	$f(x)=\sin((x-[x])\pi)$ (where [.] denote G.I.F)	(q)	Not differentiable at $x=2$
(C)	$f(x)=\begin{cases}x\sin\left(\frac{1}{x}\right) & if \quad x \ne 0 \\ 0 & if \quad x=0\end{cases}$	(r)	Not differentiable at -1 and 1
(D)	$f(x)=\lvert 2-x\rvert+[2+x]$ (where [.] denote G.I.F)	(s)	Continuous at $x=0$ but not differentiable at $x=0$

Key. (A) $\to$(p) (B) $\to$(q, r, s) ; (C) $\to$(s) (D) $\to$(q, r)

Sol.

1) We know that $[x] \in I, \ \forall x \in R$

$\therefore sin(\pi[x]) = \sin \pi x = 0 \quad \forall x \in R$

if $[x] = 0$, $x \in I$

Since every constant function is differentiable in its domain

$\therefore Sin(\pi[x])$ is differentiable every where.

2) $f(x) = \sin[(x-[x])\pi]$

Since $x-[x]$ is not differentiable at integral points

$\therefore f(x) = \sin(\pi(x-[x]))$ is not differentiable at $x \in I$

$\therefore$ It is not differentiable at $= \ -1, \ 1$

3) $\underset{x\to 0}{Lt} f(x) = 0$ (a finite quantity between -1 and 1) $= 0 = f(0)$

$\therefore f(x)$ is continuous at x = 0 and $\underset{x\to 0}{Lt} \dfrac{f(x)-f(0)}{x-0} = \underset{x\to 0}{Lt} \sin\left(\dfrac{1}{x}\right)$

Which does not exit

$\therefore f(x)$ is not differentiable at x = 0

4) $|2-x|$ is continuous every where and [2 + x] is discontinuous at all integral values of x.

$\therefore f(x)$ is discontinuous at x = 2

$\therefore f(x)$ is not differentiable at x = 2

7. Match the Following:

	Column I		Column II
(A)	The function $f(x) = \begin{cases} x^2+3x+a; & x \le 1 \\ bx+2; & x > 1 \end{cases}$ is differentiable $\forall x \in R$, then	(p)	a = 3
(B)	The function $f(x) = \begin{cases} \dfrac{1}{\lvert x\rvert}; & \lvert x\rvert \ge 1 \\ ax^2+b; & \lvert x\rvert < 1 \end{cases}$ is differentiable everywhere, then	(q)	b = 5
(C)	The function $f(x) = \begin{cases} ax^2-bx+2; & x < 3 \\ bx^2-3; & x \ge 3 \end{cases}$ is differentiable everywhere then	(r)	$a = \dfrac{35}{9}$

(D)	If $f(x)=(x-a)\lvert x-a\rvert$ then f(x) is differentiable for a=	(s)	$b=\frac{3}{2}$

Key. (A) $\to$(p, q) (B)$\to$(s) ; (C)$\to$(r) (D)$\to$(p, r)

Sol. Conceptual

8. Match the following lists:

List I		List II	
(A)	$\underset{x\to\infty}{Lt}\ x\cos\frac{\pi}{8x}.\ \sin\frac{\pi}{8x}=$	(P)	$\frac{\pi}{8}$
(B)	$\underset{x\to\infty}{Lt}\frac{\tan\left[-\pi^2\right]x^2-\left[-\pi^2\right]x^2}{\sin^2(x)}=$	(Q)	$\sqrt{2}$
(C)	$\underset{x\to\infty}{Lt}\sqrt{\frac{2x-\sin x+\cos x}{x+\cos^2 x+sin^2 x}}$	(R)	0
(D)	$\underset{x\to 1}{Lt}\left(\frac{x^n-1}{n(x-1)}\right)^{\frac{1}{x-1}}$	(S)	$e^{\frac{n-1}{2}}$

Key. (A) $\to$(p); (B) $\to$(r); (C) $\to$(q); (D) $\to$(s)

Sol.

(A) $\underset{x\to\infty}{Lt}\frac{1}{2}\frac{\sin\left(\frac{\pi}{4}\right).\frac{1}{x}}{\left(\frac{1}{x}\right)}=\frac{\pi}{8}$

(B) $\because \pi^2=9.8\Rightarrow \underset{x\to 0}{Lt}\frac{\tan^2\left(-10x^2\right)+10x^2}{x^2}=\frac{-10x^2+10x^2}{x^2}=0$

(C) $\underset{x\to\infty}{Lt}\sqrt{\frac{2-\frac{1}{x}.\sin x+\frac{1}{x}.\cos x}{1+\frac{1}{x}}}=\sqrt{\frac{2-0+0}{1+0}}=\sqrt{2}$

(D) Put x-1=h, as $x\to 1, h\to 0$

$$\underset{h\to 0}{Lt}\left(1+\frac{(n-1)}{2}h\right)^{1/h}=\underset{h\to 0}{Lt}\left(\frac{1+nh+\frac{n(n+1)}{2!}h^2....-1}{nh}\right)^{1/h}$$

$= \underset{h\to 0}{Lt}\left(1+\frac{(n-1)}{2}h\right)^{1/h} = e^{n-1/2}$

9. Let f be a polynomial of degree 4 over reals satisfying

$f'(0) = f'(1) = f'(-1) = 0$ and $f(0) = 4, f''\left(\frac{1}{2}\right) = -1$

Match the items in Column – I with those in Column II

Column – I	Column - II
A) f(x)=0 has	p) root at x = 2
B) 4 – f(x)=0 has	q) root at x = 1
C) $f'(x)+x-1=0$ has	r) 2 equal real roots
D) $x f'(x) - 4f(x) = 0$ has	s) no real roots

Key. A-S; B-R; C-Q,R; D-P

Sol. using the conditions , we get

f (x) = $x^4 - 2x^2 + 4$

10. Match the items in Column – I with those in Column - II

Column – I	Column - II
A) f(x) = $\begin{cases} e^{\frac{-1}{x^2}}, x \neq 0 \\ 0, x = 0 \end{cases}$	p) first derivative exists
B) f(x) = $\begin{cases} e^{\frac{-1}{x}}, x > 0 \\ 0, x \leq 0 \end{cases}$	q) first derivative is continuous
C) f(x) = $\begin{cases} e^{\frac{-1}{x-e}+\frac{1}{x-\pi}}, x \in (e,\pi) \\ 0, x \notin (e,\pi) \end{cases}$	r) second derivative exists
D) f(x) = $\begin{cases} x^2 \sin\frac{1}{x}, x \neq 0 \\ 0, x = 0 \end{cases}$	s) second derivative is continuous

Key. A-p, q, r, s; B-p, q, r, s; C-p, q, r, s; D - r

Sol. (A) $f'(x) = \frac{2}{x^3}e^{-\frac{1}{x^2}}, x \neq 0$

By definition $f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$ which can be shown to be zero.

so f '(x) = $\begin{cases} \dfrac{2}{x^3}e^{\frac{-1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$

(Use $\lim_{x \to 0} \dfrac{e^{\frac{-1}{x^2}}}{x} = 0$)

Then f '(x) is continuous on R

$f''(0) = 0$ can be shown

$$f''(x) = \begin{cases} e^{-\frac{1}{x^2}}\left(\dfrac{4}{x^6} - \dfrac{23}{x^4}\right) \\ 0 \end{cases}$$

We can also show that

$$f^n(x) = \begin{cases} e^{\frac{-1}{x^2}} p\left(1/x\right), x \neq 0 \\ 0 \quad , x = 0 \end{cases}$$

P being a polynomial.

(B) Similar to (A) . (B) also have both 1st and 2nd derivative and they are continuous.

(C) $f_1(x) = g(x - e)$

$f_2(x) = g(\pi - x)$ where g(x) = $e^{-\frac{1}{x}}$

So $f(x) = f_1(x) f_2(x)$

As f_1, f_2 have both 1st and 2nd derivatives existing and continuous, the function f also will.

(D) Only 1st derivative exists and its not continuous.

11. Match the items of Column – I with those of Column II

A) f(x) = $x\left(\left[\dfrac{1}{x}\right] + \left[\dfrac{2}{x}\right] + \ldots\ldots\left[\dfrac{8}{x}\right]\right)$, $x \neq 0$ p)1

= 9k, $x = 0$

The value of k such that f is continuous at x=0 is

([.]denotes the greatest integer function)

B) f(x) = $\left(1 + xe^{-1/x^2} \sin\dfrac{1}{x^4}\right)^{e^{1/x^2}}$, $x \neq 0$ q)2

= k, x = 0

The value of k such that f is continuous at x=0 is

C) f: $[0, \infty) \to R$; r)3

f(x) = $\left(2\sin\sqrt{x} + \sqrt{x}\sin\dfrac{1}{x}\right)^x$, $x > 0$

=k , x = 0

The value of k such that f is continuous at x=0 is

d) f: $\left(0,\pi\right)\to R$; $f(x)=\begin{cases}\dfrac{1-\sin x}{(\pi-2x)^2}\cdot\dfrac{\ln\sin x}{\ln(1+\pi^2-4\pi x+4x^2)};x\neq\dfrac{\pi}{2}\\ k \qquad ;x=\dfrac{\pi}{2}\end{cases}$ s)4

The value of $8\sqrt{|k|}$ such that f is continuous at x=$\dfrac{\pi}{2}$ is

Key. A-s; B-p; C-p; D-p

Sol. A) Use sandwich theorem

$$x\left(\frac{1+2+3...+8}{x}-8\right)<\left[\frac{1}{x}\right]+\left[\frac{2}{x}\right]+......+\left[\frac{8}{x}\right]\le x\left(\frac{1+2+3+...+8}{x}\right)$$

Taking limits find that $\lim\limits_{x\to0} x\left(\left[\frac{1}{x}\right]+\left[\frac{2}{x}\right]+....+\left[\frac{8}{x}\right]\right)=36$

C) $u=\left(2\sin\sqrt{x}+\sqrt{x}\sin{1/x}\right)^x$

$\ln u=x\ln(2\sin\sqrt{x}+\sqrt{x}\sin{1/x})$

$=x\ln\left(\dfrac{\left(2\sin\sqrt{x}+\sqrt{x}\sin{1/x}\right).\sqrt{x}}{\sqrt{x}}\right)$

$=x\ln\sqrt{x}+x\ln\left(\dfrac{2\sin\sqrt{x}}{\sqrt{x}}+\sin\dfrac{1}{x}\right)$

$=x\ln\sqrt{x}+xg(x)$

g is bounded

Then

$\lim\limits_{x\to0}u=\lim\limits_{x\to0}\ln\sqrt{x}+\lim\limits_{x\to0}xg(x)$

= 0 + 0

$\therefore u=e^0=1$

12. Column I lists some functions and Column II lists its properties. Match the items of Column A with those of Column B.

Column – I | Column – II

A) f(x) = $\lim\limits_{n\to\infty}\dfrac{n^x-n^{-x}}{n^x+n^{-x}}, x\in R$ P)Continuous at all points in its domain

B) f(x) = $\lim\limits_{n\to\infty}\sqrt[n]{4^n+x^{2n}+\dfrac{1}{x^{2n}}}, x\in R-\{0\}$ Q) Discontinuous at finitely many points in its domain

C) f(x) = $\lim\limits_{n\to\infty}\dfrac{\ln(e^n+x^n)}{n}, x\ge0, x\in R$ R) Not differentiable at finitely many points in its domain.

D) f(x) = $\lim\limits_{n\to\infty} \sqrt[2n]{\cos^{2n} x + \sin^{2n} x}, x \in R$ S) Not differentiable at infinitely many points in its domain.

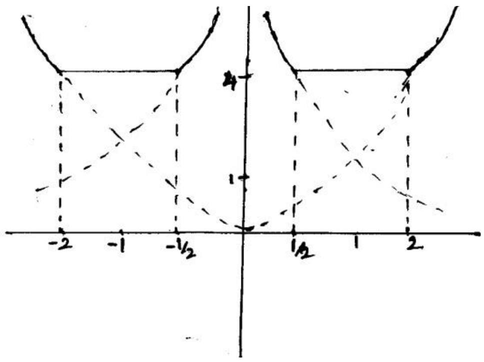
Key. A-Q,R; B-P,R; C-P,R; D-P,S

Sol. a) f (x) = 1 x > 0

= 0 x = 0

=-1 x < 0

b)



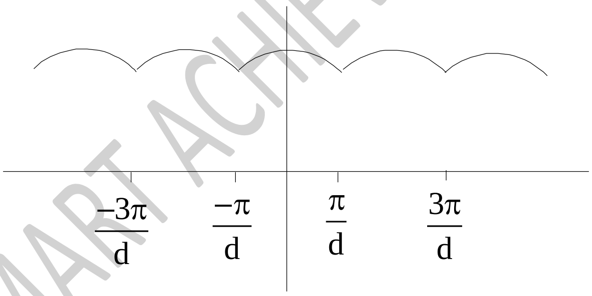
The graph shown are $y = x^2, y = \frac{1}{x^2}$ and y = 4

c) f(x) = $\lim\limits_{h\to\infty} \frac{\ln\left(e^n + x^n\right)}{h} = \lim\limits_{h\to\infty} \frac{n + \ln\left(1 + \left(\frac{x}{e}\right)^n\right)}{h}$

= 1, $o \le x \le e$

= lnx, x > e

d)



13. Match the following: -

Column – I		Column – II	
(A)	Number of points of discontinuity of f(x) = $\tan^2 x - \sec^2 x$ in $(0, 2\pi)$ is	(p)	4
(B)	Number of points at which $f(x) = 2\sin^{-1} x + \tan^{-1} x + \cot^{-1} x$ is non-differentiable in (- 1, 1) is	(q)	3
(C)	Number of points of discontinuity of	(r)	2

	$y=[\sin x], x\in[0,2\pi)$ where [.] represents greatest integer function		
(D)	Number of points where $y=\left|(x-1)^3\right|+\left|(x-2)^5\right|+\left|x-3\right|$ is non-differentiable	(s)	1
		(t)	0

Key. A $\rightarrow$ r; B $\rightarrow$ t; C $\rightarrow$ r; D $\rightarrow$ q

Sol. (A) $\tan^2 x$ is discontinuous at $x=\frac{\pi}{2},\frac{3\pi}{2}$

$\sec^2 x$ is discontinuous at $x=\frac{\pi}{2},\frac{3\pi}{2}$

$\Rightarrow$ Number of discontinuities

(B) Since $f(x)=\sin^{-1}x+\tan^{-1}x+\cot^{-1}x=\sin^{-1}x+\frac{\pi}{2}$

$\therefore$ f(x) is differentiable in (- 1, 1) $\Rightarrow$ no. of points of non-diff. = 0

(C) $$y=[\sin x]=\begin{cases}0 & , \quad 0\le x<\frac{\pi}{2}\\ 1 & , \quad x=\frac{\pi}{2}\\ 0 & , \quad \frac{\pi}{2}<x\le\pi\\ -1 & , \quad \pi<x<2\pi\\ 0 & , \quad x=2\pi\end{cases}$$

$\therefore$ points of discontinuity are $\frac{\pi}{2},\pi$

(D) $y=\left|(x-1)^3\right|+\left|(x-2)^5\right|+\left|x-3\right|$ is non differentiable at x = 3 only

14. Match the following: -

Column – I		Column – II	
(A)	Number of points where the function $f(x)=\begin{cases}1+\left[\cos\frac{nx}{2}\right] & 1<x<2\\ 1-\{x\} & 0\le x<1\\ \left|\sin\pi x\right| & -1\le x<0\end{cases}$ and f(1) = 0 is continuous but non-differentiable	(p)	0
(B)	$f(x)=\begin{cases}x^2e^{1/x} & , \; x\ne 0\\ 0 & , \; x=0\end{cases}$, then $f'(0^-)=$	(q)	1
(C)	The number of points at which $g(x)=\frac{1}{1+\frac{2}{f(x)}}$ is not	(r)	2

	differentiable where $f(x)=\frac{1}{1+\frac{1}{x}}$, is		
(D)	Number of points where tangent does not exist for the curve y = sgn (x^2-1)	(s)	3
		(t)	4

Key. A $\to$ q; B $\to$ p; C $\to$ s; D $\to$ p

Sol. (A) $f(x)=\begin{cases} 0 & , \quad 1<x\le 2 \\ 1-x & , \quad 0\le x<1 \\ -\sin x & , \quad -1\le x<0 \end{cases}$ continuous at x = 1 but not differentiable

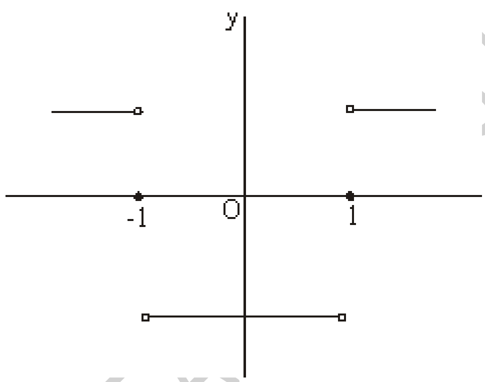
(B) $f'(0^-)=\lim\limits_{h\to 0^-}\frac{h^2e^{-1/h}-0}{-h}=\lim\limits_{h\to 0^-}\left(-he^{-1/h}\right)=0$

(C) $g(x)=\frac{1}{1+\frac{1}{x}(2+2x)}=\frac{x}{3x-2}$

Thus the points where g(x) is not differentiable are x = 0, - 1, $-\frac{2}{3}$

(D) vertical tangents exist at x = 1 and x = - 1 else where horizontal tangents exist.

$\therefore$ number of points where tangent does not exist is 0



15. Match the following: -

	Column – I		Column – II
(A)	$f(x)=\lvert x^3\rvert$ is	(p)	Continuous in (- 1, 1)
(B)	$f(x)=\sqrt{\lvert x\rvert}$ is	(q)	Discontinuous in (- 1, 1)
(C)	$f(x)=\lvert\sin^{-1}x\rvert$ is	(r)	Differentiable in (0, 1)
(D)	$f(x)=\cos^{-1}\lvert x\rvert$ is	(s)	Not differentiable atleast at one point in (- 1, 1)
		(t)	Differentiable in (-1, 1)

Key. A $\to$ p,r,t; B $\to$ p,r,s; C $\to$ p,r,s; D $\to$ p,r,s

Sol. (A) $f(x)=|x^3|$ is continuous and differentiable

(B) $f(x)=\sqrt{|x|}$ is continuous

$f'(x)=\frac{1}{2\sqrt{|x|}}\cdot\frac{x}{|x|}$ {does not exist at x = 0}

(C) $f(x)=|\sin^{-1}x|$ is continuous

$f'(x)=\frac{\sin^{-1}x}{|\sin^{-1}x|},\frac{1}{\sqrt{1-x^2}}$ {does not exist at x = 0}

(D) $f(x)=\cos^{-1}|x|$ is continuous

$f'(x)=\frac{-1}{\sqrt{1-x^2}}\cdot\frac{x}{|x|}$ {does not exist at x = 0}

16. **Column I** **Column II**

A) Let $f:R\to R$ is defined by the equation $f(x+y)=f(x).f(y)\,\forall x,y\in R$, $f(0)\neq 0$, and $f'(0)=2$, then $\frac{f'(x)}{f(x)}$ *is* — p) 1

B) Let $f(x)=\begin{cases}\frac{x^2-1}{4}, |x|<1\\ Tan^{-1}x, |x|\geq 1\end{cases}$ then $f(x)$ is not differentiable at 'x' is equal to — q) -1

C) Let $f(x)=\begin{cases}x^2+a, 0\leq x<1\\ 2x+b, 1\leq x\leq 2\end{cases}$ and $g(x)=\begin{cases}3x+b, 0\leq x<1\\ x^3\ ,\ 1\leq x\leq 2\end{cases}$. If $\frac{df}{dg}$ exists then $a=$ — r) 2

D) If y = Tan(x+y) then $\frac{d^3y}{dx^3}=-\left(\frac{6y^4+16y^2+\lambda}{y^8}\right)$ then the value of $\left[\frac{\lambda}{3}\right]$, $[\cdot]$ is greatest integer function. — s) 3

Key. **A → r; B → p,q; C → q; D → s**

Sol. A) $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$\lim_{h\to 0}\frac{f(x)[f(h)-f(0)]}{h}=f(x).f'(0)\Rightarrow\frac{f'(x)}{f(x)}=2$$

B) f(x) is not continuous at x = -1 and hence not differentiable at their points.

C) f(x) is differentiable at x = 1 if 1+a = 2+b ....(1)

g(x) is differentiable at x = 1 if 3+b = 1 ....(2)

from (1) & (2) b = -2 , a = -1

D)

$$\frac{dy}{dx} = \frac{\sec^2(x+y)}{1-\sec^2(x+y)} = \frac{1+y^2}{1-(1+y^2)}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{y^2} - 1$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{2}{y^3}\left(\frac{-1}{y^2} - 1\right) = -\frac{2}{y^5} - \frac{2}{y^3}$$

$$\Rightarrow \frac{d^3y}{dx^3} = \left(\frac{10}{y^6} + \frac{6}{y^4}\right)\left(-\frac{1}{y^2} - 1\right) = -\left(\frac{6y^4 + 16y^2 + 10}{y^8}\right)$$

$$\lambda = 10 \Rightarrow \left[\frac{\lambda}{3}\right] = \left[\frac{10}{3}\right] = [3.33] = 3$$

17. Match the following functions with their continuity and differentiability

Column I	Column II
A) $f(x) = xe^{-\lvert x\rvert}$	p) Continuous for all x
B) $f(x) = \frac{\sqrt{x+1}-1}{\sqrt{x}},\ f(0) = 0$	q) differentiable at x = 0
C) $f(x) = x Tan^{-1}\frac{1}{x},\ f(0) = 0$	r) continuous at x = 0
D) $f(x) = \frac{1}{1+e^{\frac{1}{x}}}$,f(0)=0	s) discontinuous at x = 0

Key. **A $\to$ p,q,r; B $\to$ p,r; C $\to$ p,r; D $\to$ s**

Sol. A)

f(x) is obviously continuous every where also

$$\Rightarrow f'(0) = \lim_{h\to 0^+} \frac{he^{-h} - 0}{h} = 1$$

B)

F(x) is continuous at '0' since $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{\sqrt{x}} = 0$,f(0)=0 but is not differentiable at '0'

since $\lim_{h\to 0} \frac{\sqrt{h+1}-1}{h}$ does not exist

C)

$$\lim_{x\to 0} f(x) = 0,\ f(0) = 0$$

$\Rightarrow f(x)$ is continuous at 0 but $f'(0) = \lim_{h\to 0} \frac{h\tan^{-1}\frac{1}{h}}{h}$ = 0

$$\Rightarrow \lim_{h\to 0}\tan^{-1}\frac{1}{h}=\frac{\pi}{2}\ or\ -\frac{\pi}{2}$$

$$\Rightarrow f^{'}(0)\ does\ not\ exist$$

D) $\lim_{x\to 0^+} f(x)=0, \lim_{x\to 0^-} f(x)=1$

$\Rightarrow$ f(x) is not continuous at 0

18. Match the following with their first derivatives:

Column –I	Column-II
A) $\sin^{-1}\left(2x\sqrt{1-x^2}\right),\quad \left(x<-\frac{1}{\sqrt{2}}\right)$	p) 0
B) $2\sin^{-1}\left(\sqrt{1-x}\right)+\sin^{-1}\left(2\sqrt{x(1-x)}\right),\ \left(0<x<\frac{1}{2}\right)$	q) $-\frac{2}{1+x^2}$
C) $\sin^{-1}\left(3x-4x^3\right),\ 0<x<\frac{1}{2}$	r) $\frac{3}{\sqrt{1-x^2}}$
D) $\cos^{-1}\frac{2x}{1+x^2}\ for\ \lvert x\rvert>1$	s) $-\frac{2}{\sqrt{1-x^2}}$

Key. **A $\to$ s; B $\to$ p; C $\to$ r; D $\to$ q**

Sol. (A) - s

Put $x=\sin\theta\quad \sin ce\ \sin\theta\in\left[-\frac{1}{\sqrt{2}},-1\right]$

$$\Rightarrow \theta\in\left(-\frac{\pi}{2},\frac{-\pi}{4}\right)$$

$$\Rightarrow \sin^{-1}(\sin 2\theta)=\sin^{-1}\left(\sin(-\pi-2\theta)\right)=-\pi-2\sin^{-1}x$$

$$\Rightarrow \frac{dy}{dx}=\frac{-2}{\sqrt{1-x^2}}$$

(B) – p

Proceed as in $(i)\theta\in\left(\frac{\pi}{6},\frac{\pi}{2}\right)$

(C) – r

$$\left[Put\ x=\sin\theta, \theta\in\left(0,\frac{\pi}{6}\right)\right]$$

(D) – q

$$\theta\in\left(\frac{\pi}{4},\frac{\pi}{2}\right) or \left(-\frac{\pi}{2},\frac{-\pi}{4}\right)$$

$$\Rightarrow 2\theta\in\left(\frac{\pi}{2},\pi\right)\ or\ \left(-\pi,-\frac{\pi}{2}\right)$$

$$\Rightarrow \cos^{-1}(\sin 2\theta)=\cos^{-1}\left(\frac{\pi}{2}-2\theta\right)$$

$$\Rightarrow \frac{\pi}{2} - 2\tan^{-1} x$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{1+x^2}$$