Limits

Single Correct Answer Type

1. If a_n and b_n are positive integers and $a_n + \sqrt{2}b_n = \left(2+\sqrt{2}\right)^n$, then $\lim\limits_{n\to\infty}\left(\dfrac{a_n}{b_n}\right) =$

A) $\sqrt{2}$ B) 2 C) $e^{\sqrt{2}}$ D) e^2

Key. A

Sol. We have $a_n + \sqrt{2}b_n = \left(2+\sqrt{2}\right)^n$

$$\Rightarrow a_n - \sqrt{2}b_n = \left(2-\sqrt{2}\right)^n$$

Therefore $a_n = \dfrac{1}{2}\left[\left(2+\sqrt{2}\right)^n + \left(2-\sqrt{2}\right)^n\right]$

And $b_n = \dfrac{\left[\left(2+\sqrt{2}\right)^n - \left(2-\sqrt{2}\right)^n\right]}{2\sqrt{2}}$

Therefore $\dfrac{a_n}{b_n} = \sqrt{2}\dfrac{\left[\left(2+\sqrt{2}\right)^n + \left(2-\sqrt{2}\right)^n\right]}{\left[\left(2+\sqrt{2}\right)^n - \left(2-\sqrt{2}\right)^n\right]}$

$$= \sqrt{2}\frac{\left[1+\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)^n\right]}{\left[1-\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)^n\right]}$$

Hence $\lim\limits_{n\to\infty}\left(\dfrac{a_n}{b_n}\right) = \sqrt{2}\left(\dfrac{1+0}{1-0}\right)\left(\because \dfrac{2-\sqrt{2}}{2+\sqrt{2}} < 1\right) = \sqrt{2}$

2. If $f(0)=0$ and that 'f' is differentiable at x = 0, and 'k' is a positive integer. Then $\underset{x\to 0}{\mathrm{lt}}\dfrac{1}{x}\left[f(x)+f\left(\dfrac{x}{2}\right)+f\left(\dfrac{x}{3}\right)+----+f\left(\dfrac{x}{k}\right)\right]$

(A) $K.f^1(0)$ (B) $\left(\sum\limits_{r=1}^{K}\dfrac{1}{r}.\right)f^1(0)$ (C) $\sum\limits_{r=1}^{K}\dfrac{1}{r}$ (D) does not exist

Key. B

Sol. $l = \underset{x\to 0}{\mathrm{lt}}\left\{\dfrac{f(x)-f(0)}{x-0}+\dfrac{f\left(\dfrac{x}{2}\right)-f(0)}{x-0}+---\right.$

$$---+\frac{f\left(\frac{x}{k}\right)-f(0)}{x-0}\Bigg\}$$

$$=\left(1+\frac{1}{2}+\frac{1}{3}+---+\frac{1}{k}\right)f^{1}(0).$$

3. $\underset{x\to0}{Lt}\left(\sum_{r=1}^{n}r^{\cos ec^{2}x}\right)^{\sin^{2}x}=$

A. 0 B. ∞ C. n D. $\frac{1}{n}$

Key. C

Sol. $L=\underset{x\to0}{Lt}(1^{\cos ec^{2}n}+2^{\cos ec^{2}n}+\_\_\_\_+n^{\cos ec^{2}n})^{\sin^{2}n}$

$\underset{x\to0}{Lt}((\frac{1}{n})^{\cos ec^{-}b}+(\frac{2}{n})^{\cos ec^{-}n}+\_\_\_\_(\frac{n-1}{n})^{\cos ec^{-}n}+1)^{\sin^{-}n}.n$

$=(0+0+0+\_\_\_\_\_+1)^{0}.n=n$

4. For each positive integer *n*, let $s_n=\frac{3}{1.2.4}+\frac{4}{2.3.5}+\frac{5}{3.4.6}+.....+\frac{n+2}{n(n+1)(n+3)}$. Then $\lim_{n\to\infty}s_n$ equals

A) $\frac{29}{6}$ B) $\frac{29}{36}$ C) 0 D) $\frac{29}{18}$

Key. B

Sol. Let $u_k=\frac{k+2}{k(k+1)(k+3)}$

$$=\frac{(k+2)^2}{k(k+1)(k+2)(k+3)}$$

$$=\frac{k^2+4k+4}{k(k+1)(k+2)(k+3)}$$

$$=\frac{k(k+1)+3k+4}{k(k+1)(k+2)(k+3)}$$

$$=\frac{1}{(k+2)(k+3)}+\frac{3}{(k+1)(k+2)(k+3)}+\frac{4}{k(k+1)(k+2)(k+3)}$$

$$=\left(\frac{1}{k+2}-\frac{1}{k+3}\right)-\frac{3}{2}\left[\frac{1}{(k+2)(k+3)}-\frac{1}{(k+1)(k+2)}\right]$$

$$-\frac{4}{3}\left[\frac{1}{(k+1)(k+2)(k+3)}-\frac{1}{k(k+1)(k+2)}\right]$$

Now, put $k=1,2,3,....,n$ and add. Thus

$$s_u=u_1+u_2+.....+u_n$$

$$=\left(\frac{1}{3}-\frac{1}{n+3}\right)-\frac{3}{2}\left[\frac{1}{(n+2)(n+3)}-\frac{1}{2.3}\right]$$

$$-\frac{4}{3}\left[\frac{1}{(n+1)(n+2)(n+3)}-\frac{1}{1.2.3}\right]$$

Therefore $\lim\limits_{n\to\infty}s_n=\frac{1}{3}+\frac{3}{12}+\frac{4}{18}=\frac{29}{36}$

5. $\lim\limits_{x\to 0}\frac{a^{\tan x}-a^{\sin x}}{\tan x-\sin x}$ is equal to $(a>0)$

A) $\log_e a$ B) 1 C) 0 D) ∞

Key. A

Sol. We have $\lim\limits_{x\to 0}\frac{a^{\tan x}-a^{\sin x}}{\tan x-\sin x}=\lim\limits_{x\to 0}a^{\sin x}\left(\frac{a^{\tan x-\sin x}-1}{\tan x-\sin x}\right)$

$=\lim\limits_{x\to 0}\left(a^{\sin x}\right)\times\lim\limits_{t\to 0}\left(\frac{a^t-1}{t}\right)$ (where $t=\tan x-\sin x$)

$=a^0\times\log_e a=\log_e a$

6. $\lim\limits_{x\to\frac{\pi}{2}}\frac{(1-\sin x)\left(8x^3-\pi^3\right)\cos x}{(\pi-2x)^4}$

A) $-\frac{\pi^2}{16}$ B) $\frac{3\pi^2}{16}$ C) $\frac{\pi^2}{16}$ D) $-\frac{3\pi^2}{16}$

Key. D

Sol. Let $f(x)=\frac{(1-\sin x)\left(8x^3-\pi^3\right)\cos x}{(\pi-2x)^4}$

$$=\frac{(1-\sin x)\cos x(2x-\pi)\left(4x^2+2\pi x+\pi^2\right)}{(2x-\pi)^4}$$

$$=\frac{(1-\sin x)\cos x\left(4x^2+2\pi x+\pi^2\right)}{(2x-\pi)^3}$$

Therefore $\lim\limits_{x\to\frac{\pi}{2}}f(x)=\lim\limits_{x\to\frac{\pi}{2}}\frac{(1-\sin x)\cos x}{(2x-\pi)^3}.\left(3\pi^2\right)$

$$\lim_{x\to\frac{\pi}{2}} f(x) = \lim_{x\to\frac{\pi}{2}} \frac{(1-\sin x)\cos x}{(2x-\pi)^3}.(3\pi^2) \qquad ----(1.62)$$

Put $2x-\pi = y$ so that $y \to 0$ as $x \to \pi/2$. Therefore now

$$\frac{(1-\sin x)\cos x}{(2x-\pi)^3} = \frac{\left[1-\sin\left(\frac{\pi+y}{2}\right)\right]\cos\left(\frac{\pi+y}{2}\right)}{y^3}$$

$$= \frac{\left(1-\cos\frac{y}{2}\right)\left(-\sin\frac{y}{2}\right)}{y^3}$$

$$= -\left(\frac{2\sin^2\frac{y}{4}}{y^2}\right)\left(\frac{\sin\frac{y}{2}}{y}\right)$$

$$= -2\left(\frac{\sin\frac{y}{4}}{y/4}\right)^2.\frac{1}{16}.\left(\frac{\sin\frac{y}{2}}{y/2}\right).\frac{1}{2}$$

$$= \frac{-1}{16}\left(\frac{\sin\frac{y}{4}}{y/4}\right)^2\left(\frac{\sin\frac{y}{2}}{y/2}\right) \qquad ----(1.63)$$

Therefore from Eqs. (1.62) and (1.63)

$$\lim_{x\to\frac{\pi}{2}} f(x) = \frac{-3\pi^2}{16}\times 1\times 1.$$

7. Let $f : R^+ \to R^+$ be a function satisfying the relation $f(x.f(y)) = f(xy) + x$ for all $x, y \in R^+$. Then $\lim_{x\to 0}\left(\frac{(f(x))^{1/3}-1}{(f(x))^{1/2}-1}\right) =$

(A) 1 (B) $\frac{1}{2}$ (C) $\frac{2}{3}$ (D) $\frac{3}{2}$

Key. C

Sol. Given relation is $f(x.f(y)) = f(xy) + x$ (1.56)

Interchanging x and y in Eq. (1.56), we have

$$f(y.f(x)) = f(yx) + y \qquad (1.57)$$

Again replacing *x* with $f(x)$ in Eq. (1.56) we get

$$f(f(x).f(y)) = f(y.f(x)) + f(x) \qquad (1.58)$$

Therefore, Eqs. $(1.56)-(1.58)$ imply

$$f(f(x).f(y)) = f(xy) + y + f(x) \qquad (1.59)$$

Again interchanging *x* and *y* in Eq. (1.59), we have

$$f\left(f\left(y\right).f\left(x\right)\right)=f\left(yx\right)+x+f\left(y\right) \qquad (1.60)$$

Equations (1.59) and (1.60) imply

$$f\left(xy\right)+y+f\left(x\right)=f\left(yx\right)+x+f\left(y\right) \qquad (1.61)$$

Suppose $f\left(x\right)-x=f\left(y\right)-y=\lambda$

Substituting $f\left(x\right)=\lambda+x$ in Eq. (1.56), we have

$$x.f\left(y\right)+\lambda=\left(xy+\lambda\right)+x$$

$$\Rightarrow x.f\left(y\right)=xy+x$$

Therefore $x\left(y+\lambda\right)=xy+x \qquad \left[\because f\left(y\right)=\lambda+y\right]$

$$\Rightarrow \lambda x=x$$

$$\Rightarrow \lambda=1 \qquad \left(\because x>0\right)$$

So $f\left(x\right)=x+\lambda=x+1$

Hence
$$\lim_{x\to0}\frac{\left(f\left(x\right)\right)^{1/3}-1}{\left(f\left(x\right)\right)^{1/2}-1}=\lim_{x\to0}\frac{\left(1+x\right)^{1/3}-1}{\left(1+x\right)^{1/2}-1}$$

$$=\lim_{x\to0}\left(\frac{\left(1+x\right)^{1/3}-1}{1+x-1}\right).\left(\frac{1+x-1}{\left(1+x\right)^{1/2}-1}\right)$$

$$=\frac{1/3}{1/2}=\frac{2}{3}$$

8. Let $x_1=1$ and $x_{n+1}=\dfrac{4+3x_n}{3+2x_n}$ for $n\ge1$. If $\lim\limits_{n\to\infty}x_n$ exists finitely, then the limit is equal to

(A) $\sqrt{2}$ (B) 1 (C) 2 (D) $\sqrt{2}+1$

Key. A

Sol. We have $x_1=1, x_2=\dfrac{4+3}{3+2}=\dfrac{7}{5}$

$$x_3=\frac{4+3x_2}{3+2x_2}=\frac{4+3\left(\frac{7}{5}\right)}{3+2\left(\frac{7}{5}\right)}=\frac{41}{29}>x_2$$

We can easily verify that $x_n<x_{n+1}$ and hence $\left\{x_n\right\}$ is strictly increasing sequence of positive terms. Let $\lim\limits_{n\to\infty}x_n=l$. Therefore

$$l=\lim_{n\to\infty}x_{n+1}$$

$$=\lim_{n\to\infty}\left(\frac{4+3x_n}{3+2x_n}\right)$$

$$=\frac{4+3\lim\limits_{n\to\infty}x_n}{3+2\lim\limits_{n\to\infty}x_n}$$

$$= \frac{4+3l}{3+2l}$$

Hence $3l + 2l^2 = 4 + 3l$

or $l^2 = 2 \Rightarrow l = \sqrt{2}$ $\left(\because x_n > 0\ \forall\ n\right)$.

9. Let $f(x) = x^3\left\{\sqrt{x^2+\sqrt{x^4+1}} - x\sqrt{2}\right\}$. Then $\lim_{x\to\infty} f(x)$ is equal to

(A) $\frac{1}{2\sqrt{2}}$ (B) $\frac{1}{4\sqrt{2}}$ (C) $\frac{3}{4\sqrt{2}}$ (D) does not exist

Key. B

Sol. We have $f(x) = \frac{x^3\left\{x^2+\sqrt{x^4+1}-2x^2\right\}}{\sqrt{x^2+\sqrt{x^4+1}}+x\sqrt{2}}$

$$= \frac{x^3\left\{\sqrt{x^4+1}-x^2\right\}}{\sqrt{x^2+\sqrt{x^4+1}}+x\sqrt{2}}$$

$$= \frac{x^3\left(x^4+1-x^4\right)}{\left[\sqrt{x^2+\sqrt{x^4+1}}+x\sqrt{2}\right]\left[\sqrt{x^4+1}+x^2\right]}$$

$$= \frac{x^3}{\left[\sqrt{x^2+\sqrt{x^4+1+}}+x\sqrt{2}\right]\left[\sqrt{x^4+1}+x^2\right]}$$

$$= \frac{1}{\left[\sqrt{1+\sqrt{1+\frac{1}{x^4}}}+\sqrt{2}\right]\left[\sqrt{1+\frac{1}{x^4}}+1\right]}$$

$$= \frac{1}{\left(\sqrt{1+\sqrt{1}}+\sqrt{2}\right)\left(\sqrt{1}+1\right)}$$

$$= \frac{1}{2\sqrt{2}(2)} = \frac{1}{4\sqrt{2}}.$$

10. $\lim_{x\to\frac{-1}{3}^-} \frac{1}{x}\left[\frac{-1}{x}\right]$ [.] $\rightarrow$ denotes greatest integer function

1) -9 2) -12 3) -6 4) 0

Key. 3

Sol. $x < -\frac{1}{3}$

$$\frac{1}{x} > -3 \Rightarrow -\frac{1}{x} < 3 \Rightarrow \left[-\frac{1}{3}\right] = 2$$

$$\lim_{x\to -\frac{1}{3}} \frac{1}{x}\left[-\frac{1}{x}\right] = (-3)(2) = -6$$

11. $\lim_{x\to\infty}\left(x - \log_e(\cosh x)\right) =$

1) 1 2) 0 3) $\log_e 2$ 4) ∞

Key. 3

Sol. $\lim_{x\to\infty} x - \log_e\left(\frac{e^x + e^{-x}}{2}\right)$

$\lim_{x\to\infty} x - \log_e e^x\left(\frac{1+e^{-2x}}{2}\right)$

$\lim_{x\to\infty} x - x - \log_e\left(\frac{1+e^{-2x}}{2}\right)$

$\lim_{x\to\infty} -\log_e\left(\frac{1}{2}\right) = \log_e 2$

12. If α is a root of the equation $\sin x + 1 = x$ then $\lim_{x\to\alpha}\left[\frac{\min(\sin x, \{x\})}{x-1}\right]$ is

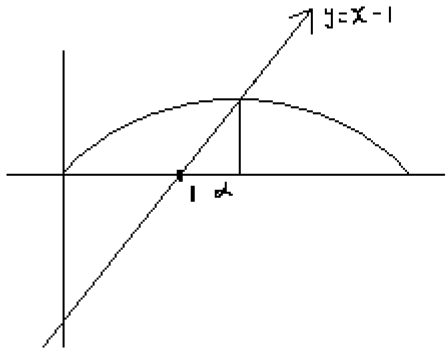
Where [.] $\rightarrow$ denotes greatest integer function

{x}$\rightarrow$ fractional part of x.

1) 1 2) 0 3) does not exist 4) -1

Key. 3

Sol. LHL :



$\lim_{x\to\alpha-}\left[\frac{\min(\sin x, x-[x]}{(x-1)}\right]$

When $1 < x < \alpha$

$\{x\} = x - 1 < \sin x$

$\min\{\sin x, x-1\} = x-1$

Required limit = $\lim_{x\to\infty-}\left[\frac{x-1}{x-1}\right] = 1$

RHL :

$\lim_{x\to\alpha+}\left[\frac{\sin x}{x-1}\right] = 0$

$x \to \alpha +$

$\sin x < x - 1$

$\frac{\sin x}{x-1} < 1$

Hence $LHL \neq RHL$ $\left[\dfrac{\sin x}{x-1}\right] = 0$

Limit does not exist

13. If a_1 is the greatest value of $f(x)$ where $f(x) = \dfrac{1}{2+[\sin x]}$ and $a_{n+1} = \dfrac{(-1)^{n+2}}{n+1} + a_n$

Then $\lim\limits_{n\to\infty} a_n =$ _______

1) 0 2) e 3) 1 4) $\log_e 2$

Key. 4

Sol. $a_1 = 1, a_2 = 1-\dfrac{1}{2}, a_3 = 1-\dfrac{1}{2}+\dfrac{1}{3} ....a_n = 1-\dfrac{1}{2}+\dfrac{1}{3}-\dfrac{1}{4}+......+(-1)^{n-1}.\dfrac{1}{n}$

$\lim\limits_{n\to\infty} a_n = \log_e 2$

14. $\lim\limits_{x\to\frac{\pi}{2}} \left[\dfrac{[\sin x]-[\cos x]+1}{3}\right] =$

[.] $\rightarrow$ denotes greatest integer function

1) 0 2) 1 3) -1 4) does not exist

Key. 1

Sol. LHL = RHL = 0

15. $\lim\limits_{x\to 0}\left(\dfrac{1+2x}{1+3x}\right)^{\frac{1}{x^2}} .e^{\frac{1}{x}} =$ ________

1) $e^{\frac{5}{2}}$ 2) e^2 3) 4) 1

Key. 1

Sol. $\lim\limits_{x\to 0} e^{\frac{1}{x^2}(\log(1+2x)-\log(1+3x)+\frac{1}{x}}$

$$e^{\lim\limits_{x\to 0}\frac{(\log(1+2x)-\log(1+3x)+x}{x^2}} = e^{\frac{5}{2}}$$

16. $\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\cot^{-1}\left(r^2+\dfrac{3}{4}\right) =$

1) $\tan^{-1}(2)$ 2) $\dfrac{\pi}{4}$ 3) $\dfrac{\pi}{2}$ 4) $\tan^{-1}(3)$

Key. 1

Sol. $\cot^{-1}\left(r^2+\frac{3}{4}\right)=\tan^{-1}\left(\frac{1}{r^2+\frac{3}{4}}\right)$

$=\tan^{-1}\left(\frac{1}{1+\left(r^2-\frac{1}{4}\right)}\right)$

$=\tan^{-1}\left(\frac{1}{1+\left(r+\frac{1}{2}\right)\left(r-\frac{1}{2}\right)}\right)$

$=\tan^{-1}\left(\frac{\left(r+\frac{1}{2}\right)-\left(r-\frac{1}{2}\right)}{1+\left(r^2+\frac{1}{4}\right)}\right)$

$=\tan^{-1}\left(r+\frac{1}{2}\right)-\tan^{-1}\left(r-\frac{1}{2}\right)$

17. $\lim_{x\to\infty}\sqrt[3]{x}\left(\sqrt[3]{(x+1)^2}-\sqrt[3]{(x-1)^2}\right)=$

1) $\frac{1}{3}$ 2) $\frac{2}{3}$ 3) 1 4) $\frac{4}{3}$

Key. 4

Sol. $\lim_{x\to\infty}x^{1/3}\left\{(x+1)^{1/3}+(x-1)^{1/3}\right\}\left\{(x+1)^{1/3}-(x-1)^{1/3}\right\}$

Rationalise $\lim_{x\to\infty}\frac{x^{1/3}\left\{(x+1)^{1/3}+(x-1)^{1/3}\right\}2}{\left\{(x+1)^{2/3}+(x^2-1)^{1/3}+(x-1)^{2/3}\right\}}$

$\lim_{x\to\infty}\frac{2.x^{2/3}\left\{\left(1+\frac{1}{x}\right)^{1/3}+\left(1-\frac{1}{x}\right)^{1/3}\right\}2}{x^{2/3}\left\{\left(1+\frac{1}{x}\right)^{2/3}+\left(1-\frac{1}{x}\right)^{1/3}+\left(1-\frac{1}{x}\right)^{2/3}\right\}}=\frac{2x2}{3}=\frac{4}{3}$

18. If $a>0, b>0$ then $\lim_{n\to\infty}\left(\frac{a-1+b^{\frac{1}{n}}}{a}\right)^n=$

1) $b^{\frac{1}{a}}$ 2) $a^{\frac{1}{b}}$ 3) a^b 4) b^a

Key. 1

Sol. Let $\frac{1}{n}=x, \quad \Rightarrow x\to 0 \ as \ n\to\infty$ then required limit $\underset{x\to 0}{Lt}\left(\frac{a-1+b^x}{a}\right)^{\frac{1}{x}} \quad =e^{\underset{x\to 0}{Lt}\frac{b^x-1}{x^a}}$

$$= e^{\frac{1}{a}\log e^{b}} = \left(b^{\frac{1}{a}}\right)$$

19. If $S_n = \frac{1}{1.2.3.4} + \frac{1}{2.3.4.5} + \ldots\ldots\ldots + \frac{1}{n(n+1)(n+2)(n+3)}$ then $\lim_{n\to\infty} S_n =$

1) $\frac{5}{18}$ 2) $\frac{1}{9}$ 3) $\frac{7}{18}$ 4) $\frac{1}{18}$

Key. 4

Sol. $S_n = c - \frac{1}{(n+1)(n+2)(n+3).3}$

$n = 1 \Rightarrow s_1 = c - \frac{1}{2.3.4.3} \Rightarrow c = \frac{1}{1.2.3.4} + \frac{1}{2.3.4.3}$

$c = \frac{1}{2.3.4}\left(1 + \frac{1}{3}\right)$

$= \frac{1}{18}$ Now as $n \to \infty,\ S_n \to c = \frac{1}{18}$

20. $\lim_{x\to\infty}\left(\frac{x^2+5x+3}{x^2+x+2}\right)^x =$

1) e^2 2) e^4 3) e^3 4) e

Key. 2

Sol. $\lim_{x\to\infty}\left(\frac{x^2+5x+3}{x^2+x+2}\right)^x = e^{\lim_{x\to 0}\left(\frac{4x+1}{x^2+x+2}\right)x} = e^4$

21. If a_n and b_n are positive integers and $a_n + \sqrt{2}b_n = \left(2+\sqrt{2}\right)^n$, then $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) =$

A) $\sqrt{2}$ B) 2 C) $e^{\sqrt{2}}$ D) e^2

Key. A

Sol. We have $a_n + \sqrt{2}b_n = \left(2+\sqrt{2}\right)^n$

$\Rightarrow a_n - \sqrt{2}b_n = \left(2-\sqrt{2}\right)^n$

Therefore $a_n = \frac{1}{2}\left[\left(2+\sqrt{2}\right)^n + \left(2-\sqrt{2}\right)^n\right]$

And $b_n = \frac{\left[\left(2+\sqrt{2}\right)^n - \left(2-\sqrt{2}\right)^n\right]}{2\sqrt{2}}$

Therefore $$\frac{a_n}{b_n}=\sqrt{2}\frac{\left[\left(2+\sqrt{2}\right)^n+\left(2-\sqrt{2}\right)^n\right]}{\left[\left(2+\sqrt{2}\right)^n-\left(2-\sqrt{2}\right)^n\right]}$$

$$=\sqrt{2}\frac{\left[1+\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)^n\right]}{\left[1-\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)^n\right]}$$

Hence $\lim\limits_{n\to\infty}\left(\frac{a_n}{b_n}\right)=\sqrt{2}\left(\frac{1+0}{1-0}\right)\left(Q\ \frac{2-\sqrt{2}}{2+\sqrt{2}}<1\right)=\sqrt{2}$

22. $\lim\limits_{n\to\infty}\frac{(n!)^{\frac{1}{n}}}{n}$ equals

a) e   b) e^{-1}

c) e^{-2} d) e^2

Key. B

$let \quad P=\frac{(n!)^{\frac{1}{n}}}{n}$

Sol. $=\left(\frac{(n!)}{n^n}\right)^{\frac{1}{n}}$

$\log P=\frac{1}{n}\sum_{r=1}^{n}\log\left(\frac{r}{n}\right)$

23. The value of $\lim\limits_{x\to 0}\frac{(1+x)^{1/x}-e}{x}$ is

a) $\frac{e}{2}$ b) $-\frac{e}{2}$

c) $\frac{3e}{2}$ d) $-\frac{2e}{3}$

Key. B

Sol. $(1+x)^{\frac{1}{x}}=e^{\frac{1}{x}\log(1+x)}$

$=e^{(1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}......)}$

24. $\underset{n\to\infty}{Lt}\left[\left(1+\frac{1}{n}\right)^{n}-\left(1+\frac{1}{n}\right)\right]^{-n}=$

1) 1 2) $\frac{1}{e-1}$ 3) $1-e^{-1}$ 4) 0

Key. 4

Sol. $\underset{n\to\infty}{Lt}\left[\left(1+\frac{1}{n}\right)^{n}-\left(1+\frac{1}{n}\right)\right]=e-1>1$

25. Let $f(x)=\frac{\tan x}{x}$, then $\log_e\left(\lim_{x\to0}\left([f(x)]+x^2\right)^{\frac{1}{\{f(x)\}}}\right)$ is equal, (where [·] denotes greatest integer function and {·} fractional part)

(A) 1 (B) 2 (C) 3 (D) 4

Key. C

Sol. $\lim_{x\to0}\left[f(x)\right]=\lim_{x\to0}\left[\frac{\tan x}{x}\right]=1$

$\lim_{x\to0}\left([f(x)]+x^2\right)^{\frac{1}{\{f(x)\}}}=\lim_{x\to0}\left(1+x^2\right)^{\frac{1}{\{f(x)\}}}\left(1^\infty \text{ form}\right)$

Again, f(x) = $\frac{\tan x}{x}=\frac{x+\frac{x^3}{3}+\frac{2}{15}x^5+\ldots}{x}$

$=1+\frac{x^2}{3}+\frac{2}{15}x^4+\ldots$

$\{f(x)\}=\frac{x^2}{3}+\frac{2}{15}x^4+\ldots$

(i) becomes,

$$\log_e\left(e^{\lim_{x\to0}x^2\times\frac{1}{\{f(x)\}}}\right)=e^{\lim_{x\to0}\frac{x^2}{\frac{x^2}{3}+\frac{2}{15}x^4+\ldots}}=3$$

∴ (C) is the correct answer.

26. Let $x>0$ then $\underset{x\to0}{Lt}\left(\sqrt{\tan x}\right)^{\sqrt{x}}+\left(\sec x\right)^{\frac{1}{x}}=$

(A) $1/e$ (B)1 (C) $\frac{1}{e^2}$ (D) 2

Key. D

Sol. $\underset{x\to0^+}{Lt}\left(\sqrt{\tan x}\right)^{\sqrt{x}}+\underset{x\to0^+}{Lt}\left(\cos x\right)^{-1/x}$

$$e^{\underset{x\to0^+}{Lt}\frac{\log_e\left(\sqrt{\tan x}\right)}{\frac{1}{\sqrt{x}}}}\left(\frac{-\infty}{\infty}\right)=e^0=1,\ \underset{x\to0^+}{Lt}\left(\cos x\right)^{-1/x}=1\ as\ \ 0<\cos x<1$$

27. $\lim\limits_{x\to 0}\frac{\sin[\cos x]}{1+[\cos x]}$, ([x] denotes the greatest integer less than or equal to)

(A) sin 1 (B) 0

(C) Does not exist (D) $\frac{\sin 1}{2}$

Key. B

Sol. $\text{LHL} = \lim\limits_{x\to 0^-} f(0-h) = \lim\limits_{h\to 0}\frac{\sin[\cosh]}{1+[\cosh]}$

$= \frac{\sin(0)}{1+0} = 0 \qquad \left(\because h>0 \atop \therefore \cosh<1\right)$

$\text{RHL} = \lim\limits_{x\to 0^+} f(x) = \lim\limits_{h\to 0} f(0+h)$

$= \lim\limits_{h\to 0}\frac{\sin[\cos h]}{1+[\cosh]}$

$= \frac{\sin(0)}{1+0} = 0 \qquad \left(\because h>0 \atop \therefore \cosh<1\right)$

$\therefore \lim\limits_{x\to 0}\frac{\sin[\cos x]}{1+[\cos x]} = 0$

28. If $\lim\limits_{x\to a}\left(2-\frac{a}{x}\right)^{a\tan\left(\frac{\pi x}{2a}\right)} = e$, then '$a$' is equal to

A) $-\pi$ B) $\frac{-\pi}{2}$ C) $\frac{\pi}{2}$ D) $\frac{-2}{\pi}$

Key. B

Sol. $\text{Lim}_{x\to a}\left(2-\frac{a}{x}\right)^{a\tan\left(\frac{\pi x}{2a}\right)} = e$

$\Rightarrow e^{\text{Lim}_{x\to a}\ a\tan\left(\frac{\pi x}{2a}\right)\left(1-\frac{a}{x}\right)}$

$\Rightarrow e^{\text{Lim}_{x\to a}\frac{a\left(1-\frac{a}{x}\right)}{\cot\left(\frac{\pi x}{2a}\right)}} = e$

$\therefore \text{Lim}_{x\to a}\frac{a\left(\frac{-x}{a}\right)\left(1-\frac{x}{a}\right)}{\text{Tan}\frac{\pi}{2}\left(1-\frac{x}{a}\right)} = 1$

$$\underset{x\to a}{\text{Lim}}\frac{\frac{-2x}{\pi}\left(1-\frac{x}{a}\right)\frac{\pi}{2}}{\text{Tan}\frac{\pi}{2}\left(1-\frac{x}{a}\right)}=1$$

$$\frac{-2a}{\pi}=1\Rightarrow a=\frac{-\pi}{2}$$

29. If $f(x)=\left(\frac{|x|}{|x|+2}\right)^{-x}$ then

A) $\lim_{x\to-\infty} f(x)=e^2$

B) $\lim_{x\to-\infty} f(x)=0$

C) $\lim_{x\to 1} f(x)=\frac{1}{3}$

D) $\lim_{x\to\infty} f(x)=e^2$

Key. D

Sol. $\lim_{x\to\infty}\left(\frac{|x|}{|x|+2}\right)^{-x}$

$$=\lim_{x\to-\infty}\left(\frac{2-x-2}{2-x}\right)^{x}$$

$$=\lim_{x\to-\infty}\left(1-\frac{2}{2-x}\right)^{x}$$

$$x\to-\infty\Rightarrow|x|=-x$$

$$x=-\frac{1}{y},\ y\to 0$$

$$\underset{y\to 0}{Lt}\left(1-\frac{2}{2+\frac{1}{y}}\right)^{\frac{1}{y}}$$

$$=\underset{y\to 0}{Lt}\left(1-\frac{y}{2y+1}\right)^{\frac{1}{y}};\ 1^{\infty}\text{ form}$$

$$=e^{\underset{y\to 0}{Lt}\frac{1}{y}\left(1-\frac{y}{2y+1}-1\right)}$$

$$=e^{\underset{y\to 0}{Lt}\left(\frac{1}{2y+1}\right)}=e^{1}$$

30. The value of $\lim_{x\to 0}\frac{\cos(\sin^2 x)-\cos(x^2)}{x^6}$ is

(A) 0 (B) 1/2

(C) 1/3 (D) 3/4

Key. C

Sol. $\lim_{x\to 0}\frac{\cos(\sin^2 x)-\cos(x^2)}{x^6}$

$$= \lim_{x\to 0}\frac{2\sin\left(\frac{\sin^2 x+x^2}{2}\right).\sin\left(\frac{x^2-\sin^2 x}{2}\right)}{x^6}$$

$$= \lim_{x\to 0}\frac{2\sin\left(\frac{\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+.....\right)^2+x^2}{2}\right).\sin\left(\frac{x^2-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+......\right)^2}{2}\right)}{x^6}$$

$$= \lim_{x\to 0}\frac{2\sin\left(\frac{2x^2-\frac{2x^4}{6}....}{2}\right)\sin\left(\frac{x^4}{6}......\right)}{x^2\times 6.\frac{x^4}{6}}$$

$$= \lim_{x\to 0}\frac{2\sin\left(x^2-\frac{x^4}{6}.....\right)}{x^2}.\frac{1}{6}=\frac{1}{3}$$

31. $\lim_{x\to 0}\frac{\sin^{-1}x-\tan^{-1}x}{x^3}$ is equal to

(A) $\frac{1}{6}$ (B) $\frac{1}{2}$

(C) 2 (D) $-\frac{1}{2}$

Key. B

Sol. $p=\lim_{x\to 0}\frac{\sin^{-1}x-\tan^{-1}x}{x^3}=\lim_{x\to 0}\left(\frac{1}{\sqrt{1-x^2}}-\frac{1}{1+x^2}\right).\frac{1}{3x^2}$

$$=\frac{1}{3}\lim_{x\to 0}\frac{1+x^2-\sqrt{1-x^2}}{x^2}.\frac{1}{\sqrt{1-x^2}(1+x^2)}$$

$$=\frac{1}{3}\lim_{x\to 0}\frac{(1+x^2)^2-(1-x^2)}{x^2}.\frac{1}{1+x^2+\sqrt{1-x^2}}.\frac{1}{\sqrt{1-x^2}(1+x^2)}$$

$$=\frac{1}{3}.3.\frac{1}{2}.\frac{1}{1}=\frac{1}{2}$$

32. Let $f(x)=\lim\limits_{n\to\infty}\dfrac{(2\sin x)^{2n}}{3^n-(2\cos x)^{2n}}; n\in I$,then which of the following is not true?

(A) at $x=n\pi\pm\dfrac{\pi}{6}$, f(x) is discontinuous (B) $f\left(\dfrac{\pi}{3}\right)=1$

(C) f(0) = 0 (D) $f\left(\dfrac{\pi}{2}\right)=1$

Key. D

Sol.

33. If $\lim\limits_{x\to e^3}\dfrac{(\ln x-3)^n}{\ln((\cos^m(\ln x-3)))}$ = –1 (n, m∈N) then n/m is equal to

(A) 3 (B) 4

(C) 9 (D) 1

Key. D

Sol. Let ln x – 3 = t

$\Rightarrow \lim\limits_{t\to 0}\dfrac{t^n}{\ln(\cos^m t)}\left(\dfrac{0}{0}\text{form}\right)$ = –1

$\Rightarrow \lim\limits_{t\to 0}\dfrac{nt^{n-1}}{-m\tan t}$ = –1

$\Rightarrow$ n – 1 = 1 & $-\dfrac{n}{m}=-1\Rightarrow$ n = m = 2.

34. $\mathop{Lt}\limits_{x\to 0}\dfrac{\tan\left(\left[-\pi^2\right]x^2\right)-x^2\tan\left(\left[-\pi^2\right]\right)}{\sin^2 x}$ where $[.]$ denote g.i.f

a) $\tan 10+10$ b) $\tan 10-10$ c) $10-\tan 10$ d) none of these

Key. B

Sol. $\pi=3.14, then\left[-\pi^2\right]=-10$

$\mathop{Lt}\limits_{x\to 0}\dfrac{\tan\left(\left[-\pi^2\right]x^2\right)-\tan\left(\left[-\pi^2\right]\right)x^2}{\sin^2 x}$ dilute by x^2 we get

$$\mathop{Lt}\limits_{x\to 0}\dfrac{\dfrac{-\tan 10x^2}{x^2}+\tan 10}{\dfrac{\sin^2 x}{x^2}}=\tan 10-10$$

35. $\mathop{\text{Lim}}\limits_{x\to 0}x^2\left(1+2+3+......\left[\dfrac{1}{|x|}\right]\right)$ is equal to, where [.] is greatest integer function

(A) 1 (B) 3/2 (C) 1/2 (D) 2

Key. C

Sol. $x^2\left(1+2+3+.....\left[\frac{1}{|x|}\right]\right)$

$$\frac{x^2\left(1+\left[\frac{1}{|x|}\right]\right)}{2}\left[\frac{1}{|x|}\right]$$

Now using the property that

$$\frac{1}{|x|}-1<\left[\frac{1}{|x|}\right]\le\frac{1}{|x|}$$

we get

$$\frac{1}{2}|x|<\frac{x^2\left(1+\left[\frac{1}{|x|}\right]\right)}{2}\left[\frac{1}{|x|}\right]\le\frac{1}{2}(1+|x|)$$

Now applying sandwich theorem the required limit is $\frac{1}{2}$

36. If 'f' be a bounded, differentiable and increasing function then

$\lim_{x\to 0}\left[f(\sin x.\tan x)-f(x^2)\right]$, where [.] is greatest integer function is equal to

(A) 1 (B) 0 (C) –1 (D) does not exists

Key. B

Sol. since sin x . tan x $> x^2\ \forall\ x\in(0,\pi/2)$

so, f(sin x . tanx) > f(x^2)

hence required limit is 0.

37. If $\underset{x\to 0}{\text{Lt}}\frac{\left(\left(a-n\right)nx-\tan x\right)\sin nx}{x^2}=0$ where n is a non zero real number then a is equal to

a) 0 b) $\frac{n+1}{n}$ c) n d) $n+\frac{1}{n}$

Key: D

Hint $\underset{x\to 0}{\text{Lt}}\left(\left(a-n\right)n-\frac{\tan x}{x}\right)\frac{\sin nx}{x}=0$

$\Rightarrow\left(\left(a-n\right)n-1\right)n=0$

$\Rightarrow a=n+\frac{1}{n}$

38. Let $x>0$ then $\underset{x\to 0}{Lt}\left(\sqrt{\tan x}\right)^{\sqrt{x}}+\left(\sec x\right)^{\frac{1}{x}}=$

(A) $1/e$ (B)1 (C) $\frac{1}{e^2}$ (D) 2

Key: D

Hint: $\underset{x\to 0^+}{Lt}\left(\sqrt{\tan x}\right)^{\sqrt{x}}+\underset{x\to 0^+}{Lt}\left(\cos x\right)^{-1/x}$

$$e^{\underset{x\to 0^+}{Lt}\frac{\log_e\left(\sqrt{\tan x}\right)}{\frac{1}{\sqrt{x}}}\left(\frac{-\infty}{\infty}\right)}=e^0=1,\ \underset{x\to 0^+}{Lt}\left(\cos x\right)^{-1/x}=1\ as\ \ 0<\cos x<1$$

39. Let $f(x)=\begin{cases}\underset{n\to\infty}{Lt}\frac{x^n-\sin\left(x^n\right)}{x^n+\sin\left(x^n\right)} & ,\ if\ \ x>0,\ x\neq 1\\ 1 & ,\ if\ \ \ \ x=1\end{cases}$. Then, at x = 1,

A) f is continuous

B) f has removable discontinuity (i.e., $\underset{x\to 1}{Lt}f(x)$ exists, but this limit is different from f(1))

C) f has finite (jump) discontinuity (i.e., f (1+) and f (1-) both exist finitely, but they are different)

D) f has infinite or oscillatory discontinuity (for eg like $\sin\frac{1}{x}$ at $x=0$ and $\tan x$ at $x=\frac{\pi}{2}$)

Key: C

Hint: $0<x<1\Rightarrow x^n\to 0\ as\ n\to\infty\Rightarrow f(x)=0$ and

$x>1\Rightarrow x^n\to+\infty\ \ as\ n\to\infty\Rightarrow f(x)=1$

$\therefore f$ has a jump (finite) discontinuity at $x=1$

40. $\underset{n\to\infty}{Lt}\left[\left(1+\frac{1}{n}\right)^n-\left(1+\frac{1}{n}\right)\right]^{-n}=$

A) 1 B) $\frac{1}{e-1}$ C) $1-e^{-1}$ D) 0

Ans: D

Hint: $\underset{n\to\infty}{Lt}\left[\left(1+\frac{1}{n}\right)^n-\left(1+\frac{1}{n}\right)\right]=e-1>1$

41. Let $f(x)=\frac{\tan x}{x}$, then $\log_e\left(\lim_{x\to 0}\left(\left[f(x)\right]+x^2\right)^{\frac{1}{\{f(x)\}}}\right)$ is equal, (where [·] denotes greatest integer function and {·} fractional part)

(A) 1 (B) 2 (C) 3 (D) 4

Key: C

Hint: $\lim\limits_{x\to 0}\left[f(x)\right]=\lim\limits_{x\to 0}\left[\dfrac{\tan x}{x}\right]=1$

$\lim\limits_{x\to 0}\left(\left[f(x)\right]+x^2\right)^{\frac{1}{\{f(x)\}}}$ = $\lim\limits_{x\to 0}\left(1+x^2\right)^{\frac{1}{\{f(x)\}}}\left(1^{\infty}\text{ form}\right)$

Again, f(x) = $\dfrac{\tan x}{x}=\dfrac{x+\dfrac{x^3}{3}+\dfrac{2}{15}x^5+\ldots}{x}$

= $1+\dfrac{x^2}{3}+\dfrac{2}{15}x^4+\ldots$

$\{f(x)\}=\dfrac{x^2}{3}+\dfrac{2}{15}x^4+\ldots$

(i) becomes,

$$\log_e\left(e^{\lim\limits_{x\to 0}x^2\times\frac{1}{\{f(x)\}}}\right)=e^{\lim\limits_{x\to 0}\frac{x^2}{\frac{x^2}{3}+\frac{2}{15}x^4+\ldots}}=3$$

$\therefore$ (C) is the correct answer.

42. If $\lim\limits_{x\to\infty}x\left(\tan^{-1}\left(\dfrac{x+\lambda}{x+\mu}\right)-\dfrac{\pi}{4}\right)=1$ then ordered pair(s) (λ,μ) can be

(A) (2000,2011) (B) (0,1)
(C) (5,3) (D) (1,0)

Key: C

Hint: $\lim\limits_{x\to ki}\dfrac{\tan^{-1}\left(\dfrac{x+\lambda}{x+\mu}\right)-\dfrac{\pi}{4}}{\dfrac{1}{x}}=1$

Apply L' hospital rule and simplifying we get

$\lim\limits_{x\to\infty}\dfrac{(\lambda-\mu)x^2}{2x^2+2x(\lambda+\mu)+\left(\mu^2+\lambda^2\right)}=1$

$\Rightarrow\dfrac{\lambda-\mu}{2}=1$

$\Rightarrow\lambda-\mu=2$

$\therefore(\lambda,\mu)$ can be (5,3)

43. Consider the function $f(x)=\begin{cases}\dfrac{p(x)}{x-2}; & x\neq 2\\ 7\ ; & x=2\end{cases}$ where P(x) is a polynomial such that $p'''(x)$ is identically equal to 0 and $p(3)=9.$ If f(x) is continuous at x = 2, then p(x) is

(A) $2x^2+x+6$ (B) $2x^2-x-6$ (C) x^2+3 (D) x^2-x+7

Key: B

Hint: Since $P'''(x)=0$

Let $p(x)=ax^2+bx+c$

$p(2)=0$

$4a+2b+c=0$ ....(1)

$9a+3b+c=9$ ....(2)

$p'(2)=7$

$\Rightarrow 4a+b=7$

Solve 1,2 and 3 to get a,b,c

44. $\lim\limits_{n\to\infty}\frac{(n!)^{\frac{1}{n}}}{n}$ equals

a) e b) e^{-1} c) e^{-2} d) e^{2}

KEY : B

$let \quad P=\frac{(n!)^{\frac{1}{n}}}{n}$

Sol. $=\left(\frac{(n!)}{n^n}\right)^{\frac{1}{n}}$

$\log P=\frac{1}{n}\sum_{r=1}^{n}\log\left(\frac{r}{n}\right)$

45. $\underset{x\to0}{\text{Lim}}\,x^2\left(1+2+3+......\left[\frac{1}{|x|}\right]\right)$ is equal to, where [.] is greatest integer function

(A) 1 (B) 3/2

(C) 1/2 (D) 2

Key. C

Sol. $x^2\left(1+2+3+.....\left[\frac{1}{|x|}\right]\right)$

$\frac{x^2\left(1+\left[\frac{1}{|x|}\right]\right)}{2}\left[\frac{1}{|x|}\right]$

Now using the property that

$\frac{1}{|x|}-1<\left[\frac{1}{|x|}\right]\le\frac{1}{|x|}$

we get

$$\frac{1}{2}|x| < \frac{x^2\left(1+\left[\frac{1}{|x|}\right]\right)}{2}\left[\frac{1}{|x|}\right] \le \frac{1}{2}(1+|x|)$$

Now applying sandwich theorem the required limit is $\frac{1}{2}$

46. If 'f' be a bounded, differentiable and increasing function then $\lim_{x\to 0}\left[f(\sin x.\tan x) - f(x^2)\right]$, where [.] is greatest integer function is equal to

(A) 1 (B) 0
(C) –1 (D) does not exists

Key. B

Sol. since sin x . tan x $> x^2\ \forall\ x \in (0, \pi/2)$
so, f(sin x . tanx) > f(x^2)
hence required limit is 0.

47. $\lim_{x\to 0}\frac{\sin[\cos x]}{1+[\cos x]}$, ([x] denotes the greatest integer less than or equal to)

(A) sin 1 (B) 0

(C) Does not exist (D) $\frac{\sin 1}{2}$

Key. B

Sol. $\text{LHL} = \lim_{x\to 0^-} f(0-h) = \lim_{h\to 0}\frac{\sin[\cos h]}{1+[\cos h]}$

$= \frac{\sin(0)}{1+0} = 0 \qquad \begin{pmatrix} \because\ h>0 \\ \therefore\ \cos h<1 \end{pmatrix}$

$\text{RHL} = \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h)$

$= \lim_{h\to 0}\frac{\sin[\cos h]}{1+[\cos h]}$

$= \frac{\sin(0)}{1+0} = 0 \qquad \begin{pmatrix} \because\ h>0 \\ \therefore\ \cos h<1 \end{pmatrix}$

$\therefore \lim_{x\to 0}\frac{\sin[\cos x]}{1+[\cos x]} = 0$

48. $\underset{x\to 0}{\text{Lt}}\left(\frac{1}{x^2} - \text{cosec}^2 x\right) =$

a) $\frac{1}{3}$ b) $\frac{2}{3}$ c) $\frac{-1}{3}$ d) $\frac{-2}{3}$

Key. C

Sol. Apply, L-H rule

49. If a_n and b_n are positive integers and $a_n + \sqrt{2}b_n = \left(2+\sqrt{2}\right)^n$, then $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) =$

A) 2 B) $\sqrt{2}$ C) $e^{\sqrt{2}}$ D) e^{2}

Key. B

Sol. We have $a_n+\sqrt{2}b_n=\left(2+\sqrt{2}\right)^n$

$$\Rightarrow a_n-\sqrt{2}b_n=\left(2-\sqrt{2}\right)^n$$

Therefore $a_n=\frac{1}{2}\left[\left(2+\sqrt{2}\right)^n+\left(2-\sqrt{2}\right)^n\right]$

And $b_n=\frac{\left[\left(2+\sqrt{2}\right)^n-\left(2-\sqrt{2}\right)^n\right]}{2\sqrt{2}}$

Therefore $\frac{a_n}{b_n}=\sqrt{2}\frac{\left[\left(2+\sqrt{2}\right)^n+\left(2-\sqrt{2}\right)^n\right]}{\left[\left(2+\sqrt{2}\right)^n-\left(2-\sqrt{2}\right)^n\right]}$

$$=\sqrt{2}\frac{\left[1+\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)^n\right]}{\left[1-\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)^n\right]}$$

Hence $\lim\limits_{n\to\infty}\left(\frac{a_n}{b_n}\right)=\sqrt{2}\left(\frac{1+0}{1-0}\right)\left(\because \frac{2-\sqrt{2}}{2+\sqrt{2}}<1\right)=\sqrt{2}$

50. The value of $\lim\limits_{x\to 0} f\left(x\right)$ where $f\left(x\right)=\frac{\cos\left(\sin x\right)-\cos x}{x^4}$, is

(A) 2 (B) $1/6$ (C) $2/3$ (D) $-1/3$

Key. B

Sol. $\lim\limits_{x\to 0} f\left(x\right)=\lim\limits_{x\to 0}\frac{\cos\left(\sin x\right)-\cos x}{x^4}$

$$=\lim_{x\to 0}\frac{-2\sin\frac{\sin x+x}{2}\sin\frac{\sin x-x}{2}}{x^4}$$

$$=-\frac{1}{2}\lim_{x\to 0}\frac{\sin\left(\frac{\sin x+x}{2}\right)}{\left(\frac{\sin x+x}{2}\right)}\frac{\sin\left(\frac{\sin x-x}{2}\right)}{\left(\frac{\sin x-x}{2}\right)}\times\frac{\sin x+x}{x}\times\frac{\sin x-x}{x^3}$$

$$=-\frac{1}{2}\lim_{u\to 0}\frac{\sin u}{u}\lim_{v\to 0}\frac{\sin v}{v}\lim_{x\to 0}\left(\frac{\sin x}{x}+1\right)$$

$$\times\frac{-\frac{x^3}{3!}+\frac{x^5}{5!}+....}{x^3}\left(u=\frac{\sin x+x}{2},v=\frac{\sin x-x}{2}\right)$$

$$= -\frac{1}{2} \times 1 \times 1 \times 2 \times \frac{-1}{3!} = \frac{1}{6}.$$

51. $\lim_{n \to \infty} \frac{\{x\} + \{2x\} + \{3x\} + ... + \{nx\}}{n^2} =$

[Where $\{x\} = x - [x]$ denotes the fractional part of x]

A) 1 B) 0 C) $\frac{1}{2}$ D) None of these

Key. B

Sol. $0 \le \{nx\} < 1,\ for\, n = 1, 2, 3, ..., n$

$$\Rightarrow 0 \le \sum_{n=1}^{n} \{nx\} < n \qquad \Rightarrow \frac{0}{n^2} \le \frac{\sum_{n=1}^{n}\{nx\}}{n^2} < \frac{1}{n}$$

$$\Rightarrow \underset{x \to \infty}{Lt} \frac{0}{n^2} \le \underset{n \to \infty}{Lt} \frac{\sum_{n=1}^{n}\{nx\}}{n^2} \le \underset{n \to \infty}{Lt} \frac{1}{n} \qquad \Rightarrow 0 \le \underset{n \to \infty}{Lt} \frac{\sum_{n=1}^{n}\{nx\}}{n^2} \le 0$$

$$\Rightarrow \underset{n \to \infty}{Lt} \frac{\{x\} + \{2x\} + ... + \{nx\}}{n^2} = 0$$

52. For x > 0; $\lim_{x \to 0} \left\{ (\sin x)^{1/x} + \left(\frac{1}{x}\right)^{\sin x} \right\}$ is __________

(1) 0 (2) -1 (3) 1 (4) 2

Key. 3

Sol. $\underset{x \to 0}{Lt} (\sin x)^{1/2} = 0 \quad \left(0 < \sin x < 1;\ \frac{1}{x} \to \infty \right)$

And $\log y = \sin x . \log x$

53. $\lim_{x \to 0} \frac{\cos(\sin x) - \cos x}{x^4} =$ ____

(1) $\frac{1}{5}$ (2) $\frac{1}{6}$ (3) $\frac{1}{4}$ (4) $\frac{1}{2}$

Key. 2

Sol. $\underset{x \to 0}{Lt} \frac{2 \sin\left(\frac{x + \sin x}{2}\right)}{\frac{\sin x + x}{2}} \left(\frac{\sin x + x}{2}\right) . \underset{x \to 0}{Lt} \frac{2 \sin\left(\frac{x - \sin x}{2}\right)}{\frac{x - \sin x}{2}} . \underset{x \to 0}{Lt} \frac{1}{2}(x - \sin x)$

$$\underset{x \to 0}{Lt} \left(\frac{\sin x + x}{2x^4}\right) . \frac{1}{2} \left[x - \left(x - \frac{x^3}{13} + \frac{x^5}{15} + ...\infty \right) \right]$$

54. $\lim_{x\to 0}\left\{\frac{7}{10}+\frac{29}{10^2}+\frac{133}{10^3}+\ldots+\frac{5^n+2^n}{10^n}\right\}=$ ______

(1) $\frac{3}{4}$ (2) 2 (3) $\frac{5}{4}$ (4) $\frac{1}{2}$

Key. 3

Sol. $\frac{5+2}{10}+\frac{5^2+2^2}{10^2}+.....+\frac{5^w+2^w}{10^n}$

(use G.P; s_∞)

55. $\lim_{x\to 0}\frac{729^x-243^x-81^x+9^x+3^x-1}{x^3}=K(\log 3)^3 \quad \Rightarrow$ K = _______

(1) 4 (2) 5 (3) 6 (4) 7

Key. 3

Sol. $\underset{x\to 0}{Lt}\frac{(3^x-1)}{x}\frac{(9^x-1)}{x}\left(\frac{27^x-1}{x}\right)$

56. $\lim_{x\to\infty}\left(1+\frac{a}{x}+\frac{b}{x^2}\right)^{2x}=e^2$ then ____

(1) $a\in R;\ b\in R$ (2) $a=1;\ b\in R$ (3) $a\in R;\ b=2$ (4) $a=1;\ b=2$

Key. 2

Sol. $Lt\ f(x)^{g(x)}$ is of form $1^\infty \Rightarrow e^{\underset{x\to 0}{Lt}\, g(x)\{f(x)-1\}}$

57. $\lim_{\theta\to 0}\left\{\left[\frac{n\sin\theta}{\theta}\right]+\left[\frac{n\tan\theta}{\theta}\right]\right\}=$ ______ where [x] is greatest integer $\le x$ and $n\in I$

(1) 2n (2) 2n + 1 (3) 2n – 1 (4) 0

Key. 3

Sol. $\frac{\sin\theta}{\theta}\to 1 as\,\theta\to 0$ but < 1

$\therefore\left[\frac{n\sin\theta}{\theta}\right]=n-1$

$\left[n\frac{Tan\theta}{\theta}\right]=n$ $\qquad \frac{Tan\theta}{\theta}\to 1\ \ as\ \ \theta\to 0$ but > 1

58. If $f(x)=\underset{n\to\infty}{Lt}\left\{\frac{x}{x+1}+\frac{x}{(x+1)(2x+1)}+\frac{x}{(2x+1)(3x+1)}+\ldots\ldots\right\}$ to n terms; then range of f(x) is _____

(1) $[0,1]$ (2) $[-1,1]$ (3) $\{0,1\}$ (4) $\{-1,0,1\}$

Key. 3

Sol. $1-\frac{1}{1+nx}$

$Lt\ nx=\infty \quad for\ \ x>0$

$Lt\ nx=-\infty \quad for\ \ x<0$

$Lt\ nx=0 \quad for\ \ x=0$

$\underset{n\to\infty}{Lt}S_w=1;\ 0$

59. $\lim_{x\to 0}\left\{Tan\left(\frac{\pi}{4}+x\right)\right\}^{1/x}=$ ____

(1) 1 (2) -1 (3) e^2 (4) e

Key. 3

Sol. 1^∞ form $\Rightarrow e^{\underset{x\to 0}{Lt}\, g(x)(f(x)-1)}$

60. $\lim_{x\to\infty} x\left\{Tan^{-1}\frac{x+1}{x+2}-Tan^{-1}\frac{x}{x+2}\right\}=$ ____

(1) 1 (2) -1 (3) $\frac{1}{2}$ (4) $-\frac{1}{2}$

Key. 3

Sol. $Tan^{-1}x-Tan^{-1}y=Tan^{-1}\frac{x-y}{1-xy}$

$$\underset{x\to\infty}{Lt}\ x\left(\frac{Tan^{-1}\frac{x+2}{2x^2+5x+4}}{\frac{x+2}{2x^2+5x+4}}\right)\left(\frac{x+2}{2x^2+5x+4}\right)$$

61. If 0 < b < a then $\lim_{n\to\infty}\frac{a^n+b^n}{a^n-b^n}=$

(1) 0 (2) 1 (3) –1 (4) none of these

Key. 2

Sol. $0<\frac{b}{a}<1;\left(\frac{b}{a}\right)^n\to 0$ as $n\to\infty$

62. If $a_1=1 and\ a_n=n(1+a_{n-1})\forall n\geq 2,$ then $\lim_{n\to\infty}\left(1+\frac{1}{a_1}\right)\left(1+\frac{1}{a_2}\right)......\left(1+\frac{1}{a^n}\right)=$

(1) e (2) $\log e^2$ (3) $e^{\frac{1}{2}}$ (4) $\log_2 e$

Key. 1

Sol. $\underset{n\to\infty}{Lt}\left(\frac{a_1+1}{a_1}\right)\left(\frac{a_2+1}{a_2}\right).......\left(\frac{a_n+1}{a_n}\right)$

$$\underset{n\to\infty}{Lt}\left(\frac{a_2}{2}\right)\left(\frac{a_3}{3}\right)\left(\frac{a_4}{4}\right)...\left(\frac{a_{n+1}}{n+1}\right)\frac{1}{a_1a_2....a_n}$$

$$=\underset{n\to\infty}{Lt}\ \frac{a_{n+1}}{(n+1)!}=\underset{n\to\infty}{Lt}\frac{1+a_n}{n!}=\underset{n\to\infty}{Lt}\left(\frac{1}{n!}+\frac{a_n}{n!}\right)$$

$= \underset{n\to\infty}{\text{Lt}}\left(\frac{1}{n!}+\frac{1}{(n-1)!}+\frac{a_{n-1}}{(n-1)!}\right) = e$

63. The integer n for which $\lim_{x\to 0}\left(\frac{(\cos x-1)(\cos x-e^x)}{x^n}\right)$ is a finite non zero number is

(1) 1 (2) 2 (3) 3 (4) 4

$$\lim_{x\to 0}\left(\left[\frac{100x}{\sin x}\right]+\left[\frac{99\sin x}{x}\right]\right)$$

Key. 3

Sol. Conceptual

64. The value of where [.] represents greatest integral function, is

(1) 199 (2) 198 (3) 0 (4) none of these

Key. 2

Sol. We know that $\lim_{x\to 0}\frac{\sin x}{x}\to I^-$ and $\lim_{x\to 0}\frac{x}{\sin x}\to I^+$

So, $\lim_{x\to 0}\left[100\frac{x}{\sin x}\right]+\lim_{x\to 0}\left[99\frac{\sin x}{x}\right]=100+98=198$

65. If $\sum_{r=1}^{k}\cos^{-1}\beta_r=\frac{k\pi}{2}$ for any $k\ge 1$ where $\beta_r\ge 0\,\forall r$ and $A=\sum_{r=1}^{k}(\beta_r)^r$. Then

$\lim_{x\to A}\frac{(1+x^2)^{1/3}-(1-2x)^{1/4}}{x+x^2}=$

A) $\frac{1}{2}$ B) 0 C) 3/2 D) $\frac{\pi}{2}$

Key. A

Sol. Given $\cos^{-1}\beta_1+\cos^{-1}\beta_2+...+\cos^{-1}\beta_k=k\frac{\pi}{2}$ We know that $\cos^{-1}x\le\frac{\pi}{2}\,\forall r\ge 0$

$\therefore \cos^{-1}\beta_r\le\frac{\pi}{2}\,\forall r=1,2,3....,k \quad\Rightarrow \sum_{r=1}^{k}\cos^{-1}\beta_r\le\frac{k\pi}{2}$

So the given equality holds only if

$\cos^{-1}\beta_1=\cos^{-1}\beta_2=....=\cos^{-1}\beta_k=\frac{\pi}{2}$

$\Rightarrow \beta_1=\beta_2=....=\beta_k=0$

Thus $A=\sum_{r=1}^{k}(\beta_r)^r=0$

Rrequired limit = $\lim\limits_{x\to 0} \dfrac{(1+x^2)^{1/3}-(1-2x)^{1/4}}{x+x^2}$

= $\lim\limits_{x\to 0} \dfrac{\frac{1}{3}(1+x^2)^{-2/3}(2x)-\frac{1}{4}(1-2x)^{3/4}(-2)}{1+2x}$ (L' Hospital Rule)

= $\dfrac{1}{2}$

66. If [x] and {x} represent integral and fractional parts of x respectively and a is any real number, then $\lim\limits_{x\to[a]} \dfrac{e^{\{x\}}-\{x\}-1}{\{x\}^2} =$

A) a B) $\{a\}$ C) $\dfrac{1}{2}$ D) Does not exist

Key. D

Sol. Let $P = \lim\limits_{x\to[a]} = \dfrac{e^{\{x\}}-\{x\}-1}{\{x\}^2}$

Put $x=[a]+h$, $h>0$

Then $P = \lim\limits_{h\to 0} \dfrac{e^{\{[a]+h\}}-\{[a]+h\}-1}{\{[a]+h\}^2}$

$P = \lim\limits_{h\to 0} \dfrac{e^h-h-1}{h^2}$

$\lim\limits_{h\to 0} \dfrac{e^h-1}{2h} = \dfrac{1}{2}$ [Using L Hospital Rule]

Next put $x=[a]-h, h>0$

then $P = \lim\limits_{h\to 0} \dfrac{e^{\{[a]-h\}}-\{[a]-h\}-1}{\{[a]-h\}^2}$

$= \lim\limits_{h\to 0} \dfrac{e^{1-h}-(1-h)-1}{(1-h)^2} = \lim\limits_{h\to 0} \dfrac{e^{1-h}+h-2}{(1-h)^2} = e-2$

$\therefore$ Limit does not exist

67. Let $f: R^+ \to R^+$ be a function satisfying the relation $f\left(x.f\left(y\right)\right) = f\left(xy\right)+x$ for all $x, y \in R^+$. Then $\lim\limits_{x\to 0}\left(\dfrac{\left(f\left(x\right)\right)^{1/3}-1}{\left(f\left(x\right)\right)^{1/2}-1}\right) =$

(A) 1 (B) $\frac{1}{2}$ (C) $\frac{2}{3}$ (D) $\frac{3}{2}$

Key. C

Sol. Given relation is $f\left(x.f\left(y\right)\right)=f\left(xy\right)+x$ (1.56)

Interchanging x and y in Eq. (1.56), we have

$$f\left(y.f\left(x\right)\right)=f\left(yx\right)+y \quad (1.57)$$

Again replacing *x* with $f\left(x\right)$ in Eq. (1.56) we get

$$f\left(f\left(x\right).f\left(y\right)\right)=f\left(y.f\left(x\right)\right)+f\left(x\right) \quad (1.58)$$

Therefore, Eqs. $(1.56)-(1.58)$ imply

$$f\left(f\left(x\right).f\left(y\right)\right)=f\left(xy\right)+y+f\left(x\right) \quad (1.59)$$

Again interchanging *x* and *y* in Eq. (1.59), we have

$$f\left(f\left(y\right).f\left(x\right)\right)=f\left(yx\right)+x+f\left(y\right) \quad (1.60)$$

Equations (1.59) and (1.60) imply

$$f\left(xy\right)+y+f\left(x\right)=f\left(yx\right)+x+f\left(y\right) \quad (1.61)$$

Suppose $f\left(x\right)-x=f\left(y\right)-y=\lambda$

Substituting $f\left(x\right)=\lambda+x$ in Eq. (1.56), we have

$$x.f\left(y\right)+\lambda=\left(xy+\lambda\right)+x$$

$$\Rightarrow x.f\left(y\right)=xy+x$$

Therefore $x\left(y+\lambda\right)=xy+x$ $\quad\left[\because f\left(y\right)=\lambda+y\right]$

$$\Rightarrow \lambda x=x$$

$$\Rightarrow \lambda=1 \quad (\because x>0)$$

So $f\left(x\right)=x+\lambda=x+1$

Hence
$$\lim_{x\to 0}\frac{\left(f\left(x\right)\right)^{1/3}-1}{\left(f\left(x\right)\right)^{1/2}-1}=\lim_{x\to 0}\frac{\left(1+x\right)^{1/3}-1}{\left(1+x\right)^{1/2}-1}$$

$$=\lim_{x\to 0}\left(\frac{\left(1+x\right)^{1/3}-1}{1+x-1}\right)\cdot\left(\frac{1+x-1}{\left(1+x\right)^{1/2}-1}\right)$$

$$=\frac{1/3}{1/2}=\frac{2}{3}$$

68. The value of $\lim_{x\to 0}f\left(x\right)$ where $f\left(x\right)=\frac{\cos\left(\sin x\right)-\cos x}{x^4}$, is

(A) 2 (B) $1/6$ (C) $2/3$ (D) $-1/3$

Key. B

Sol. $\lim_{x\to 0}f\left(x\right)=\lim_{x\to 0}\frac{\cos\left(\sin x\right)-\cos x}{x^4}$

$$=\lim_{x\to 0}\frac{-2\sin\frac{\sin x+x}{2}\sin\frac{\sin x-x}{2}}{x^4}$$

$$= -\frac{1}{2}\lim_{x\to 0}\frac{\sin\left(\frac{\sin x + x}{2}\right)}{\left(\frac{\sin x + x}{2}\right)}\frac{\sin\left(\frac{\sin x - x}{2}\right)}{\left(\frac{\sin x - x}{2}\right)}\times\frac{\sin x + x}{x}\times\frac{\sin x - x}{x^3}$$

$$= -\frac{1}{2}\lim_{u\to 0}\frac{\sin u}{u}\lim_{v\to 0}\frac{\sin v}{v}\lim_{x\to 0}\left(\frac{\sin x}{x}+1\right)$$

$$\times\frac{-\frac{x^3}{3!}+\frac{x^5}{5!}+....}{x^3}\left(u=\frac{\sin x + x}{2}, v=\frac{\sin x - x}{2}\right)$$

$$= -\frac{1}{2}\times 1\times 1\times 2\times\frac{-1}{3!}=\frac{1}{6}.$$

69. Let $x_1 = 1$ and $x_{n+1} = \frac{4+3x_n}{3+2x_n}$ for $n \geq 1$. If $\lim_{n\to\infty} x_n$ exists finitely, then the limit is equal to

(A) $\sqrt{2}$ (B) 1 (C) 2 (D) $\sqrt{2}+1$

Key. A

Sol. We have $x_1 = 1, x_2 = \frac{4+3}{3+2} = \frac{7}{5}$

$$x_3 = \frac{4+3x_2}{3+2x_2} = \frac{4+3\left(\frac{7}{5}\right)}{3+2\left(\frac{7}{5}\right)} = \frac{41}{29} > x_2$$

We can easily verify that $x_n < x_{n+1}$ and hence $\{x_n\}$ is strictly increasing sequence of positive terms. Let $\lim_{n\to\infty} x_n = l$. Therefore

$$l = \lim_{n\to\infty} x_{n+1}$$

$$= \lim_{n\to\infty}\left(\frac{4+3x_n}{3+2x_n}\right)$$

$$= \frac{4+3\lim_{n\to\infty} x_n}{3+2\lim_{n\to\infty} x_n}$$

$$= \frac{4+3l}{3+2l}$$

Hence $3l + 2l^2 = 4 + 3l$

or $l^2 = 2 \Rightarrow l = \sqrt{2}$ $(\because x_n > 0\ \forall\ n)$.

70. Let $f(x) = x^3\left\{\sqrt{x^2+\sqrt{x^4+1}} - x\sqrt{2}\right\}$. Then $\lim_{x\to\infty} f(x)$ is equal to

(A) $\frac{1}{2\sqrt{2}}$ (B) $\frac{1}{4\sqrt{2}}$ (C) $\frac{3}{4\sqrt{2}}$ (D) does not exist

Key. B

Sol. We have $f\left(x\right)=\dfrac{x^3\left\{x^2+\sqrt{x^4+1}-2x^2\right\}}{\sqrt{x^2+\sqrt{x^4+1}}+x\sqrt{2}}$

$$=\frac{x^3\left\{\sqrt{x^4+1}-x^2\right\}}{\sqrt{x^2+\sqrt{x^4+1}}+x\sqrt{2}}$$

$$=\frac{x^3\left(x^4+1-x^4\right)}{\left[\sqrt{x^2+\sqrt{x^4+1}}+x\sqrt{2}\right]\left[\sqrt{x^4+1}+x^2\right]}$$

$$=\frac{x^3}{\left[\sqrt{x^2+\sqrt{x^4+1+}}+x\sqrt{2}\right]\left[\sqrt{x^4+1}+x^2\right]}$$

$$=\frac{1}{\left[\sqrt{1+\sqrt{1+\frac{1}{x^4}}}+\sqrt{2}\right]\left[\sqrt{1+\frac{1}{x^4}}+1\right]}$$

$$=\frac{1}{\left(\sqrt{1+\sqrt{1}}+\sqrt{2}\right)\left(\sqrt{1}+1\right)}$$

$$=\frac{1}{2\sqrt{2}\left(2\right)}=\frac{1}{4\sqrt{2}}.$$

71. If a_n and b_n are positive integers and $a_n+\sqrt{2}b_n=\left(2+\sqrt{2}\right)^n$, then $\lim\limits_{n\to\infty}\left(\dfrac{a_n}{b_n}\right)=$

A) 2 B) $\sqrt{2}$ C) $e^{\sqrt{2}}$ D) e^2

Key. B

Sol. We have $a_n+\sqrt{2}b_n=\left(2+\sqrt{2}\right)^n$

$$\Rightarrow a_n-\sqrt{2}b_n=\left(2-\sqrt{2}\right)^n$$

Therefore $a_n=\dfrac{1}{2}\left[\left(2+\sqrt{2}\right)^n+\left(2-\sqrt{2}\right)^n\right]$

And $b_n=\dfrac{\left[\left(2+\sqrt{2}\right)^n-\left(2-\sqrt{2}\right)^n\right]}{2\sqrt{2}}$

Therefore $\dfrac{a_n}{b_n}=\sqrt{2}\dfrac{\left[\left(2+\sqrt{2}\right)^n+\left(2-\sqrt{2}\right)^n\right]}{\left[\left(2+\sqrt{2}\right)^n-\left(2-\sqrt{2}\right)^n\right]}$

$$= \sqrt{2}\frac{\left[1+\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)^n\right]}{\left[1-\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)^n\right]}$$

Hence $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \sqrt{2}\left(\frac{1+0}{1-0}\right)\left(\because \frac{2-\sqrt{2}}{2+\sqrt{2}} < 1\right) = \sqrt{2}$

72. If $\lim_{x\to 0}\frac{\left((a-n)nx-\tan x\right)\sin nx}{x^2} = 0$, where $n \in R \sim \{0\}$, then *a* is equal to

A) 0 B) $\frac{n}{n+1}$ C) *n* D) $n+\frac{1}{n}$

Key. D

Sol. The given limit can be written as

$$\lim_{x\to 0}\left(\frac{\sin nx}{nx}\right)(n)\left((a-n)n-\frac{\tan x}{x}\right)=0$$

$$\Rightarrow (1)(n)\left((a-n)n-1\right)=0$$

$$\Rightarrow (a-n)n-1=0 \Rightarrow a=n+1/n$$

73. For each positive integer *n*, let $s_n = \frac{3}{1.2.4}+\frac{4}{2.3.5}+\frac{5}{3.4.6}+.....+\frac{n+2}{n(n+1)(n+3)}$. Then $\lim_{n\to\infty} s_n$ equals

A) $\frac{29}{6}$ B) $\frac{29}{36}$ C) 0 D) $\frac{29}{18}$

Key. B

Sol. Let $u_k = \frac{k+2}{k(k+1)(k+3)}$

$$= \frac{(k+2)^2}{k(k+1)(k+2)(k+3)}$$

$$= \frac{k^2+4k+4}{k(k+1)(k+2)(k+3)}$$

$$= \frac{k(k+1)+3k+4}{k(k+1)(k+2)(k+3)}$$

$$= \frac{1}{(k+2)(k+3)}+\frac{3}{(k+1)(k+2)(k+3)}+\frac{4}{k(k+1)(k+2)(k+3)}$$

$$= \left(\frac{1}{k+2}-\frac{1}{k+3}\right)-\frac{3}{2}\left[\frac{1}{(k+2)(k+3)}-\frac{1}{(k+1)(k+2)}\right]$$

$$-\frac{4}{3}\left[\frac{1}{(k+1)(k+2)(k+3)}-\frac{1}{k(k+1)(k+2)}\right]$$

Now, put $k=1,2,3,....,n$ and add. Thus

$$s_u = u_1 + u_2 + ... + u_n$$

$$= \left(\frac{1}{3} - \frac{1}{n+3}\right) - \frac{3}{2}\left[\frac{1}{(n+2)(n+3)} - \frac{1}{2.3}\right]$$

$$-\frac{4}{3}\left[\frac{1}{(n+1)(n+2)(n+3)} - \frac{1}{1.2.3}\right]$$

Therefore $\lim_{n\to\infty} s_n = \frac{1}{3} + \frac{3}{12} + \frac{4}{18} = \frac{29}{36}$

74. $\lim_{x\to 0} \frac{a^{\tan x} - a^{\sin x}}{\tan x - \sin x}$ is equal to $(a>0)$

A) $\log_e a$ B) 1 C) 0 D) ∞

Key. A

Sol. We have $\lim_{x\to 0} \frac{a^{\tan x} - a^{\sin x}}{\tan x - \sin x} = \lim_{x\to 0} a^{\sin x}\left(\frac{a^{\tan x - \sin x} - 1}{\tan x - \sin x}\right)$

$= \lim_{x\to 0}\left(a^{\sin x}\right) \times \lim_{t\to 0}\left(\frac{a^t - 1}{t}\right)$ (where $t = \tan x - \sin x$)

$= a^0 \times \log_e a = \log_e a$

75. $\lim_{x\to\frac{\pi}{2}} \frac{(1-\sin x)(8x^3 - \pi^3)\cos x}{(\pi - 2x)^4}$

A) $-\frac{\pi^2}{16}$ B) $\frac{3\pi^2}{16}$ C) $\frac{\pi^2}{16}$ D) $-\frac{3\pi^2}{16}$

Key. D

Sol. Let $f(x) = \frac{(1-\sin x)(8x^3 - \pi^3)\cos x}{(\pi - 2x)^4}$

$$= \frac{(1-\sin x)\cos x(2x-\pi)(4x^2 + 2\pi x + \pi^2)}{(2x-\pi)^4}$$

$$= \frac{(1-\sin x)\cos x(4x^2 + 2\pi x + \pi^2)}{(2x-\pi)^3}$$

Therefore $\lim_{x\to\frac{\pi}{2}} f(x) = \lim_{x\to\frac{\pi}{2}} \frac{(1-\sin x)\cos x}{(2x-\pi)^3}.(3\pi^2)$

$$\lim_{x\to\frac{\pi}{2}} f(x) = \lim_{x\to\frac{\pi}{2}} \frac{(1-\sin x)\cos x}{(2x-\pi)^3}.(3\pi^2) \qquad ----(1.62)$$

Put $2x - \pi = y$ so that $y \to 0$ as $x \to \pi/2$. Therefore now

$$\frac{(1-\sin x)\cos x}{(2x-\pi)^3} = \frac{\left[1 - \sin\left(\frac{\pi+y}{2}\right)\right]\cos\left(\frac{\pi+y}{2}\right)}{y^3}$$

$$=\frac{\left(1-\cos\frac{y}{2}\right)\left(-\sin\frac{y}{2}\right)}{y^3}$$

$$=-\left(\frac{2\sin^2\frac{y}{4}}{y^2}\right)\left(\frac{\sin\frac{y}{2}}{y}\right)$$

$$=-2\left(\frac{\sin\frac{y}{4}}{y/4}\right)^2.\frac{1}{16}.\left(\frac{\sin\frac{y}{2}}{y/2}\right).\frac{1}{2}$$

$$=\frac{-1}{16}\left(\frac{\sin\frac{y}{4}}{y/4}\right)^2\left(\frac{\sin\frac{y}{2}}{y/2}\right) \qquad ----(1.63)$$

Therefore from Eqs. (1.62) and (1.63)

$$\lim_{x\to\frac{\pi}{2}} f(x)=\frac{-3\pi^2}{16}\times 1\times 1.$$

76. If a_1 is the greatest value of $f(x)$ where $f(x)=\frac{1}{2+[\sin x]}$ and $a_{n+1}=\frac{(-1)^{n+2}}{n+1}+a_n$

Then $\lim_{n\to\infty} a_n =$ _______

1) 0 2) e 3) 1 4) $\log_e 2$

Key. 4

Sol. $a_1=1, a_2=1-\frac{1}{2}, a_3=1-\frac{1}{2}+\frac{1}{3}.........a_n=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+......+(-1)^{n-1}.\frac{1}{n}$

$\lim_{n\to\infty} a_n=\log_e 2$

77. $\lim_{x\to\frac{\pi}{2}}\left[\frac{[\sin x]-[\cos x]+1}{3}\right]=$

[.] $\rightarrow$ denotes greatest integer function

1) 0 2) 1 3) -1 4) does not exist

Key. 1

Sol. LHL = RHL = 0

78. $\lim_{x\to 0}\left(\frac{1+2x}{1+3x}\right)^{\frac{1}{x^2}}.e^{\frac{1}{x}}=$ _______

1) $e^{\frac{5}{2}}$ 2) e^2 3) 4) 1

Key. 1

Sol. $\lim_{x\to 0} e^{\frac{1}{x^2}(\log(1+2x)-\log(1+3x)+\frac{1}{x}}$

$$e^{\lim_{x\to 0}\frac{(\log(1+2x)-\log(1+3x)+x}{x^2}} = e^{\frac{5}{2}}$$

79. $\lim_{n\to\infty}\sum_{r=1}^{n}\cot^{-1}\left(r^2+\frac{3}{4}\right)=$

1) $\tan^{-1}(2)$ 2) $\frac{\pi}{4}$ 3) $\frac{\pi}{2}$ 4) $\tan^{-1}(3)$

Key. 1

Sol. $\cot^{-1}\left(r^2+\frac{3}{4}\right)=\tan^{-1}\left(\frac{1}{r^2+\frac{3}{4}}\right)$

$$=\tan^{-1}\left(\frac{1}{1+\left(r^2-\frac{1}{4}\right)}\right)$$

$$=\tan^{-1}\left(\frac{1}{1+\left(r+\frac{1}{2}\right)\left(r-\frac{1}{2}\right)}\right)$$

$$=\tan^{-1}\left(\frac{\left(r+\frac{1}{2}\right)-\left(r-\frac{1}{2}\right)}{1+\left(r^2+\frac{1}{4}\right)}\right)$$

$$=\tan^{-1}\left(r+\frac{1}{2}\right)-\tan^{-1}\left(r-\frac{1}{2}\right)$$

80. $\lim_{x\to\infty}\sqrt[3]{x}\left(\sqrt[3]{(x+1)^2}-\sqrt[3]{(x-1)^2}\right)=$

1) $\frac{1}{3}$ 2) $\frac{2}{3}$ 3) 1 4) $\frac{4}{3}$

Key. 4

Sol. $\lim_{x\to\infty} x^{1/3}\left\{(x+1)^{1/3}+(x-1)^{1/3}\right\}\left\{(x+1)^{1/3}-(x-1)^{1/3}\right\}$

Rationalise $\lim_{x\to\infty}\frac{x^{1/3}\left\{(x+1)^{1/3}+(x-1)^{1/3}\right\}2}{\left\{(x+1)^{2/3}+(x^2-1)^{1/3}+(x-1)^{2/3}\right\}}$

$$\lim_{x\to\infty}\frac{2.x^{2/3}\left\{\left(1+\frac{1}{x}\right)^{1/3}+\left(1-\frac{1}{x}\right)^{1/3}\right\}2}{x^{2/3}\left\{\left(1+\frac{1}{x}\right)^{2/3}+\left(1-\frac{1}{x}\right)^{1/3}+\left(1-\frac{1}{x}\right)^{2/3}\right\}}=\frac{2x2}{3}=\frac{4}{3}$$

81. If $a>0, b>0$ then $\lim_{n\to\infty}\left(\frac{a-1+b^{\frac{1}{n}}}{a}\right)^n=$

1) $b^{\frac{1}{a}}$ 2) $a^{\frac{1}{b}}$ 3) a^b 4) b^a

Key. 1

Sol. Let $\frac{1}{n}=x, \quad \Rightarrow x\to 0 \; as \; n\to\infty$ then required limit $\underset{x\to 0}{Lt}\left(\frac{a-1+b^x}{a}\right)^{\frac{1}{x}}=e^{\underset{x\to 0}{Lt}\frac{b^x-1}{x^a}}$

$=e^{\frac{1}{a}\log e^b}=\left(b^{\frac{1}{a}}\right)$

82. If $S_n=\frac{1}{1.2.3.4}+\frac{1}{2.3.4.5}+.........+\frac{1}{n(n+1)(n+2)(n+3)}$ then $\lim_{n\to\infty}S_n=$

1) $\frac{5}{18}$ 2) $\frac{1}{9}$ 3) $\frac{7}{18}$ 4) $\frac{1}{18}$

Key. 4

Sol. $S_n=c-\frac{1}{(n+1)(n+2)(n+3).3}$

$n=1\Rightarrow s_1=c-\frac{1}{2.3.4.3}\Rightarrow c=\frac{1}{1.2.3.4}+\frac{1}{2.3.4.3}$

$c=\frac{1}{2.3.4}\left(1+\frac{1}{3}\right)$

$=\frac{1}{18}$ Now as $n\to\infty, S_n\to c=\frac{1}{18}$

83. $\lim_{x\to\infty}\left(\frac{x^2+5x+3}{x^2+x+2}\right)^x=$

1) e^2 2) e^4 3) e^3 4) e

Key. 2

Sol. $$\lim_{x\to\infty}\left(\frac{x^2+5x+3}{x^2+x+2}\right)^x=e^{\lim_{x\to 0}\left(\frac{4x+1}{x^2+x+2}\right)x}=e^4$$

84. $\lim_{x\to\frac{-1}{3}^-}\frac{1}{x}\left[\frac{-1}{x}\right]$ [.] $\to$ denotes greatest integer function

1) -9 2) -12 3) -6 4) 0

Key. 3

Sol. $x < -\frac{1}{3}$

$\frac{1}{x} > -3 \Rightarrow -\frac{1}{x} < 3 \Rightarrow \left[-\frac{1}{3}\right] = 2$

$\lim_{x \to -\frac{1}{3}} \frac{1}{x}\left[-\frac{1}{x}\right] = (-3)(2) = -6$

85. $\lim_{x \to \infty}\left(x - \log_e(\cosh x)\right) =$

1) 1 2) 0 3) $\log_e 2$ 4) ∞

Key. 3

Sol. $\lim_{x \to \infty} x - \log_e\left(\frac{e^x + e^{-x}}{2}\right)$

$\lim_{x \to \infty} x - \log_e e^x\left(\frac{1+e^{-2x}}{2}\right)$

$\lim_{x \to \infty} x - x - \log_e\left(\frac{1+e^{-2x}}{2}\right)$

$\lim_{x \to \infty} -\log_e\left(\frac{1}{2}\right) = \log_e 2$

86. If f(x) = 0 be a quadratic equation such that $f(-\pi) = f(\pi) = 0$ and $f\left(\frac{\pi}{2}\right) = \frac{-3\pi^2}{4,}$ then

$\lim_{x \to -\pi} \frac{f(x)}{\sin(\sin x)}$ is equal to

a) 0 b) π c) $+2\pi$ d) None

Key. C

Sol. From given data $f(x) = x^2 - \pi^2$

$\lim_{x \to -\pi} \frac{x^2 - \pi^2}{-\sin(\sin x)} = 2\pi$.

$\lim_{h \to o} \frac{-2h\pi + h^2}{-\sin(\sinh)} = 2\pi$.

87. If the normal to the curve y = f(x) at x = 0 be given by the equation 3x – y + 1 = 0 then the value of $\lim_{x \to 0} x^2\left\{f(x^2) - 5f(4x^2) + 4f(7x^2)\right\}^{-1}$ is

(A) $\frac{1}{3}$ (B) $\frac{2}{3}$

(C) $-\frac{2}{3}$ (D) $-\frac{1}{3}$

Key. D

SOL. SLOPE OF TANGENT AT X = 0 IS $-\frac{1}{3}$

$\Rightarrow$ $f'(x) = -\frac{1}{3}$

$$\lim_{x\to 0}\frac{x^2}{f(x^2)-5f(4x^2)+4f(7x^2)} \div \text{(USE L.H. RULE)}$$

$$\Rightarrow \quad \lim_{x\to 0}\frac{1}{f'(x^2)-20f'(4x^2)+28f'(7x^2)}=-\frac{1}{3}$$

88. $f(x)$ is a polynomial function and $(f(\alpha))^2+(f'(\alpha))^2=0$ then the value of $\underset{x\to\alpha}{\text{lt}}\frac{f(x)}{f'(x)}\left[\frac{f'(x)}{f(x)}\right]$ (where $[.]$ denotes greatest integer function) is __

a) 0 b) 1 c) -1 d) 2

Key. B

Sol. Clearly, α is repeated root of $f(x)=0$

$$\underset{x\to\alpha}{\text{lt}}\frac{f(x)}{f'(x)}\left(\frac{f'(x)}{f(x)}-\left\{\frac{f'(x)}{f(x)}\right\}\right)\Rightarrow \underset{x\to\alpha}{\text{lt}}\left(1-\frac{f(x)}{f'(x)}\left\{\frac{f'(x)}{f(x)}\right\}\right)$$

$\left(\underset{x\to\alpha}{\text{lt}}\frac{f(x)}{f'(x)}=0\ \&\ \left\{\frac{f'(x)}{f(x)}\right\}\right.$ is bounded function)

89. $\underset{x\to a^-}{\ln}\left(\frac{|x|^3}{a}-\left[\frac{x}{a}\right]^3\right)(a>0)$, [.] GIF, is

A) a^2-2 B) a^2-1 C) a^2 D) a^2+1

Key. C

Sol. For $a-1<x<a\Rightarrow\left[\frac{x}{a}\right]=0$

$$\underset{x\to a^-}{\ln}\left(\frac{|x|^3}{a}-0\right)=\frac{a^3}{a}=a^2$$

90. $\underset{x\to\frac{\pi}{2}}{Lt}\frac{\sin x-(\sin x)^{\sin x}}{1-\sin x+\ln(\sin x)}=$

(A) 1 (B) 0 (C) 2 (D) – 1

Key. C

Sol. $\underset{x\to 1}{Lt}\frac{t-t^t}{1-t+\log t}$

91. $\underset{x\to 1}{Lt}\left(\tan^{-1}x.\frac{4}{\pi}\right)^{\frac{1}{x^2-1}}=$

(A) e^{π} (B) $e^{\frac{1}{\pi}}$ (C) $\frac{1}{e^{\pi}}$ (D) $e^{-\frac{1}{\pi}}$

Key. B

Sol. $e^{\underset{x\to\pi}{Lt}\left(\frac{4}{\pi}\tan^{-1}x-1\right)\frac{1}{x^2-1}}$

92. Value of $f\left(\frac{\pi}{2}\right)$ so that the function is continuous at $x=\frac{\pi}{2}$ is, if

$$f(x)=\frac{(1-\sin x)\ln\sin x}{(\pi-2x)^2\ln\left(1+\pi^2-4\pi x+4x^2\right)}$$

a) $\frac{1}{8}$ b) $\frac{1}{16}$ c) $-\frac{1}{32}$ d) $-\frac{1}{64}$

Key. D

Sol. Put $x=\frac{\pi}{2}+h$

$$\Rightarrow \underset{h\to 0}{Lt}\frac{(1-\cosh)\ln(\cosh)}{4h^2\ln\left(1+4h^2\right)}$$

Simplify to get $-\frac{1}{64}$

93. S_1 : If $\underset{x\to a}{Lim}f(x)+g(x)$ and $\underset{x\to a}{Lim}f(x)-g(x)$ exist : then it is not necessary that $\underset{x\to a}{Lim}f(x)$ and $\underset{x\to a}{Lim}g(x)$ both exist separately

S_2 : If $\underset{x\to a}{Lim}f(x)g(x)$ exists then it is necessary that $\underset{x\to a}{Lim}f(x)$ and $\underset{x\to a}{Lim}g(x)$ both exist separately

S_3 : $\underset{x\to a}{Lim}(f(x))^{g(x)}=e^{\underset{x\to a}{Lim}g(x)(f(x)-1)}$

S_4 : $\underset{x\to 0^+}{Lt}\frac{e^{x\ln x}-e^{[\cos x]}}{x\ln x}=1$, where [ ] represents greatest integer function state in order, whether S_1,S_2,S_3,S_4 are true or false.

a) FTTT b) FFFF c) TTTT d) FFFT

Key. D

Sol. S_3 is applied only for form $(\to 1)^{\infty}$

94. $\underset{n\to\infty}{Lt}\frac{2^3-1^3}{2^3+1^3}\cdot\frac{3^3-1^3}{3^3+1^3}..........\frac{n^3-1^3}{n^3+1^3}$ is equal to

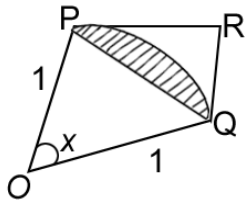
a) $\frac{1}{3}$ b) $\frac{1}{2}$ c) $\frac{2}{3}$ d) None of these

Key. C

Sol. Conceptual

95.

A circular arc of radius '1' subtends an angle of '*x*' radians, $0 < x < \frac{\pi}{2}$ as shown in the figure. The point 'R' is the point of intersection of the two tangent lines at P & Q. Let T(x) be the area of triangle PQR and S(*x*) be area of the shaded region. Then $\underset{x\to 0}{Lim}\frac{T(x)}{S(x)} =$



a) 2 b) $\frac{1}{2}$ c) $\frac{3}{4}$ d) $\frac{3}{2}$

Key. D

Sol. $T(x) = \frac{1}{2}.PR.RQ\sin(\pi - x)$

$= \frac{1}{2}\left(\tan^2\frac{x}{2}\right).\sin x = \tan\frac{x}{2} - \frac{\sin x}{2}$

$s(x) =$ **area of sector OPQ – area of** ΔOPQ

$= \frac{1}{2}(1)^2.x - \frac{1}{2}(1)^2\sin x$

$\underset{x\to 0}{Lim}\frac{\tan\frac{x}{2} - \sin\frac{x}{2}}{\frac{x - \sin x}{2}} = \frac{3}{2}$

96. $\underset{x\to 0}{Lt}\left(\frac{\sin hx}{x}\right)^{\frac{1}{x^2}}$

(a) $e^{\frac{1}{2}}$ (b) 1 (c) $e^{\frac{1}{6}}$ (d) $e^{\frac{1}{3}}$

Key. C

Sol. $Let\, l = \underset{x\to 0}{Lt}\left(\frac{\sin hx}{x}\right)^{\frac{1}{x^2}}$

$\log l = \underset{x\to 0}{Lt}\frac{1}{x^2}\log\left(\frac{\sin hx}{x}\right)$ by L' Hospital Rule $\Rightarrow l = e^{\frac{1}{6}}$

Limits

Multiple Correct Answer Type

1. If $f(x) = x\left(\dfrac{e^{|x|+[x]} - 2}{|x|+[x]}\right)$ then

A) $\lim_{x\to 0^+} f(x) = -1$ B) $\lim_{x\to 0^-} f(x) = 0$ C) $\lim_{x\to 0} f(x) = -1$ D) $\lim_{x\to 0} f(x) = 0$

Key. A,B

Sol. A) $\underset{x\to 0+}{Lt}\, x\left(\dfrac{e^{|x|+[x]}-2}{|x|+[x]}\right) = \underset{x\to 0+}{Lt}\, x\left(\dfrac{e^{x+0}-2}{x+0}\right) = \underset{x\to 0}{Lt}\left(e^{-x}-2\right)$ = 1 - 2 = -1

B) $\underset{x\to 0-}{Lt}\, x\left(\dfrac{e^{|x|+[x]}-2}{|x|+[x]}\right) = \underset{x\to 0-}{Lt}\, x\left(\dfrac{e^{-x-1}-2}{-x-1}\right) = 0$

2. If $\lim_{x\to -a}\dfrac{x^7+a^7}{x+a} = 7,$ then the value of a is

(A) 1 (B) –1 (C) 7 (D) –7

Key. A,B

Sol. $\underset{x\to -a}{\text{lam}}\dfrac{a^7-(-x)^7}{a-(-x)} = 7 \Rightarrow 7a^6 = 7 \Rightarrow a^6 = 1$

$\therefore$ a = ± 1

3. $\underset{x\to 0}{\lim}\left[\dfrac{\left|\sin^{-1}|x|\right|}{x}\right]$ (where $[.]$ denotes greatest integer function) is

(A) left hand limit is – 2 (B) left hand limit is -1

(C)right hand limit is 1 (D) limit exists and both are equal to 1

Key. A,C

Sol. $LHL = \lim_{h\to 0}\left[\dfrac{\left|\sin^{-1} h\right|}{h}\right] = \lim_{h\to 0}\left[\dfrac{\sin^{-1} h}{-h}\right] = -2$

$RHL = \underset{h\to 0}{\lim}\left[\dfrac{\left|\sin^{-1} h\right|}{h}\right] = 1$

4. Suppose 'f' is a function that satisfies the equation $f(x+y) = f(x) + f(y) + x^2y + xy^2$ for all real numbers 'x' and 'y'. If $\lim_{x\to 0}\dfrac{f(x)}{x} = 1,$ then

A) $f(x) > 0$ for $x > 0$ and $f(x) < 0$ for $x < 0$ B) $f'(0) = 1$

C) $f''(0)=1$ D) $f'''(0)=6$

Key. A,B

Sol. Observe that $f(0)=0$ and $f'(0)=\lim_{h\to 0}\frac{f(h)}{h}=1$

$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$\Rightarrow \lim_{h\to 0}\frac{f(h)+x^2h+xh^2}{h}=x^2+1$

Hence, $f(x)=\frac{x^3}{3}+x$

5. $\underset{x\to\infty}{Lt}\frac{x^p+x^{p-1}+1}{x^q+x^{q-2}+2}=$ $(p>0,q>0)$

A) 0 if $p<q$ B) 1 if $p=q$

C) infinite if $p>q$ D) 1 if $p>q$

Key: A,B,C

Hint: Conceptual

5. If $\alpha,\beta\in\left(-\frac{\pi}{2}.0\right)$ such that $(\sin\alpha+\sin\beta)+\frac{\sin\alpha}{\sin\beta}=0$ and $(\sin\alpha+\sin\beta)\frac{\sin\alpha}{\sin\beta}=-1$

and $\lambda=\lim_{n\to\infty}\frac{1+(2\sin\alpha)^{2n}}{(2\sin\beta)^{2n}}$ then

(A) $\alpha=-\frac{\pi}{6}$ (B) $\lambda=2$

(C) $\alpha=-\frac{\pi}{3}$ (D) $\lambda=1$

Key: A,B

Hint: $(\sin\alpha+\sin\beta)^2=1$

$\Rightarrow \sin\alpha+\sin\beta=-1$

$\Rightarrow \sin\alpha=\frac{-1}{2}\ \sin\beta=\frac{-1}{2}$

$\alpha=\beta=-30^0$

6. If $f(x)=\lim_{p\to\infty}\frac{x^pg(x)+h(x)+7}{7x^p+3x+1};x\neq 1$ and $f(1)=7,f(x),g(x)$ and $h(x)$ are all continuous function at $x=1$. Then which of the following statement(s) is/are correct

(A) $g(1)+h(1)=70$ (B) $g(1)-h(1)=28$

(C) $g(1)+h(1)=60$ (D) $g(1)-h(1)=-28$

Key: A,B

Hint: When $x<1$

$$f(1)=\frac{h(1)+7}{3+1}$$

$$7=\frac{h(1)+7}{4}$$

$h(1)=21$

When x > 1

$$f(1)=\underset{p\to\infty}{Lt}\frac{\cancel{x^p}\left[g(x)+\frac{h(x)}{x^p}+\frac{7}{x^p}\right]}{\cancel{x^p}\left[7+\frac{3x+1}{x^P}\right]}$$

$$7=\frac{g(1)}{7}$$

$\therefore g(1)=49$

$\therefore g(1)-h(1)=28$

$g(1)+h(1)=70$

7. The function $f(x)=\frac{1}{u^2-u-2}$ where $u=\frac{1}{x-1}$

a) has a removable discontinuous at x=1

b) has irremovable discontinuous at $x=0,\ \frac{3}{2}$

c) discontinuous at $u=-2,\ 1$

d) discontinuous at $u=-1,\ 2$

Key: A,B,D

Hint The function $u=\frac{1}{x-1}$ is discontinuous at x = 1

$f(x)=\frac{1}{(u+1)(u-2)}$ is discontinuous at $u=-1,2$

i.e. at $x=0,\frac{3}{2}$

also we have $\underset{x\to0}{Lt}f(x)=\underset{u\to-1}{Lt}f(x)=\infty$

$\underset{x\to\frac{3}{2}}{Lt}f(x)=\underset{u\to2}{Lt}f(x)=\infty$

$\underset{x\to1}{Lt}f(x)=\underset{x\to\infty}{Lt}f(x)=0$

8. Let $f(x)=\begin{cases}\frac{x^{2^{32}}-2^{32}.x+4^{16}-1}{(x-1)^2} & ,x\neq1\\ k & ,x=1\end{cases}$.

Then value of k so that the function is continuous at x = 1 is

(A) $2^{63}-2^{31}$ (B) $2^{65}-2^{33}$

(C) $\left(2^{16}+1\right)\left(2^{8}+1\right)\left(2^{4}+1\right)\left(2^{2}+1\right)\left(2^{32}+2^{31}\right)$

(D) $\left(2^{32}+1\right)\left(2^{16}+1\right)\left(2^{8}+1\right)\left(2^{4}+1\right)\left(2^{2}+1\right)\left(2^{33}+2^{32}\right)$

Key: A,C

Hint Conceptual

9. The function $f(x)=\begin{cases}|x-3| , & x\ge 1\\ \left(\dfrac{x^2}{4}\right)-\left(\dfrac{3x}{2}\right)+\left(\dfrac{13}{4}\right), & x<1\end{cases}$ is

(A) continuous at x = 1 (B) differentiable at x =1

(C) continuous at x=3 (D) differentiable at x = 3

Key: A,B,C

Hint: $f(1+h)=2=f(1-h)$

$f(3+)=f(3-)=0$

Continues at $x=1,3$

$f'(1+)=f'(1-)=-1$

Not differentiable at x = 3

10. $\underset{x\to\infty}{Lt}\dfrac{x^p+x^{p-1}+1}{x^q+x^{q-2}+2}=\ (p>0,q>0)$

1) 0 if $p<q$ 2) 1 if $p=q$

3) infinite if $p>q$ 4) 1 if $p>q$

Key. 1,2,3

Sol. Conceptual

11. If $\lim\limits_{x\to 0}\dfrac{a\sin x-bx+cx^2+x^3}{2x^2 ln(1+x)-2x^3+x^4}$ exists and is finite, then

A) $a=6$ B) $b=0$ C) $c=0$ D) The limit = $\dfrac{3}{40}$

Key. A,C,D

Sol. Givin limit

$$=\lim_{x\to 0}\frac{a\left(x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-....\right)-bx+cx^2+x^3}{2x^2\left(x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-...\right)-2x^3+x^4}$$

$$= \lim_{x\to 0} \frac{(a-b)x + cx^2 + \left(1-\frac{a}{6}\right)x^3 + \frac{ax^5}{120}....}{2\frac{x^5}{3} - \frac{x^6}{2} + ...}$$

For this limit to exist, we must have

$a = b, c = 0, a = 6$

and given limit = $\frac{a}{120} \times \frac{3}{2} = \frac{6\times 3}{120\times 2} = \frac{3}{40}$

12. If $\lim_{x\to 0} \sin\left(\frac{\pi\left(1-\cos^m x\right)}{x^n}\right)$ exists, where $m, n \in N$, then

a) $m \in N, n = 3$ b) $m \in N, n \in N$ c) $m \in N, n = 2$ d) $m \in N, n = 1$

Key. C,D

Sol. $\lim_{x\to 0}\sin\left(\frac{\pi\left(1-\cos^m x\right)}{x^n}\right)$

$= \sin \lim_{x\to 0}\left(\frac{\pi\left(1-\cos^m x\right)}{x^n}\right)$

Possible when $m \in N$ and n = 1 or 2 .

13. α, β are roots of equation $ax^2 + bx + c = 0$ where 1 < α < β if $\lim_{x\to m} \frac{|ax^2+bx+c|}{ax^2+bx+c} = 1$ then which of the following are true

(A) a < 0 and α < m < β (B) a > 0 and m < 1

(C) a > 0 and α < m < β (B) a > 0 and m > 1

Key. A,B

SOL. $\lim_{x\to m}\frac{|a(x-\alpha)(x-\beta)|}{a(x-\alpha)(x-\beta)} = 1$

WHEN A > 0, M < 1 ⇒ (M – α) (M – β) > 0

⇒ A(M – α) (X – β) > 0

⇒ $\lim_{x\to m}\frac{a(x-\alpha)(x-\beta)}{a(x-\alpha)(x-\beta)} = 1$

WHEN A < 0, α < M < β ⇒ (M – α) (M – β) < 0

⇒ A(M – α) (M – β) > 0

⇒ $\lim_{x\to m}\frac{a(x-\alpha)(x-\beta)}{a(x-\alpha)(x-\beta)} = 1$

14. $\underset{x\to\infty}{Lt} \frac{x^p + x^{p-1} + 1}{x^q + x^{q-2} + 2}$ where $p > 0 \quad q > 0$ is

(A) 0 if $p < q$ (B) 1 if $p = q$ (C) Infinite if $p > q$ (D) 1 if $p > q$

Key. A,B,C

Sol. $SS^1 > k$

Limits

Assertion Reasoning Type

1. Statement-1: $\lim_{x\to\infty} \frac{(x+1)^{10}+(x+2)^{10}+.....+(x+100)^{10}}{x^{10}+9^{10}}=100$

Statement-2: If $p(x)$ and $q(x)$ are polynomials of same degree, then

$$\lim_{x\to\infty}\frac{p(x)}{q(x)}=\frac{\text{leading coefficient of } p(x)}{\text{leading coefficient of } q(x)}$$

Key. A

Sol. Let $p(x)=a_0x^n+a_1x^{n-1}+...+a_n$ and $q(x)=b_0x^n+b_1x^{n-1}+...+b_n$ then

$$\lim_{x\to\infty}\frac{p(x)}{q(x)}=\lim_{x\to\infty}\frac{a_0+a_1/x+a_2/x^2+...+a_n/x^n}{b_0+b_1/x+b_2/x^2+...+b_n/x^n}=\frac{a_0}{b_0}$$

$(x+1)^{10}+...+(x+100)^{10}$ is a polynomial of degree 10 with leading coefficient 100 and $x^{10}+9^{10}$ is a polynomial of degree 10 with leading coefficient 1. So statement 1 follows from statement 2.

2. Statement-1: $\lim_{x\to\infty}\left(\frac{x-3}{x+2}\right)^x$ is equal to e^{-5}.

Statement-2: $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$.

Key. A

Sol. Clearly Statement II is true. Now

$$\lim_{x\to\infty}\left(\frac{x-3}{x+2}\right)^x=\lim_{x\to\infty}\left(\frac{1-\frac{3}{x}}{1+\frac{2}{x}}\right)^x$$

$$=\frac{\lim_{x\to\infty}\left[\left(1-\frac{3}{x}\right)^{-x/3}\right]^{-3}}{\lim_{x\to\infty}\left[\left(1+\frac{2}{x}\right)^{x/2}\right]^{2}}$$

$$=\frac{e^{-3}}{e^2}=e^{-5}.$$

Statement I is also true and Statement II is a correct explanation of Statement I.

3. Statement-1: Let $f(x)=\begin{cases}\frac{\sin x}{x} & if \quad x<0\\ ax+b & if \quad x>0\end{cases}$

If $\lim_{x\to 0} f(x)$ exists, then $a=1$ and $b=0$.

Statement-2: $\lim_{x\to 0}(px+q)=q$ where *p* and *q* are any real constants.

Key. D

Sol. Statement II is clearly true. Since $\lim\limits_{x\to 0} f(x)$ exists, the left and right limits of $f(x)$ at $x=0$ must be equal. So

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)=1$$

$$\Rightarrow \lim_{x\to 0-0} f(x)=1$$

Now $\lim\limits_{x\to 0+0} f(x)=\lim\limits_{x\to 0+0}(ax+b)=b$.

Therefore $b=1$ whereas *a* may be any real number. Hence Statement I is false.

4. Statement-1: If *a* and *b* are positive and $[x]$ denotes the greatest integer $\le x$, then

$$\lim_{x\to 0+}\frac{x}{a}\left[\frac{b}{x}\right]=\frac{b}{a}.$$

Statement-2: $\lim\limits_{x\to\infty}\frac{\{x\}}{x}=0$, where $\{x\}$ denotes fractional part of *x*.

Key. A

Sol. $$\lim_{x\to 0+}\frac{x}{a}\left[\frac{b}{x}\right]=\lim_{x\to 0+}\frac{x}{a}\left(\frac{b}{x}-\left\{\frac{b}{x}\right\}\right)$$

$$=\lim_{x\to 0+}\frac{x}{a}.\frac{b}{x}\left(1-\frac{\left\{\frac{b}{x}\right\}}{\left(\frac{b}{x}\right)}\right)$$

$$=\lim_{x\to 0+}\frac{b}{a}\left(1-\frac{\left\{\frac{b}{x}\right\}}{\left(\frac{b}{x}\right)}\right)$$

$$=\frac{b}{a}-\frac{b}{a}\lim_{y\to\infty}\frac{\{y\}}{y} \qquad \text{where } y=\frac{b}{x}$$

$$=\frac{b}{a}.$$

Since, $0\le\{x\}<1$ so $\frac{\{x\}}{x}\le\frac{1}{x}$ for $x>0$.

Hence $\lim\limits_{x\to\infty}\frac{\{x\}}{x}=0$.

5. Let $[x]$ denote the integral part of *x*.

Statement-1: Let $f(x)=\begin{cases}\dfrac{\sin[x]}{[x]} & if\ [x]\ne 0\\ 0 & if\ [x]=0\end{cases}$

Then $\lim_{x\to 0} f(x)$ does not exist.

Statement-2: $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)=1$.

Key. B

Sol. Statement-II is true.

Now $\lim_{x\to 0-0} f(x)=\lim_{\substack{h\to 0\\ h>0}} = \frac{\sin[0-h]}{[0-h]}$

$=\lim_{h\to 0}\frac{\sin(-1)}{(-1)}$

$=\sin 1$.

Also $\lim_{x\to 0+0} f(x)=0$ because $[0+h]=0$. Therefore

$\lim_{x\to 0-0} f(x)\neq \lim_{x\to 0+0} f(x)$

$\Rightarrow \lim_{x\to 0} f(x)$ does not exist.

Statement-I is also true. That is both statements are true and Statement-II is not a correct explanation of Statement-I.

6. Statement-1: If $\lim_{x\to 0}\left(\frac{\sin 2x+a\sin x}{x^3}\right)$ exists finitely, then the value of *a* is -2.

Statement-2: If $\lim_{x\to a}\frac{f(x)}{g(x)}$ exists finitely and $\lim_{x\to a} g(x)=0$ then $\lim_{x\to a} f(x)=0$.

Key. A

Sol. Statement-II is clearly true.

Now $\lim_{x\to 0}\left(\frac{\sin 2x+a\sin x}{x^3}\right)=\lim_{x\to 0}\left(\frac{\sin x}{x}\right)\frac{(2\cos x+a)}{x^2}$

$=\lim_{x\to 0}\frac{2\cos x+a}{x^2}$ exists finitely.

Therefore $\lim_{x\to 0}(2\cos x+a)=0$.

So $a=-2$.

A) Statement-1is True, Statement-2 is True; Statement-2 is a correctExplanation forStatement-1

B) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1

C) Statement-1 is True, Statement-2 is False

D) Statement-1 is False, Statement-2 is True

7. STATEMENT-1: $\lim_{x\to\infty} x^2\left(1-\cos\frac{1}{x}\right)=\frac{1}{2}$

STATEMENT–2: $\lim_{x\to 0} \cos\left(\frac{1}{x}\right)$ does not exist

Key: B

Hint: $\lim_{x\to\infty} x^2\left(1-\cos\frac{1}{x}\right) = \lim_{x\to\infty} x^2 \times 2\sin^2\left(\frac{1}{2x}\right) = \lim_{x\to\infty} \frac{2x^2\times\sin^2\left(\frac{1}{2x}\right)}{\frac{1}{4x^2}\times 4x^2}$

as $x\to\infty, \frac{1}{2x}\to 0$

So, $\lim_{x\to\infty} x^2\left(1-\cos\frac{1}{x}\right)=\frac{1}{2}\times 1^2=\frac{1}{2}$

Also, $\lim_{x\to 0} \cos\left(\frac{1}{x}\right)$ does not exist because $\cos\left(\frac{1}{x}\right)$ does not approach to a definite value as $x\to 0$

8. STATEMENT- 1: $\lim_{n\to\infty}\left[1-\frac{(-1)^n}{n+1}\right]$, ([ ] denotes the G.I.F.) does not exist.

STATEMENT 2: $\lim_{n\to\infty}\left(1-\frac{(-1)^n}{n+1}\right)$ = 1

Key: B

Hint: Conceptual

9. Let $f(x)=\cos\left(\frac{\pi x}{[x-3]}\right)\left(where\,[\bullet]\text{ denotes greatest integer function}\le x\right)$ then

Statement I : $f(x)$ is continuous at atleast one integer in the domain of $f(x)$

Statement II : $f(x)$ is discontinuous at all integers in its domain.

Key: C

Hint: Domain = $(-\infty, 3)\cup[4,\infty)$

Let $k\in$ Domain of f(x), $k\in I$

$$\underset{x\to k^+}{Lt} f(x)=\cos\left(\frac{\pi k}{k-3}\right)$$

$$\underset{x\to k^-}{Lt} f(x)=\cos\left(\frac{\pi k}{k-4}\right)$$

$$f(k)=\cos\left(\frac{\pi k}{k-3}\right)$$

At k = 4, 0 f(x) is continuous

10. Statement-1: If *a* and *b* are positive and $[x]$ denotes the greatest integer $\le x$, then

$$\lim_{x\to 0+}\frac{x}{a}\left[\frac{b}{x}\right]=\frac{b}{a}.$$

Statement-2: $\lim_{x\to\infty}\frac{\{x\}}{x}=0$, where $\{x\}$ denotes fractional part of *x*.

Key. A

Sol. $$\lim_{x\to 0+}\frac{x}{a}\left[\frac{b}{x}\right]=\lim_{x\to 0+}\frac{x}{a}\left(\frac{b}{x}-\left\{\frac{b}{x}\right\}\right)$$

$$=\lim_{x\to 0+}\frac{x}{a}.\frac{b}{x}\left(1-\frac{\left\{\frac{b}{x}\right\}}{\left(\frac{b}{x}\right)}\right)$$

$$=\lim_{x\to 0+}\frac{b}{a}\left(1-\frac{\left\{\frac{b}{x}\right\}}{\left(\frac{b}{x}\right)}\right)$$

$$=\frac{b}{a}-\frac{b}{a}\lim_{y\to\infty}\frac{\{y\}}{y} \qquad \text{where } y=\frac{b}{x}$$

$$=\frac{b}{a}.$$

Since, $0\le\{x\}<1$ so $\frac{\{x\}}{x}\le\frac{1}{x}$ for $x>0$.

Hence $\lim_{x\to\infty}\frac{\{x\}}{x}=0$.

11. Let $[x]$ denote the integral part of *x*.

Statement-1: Let $f(x)=\begin{cases}\frac{\sin[x]}{[x]} & if\ [x]\ne 0\\ 0 & if\ [x]=0\end{cases}$

Then $\lim_{x\to 0}f(x)$ does not exist.

Statement-2: $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)=1.$

Key. B

Sol. Statement-II is true.

Now $\lim_{x\to 0-0}f(x)=\lim_{\substack{h\to 0\\ h>0}}=\frac{\sin[0-h]}{[0-h]}$

$$=\lim_{h\to 0}\frac{\sin(-1)}{(-1)}$$

$$= \sin 1.$$

Also $\lim\limits_{x \to 0+0} f(x) = 0$ because $[0+h] = 0$. Therefore

$$\lim_{x \to 0-0} f(x) \neq \lim_{x \to 0+0} f(x)$$

$$\Rightarrow \lim_{x \to 0} f(x) \text{ does not exist.}$$

Statement-I is also true. That is both statements are true and Statement-II is not a correct explanation of Statement-I.

12. Statement-1: If $\lim\limits_{x \to 0}\left(\dfrac{\sin 2x + a\sin x}{x^3}\right)$ exists finitely, then the value of *a* is -2.

Statement-2: If $\lim\limits_{x \to a}\dfrac{f(x)}{g(x)}$ exists finitely and $\lim\limits_{x \to a} g(x) = 0$ then $\lim\limits_{x \to a} f(x) = 0$.

Key. A

Sol. Statement-II is clearly true.

Now $\lim\limits_{x \to 0}\left(\dfrac{\sin 2x + a\sin x}{x^3}\right) = \lim\limits_{x \to 0}\left(\dfrac{\sin x}{x}\right)\dfrac{(2\cos x + a)}{x^2}$

$= \lim\limits_{x \to 0}\dfrac{2\cos x + a}{x^2}$ exists finitely.

Therefore $\lim\limits_{x \to 0}(2\cos x + a) = 0$.

So $a = -2$.

13. Assertion (A): For $|x| < 1$, then $\ln\limits_{n \to \infty}\dfrac{\ln(2+x) - x^{2n}\sin x}{1 + x^{2n}} = \ln(2+x)$.

Reason (R): For $-1 < x < 1$, $\ln\limits_{n \to \infty} x^{2n} = 0$

Key. A

Sol. Conceptual

14. Statement – I : The period of $[x] + [2x] + ... + [nx] - \dfrac{n(n+1)}{2}x$. where $n \in N$.(Where [.] denotes the greatest integer function) is $\underline{|n}$

Statement - II : The period of $\{x\} + \{2x\} + ... + \{nx\}$ when $n \in N$ is L.C.M of the periods of $\{x\}, \{2x\}, ...\{nx\}$, (where $\{.\}$ denotes the fractional part of x)

Key. D

Sol. Conceptual

15. Assertion (A) : $\mathop{\mathrm{Lt}}\limits_{n \to \infty}\dfrac{x^n}{n!} = 0$ for every $x > 0, n \in N$

Reason(R) : every sequence whose nth term contains n! in the denominator converges to 0.

Key. C

Sol. Conceptual

16. **STATEMENT 1**: If *a* and *b* are positive and [x] denotes greatest integer $\leq x$, then $\lim\limits_{x \to 0^+}\dfrac{x}{a}\left[\dfrac{b}{x}\right] = \dfrac{b}{a}$

STATEMENT 2 : $\lim_{x\to\infty}\frac{\{x\}}{x}\to 0$ where {x} denotes fractional part of *x*.

Key. A

Sol. The Assertion A is true and follows from Reason R.

Since $\lim_{x\to 0^+}\frac{x}{a}\left[\frac{b}{x}\right]=\lim_{x\to 0^+}\frac{x}{a}\left(\frac{b}{x}-\left\{\frac{b}{x}\right\}\right)$

$\lim_{x\to 0^+}\left(\frac{b}{a}-\left(\frac{b}{a}\right)\frac{\{b/x\}}{b/x}\right)=\frac{b}{a}-0=\frac{b}{a}$.

Limits

Comprehension Type

Passage – 1

Let $n \in N$, The A.M, G.M, H.M of the n numbers n+1, n+2, n+3, ----, n+n are An, Gn, Hn respectively then attempt the following.

1. $\lim\limits_{n\to\infty} \frac{An}{n}$

A. 1 **B.** 1/2 **C.** 3/2 **D.** 2

Key. C

Sol. $$A_n = \frac{(n+1)+(n+2)+----+(n+n)}{n} = \frac{n^2+\frac{n(n+1)}{2}}{n} = \frac{3n+1}{2}$$

$$\frac{An}{n} = \frac{3}{2}+\frac{1}{2n} \to \frac{3}{2} \ \ as\ n \to \alpha$$

2. $\lim\limits_{n\to\infty} \frac{Gn}{n}$

A. 1/e **B.** 2/e **C.** 3/e **D.** 4/e

Key. D

Sol.
$$Gn = ((n+1)(n+2)----(n+n))^{\frac{1}{n}}$$

$$= n((1+\frac{1}{n})(1+\frac{2}{n})----(1+\frac{n}{n}))^{\frac{1}{n}}$$

$$\frac{Gn}{n} = ((1+\frac{1}{n})(1+\frac{2}{n})----(1+\frac{n}{n}))^{\frac{1}{n}}$$

$$\log\frac{Gn}{n} = \frac{1}{n}\sum_{r=1}^{n}\log(1+\frac{r}{n}) \Rightarrow \lim_{n\to\infty}\log\frac{Gn}{n} = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\log(1+\frac{r}{n})$$

$$= \int_0^1 \log(1+x)dx = \log\frac{4}{e}$$

$$\lim_{n\to\infty}\frac{Gn}{n} = \frac{4}{e}$$

3. $\lim\limits_{n\to\infty} \frac{Hn}{n}$

A. 1/e **B.** 1/log2 **C.** 2/e **D.** 1/log4

Key. B

Sol. $$Hn=\frac{n}{\frac{1}{n+1}+\frac{1}{n+2}+---+\frac{1}{n+n}} \Rightarrow \frac{n}{Hn}=\sum_{r=1}^{n}\frac{1}{n+r}=\frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+\frac{r}{n}}$$

$$=\int_0^1 \frac{dx}{1+x}=\log(1+x)\int_0^1=\log 2$$

Passage – 2

If *f, g* and *h* are functions having a common domain *D* and $h(x)\le f(x)\le g(x), x\in D$ and if

$\lim_{x\to a} h(x)=\lim_{x\to a} g(x)=l$ then $\lim_{x\to a} f(x)=l$. This is known as Sandwich Theorem. Using this result, compute the following limits.

4. The value of $\lim_{x\to 0}\frac{|x|}{\sqrt{x^4+4x^2+7}}$

(A) 1 (B) 0 (C) 1/2 (D) doesn't exist

Key. B

Sol. Since $\sqrt{x^4+4x^2+7}\ge 1$, so

$0\le \frac{|x|}{\sqrt{x^4+4x^2+7}}\le |x|$. But $\lim_{x\to 0}|x|=0$,

Hence $0\le \lim_{x\to 0}\frac{|x|}{\sqrt{x^4+4x^2+7}}\le \lim_{x\to 0}|x|=0$.

$\therefore \lim_{x\to 0}\frac{|x|}{\sqrt{x^4+4x^2+7}}=0$

5. $\lim_{x\to 0} x^4\sin\left(\frac{1}{3\sqrt{x}}\right)$ is

(A) 0 (B) 1 (C) 1/3 (D) does not exist

Key. A

Sol. Since $-1\le \sin\left(\frac{1}{3\sqrt{x}}\right)\le 1$, so $-x^4\le x^4\sin\left(\frac{1}{3\sqrt{x}}\right)\le x^4$. But $\lim_{x\to 0}x^4=0$,

Hence $0\le x^4\sin\left(\frac{1}{3\sqrt{x}}\right)\le 0$,

$\therefore \lim_{x\to 0} x^4\sin\left(\frac{1}{3\sqrt{x}}\right)=0$.

6. Let $f(x)=x^2\left(\frac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}\right), x\ne 0$, then

(A) $\lim_{x\to 0+} f(x)$ doesn't exist (B) $\lim_{x\to 0} f(x)$ doesn't exist

(C) $\lim_{x\to 0} f(x)$ exist (D) $\lim_{x\to 0} f(x)=1$

Key. C

Sol. $0 \le x^2 \dfrac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}} = x^2 \dfrac{1-e^{-2/x}}{1+e^{-2/x}} \le x^2$ for $x>0$.

So $\lim_{x\to 0+} f(x)=0$. Also $\lim_{x\to 0-} f(x)=\lim_{x\to 0-} x^2 \dfrac{e^{2/x}-1}{e^{2/x}+1}=0$.

Hence $\lim_{x\to 0} f(x)=0$.

Passage – 3

Let $f(x)=\lim_{n\to\infty}\left(\cos\sqrt{\frac{x}{n}}\right)^n$, $g(x)=\lim_{n\to\infty}\left(1-x+x\sqrt[n]{e}\right)^n$. Now, consider the function y = h(x), where h(x) = $\tan^{-1}\left(g^{-1}f^{-1}(x)\right)$.

7. $\lim_{x\to 0^+}\dfrac{\ln(f(x))}{\ln(g(x))}$ is equal to

(A) $\frac{1}{2}$ (B) $-\frac{1}{2}$

(C) 0 (D) 1

Key. B

8. Domain of the function y = h(x) is

(A) $(0,\infty)$ (B) R

(C) (0, 1) (D) [0, 1]

Key. C

9. Range of the function y = h(x) is

(A) $\left(0,\frac{\pi}{2}\right)$ (B) $\left(-\frac{\pi}{2},0\right)$

(C) R (D) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

Key. D

Sol. Conceptual

Passage – 4

In the evaluation of limits following the paragraph, one may use one or the other of the following results:

1) If $f(x)$ and $g(x)$ are functions defined in some deleted neighbourhood N of 'a' such that $g(x)$ never vanishes in N and $\lim_{x\to a}\frac{f(x)}{g(x)}$ exists, then $\lim_{x\to a} g(x)=0$ implies $\underset{x\to a}{Lt} f(x)=0$

2) If $f(x)\to 1$ as $x\to a, g(x)\to +\infty$ as $x\to a$ and $\lim_{x\to a}\{f(x)\}^{g(x)}$ exists, then this limit equals $e^{\underset{x\to a}{Lt} g(x)\{f(x)-1\}}$

10. If $\lim_{x\to 0}\frac{\sin ax-\log_e(e^x\cos x)}{x\sin bx}=\frac{1}{2}$, then

a) $a=1,b=1$ b) $a=-1,b=\frac{1}{2}$ c) $a=-1,b=1$ d) $a=-1,b=\frac{1}{2}$

Key. A

11. Let $f(x)$ be a function that is defined in a deleted neighbourhood of '0' such that $\lim_{x\to 0}\frac{f(x)}{x}=1$. If $\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{\{f(x)\}^3}=1$ then

a) $a=-\frac{3}{2},b=-\frac{1}{2}$ b) $a=\frac{1}{2},b=\frac{3}{2}$ c) $a=-\frac{5}{2},b=-\frac{3}{2}$ d) $a=\frac{3}{2},b=\frac{5}{2}$

Key. C

12. Let $L=\lim_{x\to 0}\frac{a-\sqrt{a^2-x^2}-\frac{x^2}{2}}{x^2\sin^2 x}, a>0$. If L is finite, then L =

a) $\frac{1}{4}$ b) $-\frac{1}{4}$ c) $\frac{1}{8}$ d) $-\frac{1}{8}$

Key. C

Sol. 4. $\frac{b}{2}=\underset{x\to 0}{Lt}\frac{a\cos ax-1+\tan x}{2x}$

$\Rightarrow b=\underset{x\to 0}{Lt}\frac{a\cos ax-1}{x}+1$

$\Rightarrow a=b=1$

5. $\lim_{x\to 0}\frac{1+a\cos x-ax\sin x-b\cos x}{3x^2}=1$

implies $a-b=-1$ and in this case,

$1=\lim_{x\to 0}\frac{(1-\cos x)-ax\sin x}{3x^2}=\frac{1}{3}\left(\frac{1}{2}-a\right)$

This gives $a=-\frac{5}{2}$ and $b=a+1=-\frac{3}{2}$

6. $L = \lim\limits_{x\to 0} \dfrac{\dfrac{-x}{\sqrt{a^2-x^2}} + x}{4x^3} = \underset{x\to 0}{Lt} \dfrac{1-\dfrac{1}{\sqrt{a^2-x^2}}}{4x^2}$

$\Rightarrow a = 1$ and $L = \underset{x\to 0}{Lt} \dfrac{\sqrt{1-x^2}-1}{4x^2} = -\dfrac{1}{8}$

Passage – 5

If f, g and h are functions having a common domain D and $h(x) \le f(x) \le g(x),\ x \in D$ and if $\lim\limits_{x\to a} h(x) = \lim\limits_{x\to a} g(x) = l$ then $\lim\limits_{x\to a} f(x) = l$. This is known as Sandwich Theorem.

13. $\lim\limits_{x\to 0} x^4 \sin\left(\dfrac{1}{3\sqrt{x}}\right)$ is

(a) 0 (b) 1 (c) $\dfrac{1}{3}$ (d) does not exist

Key: A

Hint: Since $-1 \le \sin\left(\dfrac{1}{\sqrt{3}\,x}\right) \le 1$, so $-x^4 \le x^4 \sin\left(\dfrac{1}{\sqrt{3}\,x}\right) \le x^4$. But $\lim\limits_{x\to 0} x^4 = 0$.

$\therefore \lim\limits_{x\to 0} x^4 \sin\left(\dfrac{1}{3\sqrt{x}}\right) = 0$

14. Let $f(x) = x^2 \dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}, x \ne 0$ and f(0) = 1 then

(a) $\lim\limits_{x\to 0^+} f(x)$ does not exist (b) $\lim\limits_{x\to 0} f(x)$ does not exist

(c) $\lim\limits_{x\to 0} f(x)$ exists (d) f is continuous at x = 0.

Key: c

Hint: $0 \le x^2 \dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}} = x^2 \dfrac{1-e^{-2/x}}{1+e^{-2/x}} \le x^2$ for x > 0

So $\lim\limits_{x\to 0^+} f(x) = 0$. Also $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} x^2 \dfrac{e^{2/x}-1}{e^{2/x}+1} = 0$

$\lim\limits_{x\to 0} f(x) = 0$

15. Let $f(x) = x^5 \left[\dfrac{1}{x^3}\right],\ x \ne 0$ and f(0) = 0 ([.] denotes the greatest integer function)

(a) $\lim\limits_{x\to 0} f(x)$ does not exist (b) f is not continuous at x = 0

(c) $\lim\limits_{x\to 0} f(x)$ = 1 (d) $\lim\limits_{x\to 0} f(x)$ = 0

Key: d

Hint: Since $x - 1 < [x] \le x$ for $x \in R$

$\dfrac{1}{x^3} - 1 < \left[\dfrac{1}{x^3}\right] \le \dfrac{1}{x^3} \quad \Rightarrow \quad x^5\left(\dfrac{1}{x^3} - 1\right) \le x^5\left[\dfrac{1}{x^3}\right] \le x^2$

so $\lim\limits_{x\to 0} x^5 \left[\dfrac{1}{x^3}\right] = 0$

Passage – 6

$f: R \to R$ is a function satisfying the following three conditions:

(a) $f(-x) = -f(x), \forall x \in R$

(b) $f(x+1) = f(x) + 1, \forall x \in R$

(c) $f\left(\frac{1}{x}\right) = \frac{f(x)}{x^2}, \forall x \neq 0$

Answer the following questions.

16. $\lim_{x \to 2} \frac{2^{f(x+1)} - 2^{f(x)}}{x}$ is

A) 2 B) $\log 2$ C) $2\log 2$ D) $\frac{2}{\log 2}$

Key. A

17. $\lim_{x \to 1} \left(f^{-1}(x)\right)$ is

A) 1 B) 0 C) does not exist D) e

Key. A

18. The number of common points of the graph of $y = f(x)$ with the line $y = x$ is

A) 2 B) 4 C) 8 D) infinite

Key. D

Sol. 16-18. Hint: $f(x) = x, \forall x \in R$.

Passage – 7

Consider two functions

$$f(x) = \lim_{n \to \infty} \left(\cos \frac{x}{\sqrt{n}}\right)^n \;\&\; g(x) = -x^{4b}, where\;\; b = \lim_{x \to \infty}\left(\sqrt{x^2 + x + 1} - \sqrt{x^2 + 1}\right), \text{ then}$$

19. $f(x)$ is

A) e^{-x^2} B) $e^{-x^2/2}$ C) e^{x^2} D) $e^{x^2/2}$

Key. B

20. Number of solutions of $f(x) + g(x) = 0$ is

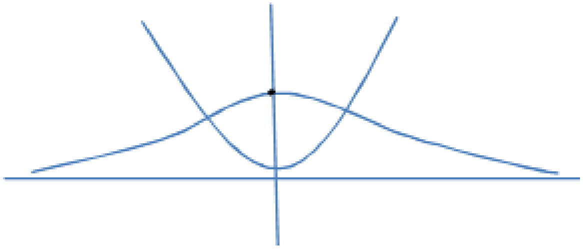
A) 2 B) 4 C) 0 D) 1

Key. A

Sol. $f(x) = \lim_{n \to \infty} e^{\left(\cos \frac{x}{\sqrt{n}} - 1\right) \times n} = \lim_{n \to \infty} e^{-\left(\frac{1 - \cos \frac{x}{\sqrt{n}}}{\left(\frac{x}{\sqrt{n}}\right)^2}\right) \times x^2} = e^{-x^2/2}$

& b = $\lim_{x \to \infty} \frac{(x^2 + x + 1) - (x^2 + 1)}{\sqrt{x^2 + x + 1} + \sqrt{x^2 + 1}} = \frac{1}{2}$

$\Rightarrow f(x) + g(x) = e^{-x^2/2} - x^2 = 0 \qquad \Rightarrow e^{-x^2/2} = x^2$



Two solutions

Passage – 8

If *f, g* and *h* are functions having a common domain *D* and $h(x) \le f(x) \le g(x), x \in D$ and if $\lim_{x \to a} h(x) = \lim_{x \to a} g(x) = l$ then $\lim_{x \to a} f(x) = l$. This is known as Sandwich Theorem. Using this result, compute the following limits.

21. The value of $\lim_{x \to 0} \frac{|x|}{\sqrt{x^4 + 4x^2 + 7}}$

(A) 1 (B) 0 (C) 1/2 (D) doesn't exist

Key. B

22. $\lim_{x \to 0} x^4 \sin\left(\frac{1}{3\sqrt{x}}\right)$ is

(A) 0 (B) 1 (C) 1/3 (D) does not exist

Key. A

23. Let $f(x) = x^2\left(\frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}\right), x \ne 0$, then

(A) $\lim_{x \to 0+} f(x)$ doesn't exist (B) $\lim_{x \to 0} f(x)$ doesn't exist

(C) $\lim_{x \to 0} f(x)$ exist (D) $\lim_{x \to 0} f(x) = 1$

Key. C

Sol. 21. Since $\sqrt{x^4 + 4x^2 + 7} \ge 1$, so

$0 \le \frac{|x|}{\sqrt{x^4 + 4x^2 + 7}} \le |x|$. But $\lim_{x \to 0} |x| = 0$,

Hence $0 \le \lim_{x \to 0} \frac{|x|}{\sqrt{x^4 + 4x^2 + 7}} \le \lim_{x \to 0} |x| = 0$.

$\therefore \lim_{x \to 0} \frac{|x|}{\sqrt{x^4 + 4x^2 + 7}} = 0$

22. Since $-1 \le \sin\left(\frac{1}{3\sqrt{x}}\right) \le 1$, so $-x^4 \le x^4 \sin\left(\frac{1}{3\sqrt{x}}\right) \le x^4$. But $\lim_{x \to 0} x^4 = 0$,

Hence $0 \le x^4 \sin\left(\frac{1}{3\sqrt{x}}\right) \le 0$,

$\therefore \lim_{x\to 0} x^4 \sin\left(\frac{1}{3\sqrt{x}}\right) = 0$.

23. $0 \le x^2 \frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}} = x^2 \frac{1-e^{-2/x}}{1+e^{-2/x}} \le x^2$ for $x > 0$.

So $\lim_{x\to 0+} f(x) = 0$. Also $\lim_{x\to 0-} f(x) = \lim_{x\to 0-} x^2 \frac{e^{2/x}-1}{e^{2/x}+1} = 0$.

Hence $\lim_{x\to 0} f(x) = 0$.

Passage – 9

Let $f(x) = {}^nC_x p^x q^{n-x}; x = 0,1,2,.....n$ where p+q=1, $0 < p < 1$. Here ${}^nC_x = \frac{n!}{x!(n-x)!}$ and $n! = 1\times 2\times 3\times ....\times n$. $np = \lambda$ (finite) when $p \to 0$ and $n \to \infty$. Now answer the following questions

24. $\underset{n\to\infty}{Lt} \sum_{x=0}^{n} x f(x) =$

a) λ b) $\frac{\lambda}{2}$ c) 0 d) 1

Key. A

25. $\underset{n\to\infty}{Lt} \sum_{x=0}^{n} x^2 f(x) =$

a) $1+\lambda$ b) $\lambda + \lambda^2$ c) λ^2 d) 0

Key. B

26. $\underset{n\to\infty}{Lt} \sum_{x=0}^{n} (x-\lambda)^2 f(x) =$

a) λ b) 0 c) 2λ d) $\lambda^2 + \lambda$

Key. A

Sol. 24,25,26 : $\lim_{n\to\infty} f(x) = \lim_{n\to\infty} \frac{n(n-1)(x-2)........(n-x+1)}{x!}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x}$

$= \frac{e^{-\lambda}\lambda^x}{x!}$

$\sum_{x=0}^{\infty} xf(x) = \sum_{x=0}^{\infty} x.e^{-\lambda}\frac{\lambda^{x-1}\lambda}{(x-1)!x} = \lambda e^{-\lambda}\sum_{x=1}^{\infty}\frac{\lambda^{x-1}}{(x-1)!} = e^{-\lambda}e^{\lambda}\lambda = \lambda$

Similarly $\sum_{x=0}^{\infty} x^2 f(x) = \sum_{x=0}^{\infty} x(x-1)f(x) + \sum_{x=0}^{\infty} xf(x) = \lambda^2 + \lambda$

$\sum_{x=0}^{\infty}(x-\lambda)^2 f(x) = \sum x^2 f(x) + \lambda^2 \sum f(x) - 2\lambda \sum xf(x)$

$= \lambda^2 + \lambda - \lambda^2 = \lambda$

Passage – 10

$\underset{x\to a}{Lt} f(x)$ exists if $\underset{x\to a^-}{Lt} f(x)\, and \underset{x\to a^+}{Lt} f(x)$ both exist and are equal. Their common value is the value of $\underset{x\to a}{Lt} f(x)$. Answer the following questions

27. If

$$f(x)=\frac{\sin(1+[x])}{[x]} \quad [x]\neq 0$$
$$=0 \qquad [x]=0 \quad (\text{where } [.] \text{ denotes the greatest integer function}) \ then$$

(A) $\underset{x\to 0-}{Lt} f(x)$ exists and equal to zero (B) $\underset{x\to 0+}{Lt} f(x)$ exists
(C) $\underset{x\to 0}{Lt} f(x)$ exists (D) none of these

Key. A

28. If $f(x)=3+\left(1+7^{\frac{1}{1-x}}\right)^{-1}$ then

(A) $\underset{x\to 1-}{Lt} f(x)=4$ (B) $\underset{x\to 1+}{Lt} f(x)=3$ (C) $\underset{x\to 1+}{Lt} f(x)=5$ (D) $\underset{x\to 1}{Lt} f(x)$ does not exist

Key. D

29. If $[x]$ denotes the greatest integer $\leq x$ then $\underset{x\to 1}{Lt}(1-x+[x-1]+[1-x])$ is

(A) 0 (B) 1 (C) – 1 (D) none of these

Key. C

Sol. Conceptual

Passage – 11

Let $\underset{x\to c}{Lt} f(x)=l=\underset{x\to c}{Lt} h(x)\, and\ f(x)\leq g(x)\leq h(x) \quad \forall x\in(c-\delta,c)\cup(c,c+\delta)$ for δ >0 then $\underset{x\to c}{Lt} g(x)=l$ this is called squeeze principle or sandwich principle. Then answer the following questions.

30.

$$\underset{x\to 0}{Lt}\,\underset{n\to\infty}{Lt}\left(\frac{[1^2x^x]+[2^2x^x]+.......+[n^2x^x]}{n^3}\right)= \quad (\text{where } [.] \text{ denotes the greatest integer function})$$

(A) 0 (B) $\frac{1}{6}$ (C) $\frac{1}{3}$ (D) $\frac{1}{2}$

Key. C

31. $\underset{x\to 0}{Lt}\left(\left[\frac{\sin x}{x}\right]+\left[\frac{\tan x}{x}\right]\right)= \quad (\text{where } [.] \text{ denotes the greatest integer function})$

(A) 0 (B) 3 (C) 2 (D) 1

Key. D

32. $\underset{n\to\infty}{Lt}\left(\frac{\{x\}+\{2x\}+\{3x\}+.......+\{nx\}}{n^2}\right)$ (where $\{x\}$ denotes the fractional part of x) is equal to

(A) 1 (B) – 1 (C) 0 (D) none of these

Key. C

Sol. Conceptual

Passage – 12

Left hand derivative and right hand derivative of a function $f(x)$ at a point $x = a$ are defined as

$$f^1(a^-) = \lim_{h\to 0^+} \frac{f(a)-f(a-h)}{h} = \lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h} \text{ and}$$

$$f^1(a^+) = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0^-} \frac{f(a-h)-f(a)}{-h} = \operatorname*{Lt}_{h\to a^+} \frac{f(x)-f(a)}{x-a}$$

respectively

33. If f is odd, which of the following is left hand derivate of f at x = - a

a) $\operatorname*{Lt}_{h\to 0^+} \frac{f(a-h)-f(a)}{-h}$ b) $\operatorname*{Lt}_{h\to 0^-} \frac{f(h-a)-f(a)}{h}$

c) $\operatorname*{Lt}_{h\to 0^+} \frac{f(a)+f(a-h)}{-h}$ d) $\operatorname*{Lt}_{h\to 0^-} \frac{f(-a)-f(-a-h)}{-h}$

Key. A

34. If f is even which of the following is right hand derivative of f^1 at x = a

a) $\operatorname*{Lt}_{h\to 0^-} \frac{f^1(a)+f^1(-a+h)}{h}$ b) $\operatorname*{Lt}_{h\to 0^+} \frac{f^1(a)+f^1(-a-h)}{h}$

c) $\operatorname*{Lt}_{h\to 0^-} \frac{-f^1(-a)+f^1(-a-h)}{-h}$ d) $\operatorname*{Lt}_{h\to 0^+} \frac{f^1(a)+f^1(-a+h)}{-h}$

Key. A

35. The statement $\operatorname*{Lt}_{h\to 0} \frac{f(-x)-f(-x-h)}{h} = \operatorname*{Lt}_{h\to 0} \frac{f(x)-f(x-h)}{-h}$ implies that

a) f is odd b) f is even

c) f is neither odd nor even d) nothing can be concluded.

Key. B

Sol. 33. LHD $= \operatorname*{Lt}_{h\to 0^-} \frac{f(-a)-f(-a+h)}{-h}$

34. RHD $= \operatorname*{Lt}_{h\to 0^-} \frac{f^1(a-h)-f^1(a)}{-h}$

35. LHL = $f^1(-x) = -f^1(x) \Rightarrow f^1$ is odd

$\Rightarrow$ is even

Limits

Integer Answer Type

1. If $f(n+1)=\frac{1}{2}\left\{f(n)+\frac{9}{f(n)}\right\}$ where $n\in N$ and $f(n)>0\forall n\in N$ and $\underset{n\to\infty}{Lt} f(n)$ exist then the value of $\underset{n\to\infty}{Lt} f(n)=$

Key. 3

Sol. Let $\underset{n\to\infty}{Lt} fn=l\Rightarrow \underset{x\to\infty}{Lt} f(n+1)=l$

$$\underset{n\to\infty}{Lt} f(n+1)=\frac{1}{2}\underset{n\to\infty}{Lt}\left[f(n)+\frac{9}{f(n)}\right]$$

$$\Rightarrow l=\frac{1}{2}\left[l+\frac{9}{l}\right]$$

$$2l=\frac{l^2+9}{l}\Rightarrow 2l^2=l^2+9\Rightarrow l^2=9 \qquad \Rightarrow l=3$$

$$\because f(n)>0\forall n\in N \qquad \therefore \underset{x\to\infty}{Lt} f(n)=3$$

2. If $\{x\},[x]$ are fractional part function and greatest integer functions of x respectively then for any real number a , the value of $\underset{x\to[a]}{Lt}-\frac{e^{\{x\}}-\{x\}-1}{\{x\}^2}$ $is\ e-K\Rightarrow K=$ _____

Key. 2

Sol. As

$x\to[a],\{x\}\to 1$

$$\therefore G.L=\frac{e^1-1-1}{1^2}=e-2$$

3. If $f(n+1)=\frac{1}{2}\left\{f(n)+\frac{9}{f(n)}\right\}$ where $n\in N$ and $f(n)>0\forall n\in N$ and $\underset{n\to\infty}{Lt} f(n)$ exist then the value of $\underset{n\to\infty}{Lt} f(n)=$

Key. 3

Sol. Let $\underset{n\to\infty}{Lt} fn=l\Rightarrow \underset{x\to\infty}{Lt} f(n+1)=l$

$$\underset{n\to\infty}{Lt} f(n+1)=\frac{1}{2}\underset{n\to\infty}{Lt}\left[f(n)+\frac{9}{f(n)}\right]$$

$\Rightarrow l = \frac{1}{2}\left[l + \frac{9}{l} \right]$

$2l = \frac{l^2 + 9}{l} \Rightarrow 2l^2 = l^2 + 9 \Rightarrow l^2 = 9$

$l = 3$

$\because f(n) > 0 \forall n \in N$

$\therefore \underset{x \to \infty}{Lt} f(n) = 3$

4. The integer 'n' for which $\underset{x \to 0}{Lt} \left[\frac{(\cos x - 1)(\cos x - e^x)}{x^n} \right]$ is a finite non zero number, is

Key. 3

Sol. Let $\underset{x \to 0}{Lt} \frac{(\cos x - 1)(\cos - e^n)}{n^x} = k\,(\text{finite, non-zero})$

$$\Rightarrow \underset{x \to 0}{Lt} \frac{\left[\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ...\right) - 1\right]\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... + \right) - \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} = 1\right)}{x^n} = K$$

As the limit in finite, non zero we have degree of denominator = least power of x

$\Rightarrow$ n =3

5. If $A = \underset{x \to -2}{Lt} \frac{\tan \pi x}{x + 2} + \underset{x \to \infty}{Lt} \left(1 + \frac{1}{x^2}\right)^x$ then $[A]$ is, where [.] denotes g.i.f

Key. 4

Sol. Give A = $\underset{x \to -2}{Lt} \frac{\tan \pi x}{x + 2} + \underset{\frac{1}{x} \to 0}{Lt} \left(1 + \frac{1}{x^2}\right)^{x^2} \frac{1}{x}$

$\underset{x \to -2}{Lt} \frac{\pi \sec^2 \pi x}{1} + \underset{\frac{1}{x} \to 0}{Lt} e^{1/x}$

= $\pi + 1 = 3.14 + 1 = 4.14$

$\therefore A = 4.14$

$[A] = 4$

6. If $\underset{x \to 0}{Lt} \frac{ae^x - b\cos x + ce^{-x}}{x \sin x} = 2$ then the value of a + b + c =

Key. 3

Sol. $\underset{x \to 0}{Lt} \frac{ae^x - b\cos x + c.e^{-x}}{x \sin x} = 2 \Rightarrow a - b + c = 0$ .....(i)

Apply LH Rule

$\underset{x \to 0}{Lt} \frac{ae^x + b\sin x - c.e^{-x}}{\sin x + x\cos x} = 2 \Rightarrow a + 0 - c = 0 \quad \Rightarrow a = c$ ....(ii)

Apply LH rule

$\underset{x \to 0}{Lt} \frac{ae^x + b\cos x + ce^{-x}}{\cos x + \cos x - x\sin x} = 2 \Rightarrow a + b + c = 4$

$\therefore a+b+c=4$

7. If

$$f(x)=\begin{cases}\dfrac{1-\sin^3 x}{3\cos^2 x} & x<\dfrac{\pi}{2}\\ a & x=\dfrac{\pi}{2}\\ \dfrac{b(1-\sin x)}{(\pi-2x)^2} & x>\dfrac{\pi}{2}\end{cases}$$

If $f(x)$ is continuous $x=\dfrac{\pi}{2}$ then $\dfrac{b}{a}=$

Ans: 8

Hint: $LHL=\dfrac{1}{2}.RHL=\dfrac{b}{8}$

$\therefore \dfrac{1}{2}=a=\dfrac{b}{8}$

8. If $\lim\limits_{x\to 0}\dfrac{\log(1+x)^{1+x}}{x^2}-\dfrac{1}{x}=k$ then value of 12 k is

Key. 6

Sol. $k=\lim\limits_{x\to 0}\dfrac{(1+x)\ln(1+x)-x}{x^2}=\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{2x}=\dfrac{1}{2}$

(on using L' Hopital rule) $\therefore 12k=6$

9. The value of $\lim\limits_{x\to\frac{\pi}{2}}\sqrt{\dfrac{\tan x-\sin(\tan^{-1}(\tan x))}{\tan x+\cos^2(\tan x)}}$ is

Key. 1

Sol. We have

$$LHL=\lim_{x\to\frac{\pi}{2}-}\sqrt{\frac{\tan x-\sin\tan^{-1}(\tan x)}{\tan x+\cos^2(\tan x)}}$$

$$=\lim_{x\to\frac{\pi}{2}-}\sqrt{\frac{\tan x-\sin x}{\tan x+\cos^2(\tan x)}}$$

$$= \lim_{x \to \frac{\pi}{2}-} \sqrt{\frac{1 - \frac{\sin x}{\tan x}}{1 + \frac{\cos^2(\tan x)}{\tan x}}} = \sqrt{\frac{1-0}{1+0}} = 1$$

At $x \to \frac{\pi}{2}-, o < x < \frac{\pi}{2}$ $\quad \therefore \tan^{-1}(\tan x) = x$

Further as, $x \to \frac{\pi}{2}-, \tan x \to \infty$ and $\cos^2(\tan x)$ is real number between 0 and 1]

$$\text{RHL} = \lim_{x \to \frac{\pi}{2}+} \sqrt{\frac{\tan x - \sin \tan^{-1}(\tan x)}{\tan x + \cos^2(\tan x)}}$$

$$= \lim_{x \to \frac{\pi}{2}+} \sqrt{\frac{\tan x + \sin x}{\tan x + \cos^2 x(\tan x)}}$$

$$= = \lim_{x \to \frac{\pi}{2}+} \sqrt{\frac{1 + \frac{\sin x}{\tan x}}{1 + \frac{\cos^2(\tan x)}{\tan x}}} = \sqrt{\frac{1+0}{1-0}} = 1$$

(As $x \to \frac{\pi}{2}+, x > \frac{\pi}{2} \Rightarrow \tan^{-1} \tan x$

$= \tan^{-1} \tan(x - \pi) = x - \pi$

$\therefore \sin \tan^{-1}(\tan x) = \sin(x - \pi) = -\sin x$

Further as $x \to \frac{\pi}{2}+; \tan x \to -\infty$ and $\cos^2(\tan x)$ is a real number between 0 and 1)

LHL = RHL = 1 $\therefore$ required limit = 1

10. Let $(\tan\alpha)x + (\sin\alpha)y = \alpha$ and $(\alpha\cos ec\alpha)x + \cos\alpha y = 1$ be two variable straight lines, α being the parameter. Let P be the point of intersection of the lines. If the coordinates of P in the limiting position when $\alpha \to 0$ be (h, k) then is $h - k$ equal to

Key. 3

Sol. Here two straight line, $(\tan\alpha)x + (\sin\alpha)y = \alpha$ and

$(\alpha\cos ec\,\alpha)x + (\cos\alpha)y = 1$ have their point of intersection as,

$x = \dfrac{\alpha\cos\alpha - \sin\alpha}{\sin\alpha - \alpha}$ and $y = \dfrac{\alpha - x\tan\alpha}{\sin\alpha}$

$\therefore$ when $\alpha \to o$, we obtain the pointP.

i.e., $\lim\limits_{\alpha \to 0} x = \lim\limits_{\alpha \to 0} \dfrac{\alpha\cos\alpha - \sin\alpha}{\sin\alpha - \alpha} \left(\dfrac{0}{0}\, form\right)$

$$= \lim_{\alpha\to 0}\frac{-\alpha\sin\alpha+\cos\alpha-\cos\alpha}{\cos\alpha-1}$$

(applying L-Hospital's rule)

$$= \lim_{\alpha\to 0}\frac{-\alpha\sin\alpha}{-2\sin^2\alpha/2} = \lim_{\alpha\to 0}\frac{\alpha\left(2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right)}{2\sin^2\frac{\alpha}{2}}$$

$$\lim_{\alpha\to 0}\frac{\alpha}{\tan\alpha/2} = \lim_{\alpha\to 0}\frac{2\frac{\alpha}{2}}{\tan\frac{\alpha}{2}} = 2$$

Again, $\lim_{\alpha\to 0} y = \lim_{\alpha\to 0}\frac{\alpha - x\tan\alpha}{\sin\alpha} = \lim_{x\to 0}\left(\frac{\alpha}{\sin\alpha}-\frac{x}{\cos\alpha}\right)$

$$\lim_{\alpha\to 0}\frac{\alpha}{\sin\alpha} - \lim_{\alpha\to 0}\frac{x}{\cos\alpha} = 1-2=-1 \qquad \left[\because \lim_{\alpha\to 0} x = 2\right]$$

$$\Rightarrow \lim_{\alpha\to 0} y = -1$$

Hence, in limiting position $P(2-1) \Rightarrow h-k = 2+1 = 3$

11. $\lim_{n\to\infty} 2\left(\prod_{r=2}^{n}\frac{r^3+1}{r^3-1}\right) =$

Key. 3

Sol. $2\lim_{n\to\infty}\prod_{r=2}^{r=n}\left(\frac{r^3+1}{r^3-1}\right) = 2\lim_{n\to\infty}\prod_{r=2}^{r=n}\left(\frac{r+1}{r-1}\right)\left(\frac{r^2-r+1}{r^2+r+1}\right) = 2\lim_{n\to\infty}\frac{3\times\left(n^2-n+1\right)}{1\times 2\times (n-1)n} = 3$

Limits

Matrix-Match Type

1. Let $f(x)=\lim_{n\to\infty}\frac{x^{2n}-1}{x^{2n}+1}$

Column-1		Column-2	
(A)	$f=1$ on	(P)	$(1,\infty)$
(B)	$f=-1$ on	(Q)	$(2,\infty)$
(C)	$f\ge 0$ on	(R)	$[2,\infty)$
(D)	$f(x)=\operatorname{sgn}(\lvert x\rvert-1)$ on	(S)	$\left(-\frac{1}{2},\frac{1}{2}\right)$

Key. A-p,q,r; B-s; C-p,q,r; D-p,q,r,s

Sol. If $|x|>1$, then $\lim_{n\to\infty}x^{2n}=\infty$, so

$$f(x)=\lim_{n\to\infty}\frac{1-x^{-2n}}{1+x^{-2n}}=1$$

If $|x|<1$ then $\lim_{n\to\infty}x^{2n}=0$, therefore $f(x)=-1$. If $x=\pm1$ then $x^{2n}=1$ for any *n*, therefore $f(x)=0$. Thus

$$f(x)=\begin{cases}1 & if\ |x|>1 i.e. x\in(-\infty,-1)\cup(1,\infty)\\ -1 & if\ |x|<1 i.e. -1<x<1\\ 0 & if\ x=\pm1\end{cases}$$ or $f(x)=\operatorname{sgn}(|x|-1)$

2.

COLUMN-I		COLUMN-II	
(A)	$f(x)=x\ \operatorname{sgn}(x-1)$	(p)	$\lim_{x\to1}f(x)$ doesn't exist
(B)	$f(x)=\frac{\sin\left(\sin\left(\tan\left(x^2/2\right)\right)\right)}{\log\cos 3x}$	(q)	$\lim_{x\to0}f(x)$ doesn't exist
(C)	$f(x)=\frac{\sqrt[3]{1+\tan^{-1}3x}-\sqrt[3]{1-\sin^{-1}3x}}{\sqrt{1-\sin^{-1}2x}-\sqrt{1+\tan^{-1}2x}}$	(r)	$\lim_{x\to0}f(x)=-1/9$
(D)	$f(x)=\frac{e^{1/x}-1}{e^{1/x}+1}$	(s)	$\lim_{x\to0}f(x)=-1$

Key. A-p; B-r; C-s; D-q

Sol. For function f in (A)

$$f(x)=\begin{cases} x, & x>1 \\ 0, & x=1 \\ -x, & x<1 \end{cases}$$ so $\lim_{x\to 1} f(x)$ does not exist.

$$\lim_{x\to 0}\frac{\sin\left(\sin\left(\tan\left(x^2/2\right)\right)\right)}{\log\cos 3x}$$

$$=\lim_{x\to 0}\frac{\sin\left(\sin\left(\tan\left(x^2/2\right)\right)\right)}{\sin\left(\tan x^2/2\right)}\times\frac{\sin\left(\tan x^2/2\right)}{\tan x^2/2}\times\frac{\tan\left(x^2/2\right)}{x^2/2}\times\frac{x^2/2}{\log\cos 3x}$$

$$=\frac{1}{2}\lim_{x\to 0}\frac{2x}{-3\tan 3x}=\frac{-1}{9}$$

$$\lim_{x\to 0}\frac{\sqrt[3]{1+\tan^{-1}3x}-\sqrt[3]{1-\sin^{-1}3x}}{\sqrt{1-\sin^{-1}2x}-\sqrt{1+\tan^{-1}2x}}$$

$$=\lim_{x\to 0}\frac{\left(1+\frac{1}{3}\tan^{-1}3x+.......\right)-\left(1-\frac{1}{3}\sin^{-1}3x+....\right)}{\left(1-\frac{1}{2}\sin^{-1}2x-.....\right)-\left(1+\frac{1}{2}\tan^{-1}2x+.....\right)}$$

$$=\frac{\frac{1}{3}\frac{\tan^{-1}3x}{x}+\frac{1}{3}\frac{\sin^{-1}3x}{x}+\frac{1}{x}(higher\ power\ of\ \sin^{-1}3x\ and\ \tan^{-1}3x)}{-\frac{1}{2}\frac{\sin^{-1}2x}{x}-\frac{1}{2}\frac{\tan^{-1}2x}{x}+\frac{1}{x}\left(higher\ power\ of\ \sin^{-1}2x\ and\ \tan^{-1}2x\right)}$$

$$=\frac{1+1}{-1-1}=-1$$

$\lim_{x\to 0-}\frac{e^{1/x}-1}{e^{1/x}+1}=-1$ and $\lim_{x\to 0+}\frac{e^{1/x}-1}{e^{1/x}+1}=1$.

3.

COLUMN-I		COLUMN-II	
(A)	$\lim_{x\to 3}\frac{\left(x^3+27\right)\log\left(x-2\right)}{x^2-9}=$	(p)	12
(B)	$\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)^{\left(\frac{x}{x+1-e^x}\right)}=$	(q)	8
(C)	If $\lim_{x\to 0}\frac{x\left(a+\cos x\right)-b\sin x}{x^3}=1$ then *a* and *b* are respectively	(r)	9
(D)	If $f(x)$ is a thrice differentiable function such that	(s)	e^{-1}

	$\lim\limits_{x\to 0}\frac{f(4x)-3f(3x)+3f(2x)-f(x)}{x^3}=12$, then $f'''(0)$ is equal to		

Key. A-r; B-s; C-q, r; D-p

Sol. (A) By L' Hospital Rule

$$\lim_{x\to 3}\frac{(x^3+27)\frac{1}{x-2}+\log(x-2).3x^2}{2x}=\frac{54}{6}=9$$

(B) $L=\lim\limits_{x\to 0}\left(\frac{e^x-1}{x}\right)^{\left(\frac{x}{x+1-e^x}\right)}$

$$=\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)^{\left(\frac{1}{1-\frac{e^x-1}{x}}\right)}$$

Put $\frac{e^x-1}{x}=t$. As $x\to 0, t\to 1$.

$\therefore L=\lim\limits_{t\to 1}t^{\frac{1}{1-t}}$ $\quad [1^\infty \text{ form}]$

Taking logarithm,

$\log L=\lim\limits_{t\to 1}\frac{1}{1-t}\log t \quad \left(\frac{0}{0}\text{ form}\right)$

$=\lim\limits_{x\to 1}\frac{\frac{1}{t}}{-1}$ (by L. Hospital's rule) $=-1\Rightarrow L=e^{-1}$.

(C) We know that $\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots$

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots.$$

$$\therefore L=\lim_{x\to 0}\frac{ax+x\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots.\right)-b\left(x-\frac{x^3}{3!}+\ldots.\right)}{x^3}$$

$$=\lim_{x\to 0}(a-b+1)\times\frac{1}{x^2}+\left(\frac{b}{3!}-\frac{1}{2!}\right)$$

+ terms containing *x*

As L = 1, we must have

$a-b+1=0$ and $\frac{b}{3!}-\frac{1}{2!}=1$

$\Rightarrow b=9$ and $a=8$.

(D) $L=\lim\limits_{x\to 0}\dfrac{-f(x)+3f(2x)-3f(3x)+f(4x)}{x^3}\left(\dfrac{0}{0}\ form\right)$

$$=\lim_{x\to 0}\frac{-f'(x)+6f'(2x)-9f'(3x)+4f'(4x)}{3x^2}$$

$$=\lim_{x\to 0}\frac{-f''(x)+12f''(2x)-27f''(3x)+16f''(4x)}{6x}\left(\frac{0}{0}\ form\right)$$

$$=\lim_{x\to 0}\frac{-f'''(x)+24f'''(2x)-81f'''(3x)+64f'''(4x)}{6}\left(\frac{0}{0}\ form\right)$$

$$=\frac{6.f'''(0)}{6}=f'''(0).$$

But $L=12\Rightarrow f'''(0)=12$.

4. Column I — Column II

(A) $\underset{x\to 0}{Lt}\dfrac{\int_0^x(1-\cos 2t)\,dt}{x\int_0^x \tan t.dt}$ — (p) $\dfrac{3}{4}$

(B) $\underset{n\to\infty}{\mathrm{Lt}}\dfrac{1+\sqrt[3]{2}+\sqrt[3]{3}+.....+\sqrt[3]{n-1}}{\sqrt[3]{n^4}}$ — (q) $\dfrac{4}{3}$

(C) $\underset{x\to 0}{Lt}\dfrac{e^x.\sin x-x(1+x)}{\tan^3 x}$ — (r) $\dfrac{2}{3}$

(D) $\underset{x\to 0}{Lt}\dfrac{\cos(\sin x)-\cos x}{x^4}=L$ — (s) $\dfrac{1}{3}$

Then 4 L is

Key. $A\to q, B\to p$
$C\to s, D\to r$

Sol. (A) $\underset{x\to 0}{Lt}\dfrac{-\cos 2x.1}{x.\tan x+\int_0^x \tan t\,dt}dt$

$$\underset{x\to 0}{Lt}\frac{\sin 2x.2}{2\tan x+x\sec^2 x}=\underset{x\to 0}{Lt}\frac{4\cos 2x}{2\sec^2 x+\sec^2 x+x.2\sec^2 x\tan x}$$

(B) $\underset{x\to\infty}{Lt}\sum\limits_{r=1}^{n-1}\dfrac{\sqrt[3]{r}}{n.\sqrt[3]{n}}=\int_0^1\sqrt[3]{x}\ \ dx$

(C) $\underset{x\to 0}{Lt}\frac{e^x\sin x-x-x^2}{x^3\frac{\tan^3 x}{x^3}}$

$\underset{x\to 0}{Lt}\frac{e^x\sin x-x-x^2}{x^3\frac{\tan^3 x}{x^3}}$

$\underset{x\to 0}{Lt}\frac{e^x\cos x+e^x\sin x-1-2x}{3x^2}$

$\underset{x\to 0}{Lt}\frac{e^x\cos x-e^x\sin x+e^x\sin x+e^x\cos x-2}{6x}$ = $\underset{x\to 0}{lt}\frac{2e^x\cos x-2}{6x}$

$\frac{2e^x\cos x-2\ e^x\sin x}{6}=\frac{1}{3}$

(D) $\underset{x\to 0}{Lt}\frac{-2\sin\frac{\sin x+x}{2}.\sin\frac{\sin x-x}{2}}{\frac{\sin x+x}{2}.\frac{\sin x-x}{2}}.\frac{\sin^2 x-.x^2}{4x^4}$

$\underset{x\to 0}{Lt}\ -2.\frac{\sin 2x-2x}{4.4x^3}$

$\underset{x\to 0}{Lt}\ \frac{-2}{4}.\frac{2\cos 2x-2}{4.3x^2}=\underset{x\to 0}{Lt}\ \frac{1}{12}.\frac{1-\cos 2x}{x^2}$

$=\frac{1}{12}.\frac{2^2}{2}=\frac{1}{6}=L$

$\therefore 4L=\frac{2}{3}$

5. Match the following.

	Column – I		Column – II
A	If $L=\underset{x\to -1}{Lt}\frac{\sqrt[3]{7-x}-2}{x+1}$ then $12L=$	P	-2
B	If $L=\underset{x\to\frac{\pi}{4}}{Lt}\frac{\tan^3 x-\tan x}{\cos\left(x+\frac{\pi}{4}\right)}$ then -L/4 =	Q	2
C	If $L=\underset{x\to 1}{Lt}\frac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3}$ then 20L =	R	1
D	If $L=\underset{x\to\infty}{Lt}\frac{\log x^n-[x]}{[x]}$ where $n\in N$.([x] denotes g.i.f then -2L =	S	-1

Key. (A) S; (B) R; (C) P; (D) Q

Sol. A) $L = \underset{x\to -1}{Lt}\frac{\sqrt[3]{7-x-2}}{x+1} = \underset{h\to 0}{Lt}\frac{(8-h)^{1/3}-2}{h}$ (put $x+1=h$, as $x\to -1\Rightarrow h\to 0$)

$$= \underset{x\to 0}{Lt}\frac{2\left[1-\frac{h}{8}\right]^{\frac{1}{3}}-2}{h} = 2\left[\underset{x\to 0}{Lt}\frac{\left(1-\frac{1}{3},\frac{h}{8}\right)-l}{h}\right]$$

$$= 2\left(-\frac{1}{24}\right) = \frac{-1}{12}$$

$$\therefore L = \frac{-1}{12}$$

$$\Rightarrow 12L = -1$$

B) $L = \underset{x\to\frac{\pi}{4}}{Lt}\frac{\tan^3 x-\tan x}{\cos\left(x+\frac{\pi}{4}\right)} = \underset{x\to\frac{\pi}{4}}{Lt}\tan x\frac{(\tan x-1)(\tan x+1)}{\cos\left(x+\frac{\pi}{4}\right)}$

$$= \underset{x\to\frac{\pi}{4}}{Lt}\frac{\tan x(\sin x-\cos x)(\tan x+1)}{\cos x\cos\left(x+\frac{\pi}{4}\right)} = -\sqrt{2}\ \underset{x\to\frac{\pi}{4}}{Lt}\frac{\tan x\left[\cos x\frac{1}{2}-\sin x\frac{1}{2}\right](\tan x+1)}{\cos x\left(\cos\left(x+\frac{\pi}{4}\right)\right)}$$

$$= \sqrt{2}\ \underset{x+\frac{\pi}{4}}{Lt}\frac{\tan x\cos\left(x+\frac{\pi}{4}\right)(\tan x+1)}{\cos x.\cos\left(x+\frac{\pi}{4}\right)}$$

$$= -\sqrt{2}\times 2\times\sqrt{2} = -4$$

$$\therefore L = -4 \Rightarrow \frac{-L}{4} = 1$$

C) $L = \underset{x\to 1}{Lt}\frac{(2x-3)\left(\sqrt{x}-1\right)}{(2x+3)\left(\sqrt{x}+1\right)\left(\sqrt{x}-1\right)} = \frac{-1}{10}$

$$20L = -2$$

D) $L = \underset{x\to\infty}{Lt}\frac{\log x^n-[x]}{[x]} = \underset{x\to\infty}{Lt}\frac{\log x^n}{[x]} - \underset{x\to\infty}{Lt}\frac{[x]}{[x]}$

= 0-1 = - 1

$$L = -1$$

$$\therefore -2L = 2$$

6.

COLUMN-I		COLUMN-II	
(A)	$f(x) = x\ \text{sgn}\ (x-1)$	(p)	$\lim\limits_{x\to 1} f(x)$ doesn't exist

(B)	$f(x)=\dfrac{\sin\left(\sin\left(\tan\left(x^2/2\right)\right)\right)}{\log\cos 3x}$	(q)	$\lim\limits_{x\to 0} f(x)$ doesn't exist
(C)	$f(x)=\dfrac{\sqrt[3]{1+\tan^{-1}3x}-\sqrt[3]{1-\sin^{-1}3x}}{\sqrt{1-\sin^{-1}2x}-\sqrt{1+\tan^{-1}2x}}$	(r)	$\lim\limits_{x\to 0} f(x)=-1/9$
(D)	$f(x)=\dfrac{e^{1/x}-1}{e^{1/x}+1}$	(s)	$\lim\limits_{x\to 0} f(x)=-1$

Key. A-p; B-r; C-s; D-q

Sol. For function f in (A)

$f(x)=\begin{cases} x, & x>1 \\ 0, & x=1 \\ -x, & x<1 \end{cases}$ so $\lim\limits_{x\to 1} f(x)$ does not exist.

$$\lim_{x\to 0}\frac{\sin\left(\sin\left(\tan\left(x^2/2\right)\right)\right)}{\log\cos 3x}$$

$$=\lim_{x\to 0}\frac{\sin\left(\sin\left(\tan\left(x^2/2\right)\right)\right)}{\sin\left(\tan x^2/2\right)}\times\frac{\sin\left(\tan x^2/2\right)}{\tan x^2/2}\times\frac{\tan\left(x^2/2\right)}{x^2/2}\times\frac{x^2/2}{\log\cos 3x}$$

$$=\frac{1}{2}\lim_{x\to 0}\frac{2x}{-3\tan 3x}=\frac{-1}{9}$$

$$\lim_{x\to 0}\frac{\sqrt[3]{1+\tan^{-1}3x}-\sqrt[3]{1-\sin^{-1}3x}}{\sqrt{1-\sin^{-1}2x}-\sqrt{1+\tan^{-1}2x}}$$

$$=\lim_{x\to 0}\frac{\left(1+\frac{1}{3}\tan^{-1}3x+.......\right)-\left(1-\frac{1}{3}\sin^{-1}3x+....\right)}{\left(1-\frac{1}{2}\sin^{-1}2x-.....\right)-\left(1+\frac{1}{2}\tan^{-1}2x+.....\right)}$$

$$=\frac{\frac{1}{3}\frac{\tan^{-1}3x}{x}+\frac{1}{3}\frac{\sin^{-1}3x}{x}+\frac{1}{x}(higher\ power\ of\ \sin^{-1}3x\ and\ \tan^{-1}3x)}{-\frac{1}{2}\frac{\sin^{-1}2x}{x}-\frac{1}{2}\frac{\tan^{-1}2x}{x}+\frac{1}{x}\left(higher\ power\ of\ \sin^{-1}2x\ and\ \tan^{-1}2x\right)}$$

$$=\frac{1+1}{-1-1}=-1$$

$\lim\limits_{x\to 0-}\dfrac{e^{1/x}-1}{e^{1/x}+1}=-1$ and $\lim\limits_{x\to 0+}\dfrac{e^{1/x}-1}{e^{1/x}+1}=1$.

7. Match the following:-

COLUMN-I		COLUMN-II	
(A)	$\lim_{x\to 3}\frac{(x^3+27)\log(x-2)}{x^2-9}=$	(p)	12
(B)	$\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)^{\left(\frac{x}{x+1-e^x}\right)}=$	(q)	8
(C)	If $\lim_{x\to 0}\frac{x(a+\cos x)-b\sin x}{x^3}=1$ then *a* and *b* are respectively	(r)	9
(D)	If $f(x)$ is a thrice differentiable function such that $\lim_{x\to 0}\frac{f(4x)-3f(3x)+3f(2x)-f(x)}{x^3}=12$, then $f'''(0)$ is equal to	(s)	e^{-1}

Key. A-r; B-s; C-q, r; D-p

Sol. (A) By L' Hospital Rule

$$\lim_{x\to 3}\frac{(x^3+27)\frac{1}{x-2}+\log(x-2).3x^2}{2x}=\frac{54}{6}=9$$

(B) $L=\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)^{\left(\frac{x}{x+1-e^x}\right)}$

$$=\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)^{\left(\frac{1}{1-\frac{e^x-1}{x}}\right)}$$

Put $\frac{e^x-1}{x}=t$. As $x\to 0, t\to 1$.

$\therefore L=\lim_{t\to 1}t^{\frac{1}{1-t}}$ $\quad\left[1^\infty\ form\right]$

Taking logarithm,

$\log L=\lim_{t\to 1}\frac{1}{1-t}\log t\quad\left(\frac{0}{0}\ form\right)$

$=\lim_{x\to 1}\frac{\frac{1}{t}}{-1}$ (by L. Hospital's rule) $=-1\Rightarrow L=e^{-1}$.

(C) We know that $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ....$$

$$\therefore L = \lim_{x \to 0} \frac{ax + x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ....\right) - b\left(x - \frac{x^3}{3!} + ...\right)}{x^3}$$

$$= \lim_{x \to 0} (a - b + 1) \times \frac{1}{x^2} + \left(\frac{b}{3!} - \frac{1}{2!}\right)$$

+ terms containing *x*

As L = 1, we must have

$a - b + 1 = 0$ and $\frac{b}{3!} - \frac{1}{2!} = 1$

$\Rightarrow b = 9$ and $a = 8$.

(D) $L = \lim_{x \to 0} \frac{-f(x) + 3f(2x) - 3f(3x) + f(4x)}{x^3} \left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{x \to 0} \frac{-f'(x) + 6f'(2x) - 9f'(3x) + 4f'(4x)}{3x^2}$$

$$= \lim_{x \to 0} \frac{-f''(x) + 12f''(2x) - 27f''(3x) + 16f''(4x)}{6x} \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{-f'''(x) + 24f'''(2x) - 81f'''(3x) + 64f'''(4x)}{6} \left(\frac{0}{0} \text{ form}\right)$$

$$= \frac{6.f'''(0)}{6} = f'''(0).$$

But $L = 12 \Rightarrow f'''(0) = 12$.

8. Match the following:-

Column – I	Column – II
a) $\underset{n \to \infty}{Lt} \left(\frac{1}{n} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + ... + \frac{1}{8n}\right)$	p) 1/24
b) $\underset{n \to \infty}{Lt} \left(\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{4n}\right)$	q) $\pi/2$
c) $\underset{n \to \infty}{Lt} \sum_{r=1}^{3n} \left(\frac{n^2}{(3n+r)^3}\right)$	r) $\log 4$
d) $\underset{n \to \infty}{Lt} \left(\frac{1}{\sqrt{2n - 1^2}} + \frac{1}{\sqrt{4n - 2^2}} + \frac{1}{\sqrt{6n - 3^2}} + ... + \frac{1}{n}\right)$	s) 3/8

Key. a) s b) r c) p d) q

Sol. Conceptual