Functions

Single Correct Answer Type

1. Types of Functions

1. $f:\mathbf{R}\to\mathbf{R}, f(x)=x|x|$ is

1) one-one but not onto
2) onto but not one-one
3) Both one-one and onto
4) neither one-one nor onto

Key. 3

Sol. Give that $f(x)=\begin{cases} x^2 \text{ if } x>0 \\ 0 \text{ if } x=0 \\ -x^2 \text{ if } x<0 \end{cases}$

2. Let $f:\left[0,\sqrt{3}\right]\to\left[0,\frac{\pi}{3}+\log_e 2\right]$ defined by $f(x)=\log_e\sqrt{x^2+1}+\tan^{-1}x$ then $f(x)$ is

A) one – one and onto
B) one – one but not onto
C) onto but not one – one
D) neither one – one nor onto

Key. A

Sol. $f'(x)=\frac{x+1}{x^2+1}\Rightarrow f(x)$ is increasing in $\left[0,\sqrt{3}\right]$

3. If $f:N\to N$ is defined by $f(n)=n-(-1)^n$, then

(A) *f* is one-one but not onto
(B) *f* is both one-one and onto
(C) *f* is neither one-one nor onto
(D) *f* is onto but not one-one

Key. B

Sol. This function *f* maps

$1\to2,\ 2\to1$
$3\to4,\ 4\to3$
$5\to6,\ 6\to5$

i.e., $2m-1\to2m$ and $2m\to2m-1$

So *f* is one-one and onto.

4. Given A={x,y,z}, B={u,v,w}, the function $f:A\to B$ defined by $f(x)=u, f(y)=v, f(z)=w$ is

1) Surjective
2) bijective
3) injective
4) all of the above

Key. 4

Sol. Conceptual

2. Domain & Range

6. The domain of $\sqrt{\sin(\cos x)}$

1) $\left[2n\pi, 2n\pi+\frac{\pi}{2}\right], n\in I$
2) $\left[2n\pi+\frac{\pi}{2}, 2n\pi+\pi\right], n\in I$
3) $\left[2n\pi+\pi, 2n\pi+\frac{3\pi}{2}\right], n\in I$
4) $\left[2n\pi-\frac{\pi}{2}, 2n\pi+\frac{\pi}{2}\right], n\in I$

Key. 4

Sol. F(x) is defined when $\sin(\cos x) \ge 0$

$$\cos x \ge \sin^{-1} 0 \Rightarrow \cos x \ge 0$$

X lies on I and IV quadrant

$$2n\pi - \frac{\pi}{2} \le x \le 2n\pi + \frac{\pi}{2},\ n \in I$$

7. The domain of the function $f(x) = \sin^{-1}\left(\log_2\left(\frac{x^2}{2}\right)\right)$ is

1) $[-2,\ 2]$ 2) $[-2, -1]$ 3) $[1, 2]$ 4) $[-2, -1] \cup [1, 2]$

Key. 4

Sol.

$$f(x) = \sin^{-1}\left(\log_2\left(\frac{x^2}{2}\right)\right) \in \mathbf{R} \Leftrightarrow -1 \le \log_2\left(\frac{x^2}{2}\right) \le 1 \Leftrightarrow \frac{1}{2} \le \frac{x^2}{2} \le 2 \Leftrightarrow 1 \le x^2 \le 4 \Leftrightarrow x \in [-2, -1] \cup [1, 2]$$

8. The domain of definition of the function, f(x) given by the equation $2^x + 2^y = 2$ is

(A) $0 < x \le 1$ (B) $0 \le x \le 1$ (C) $-\infty < x \le 0$ (D) $-\infty < x < 1$

Key. D

Sol. It is given that $2^x + 2^y = 2 \forall x,\ y \in R$

Therefore, $2^x = 2 - 2^y < 2 \Rightarrow 0 < 2^x < 2$

Taking log for both side with base 2.

$\Rightarrow \log_2 0 < \log_2 2^x < \log_2 2$

Hence domain is $-\infty < x < 1$.

9. The domain of the function $f(x) = \frac{1}{x} + \sin^{-1} x + \frac{1}{\sqrt{x-2}}$ is

1) $[-1,1] \setminus \{0\}$ 2) $[-1,1]$ 3) $(-1,0)$ 4) ϕ

Key. 4

Sol. $x \ne 0,\ -1 \le x \le 1,\ x - 2 > 0$

10. If $f : \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = \frac{\sin[x]\pi + \tan[x]\pi}{1 + [x]^2}$, then the range of f= (where [x] denotes integral part of x)

1) $[-1,1]$ 2) $\{-1,1\}$ 3) $\{1\}$ 4) $\{0\}$

Key. 4

Sol. $[x] = n \in Z \Leftrightarrow \sin[x]\pi = \tan[x]\pi = 0$

11. The range of $f(x) = \frac{3}{5 + 4\sin 3x}$ is

1) $\left[\frac{1}{3}, 3\right]$ 2) $\left[\frac{1}{3}, 1\right]$

3) $[1,3]$ 4) $\left(-\infty,\frac{1}{3}\right)\cup(3,\infty)$

Key. 1

Sol. $-1\le\sin 3x\le 1$

12. Let $f:R\to[0.\frac{\pi}{2})$ be defined by $f(x)=Tan^{-1}(x^2+x+a)$. Then the set of values of a for which f is onto is

1) $[0,\infty)$ 2) $[\frac{1}{4},\infty)$ 3) $[\frac{1}{4},(-\infty,\frac{1}{4}]$ 4) $\left\{\frac{1}{4}\right\}$

Key. 4

Sol. $x^2+x+a=0$ has a real solution

$\Rightarrow 1-4a\ge 0$

13. The range of $x^2+4y^2+9z^2-6yz-3xz-2xy$ is

1) $\varnothing$ 2) R 3) $[0,\infty)$ 4) $(-\infty,0)$

Key. 3

Sol. $x^2+4y^2+9z^2-6yz-3xz-2xy=(x)^2+(2y)^2+(3z)^2-(2y)(3z)-(x)(3z)-(x)(2y)\ge 0$

$\therefore$ Range = $[0,\infty)$.

14. The range of $\frac{x^2-x+1}{x^2+x+1}$ is

1) $\left[\frac{1}{3},3\right]$ 2) $\left[\frac{1}{3},1\right]$

3) $[1,3]$ 4) $(-\infty,\frac{1}{3}]\cup[3,\infty)$

Key. 1

Sol. Let $y=\frac{x^2-x+1}{x^2+2x+7}\Rightarrow yx^2+yx+y=x^2-x+1\Rightarrow(y-1)x^2+(y+1)x+(y-1)=0$

$x\in R\Rightarrow$ Discriminant $\ge 0\Rightarrow(y+1)^2-4(y-1)^2\ge 0\Rightarrow -3y^2+10y-3\ge 0$

$\Rightarrow 3y^2-10y+3\le 0\Rightarrow(3y-1)(y-3)\le 0\Rightarrow\frac{1}{3}\le y\le 3$

Range = $\left[\frac{1}{3},3\right]$

15. The range of $|x-2|+|x-5|$ is

1) $[2,\infty)$ 2) $[3,\infty)$ 3) $[4,\infty)$ 4) $[5,\infty)$

Key. 2

Sol. $f(x)=|x-2|+|x-5|$ and domain f=R

For x<2, $f(x)=2-x+5-x=7-2x>3;$

For 2<x<5, $f(x)=x-2+5-x=3;$

For x>5, $f(x)=x-2+x-5=2x-7>3;$

Range $f=[3,\infty)$

16. The range of the function $f(x) = {}^{7-x}P_{x-3}$ is

1) $\{1,2,3\}$ 2) $\{1,2,3,4,5\}$ 3) $\{1,2,3,4\}$ 4) $\{1,2,3,4,5,6\}$

Key. 1

Sol. f(x) is defined $\Rightarrow x-3 \ge 0, x-3 \le 7-x \Rightarrow x \ge 3, 2x \le 10 \Rightarrow 3 \le x \le 5 \Rightarrow x = 3\,or\,4\,or\,5$

Range = $\{f(3), f(4), f(5)\} = \{{}^4P_0, {}^3P_1, {}^2P_2\} = \{1,3,2\}$

17. The range of $\sin^{-1}x - \cos^{-1}x$ is

1) $\left[\frac{-3\pi}{2}, \frac{\pi}{2}\right]$ 2) $\left[\frac{-5\pi}{2}, \frac{\pi}{3}\right]$ 3) $\left[\frac{-3\pi}{2}, \pi\right]$ 4) $\left[0, \frac{\pi}{2}\right]$

Key. 1

Sol. $\sin^{-1}x - \cos^{-1}x = \frac{\pi}{2} - \cos^{-1}x - \cos^{-1}x = \frac{\pi}{2} - 2\cos^{-1}x$

$0 \le \cos^{-1}x \le \pi \Rightarrow 0 \le 2\cos^{-1}x \le 2\pi \Rightarrow -2\pi \le -2\cos^{-1}x \le 0 \Rightarrow \frac{-3\pi}{2} \le \frac{\pi}{2} - 2\cos^{-1}x \le \frac{\pi}{2}$

$\therefore$ Range = $\left[\frac{-3\pi}{2}, \frac{\pi}{2}\right]$

18. The range of the function $f(x) = \frac{2+x}{2-x}, x \ne 2$ is

1) R 2) R-{-1} 3) R-{1} 4) R-{2}

Key. 2

Sol. $y = \frac{2+x}{2-x} \Rightarrow 2y - yx = 2 + x \Rightarrow x(y+1) = 2y - 2 \Rightarrow x = \frac{2y-2}{y+1} \Rightarrow f^{-1}(x) = \frac{2x-2}{x+1}$

$\therefore$ Range = f = Domain $f^{-1} = R - \{-1\}$

19. The domain of $f(x) = \frac{3}{4-x^2} + \log_{10}(x^3 - x)$ is

1) (1, 2) 2) $(-1,0) \cup (1,2)$

3) $(-1,0) \cup (2,\infty)$ 4) $(-1,0) \cup (1,2) \cup (2,\infty)$

Key. 4

Sol. $f(x) = \frac{3}{4-x^2} + \log_{10}(x^3 - x)$ is defined $\Rightarrow 4 - x^2 \ne 0, x^3 - x > 0 \Rightarrow x \ne \pm 2, (x+1)x(x-1) > 0$

$\therefore$ Domain = $(-1,0) \cup (1,2) \cup (2,\infty)$

20. The domain of $\frac{\sqrt{2+x} + \sqrt{2-x}}{x}$ is

1) [-2, 2] 2) (-2, 2) 3) $[-2,0) \cup (0,2]$ 4) $R - \{0\}$

Key. 3

Sol. $\frac{\sqrt{2+x} + \sqrt{2-x}}{x}$ is defined $\Rightarrow 2 + x \ge 0, x - x \ge 0, x \ne 0 \Rightarrow x \ge -2, x \le 2, x \ne 0$

$\therefore$ Domain = $[-2,0)\,\mathrm{U}\,(0,2]$

21. The domain of the function $f(x)=\left[9^{x}+27^{\frac{2}{3}(x-2)}-219-3^{2(x-1)}\right]^{1/4}$

A) $[-3,3]$ B) $[3,\infty)$ C) $\left[\frac{5}{2},\infty\right)$ D) $[0,1]$

Key. C

Sol. We must have

$9^{x}+27^{\frac{2}{3}(x-2)}-219-3^{2(x-1)}\geq 0$

$\left(3^{x}\right)^{2}+3^{2(x-2)}-219-3^{2x-2}\geq 0$

$\left(3^{2x}\right)+\frac{3^{2x}}{81}-219-\frac{3^{2x}}{9}\geq 0$

$\left(1+\frac{1}{81}-\frac{1}{9}\right)3^{2x}\geq 219$

$\frac{73}{81}\times 3^{2x}\geq 219$

$3^{2x}\geq 3\times 81=3^{5}$

$2x\geq 5$

$x\geq\frac{5}{2}$

$\Rightarrow$ Domain is $\left[\frac{5}{2},\infty\right)$.

22. The domain of the function $f(x)=\sqrt{10-\sqrt{x^{4}-21x^{2}}}$ is

(A) $[5,\infty)$ (B) $\left[-\sqrt{21},\sqrt{21}\right]$

(C) $\left[-5,-\sqrt{21}\right]\cup\left[\sqrt{21},5\right]\cup\{0\}$ (D) $(-\infty,-5]$

Key. C

Sol. We must have $x^{4}-21x^{2}\geq 0$ and $10-\sqrt{x^{4}-21x^{2}}\geq 0$

$\Rightarrow x^{2}\left(x^{2}-21\right)\geq 0$ ----(1)

and $100\geq x^{4}-21x^{2}$ ------- (2)

(1) gives $x=0$ or $x\leq-\sqrt{21}$ or $x\geq\sqrt{21}$

(2) $\Rightarrow x^{4}-21x^{2}-100\leq 0$

$\Rightarrow\left(x^{2}-25\right)\left(x^{2}+4\right)\leq 0$

$\Rightarrow x^{2}-25\leq 0$ (as $x^{2}+4>0$ always)

$\Rightarrow -5\leq x\leq 5$

Domain is given by $\left[-5,-\sqrt{21}\right]\cup\left[\sqrt{21},5\right]$ and $x=0$.

23. $f(x)=\frac{x^{2}-3x+4}{x^{2}+3x+4}$ then range of $f(x)$ is

1) $\left[0,\frac{1}{7}\right]$ 2) $\left(-\infty,\frac{1}{7}\right)U\left(7,\infty\right)$ 3) ¡ 4) $\left[\frac{1}{7},7\right]$

Key. 4

Sol. $y=\frac{x^2-3x+4}{x^2+3x+4}$

$yx^2+3xy+4y=x^2-3x+4$

$x^2(y-1)+3x(y+1)+4(y-1)=0$

Dis $1\geq 0 \Rightarrow 9(y+1)^2-4x4(y-1)^2\geq 0$

$(3(y+1)-4(y-1)(3(y+1)4-4(y-1)\geq 0$

$(-y+7)(7y-1)\geq 0\backslash$

$(y-7)\left(y-\frac{1}{7}\right)\leq 0$

$\frac{1}{7}\leq y\leq 7$

24. If $2f(\sin x)+f(\cos x)=x\forall x\in$ ¡ then range of $f(x)$ is

1) $\left[\frac{-\pi}{3},\frac{\pi}{3}\right]$ 2) $\left[\frac{-2\pi}{3},\frac{\pi}{3}\right]$ 3) $\left[\frac{-2\pi}{3},\frac{\pi}{6}\right]$ 4) $\left[\frac{-\pi}{6},\frac{\pi}{6}\right]$

Key. 2

Sol. Put $x=\sin^{-1}x$

$2f(x)+f\left(\sqrt{1-x^2}\right)=\sin^{-1}x \ \rightarrow(1)$

$x=\cos^{-1}x$

$\Rightarrow 2f\left(\sqrt{1-x^2}\right)+f(x)=\cos^{-1}x \ \rightarrow(2)$

(1) x (2) $\Rightarrow 4f(x)+2f\left(\sqrt{1-x^2}\right)=2\sin^{-1}x$

$f(x)+2f\left(\sqrt{1-x^2}\right)=\cos^{-1}x$

$3f(x)=2\sin^{-1}x-\cos^{-1}x$

$f(x)=\frac{2}{3}\sin^{-1}x-\frac{1}{3}\left(\frac{\pi}{3}-\sin^{-1}x\right)$

$=\sin^{-1}x-\frac{\pi}{6}$

$f_{\max}=\frac{\pi}{2}-\frac{\pi}{6}=\frac{\pi}{3},\ f_{\min}=-\frac{\pi}{2}-\frac{\pi}{6}=\frac{-4\pi}{6}=\frac{-2\pi}{3}$

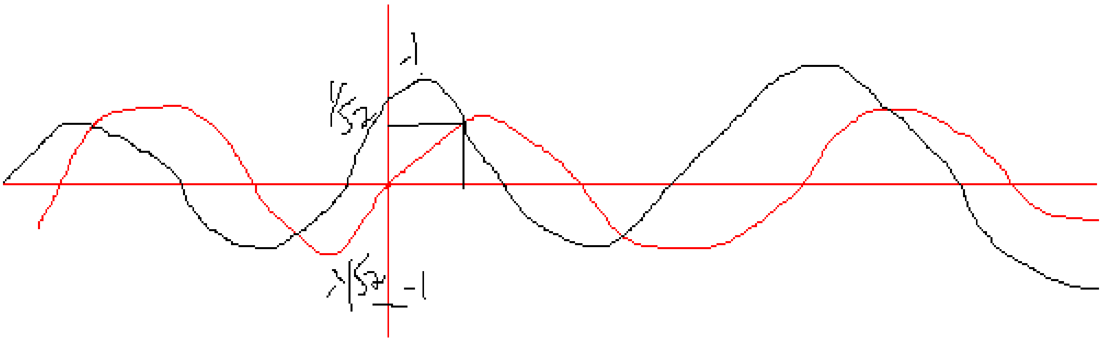
$=\left[\frac{-2\pi}{3},\frac{\pi}{3}\right]$

25. $f(x)=Max\{\sin x,\cos x\}\ \forall x\in$ ¡ then Range of $f(x)$ is.

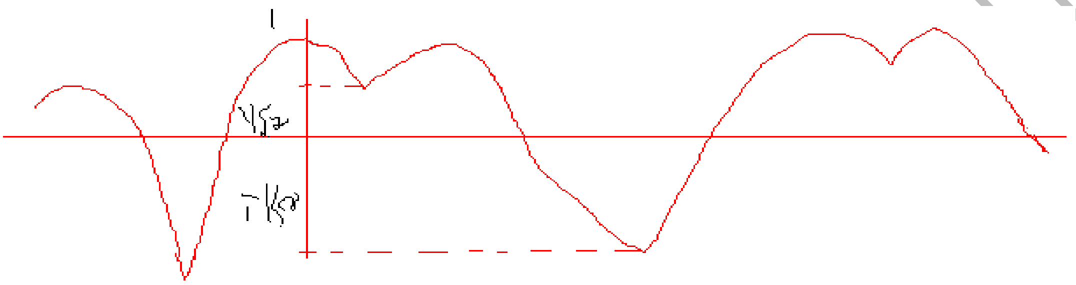
1) $\left[\frac{-1}{\sqrt{2}},1\right]$ 2) $\left[\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right]$ 3) $\left[-1,1\right]$ 4) ϕ

Key. 1

Sol.



$f(x) = \max\{\sin x, \cos x\}$



Required range = $\left[-\frac{1}{\sqrt{2}}, 1\right]$

26. The range of $f(x) = \tan^{-1}(x^2 + x + a) \forall x \in \mathbb{R}$ is a subset of $[0, \frac{\pi}{2})$ then range of a is

1) $\mathbb{R}$ 2) $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ 3) $\left[-\sqrt{3}, -1\right]$ 4) $[\frac{1}{4}, \infty)$

Key. 4

Sol. $\tan^{-1}(x^2 + x + a) \ge 0 \Rightarrow x^2 + x + a \ge 0$

$\Rightarrow disc \le 0 \Rightarrow 1 - 4a \le 0 \Rightarrow a \ge \frac{1}{4}$

$\Rightarrow a \in [\frac{1}{4}, \infty)$

27. The domain of the function $f(x) = \frac{1}{x - [x]}$.

(A) N (B) $(0, \infty)$ (C) $R - \{0, \pm 1, \pm 2, \pm 3, ...\}$ (D) $R - N$

Key. C

Sol. Observe that when x is an integer x = [x]. Hence, f(x) is not defined when x is an integer. Domain is R excluding $0, \pm 1, \pm 2$, …..

28. Domain of the function f(x) = $\log_2\left(\log_4\left(\log_2\left(\log_3(x^2 + 4x - 23)\right)\right)\right)$ is

(A) (–8, 4) (B) $(-\infty, -8) \cup (4, \infty)$

(C) $(-4, 8)$ (D) $(-\infty, -4) \cup (8, \infty)$

Key. B

Sol. The given function is defined when

$\log_2 \log_3 (x^2 + 4x - 23) > 1$

i.e., when $\log_3 (x^2 + 4x - 23) > 2$

i.e., when $x^2 + 4x - 23 > 3^2$

i.e., when $x^2 + 4x - 32 > 0$

i.e., when $x < -8$ or $x > 4$

29. Domain of the function $f(x) = \sqrt{5|x| - x^2 - 6}$ is

(A) $(-\infty, 2) \cup (3, \infty)$ (B) $[-3, -2] \cup [2, 3]$ (C) $(-\infty, -2) \cup (2, 3)$ (D) $R - \{-3, -2, 2, 3\}$

Key. B

Sol. $5|x| - x^2 - 6 \geq 0$ $\Rightarrow$ $x^2 - 5|x| + 6 \leq 0$

when $x < 0$, $x^2 + 5x + 6 \leq 0$, $-3 \leq x \leq -2$

when $x > 0$, $x^2 - 5x + 6 \leq 0$, $2 \leq x \leq 3$

x = 0 will not satisfy the condition.

Domain is $[-3, -2] \cup [2, 3]$.

30. Range of the function $y = \dfrac{2^x - 2^{-x}}{2^x + 2^{-x}}$ is

(A) R (B) (–1, 1) (C) [–1, 1] (D) (0, 1)

Key. B

Sol. $2^x + 2^{-x}$ is always > 0 i.e., domain is R

$y = \dfrac{2^x - 2^{-x}}{2^x + 2^{-x}} = \dfrac{2^{2x} - 1}{2^{2x} + 1}$

$\Rightarrow \dfrac{1+y}{1-y} = \dfrac{2.2^{2x}}{2}$ (Componendo Dividendo)

$= 2^{2x} > 0$

$\Rightarrow \dfrac{1+y}{1-y} > 0$ i.e., $\dfrac{(1+y)^2}{1-y^2} > 0$

$\Rightarrow 1 - y^2 > 0$ $\Rightarrow$ $-1 < y < 1$

31. The range of the function $f(x) = \dfrac{x+3}{|x+3|}$, $x \neq -3$ is

(A) {3, –3} (B) R – {–3} (C) all positive integers (D) {–1, 1}

Key. D

Sol. f(x) = 1 when x + 3 > 0

f(x) = –1 when x + 3 < 0

Range = {–1, 1}

32. The range of the function $f(x) = \cos^2\frac{x}{4} + \sin\frac{x}{4},\ x \in R$ is

(A) $\left[0, \frac{5}{4}\right]$ (B) $\left[1, \frac{5}{4}\right]$ (C) $\left(-1, \frac{5}{4}\right)$ (D) $\left[-1, \frac{5}{4}\right]$

Key. D

Sol. $f(x) = 1 - \sin^2\frac{x}{4} + \sin\frac{x}{4} = -\left\{\sin^2\frac{x}{4} - \sin\frac{x}{4}\right\} + 1 = -\left\{\left(\sin\frac{x}{4} - \frac{1}{2}\right)^2 - \frac{1}{4}\right\} + 1$

$= \frac{5}{4} - \left(\sin\frac{x}{4} - \frac{1}{2}\right)^2$

Maximum f(x) $= \frac{5}{4}$

Minimum f(x) $= \frac{5}{4} - \left(-1 - \frac{1}{2}\right)^2 = \frac{5}{4} - \frac{9}{4} = -1$

Range of f(x) $= \left[-1, \frac{5}{4}\right]$

33. The domain of the function $f(x) = \log_e(x^2 + x + 1) + \sin\sqrt{x-1}$ is

(A) (–2, 1) (B) $(-2, \infty)$ (C) $[1, \infty)$ (D) None of these

Key. C

Sol. We must have $x - 1 \ge 0$.

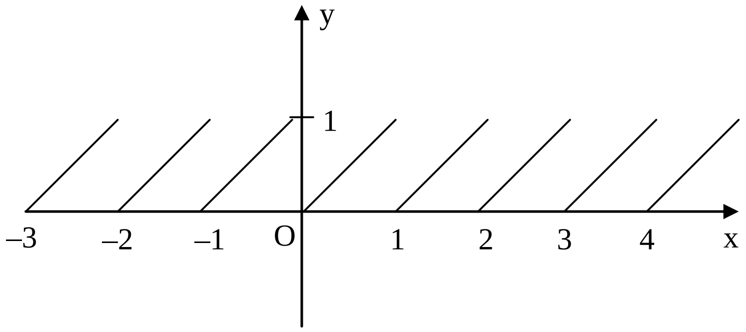
Note that $(x^2 + x + 1)$ is always positive combining, the domain is $[1, \infty)$.

34. Let $f(x) = \frac{x - [x]}{1 + x - [x]},\ x \in R$, where [ ] denotes the greatest integer function. Then, the range of f is

(A) (0, 1) (B) $\left[0, \frac{1}{2}\right)$ (C) [0, 1] (D) $\left[0, \frac{1}{2}\right]$

Key. B

Sol. The graph of y = x – [x] is as shown below



When x is an integer, x – [x] = 0

Hence, f(x) = 0 when x is an integer

$x \to [x]$ as x tends to an integer.

As $x \to 1, \frac{x}{1+x} \to \frac{1}{2}$

Hence, the range of f(x) is $\left[0, \frac{1}{2}\right)$.

35. Let $f(x) = [x]\cos\left(\frac{\pi}{[x+2]}\right)$ where, [ ] denotes the greatest integer function. Then, the domain of f is

(A) $x \in R$, x not an integer (B) $(-\infty, -2) \cup [-1, \infty)$

(C) $x \in R, x \neq -2$ (D) $(-\infty, -1]$

Key. B

Sol. $[x+2] \neq 0$

$[x] + 2 \neq 0$

$[x] \neq -2$

x should not belong to [–2, –1)

Domain of f is $(-\infty, -2) \cup [-1, \infty)$.

36. Range of $f(x) = \frac{\tan\left(\pi[x^2 - x]\right)}{1+\sin(\cos x)}$ is (where [x] denotes the greatest integer function)

(A) $(-\infty, \infty) \sim [0, \tan 1]$ (B) $(-\infty, \infty) \sim [\tan 2, 0)$

(C) $[\tan 2, \tan 1]$ (D) {0}

Key. D

Sol. $f(x) = \frac{\tan\left(\pi[x^2 - x]\right)}{1+\sin(\cos x)}$ = {0} because of $[x^2 - x]$ is integer.

37. Range of the function $f(x) = x^2 + \frac{1}{x^2+1}$, is

(A) $[1, \infty)$ (B) $[2, \infty)$ (C) $\left[\frac{3}{2}, \infty\right)$ (D) $(-\infty, \infty)$

Key. A

Sol. $f(x) = x^2 + 1 + \frac{1}{x^2+1} - 1$

$x^2 + 1 + \frac{1}{x^2+1} \geq 2$ [$\because$ AM $\geq$ GM]

$x^2 + \frac{1}{x^2+1} \geq 1$

$\therefore f(x) \in [1, \infty)$

38. If $f(x)=\ln\left(\frac{x^2+e}{x^2+1}\right)$, then range of f(x) is

(A) (0, 1) (B) (0, 1] (C) [0, 1) (D) {0, 1}

Key. B

Sol. $f(x)=\ln\left(\frac{x^2+e}{x^2+1}\right)=\ln\left(\frac{x^2+1-1+e}{x^2+1}\right)=\ln\left(1+\frac{e-1}{x^2+1}\right)$

Clearly range is (0, 1]

Hence (B) is correct answer.

39. The inverse of $f(x)=\left(5-(x-8)^5\right)^{\frac{1}{3}}$ is

(A) $5-(x-8)^5$ (B) $8+(5-x^3)^{1/5}$

(C) $8-(5-x^3)^{1/5}$ (D) $\left(5-(x-8)^{1/5}\right)^3$

Key. B

Sol. Let y = f(x) = $\left(5-(x-8)^5\right)^{1/3}$, then

$y^3 = 5-(x-8)^5 \quad \Rightarrow \quad (x-8)^5 = 5-y^3$

$\Rightarrow$ x = 8 + $(5-y^3)^{1/5}$

Let, z = $g(x)=8+(5-x^3)^{1/5}$

Now, $f\left(g(x)\right)$ = $\left[5-(x-8)^5\right]^{1/3}$

= $\left(5-\left[(5-x^3)^{1/5}\right]^5\right)^{1/3} = (5-5+x^3)^{1/3}=x$

Similarly, we can show that $g\left(f(x)\right)=x$.

Hence, g(x) = 8 + $\left(5-x^3\right)^{1/5}$ is the inverse of f(x).

40. The range of the function $f(x)=|x-1|+|x-2|+|x+1|+|x+2|$ where, $x\in[-2, 2]$ is

(A) [6, 8] (B) [2, 4] (C) [0, 4] (D) {1, 2}

Key. A

Sol. $f(x)=|x-1|+|x-2|+|x+1|+|x+2|$

when $x\in[-2, -1]$

f(x) $= -(x-1)-(x-2)-(x+1)+x+2$ = –2x + 4

when $x\in[-1, 1]$, f(x) = $-(x-1)-(x-2)+x+1+x+2$

$= -x+1-x+2+x+1+x+2=6$

when $x\in[1, 2]$, f(x) = $(x-1)-(x-2)+x+1+x+2$ = 2x + 4

Plotting the graph of the function, range of f(x) = [6, 8]

41. Range of the function $f(x) = \frac{x^2 + x + 2}{x^2 + x + 1};\ x \in R$ is

(A) $(1, \infty)$ (B) $\left(1, \frac{11}{7}\right]$ (C) $\left(1, \frac{7}{3}\right]$ (D) $\left[1, \frac{7}{5}\right]$

Key. C

Sol. We have, f(x) $= \frac{x^2 + x + 2}{x^2 + x + 1} = \frac{(x^2 + x + 1) + 1}{x^2 + x + 1} = 1 + \frac{1}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}}$

We can see here that as $x \to \infty$, f(x) $\to$ 1 which is the min value of f(x). Also f(x) is max when $\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$ is min which is so when x = $-\frac{1}{2}$ and then $\frac{3}{4}$.

$\therefore$ $f_{max} = 1 + \frac{1}{3/4} = \frac{7}{3}$

$\therefore$ $R_j = \left(1, \frac{7}{3}\right]$

42. Domain of the function $f(x) = \sqrt{\log_e \frac{1}{|\sin x - 1|}}$ is

(A) $n\pi + (-1)^n \alpha$ where n is any integer and $\alpha \in \left[0, \frac{\pi}{2}\right)$

(B) $n\pi + (-1)^n \frac{\pi}{2}$, n = 1, 2, 3, ... (C) $2n\pi - \alpha$ where $\alpha \in \left(0, \frac{\pi}{2}\right)$, n any integer

(D) $\frac{(2n+1)\pi}{2}$, n any integer

Key. A

Sol. $|\sin x - 1| \neq 0$ …(i)

$\sin x \neq 1$

$|\sin x - 1| \leq 1$

$-1 \leq \sin x - 1 \leq 1$

$0 \leq \sin x \leq 2$ …(ii)

From (i) and (ii), $\sin x \in [0, 1)$

$\sin x \in [0, 1)$

$\sin x = 0 \to x = n\pi$

$\sin x = 1 \to x = n\pi + (-1)^n \frac{\pi}{2}$

$\Rightarrow$ Domain of f(x) is

x $= n\pi$ (n any integer)

$\sin x \le 1$

$x \in \left[0, \frac{\pi}{2}\right)$

General solution is

. $x = n\pi + (-1)^n \alpha$

where, $\alpha \in \left[0, \frac{\pi}{2}\right)$.

43. If the range of $f(x) = 2+\sqrt[3]{x}, -3 \le x < -1$ is $\left[0, \sqrt[3]{n}\right]$ where $n \in N$ then n=

$= x^{\frac{2}{3}} \quad , -1 \le x \le 2$

(A) 1 (B) 2
(C) 4 (D) 6

Key. C

Sol. The given function has local maximum at x= -1 ,minimum at x=0 and F(0) = 0,F(-1) =1,

F(-3) = $2-\sqrt[3]{3}$ $f(2) = 2^{\frac{2}{3}} = \sqrt[3]{4}$

$\therefore$ range of f(x) =[0, $\sqrt[3]{4}$]

44. If $2f(\sin x) + f(\cos x) = x \forall x \in \mathbb{R}$ then range of $f(x)$ is

1) $\left[\frac{-\pi}{3}, \frac{\pi}{3}\right]$ 2) $\left[\frac{-2\pi}{3}, \frac{\pi}{3}\right]$ 3) $\left[\frac{-2\pi}{3}, \frac{\pi}{6}\right]$ 4) $\left[\frac{-\pi}{6}, \frac{\pi}{6}\right]$

Key. 2

Sol. Put $x = \sin^{-1} x$

$2f(x) + f\left(\sqrt{1-x^2}\right) = \sin^{-1} x \to (1)$

$x = \cos^{-1} x$

$\Rightarrow 2f\left(\sqrt{1-x^2}\right) + f(x) = \cos^{-1} x \to (2)$

(1) x (2) $\Rightarrow 4f(x) + 2f\left(\sqrt{1-x^2}\right) = 2\sin^{-1} x$

$f(x) + 2f\left(\sqrt{1-x^2}\right) = \cos^{-1} x$

$3f(x) = 2\sin^{-1} x - \cos^{-1} x$

$f(x) = \frac{2}{3}\sin^{-1} x - \frac{1}{3}\left(\frac{\pi}{3} - \sin^{-1} x\right)$

$= \sin^{-1} x - \frac{\pi}{6}$

$f_{\max} = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}, \; f_{\min} = -\frac{\pi}{2} - \frac{\pi}{6} = \frac{-4\pi}{6} = \frac{-2\pi}{3}$

$= \left[\frac{-2\pi}{3}, \frac{\pi}{3}\right]$

45. Range of $f\left(x\right)=\tan^{-1}\left[\frac{2}{\pi}\left(2\tan^{-1}x-\sin^{-1}x+\cot^{-1}x-\cos^{-1}x\right)\right]$ contains

(A) Only one integer
(B) More than 2 integers
(C) Only two integers
(D) No integer

Key. A

Sol. $y=\tan^{-1}\left(\frac{2}{\pi}\tan^{-1}x\right),\ -1\le x\le 1$

$-\frac{\pi}{4}\le\tan^{-1}x\le\frac{\pi}{4}$

$-\frac{1}{2}\le\frac{2}{\pi}\tan^{-1}x\le\frac{1}{2}$

$-\tan^{-1}\frac{1}{2}\le\tan^{-1}\left(\frac{2}{\pi}\tan^{-1}x\right)\le\tan^{-1}\left(\frac{1}{2}\right)$

y = 0, is only integer hence one integer

46. If $f\left(x\right)=\sqrt{\cos(\sin x)}+\sqrt{\sin(\cos x)}$, then the range of $f(x)$ is

(A) $\left[\sqrt{\cos 1},\ \sqrt{\sin 1}\right]$
(B) $\left[\sqrt{\cos 1}, 1+\sqrt{\sin 1}\right]$
(C) $\left[1-\sqrt{\cos 1},\ \sqrt{\sin 1}\right]$
(D) $\left[\sqrt{\cos 1},1\right]$

Key. B

Sol. Period of f(x) is 2π, but f(x) is not defined for $x\in(\pi/2, 3\pi/2)$. Hence it suffices to consider $x\in[-\pi/2, \pi/2]$. Further since f(x) is even, we consider $x\in[0, \pi/2]$.

Now $\sqrt{\cos(\sin x)}$ and $\sqrt{\sin(\cos x)}$ are decreasing functions for $x\in[\pi,\pi/2]$.

$\Rightarrow R_f=\left[f(\pi/2),\ f(0)\right]=\left[\sqrt{\cos 1},\ 1+\sqrt{\sin 1}\right]$

47. The range of $f\left(x\right)=-x^3+x^2-x+\cos^{-1}x,$ is

A) $\left[-1,3+\pi\right]$
B) $\left[0,\pi-1\right]$
C) $\left[-1,2+\pi\right]$
D) $\left[-1,\pi\right]$

Key. A

Sol. $f\left(x\right)=-x^3+x^2-x+\cos^{-1}x$

Domain $=\left[-1,1\right]$

$f'\left(x\right)=-3x^2+2x-1-\frac{1}{\sqrt{1-x^2}}<0$

'f' is a decreasing function

$\therefore$ Min of $f\left(x\right)$ is $f\left(1\right)=-1+1-1+0=-1$

Max of f(x) is $f\left(-1\right)=1+1+1+\pi=3+\pi$

Range $=[-1,3+\pi]$

48. The domain of the function

$f(x)=\log_e\{\text{sgn}(9-x^2)\}+\sqrt{[x]^3-4[x]}$ where [.] = G.I.F

A) $[-2,1)\cup[2,3)$ B) $[-4,1)\cup[2,3)$

C) $[4,1)\cup[2,3)$ D) $[2,1)\cup[2,3)$

Key. A

Sol. Given $f(x)=\log_e\{\text{sgn}(9-x^2)\}+\sqrt{[x]^3-4[x]}=y_1+y_2(say)$

Now, y_1 is defined if $\text{sgn}(9-x^2)>0$

But sgn x = 1(i.e. > 0) if x > 0

$\therefore \text{sgn}(9-x^2)>0 \Rightarrow 9-x^2>0 \Rightarrow x^2-9<0 \Rightarrow (x-3)(x+3)<0 \Rightarrow -3<x<3$ ...(A)

Again, y_2 is defined if $[x]^3-4[x]\ge 0 \Rightarrow [x]\{[x]^2-4\}\ge 0 \Rightarrow [x]([x]-2)([x]+2)\ge 0$.

Following the wavy curve method, we find

Thus $[x]\ge 2$ *or* $[x]$ lies between – 2 and 0, i.e. [x] = - 2, - 1 or 0

Now, $[x]\ge 2 \Rightarrow x\ge 2$ ...(B)

[x] = - 2 $\Rightarrow$ - 2 $\le$ x < 1

[x] = - 1 $\Rightarrow$ - 1 $\le$ x < 0

[x] = 0 $\Rightarrow$ 0 $\le$ x < 1.

Hence [x] = - 2, - 1, 0 $\Rightarrow -2\le x<1$

$\therefore (B)\cup(C)=(x\ge 2)$ *or* $(-2\le x<1)$ ...(C)

Hence $D_f=(A)\cup(C)=[-2,1)\cup[2,3)$.

49. The Range of the function

$f(x)=\log_{10}\left\{\sin^{-1}\left(\sqrt{x-5}\right)+\frac{3\pi}{2}\right\}$ is

A) $\left[\log\frac{3\pi}{2},\log 2\pi\right]$ B) $\left[\log\frac{3\pi}{2},\log 3\pi\right]$

C) $\left[\log\frac{3\pi}{2},\log \pi\right]$ D) $\left[\log\frac{3\pi}{4},\log 2\pi\right]$

Key. A

Sol. Let $f(x)=\log_{10}\left\{\sin^{-1}\left(\sqrt{x-5}\right)+\frac{3\pi}{2}\right\}$.

The function is defined if (i) $x-5\ge 0$ (ii) $-1\le\sqrt{x-5}\le 1$ and

(iii) $\sin^{-1}\left(\sqrt{x-5}\right)+\frac{3\pi}{2}>0$.

Now (i) $\Rightarrow x\ge 5$

(ii) $\Rightarrow 0\le x-5\le 1 \Rightarrow 6\le x\le 6$.

(iii) is satisfied by virtue of (ii).

Hence, considering (i) and (ii), we find that the domain of the function viz. $D_f = [5,6]$.

Let $y_1 = \sin^{-1}\left(\sqrt{x-5}\right)$ and $y_2 = \sin^{-1}\left(\sqrt{x-5}\right) + \frac{3\pi}{2}$ so that $y = \log_{10}\left(y_2\right)$ where $y_2 = y_1 + \frac{3\pi}{2}$

Now, for y_1 since $x \in [5,6], y_1 \ge 0$ so that $0 \le y_1 \le \frac{\pi}{2}\left(\because -\frac{\pi}{2} \le \sin^{-1}(z) \le \frac{\pi}{2}\right)$

Consequently $0 + \frac{3\pi}{2} \le y_1 + \frac{3\pi}{2} \le \frac{\pi}{2} + \frac{3\pi}{2} \Rightarrow \frac{3\pi}{2} \le y_2 \le 2\pi$

$\Rightarrow \quad \log\left(\frac{3\pi}{2}\right) \le \log\left(y_2\right) \le \log(2\pi)$, since u = log z is an increasing function

$\Rightarrow \quad \log\left(\frac{3\pi}{2}\right) \le \log\left(y_2\right) \le \log(2\pi)$.

Hence the range of f(x) is $\left[\log\frac{3\pi}{2}, \log 2\pi\right]$.

50. The domain of $f(x) = \sqrt{x-2-2\sqrt{x-3}} - \sqrt{x-2+2\sqrt{x-3}}$, is

A) $[3,5]$ B) $(3,5)$ C) $[5,\infty)$ D) $[3,\infty)$

Key. D

Sol. $x-3 \ge 0 \Rightarrow x \ge 3$

$x-2-2\sqrt{x-3} \ge 0$ For $x \ge 3$

$\Rightarrow x-2 \ge 2\sqrt{x-3}$ and $x-2+2\sqrt{x-3} \ge 0$

$\Rightarrow x^2 - 8x + 16 \ge 0 \Rightarrow (x-4)^2 \ge 0 \;\forall\; x \in R$

Domain $= [3,\infty)$

51. Minimum value of function $f(x) = x^3(x^3+1)(x^3+2)(x^3+3) : x \in R,$ is

(A) –2 (B) –1 (C) 1 (D) none

Key. B

Sol. Let $t = x^3(x^3+3); t = (x^3+\frac{3}{2})^2 - \frac{9}{4} \in [-\frac{9}{4},\infty)$

$f(x) = g(t) = t(t+2) = (t+1)^2 - 1$ is least when t = –1

and $-1 \in [-\frac{9}{4},\infty) \therefore$ min $f(x) = -1$

52. The domain of the function $f(x) = \sqrt{[x]^2 - 6[x] + 8}$ where [.] = G. I. F

A) (- 4, 4) B) $(-\infty,3) \cup [4,\infty)$ C) (3, 4) D) $(3,4) \cup (5,\infty)$ Key. B

Sol. (i) The function is defined if sin x $-\frac{1}{2} \ge 0$

$\Rightarrow \sin x \ge \frac{1}{2} \Rightarrow x \in \left[\frac{\pi}{6}, \frac{5\pi}{6}\right] \Rightarrow x \in \left[2n\pi + \frac{\pi}{6}, 2n\pi + \frac{5\pi}{6}\right]$

(ii) The function is defined if $-1 \le \frac{1}{|x-1|} - 2 \le 1;\ x \ne 1$

$\Rightarrow \quad 1 \le \frac{1}{|x-1|} \le 3 \Rightarrow \frac{1}{|x-1|} \ge 1$ ...(1)

And $\frac{1}{|x-1|} \le 3$ ...(2)

(1) $\Rightarrow \quad |x-1| \le 1 \Rightarrow -1 \le x-1 \le 1 \Rightarrow 0 \le x \le 2$ ...(A)

(2) $\Rightarrow \quad |x-1| \ge \frac{1}{3} \Rightarrow -\frac{1}{3} \le x-1 \le \frac{1}{3} \Rightarrow \frac{2}{3} \le x \le \frac{4}{3}$ ...(B)

Combining (A) and (B), we find that $x \in \left[0, \frac{2}{3}\right] \cup \left[\frac{4}{3}, 2\right]$ with is the domain of the given function.

53. The domain of the function of $f(x) = \log_{[x]}\{\text{sgn}(x^2)\}$

(where [.] G.I.F) is

A) $[2, \infty)$ B) (- 2, 2) C) $(-\infty, 2)$ D) None

Key. A

Sol. (i) f(x) is defined if (i) (4 - |x|) > 0 (iii) $[x^2] > 0$ but $[x^2] \ne 1$

Now, (i) $\Rightarrow |x| < 4 \Rightarrow -4 < x < 4$ ...(A)

From (iii), $[x^2] > 0 \Rightarrow [x^2] = 1, 2, 3, ...$

But $[x^2] \ne 1$.

$\therefore \quad [x^2] = 2, 3, 4, ...\ i.e.\ [x^2] \ge 2$

$\Rightarrow \quad x^2 \ge 2; Q[f(x)] \ge n \Rightarrow f(x) \ge n$.

$\Rightarrow \quad x \le -\sqrt{2}\ \ or\ \ x \ge \sqrt{2}$

Combining (A) and (B), we find that $-4 < x \le -\sqrt{2}\ \ or\ \ \sqrt{2} \le x < 4$. ...(B)

Hence the domain of the given function is $(-4, -\sqrt{2}] \cup [\sqrt{2}, 4)$.

(ii) The function is defined if (*i) sgn $(x^2) > 0$ and (ii) [x] > 0 but [x] $\ne$ 1.

We know that sgn $(x^2) = \begin{cases} 1\ if\ \ x^2 > 0 \\ 0\ if\ \ x = 0 \\ -1\ if\ \ x^2 < 0 \end{cases}$

(i) Since sgn (x^2) is non-negative, we have $x^2 > 0 \Rightarrow x \in R - \{0\}$. ...(A)

(ii) $\Rightarrow$ [x] = 2, 3, 4, ... $\therefore x \in [2, \infty)$ ...(B)

Hence, $D_f = A \cap B = [2, \infty)$.

54. The domain of the function

$f(x) = \log_{10}\{1 - \log_{10}(x^2 - 5x + 10)\}$ is

A) $(0,\infty)$ B) (0, 5) C) $(-\infty,0)$ D) None

Key. B

Sol. (a) The function f(x) is defined if (i) $x^2-5x+10>0$, (ii) $1-\log_{10}\left(x^2-5x+10\right)>0$

Now, (ii) $\Rightarrow \log_{10}\left(x^2-5x+10\right)<1 \Rightarrow x^2-5x+10<10$

$\Rightarrow x^2-5x<0 \Rightarrow x(x-5)<0 \Rightarrow 0<x<5$ ...(A)

Again, $x^2-5x+10>0$ for all x, ...(B)

Since the discriminant of the corresponding equation $x^2-5x+10=0$ is negative, so that the roots of the equation are imaginary.

Combining (A) and (B), we find that the domain of f(x) is (0, 5).

(b) The function g(x) is defined if (i) $(x-4)^2>0$, (ii) $\log_4(x-4)^2>0$

(iii) $\log_3\left\{\log_4(x-4)^2\right\}>0$

(i) is true for all x. ...(A)

(ii) is true if $(x-4)^2>1 \Rightarrow x^2-8x+15>0 \Rightarrow (x-3)(x-5)>0 \Rightarrow x<3 \text{ or } x>5$ ...(B)

(iii) is true if $\log_4(x-4)^2>1 \Rightarrow (x-4)^2>4 \Rightarrow x^2-8x+12>0$

$\Rightarrow$ (x – 2)(x – 6) > 0 $\Rightarrow$ x < 2 or x > 6 ...(C)

Hence combining (A), (B) and (C), we find that the domain of g(x) is $(-\infty,2)\cup(6,\infty)$.

55. The domain of the function

$f(x)=\log_e\left\{\text{sgn}\left(9-x^2\right)\right\}+\sqrt{[x]^3-4[x]}$ where [x] = G.I.F

A) $[-2,1)\cup[2,3]$ B) $[-4,1)\cup[2,3]$ C) $[4,1)\cup[2,3]$ D) $[2,1)\cup[2,3]$

Key. A

Sol. Given $f(x)=\log_e\left\{\text{sgn}\left(9-x^2\right)\right\}+\sqrt{[x]^3-4[x]}=y_1+y_2\,(say)$

Now, y_1 is defined if $\text{sgn}\left(9-x^2\right)>0$

But sgn x = 1(i.e. > 0) if x > 0

$\therefore \text{sgn}\left(9-x^2\right)>0 \Rightarrow 9-x^2>0 \Rightarrow x^2-9<0 \Rightarrow (x-3)(x+3)<0 \Rightarrow -3<x<3$ ...(A)

Again, y_2 is defined if $[x]^3-4[x]\geq 0 \Rightarrow [x]\left\{[x]^2-4\geq 0 \Rightarrow [x]\left([x]-2\right)\geq 0\right.$.

Following the wavy curve method, we find

Thus $[x]\geq 2$ *or* $[x]$ lies between – 2 and 0, i.e. [x] = - 2, - 1 or 0

Now, $[x]\geq 2 \Rightarrow x\geq 2$ ...(B)

[x] = - 2 $\Rightarrow$ - 2 $\leq$ x < 1

[x] = - 1 $\Rightarrow$ - 1 $\leq$ x < 0

[x] = 0 $\Rightarrow$ 0 $\leq$ x < 1.

Hence [x] = - 2, - 1, 0 $\Rightarrow -2\leq x<1$

$\therefore (B)\cup(C)=(x\geq 2)$ *or* $(-2\leq x<1)$ ...(C)

Hence $D_f=(A)\cup(C)=[-2,1)\cup[2,3)$.

56. The range of the function $f(x)=3x^4-4x^3-12x^2+1$ is

A) $[31,\infty)$ B) $[-31,\infty)$ C) $[3,\infty)$ D) $[-3,\infty)$

Key. B

Sol. Given that $y=f(x)=3x^4-4x^3-12x^2+1$.

It cuts the y-axis at the point (x = 0, x = 1).

Differentiating, we get $\frac{dy}{dx}=12x^3-12x^2-24x$

i.e. $\frac{dy}{dx}=12x\left(x^2-x-2\right)=12x(x-2)(x+1)$.

Now, $\frac{dy}{dx}=0\Rightarrow x(x-2)(x+1)=0\Rightarrow x=0,2,-1$

Also, $\frac{dy}{dx}>0\Rightarrow x(x-2)(x+1)>0$.

Using wavy-curve method, we have

Thus $\frac{dy}{dx}>0$ when x > 2 or $x\in(-1,0)$.

Similarly, $\frac{dy}{dx}<0$ when 0 < x < 2 or x < - 1.

Hence the graph of the curve will be as follows:

At x = 2, f(x) = 3 × 16 – 4 × 8 – 12 × 4 + 1 = 48 – 32 – 48 + 1 = - 31.

At x = - 1, f(x) = 3. 1 + 4. 1 – 12.1 + 1 = - 4.

$\therefore$ The least value of the function is – 31.

Hence the range of the function is $[-31,\infty)$.

57. The range of the function $f(x)=\sqrt{e^{\cos^{-x}\left(\log_4^{x^2}\right)}}$ is

A) $\left[1,\sqrt{e^{\pi}}\right]$ B) $\left[4,\sqrt{e^{\pi}}\right]$ C) $\left[2,\sqrt{e^{\pi}}\right]$ D) $\left[3,\sqrt{e^{\pi}}\right]$

Key. A

Sol. (iii) Given that $y(x)=\sqrt{e^{\cos^{-1}\left(\log_4 x^2\right)}}$.

The function is defined if (i) $x^2>0$ which is true for all x (ii) $-1\le\log_4 x^2\le 1$.

Now, (ii) $\Rightarrow 4^{-1}\le x^2\le 4\Rightarrow\frac{1}{4}\le x^2\le 4\Rightarrow x\in\left[\frac{1}{2},2\right] or x\in\left[-2,-\frac{1}{2}\right]$.

Hence the domain of the function is $\left[-2,-\frac{1}{2}\right]\cup\left[\frac{1}{2},2\right]$.

To find out the range, let $y_1=\log_4 x^2$ so that $y=\sqrt{e^{\cos^{-1}(y_1)}}$.

Again, let $y_2=\cos^{-1}(y_1)$.

$\therefore y=\sqrt{e^{y_2}}$ where $y_2=\cos^{-1}(y_1)$ and $y_1=\log_4 x^2$.

Now, for $x=\frac{1}{2}\left(or-\frac{1}{2}\right)y_1=\log_4\left(\frac{1}{4}\right)=\log_4\left(4^{-1}\right)=-1$

And for x = 2 (or – 2), $y_1 = \log_4(4) = 1$.

Hence y_1 lies between -1 and 1 i.e. $-1 \le y_1 \le 1 \Rightarrow \cos^{-1}(-1) \ge \cos^{-1}(y_1) \ge \cos^{-1}(1)$

$\Rightarrow \pi \ge y_2 \ge 0 \Rightarrow 0 \le y_2 < \pi$.

Again $0 \le y_2 \le \pi \Rightarrow e^{y_2} \le e^{\pi} \Rightarrow 1 \le e^{y_2} \le e^{\pi} \Rightarrow 1 \le \sqrt{e^{y_2}} \le \sqrt{e^{\pi}} \Rightarrow 1 \le y \le \sqrt{e^{\pi}}$.

Hence the range of the function is $\left[1, \sqrt{e^{\pi}}\right]$.

58. The Range of the function

$f(x) = \log_{10}\left\{\sin^{-1}\left(\sqrt{x-5}\right) + \frac{3\pi}{2}\right\}$ is

A) $\left[\log\frac{3\pi}{2}, \log 2\pi\right]$ B) $\left[\log\frac{3\pi}{2}, \log 3\pi\right]$ C) $\left[\log\frac{3\pi}{2}, \log \pi\right]$ D) $\left[\log\frac{3\pi}{4}, \log 2\pi\right]$ Key. A

Sol. Let $f(x) = \log_{10}\left\{\sin^{-1}\left(\sqrt{x-5}\right) + \frac{3\pi}{2}\right\}$.

The function is defined if (i) $x - 5 \le 0$ (ii) $-1 \le \sqrt{x-5} \le 1$ and (iii) $\sin^{-1}\left(\sqrt{x-5}\right) + \frac{3\pi}{2} > 0$.

Now (i) $\Rightarrow x \ge 5$

(ii) $\Rightarrow 0 \le x - 5 \le 1 \Rightarrow 6 \le x \le 6$.

(iii) is satisfied by virtue of (ii).

Hence, considering (i) and (ii), we find that the domain of the function viz. $D_f = [5,6]$.

Let $y_1 = \sin^{-1}\left(\sqrt{x-5}\right)$ and $y_2 = \sin^{-1}\left(\sqrt{x-5}\right) + \frac{3\pi}{2}$ so that $y = \log_{10}(y_2)$ where $y_2 = y_1 + \frac{3\pi}{2}$

Now, for y_1 since $x \in [5,6], y_1 \ge 0$ so that $0 \le y_1 \le \frac{\pi}{2}\left(Q - \frac{\pi}{2} \le \sin^{-1}(z) \le \frac{\pi}{2}\right)$

Consequently $0 + \frac{3\pi}{2} \le y_1 + \frac{3\pi}{2} \le \frac{\pi}{2} + \frac{3\pi}{2} \Rightarrow \frac{3\pi}{2} \le y_2 \le 2\pi$

$\Rightarrow \quad \log\left(\frac{3\pi}{2}\right) \le \log(y_2) \le \log(2\pi)$, since u = log z is an increasing function

$\Rightarrow \quad \log\left(\frac{3\pi}{2}\right) \le \log(y_2) \le \log(2\pi)$.

Hence the range of f(x) is $\left[\log\frac{3\pi}{2}, \log 2\pi\right]$.

59. The range of the function $f(x) = \cos^{-1}\sqrt{\log_{[x]}\frac{|x|}{x}}$ is where [.] = G.I.F

A) $\left[\frac{\pi}{2}\right]$ B) {0} C) $\{\pi\}$ D) $\{2\pi\}$

Key.

Sol. The function is defined if (i) [x] > 0 and [x] $\ne$ 1 (ii) $\frac{|x|}{x} > 0$

(iii) $\log_{[x]}\frac{|x|}{x} \ge 0$ (iv) $0 \le \sqrt{\log_{[x]}\frac{|x|}{x}} \le 1$.

Now, (i) [x] = 2, 3, ... i.e. $[x] \ge 2 \Rightarrow x \ge 2$ i.e. the domain of the function is $[2, \infty)$.

For this value of x ($\ge$ 2) (ii) is true,

(iii) is also true and $\sqrt{\log_{[x]}\frac{|x|}{x}} = \sqrt{\log_{[x]}1} = 0$.

Hence $f(x) = \cos^{-1}(0) = \frac{\pi}{2}$.

Hence the range of the function is $\left\{\frac{\pi}{2}\right\}$.

60. The range of the function

$f(x) = \sin^{-1}\left[x^2 + \frac{1}{2}\right] + \cos^{-1}\left[x^2 - \frac{1}{2}\right]$ where [.] = G. I. F

A) $\{\pi\}$ B) $\left\{\frac{\pi}{2}\right\}$ C) $\{2\pi\}$ D) {0}

Key. A

Sol. Let $y_1 = \sin^{-1}\left[x^2 + \frac{1}{2}\right]$ *and* $y_2 = \cos^{-1}\left[x^2 - \frac{1}{2}\right]$. Then $y = y_1 + y_2$.

Now, $y_1 = \sin^{-1}\left[x^2 + \frac{1}{2}\right]$ is defined

if $-1 \le \left[x^2 + \frac{1}{2}\right] \le 1 \Rightarrow -1 \le x^2 + \frac{1}{2} < 2 \Rightarrow -\frac{3}{2} \le x^2 < \frac{3}{2}$ ...(1)

Again $y_2 = \cos^{-1}\left[x^2 - \frac{1}{2}\right]$ is defined

If $-1 \le \left[x^2 - \frac{1}{2}\right] \le 1 \Rightarrow -1 \le x^2 - \frac{1}{2} < 2 \Rightarrow -\frac{1}{2} \le x^2 < \frac{5}{2}$ ...(2)

Taking the intersection of (1) and (2), we find that

$-\frac{1}{2} \le x^2 < \frac{3}{2} \Rightarrow 0 \le x^2 < \frac{3}{2}$, since x^2 cannot be negative.

Now, for x^2 so that $\frac{1}{2} \le x^2 + \frac{1}{2} \le 1$ and $-\frac{1}{2} \le x^2 - \frac{1}{2} \le 0$, we have

$$y = \sin^{-1}(0) + \cos^{-1}(-1) = 0 + \pi - \cos^{-1}(1) = 0 + \pi - 0 = \pi.$$

Similarly for $\frac{1}{2} \le x^2 < \frac{3}{2}$, we have $y = \sin^{-1}(1) + \cos^{-1}(0) = \frac{\pi}{2} + \frac{\pi}{2} = \pi$.

Hence the range of the given function is $[\pi]$.

61. Consider the real-valued function satisfying 2f(sin x) + f(cos x) = x. Find the domain and range of f(x).

Sol. Given 2f(sin x) + f(cos x) = x ...(1)

Replacing x by $\frac{\pi}{2} - x$, we have 2f(cos x) + f(sin x) $= \frac{\pi}{2} - x$ ...(2)

(1) + (2) $\Rightarrow 3f(\sin x) + 3f(\cos x) = \frac{\pi}{2} \Rightarrow f(\sin x) + f(\cos x) = \frac{\pi}{6}$ ...(3)

(1) – (3) $\Rightarrow f(\sin x) = x - \frac{\pi}{6} \Rightarrow f(x) = \sin^{-1} x - \frac{\pi}{6}$.

Hence $D_f = [-1,1]$ and $R_f = \left[-\frac{\pi}{2} - \frac{\pi}{6}, \frac{\pi}{2} - \frac{\pi}{6}\right] = \left[-\frac{2\pi}{3}, \frac{\pi}{3}\right]$.]

62. If $f(x) = x^2 + x + \frac{3}{4}$ and $g(x) = x^2 + ax + 1$ be two real functions, then the range of a for which $g(f(x)) = 0$ has no real solution is ______

A) $(-\infty, -2)$ B) $(-2, 2)$ C) $(-2, \infty)$ D) $(2, \infty)$

Key. C

Sol. $f(x) = x^2 + x + \frac{3}{4} = (x + \frac{1}{2})^2 + \frac{1}{2} \geq \frac{1}{2}$

$g(f(x)) = (f(x))^2 + af(x) + 1$, for $g(f(x)) = 0$ $a = -\left(f(x) + \frac{1}{f(x)}\right) \leq -2$

$\therefore$ If $a > -2$, $g(f(x)) = 0$ has no solutions

63. The number of integers in the domain of real function $f(x) = \log_{10} \sin(x-3) - \sqrt{16 - x^2}$ is

A) 4 B) 8 C) 9 D) infinite

Key. A

Sol. The domain of the given function is $(3 - 2\pi, 3 - \pi) \cup (3, 4]$. The integers in the domain are $\{-3, -2, -1, 4\}$

64. if $f(x)$ is a polynomial function such that $|f(x)| \leq 1 \forall x \in R$ and $g(x) = \frac{e^{f(x)} - e^{|f(x)|}}{e^{f(x)} + e^{|f(x)|}}$, then the range of $g(x)$ is

A) [0, 1] B) $\left[0, \frac{e^2+1}{e^2-1}\right]$

C) $\left[0, \frac{e^2-1}{e^2+1}\right]$ D) $\left[\frac{1-e^2}{1+e^2}, 0\right]$

Key. D

Sol. For $0 \leq f(x) < 1$ $g(x) = 0$

For $-1 < f(x) < 0$

$g(x) = \frac{e^{2f(x)} - 1}{e^{2f(x)} + 1} \Rightarrow g(x) \in \left[\frac{1-e^2}{1+e^2}, 0\right)$

$\therefore$ range of $g(x) = \left[\frac{1-e^2}{1+e^2}, 0\right]$

65. The domain of definition of the function, f(x) given by the equation $2^x + 2^y = 2$ is

(A) $0 < x \le 1$ (B) $0 \le x \le 1$ (C) $-\infty < x \le 0$ (D) $-\infty < x < 1$

Key. D

Sol. It is given that $2^x + 2^y = 2 \forall x,\ y \in R$

Therefore, $2^x = 2 - 2^y < 2 \Rightarrow 0 < 2^x < 2$

Taking log for both side with base 2.

$\Rightarrow \log_2 0 < \log_2 2^x < \log_2 2$

Hence domain is $-\infty < x < 1$

66. The range of the function f(x) = $\log_{\{x\}}[x]$, where [x] and {x} respectively denote the integral and fractional parts of x, is

(A) $(-\infty, 0]$ (B) $(-\infty, 0)$
(C) $\{\ln k;\ k \in N\}$ (D) $(-\infty, -1)$

Key. A

Sol. For $x \in (1, 2)$, f(x) = 0

for $x \in (2, 3)$, f(x) = $\log_{\{x\}} 2 = \dfrac{1}{\log_2\{x\}} \in (-\infty, 0)$

67. If $f(x) = \sin^{-1}x + \cos^{-1}x + \tan^{-1}x$ then range of the function $f(x)$ is _______

(1) $\left[\dfrac{\pi}{4}, \dfrac{3\pi}{4}\right]$ (2) $[0, \pi]$ (3) $(0, \pi)$ (4) R

Key. 1

Sol. Domain [-1, 1] and Range [f(-1), f(1)]

68. Let R be set of real numbers, the function $f : R^- \to R,\ f(x) = \log_2 \log_2 \left|x + \sqrt{1+x^2}\right|$, then range of f(x) is

(1) ϕ (2) R (3) R+ (4) R–

Key. 1

Sol. for $x < 0$

$0 < x + \sqrt{1+x^2} < 1$

so $\log_2\left(x + \sqrt{x+x^2}\right) < 0$

69. If $e^x + e^{f(x)} = e$, then for f(x)

(1) domain $= (-\infty, 1)$, range $= (-\infty, 1)$

(2) domain $= (-\infty, -1]$, range $= (-\infty, 1]$

(3) domain $= (-\infty, 0]$, range $= (-\infty, 1]$

(4) domain $= (-\infty, 0]$, range $= (-\infty, 3]$

Key. 1

Sol. $e^{f(x)} = e - e^x \Rightarrow f(x) = \log_e\left(e - e^x\right)$

$e - e^x > 0 \Rightarrow e^1 > e^x \Rightarrow x < 1$

$D_f = (-\infty, 1)$

Let $y = f(x) = \log_e(e - e^x) \Rightarrow e^y = e - e^x$

$\Rightarrow \quad e^x = e - e^y \Rightarrow x = \log_e\left(e - e^y\right)$

$\Rightarrow \quad e - e^y > 0 \Rightarrow e^1 > e^y$

$\therefore \quad y < 1$

$R_f = (-\infty, 1)$

70. The domain of $f(x) = \sin\left(\log\left(\dfrac{\sqrt{4-x^2}}{1-x}\right)\right)$ is

(1) $(0,5)$ (2) $(1,5)$ (3) $(-2,1)$ (4) $(2,3)$

Key. 3

Sol. $4 - x^2 > 0$ and $1 - x > 0$

$\therefore -2 < x < 2$ and $x < 1$

71. The range of the function $Y = \left[x^2\right] - [x]^2, x \in [0,2]$ where [.] denotes the integral part, is

(1) $\{0\}$ (2) $\{0,1\}$ (3) $\{1,2\}$ (4) $\{0,1,2\}$

Key. 4

Sol. We have, $y = \left[x^2\right] - [x]^2, \quad x \in [0,2]$

i.e., $y = \left[x^2\right], \qquad 0 \le x < 1$

$y = \left[x^2\right] - 1, \quad 1 \le x < 2$

$= \left[x^2\right] - 1, \quad x = 2$

$= 0 \qquad x = 2$

i.e., y = 0, $\quad 0 \le x < 1$

$= 1 - 1 = 0 \quad 1 \le x < \sqrt{2}$

$= 2 - 1 = 1, \quad \sqrt{2} \le x < \sqrt{3}$

$= 3 - 1 = 2, \quad \sqrt{3} \le x < 2$

$= 0 \qquad x = 2$

Hence, the range is {0, 1, 2 }

72. Let $f(x) = \left(\sin^{-1} x\right)^2 + \left(\cos^{-1} x\right)^2$ then

(1) Greatest value of $f(x)$ is $\dfrac{5\pi^2}{8}$ (2) Greatest value of $f(x)$ is $\dfrac{7\pi^2}{4}$

(3) Least value of $f(x)$ is $\frac{\pi^2}{8}$ (4) Least value of $f(x)$ is $\frac{\pi^2}{12}$

Key. 3

Sol. $f(x)=2\left(\sin^{-1}x\right)^2-\pi\sin^{-1}x+\frac{\pi^2}{4}$

$$=2\left(\sin^{-1}x-\frac{\pi}{4}\right)^2+\frac{\pi^2}{8}$$

$$\Rightarrow f(x)\in\left[\frac{\pi^2}{8},\frac{5\pi^2}{4}\right]$$

73. If [a , b] be the range of $\frac{1}{\pi^2}\left((\cos^{-1}x)^2+(\sin^{-1}x)^2\right)$ then b – a =

A. 1 B. $\frac{9}{8}$ C. $\frac{3}{4}$ D. $\frac{5}{4}$

Key. B

Sol. $(\cos^{-1}x)^2+(\sin^{-1}x)^2=\frac{1}{2}\left\{(\cos^{-1}x+\sin^{-1}x)^2+(\cos^{-1}x-\sin^{-1}x)^2\right\}$

$$=\frac{1}{2}\left\{\left(\frac{\pi}{2}\right)^2+\left(\frac{\pi}{2}-2\sin^{-1}x\right)^2\right\}\geq\frac{\pi^2}{8}$$

$$a=\frac{1}{\pi^2}\left(\frac{\pi^2}{8}\right)=\frac{1}{8}$$

$$b=\frac{1}{2\pi^2}\left\{\frac{\pi^2}{4}+\left(\frac{\pi}{2}+\pi\right)^2\right\}, at\ x=\frac{-\pi}{2}$$

$$=\frac{5}{4}$$

$$\therefore b-a=\frac{9}{8}$$

74. The domain of the function $\sqrt{\log_{10}\left(\frac{5x-x^2}{4}\right)}$ is

A. (0,5) B. (1,4) C. [0,5] D. [1,4]

Key. D

Sol. $\frac{5x-x^2}{4}\geq 1\Rightarrow x^2-5x+4\leq 0\Rightarrow x\in[1,4]$

75. The greatest and least values of $(\sin^{-1}x)^3+(\cos^{-1}x)^3$ are

A. $\frac{\pi^3}{32}, \frac{7\pi^3}{32}$ B. $\frac{7\pi^3}{8}, \frac{\pi^3}{32}$ C. $\frac{-\pi^3}{8}, \frac{7\pi^3}{8}$ D. $\frac{\pi^3}{8}, \frac{\pi^3}{32}$

KEY. B

SOL. $(\sin^{-1} x)^3 + (\cos^{-1} x)^3 = \left(\frac{\pi}{2}\right)^3 - 3\sin^{-1} x \cos^{-1} x \left(\frac{\pi}{2}\right)$

$$= \frac{\pi^3}{8} - \frac{3\pi}{2}\sin^{-1} x\left(\frac{\pi}{2} - \sin^{-1} x\right)$$

$$= \frac{\pi^3}{32} + \frac{3\pi}{2}\left\{\sin^{-1} x - \frac{\pi}{4}\right\}^2$$

$$\min = \frac{\pi^3}{32}, \max = \frac{7\pi^3}{8}$$

3.Odd & Even Functions

76. Let $f(x) = e^x + \sin x$ be defined on the interval [-4, 0], the odd extension of $f(x)$ in the interval [-4, 4]

1) $e^{-x} + \sin x,\ x \in (0,4)$ 2) $-e^{-x} + \sin x,\ x \in (0,4)$

3) $e^{-x} - \sin x,\ x \in (0,4)$ 4) $-e^{-x} - \sin x,\ x \in (0,4)$

Key. 2

Sol. $f(x) = -f(-x)$

77. The function $f(x) = \frac{2x(\sin x + \tan x)}{2\left[\frac{x+21\pi}{\pi}\right] - 41}$ is (where [.] = G.I.F)

A) An odd function. B) An even function

C) Neither even nor odd function D) None of these

Key. A

Sol. The denominator is $= 2\left[\frac{x+21\pi}{\pi}\right] - 41 = 2\left[\frac{x}{\pi} + 21\right] - 41$

$$\therefore f(x) = \frac{x(\sin x + \tan x)}{\left[\frac{x}{\pi}\right] + \frac{1}{2}}$$

$$\Rightarrow f(-x) = \frac{-x\{\sin(-x) + \tan(-x)\}}{\left[-\frac{x}{\pi}\right] + \frac{1}{2}} = \frac{x(\sin x + \tan x)}{-1 - \left[\frac{x}{\pi}\right] + \frac{1}{2}} \ (if\ \ x \neq n\pi)$$

$$= -\frac{x(\sin x + \tan x)}{\left[\frac{x}{\pi}\right] + \frac{1}{2}} = -f(x).\left(if \ \ x = n\pi \quad then \ \ f(x) = 0\right)$$

78. If $f:[-20,20] \to R$ defined by $f(x) = \left[\frac{\pi^2}{a}\right]\sin x + \cos x$ is an even function, then the set of values of 'a' is [.] G.I.F

A) $a \in (500, \infty)$ B) $a \in (400, \infty)$

C) $a \in (-400, 0)$ D) $a \in (-300, 900)$

Key. B

Sol. If $f:[-20,20] \to R$ defined by $f(x) = \left[\frac{x^2}{a}\right]\sin x + \cos x$ is an even function , find the set of values of a; [.] = G. I. F.

[Hint: If f(x) is an even function, then $f(x) = f(-x) \Rightarrow f(x) - f(-x) = 0 \Rightarrow \left[\frac{x^2}{a}\right]\sin x = 0$

$\Rightarrow \left[\frac{x^2}{a}\right] = a \Rightarrow 0 \le \frac{x^2}{a} < 1 \Rightarrow a > x^2 \Rightarrow a > 400$ (since $-20 \le x \le 20$) $\therefore a \in (400, \infty)$.]

79. If $f:[-4,4] - \{-\pi, o, \pi\} \to R$, such that $f(x) = \cot(\sin x) + \left[\frac{x^2}{|a|}\right] + \frac{\sin 2x}{x^2}$,([.] denotes greatest integer function) is an odd function, then the complete set of values of 'a' , is

A) $[-\infty, -4] \cup [4, \infty)$ B) $(-\infty, -16) \cup (16, \infty)$

C) $[-16, -16]$ D) $(-\infty, -16] \cup [16, \infty)$

Key. B

Sol. For $f(x)$ to be odd, $\left[\frac{x^2}{|a|}\right]$ should not depend on value of x.

Since $x \in [-4, 4] \Rightarrow 0 \le x^2 \le 16$

$\Rightarrow \left[\frac{x^2}{|a|}\right] = 0$ if $|a| > 16 \Rightarrow a \in (-\infty, -16) \cup (16, \infty)$

4. Periodic Functions

80. If $f: R \to R$ is a function satisfying the property $f(x+1) + f(x+3) = 2$ for all $x \in R$ than f is _____

(1) Periodic with period 3 (2) Periodic with period 4

(3) non periodic (4) Periodic with period 5

Key. 2

Sol. $f(x+1) = f(x+5)$

81. A real valued function f satisfies $f(10+x) = f(10-x)$ and $f(20-x) = -f(20+x)$, for all $x \in R$ which of the following statements is true?

(1) f is an even function (2) f is an odd function

(3) f is a constant function (4) f is a non-periodic function

Key. 2

Sol. Change x to 10 – x to obtain

$f(20-x)=f(x)$

We have $\mathrm{f}(20-\mathrm{x})=-\mathrm{f}(20+\mathrm{x})$

$\Rightarrow \mathrm{f}(\mathrm{x})=-\mathrm{f}(20+\mathrm{x})$

Now change x to 20 + x

$\mathrm{f}(20+\mathrm{x})=-\mathrm{f}(40+\mathrm{x})$

$-\mathrm{f}(\mathrm{x})=-\mathrm{f}(40+\mathrm{x})$

$\mathrm{f}(\mathrm{x})=\mathrm{f}(40+\mathrm{x})$, so f is periodic

Again $\mathrm{f}(-\mathrm{x})=-\mathrm{f}(20-\mathrm{x})=-\mathrm{f}(\mathrm{x})$

Thus f is odd

82. The period of the function $\mathrm{f}(\mathrm{x})=\frac{1}{2}\left(\frac{|\sin \mathrm{x}|}{\cos \mathrm{x}}+\frac{\sin \mathrm{x}}{|\cos \mathrm{x}|}\right)$ is

(A) π (B) 2π (C) $\frac{\pi}{2}$ (D) $\frac{\pi}{3}$

Key. B

Sol. Since $|\sin \mathrm{x}|$ and $\cos \mathrm{x}$ are periodic function with period π and 2π respectively.

Therefore, $\frac{|\sin \mathrm{x}|}{\cos \mathrm{x}}$ is periodic with period 2π.

Similarly, $\frac{|\cos \mathrm{x}|}{\sin \mathrm{x}}$ is periodic with period 2π.

So, period of f(x) is L.C.M. of $\{2\pi, 2\pi\}=2\pi$.

83. Let $f: R \rightarrow R-\{3\}$ be a function such that for some $p>0$, $f(x+p)=\frac{f(x)-5}{f(x)-3}$ for all $x \in R$.

Then, period of f is

(A) $2p$ (B) $3p$ (C) $4p$ (D) $5p$

Key. C

Sol. 3 does not belong to the range of *f* implies 2 also cannot belong to range of *f* because, if $f(x)=2$ for some $x \in R$. Then $f(x+p)=\frac{2-5}{2-3}=3$ which is not in the range of *f*. Hence 2 and 3 are not in the range of *f*. If $f(x+2p)=f(x)$, this implies

$$f(x)=f(x+p+p)$$
$$=\frac{f(x+p)-5}{f(x+p)-3}$$

$$=\frac{\frac{f(x)-5}{f(x)-3}-5}{\frac{f(x)-5}{f(x)-3}-3}$$

$$=\frac{-4f(x)+10}{-2f(x)+4}=\frac{2f(x)-5}{f(x)-2}$$

so that $\left[f(x)-2\right]^2=-1$ which is absurd. Therefore, $2p$ is not a period. Again

$$f(x+3p)=\frac{2f(x+p)-5}{f(x+p)-2}$$

$$=\frac{3f(x)-5}{f(x)-1}\neq f(x).$$

Now $f(x+4p)=f(x+3p+p)$

$$=\frac{f(x+3p)-5}{f(x+3p)-3}$$

$$=\frac{\frac{3f(x)-5}{f(x)-1}-5}{\frac{3f(x)-5}{f(x)-1}-3}$$

$$=\frac{-2f(x)}{-2}=f(x).$$

Therefore $4p$ is a period.

84. Period of the function f(x) = [x] + [2x] + [3x] + [4x] + … + [nx] – $\frac{n(n+1)x}{2}$, where $n\in N$ and [ ] denotes the greatest integer function, is

(A) 1 (B) n (C) $\frac{1}{n}$ (D) 2n

Key. A

Sol. $f(x)=[x]+[2x]+.....+[nx]-(x+2x+.....+nx) = [x]-x+[2x]-2x+....+[nx]-(nx)$

$= -[\{x\}+\{2x\}+.....+\{nx\}]$

Period of $\{rx\}$ = $\frac{1}{r}$

$\therefore$ Period of f(x) = $\text{LCM}\left(1,\frac{1}{2},\frac{1}{3},......,\frac{1}{n}\right)=1$

85. Which of the following is non-periodic

(A) $\frac{\tan x}{\tan x}$ (B) $\sin\sqrt{x}$ (C) $\cos|x|$ (D) $\frac{\sin x}{\sin x}$

Key. B

Sol. $f(x)=\sin\sqrt{x}$ is non-periodic because f(T) = f(0) = f(–T) is not satisfied.

86. If $f(2+x)=a+\left[1-(f(x)-a)^4\right]^{1/4}$ for all $x\in R$, then f(x) is periodic with period

(A) 1 (B)2 (C) 4 (D) 8

Key. C

Sol. $f(2+x)-a = \{1-[f(x)-a]^4\}^{1/4}$

$\Rightarrow$ $[f(2 + x) – a]^4 = 1 – [f(x) – a]^4$

$[f(2 + x) – a]^4 + [f(x) – a]^4 = 1$ …(i)

(i) is true for all x

Replace x by (x + 2) in (i)

$[f(x + 4) – a]^4 + [f(x + 2) – a]^4 = 1$ …(ii)

(i) and (ii) gives, $f(x) – a]^4 = [f(x + 4) – a]^4$

$\Rightarrow$ f(x + 4) – a = f(x) – a

$\Rightarrow$ f(x + 4) = f(x)

87. Period of f(x) = sgn([x] + [–x]) is equal to
(where [.] denotes greatest integer function)

(A) 1 (B) 2

(C) 3 (D) does not exist

Key. A

Sol. Let f(x) = sgn([x] + [–x])

$$=\begin{cases}0; & x\in I\\ -1; & x\notin I\end{cases}$$

Hence f(x) is periodic with period 1.

88. The period of the function

$$f(x)=\exp\left[x-[x]+\sqrt{x-[x]}+(x-[x])^2\right.$$
$$\left.+|\sin\pi x|+|\cos\pi x|+|\tan\pi x|\right]$$

A) 1 B) 2 C) 3 D) 4

Key. A

Sol. The period of x – [x] is 1.

The period of $\sqrt{x-[x]}$ is 1.

The period of $(x-[x])^2$ is 1.

The period of $|\sin\,\pi x|=\frac{\pi}{\pi}=1$.

The period of $|\cos\pi x|=1$.

The period of $|\tan\pi x|=1$.

Thus each of the above functions is a periodic function with period 1. Therefore their L.C.M. is 1. Hence the function f(x) is periodic with fundamental period = 1.

89. The period of the function $f(x)=\sin 3x\cos[3x]-\cos 3x\sin[3x]$, where $[\cdot]$ denotes the greatest integer function is

(1) 6 (2) 3 (3) 1/3 (4) 1/6

Key. 3

Sol. f(x) = sin 3{x}, where {.} is a fractional part function.

90. If $f(2+x)=a+\left[1-(f(x)-a)^4\right]^{1/4}$ for all $x\in R$, then f(x) is periodic with period

(A) 1 (B) 2 (C) 4 (D) 8

Key. C

Sol. $f(2+x)-a = \{1-[f(x)-a]^4\}^{1/4}$

$\Rightarrow$ $[f(2+x)-a]^4 = 1-[f(x)-a]^4$

$[f(2+x)-a]^4+[f(x)-a]^4 = 1$ …(i)

(i) is true for all x

Replace x by (x + 2) in (i)

$[f(x+4)-a]^4+[f(x+2)-a]^4 = 1$ …(ii)

(i) and (ii) gives, $f(x)-a]^4 = [f(x+4)-a]^4$

$\Rightarrow$ $f(x+4)-a = f(x)-a$

$\Rightarrow$ $f(x+4) = f(x)$

91. The period of the function $f(x)=(-1)^{[x]}$ where [.] = G.I.F

A) 2 B) 1 C) 3 D) 4

Key. A

Sol. Given: $f(x)=(-1)^{[x]}$.

First of all, we sketch the graph of f(x) with the help of piecewise defined functions as follows:

$$f(x)=(-1)^{[x]}=\begin{cases}1; & -2x<-1\\ -1; & -1x<0\\ 1; & 0\le x<1\\ -1: & 1\le x<2\\ 1; & 2\le x<3.\end{cases}$$

The graph of f(x) is given by

From the above graph of f(x), we see that the function f(x) repeats its value after the least interval of 2.

Therefore the function f(x) is periodic with period 2.

92. If $2f(x)+3.f\left(\frac{1}{x}\right)=x^2-1$ then f(x) is _______

(1) Periodic function (2) an even function

(3) an odd function (4) one one function on domain R

Key. 2

Sol. replace x by $\frac{1}{x}$

5. Inverse, Composition Functions

93. If the functions of *f* and *g* are defined by $f(x)=3-x, g(x)=2+3x$ *for* $x\in R$ respectively, then $g^{-1}(f^{-1}(2))=$

A. 1 B. $-\frac{1}{3}$ C. $-\frac{4}{3}$ D. $\frac{1}{4}$

Key. B

Sol. $f^{-1}(x)=3-x, g^{-1}(x)=\frac{x-2}{3}$

$g^{-1}[f^{-1}(2)]=g^{-1}(1)=-\frac{1}{3}$

94. Which among the functions is inverse of itself?

(A) $y=a^{2\log x}$ (B) $y=5^{x^2+2}$ (C) $y=\frac{1+x^2}{1-x^2}$ (D) $y=\frac{1-x}{1+x}$

Key. D

Sol. Out of 4 choices, if f(x) = $\frac{1-x}{1+x}$.

$$f[f(x)] = \frac{1-\frac{(1-x)}{(1+x)}}{1+\frac{(1-x)}{(1+x)}} = x$$

$\therefore$ $\frac{1-x}{1+x}$ is the inverse of itself.

95. The inverse of $f(x)=\left(5-(x-8)^5\right)^{\frac{1}{3}}$ is

(A) $5-(x-8)^5$ (B) $8+(5-x^3)^{1/5}$

(C) $8-(5-x^3)^{1/5}$ (D) $\left(5-(x-8)^{1/5}\right)^3$

Key. B

Sol. Let y = f(x) = $\left(5-(x-8)^5\right)^{1/3}$, then

$y^3 = 5-(x-8)^5 \quad \Rightarrow \quad (x-8)^5=5-y^3$

$\Rightarrow$ x = 8 + $(5-y^3)^{1/5}$

Let, z = $g(x)=8+(5-x^3)^{1/5}$

Now, $f\left(g(x)\right)$ = $\left[5-(x-8)^5\right]^{1/3}$

$= \left(5-\left[(5-x^3)^{1/5}\right]^5\right)^{1/3} = (5-5+x^3)^{1/3}=x$

Similarly, we can show that $g(f(x)) = x$.

Hence, g(x) = 8 + $(5-x^3)^{1/5}$ is the inverse of f(x).

96. If $f(x) = x - x^2 + x^3 - x^4 + ...\infty$ when $|x| < 1$ then $f^{-1}(x) =$

1) $\frac{x}{1-x}$ 2) $\frac{x}{1+x}$ 3) $\frac{1}{1-x}$ 4) $\frac{1}{1+x}$

Key. 1

Sol. $f(x) = x - x^2 + x^3 - x^4 + ... = \frac{x}{1+x}$

$$f^{-1}(x) = t \Rightarrow f(t) = \frac{t}{1+t} \Rightarrow x + xt = t \Rightarrow x = t(1-x) \Rightarrow t = \frac{x}{1-x} \Rightarrow f^{-1}(x) = \frac{x}{1-x}$$

97. If $f:(0,\infty) \to R$ defined by $f(x) = \log_{10}^{x}$ then $f^{-1}(x) =$

1) $\log_x^{10}$ 2) x^{10} 3) 10^x 4) None

Key. 3

Sol. $f^{-1}(x) = y \Rightarrow x = f(y) \Rightarrow x = \log_{10}^{y} \Rightarrow y = 10^x \Rightarrow f^{-1}(x) = 10^x$

98. If $f(x) = (1-x^n)^{1/n}, 0 < x < 1, n$ being an odd positive integer and $h(x) = f(f(x))$, then $h'(1/2)$

A. 2^n B. 2 C. $n.2^{n-1}$ D. 1

Key. D

Sol. $h(x) = (1 - f(x)^n)^{\frac{1}{n}} = (1-(1-x^n)^{\frac{1}{n}} = x \quad \therefore h^1(\frac{1}{2}) = 1$

99. If $f(x) = x - \frac{1}{x}$ then number of solutions of $f(f(f(x)) = 1$.

1) 1 2) 4 3) 6 4) 2

Key. 2

Sol. $f(x) = x - \frac{1}{x}, \quad \Rightarrow f(f(x)) = \frac{x^4 - 3x^2 + 1}{x(x^2-1)}$

$\Rightarrow f(f(f(x))) = 1 \Rightarrow f(f(x)) = f^{-1}(1) = \frac{1+\sqrt{5}}{2}$ $\to$ 2 values exist

Or $f^{-1}(1) = \frac{1-\sqrt{5}}{2}$ $\to$ 2 values exist

100. Which among the functions is inverse of itself?

(A) $y = a^{2\log x}$ (B) $y = 5^{x^2+2}$ (C) $y = \frac{1+x^2}{1-x^2}$ (D) $y = \frac{1-x}{1+x}$

Key. D

Sol. Out of 4 choices, if f(x) = $\frac{1-x}{1+x}$.

$$f[f(x)] = \frac{1-\frac{(1-x)}{(1+x)}}{1+\frac{(1-x)}{(1+x)}} = x$$

$\therefore \ \frac{1-x}{1+x}$ is the inverse of itself.

101. If $f(x)=x(x-1)$ is a function from $\left[\frac{1}{2},\infty\right)$ to $\left[-\frac{1}{4},\infty\right)$, then $\{x \in R / f^{-1}(x)=f(x)\}$ is

(A) null set
(B) {1}
(C) {0, 2}
(D) a set containing 3 elements

Key. C

Sol. $\{x \in R / f^{-1}(x)=f(x)\} = \{x \in R / f\,f(x)=x\}$

$f(f(x)) = f(x(x-1)) = [x(x-1)][x(x-1)-1] = x(x-1)[x^2-x-1]$

$f(f(x)) = x \quad \Rightarrow \quad x(x-1)(x^2-x-1)=x$

$\Rightarrow \quad x(x^3-2x^2) = 0 \quad \Rightarrow \quad x = 0, 2$

102. Let $f(x)=3x^2-7x+c$, where 'c' is a variable co-efficient and $x>\frac{7}{6}$. The value of 'c' such that $f(x)$ touches $f^{-1}(x)$ is……

(A) 6
(B) 7
(C) $\frac{16}{3}$
(D) $\frac{4}{3}$

Key. C

Sol. $f(x)$ and $f^{-1}(x)$ can only intersect on the line $y=x$

$\therefore y=x$ must be tangent

Solving $3x^2-7x+c=x$ p

$\Rightarrow 3x^2-8x+c=0$

The above equation has real and equal roots

$\Rightarrow 64-12c=0$

$c=\frac{16}{3}$

103. Let $f:\left[\frac{-\pi}{3},\frac{2\pi}{3}\right]\to[0,4]$ be a function defined as $f(x)=\sqrt{3}\sin x-\cos x+2$

then $f^{-1}(x)$ is given by

(1) $\sin^{-1}\left(\frac{x-2}{2}\right)-\frac{\pi}{6}$
(2) $\sin^{-1}\left(\frac{x+2}{2}\right)+\frac{\pi}{6}$
(3) $\frac{2\pi}{3}-\cos^{-1}\left(\frac{x-2}{2}\right)$
(4) Does not exist

Key. 3

Sol. $f(x)=2\sin\left(x-\frac{\pi}{6}\right)+2$

Since f is one – one onto

f is invertible

Now $fof^{-1}(x)=x$

$\Rightarrow 2\sin\left(f^{-1}(x)-\frac{\pi}{6}\right)+2=x$

$f^{-1}(x)=\sin^{-1}\left(\frac{x}{2}-1\right)+\frac{\pi}{6}$ $\left(\because\left|\frac{x}{2}-1\right|\le 1\forall x\in[0,4]\right)$

Also using $\sin^{-1}\alpha+\cos^{-1}\alpha=\frac{\pi}{2}$

$f^{-1}(x)=\frac{\pi}{2}-\cos^{-1}\frac{x-2}{2}+\frac{\pi}{6}=\frac{2\pi}{3}-\cos^{-1}\left(\frac{x-2}{2}\right)$

104. The value of the parameter α, for which the function $f(x)=1+\alpha x,\ \alpha\neq 0$ is the inverse of itself, is

(A) –2 (B) –1 (C) 1 (D) 2

Key. B

Sol. $y = 1+\alpha x \Rightarrow x = \frac{y-1}{\alpha}$

$f^{-1}(x) = \frac{x-1}{\alpha} = f(x)=1+\alpha x$

$\Rightarrow \frac{x-1}{\alpha} = 1+\alpha x \quad \Rightarrow x-1 = \alpha+\alpha^2 x$

Equating the coefficient of x

$\alpha^2 = 1$ and $\alpha = -1$

$\alpha = \pm 1$

$\alpha = -1$

6. Functional Equations

105. If f is real function satisfying the relation $f(x+y)=f(x).f(y)$ for all $x,y\in R$ and f(1) = 2 and $a\in N$, for which $\sum_{K=1}^{n} f(a+k)=16(2^n-1)$ then a = _______

(1) 2 (2) 4 (3) 3 (4) 8

Key. 3

Sol. $f(a)=a^n$; $\sum f(a+K)=\sum f(a)f(K)$

$=2^a\sum_{K=1}^{n}2^K=2^a(2^n-1)$

$\therefore a=3$

106. A real valued function f(x) satisfies the functional equation

$f(x-y)=f(x).f(y)-f(a-x)f(a+y)$ for some given constant a and f(0) = 1 then f(2a-x) =

(1) $f(x)$ (2) $-f(x)$ (3) $f(-x)$ (4) $f(a)+f(a-x)$

Key. 2

Sol. Put $x=y=0 \Rightarrow f(a)=0$

$$f(a-x)=f(a-(x-a))=f(a).f(x-a)-f(a-a)$$

107. If $f:R\to R$ is a function satisfying $f(x+y)=f(xy)$ for all $x,y\in R$ and $f\left(\frac{3}{4}\right)=\left(\frac{3}{4}\right)$, then $f\left(\frac{9}{16}\right)=$

1) $\frac{3}{4}$ 2) $\frac{9}{16}$ 3) $\frac{\sqrt{3}}{2}$ 4) 0

Key. 1

Sol. Let f(0)=k, then f(x) = f(x+0) = f(0) = k, f is a constant function. But $f\left(\frac{3}{4}\right)=\left(\frac{3}{4}\right)$

$\therefore f(x)=\left(\frac{3}{4}\right)$ for all x and hence $f\left(\frac{9}{16}\right)=\left(\frac{3}{4}\right)$

108. If for nonzero x, $2f(x^2)+3f\left(\frac{1}{x^2}\right)=x^2-1$, then $f(x^2)=$

1) $\frac{3+2x^4-x^2}{5x^2}$ 2) $\frac{3-2x^4+x^2}{5x^2}$ 3) $\frac{3-2x^4-x^2}{5x^2}$ 4) $\frac{3+2x^4+x^2}{5x^2}$

Key. 3

Sol. $2f(x^2)+3f\left(\frac{1}{x^2}\right)=x^2-1\Rightarrow 4f(x^2)+6f\left(\frac{1}{x^2}\right)=2x^2-2$ ....(1)

$2f\left(\frac{1}{x^2}\right)+3f(x^2)=\frac{1}{x^2}-1\Rightarrow 9f(x^2)+6f\left(\frac{1}{x^2}\right)=\frac{3}{x^2}-3$ ....(2)

$(2)-(1)\Rightarrow 5f(x^2)=\frac{3}{x^2}-2x^2-1\Rightarrow f(x^2)=\frac{3-2x^4-x^2}{5x^2}$

109. If $f:\mathbf{R}\to\mathbf{R}$ satisfies $f(x+y)=f(x)+f(y)$ for all $x,y\in\mathbf{R}$ and f (1) =7, then $\sum_{r=1}^{n}f(r)$ is

1) $\frac{7(n+1)}{2}$ 2) $\frac{7n(n+1)}{2}$ 3) $\frac{7n}{2}$ 4) $7n(n+1)$

Key. 2

Sol. $f(1)=7, f(2)=f(1+1)=f(1)+f(1)=2f(1), f(n)=nf(1)$

110. If f is a real valued function satisfying $f(x)+f(x+6)=f(x+3)+f(x+9)$, then $f(x)=$

1) $f(x+3)$ 2) $f(x+6)$ 3) $f(x+9)$ 4) $f(x+12)$

Key. 4

Sol. Replace x with x+3

111. If $f(x)f\left(\frac{1}{x}\right)=f(x)+f\left(\frac{1}{x}\right)\forall x\in R-\{0\}$, where f(x) be a polynomial function and f(5) = 126, then f(3) =

(A) 28 (B)26 (C) 27 (D) 25

Key. A

Sol. $f(x) = 1 \pm x^n$ or $f(5) = 1 \pm 5^n$

or, $126 = 1 \pm 5^n$ or $\pm 5^n = 125 \Rightarrow \pm 5^n = 5^3$

$n = 3$

$f(3) = 1 + 3^3 = 28$

112. If g(x) is a polynomial satisfying g(x) g(y) = g(x) + g(y) + g(xy) –2 for all real x and y and g(2) = 5, then g(3) is equal to

(A) 10 (B) 24

(C) 21 (D) 15

Key. A

Sol. Putting x = 1, y = 2, then

g(1) g(2) = g(1) + g(2) + g(2) – 2

$\Rightarrow$ 5g(1) = 8 + g(1)

$\therefore$ g(1) = 2

Also, replacing y by $\frac{1}{x}$ in the given relation, then

$$g(x)g\left(\frac{1}{x}\right)=g(x)+g\left(\frac{1}{x}\right)+g(1)-2$$

or $g(x)g\left(\frac{1}{x}\right)=g(x)+g\left(\frac{1}{x}\right)$

$\Rightarrow$ $g(x)=1\pm x^n$

$\Rightarrow$ $\pm 2^n = 2^2$

Taking +ve sign

$2^n = 2^2$

$\therefore$ n = 2

$\Rightarrow$ $g(x) = 1 + x^2$

$\therefore$ $g(3) = 1 + 3^2 = 10$

113. Let $f\left(\frac{x+y}{2}\right)=\frac{1}{2}(f(x)+f(y))$ for real x and y. If $f'(0)$ exists and equals to -1 and f(0)=1 then the value of f(2) is

a) 1 b) -1 c) $\frac{1}{2}$ d) 2

Key. B

Sol.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{f(2x)+f(2h)}{2}-f(x)}{h}$$

$f'(x)=-1$; $f(2x)=2f(x)$-1

$\Rightarrow f(x)=1-x$

114. A function f : R → R satisfies the equation f(x) f(y) – f(xy) = x + y ∀ x, y ∈ R and f(1) > 0, then

(A) $f(x)\ f^{-1}(x)=x^2-4$ (B) $f(x)\ f^{-1}(x)=x^2-6$

(C) $f(x)\ f^{-1}(x)=x^2-1$ (D) none of these

Key. C

Sol. Taking x = y = 1, we get

$f(1)f(1)-f(1)=2$

$\Rightarrow\ f^2(1)-f(1)-2=0\ \Rightarrow\ (f(1)-2)(f(1)+1)=0$

$\Rightarrow\ f(1)=2$ (as f(1) > 0)

Taking y = 1, we get

f(x). f(1) – f(x) = x + 1

$\Rightarrow$ f(x) = x + 1 $\Rightarrow\ f^{-1}(x)=x-1$

$\therefore\ f(x).f^{-1}(x)=x^2-1$

$\therefore$ (C) is the correct answer.

115. A function f satisfies the equation $3f(x)+2f\left(\frac{x+59}{x-1}\right)=10x+30,\ (x\neq 1)$

then the value of $\frac{f(11)}{f(7)}$ is

A) 7 B) 11 C) -7 D) -11

Key. B

Sol. At $x=11$

$3f(11)+2f(7)=140$.(1)

but $x=7$ to get

$3f(7)+2f(11)=100$ ..(2)

$\frac{(1)}{(2)}\Rightarrow\frac{3f(1)+2f(7)}{3f(7)+2f(11)}=\frac{7}{5}$

Using componendo and dividendo

$\frac{5(f(11)+f(7))}{(f(11)-f(7))}=\frac{6}{1} \Rightarrow \frac{(f(11)+f(7))}{(f(11)-f(7))}=\frac{6}{5}$

$\frac{f(11)}{f(7)}=11$

116. Let f be a real-valued function with domain R. If for some positive constant a, the equation

$$f(x + a) = 1 + (1 - 3f(x) + 3(f(x))^2 - (f(x))^3\}^{1/3}$$

holds good for all $x \in R$, prove that f(x) is a periodic function with period 2a.

Sol. Given $f(x+a)=1+\left\{1-3f(x)+3(f(x))^2-(f(x))^3\right\}^{1/3}$

$\Rightarrow \quad f(x+a)-1=\left\{(1-f(x))^3\right\}^{1/3} \Rightarrow \{f(x+a)-1\}^3=\{1-f(x)\}^3$

$\Rightarrow \quad f(x + a) - 1 = 1 - f(x)\ \ f(x + a) + f(x) = 2$ ...(1)

Replacing x by x – a, the equation (1) becomes f(x) + f(x – a) = 2 ...(2)

Subtracting (2) from (1), we get f(x + a) – f(x – a) = 0.

Finally replacing x by x + a, we get f(x + 2a) – f(x) = 0

$\Rightarrow$ f(x + 2a) = f(x) and hence f is periodic with period 2a.

117. Let $f: R \to R$ be continuous and periodic with period T > 0. then

(A) $f(x_0+T/2)=f(x_0)$ for some $x_0 \in [k, k+T/2], K \in R$

(B) $f(x_0+T/2)=f(x_0)$ for some $x_0 \in (k, k+T/4), K \in R$

(C) $f(x_0+T/2)=f(x_0)$ for some $x_0 \in (k, k+T/3), K \in R$

(D) $f(x_0+T/2)=f(x_0)$ for some $x_0 \in (k, k+T/6), K \in R$

Key. A

Sol. Let $g(x)=f(x+T/2)-f(x)$

then $g(k)=f(k+T/2)-f(k)$ ….. (1)

and $g(k+T/2)=f(k+T)-f(k+T/2)$

$=f(k)-f(k+T/2)$

$=-g(k)$

Hence by intermediate value property there exist an $x_0 \in [k, k+T/2]$ for which g(x) = 0

118. A function f : R → R satisfies the equation f(x) f(y) – f(xy) = x + y $\forall$ x, y $\in$ R and f(1) > 0, then

(A) $f(x)\, f^{-1}(x)=x^2-4$ (B) $f(x)\, f^{-1}(x)=x^2-6$

(C) $f(x)\, f^{-1}(x)=x^2-1$ (D) none of these

Key: C

Hint: Taking x = y = 1, we get

$f(1)f(1)-f(1)=2$

$\Rightarrow \ f^2(1)-f(1)-2=0 \ \Rightarrow \ (f(1)-2)(f(1)+1)=0$

$\Rightarrow\ f(1)=2$ (as f(1) > 0)

Taking y = 1, we get

f(x). f(1) – f(x) = x + 1

$\Rightarrow$ f(x) = x + 1 $\Rightarrow\ f^{-1}(x)=x-1$

$\therefore\ f(x).f^{-1}(x)=x^2-1$

$\therefore$ (C) is the correct answer.

119. Let f be a function such that $f(x+f(y))=f(x)+y;\ \forall x,y\in R$ then $f(2013)=$ ___

(1) 0 (2) 1 (3) 2013 (4) 4026

Key. 3

Sol. Put $y=x\Rightarrow f(x+f(x))=f(x)+x$

$\Rightarrow f(t)=t$ (Identity function)

120. If f(x) is a polynomial function satisfying the condition f(x).f(1/x) = f(x) + f(1/x), x$\in$R-{0} and f(2) = 9 then

(1) 2 f(4) = 3 f(6) (2) 7 f(1) = f(3) 3) 9 f(3) = 2 f(5) (4) f(10) = f(11)

Key. 3

Sol. f(x) = 1 + x^n put x = 2, we get n = 3

$\therefore$ f(x) = 1 + x^3

$\therefore$ 2 f(4) = 130 $\neq$ 3 f(6)

14 f(1) = 28 = 3 f(3)

9 f(3) = 252 = 2 f(5)

f(10) $\neq$ f(11)

121. If $f(x)f\left(\frac{1}{x}\right)=f(x)+f\left(\frac{1}{x}\right)\forall x\in R-\{0\}$, where f(x) be a polynomial function and f(5) = 126, then f(3) =

(A) 28 (B)26 (C) 27 (D) 25

Key. A

Sol. f(x) = 1 $\pm$ x^n or f(5) = 1 $\pm$ 5^n

or, 126 = 1 $\pm$ 5^n or $\pm 5^n$ = 125 $\Rightarrow$ $\pm$ $5^n = 5^3$

n = 3

f(3) = 1 + 3^3 = 28

7. Different Types of Functions

122. Let $f(x)=a_1\tan x+a_2\tan\left(\frac{x}{2}\right)+a_3\tan\left(\frac{x}{3}\right)+---$

$+a_n.\tan\left(\frac{x}{n}\right) where\, a_1,a_2,a_3---a_n$ are real numbers and

$n \in Z^{+}, \left|f(x)\right| \le \left|\tan x\right|$ for $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ then $\left|a_1 + \frac{a_2}{2} + \frac{a_3}{3} + ---- + \frac{a_n}{n}\right|$ is

(A) 1 (B) ≤ 1 (C) >1 (D) $\ge \frac{1}{2}$

Key. B

Sol. Clearly $f^{1}(0)$ is required $\left|f^{1}(0)\right| = \left|\underset{n\to 0}{\text{lt}} \frac{f(h)}{h}\right|$

$= \underset{h\to 0}{\text{lt}} \frac{|f(h)|}{|h|} \le \underset{h\to 0}{\text{lt}} \left|\frac{\tan h}{h}\right| = 1$

123. If $[x]$ denotes the integral part of x. for real x. then the value of

$\left[\frac{1}{4}\right] + \left[\frac{1}{4} + \frac{1}{200}\right] + \left[\frac{1}{4} + \frac{1}{100}\right] + \left[\frac{1}{4} + \frac{3}{200}\right] + \_\_\_\_\_\_ + \left[\frac{1}{4} + \frac{199}{200}\right]$

1) 50 2) 100 3) 25 4) 75

Key. 1

Sol. $\left[200.\frac{1}{4}\right] = [50] = 50$

124. If $g = \{(1,1),(2,3),(3,5),(4,7)\}$ is described by the formula $g(x) = \alpha x + \beta$, then $(\alpha, \beta) =$

1) (2, 1) 2) (2,-1) 3) (-2, 1) 4) (-2,-1)

Key. 2

Sol. $g(1) = \alpha + \beta = 1$

$g(2) = 2\alpha + \beta = 3$

125. If $f(x) = \cos\left[\pi^2\right]x + \cos\left[-\pi^2\right]x$, then (where $[\alpha]$ is integral part of α)

1) $f\left(\frac{\pi}{2}\right) = -1$ 2) $f(\pi) = 1$ 3) $f(-\pi) = 1$ 4) $f\left(\frac{\pi}{4}\right) = 2$

Key. 1

Sol. $f(x) = \cos 9x + \cos 10x, \because 9 < \pi^2 < 10$

126. Set A has 3 elements and set B has 4 elements. The number of injections that can be defined from A to B is

1) 144 2) 12 3) 24 4) 64

Key. 3

Sol. ${}^{n(B)}P_{n(A)} = {}^{4}P_{3} = 4.3.2 = 24$

127. $f : \mathbf{N} \to \mathbf{Z}$ is defined by $f(n) = \begin{cases} 2, \text{if } n = 3k, k \in Z \\ 10 - n, \text{if } n = 3k+1, k \in \mathbf{Z}. \\ 0, \text{if } n = 3k+2, k \in \mathbf{Z} \end{cases}$ Then $\{n \mid f(n) > 2\} =$

1) $\{3,6,3\}$ 2) $\{1,4,7\}$ 3) $\{4,7\}$ 4) $\{7\}$

Key. 2

Sol. $\{n \backslash (f(n) > 2)\} = \{n \backslash 10 - n > 2, n = 3k+1\}$

$$= \{n \backslash n < 8, n = 3k+1\}$$

128. Let $f_1(x) = \begin{cases} x, 0 \le x \le 1 \\ 1, \quad x > 1 \\ 0, \text{ otherwise} \end{cases}$ and $f_2(x) = f_1(-x)$ for all x

$f_3(x) = -f_2(x)$ for all x

$f_4(x) = f_3(-x)$ for all x

Which of the following is necessarily true?

(A) $f_4(x) = f_1(x)$ for all x

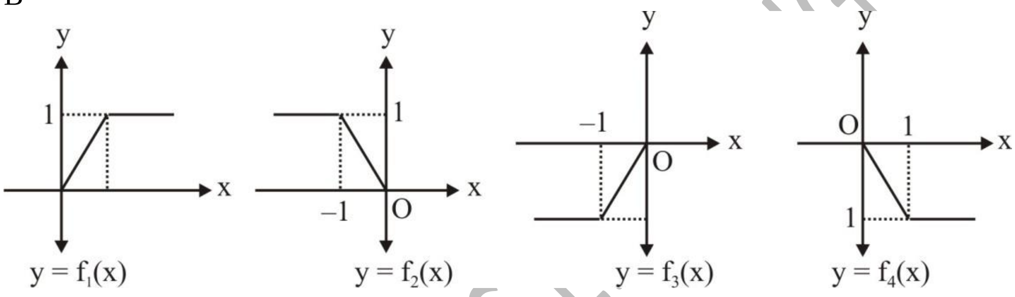
(B) $f_1(x) = -f_3(-x)$ for all x

(C) $f_2(-x) = f_4(x)$ for all x

(D) $f_1(x) = f_3(x) = 0$ for all x

Key. B

Sol.



129. If $\log_{(2x+3)}(6x^2 + 23x + 21) = 4 - \log_{(3x+7)}(4x^2 + 12x + 9)$, then the value of $-4x$ is

A) 0 B) 1 C) 2 D) $-\frac{1}{4}$

Key. B

Sol. First note that $2x+3>0$ and $2x+3 \ne 1$, that is, $x > -3/2$ and $x \ne -1$. Also, $3x+7>0$ and $3x+7 \ne 1$, that is, $x > -7/3$ and $x \ne -2$. Suppose $x > -3/2,\ x \ne -1$. Then the given equation can be written as

$$\frac{\log[(2x+3)(3x+7)]}{\log(2x+3)} = 4 - \frac{2\log(2x+3)}{\log(3x+7)}$$

$$1 + \frac{\log(3x+7)}{\log(2x+3)} = 4 - \frac{2\log(2x+3)}{\log(3x+7)}$$

Put $\frac{\log(3x+7)}{\log(2x+3)} = y$

Then $1 + y = 4 - \frac{2}{y}$

Therefore $y = 3 - \frac{2}{y}$

$$y^2 - 3y + 2 = 0$$

$$(y-1)(y-2) = 0$$

This gives $y=1$ or 2

Case 1: suppose that $y=1$. Then

$$\log(3x+7)=\log(2x+3)$$
$$3x+7=2x+3$$
$$x=-4$$

This is rejected because $x>-3/2$.

Case 2: Suppose that $y=2$. Then

$$\log(3x+7)=2\log(2x+3)=\log(2x+3)^2$$

Therefore $3x+7=4x^2+12x+9$

$$4x^2+9x+2=0$$
$$(4x+1)(x+2)=0$$
$$x=-1/4 \text{ or } -2$$

Here $x=-1/4$ (since $x>-3/2$) so $-4x=1$

130. If f and g are two functions defined on N, such that $f(n)=\begin{cases}2n-1 \; if \; n \, is \, even\\ 2n+2 \, if \; n \, is \, odd\end{cases}$ and $g(n)=f(n)+f(n+1)$. Then range of g is

A) $\{m\in N/m=$ multiple of 4$\}$

B) { set of even natural numbers}

C) $\{m\in N/m=4k+3, k$ is a natural number}

D) $\{m\in N/m=$ multiple of 3 or multiple of 4}

Key. C

Sol. $g(n)=f(n)+f(n+1)$

If n is even, n+1 is odd.

$\therefore g(n)=2n-1+2(n+1)+2=4n+3$

If n is odd, n+1 is even.

$\therefore g(n)=2n+2+2(n+1)-1=4n+3$.

131. The number of solution of $y=\frac{1}{3}\left[\sin x+\left[\sin x+[\sin x]\right]\right]$ and $\left[y+[y]\right]=2\cos x$ where [.] denotes the greatest integer function is

a) 4 b) 0 c) 2 d) 7

Key. B

Sol. $y=[\sin x]$ and $2\cos x=2[y]$ is impossible for every $x\in R$.

132. Let W be the set of whole numbers and $f:W\to W$ be defined by

$$f(x)=\begin{cases}\left(x-10\left[\dfrac{x}{10}\right]\right)10^{[\log_{10}x]}+f\left(\left[\dfrac{x}{10}\right]\right) & if \;\; x>0\\ 0 \quad if \quad x=0\end{cases}$$

where $[y]$ denotes the largest integer $\le y$. Then $f(7752)=$

(A) 7527 (B) 5727 (C) 7257 (D) 2577

Key. D

Sol. This function simply writes the digits of the given number in the reverse order.

133. $f(x)=\sin[x]+[\sin x], 0<x<\frac{\pi}{2}$, where [ ] represents the greatest integer function, can also be represented as

(A) $\begin{cases}0 & , 0<x<1 \\ 1+\sin 1 & , 1\le x<\frac{\pi}{2}\end{cases}$

(B) $\begin{cases}\frac{1}{\sqrt{2}} & , 0<x<\frac{\pi}{4} \\ 1+\frac{1}{2}+\frac{1}{\sqrt{2}}+\frac{\sqrt{3}}{2} & , \frac{\pi}{4}\le x<\frac{\pi}{2}\end{cases}$

(C) $\begin{cases}0 & , 0<x<1 \\ \sin 1 & , 1\le x<\frac{\pi}{2}\end{cases}$

(D) $\begin{cases}0 & , 0<x<\frac{\pi}{4} \\ 1 & , \frac{\pi}{4}<x<1 \\ \sin 1 & , 1\le x<\frac{\pi}{2}\end{cases}$

Key. C

Sol. $0<x<\frac{\pi}{2}$

$\therefore [x]=\begin{cases}0 \text{ if } 0<x<1 \\ 1 \text{ if } 1\le x<\frac{\pi}{2}\end{cases}$

$\Rightarrow \sin[x]=\begin{cases}\sin 0=0 \text{ if } 0<x<1 \\ \sin 1 \quad \text{if } 1\le x<\frac{\pi}{2}\end{cases}$

We have $0<\sin x<1$ when $0<x<\frac{\pi}{2}$.

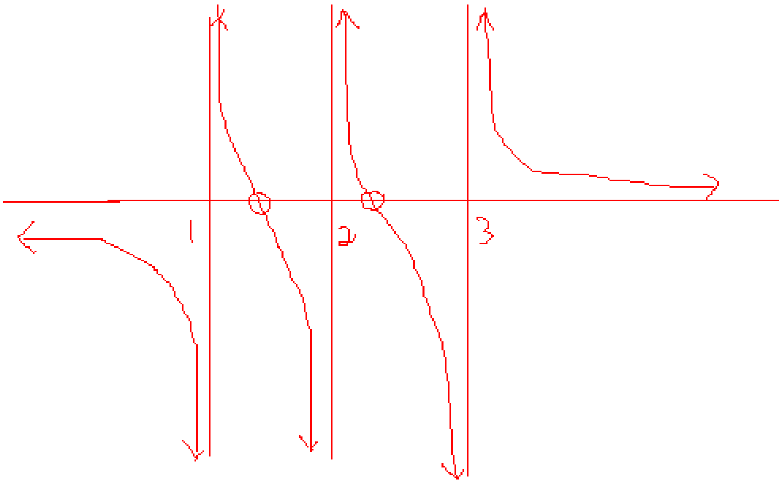
$\therefore [\sin x]=0$ *for* $0<x<\frac{\pi}{2}$

$\therefore \sin[x]+[\sin x]=\begin{cases}0 & \text{if } 0<x<1 \\ \sin 1 & \text{if } 1\le x<\frac{\pi}{2}\end{cases}$

134. $f(x)=\frac{1}{x-1}+\frac{1}{x-2}+\frac{1}{x-3}$ then number of points where $f(x)=0$

1) 1 2) 2 3) 3 4) 4

Key. 2



Sol.

135. $f(x) = x^2 + \lambda x + \mu \cos x$, $\lambda \in$ ¢, $\mu \in$ ¡ . The number of ordered pairs (λ, μ) for which $f(x) = 0$ and $f(f(x)) = 0$ have same set of real roots.

1) 4 2) 6 3) 8 4) 10

Key. 1

Sol. $f(x) = x^2 + \lambda x + u \cos x$

Let α be the root of $f(x) = 0 \Rightarrow f(\alpha) = 0$

$\Rightarrow f(f(\alpha)) = f(0) = 0$ ((Q α is root of $f(f(x) = 0$ also)

Now $f(0) = \mu = 0$

$f(x) = x^2 + \lambda x = 0 \Rightarrow x = 0, x = -\lambda$

$f(f(x)) = f(x^2 + \lambda x) = (x^2 + \lambda x)^2 + \lambda(x^2 + \lambda x)$

$= (x^2 + \lambda x)\{x^2 + \lambda x + \lambda\} = 0$

Will have same root $x = 0,\ x = -\lambda$ If

$x^2 + \lambda x + \lambda = 0$ have no real roots

$\Rightarrow \lambda^2 - 4\lambda < 0$

$\Rightarrow 0 < \lambda < 4 \Rightarrow \lambda = 1, 2, 3$

But $\lambda = 0$ is also satisfy

$(0,0), (0,1), (,2)(0,3)$ are 4 or diff. (λ, μ) does exist.

136. $f(x) = x^5 + x^2 + 1$ has roots x_1, x_2, x_3, x_4, x_5 *and* $g(x) = x^2 - 2$ then $g(x_1)g(x_2)g(x_3)g(x_4)g(x_5) - 30g(x_1x_2x_3x_4x_5) =$ ____

1) 2 2) 5 3) 7 4) 11

Key. 3

Sol. Put $g(x) = y = x^2 - 2 \Rightarrow x = \sqrt{y+2} \Rightarrow f(\sqrt{y+2}) = 0$

$\Rightarrow y^5 + 20y^4 + 40y^3 + 79y^2 + 74y + 23 = 0$

Roots are $g(x_1), g(x_2), g(x_3), g(x_4), g(x_5)$

$g(x_1).g(x_2).g(x_3).g(x_4).g(x_5) = -23$

And $x_1x_2x_3x_4x_5 = -1$

$g(x_1x_2x_3x_4x_5) = g(-1) = -1$

$\therefore g(x_1).g(x_2).g(x_3).g(x_4).g(x_5) - 30g(x_1x_2x_3x_4x_5)$

$= -23 + 30 = 7$

137. $f(x) = \cos^{-1}\left(\dfrac{2[|\sin x| + |\cos x|]}{\sin^2 x + 2\sin x + \frac{11}{4}}\right)$

([ ] denotes greatest integer function}. Then domain of $f(x)$ is the internal $\left[0, 2\pi\right]$ is.

1) $\left[0, \dfrac{7\pi}{6}\right] U \left[\dfrac{11\pi}{6}, 2\pi\right]$ 2) $\left[0, 2\pi\right]$

3) $\left[\dfrac{7\pi}{6}, \dfrac{11\pi}{6}\right]$ 4) $\left[\dfrac{3\pi}{2}, \dfrac{11\pi}{6}\right]$

Key. 1

Sol. $| \le |\sin x| + |\cos x| \le \sqrt{2}$

$\left[|\sin x| + |\cos x|\right] = 1 \quad \forall x \in$ ¡

Now $\sin^2 x + 2\sin x + \dfrac{11}{4} = (\sin x + 1)^2 + \dfrac{7}{4}$

For f to be well defined $(\sin x + 1)^2 + \dfrac{7}{4} \ge 2$

$$(\sin x + 1)^2 \ge \frac{1}{4}$$

$\Rightarrow \sin x + 1 \ge \dfrac{1}{2}, \quad \sin x + 1 \le -\dfrac{1}{2}$

$\sin x \ge -\dfrac{1}{2}, \quad \sin x \le -\dfrac{3}{2}$ (This is impossible

$\Rightarrow x \in \left[0, \dfrac{7\pi}{6}\right] U \left[\dfrac{11\pi}{6}, 2\pi\right]$ Hence (A) is correct

138. If $f(x)$ is a polynomial of degree 4 with leading coefficient one satisfying $f(1) = 1$, $f(2) = 2$, $f(3) = 3$ then $\left[\dfrac{f(-1) + f(5)}{f(0) + f(4)}\right] =$ ([.] denotes GIF)

1) 0 2) 5 3) 1 4) -1

Key. 2

Sol. $f(x) - x = (x-1)(x-2)(x-3)(x-\alpha)$

$f(-1) = 24(1+\alpha) - 1$
$f(0) = 6\alpha$
$f(4) = 6(4-\alpha) + 4$
$f(5) = 24(5-\alpha) + 5$

$$\left[\frac{f(-1) + f(5)}{f(0) + f(4)}\right] = \left[\frac{148}{28}\right] = 5$$

139. A function 'f' defined as $f(\alpha) = (-1)^{\alpha_1} + (-1)^{\alpha_2} + (-1)^{\alpha_3} + ... + (-1)^{\alpha_k}$ where $\alpha \in$ ¥ , and α_1, α_2, α_k are all divisors of α including 1 and itself such that $\alpha_1, \alpha_2 ...... \alpha_k = \alpha$ and $\alpha_1, \alpha_2 ...... \alpha_k \in$ ¥

If $f(\alpha)=4$ and $\alpha<60$ then number of possible values of α .

1) 3 2) 6 3) 10 4) 4

Key. 1

Sol. $4=(-1)^2+(-1)^2+(-1)^2+(-1)^2=\quad \alpha=16<60$

$4=(-1)^4+(-1)^2+(-1)^2+(-1)^2=\quad \alpha=32<60$

$4=(-1)^6+(-1)^2+(-1)^2+(-1)^2=\quad \alpha=48<60$

140. $f(x+1)=(-1)^{x+1}x-1f(x)$ for $x\in ¥$ and $f(1)=f(1986)$. Then sum of digits of $(f(1)+f(2)+........+f(1985))$ is

1) 4 2) 3 3) 7 4) 11

Key. 3

Sol. $\sum_{x=1}^{1985} f(x+1)=\sum_{x=1}^{1985}(-1)^{x+1}x-2\sum_{x=1}^{1985} f(x)$

Since $f(1)=f(1986)$

$3\sum_{x=1}^{1985} f(x)=1-2+3-4+5.........+1985$

$=(1+3+.......+1985)-2(1+2+3+......992)$

$=\frac{993}{2}(1986)-2\left(\frac{992\text{x}993}{2}\right)$

$=(993)^2-993\,\text{x}\,992$

= 993

$\therefore \sum_{x=1}^{1985} f(x)=\frac{993}{3}=331$

Sum of digits = 3+3+1= 7

141. $f(x)=ax^2-C^{satisfy}-4\le f(1)\le -1$ and $-1\le f(2)\le 5$ then which of the following is true

1) $-7\le f(3)\le 26$ 2) $-4\le f(3)\le 15$

3) $-1\le f(3)\le 20$ 4) $\frac{-28}{3}\le f(3)\le \frac{35}{3}$

Key. 1

Sol. $f(x)=ax^2-c$

$-4\le f(1)\le -1 \Rightarrow -4\le a-c\le -1;$

$1\le c-a\le 4\rightarrow (1)$

$-1\le f(2)\le 5 \Rightarrow -1\le 4a-c\le 5\rightarrow (2)$

$(1)+(2)\Rightarrow 0\le 3a\le a$

$0\le a\le 3\rightarrow (3)$

From (1)

$-16\le 4a-4c\le -4$

$\Rightarrow 4\le 4c-4a\le 16$

From (2)

$-1\le 4a-c\le 5$

$3 \le 3c \le 21 \Rightarrow 1 \le c \le 7 \to (4)$

Now $f(3) = 9a - c$ is max of a is max and c is min

$f(3)_{\max} = 9(3) - 1 = 26$

$f(3)_{\min} = 9(0) - 7 = -7$

$\therefore -7 \le f(3) \le 26$

142. Let $f(x) = 5 + \sum_{r=1}^{2010} a_{2r-1}x^{2r-1}$ and $f(-1) = 4$ then $f(1) =$

1) 2 2) 6 3) 5 4) 4

Key. 2

Sol. $f(x) = 5 + a_1x + a_3x^3 + a_5x^5 + ... + a_{4019}x^{4019}$

$f(-1) = 5 - a_1 - a_3 - a_5 .... - a_{4019} = 4$

$f(1) = 5 + a_1 + a_3 + a_5 .... + a_{4019} = \lambda$ *say*

$10 = 4 + \lambda \Rightarrow \lambda = 6$

143. Let $f(x) = ax + b$ where a and b are rational numbers (where $b \ne 0$). Such that $f(1) \le f(2)$, $f(3) \ge f(4)$ then value of $\left(\dfrac{\sum_{r=1}^{2n-1} f\left(\sqrt{2}r\right)}{f(\sqrt{3})} \right)$ $(where\ n \in ¥)$ is

1) n 2) 1 3) 0 4) n^2

Key. 4

Sol. For fixed values of a and b $f(x) = ax + b$ is a straight line

But given $f(1) \le f(2)$ and $f(3) \ge f(4)$

$\therefore f(1) = f(2) = f(3) = f(4) = \lambda$

$\Rightarrow f(x)$ should be constant function $\Rightarrow a = 0$

$\Rightarrow f(x) = b \Rightarrow f(\sqrt{2}r) = b$ and $f\left(\sqrt{3}\right) = b$

Given expression is $\dfrac{n^2 b}{b} = n^2$

144. A linear function that map the set {–2, 2} onto the set {0, 4} is

(A) f(x) = (x – 2) (B) f(x) = (2 – x) (C) f(x) = (2 + x) (D) (B) and (C)

Key. D

Sol. Let the linear function be

f(x) = ax + b

Let f(–2) = 0 and f(2) = 4 $\Rightarrow$ f(x) = x + 2

Let f(–2) = 4 and f(0) = 0 $\Rightarrow$ f(x) = –x + 2

The two linear function as are

f(x) = (x + 2) and f(x) = (2 – x)

145. Suppose $f(x) = (x + 2)^2$ for $x \geq -2$. If g(x) is the function whose graph is the reflection of the graph of f(x) in the line y = x, then g(x) equals

(A) $-\sqrt{x}-2,\ x\geq 0$ (B) $\sqrt{x}-2,\ x\geq 0$ (C) $\frac{1}{(x+2)^2},\ x>2$ (D) $\sqrt{x+2},\ x>-2$

Key. B

Sol. $y = (x + 2)^2$

Equation of the reflection curve in y = x is obtained by interchanging x and y in $y = (x + 2)^2$

$\Rightarrow$ reflection curve is

$x = (y + 2)^2$

$y + 2 = \sqrt{x}$

$y = \sqrt{x}-2,\ x\geq 0$

Since x is always ≥ 2.

146. If $\log_4\left(\log_3\left(\log_2 x\right)\right)=1$, then x is

(A) 2^{3^4} (B) 9 (C) 24 (D) 4^{3^2}

Key. A

Sol. $\log_4\left[\log_3 \log_2 x\right] = 1 \Rightarrow \log_3 \log_2 x = 4$

$\Rightarrow \log_2 x = 3^4 \qquad \Rightarrow x = 2^{3^4}$

147. The value of the parameter α, for which the function $f(x)=1+\alpha x,\ \alpha\neq 0$ is the inverse of itself, is

(A) –2 (B) –1 (C) 1 (D) 2

Key. B

Sol. $y = 1+\alpha x \Rightarrow x = \frac{y-1}{\alpha}$

$f^{-1}(x) = \frac{x-1}{\alpha} = f(x) = 1+\alpha x$

$\Rightarrow \frac{x-1}{\alpha} = 1+\alpha x \qquad \Rightarrow x - 1 = \alpha+\alpha^2 x$

Equating the coefficient of x

$\alpha^2 = 1$ and $\alpha = -1$

$\alpha = \pm 1$

$\alpha = -1$

148. If $f(x)=x^2+2bx+2c^2$ and $g(x)=-x^2-2cx+b^2$ such that min f(x) > max g(x), then the relation between b and c, is

(A) no real value of b and c (B) $0<c<b\sqrt{2}$

(C) $|c| < |b|\sqrt{2}$ (D) $|c| > |b|\sqrt{2}$

Key. D

Sol. We have, $f(x) = x^2 + 2bx + 2c^2$; $g(x) = -x^2 - 2cx + b^2$

$\Rightarrow$ $f(x) = (x+b)^2 + 2c^2 - b^2$

and, $g(x) = -(x + c)^2 + b^2 + c^2$

$\Rightarrow$ $f_{min} = 2c^2 - b^2$ and $g_{max} = b^2 + c^2$

for, $f_{min} > g_{max} \Rightarrow$ $2c^2 - b^2 > b^2 + c^2$

$\Rightarrow$ $c^2 > 2b^2$ $\Rightarrow$ $|c| > |b|\sqrt{2}$

149. Let $A_1, A_2, A_3, \ldots, A_{40}$ are 40 sets each with 7 elements and $B_1, B_2, \ldots, B_n$ are n sets each with 7 elements. If $\bigcup_{i=1}^{40} A_i = \bigcup_{j=1}^{n} B_j = S$ and each element of S belongs to exactly ten of A_i's and exactly 9 of B_j's, then n equals

(A) 42 (B) 35

(C) 28 (D) 36

Key. D

Sol. $n(S) \times 10 = 40 \times 7$

$n(S) = 28$

$28 \times 9 = n \times 7$

$n = 36$

150. The number of functions f from the set A={0,1,2} in to the set B={0,1,2,3,4,5,6,7} such that $f(i) \le f(j)$ for i<j and $i, j \in A$ is

a) ${}^8C_3$ b) ${}^8C_3 + 2({}^8C_2)$

c) ${}^{10}C_3$ d) ${}^{10}C_4$

Key. C

Sol.

$0 < 1 < 2$

$\Rightarrow f(0) \le f(1) \le f(2)$

$f(0)<f(1)<f(2) \Rightarrow {}^8C_3$

$f(0)<f(1)=f(2) \Rightarrow {}^8C_2$

$f(0)=f(1)<f(2) \Rightarrow {}^8C_2$

$f(0)=f(1)=f(2) = {}^8C_1$

151. Find the value of $\sum_{r=1}^{n}\sum_{s=1}^{n} \delta_{rs}\, 2^r 3^s$ where $\begin{cases} \delta_{rs} = 0, \text{ if } r \ne s \\ \delta_{rs} = 1, \text{ if } r = s \end{cases}$

a) $\frac{6}{5}\left(6^n - 1\right)$ b) $6^n - 1$ c) $\frac{1}{5}\left(6^n - 1\right)$ d) none

Key. A

Sol. $\sum_{r=1}^{n} 2^r \left\{ \sum_{s=1}^{n} \delta_{rs}\, 3^s \right\}$

$$= \sum_{r=1}^{n} 2^r \left\{\delta_{r1}3^1 + \delta_{2r}3^2 + \delta_{r3}3^3 + ...+ \delta_{rn}3^n\right\}$$

$$= 2^1 3^1 + 2^2 3^2 + 2^3 3^3 + ...+ 2^n 3^n = 6 + 6^2 + ...+ 6^n = \frac{6}{5}\left(6^n - 1\right)$$

152. Consider $\int_0^x \left(t^2 - 8t + 13\right)dt = x\sin\left(\frac{a}{x}\right)$ and $\left(a, x \in R - \{0\}\right)$ x takes the values for which the equation has a solution, then the number of values of a $\in [0,100]$ is ___

a) 1 b) 2 c) 3 d) 4

Key. C

Sol. $\left(\frac{t^3}{3} - \frac{8t^2}{2} + 13t\right)_0^x = x\sin\left(\frac{a}{x}\right)$

$$x\left\{\frac{x^2}{3} - 4x + 13 - \sin\left(\frac{a}{x}\right)\right\} = 0$$

Here $x \neq 0 \Rightarrow \frac{1}{3}\left(x^2 - 12x + 39\right) = \sin\left(\frac{a}{x}\right)$

$$\Rightarrow \frac{1}{3}(x-6)^2 + 1 = \sin\left(\frac{a}{x}\right)$$

$$\Rightarrow \frac{a}{6} = \frac{\pi}{2} \text{ or } \frac{5\pi}{2} \text{ or } \frac{9\pi}{2}$$

$\therefore \quad a = 3\pi, 15\pi, 27\pi$ (3 values)

153. Let f be a function defined on the set of non-negative integers and taking values in the same set. Given that

i) $x - f(x) = 19\left[\frac{x}{19}\right] - 90\left[\frac{f(x)}{90}\right]$ $\forall$ non-negative integers x. [x] denotes greatest integer functions.

ii) $1900 \le f(1990) \le 2000$. Then possible values of $f(1990)$ can take.

a) 2004, 2094 b) 1804, 1994 c) 1904, 1994 d) 1894

Key. C

Sol. Since $1900 \le f(1990) \le 2000$

$$\Rightarrow \left[\frac{1900}{90}\right] \le \left[\frac{f(1990)}{90}\right] \le \left[\frac{2000}{90}\right] \Rightarrow 21 \le \left[\frac{f(1990)}{90}\right] \le 22$$

Case - I

If $\left[\frac{f(1990)}{90}\right] = 21$, $x - f(x) = 19\left[\frac{x}{19}\right] - 90\left[\frac{f(x)}{90}\right]$

Substitute $x = 1990$

$$1990-f(1990)=19\left[\frac{1990}{19}\right]-90\left[\frac{f(1990)}{90}\right]$$

$$1990-f(1990)=19\times104-90\times21\Rightarrow f(1990)=1904$$

Case – II

If $\left[\frac{f(1990)}{90}\right]=22$

$$\Rightarrow 1990-f(1990)=19\times104-90\times22\Rightarrow f(1990)=1994$$

154. If $g:[-1,1]\to\mathbf{R}$ is a function and the area of the equilateral triangle with two of its vertices at (0,0) and (x,g(x)) is $\frac{\sqrt{3}}{4}$, then g(x)=

1) $+\sqrt{x^2-1}$ 2) $+\sqrt{1-x^2}$ 3) $+\sqrt{1+x^2}$ 4) $+x$

Key. 2

Sol. If a = length of a side = $\sqrt{(x-0)^2+(g(x)-0)^2}=\sqrt{x^2+g^2(x)}$

Area of an equilateral triangle = $\frac{\sqrt{3}}{4}a^2=\frac{\sqrt{3}}{4}\Rightarrow x^2+g^2(x)=1\Rightarrow g(x)=+\sqrt{1-x^2}$

155. Let $S=\sum_{r=1}^{117}\frac{1}{2[\sqrt{r}]+1}$ where [.] denotes the greatest integer function. The value of S is

(A) $\frac{69}{7}$ (B) $\frac{206}{21}$ (C) $\frac{76}{7}$ (D) $\frac{227}{21}$

Key. A

Sol. $[\sqrt{117}]=10$; If $r\in[n^2,(n+1)^2):n\in¥$ then $[\sqrt{r}]=n$

The interval $[n^2,(n+1)^2)$ has 2n + 1 integers

$$S=\frac{1}{2.1+1}.3+\frac{1}{2.2+1}.5+....+\frac{1}{2.9+1}.19+\frac{1}{2.10+1}.18$$

$$=9+\frac{18}{21}=\frac{69}{7}.$$

156. If $x+[y]+\{z\}=1.1$ [.] is G.I.F and {.} is fractional part

$[x]+\{y\}+z=2.2$

$\{x\}+y+[z]=3.3$ then

(A) $x+y+z=3.3$ (B) $y-2x=1$

(C) $2(z+1)=5y$ (D) $\{x\}+\{y\}+\{z\}=0.3$

Key. A, B, C, D

Sol. $x+[y]+\{z\}=1.1$ (1)

$[x]+\{y\}+z=2.2$ (2)

$\{x\}+y+[z]=3.3$ (3)

(1) + (2) + (3)

$\Rightarrow$ $2(x+y+z)=6.6$

$\Rightarrow \quad x+y+z=3.3 \qquad (4)$

$(4)-(1)$

$\{y\}+[z]=2.2 \quad \Rightarrow \{y\}=0.2 \,\&\, [z]=2$

$(4)-(2)$

$\{x\}+[y]=1.1 \quad \Rightarrow \{x\}=0.1,\ [y]=1$

$(4)-(3)$

$[x]+\{z\}=0 \quad \Rightarrow [x]=0 \ \&\ \{z\}=0$

$$\begin{cases} x=0.1 \\ y=1.2 \\ z=2 \end{cases}$$

157. If $0<x<1000$ and $\left[\frac{x}{2}\right]+\left[\frac{x}{3}\right]+\left[\frac{x}{5}\right]=\frac{31}{30}x,$ where [x] is the greatest integer less than or equal to x, the number of possible values of x is

(A) 34 (B) 33

(C) 32 (D) none of these

Key : B

Sol : Q LHS is an integer

$\therefore$ RHS is must be an integer for which x is multiple of 30.

$\therefore \ x=30,60,90,120,.....,990$

$\Rightarrow$ Number of possible values of x is 33.

158. If $f(x)=[x^2]-[x]^2$, [ ] denotes greatest integer function and $x\in[0,n]$, $n\in N$, then the number of elements in the range of $f(x)$ is

A) 1 B) $n-1$ C) n D) $2n-1$

Key. D

Sol. If $x\in(n-1,n)$ then $[x]=n-1 \Rightarrow [x]^2=(n-1)^2$

and $(n-1)^2 \le [n^2] \le n^2-1$

$0 \le [x^2]-[x]^2 \le n^2-1-(n-1)^2$

$0 \le f(x) \le 2n-2$

Since $f(x)$ has to be integer, range of $f(x)=\{0,1,2,3,.....2n-2\}$

$\therefore$ The number of elements in range of f is $(2n-1)$

159. If $\frac{5^m+3}{40}-\left[\frac{5^m+3}{40}\right]=\lambda$ (m$\in$N, m $\ge$ 3) and [ ] denote the G.I.F., then λ can take

(A) two values (B) one value

(C) infinite values (D) four values

Key: A

Hint: $\frac{5^m+3}{40}=\frac{1}{10}$ $(5+5^2+5^3+\ldots+5^{m-1}+2) \Rightarrow \lambda = \frac{1}{5},\frac{7}{10}$

160. The sum of all positive integral values of $'a'$, $a\in[1,500]$ for which the equation $[x]^3+x-a=0$ has solution is ([.] denote G.I.F)

(A) 462 (B) 512 (C) 784 (D) 812

Key: D

Hint: a is integer then x must be integer, i.e., [x] = x

$a=x^3+x$

$1\le a\le 500 \Rightarrow 1\le x\le 7,\ x\in I$

$\sum a_i=\sum_{x=1}^{7}\left(x^3+x\right)=\left(\frac{7.8}{2}\right)^2+\left(\frac{7.8}{2}\right)=812$

161. $f:[0,1]\to R$ is a differentiable function such that $f(0)=0$ and $|f(x)|\le k|f(x)|$ for all x $\in[0,1],(k>0)$, then which of the following is/are always true ?

(A) $f(x)=0,\forall x\in R$ (B) $f(x)=0,\forall x\in[0,1]$

(C) $f(x)\ne 0,\forall x\in[0,1]$ (D) $f(1)=k$

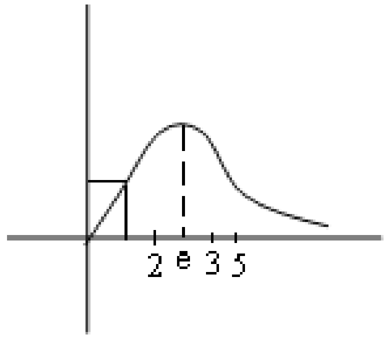
Key: B

Hint: $(f(x))^2-k^2(f(x))^2\le 0$

$\Rightarrow (f'(x)-kf(x))(f'(x)+kf(x))\le 0$

$\Rightarrow \left(f(x)e^{-kx}\right)'\left(f(x)e^{kx}\right)'\le 0$

$\Rightarrow$ Exactly one of the functions $g_1(x)=f(x)e^{-kx}$ or $g_2(x)=f(x)e^{kx}$ is non decreasing.



But $f(0)=0\Rightarrow$ both function g_1 and g_2 have a value zero at x = 0

$\forall x\in[0,1], g_1(0)=0$ and g_1 increasing $\Rightarrow g_1(x)\ge 0\Rightarrow f(x)\ge 0$

$g_2(0)=0$ and g_2 decreasing $\Rightarrow g_2(x)\le 0\Rightarrow f(x)\le 0$

$\Rightarrow f(x)=0\forall x\in[0,1]$

162. Let f be a one one function with domain = {x, y, z} and range = {1, 2, 3}. It is given that exactly one of the following statements is true and the remaining two are false : $f(x)=1,\ f(y)\ne 1,\ f(z)\ne 2$, then $f^{-1}(1)=$ ____

(1) x (2) y (3) z (4) 1

Key. 2

Sol. $f(x)=1\,(F)\Rightarrow f(x)=2$ *or* 3

$f(y) \neq 1\ (F) \Rightarrow f(y) = 1$

$f(z) \neq 2\ (T) \Rightarrow f(z) = 1\ or\ 3$

163. If $f(x) = 1 + x; \quad x \geq 0$
$\quad = 1 - x; \quad x < 0$

Which of the following are true ?

(1) Range of f(x) is $[2, \infty)$

(2) $f(f(x))$ is not a one one function

(3) Graph of $y = f(f(x))$ is symmetric about y axis.

(4) All the above

Key. 4

Sol. $f(x) = 1 + |x|;\ x \in R$

$f(f(x)) = f(1+|x|) = 1 + 1 + |x| = 2 + |x| \quad \forall x \in R$

164. If $[x]$ is the greatest integral function, then $\sum_{k=1}^{4020}\left[\frac{1}{2} + \frac{k-1}{4020}\right]$ is equal to

(1) 2010 (2) 2009 (3) 2011 (4) 2005

Key. 1

Sol. For $k = 1, 2, 3......$ upto 2010, the value of $\left[\frac{1}{2} + \frac{k-1}{4020}\right]$ is equal to zero

For $k = 2011$, 2010, ----- 4020, the value of $\left[\frac{1}{2} + \frac{k-1}{4020}\right]$ = 1

$\therefore$ The sum value is 2010.

165. Let $f(x) = [x]$ and $g(x) = x + [x]$. Then the number of solutions of the equality ($[\cdot]$ is G.I.F)

$4(x - f(x)) = g(x)$ is

(1) 2 (2) 3 (3) 4 (4) 0

Key. 1

Sol. The given equation is

$4(x - [x]) = x + [x] = 2[x] + \{x\}$

$4\{x\} = 2[x] + \{x\}$

$\therefore 0 \leq \frac{2[x]}{3} < 1 \Rightarrow x = 0, 5/3$

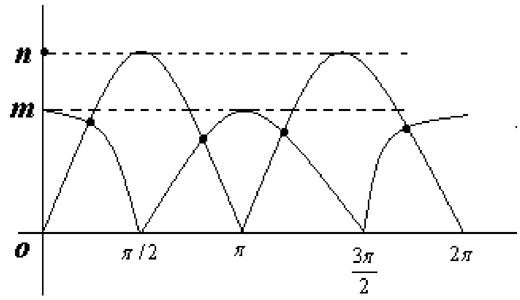
166. If m, n (n > m) are positive integers, then number of solutions of the equation

$n\ |\sin x| = m\ |\cos x|\ in\ [0, 2\pi]$ is

(1) 2 (2) 4 (3) m (4) n

Key. 2

Sol. No.of solutions = 4



167. Let $f(x) = ln\left(\frac{1-x}{1+x}\right)$. The set of values of $'\alpha'$ for which $f(\alpha) + f(\alpha^2) = f\left(\frac{\alpha}{\alpha^2 - \alpha + 1}\right)$ is satisfied are

A) $(-\infty, -1) \cup (1, \infty)$ B) $(-1,1)$ C) $(0,1)$ D) $(1,2)$

Key. B

Sol. $$f(\alpha) + f(\alpha)^2 = ln\left[\left(\frac{1-\alpha}{1+\alpha}\right)\left(\frac{1-\alpha^2}{1+\alpha^2}\right)\right] = ln\left[\frac{(1-\alpha)^2}{1+\alpha^2}\right]$$

$$f\left(\frac{\alpha}{\alpha^2 - \alpha + 1}\right) = ln\left[\frac{1 - \frac{\alpha}{\alpha^2 - \alpha + 1}}{1 + \frac{\alpha}{\alpha^2 - \alpha + 1}}\right] = ln\left[\frac{(1-\alpha)^2}{1+\alpha^2}\right]$$

$\therefore\ f(\alpha) + f(\alpha^2) = f\left(\frac{\alpha}{\alpha^2 - \alpha + 1}\right)$ for all values of α for which the functions are defined, therefore

(i) $\frac{1-\alpha}{1+\alpha} > 0 \Rightarrow -1 < \alpha < 1....(1)$

(ii) $\frac{1-\alpha^2}{1+\alpha^2} > 0 \Rightarrow 1 - \alpha^2 > 0 \Rightarrow -1 < \alpha < 1....(2)$

From (1) and (2), we have $-1 < \alpha < 1$

$\therefore$ The set of values of $\alpha = (-1,1)$.

168. If $e^{f(x)} = \frac{10+x}{10-x}, x \in (-10,10)$ and $f(x) = kf\left(\frac{200x}{100+x^2}\right)$, then k =

A. 2 B. 10 C. $\frac{1}{2}$ D. $\frac{1}{10}$

Key. C

Sol. $$f(x) = \log_e\left(\frac{10+x}{10-x}\right)$$

$$f\left(\frac{200x}{100+x^2}\right)=\log_e\left(\frac{10+\frac{200x}{100+x^2}}{10-\frac{200x}{100+x^2}}\right)=2\log\left(\frac{10+x}{10-x}\right)=2f(x)$$

169. The number of solutions of $\sin\{x\}=\cos\{x\}$ (where {.} denotes fractional part) in [0, 2π] is equal to

A. 5 B. 6 C. 7 D. 8

KEY. B

SOL. Draw Graph

170. A linear function that map the set {–2, 2} onto the set {0, 4} is

(A) f(x) = (x – 2) (B) f(x) = (2 – x) (C) f(x) = (2 + x) (D) (B) and (C)

Key. D

Sol. Let the linear function be

f(x) = ax + b

Let f(–2) = 0 and f(2) = 4 $\Rightarrow$ f(x) = x + 2

Let f(–2) = 4 and f(0) = 0 $\Rightarrow$ f(x) = –x + 2

The two linear function as are

f(x) = (x + 2) and f(x) = (2 – x)

171. Suppose $f(x) = (x + 2)^2$ for $x\ge -2$. If g(x) is the function whose graph is the reflection of the graph of f(x) in the line y = x, then g(x) equals

(A) $-\sqrt{x}-2,\ x\ge 0$ (B) $\sqrt{x}-2,\ x\ge 0$ (C) $\frac{1}{(x+2)^2},\ x>2$ (D) $\sqrt{x+2},\ x>-2$

Key. B

Sol. $y = (x + 2)^2$

Equation of the reflection curve in y = x is obtained by interchanging x and y in $y = (x + 2)^2$

$\Rightarrow$ reflection curve is

$x = (y + 2)^2$

$y + 2 = \sqrt{x}$

$y = \sqrt{x}-2,\ x\ge 0$

Since x is always ≥ 2.

172. If f and g are two functions defined on N, such that $f(n)=\begin{cases}2n-1 \text{ if } n \text{ is even}\\ 2n+2 \text{ if } n \text{ is odd}\end{cases}$ and

$g(n)=f(n)+f(n+1)$. Then range of g is

A) $\{m\in N/m=$ multiple of 4}

B) { set of even natural numbers}

C) $\{m\in N/m=4k+3, k$ is a natural number}

D) $\{m\in N/m=$ multiple of 3 or multiple of 4}

Key. C

Sol. $g(n)=f(n)+f(n+1)$

If n is even, n+1 is odd.

$\therefore g(n)=2n-1+2(n+1)+2=4n+3$

If n is odd, n+1 is even.

$\therefore g(n) = 2n + 2 + 2(n+1) - 1 = 4n + 3$.

173. A function 'f' defined as $f(\alpha) = (-1)^{\alpha_1} + (-1)^{\alpha_2} + (-1)^{\alpha_3} + \text{.........} + (-1)^{\alpha_k}$ where $\alpha \in ¥$, and α_1, α_2, α_k are all divisors of α including 1 and itself such that $\alpha_1, \alpha_2 \text{......} \alpha_k = \alpha$ and $\alpha_1, \alpha_2 \text{......} \alpha_k \in ¥$ If $f(\alpha) = 4$ and $\alpha < 60$ then number of possible values of α .

1) 3 2) 6 3) 10 4) 4

Key. 1

Sol. $4 = (-1)^2 + (-1)^2 + (-1)^2 + (-1)^2 = \quad \alpha = 16 < 60$

$4 = (-1)^4 + (-1)^2 + (-1)^2 + (-1)^2 = \quad \alpha = 32 < 60$

$4 = (-1)^6 + (-1)^2 + (-1)^2 + (-1)^2 = \quad \alpha = 48 < 60$

174. $f(x+1) = (-1)^{x+1} x - 1 f(x)$ for $x \in ¥$ and $f(1) = f(1986)$. Then sum of digits of $(f(1) + f(2) + \text{........} + f(1985))$ is

1) 4 2) 3 3) 7 4) 11

Key. 3

Sol. $\sum_{x=1}^{1985} f(x+1) = \sum_{x=1}^{1985} (-1)^{x+1} x - 2 \sum_{x=1}^{1985} f(x)$

Since $f(1) = f(1986)$

$3 \sum_{x=1}^{1985} f(x) = 1 - 2 + 3 - 4 + 5 \text{.........} + 1985$

$= (1 + 3 + \text{.......} + 1985) - 2(1 + 2 + 3 + \text{......} 992)$

$= \frac{993}{2}(1986) - 2\left(\frac{992x993}{2}\right)$

$= (993)^2 - 993 \text{x} 992$

= 993

$\therefore \sum_{x=1}^{1985} f(x) = \frac{993}{3} = 331$

Sum of digits = 3+3+1= 7

175. $f(x) = ax^2 - C^{satisfy} - 4 \le f(1) \le -1$ and $-1 \le f(2) \le 5$ then which of the following is true

1) $-7 \le f(3) \le 26$ 2) $-4 \le f(3) \le 15$

3) $-1 \le f(3) \le 20$ 4) $\frac{-28}{3} \le f(3) \le \frac{35}{3}$

Key. 1

Sol. $f(x) = ax^2 - c$

$-4 \le f(1) \le -1 \Rightarrow -4 \le a - c \le -1;$

$1 \le c - a \le 4 \to (1)$

$-1 \le f(2) \le 5 \Rightarrow -1 \le 4a - c \le 5 \to (2)$

$(1) + (2) \Rightarrow 0 \le 3a \le a$

$0 \le a \le 3 \to (3)$

From (1)

$-16 \le 4a - 4c \le -4$

$\Rightarrow 4 \le 4c - 4a \le 16$

From (2)

$-1 \le 4a - c \le 5$

$3 \le 3c \le 21 \Rightarrow 1 \le c \le 7 \to (4)$

Now $f(3) = 9a - c$ is max of a is max and c is min

$f(3)_{\max} = 9(3) - 1 = 26$

$f(3)_{\min} = 9(0) - 7 = -7$

$\therefore -7 \le f(3) \le 26$

176. Let $f(x) = 5 + \sum_{r=1}^{2010} a_{2r-1} x^{2r-1}$ and $f(-1) = 4$ then $f(1) =$

1) 2 2) 6 3) 5 4) 4

Key. 2

Sol. $f(x) = 5 + a_1 x + a_3 x^3 + a_5 x^5 + ... + a_{4019} x^{4019}$

$f(-1) = 5 - a_1 - a_3 - a_5 .... - a_{4019} = 4$

$f(1) = 5 + a_1 + a_3 + a_5 .... + a_{4019} = \lambda \ say$

$10 = 4 + \lambda \Rightarrow \lambda = 6$

177. Let $f(x) = ax + b$ where a and b are rational numbers (where $b \ne 0$). Such that $f(1) \le f(2)$, $f(3) \ge f(4)$ then value of $\left(\frac{\sum_{r=1}^{2n-1} f\left(\sqrt{2r}\right)}{f(\sqrt{3})} \right)$ $\left(where \ n \in ¥ \right)$ is

1) n 2) 1 3) 0 4) n^2

Key. 4

Sol. For fixed values of a and b $f(x) = ax + b$ is a straight line

But given $f(1) \le f(2)$ and $f(3) \ge f(4)$

$\therefore f(1) = f(2) = f(3) = f(4) = \lambda$

$\Rightarrow f(x)$ should be constant function $\Rightarrow a = 0$

$\Rightarrow f(x) = b \Rightarrow f(\sqrt{2r}) = b$ and $f\left(\sqrt{3}\right) = b$

Given expression is $\frac{n^2 b}{b} = n^2$

178. If α is a root of the equation $\sin x + 1 = x$ then $\lim_{x \to \alpha} \left[\frac{\min\left(\sin x, \{x\}\right)}{x - 1} \right]$ is

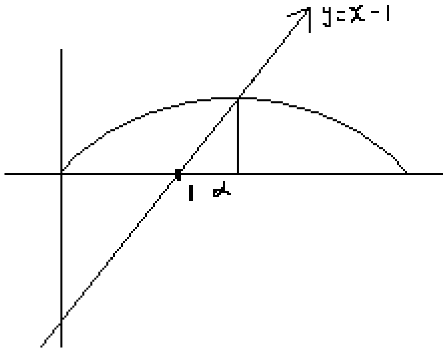
Where [.] $\to$ denotes greatest integer function

{x} $\to$ fractional part of x.

1) 1 2) 0 3) does not exist 4) -1

Key. 3

Sol. LHL :



$$\lim_{x\to\alpha-}\left[\frac{\min(\sin x,\ x-[x]}{(x-1)}\right]$$

When $1 < x < \alpha$

$\{x\} = x-1 < \sin x$

$\min\{\sin x,\ x-1\} = x-1$

Required limit = $\lim_{x\to\infty-}\left[\frac{x-1}{x-1}\right] = 1$

$x\to\alpha+$ $\sin x < x-1$ $\frac{\sin x}{x-1} < 1$ $\left[\frac{\sin x}{x-1}\right] = 0$

RHL :

$$\lim_{x\to\alpha+}\left[\frac{\sin x}{x-1}\right] = 0$$

Hence $LHL \neq RHL$

Limit does not exist

179. $f(x) = x^5 + x^2 + 1$ has roots x_1, x_2, x_3, x_4, x_5 *and* $g(x) = x^2 - 2$ then
$g(x_1)g(x_2)g(x_3)g(x_4)g(x_5) - 30g(x_1x_2x_3x_4x_5) =$ ____

1) 2 2) 5 3) 7 4) 11

Key. 3

Sol. Put $g(x) = y = x^2 - 2 \Rightarrow x = \sqrt{y+2} \Rightarrow f(\sqrt{y+2}) = 0$

$\Rightarrow y^5 + 20y^4 + 40y^3 + 79y^2 + 74y + 23 = 0$

Roots are $g(x_1), g(x_2), g(x_3), g(x_4), g(x_5)$

$g(x_1).g(x_2).g(x_3).g(x_4).g(x_5) = -23$

And $x_1x_2x_3x_4x_5 = -1$

$g(x_1x_2x_3x_4x_5) = g(-1) = -1$

$\therefore g(x_1).g(x_2).g(x_3).g(x_4).g(x_5) - 30g(x_1x_2x_3x_4x_5)$

$= -23 + 30 = 7$

180. $f(x) = x^2 + \lambda x + \mu\cos x$, $\lambda \in ¢$, $\mu \in ¡$. The number of ordered pairs $(\lambda,\ \mu)$ for which $f(x) = 0$ and $f(f(x)) = 0$ have same set of real roots.

1) 4 2) 6 3) 8 4) 10

Key. 1

Sol. $f(x) = x^2 + \lambda x + u\cos x$

Let α be the root of $f(x) = 0 \Rightarrow f(\alpha) = 0$

$\Rightarrow f(f(\alpha)) = f(0) = 0$ $((Q\,\alpha$ is root of $f(f(x) = 0$ also)

Now $f(0)=\mu=0$

$$f(x)=x^2+\lambda x=0 \Rightarrow x=0, x=-\lambda$$

$$f(f(x))=f(x^2+\lambda x)=(x^2+\lambda x)^2+\lambda(x^2+\lambda x)$$

$$=(x^2+\lambda x)\left\{x^2+\lambda x+\lambda\right\}=0$$

Will have same root $x=0,\ x=-\lambda$ If

$x^2+\lambda x+\lambda=0$ have no real roots

$\Rightarrow \lambda^2-4\lambda<0$

$\Rightarrow 0<\lambda<4 \Rightarrow \lambda=1,2,3$

But $\lambda=0$ is also satisfy

$(0,0),(0,1),(,2)(0,3)$ are 4 or diff. (λ,μ) does exist.

181. A polynomial of 6th degree $f(x)$ satisfies $f(x)=f(2-x)\ \forall\ x \in R$.

If $f(x)=0$ has

four distinct and two equal roots then sum of roots of $f(x)=0$ is

a) 4 b) 5 c) 6 d) 7

Key. C

Sol. Let α be a root of f(x) = 0 $\therefore$ $f(\alpha)=f(2-\alpha)$

f(x) has 4 distinct and two equal roots $\therefore$ Sum of roots = 6

182. The number of the functions f from the set X = {1, 2, 3} to the Y = {1, 2, 3, 4, 5, 6, 7} such that f(i) ≤ f(j) for i < j and i, j $\in$ X is

(A) $^6C_3$ (B) $^7C_3$ (C) $^8C_3$ (D) $^9C_3$

Key: D

Hint $^7C_3 + 2 \times {}^7C_2 + {}^7C_1 = {}^9C_3$.

183. $f(x)=\sin[x]+[\sin x], 0<x<\frac{\pi}{2}$, where $[\ ]$ represents the greatest integer function, can also be represented as

(A) $\begin{cases} 0 & , 0<x<1 \\ 1+\sin 1 & , 1\le x<\frac{\pi}{2} \end{cases}$

(B) $\begin{cases} \frac{1}{\sqrt{2}} & , 0<x<\frac{\pi}{4} \\ 1+\frac{1}{2}+\frac{1}{\sqrt{2}}+\frac{\sqrt{3}}{2} & , \frac{\pi}{4}\le x<\frac{\pi}{2} \end{cases}$

(C) $\begin{cases} 0 & , 0<x<1 \\ \sin 1 & , 1\le x<\frac{\pi}{2} \end{cases}$

(D) $\begin{cases} 0 & , 0<x<\frac{\pi}{4} \\ 1 & , \frac{\pi}{4}<x<1 \\ \sin 1 & , 1\le x<\frac{\pi}{2} \end{cases}$

Key. C

Sol. $0 < x < \frac{\pi}{2}$

$\therefore [x] = \begin{cases} 0 & if \quad 0 < x < 1 \\ 1 & if \quad 1 \le x < \frac{\pi}{2} \end{cases}$

$\Rightarrow \sin[x] = \begin{cases} \sin 0 = 0 & if \quad 0 < x < 1 \\ \sin 1 & if \quad 1 \le x < \frac{\pi}{2} \end{cases}$

We have $0 < \sin x < 1$ when $0 < x < \frac{\pi}{2}$.

$\therefore [\sin x] = 0 \ \ for \ \ 0 < x < \frac{\pi}{2}$

$\therefore \sin[x] + [\sin x] = \begin{cases} 0 & if \quad 0 < x < 1 \\ \sin 1 & if \quad 1 \le x < \frac{\pi}{2} \end{cases}$

184. Domain of function $f(x) = \ln\left|\frac{2b^2 + x^2}{b^3 - x^3} - \frac{2x}{bx + b^2 + x^2} - \frac{1}{b - x}\right|$ is

(A) R

(B) R^+

(C) $R - \left\{\frac{b}{2}\right\}$

(D) $R - \left\{b, \frac{b}{2}\right\}$

Key. D

Sol. $\left|\frac{2b^2 + x^2}{b^3 - x^3} - \frac{2x}{bx + b^2 + x^2} - \frac{1}{b - x}\right| > 0$

$\Rightarrow \quad \frac{2b^2 + x^2}{b^3 - x^3} - \frac{2x}{bx + b^2 + x^2} - \frac{1}{b - x} \ne 0$

$\Rightarrow \quad \frac{2x^2 - 3bx + b^2}{b^3 - x^3} \ne 0 \qquad x \ne b$

$\Rightarrow \quad 2x^2 - 3bx + b^2 \ne 0 \qquad \Rightarrow \qquad x \ne b, \frac{b}{2}$

185. Which of the following is a function ([.] denotes the greatest integer function, {.} denotes the fractional part function)?

(A) $\frac{1}{\log[1 - |x|]}$

(B) $\frac{x!}{\{x\}}$

(C) $x!\{x\}$

(D) $\frac{\log(x - 1)}{\sqrt{1 - x^2}}$

Key. C

Sol. For a, b & d Domain is Null set.

$\therefore$ they are not functions.

186. If $f(x)=x^{101}-2x^{11}+2x+1$ and 'g' be the inverse of 'f', then $g'(1)$ is equal to

(A) $-\frac{1}{2}$ (B) $\frac{1}{2}$

(C) $-\frac{1}{2}<g(1)<\frac{1}{2}$ (D) none of these

Key. B

Sol. $g[f(x)]=x$

$\Rightarrow g'[f(x)].f'(x)=1$

$\Rightarrow$ on putting x = 0

$\Rightarrow g'[f(0)].f'(0)=1$

$\Rightarrow g'(1).2=1$

$\Rightarrow g'(1)=\frac{1}{2}$

187. Let g (x) be a polynomial of degree 8 satisfying $g(r)=\frac{1}{r},\ r=1,2,3,.....,9,$ and

$$f(x)=\begin{cases}\dfrac{\left(\frac{x}{1}-1\right)\left(\frac{x}{2}-1\right)\left(\frac{x}{3}-1\right)........\left(\frac{x}{9}-1\right)}{x}+\dfrac{1}{x}, & x\neq 0\\ 1+\frac{1}{2}+\frac{1}{3}+.........+\frac{1}{9}, & x=0\end{cases}$$ Then $\frac{f(-1)}{g(10)}=$

a) 50 b)45 c)5 d)50

Key. B

Sol. Notes that $f(x)=g(x)$

$\Rightarrow \frac{f(-1)}{g(10)}=\frac{f(-1)}{f(10)}$

$\Rightarrow \frac{9}{\frac{1}{5}}=45$

188. The range of function f(x) defined by $f(x)=\frac{x^2+1}{\log_e(x^2+1)},\ x\in R-\{0\}$ *is*

a) $(-0,1)$ b) $(0,2)$ c) $[e,\infty)$ d) $(-\infty,\infty)$

Key. C

Sol. Let $t=x^2+1$, then $(t>1)$ and $g(t)=\frac{t}{\log t}\ \Rightarrow g'(t)=\frac{\log t-1}{\log^2 t}$.

Thus g(t) decreases for $t\in[1,e]$ and

increases for $t\in[e,\infty), g(e)=\frac{e}{\log e}=e$.

We observe that $\lim_{t\to 1^+} g(t)\to\infty$ and $\lim_{t\to\infty} g(t)\to\infty$

Thus range of g is $[e,\infty)$. Hence range of f is $[e,\infty)$.

189. Let f(x) be a polynomial one – one function such that

$f(x)f(y)+2=f(x)+f(y)+f(xy), \forall x,y \in R-\{0\} \quad f(1)\neq 1, f'(1)=3$.

Let $g(x)=\frac{x}{4}(f(x)+3)-\int_0^x f(x)dx,$ then

a) $g(x)=0$ has exactly one root for $x\in(0,1)$

b) $g(x)=0$ has exactly two roots for $x\in(0,1)$

c) $g(x)\neq 0 \;\; \forall x\in R-\{0\}$

d) $g(x)=0 \;\; \forall \; x\in R-\{0\}$

Key. D

Sol. Put x = y = 1 $\Rightarrow f(1)=2$ again put $y=\frac{1}{x} \Rightarrow f(x)+f\left(\frac{1}{x}\right)=f(x)f\left(\frac{1}{x}\right)$

$\Rightarrow f(x)=x^3+1 \Rightarrow g(x)=0 \;\; \forall x\in R-\{0\}$.

190. Let 'm' be the least value of the function f(x) = |x. lnx| , x∈ [e, ∞), then the number of values of x for which $e^{|x^2-4x+5|}$ = m is true is

(A) 2 (B) 4

(C) 1 (D) zero

Key. D

Sol. f(x) = ⇒ |x ln x|

Graph of f(x):

Obviously least value

Ocurs at x = e

∴ m = |e ln e| = e.

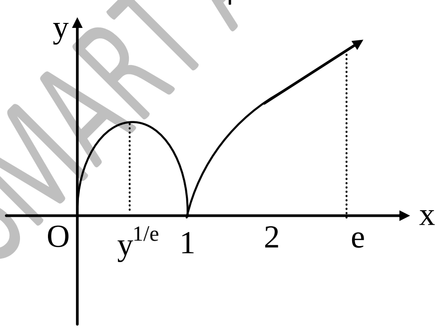
∴ $e^{|x^2-4x+5|} = e^1$

⇒ $x^2 - 4x + 4 = 0$ and $x^2 - 4x + 6 = 0$

⇒ x = 2 and no solution

But x = 2 ∉ [e, ∞)

⇒ No value of x is possible.



191. If f(x) = 2x + |x| , g(x) = $\frac{1}{3}$ (2x – |x|) and h(x) = f(g(x)) then $\sin^{-1}$ (h (h(h(h .... h.h(x))))) is n times

(A) $\sin^{-1}$ (sinx) (B) x

(C) $\sin^{-1}$ x (D) $\sin^{-1}$ (|x| + 2x)

Key. C

Sol. Since f[g(x)] = x, $\forall x \in R \Rightarrow$ h(x) = x

$\Rightarrow \sin^{-1}$ [h (h(h... h(x)))] = $\sin^{-1}$x

192. The range of the function defined as f(x) = $\cos^{-1}(-\{-x\})$ is (where {x} is fractional part of x)

(A) $\left[\frac{\pi}{2}, \pi\right)$ (B) $(0, \pi)$

(B) $\left[0, \frac{\pi}{2}\right)$ (D) $\left(\frac{\pi}{2}, \pi\right]$

Key. A

Sol. $0 \le \{-x\} < 1 \ \forall \ x \in R \Rightarrow \ -1 < -\{-x\} \le 0$, so range of f(x) is $\left[\frac{\pi}{2}, \pi\right)$

193. If $f(x) \, . \, f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$, then f(x) can be

(A) $1 \pm x^n$ (B) $\frac{2}{1+k \ \ln|x|}$, where k is a fixed real number

(C) $\frac{\pi}{2\tan^{-1}|x|}$ (D) All of these

Key. D

Sol. Consider f(x) = $1 \pm x^n \Rightarrow (f(x) - 1)(f(1/x) - 1) = (\pm x^n)\left(\pm \frac{1}{x^n}\right) = 1$

$\Rightarrow f(x) \, . \, f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$

Consider f(x) = $\frac{2}{1+k\ln|x|}$

$f(x). \, f\left(\frac{1}{x}\right) = \frac{2}{1+k\ln|x|} \times \frac{1}{1-k\ln|x|} = \frac{4}{1-k^2\ln^2|x|}$

$\Rightarrow f(x) \ f\left(\frac{1}{x}\right) = f(x) f\left(\frac{1}{x}\right)$

Consider f(x) = $\frac{\pi}{2\tan^{-1}|x|}$

$f(x) \, . \, f\left(\frac{1}{x}\right) = \frac{\pi}{2\tan^{-1}|x|} . \frac{\pi}{2\tan^{-1}\left|\frac{1}{x}\right|} = \frac{\pi^2}{4\tan^{-1}|x|.\cot^{-1}|x|}$

$f(x) + f\left(\frac{1}{x}\right) = \frac{\pi}{2\tan^{-1}|x|} + \frac{\pi}{2\tan^{-1}\frac{1}{|x|}} = \frac{\pi}{2}\left(\frac{1}{\tan^{-1}|x|} + \frac{1}{\cot^{-1}|x|}\right)$

$= \frac{\pi}{2}\frac{\cot^{-1}|x| + \tan^{-1}|x|}{\cot^{-1}|x|.\tan^{-1}|x|} = \frac{\pi^2}{4\tan^{-1}|x|\cot^{-1}|x|}$

$\Rightarrow$ f(x) f(1/x) = f(x) + f(1/x)

194. A function f: R $\rightarrow$ R is defined by $f(x)=x^4-10x^3+9x^2-x+1$ then f is

A) A bijection B) one-one but not onto

C) Onto but not one-one D) Neither one-one nor onto

Key. D
Sol. Conceptual

195. If $f:R\rightarrow R$ and $f(x)=\frac{x^2+4x+7}{x^2+x+1}$ then $f(x)$ is

(A) one-one function (B) bijective function
(C) many one function (D) Identity function

Key. C
Sol. $f(x)$ is monotonic function

196. If $f(x)=x^3+x^2 \quad 0\le x\le 2$
$=x+2 \quad 2<x\le 4$ and $g(x)$ is even extension of $f(x)$ then

(A) $g(x)=-x+2 \quad -4\le x<-2$
$=-x^3+x^2 \quad -2\le x\le 0$

(B) $g(x)=x-2 \quad -4\le x<-2$
$=x^3-x^2 \quad -2\le x\le 0$

(C) $g(x)=-x+2 \quad -4\le x<-2$
$=x^3-x^2 \quad -2\le x\le 0$

(D) $f(x)=x-2 \quad -4\le x<-2$
$=-x^3+x^2 -2\le x\le 0$

Key. A
Sol. Conceptual

197. $f(x)=\frac{\cos x}{\left[\frac{2x}{\pi}\right]+\frac{1}{2}}$ (where x is not integral multiple of π and [.] denotes the greatest integer function)

is

(A) an odd function (B) an even function (C) neither odd nor even (D) none of these

Key. A
Sol. $\left[\frac{-2x}{\pi}\right]+\frac{1}{2}=-\left(\left[\frac{2x}{\pi}\right]+\frac{1}{2}\right)$

198. If $f(x)+2f(1-x)=x^2+2 \ \forall x\in R$ then $f(x)$ is given as

(A) $\frac{(x-2)^2}{3}$ (B) x^2-2 (C) 1 (D) none of these

Key. A
Sol. Replace x with $(1-x)$ in the given expression

199. If $f(x)=\frac{x-[x]}{1+x-[x]}$ $x\in R$ (where [.] denotes the greatest integer function) then $f(R)$ can not

contain

(A) 1 (B) $\frac{3}{4}$ (C) $\frac{1}{4}$ (D) $-\frac{1}{2}$

Key. A,B,D
Sol. Find the range of $f(x)$

200. Equation of the locus of points equidistant from two points $(f(-1), f(0), f(1))$ and $(f'(1), f'(-2), f'(2))$ where 'f' is a differentiable function satisfying the equation $f(x-f(y)) = f(f(y)) + x\,f(y) + f(x) - 1, \forall x, y \in R$

(a) $6x-4y+10z+15=0$ (b) $3x-2y+5z+15=0$

(c) $6x+4y+10z-15=0$ (d) $3x+2y-5z-15=0$

Key. A

Sol.

$$f(x-f(y)) = f(f(y)) + xf(y) + f(x) - 1 \rightarrow \quad I$$

$$put\ \ x = f(y) = 0$$

$$\Rightarrow f(0) = f(0) + 0 + f(0) - 1$$

$$\Rightarrow f(0) = 1$$

$$put\ \ x = f(y) = k\ \ in \quad I$$

$$f(0) = f(k) + k(k) + f(k) - 1$$

$$1 = k^2 + 2f(k) - 1$$

$$\Rightarrow 2f(k) = 2 - k^2$$

$$\Rightarrow f(k) = 1 - \frac{k^2}{2}$$

$$\Rightarrow f(x) = 1 - \frac{x^2}{2}$$

$$\Rightarrow f^1(x) = -x$$

$$A\left(\frac{1}{2}, 1, \frac{1}{2}\right), B(-1, 2, -2)$$

Let $p(x, y, z)$ be the point on the locus

$$\Rightarrow PA^2 = PB^2$$

$$\left(x-\frac{1}{2}\right)^2 + (y-1)^2 + \left(z-\frac{1}{2}\right)^2 = (x+1)^2 + (y-2)^2 + (z+2)^2$$

$$\Rightarrow 6x - 4y + 10z + 15 = 0$$

201. The function defined by

$$f(x) = \begin{cases} x|x| & x \le -1 \\ [1+x] + [1-x] & -1 < x < 1 \\ -x|x| & x \ge 1 \end{cases}$$ is

a) an odd function b) an even function c) neither even nor odd d) even as well as odd

Key. B

Sol. Draw graph.

202. The range of the function $\dfrac{2+x^2}{5+4x^2+x^4}$

a) $(0,1)$ b) $\left(0, \dfrac{3}{4}\right)$ c) $\left(0, \dfrac{2}{3}\right)$ d) None of these

Key. D

Sol. $f(x)=\dfrac{1}{(x^2+x)+\dfrac{1}{(x^2+2)}}, x^2+2=t,\ t\ge 2$

$\max f(x)=\dfrac{1}{2+\dfrac{1}{2}}=\dfrac{2}{5}$

$\min f(x)\to 0 \Rightarrow \text{range}\in\left(0,\dfrac{2}{5}\right]$

203. The function f(x) is defined on [0, 1] as following $f(x)=\begin{cases} x, & \text{if } x \text{ is rational} \\ 1-x & \text{if } x \text{ is irrational}\end{cases}$ then for all $x\in[0,1]$ f f(x) is equal to

a) 0 b) 1 + x c) x d) 1

Key. C

Sol. $f(f(x))=f(x)=x$, if x is rational.

$=1-f(x)=1-(1-x)$ if x is irrational

204. If $f(x)=\log_{100x}\left(\dfrac{2(\log_{10}x)+2}{-x}\right), g(x)=\{x\}$ where {} denotes the fractional part of x, then range of g(x) for existence of fog(x) is

a) $(0,100)\cup(100,200)$ b) $\left(0,\dfrac{1}{100}\right)\cup\left(\dfrac{1}{100},1\right)$

c) $\left(0,\dfrac{1}{100}\right)\cup\left(\dfrac{1}{100},\dfrac{1}{10}\right)$ d) None of these

Key. C

Sol. Range of g(x) $\subseteq$ domain of f(x)

Domain of f(x) is $\left(0,\dfrac{1}{100}\right)\cup\left(\dfrac{1}{100},\dfrac{1}{10}\right)$

205. If $f(x)=\dfrac{9^x}{9^x+3}$ for all $x\in R$ then the sum $f\left(\dfrac{1}{1996}\right)+f\left(\dfrac{2}{1996}\right)+.........+f\left(\dfrac{1995}{1996}\right)$ is

a) 997.5 b) 997 c) 996.5 d) 996

Key. A

Sol. $f(x)=\dfrac{9x}{9^x+3}\Rightarrow f(1-x)=\dfrac{3}{3+3^x}\therefore f(x)+f(x)=1$

G.E. $\left[f\left(\dfrac{1}{1996}\right)+f\left(\dfrac{1995`}{1996}\right)\right]+\left[f\left(\dfrac{2}{1996}\right)+f\left(\dfrac{1994}{1996}\right)\right]+$

$$........+\left[f\left(\frac{997}{1996}\right)+f\left(\frac{999}{1996}\right)\right]+f\left(\frac{998}{1996}\right)$$

$$=[1+1+.............\text{upto 99 times}]+\frac{1}{2}=997.5$$

206. The function $f:R\to R$ defined by $f(x)=\frac{x^2+4x+30}{x^2-8x+18}$

a) injective but not surjective b) Surjection but not injective

c) Both injective and surjective d) Neither injective nor surjective

Key. D

Sol. $x^2-8x+18$ is not zero for any real number because $x^2-8x+18$ can be written as $(x-4)^2+2$ and numerator $x^2+4x+30$ is also +ve because $(x+2)^2+26\equiv x^2+4x+30$ since f take values which are only $+$ ve for real $'x'$. Range f is a sub set of $(0,\infty)$ $\therefore$ range $f\neq R\Rightarrow f$ is not on to also

$$f(0)=\frac{5}{3}\Rightarrow\frac{x^2+4x+30}{x^2-8x+18}=\frac{5}{3}\Rightarrow 2x^2=52x\Rightarrow x=26 \text{ if } x\neq 0$$

$\therefore f(10)=\frac{5}{3}=26$ $\therefore f$ is not injective.

207. If $f\left(x+\frac{y}{8},x-\frac{y}{8}\right)=xy$, then f(m, n) + f(n, m) = 0

A) only when m = n B) only when $m\neq n$ C) only when m = - n D) for all m & n

Key. D

Sol. Let $x+\frac{y}{8}=m$ $x-\frac{y}{8}=n$

$x=\frac{m+n}{2},\ y=4(m-n)$

F(m, n) = $2(m^2-n^2)$

Similarly f(n, m) = $2(n^2-m^2)$

= f(m, n) + f(n, m) = 0 $\forall$ m, n

208. If $y=\sqrt{\log_{\sin x}\left(\frac{|x|}{x}\right)}$ then the possible set of values of x and y are

A) $x\in[2n\pi,2n\pi+\pi],\ y\in\{0,1\}$

B) $x\in(0,\infty),y\in\{1\}$

C) $x\in\bigcup_{n\in W}\left(2n\pi,2n\pi+\frac{\pi}{2}\right)\cup\left(2n\pi+\frac{\pi}{2},(2n+1)\pi\right)$ and $y\in\{0\}$

D) $x\in\bigcup_{n\in W}(2n\pi,(2n+1)\pi)$ and $y\in\{0,1\}$

Key. C

Sol. $\log_{\sin x}\frac{|x|}{x}$ $\Rightarrow$ $\sin x\in(0,1)$ and $x\in(0,\infty)$

$\therefore \quad x \in \bigcup_{n \in W}\left(2n\pi, 2n\pi+\frac{\pi}{2}\right) \cup \left(2n\pi+\frac{\pi}{2},(2n+1)\pi\right)$ and $y \in \{0\}$

209. Let S be the set of all triangles and R^+ be the set of positive real numbers. Then the function, $f: S \to R^+, F(\Delta) =$ area of Δ, where $\Delta \in S$ is

A) injective but not surjective
B) surjective but not injective
C) injective as well as surjective
D) neither injective nor surjective

Key. B

Sol. Two triangle may have equal areas

$\therefore$ f is not one-one

Since each positive real number can represent area of a triangle

$\therefore$ f is onto

210. $f(x) = |x-1|, f: R^+ \to R$ and $g(x) = e^x, g: [-1, \infty) \to R$ if the function fog(x) is defined, then its domain and range respectively are

A) $(0,\infty) \& [0,\infty)$
B) $[-1,\infty) \& [0,\infty)$
C) $[-1,\infty) \& \left[1-\frac{1}{e},\infty\right)$
D) $[-1,\infty) \& \left[\frac{1}{e},-1,\infty\right)$

Key. B

Sol. $f(x) = |x-1| = \begin{cases} 1-x & 0 < x < 1 \\ x-1 & x \ge 1 \end{cases}$

$g(x) = e^x \qquad x \ge -1$

$(fog)(x) = \begin{cases} 1-g(x) & 0 < g(x) < 1 \quad \text{ie } -1 \le x < 0 \\ g(x)-1 & g(x) \ge 1 \quad \text{ie } \quad 0 \le x \end{cases}$

$= \begin{cases} 1-e^x & -1 \le x < 0 \\ e^x - 1 & x \ge 0 \end{cases}$

$\therefore$ domain $= [-1, \infty)$

Fog is decreasing in [- 1, 0] and increasing in $(0, \infty)$

$fog(-1) = 1-\frac{1}{e}$ and fog (0) = 0

lie $x \to \infty$ fog (x) = ∞ $\therefore$ range $[0,\infty)$

211. The range of the function $f(x) = \log_{\sqrt{2}}(2-\log_2)\left(2-\log_2\left(16\sin^2 x+1\right)\right)$ is

A) $(-\infty,1)$
B) $(-\infty,2)$
C) $(-\infty,1]$
D) $(-\infty,2]$

Key. D

Sol. $f(x) = \log_{\sqrt{2}}\left(2-\log_2\left(16\sin^2 x+1\right)\right)$

$1 \le 16\sin^2 x + 1 \le 17$

$\therefore \quad 0 \le \log_2\left(16\sin^2 x+1\right) \le \log_2 17$

$\therefore \quad 2-\log_2 17 \le 2-\log_2\left(16\sin^2 x+1\right) \le 2$

Now consider

$0 < 2-\log_2\left(16\sin^2 x+1\right) \le 2$

$-\infty < \log_{\sqrt{2}}\left[2-\log_2\left(16\sin^2 x+1\right)\right] \le \log_{\sqrt{2}} 2 = 2$

the range is $(-\infty, 2]$

212. If $f(x).f(y)=f(x)+f(y)+f(xy)-2\ \forall x,y\in R$ and iff(x) is not a constant function, then the value of f(1) is equal to
A) 1 B) 2 C) 0 D) – 1

Key. B

Sol. Put x = y = 1, $\left(f(1)^2 = 3f(1)-2\right)$ $\Rightarrow$ $f(1)=1$ or 2

Let f(1) = 1, then put y = 1

f(x). f(1) = f(x) + f(1) + f(x) – 2

$\Rightarrow$ f(x) = 1 constant function

$\therefore$ $f(1)\neq 1$, hence f(1) = 2

213. Let $f(x)=\tan x,\ g(f(x))=f\left(x-\frac{\pi}{4}\right)$, where f(x) and g(x) are real valued functions. For all possible value of x, f(g(x)) =

A) $\tan\left(\frac{x-1}{x+1}\right)$ B) tan(x – 1) – tan (x + 1)

C) $\frac{f(x)+1}{f(x)-1}$ D) $\frac{x-\pi/4}{x+\pi/4}$

Key. A

Sol. $g(f(x))=\tan\left(x-\frac{\pi}{4}\right)=\frac{\tan x-1}{\tan x+1}$ $\Rightarrow$ $g(x)=\frac{x-1}{x+1}$

$f(g(x))=\tan\left(\frac{x-1}{x+1}\right)$.

214. Let h(x) = |kx + 5|, domain of f(x) is [- 5, 7], domain of f(h(x)) is [- 6, 1] and range of h(x) is the same as the domain of f(x), then value of k is

A) $\frac{1}{3}$ B) $\frac{4}{5}$ C) 1 D) none of these

Key. D

Sol. $-5\le|kx+5|\le 7$ $\Rightarrow$ $-12\le kx\le 2$ where $-6\le x\le 1$

$\Rightarrow$ $-6\le\frac{k}{2}x\le 1$ where $-6\le x\le 1$

$\therefore$ k = 2 { Q range of h(x) = domain of f(x)}

215. The function $(x)=\frac{x}{e^x-1}+\frac{x}{2}+1$ is

A) an odd function B) an even function

C) neither an odd nor an even function D) a periodic function

Key. B

Sol. $f(x)=\frac{x}{e^x-1}+\frac{x}{2}+1=\frac{2x+xe^x-x}{2(e^x-1)}+1=\frac{x+xe^x}{2(e^x-1)}+1$

$f(-x)=\frac{-x-xe^{-x}}{2(e^{-x}-1)}+1=\frac{x+xe^x}{2(e^x-1)}+1$

$\therefore$ f(-x) = f(x) for all x

$\therefore$ f(x) is an even function.

216. Let f: {x, y, z} $\to$ {1, 2, 3} be a one-one mapping such that only one of the following three statements is true and remaining two are false: $f(x)\neq 2, f(y)=2, f(z)\neq 1$, then

A) f(x) > f(y) > f(z) B) f(x) < f(y) < f(z) C) f(y) < f(x) < f(z) D) f(y) < f(z) < f(x)

Key. C

Sol. **Case – I** f(x) $\neq$ 2 is true, f(y) = 2 and f(z) $\neq$ 1 are false, then
f(x) = 1 or 3, f(y) = 1 or 3 and f(z) = 1
$\Rightarrow$ f is not one-one
Case – II f(x) $\neq$ 2 is false, f(y) = 2 is true, f(z) $\neq$ 1 is false, then
f(x) = 2, f(y) = 2, f(z) = 1
$\Rightarrow$ not possible
Case – III f(x) $\neq$ 2 is false, f(y) = 2 is false, f(z) $\neq$ 1 is true, then
f(x) = 2, f(y) = 1 or 3 f(z) = 2 or 3
$\Rightarrow$ f(x) = 2, f(z) = 3, f(y) = 1

217. The image of the interval [- 1, 3] under the mapping specified by the function $f(x)=4x^3-12x$ IS

A) [f(+ 1), f(- 1)] B) [f(-1), f(3)] C) [- 8, 16] D) [- 8, 72]

Key. D

Sol. f(x) = 4x (x^2-3)

$f'(x)=12x^2-12=0$

or $x=\pm 1$

f(x) $\in$ [f(1), max (f(- 1), f(3))] = [- 8, 72]

218. If $f(x)=2\sin^2\theta+4\cos(x+\theta)\sin x.\sin\theta+\cos(2x+2\theta)$ then value of $f^2(x)+f^2\left(\frac{\pi}{4}-x\right)$

A) 0 B) 1 C) – 1 D) x^2

Key. B

Sol. $f(x)=2\sin^2\theta+4\cos(x+\theta)\sin x.\sin\theta+\cos(2x+2\theta)$

$=2\sin^2\theta+\cos(2x+2\theta)+2\cos(x+\theta)\cos(x-\theta)-2\cos^2(x+\theta)$

$=2\sin^2\theta+2\cos^2(x+\theta)-1+2\cos^2 x-2\sin^2\theta-2\cos^2(x+\theta)$

= cos 2x

$\therefore f^2(x)+f^2\left(\frac{\pi}{4}-x\right)=\cos^2 2x+\sin^2 2x=1$

219. Let $G(x)=\left(\frac{1}{a^x-1}+\frac{1}{2}\right)$ F9x), where ‘a’ is a positive real number not equal to 1 and F(x) is an odd function, Which of the following statements is true?

A) G(x) is an odd function
B) G(x) is an even function
C) G(x) is neither even nor odd function
D) Whether G(x) is an odd or even function depends on the value of ‘a’

Key. B

Sol. $G(x)=\left(\frac{1}{a^x-1}+\frac{1}{2}\right)F(x)$

$G(-x)=\left(\frac{1}{a^{-x}-1}+\frac{1}{2}\right)F(-x)=-\left(\frac{a^x}{1-a^x}+\frac{1}{2}\right)F(x)=\left(\frac{a^x}{a^x-1}-\frac{1}{2}\right)F(x)$

$=\left(\frac{a^x-1+1}{a^x-1}-\frac{1}{2}\right)F(x)=\left(1+\frac{1}{a^x-1}-\frac{1}{2}\right)F(x)=\left(\frac{1}{a^x-1}+\frac{1}{2}\right)F(x)=G(x)$

$\therefore$ G(x) is an even function.

Functions

Multiple Correct Answer Type

1. Let $f(x)$ be a non constant polynomial satisfying the relation $f(x).f(y)=f(x)+f(y)+f(xy)-2$ for all real x and y and $f(0) \neq 1$, suppose $f(4)=65$. Then

(A) $f^1(x)$ is a polynomial of degree 2 (B) roots of $f^1(x)=2x+1$ are real

(C) $xf^1(x)=3[f(x)-1]$ (D) $f^1(-1)=3$

Key. A,B,C,D

Sol. Given $f(x).f(y)=f(x)+f(y)+f(xy)-2 \forall x,y \in R.$

Put $x=y=1$ then $f(1)^2-3f(1)+2=0$

$\Rightarrow f(1)=1(or)f(1)=2$

If $f(1)=1$, then $f(x)=1 \forall x \in R.$

A contradiction $\because$ degree of $f(x)$ is positive.

$\therefore f(1)\neq 1.$ hence $f(1)=2$.

Replace '$\underline{y}$' with $\frac{1}{x}$ $then\ f(x).f\left(\frac{1}{x}\right)=f(x)+f\left(\frac{1}{x}\right)\left(\because f(1)=2\right)$

$\therefore f(x)$ must be in the form $x^n+1(or)x^n+1$.

$\because f(4)=65, f(x)=x^3+1 \Rightarrow f^1(x)=3x^2.$

2. f : R → [–1, ∞) and f(x) = ln([|sin2x| + |cos2x|]) (where [·] is greatest integer function).

(A) R^- ∩ range of f is null set

(B) f(x) is periodic but fundamental period not defined

(C) f(x) is invertible in $\left[0, \frac{\pi}{4}\right]$ (D) f(x) is into function.

Key. A, B, D

Sol. Period of f(x) = |sin2x| + |cos2x| is π/4

but f(x) = ln ([|sin2x| + |cos2x|])

Max. value of |sin2x| + |cos2x| = $\sqrt{2}$

f(x) = ln ([$\sqrt{2}$]) = ln (1) = 0

⇒ it is periodic function but fundamental period not defined.

f(x) is many one and into function

3. $f:[0,1]\rightarrow R$ is a differentiable function such that $f(0)=0$ and $|f(x)|\leq k|f(x)|$ for all x $\in[0,1],(k>0)$, then which of the following is/are always true ?

(A) $f(x)=0, \forall x \in R$ (B) $f(x)=0, \forall x \in [0,1]$

(C) $f(x) \neq 0, \forall x \in [0,1]$ (D) $f(1)=k$

Key. B

Sol. $(f(x))^2 - k^2(f(x))^2 \leq 0$

$\Rightarrow (f'(x)-kf(x))(f'(x)+kf(x)) \leq 0$

$\Rightarrow \left(f(x)e^{-kx}\right)'\left(f(x)e^{kx}\right)' \leq 0$

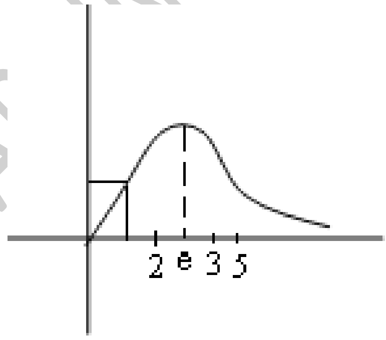
$\Rightarrow$ Exactly one of the functions $g_1(x)=f(x)e^{-kx}$ or $g_2(x)=f(x)e^{kx}$ is non decreasing.

But $f(0)=0 \Rightarrow$ both function g_1 and g_2 have a value zero at x = 0

$\forall x \in [0,1], g_1(0)=0$ and g_1 increasing $\Rightarrow g_1(x) \geq 0 \Rightarrow f(x) \geq 0$

$g_2(0)=0$ and g_2 decreasing $\Rightarrow g_2(x) \leq 0 \Rightarrow f(x) \leq 0$

$\Rightarrow f(x)=0 \forall x \in [0,1]$



4. If a function satisfies $(x-y)f(x+y)-(x+y)f(x-y)=2(x^2y-y^3) \forall x,y \in R$ and $f(1)=2$, then

(A) f(x) must be polynomial function (B) f(3) = 12

(C) f(0) = 0 (D) f(x) may not be differentiable

Key. A,B,C

Sol. $(x-y)f(x+y)-(x+y)f(x-y)=2y(x-y(x+y))$

Let $x-t=u; x+y=v$

$uf(v)-vf(u)=2uv(v-u)$

$\frac{f(v)}{v}-\frac{f(u)}{u}=v-u$

$\left(\frac{f(v)}{v}-v\right)=\left(\frac{f(u)}{u}-u\right)=$ constant

Let $\frac{f(x)}{x}-x=\lambda$

$\Rightarrow f(x)=(\lambda x+x^2)$

$f(1)=2$

$\lambda+1=2 \Rightarrow \lambda=1 \qquad f(x)=x^2+x$

5. Let $f: R-\{0,1\} \to R$ satisfying $f(x)+f\left(\frac{1}{1-x}\right)=\frac{2(1-2x)}{x(1-x)}$ then

(A) $\int_3^4 f(x)dx = In\left(\frac{9e}{4}\right)$

(B) the graph of $y=f(x)$ crosses x-axis at $x=-1$

(C) $f(2)+f(3)=5$

(D) $f(2)+f(3)=6$

Key. A,B,C

Sol. $f(x)+f\left(\frac{1}{1-x}\right)=\frac{2}{x}-\frac{2}{1-x}$

Replacing x by $\frac{1}{1-x}$ and by $\left(1-\frac{1}{x}\right)$ in equation (1) one by one and on solving, we get $f(x)=\frac{x+1}{x-1}$

6. Let $f(x)$ be an onto function defined from $[-3,2]$ to $[2,7]$ then

A) Domain of $f(\sin x-\cos x)$ is R

B) Range of $f(x)$ & $f(x+7)$ is same

C) Domain of $f(x)$ & $f(x-2)$ is same

D) Range of $\frac{5}{f(x-2)+3}$ is $\left[\frac{1}{2},1\right]$

Key. A,B,D

Sol. $-3 \le x \le 2$

Now $-3 \le \sin x - \cos x \le 2$ which is True $\forall x$

$\therefore \sin x - \cos x$ also lies in $\left[-\sqrt{2},\sqrt{2}\right]$

Range of f(x) and f(x + 7) is same

Range of $\frac{5}{f(x-2)+3}$ is $[\varepsilon,1]$

$\therefore 2 \le f(x-2) \le 7$

7. Let $f: R \to R$, such that $f''(x) - 2f'(x) + f(x) = 2e^x$ and $f'(x) > 0, \forall\ x \in R$, then which of the following can be correct

A) $|f(x)| = -f(x), \forall\ x \in R$ B) $|f(x)| = f(x), \forall\ x \in R$

C) $f(3) = -5$ D) $f(3) = 7$

Key. B,D

Sol. The equation can be written as $\frac{d}{dx}\left(e^{-x}\left(f'(x) - f(x)\right)\right) = 2$

$\Rightarrow e^{-x}\left(f'(x) - f(x)\right) = 2x + c_1$

$\Rightarrow f(x) = \left(x^2 + c_1 x + c_2\right)e^x$ and $f'(x) = \left(x^2 + (c_1+2)x + c_1 + c_2\right)e^x$

Given that $f'(x) > 0 \Rightarrow c_1^2 - 4c_2 + 4 < 0 \Rightarrow c_1^2 - 4c_2 < 0 \Rightarrow f(x) > 0$

8. Let $f: R \to [\alpha, \infty), f(x) = x^2 + 3ax + b, g(x) = \sin^{-1}\frac{x}{4}, (\alpha \in R)$ then

(A) The number of possible integral values of 'a' for which $f(x)$ is many to one in $[-3, 5]$ is 5

(B) If a = -1 and $gof(x)$ is defined for $x \in [-1, 1]$ then number of possible integral values of 'b' can be 3.

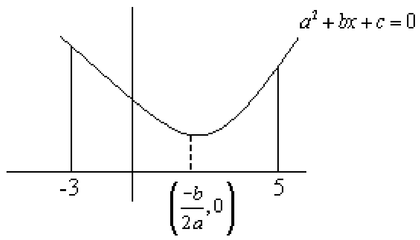
(C) If $a = 2, \alpha = -8$ then the value of 'b' for which $f(x)$ is surjective is 1

(D) If $a = 1, b = 2$ then exact number of integers in the range of $fog(x)$ is 3

Key. A,B,C

Sol. (A) $-3 < -\frac{3a}{2} < 5 \Rightarrow -2 < -a < \frac{10}{3} \Rightarrow -\frac{10}{3} < a < 2$

$\Rightarrow a \in \{-3, -2, -1, 0, 1\}$



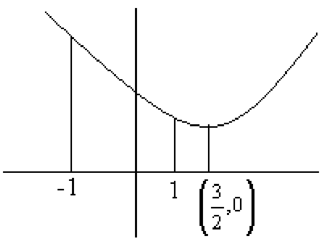
(B) $f(x) = x^2 - 3x + b$

$gof(x) = \sin^{-1}\left(\frac{f(x)}{4}\right)$ is defined for $\left|\frac{f(x)}{4}\right| \le 1$, $|x| \le 1$

$\Rightarrow -1 \le \frac{f(x)}{4} \le 1 \quad , |x| \le 1$

$\Rightarrow -4 \le f(x) \le 4 \quad , |x| \le 1$

$\Rightarrow -4 \le x^2 - 3x + b \le 4 \quad , |x| \le 1$



$f\min = f(1) = b-2$

$f\max = f(-1) = b+4$

$\Rightarrow b-2 \geq -4 \ and \ b+4 \leq 4$

$b \geq -2 \ and \ b \leq 0$

$\Rightarrow Integral \ values \ of \ b = -2,-1,0$

(C) $f: R \to [-8,\infty)$ $\quad f(x) = x^2+6x+b$

$f\min = f(-3) = b-9$

$\Rightarrow b-9 = -8$

$\Rightarrow b = 1$

(D) $h(x) = fog(x) = \left(\sin^{-1}\frac{x}{4}\right)^2 + 3\left(\sin^{-1}\frac{x}{4}\right) + 2$

$h(t) = t^2+3t+2$ where $t = \sin^{-1}\frac{x}{4},\ t \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$\left(t+\frac{3}{2}\right)^2 - \frac{1}{4}$

max $(h(t))$ occurs when $t = \frac{\pi}{2}$ and min $h(t)$ occurs at $t = \frac{-3}{2}$

min $h(t) = \frac{-1}{4}$ and max of $h(t) = \frac{\pi^2}{4}+\frac{3\pi}{2}+2$

Range of $(fog)(x) = \left[-\frac{1}{4},\frac{\pi^2}{4}+\frac{3\pi}{2}+2\right]$

9. Range of the function $f(x) = \sqrt{\frac{1-|x|}{2-|x|}}$ contains

(A) $[0,1)$ (B) $[1,2)$

(C) $(1,2)$ (D) $(2,\infty)$

Key. C,D

Sol. Domain $= \{x \in R : |x| \leq 1 \text{ or } |x| > 2\}$

If $y_0 \in$ range then $y_0 \geq 0$ and

$y_0^2(2-|x|) = 1-|x|$

$0 \leq |x| = \frac{2y_0^2-1}{y_0^2-1} = 2+\frac{1}{y_0^2-1}$

$\therefore\ y_0^2-1>0$ or $\frac{1}{y_0^2-1} \leq -1$, ; also $y_0^2 \leq \frac{1}{2}$ or >1.

$y_0 \in (1,\infty)$ or $y_0^2-1<0$ and $y_0^2 \leq \frac{1}{2}$

$y_0 \in \left[0,\frac{1}{\sqrt{2}}\right] \cup (1,\infty)$.

10. Let f(x) = $\frac{\sin x}{\sqrt{1+\tan^2 x}} + \frac{\cos x}{\sqrt{1+\cot^2 x}}$, then

(A) fundamental period of f(x) is 2π
(B) fundamental period of f(x) is π
(C) domain of f(x) is R
(D) range of f(x) is [–1, 1]

Key. A, D

Sol. f(x) = $\begin{cases} \sin 2x & 0 < x < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x < \pi \\ -\sin 2x & \pi < x < \frac{3\pi}{2} \\ 0 & \frac{3\pi}{2} < x < 2\pi \end{cases}$

So f(x) is periodic with fundamental period 2π and range [–1, 1].

11. Solutions of the equations [x] + [y] = [x] [y] is/are where [.] denotes the greater integer function

A) $2 \le x < 3$ *and* $2 \le y < 3$
B) $0 \le x < 1$ *and* $0 \le y < 1$
C) $0 < x \le 2$ *and* $0 < y \le 2$
D) None

Key. A,B

Sol. Let a = [x] + [y] = [x]. [y].

Then from the given equation, we have a + b = a. b $\Rightarrow$ ab – a – b = 0

$\Rightarrow$ ab – a – b + 1 = 1 $\Rightarrow$ (a – 1)(b – 1) = 1.

This is possible if (i) a – 1 = 1, b – 1 = 1 or (ii) a – 1 = - 1, b – 1 = - 1.

Now, for (i), a – 1 = 1 $\Rightarrow$ a = 2 and b – 1 = 1 $\Rightarrow$ b = 2

And for (ii) a – 1 = - 1 $\Rightarrow$ a = 0 and b – 1 = - 1 $\Rightarrow$ b = 0.

Thus (a = 2 and b = 2) or (a = 0, b = 0)

i.e., ([x] = 2 and [y] = 2) or ([x] = 0, [y] = 0).

But $[x] = 2 \Rightarrow 2 \le x < 3$ and $[y] = 2 \Rightarrow 2 \le y < 3$.

Again $[x] = 0 \Rightarrow 0 \le x < 1$ and $[y] = 0 \Rightarrow 0 \le y < 1$.

Thus, the solution sets are (i) $0 \le x < 1$ and $0 \le y < 1$ (ii) $2 \le x < 3$ and $2 \le y < 3$.

12. For real number 'x', let $[x]$ denote the largest integer smaller than or equal to 'x' and (x) denote the smallest integer greater than or equal to 'x'. Also, let $f(x) = \min(x - [x], (x) - x)$ for $0 \le x \le 4$. Then

A) $f(x)$ is not periodic
B) $f(x)$ is periodic with period 1
C) $\int_0^4 f(x)dx = 1$
D) $\int_0^4 f(x)dx = 2$

Key. B,C

Sol.

$$f(x)=\begin{cases} x & if \quad 0\le x\le \frac{1}{2} \\ 1-x & if \quad \frac{1}{2}\le x\le 1 \end{cases}.$$

So $'f'$ periodic with period 1

Also, $\int_0^1 f(x)dx=4\int_0^1 f(x)dx=4\left(\frac{1}{8}+\frac{1}{8}\right)=1$

13. Let $f:R\to R$ be a function defined by $f(x+1)=\frac{f(x)-5}{f(x)-3}\,\forall x\in R$.

Then which of the following statement(s) is/are true

A) $f(x+2)=f(x)$

B) $f(x+4)=f(x)$

C) $f(x+6)=f(x)$

D) $f(x+8)=f(x)$

Key. B,D

Sol. $f(x+1)=\frac{f(x)-5}{f(x)-3}$ ....(1)

$\Rightarrow f(x)=\frac{3f(x+1)-5}{f(x+1)-1}$

Replace $'x'$ by $x-1$

$\Rightarrow f(x-1)=\frac{3f(x)-5}{f(x)-1}$ ....(2)

Replace $'x'$ by $x+1$ in (1)

$f(x+2)=\frac{f(x+1)-5}{f(x+1)-3}=\frac{2f(x)-5}{f(x)-2}$ ....(3)

Similarly, $f(x-2)=\frac{2f(x)-5}{f(x)-2}$

$\therefore\ f(x+2)=f(x-2)\Rightarrow f(x+4)=f(x)$

14. Let $f(x)=a_1\cos(\alpha_1+x)+a_2\cos(\alpha_2+x)+.....+a_n\cos(\alpha_n+x)$.

If $f(x)$ vanishes for $x=0$ and $x=x_1$ (where $x_1\ne k\pi,\,k\in Z$), then

A) $a_1\cos\alpha_1+a_2\cos\alpha_2+......+a_n\cos\alpha_n=0$

B) $a_1\sin\alpha_1+a_2\sin\alpha_2+......+a_n\sin\alpha_n=0$

C) $f(x)=0$ has only two solutions $0, x_1$ D) $f(x)$ is identically zero $\forall\ x$

Key. A,B,D

Sol. $f(0)=0 \Rightarrow a_1\cos\alpha_1+a_2\cos\alpha_2+.....+a_n\cos\alpha_n=0$

$f(x_1)=0 \Rightarrow (a_1\cos\alpha_1+a_2\cos\alpha_2+.....+a_n\cos\alpha_n)\cos x_1$

$+(a_1\sin\alpha_1+a_2\sin\alpha_2+.....+a_n\sin\alpha_n)\sin x_1=0$

$\Rightarrow a_1\sin\alpha_1+a_2\sin\alpha_2+.....+a_n\sin\alpha_n=0 \because (x_1\neq n\pi)$

$\therefore\ a_1\cos\alpha_1+a_2\cos\alpha_2+.....+a_n\cos\alpha_n=0$

$\&\ a_1\sin\alpha_1+a_2\sin\alpha_2+.....+a_n\sin\alpha_n=0 \Rightarrow f(x)=0\ \forall\ x$

15. Let $f: R\to R,$ such that $f''(x)-2f'(x)+f(x)=2e^x$ and $f'(x)>0, \forall\ x\in R,$ then which of the following can be correct

A) $|f(x)|=-f(x), \forall\ x\in R$ B) $|f(x)|=f(x), \forall\ x\in R$

C) $f(3)=-5$ D) $f(3)=7$

Key. B,D

Sol. The equation can be written as $\frac{d}{dx}\left(e^{-x}\left(f'(x)-f(x)\right)\right)=2$

$\Rightarrow e^{-x}\left(f'(x)-f(x)\right)=2x+c_1$

$\Rightarrow f(x)=\left(x^2+c_1x+c_2\right)e^x$ and $f'(x)=\left(x^2+(c_1+2)x+c_1+c_2\right)e^x$

Given that $f'(x)>0 \Rightarrow c_1^2-4c_2+4<0 \Rightarrow c_1^2-4c_2<0 \Rightarrow f(x)>0$

16. If $2f(x)+xf\left(\frac{1}{x}\right)-2f\left(\left|\sqrt{2}\sin\pi\left(x+\frac{1}{4}\right)\right|\right)=4\cos^2\left(\frac{\pi x}{2}\right)+x\cos\frac{\pi}{x}, \forall x\in R-\{0\}$ then which of the following statement(s) is/are true?

a) $f(2)+f\left(\frac{1}{2}\right)=1$ b) $f(2)+f(1)=0$ c) $f(2)+f(1)=f\left(\frac{1}{2}\right)$ d) $f(1)\ f\left(\frac{1}{2}\right)\ f(2)=1$

Ans: a,b,c

Replace x by 2, $2f(2)+2f\left(\frac{1}{2}\right)-2f(1)=4 \Rightarrow f(2)+f\left(\frac{1}{2}\right)=2+f(1)-----(1)$

Replace x by 1, $f(1)=-1-----(2)$

Replace x by $\frac{1}{2}$, $2f\left(\frac{1}{2}\right)+\frac{1}{2}f\left(\frac{1}{2}\right)+2=\frac{5}{2}-----(3)$

Solve (1) & (3) $\Rightarrow f\left(\frac{1}{2}\right)=0; f(2)=1$

17. f : R $\rightarrow$ [–1, ∞) and f(x) = ln([|sin2x| + |cos2x|]) (where [·] is greatest integer function).
(A) $R^{-} \cap$ range of f is null set
(B) f(x) is periodic but fundamental period not defined
(C) f(x) is invertible in $\left[0, \frac{\pi}{4}\right]$ (D) f(x) is into function.

Key: A,B,D

Hint. Period of f(x) = |sin2x| + |cos2x| is $\pi/4$
but f(x) = ln ([|sin2x| + |cos2x|])
Max. value of |sin2x| + |cos2x| = $\sqrt{2}$
f(x) = ln ([$\sqrt{2}$]) = ln (1) = 0
$\Rightarrow$ it is periodic function but fundamental period not defined.
f(x) is many one and into function

18. Let f and g be functions satisfying the conditions that
$f(0)=g(0)=1,\ \ g(x)=f'(x),\ \ g'(x)=-f(x)$ then
a) f(x) is periodic function b) $f''(x)=-f(x)$
c) Range of f(x) is [-1, 1] d) Range of f(x) is $\left[-\sqrt{2},\sqrt{2}\right]$

Key: A, B, D

Hint:

$$g^{1}(x)=f^{11}(x).=-f(x),\quad g(0)=f^{1}(0)=1>0$$

$$f^{11}(x).f^{1}(x)=-f(x).f^{1}(x)\Rightarrow\int f^{11}(x)f^{1}(x)dx=-\int f(x).f^{1}(x)dx$$

$$\Rightarrow\frac{\left(f^{1}(x)\right)^{2}}{2}=\frac{-\left(f(x)\right)^{2}}{2}+c\Rightarrow c=1$$

$$f^{1}(x)=\sqrt{2-\left(f(x)\right)^{2}}\Rightarrow\int\frac{f^{1}(x)dx}{\sqrt{2-\left(f(x)\right)^{2}}}=\int 1dx\Rightarrow\sin^{-1}\frac{f(x)}{\sqrt{2}}=x+c^{1},\quad f(0)=0\Rightarrow c^{1}=\pi/4$$

$$f(x)=\sqrt{2}\sin\left(\frac{\pi}{4}+x\right)=\sin x+\cos x$$

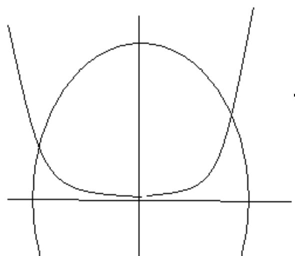
19. Let $R=\left\{(x,y);x,y\in R,x^{2}+y^{2}\le 25\right\}$ and $R'-\left\{(x,y):x,y\in R,y\ge\frac{4}{9}x^{2}\right\}$ then
(A) domain of $R\cap R'=[-3,3]$ (B) Range of $R\cap R'=[0,4]$
(C) Range of $R\cap R'=[0,5]$ (D) $R\cap R'$ defines a function

Key: A,C

Hint: $x^{2}+y^{2}\le 25$

$9y\ge 9x^{2}$

$x^{2}\le\frac{9}{4}y$



$x^2 = \frac{9y}{4}$

$\frac{9y}{4} + y^2 - 25 = 0$

$4y^2 + 9y - 1 = 0$

$(y-4)(47+21) = 0$

$y = 4$

Domain $R \cap R'$ is $\{-3,3\}$

Range of $R \cap R' = [0,5]$

20. Consider the real valued function satisfying $2f(\sin x) + f(\cos x) = x$ then

A) domain of $f(x)$ is R B) domain of $f(x)$ is [-1 1]

C) range of $f(x)$ is $\left[\frac{-2\pi}{3}, \frac{\pi}{6}\right]$ D) range of $f(x)$ is $\left[\frac{-2\pi}{3}, \frac{\pi}{3}\right]$

Key. B,D

Sol. given $2f(\sin x) + f(\cos x) = x$ replace x by $\frac{\pi}{2} - x$,

$\Rightarrow 2f(\cos x) + f(\sin x) = \frac{\pi}{2} - x$

$\Rightarrow f(\sin x) = x - \frac{\pi}{6} \Rightarrow f(x) = \sin^{-1} x - \frac{\pi}{6}$

$\therefore$ domain of $f(x)$ is [-1, 1] and range of $f^{-1}(x)$ is $\left[\frac{-\pi}{2} - \frac{\pi}{6}, \frac{\pi}{2} - \frac{\pi}{6}\right]$ i.e., $\left[\frac{-2\pi}{3}, \frac{\pi}{3}\right]$

21. Let $f(x) = \begin{cases} x^2 - 4x + 3 & x < 3 \\ x - 4 & x \ge 3 \end{cases}$ and $g(x) = \begin{cases} x - 3 & x < 4 \\ (x+1)^2 + 1 & x \ge 4 \end{cases}$ then

A) $(f-g)\left(\frac{7}{2}\right) = -1$ B) $fog(3) = 3$

C) $(fg)(2) = 1$ D) $(f+g)\left(\frac{7}{2}\right) - (f-g)(4) = 26$

Key. A,B,C,D

Sol. $f\left(\frac{7}{2}\right) = -0.5$, $g\left(\frac{7}{2}\right) = 0.5$, $g(3) = 0$, $f(0) = 3$;

$f(2) = -1, g(2) = -1;$ $f(4) = 0$; $g(4) = 26$

22. $2f(x)+xf\left(\frac{1}{x}\right)-2f\left(\left|\sqrt{2}\sin(\pi(x+\frac{1}{4}))\right|\right)=4\cos^2\frac{\pi x}{2}+x\cos\frac{\pi}{x},\ \forall x\in R, x\neq 0$ for some real function f, then

A) $f(2)+f\left(\frac{1}{2}\right)=1$

B) $f(2)+f(1)=f\left(\frac{1}{2}\right)$

C) $f(2)+f(1)=0$

D) $f(1)=-1$ but $f(2)$ and $f\left(\frac{1}{2}\right)$ cannot be obtained

Key. A,B,C

Sol. replacing x by 2, we get $2f(2)+2f\left(\frac{1}{2}\right)-2+\left|\sqrt{2}\sin\frac{9\pi}{4}\right|=4\cos^2\pi+2\cos\frac{\pi}{2}$

$\Rightarrow 2f(2)+2f\left(\frac{1}{2}\right)-2f(1)=4$

$f(2)+f\left(\frac{1}{2}\right)=2+f(1)$ ......(1)

Now replacing x by 1, we get $2f(1)+f(1)-2f(1)=4.0-1$

$\Rightarrow f(1)=-1$ ......(2)

Now replace x by $\frac{1}{2}$,

$2f\left(\frac{1}{2}\right)+\frac{1}{2}f(2)-2f(1)=2+\frac{1}{2}$

$\Rightarrow \quad 2f\left(\frac{1}{2}\right)+\frac{1}{2}f(2)+2=\frac{5}{2}$ ....(3)

$\Rightarrow f(2)$ and $f\left(\frac{1}{2}\right)$ can be obtained from (1) and (3)

23. Let f and g be functions satisfying the conditions that

$f(0)=g(0)=1,\ \ g(x)=f'(x),\ \ g'(x)=-f(x)$ then

a) f(x) is periodic function

b) $f''(x)=-f(x)$

c) Range of f(x) is [-1, 1]

d) Range of f(x) is $\left[-\sqrt{2},\sqrt{2}\right]$

Key. A, B, D

$g^1(x) = f^{11}(x). = -f(x), \quad g(0) = f^1(0) = 1 > 0$

$f^{11}(x).f^1(x) = -f(x).f^1(x) \Rightarrow \int f^{11}(x) f^1(x) dx = -\int f(x).f^1(x) dx$

Sol. $\Rightarrow \frac{(f^1(x))^2}{2} = \frac{-(f(x))^2}{2} + c \Rightarrow c = 1$

$f^1(x) = \sqrt{2-(f(x))^2} \Rightarrow \int \frac{f^1(x) dx}{\sqrt{2-(f(x))^2}} = \int 1 dx \Rightarrow \sin^{-1}\frac{f(x)}{\sqrt{2}} = x + c^1, \quad f(0) = 0 \Rightarrow c^1 = \pi/4$

$f(x) = \sqrt{2}\sin\left(\frac{\pi}{4} + x\right) = \sin x + \cos x$

24. A function $f : R \to R$ is defined by $f(x+y) - kxy = f(x) + 2y^2 \forall x.y \in R$ and $f(1) = 2; f(2) = 8,$ where k is some constant, then $f(x+y).f\left(\frac{1}{x+y}\right)$ equal to (where $x + y \neq 0$)

A) 1 B) 4 C) k D) $f(1)$

Key. B,C

Sol. Given $f(x+y) - kxy = f(x) + 2y^2$.

Replace $y by -x,$ then $f(0) + kx^2 = f(x) + 2x^2$

$\Rightarrow f(x) = f(0) + kx^2 - 2x^2$.....(1)

Now $f(1) = f(0) + k - 2 = 2 \Rightarrow f(0) = -k + 4$

and $f(2) = f(0) + 4k - 8 = 8 \Rightarrow f(0) = -4k + 16$

Which give $k = 4$ and $f(0) = 0$

Thus, from (1) $f(x) = 2x^2$

$\therefore f(x+y) f\left(\frac{1}{x+y}\right) = 4 = k$

25. π is the FUNDAMENTAL period of

A) $\frac{1+\sin x}{\cos x(1+\cos ecx)}$ B) $|\sin x| + |\cos x|$ C) $\sin 2x + \cos 2x$ D) $\cos(\sin x) + \cos(\cos x)$

Key. A,C

Sol. Let $f(x) = \frac{1+\sin x}{\cos x(1+cosec\, x)}$

$= \frac{(1+\sin x)\sin x}{\cos x(1+\sin x)} = \tan x, x \neq \frac{n\pi}{2}$

Clearly, $f(x)$ has π as fundamental period

Let $f(x) = |\sin x| + |\cos x|$

$f(\pi + x) = |\sin x| + |\cos x| = f(x)$

$f\left(\frac{\pi}{2}+x\right)=|\cos x|+|\sin x|=f(x)\Rightarrow\frac{\pi}{2}$ is the

Fundamental period

Let $f(x)=\sin 2x+\cos 2x$

$f(\pi+x)=\sin 2x+\cos 2x=f(x)$

$f\left(\frac{\pi}{2}+x\right)=-\sin 2x-\cos 2x\neq f(x)$

$\Rightarrow \pi$ is the fundamental period

Let $f(x)=\cos(\sin x)+\cos(\cos x)$

$f(\pi+x)=\cos(\sin x)+\cos(\cos x)$

$f\left(\frac{\pi}{2}+x\right)=\cos(\cos x)+\cos(\sin x)$

$\Rightarrow$ fundamental period is $\frac{\pi}{2}$

26. The function $f(x)=\frac{|x-1|}{x^2}$ is

A) one-one in $(2,\infty)$ B) one-one in $(0,1)$ C) one-one in $(-\infty,0)$ D) one-one in (1, 2)

Key. A,B,C,D

Sol. $f(x)=\frac{|x-1|}{x^2}=\begin{cases}\frac{1-x}{x^2}, & x<1, x\neq 0\\ \frac{x-1}{x^2}, & x\geq 1\end{cases}$

$f'(x)=\begin{cases}\frac{x-2}{x^3}, & x<1, x\neq 0\\ \frac{2-x}{x^3}, & x>1\end{cases}$

$f'(x)<0\Rightarrow \left.\begin{matrix}\frac{x-2}{x^3}, & x<1, x\neq 0\\ \frac{2-x}{x^3}, & x>1\end{matrix}\right\}$

$\Rightarrow 0<x<1$ or $x>2$

27. Let f be a function defined by $f(x)=\frac{x-5}{x-3}, x\neq 3,\ 2;\ f^k(x)$ denote the composition of f with itself taken k times i.e., $f^3(x)=f(f(f(x)))$ then

A. $f^{2012}(2009)=2009$

B. $f^{2009}(2010)=\frac{2006}{2007}$

C. $f^{2009}(2011)=\frac{1003}{1004}$

D. $f^{2012}(2012)=2012$

Key. A,C,D

Sol. $f(x)=\frac{x-5}{x-3}\Rightarrow f^2(x)=\frac{2x-5}{x-2}=g(x)$

Then $g^{2n}(x)=x \ and \ g^{2n+1}=g(x)$

$f^{2012}(2009)=g^{1006}(2009)=2009$

$f^{2009}(2010)=f[g^{1004}(2010)]=f(2010)=\frac{2005}{2007}$

$f^{2009}(2011)=f[g^{1004}(2011)]=f(2011)=\frac{2006}{2008}$

$f^{2012}(2012)=g^{1006}(2012)=2012$

28. If $f:R\to R$ given by $f(x)=x|x|$, then f is

A. one-one B. Onto C. Bijection D. one-one but not onto

Key. A,B,C

Sol. $f(x)=x^2, x\ge 0$

$=-x^2, x<0$

$x^2\ge 0, -x^2<0\Rightarrow$ range =R $\Rightarrow f\ is\ onto$

$x>0, x_1^2=x_2^2\Rightarrow x_1=x_2$

$x<0, -x_1^2=-x_2^2\Rightarrow x_1=x_2$ ______ f is one –one

29. The number of functions *f* from the set A={0,1,2} in to the set B={0,1,2,3,4,5,6,7} such that $f(i)\le f(j)$ for i<j and $i,j\in A$ is

a) ${}^8C_3$ b) ${}^8C_3+2({}^8C_2)$ c) ${}^{10}C_3$ d) ${}^{10}C_4$

KEY : C

Sol.

$0<1<2$

$\Rightarrow f(0)\le f(1)\le f(2)$

f(0)<f(1)<f(2) $\Rightarrow {}^8C_3$

f(0)<f(1)=f(2) $\Rightarrow {}^8C_2$

f(0)=f(1)<f(2) $\Rightarrow {}^8C_2$

f(0)=f(1)=f(2) $={}^8C_1$

30. If f : R $\to$ R, f(x) is a differentiable function such that $\{f(x)\}^2 = e^2 + \int_0^x \left[f(t)^2 + \{f'(t)\}^2 \right] dt$, $\forall$ x $\in$ R. The value of f(1) can take is/are

(A) e^2 (B) $-e^2$

(C) 1 (D) -1

Key. A,B

Sol. $2f(x)\, f'(x) = \{f(x)\}^2 + \{f'(x)\}^2$

$\Rightarrow [f(x) - f'(x)]^2 = 0$

$\Rightarrow f(x) = f'(x)$

$\Rightarrow f(x) = Ae^x = \pm e^{(x+1)}$

$\Rightarrow f(1) = \pm e^2$.

31. If $|f''(x)| \le 1$, $\forall$ x $\in$ R and f(0) = 0 = f ′(0), then which of the following can't be true ?

(A) $f\left(-\frac{1}{2}\right) = \frac{1}{5}$ (B) $f(2) = -5$

(C) $f(-2) = 5$ (D) $f\left(\frac{1}{2}\right) = -\frac{1}{5}$

Key. A,B,C,D

Sol. $-1 \le f''(x) \le 1$, On integrating it twice with limits 0 to x, we get

$|f(x)| \le \frac{x^2}{2} \Rightarrow \left|f\left(\pm\frac{1}{2}\right)\right| \le \frac{1}{8}$ and $|f(\pm 2)| \le 2$

32. Let $f(x)$ be a real valued function such that $f(0) = \frac{1}{2}$ and

$f(x+y) = f(x)f(a-y) + f(y)f(a-x)\ \forall\ x, y \in R$ then for some real 'a'

a) $f(x)$ is a periodic function b) $f(x)$ is constant function

c) $f(x) = \frac{1}{2}$ d) $f(x) = \frac{\cos x}{2}$

Key. A,B,C

Sol. $f(x+y) = f(x)f(a-y) + f(y)f(a-x) \to (1)$

Put x = y = 0 we get $f(a) = \frac{1}{2}$

Put $y = 0$ we get

$f(x) = f(x).f(a) + f(0)f(a-x) \Rightarrow f(x) = f(a-x)$

Put $y = a - x$ is equation (1)

$f(a) = (f(x))^2 + (f(a-x))^2$

$f(x) = \pm\frac{1}{2}\ \left\{f(x) \ne -\frac{1}{2}\right\}$

Hence $f(x)=\frac{1}{2}$

33. $f(x)=\operatorname{sgn} x$, $g(x)=x(x^2-1)$, $h(x)=(x^2-1)\sin x$ then which of the following is/are periodic functions

a) $g(f(x))$ b) $f(g(x))$

c) $h(f(x))$ d) $g(h(x))$

Key. A,C

Sol. $g(f(x))=0=hf(x)$

34. Which of the following statements are true for the function

$$f(x)=\sqrt{\sin\log\left(\frac{x^2+e}{x^2+1}\right)}+\sqrt{\cos\log\left(\frac{x^2+e}{x^2+1}\right)}$$

a) range of the function is $\left(1,64^{1/8}\right]$ b) domain of the function $\in R$

c) function is even d) function is invertible

Key. A,B,C

Sol. Domain $\in R$

Range of $\frac{x^2+e}{x^2+1}\in(1,e]$

$\Rightarrow$ range of $\log\left(\frac{x^2+e}{x^2+1}\right)\in(0,1]$

Many one function hence not invertible

35. If $f(x)=f(x+T), T>0$ then f is a periodic function i.e., (for each x there exist x + T). Which of the following statements is/are correct.

a) $f_1(x)=|\sin x|+|\cos x|$ has a period of $\frac{\pi}{2}$ if $-4\pi\le x\le 4\pi$

b) $f_2(x)=\sqrt{\sin x+\cos x}, x\ge 0$ is a periodic function

c) The period of $f_3(x)=\frac{(1+\sin x)(1+\sec x)}{(1+\cos x)(1+\operatorname{cosec} x)}$ is π.

d) If derivative of a function is periodic, then the function may be periodic.

Key. C,D

Sol. $f_3(x)=\tan x$

36. If $f(x)=\cos\left[\pi^2\right]x+\cos\left[-\pi^2\right]x$ *where* $[.]$ is greatest integer function

a) $f\left(\frac{\pi}{2}\right)=-1$ b) $f(\pi)=1$ c) $f(-\pi)=0$ d) $f\left(\frac{\pi}{4}\right)=1$

Key. A,C

Sol. $9<\pi^2<10 \Rightarrow [\pi^2]=9$

$-10<-\pi^2<-9 \Rightarrow [-\pi^2]=-10$

$f(x)=\cos 9x+\cos(-10)x=\cos 9x+\cos 10x$

$f\left(\frac{\pi}{2}\right)=\cos 5\pi=-1$ $\quad f(\pi)=0$ $\quad f(-\pi)=0$ $\quad f\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$

37. If f is an even function defined in the interval [- 5, 5] Then the real values of 'x' satisfying the equation $f(x)=f\left(\frac{x+1}{x+2}\right)$ are

a) $\frac{-1-\sqrt{5}}{2}$ b) $\frac{-3-\sqrt{5}}{2}$ c) $\frac{-3+\sqrt{5}}{2}$ d) $\frac{-1+\sqrt{5}}{2}$

Key. A,B,C,D

Sol. for even function $f(-x)=f(x)$

So $x=\frac{x+1}{x+2}$ & $-x=\frac{x+1}{x+2}$

38. Which of the following functions are periodic ?

A) $f(x)=\text{sgn}(e^{-x})$

B) $f(x)=\begin{cases}1 \text{ if } x \text{ is a rational number}\\ 0 \text{ if } x \text{ is an irrational number}\end{cases}$

C) $f(x)=\sqrt{\frac{8}{1+\cos x}+\frac{8}{1-\cos x}}$

D) $f(x)=\left[x+\frac{1}{2}\right]+\left[x-\frac{1}{2}\right]+2[-x]$ (where [] denotes greatest integer function).

Key. A,B,C,D

Sol. (A) f(x) = sgn (e^{-x})

$e^{-x}>0$ for every real x

Every constant function is a periodic function.

(B) Let T > 0 be a rational number then f(x + T) $=\begin{cases}1 & x\in Q\\ 0 & x\notin Q\end{cases}=f(x)$

$\therefore$ f(x) is periodic function

(C) $f(x)=\sqrt{\frac{8}{1-\cos x}+\frac{8}{1-\cos x}}=\sqrt{\frac{16}{1-\cos^2 x}}=\frac{4}{|\sin x|}=4|\text{cosec } x|$

this is a periodic function.

(D) $f(x)=\left[x+\frac{1}{2}\right]+\left[x-\frac{1}{2}\right]+2[-x]$

$=\left(x+\frac{1}{2}-\left\{x+\frac{1}{2}\right\}+x-\frac{1}{2}-\left\{x-\frac{1}{2}\right\}\right)+2(-x-\{-x\})=-\left\{x+\frac{1}{2}\right\}-\left\{x-\frac{1}{2}\right\}-2\{-x\}$

Since {.} is a periodic function hence this function is periodic.

39. The graph of the function y = f(x) is as shown in the figure. Then which one of the following graphs are correct?

A) $|y| = \text{sgn}(f(x))$
B) $|y| = \text{sgn}(-f(|x|))$
C) $|y| = |f|x||$
D) $y = x^{\text{sgn}(f(x))}$

Key. A,B,C

Sol. $y = f(x)$

(A) $|y| = \text{sgn}\, f(x)$
$f(x) > 0$ if $-2 < x < 1$ and $x > 3$
$\Rightarrow$ $|y| = 1$
$F(x) < 0$, $1 < x < 3$
$\Rightarrow$ $|y| = -1$ (not applicable)
at $x = -2, 1, 3$, $f(x) = 0$
$\Rightarrow$ $|y| = 0$

(B) $|y| = \text{sgn}(-f(|x|))$
$f|x|$

(C) $|y| = |f(x|x|)|$
$|f(x)| =$
& $|y| = |f(|x|) =$

(D) $y = x^{\text{sgn}\, f(x)}$
$f(x) > 0, -2 < x < 1$ $\Rightarrow$ $y = x$
$f(x) > 0, 1 < x < 3$ $\Rightarrow$ $y = \dfrac{1}{x}$
$x = -2, 1, 3$ $\Rightarrow$ $y = 1$
$x > 3, f(x) > 0$ $\Rightarrow$ $y = x$

40. $f(x) = \sin\left(2\left(\sqrt{[a]}.\right)x\right)$, where [.] denote the greatest integer function, has fundamental period π for

A) $a = \dfrac{3}{2}$ B) $a = \dfrac{5}{4}$ C) $a = \dfrac{2}{3}$ D) $a = \dfrac{4}{5}$

Key. A

Sol. Since fundamental period of f(x) is π, therefore, [a] = 1
$\therefore$ $1 \le a < 2$

41. Let $f(x) = [x]^2 + [x+1] - 3$, where [x] = the greatest integer $\le x$. Then

A) f(x) is a many-one and into function
B) f(x) = 0 for infinite number of values of x
C) f(x) = 0 for only two real values
D) none of these

Key. A,B

Sol. $f(x) = [x]^2 + [x+1] - 3 = \{[x]+2\}\{[x]-1\}$
So, x = 1, 1.1, 1.2, … $\Rightarrow$ f(x) = 0 $\therefore$ f(x) is many one
Only integral values will be attained
$\therefore$ f(x) is into

42. If $f : R \to R, f(x) = e^{-|x|} - e^{x}$ is a given function, then which of the following are correct

A) f is many-one into function
B) f is many one onto function
C) range of f is $[0, \infty]$
D) range of f is $(-\infty, 0]$

Key. A,D

Sol. $f(x) = \begin{cases} 0 & x < 0 \\ e^{-x} - e^{x} & x \ge 0 \end{cases}$

range $(-\infty, 0]$

∴ many one into

43. Which of the following pair(s) of functions are identical?

A) $f(x)=\cos(2\tan^{-1}x), g(x)=\frac{1-x^2}{1+x^2}$ B) $f(x)=\frac{2x}{1+x^2}, g(x)=\sin(2\cot^{-1}x)$

C) f(x) = tan x + cot x, g(x) = 2 cosec 2x D) $f(x)=e^{ln(\text{sgn}\cot^{-1}x)}, g(x)=e^{ln[1+\{x\}]}$

Where sgn(.), [.], {.} denotes signum, greatest integer and fractional part functions respectively)

Key. A,B,C,D

Sol. (A) Domain of f and g both are 'R'.

$$f(x)=\cos(2\tan^{-1}x)=\frac{1-\tan^2(\tan^{-1}x)}{1+\tan^2(\tan^{-1}x)}=\frac{1-x^2}{1+x^2}=g(x).$$

(B) Domain of f and g both are 'R'

$$g(x) = \sin\ (2\cot^{-1}x)=\frac{2\tan(\cot^{-1}x)}{1+\tan^2(\cot^{-1}x)}=\frac{2\times\frac{1}{x}}{1+\frac{1}{x^2}}=\frac{2x}{1+x^2}=f(x)$$

(C) Domain of f and g are $R-\left\{n\pi,(2n+1)\frac{\pi}{2}\right\}, n\in I$

$$f(x)=\tan x+\cot x=\tan x+\frac{1}{\tan x}=\frac{\sec^2 x}{\tan x}=\frac{\cos x}{\sin x.\cos^2 x}=\frac{1}{\cos x\sin x}=2\text{cosec}2x=g(x)$$

(D) Domain of f:

$\text{sgn}(\cot^{-1}x)>0 \Rightarrow \text{sgn}(\cot^{-1}x)=1$

$\Rightarrow \cot^{-1}x>0 \Rightarrow x\in R$

Domain of g:

$[1+\{x\}]>0 \Rightarrow [\{x\}]>0 \Rightarrow 0\le\{x\}<1 \Rightarrow x\in R$

now, $f(x)=e^{ln(\text{sgn}\cot^{-1}x)}=\text{sgn}(\cot^{-1}x)$ $(\because 0<\cot^{-1}x<\pi)=1$

$g(x)=e^{\ln[1+(x)]}=[1+\{x\}]=1+[\{x\}]$ ∵ $0\le\{x\}<1=1$

∴ f(x) and g(x) are identical functions.

44. If $f(x)=\sin\{[x+5]+\{x-\{x\}\}\}$ for $x\in\left(0,\frac{\pi}{4}\right)$ is invertible, where {.} and [.] represent fractional part and greatest integer functions respectively, then $f^{-1}(x)$ is

A) $\sin^{-1}x$ B) $\frac{\pi}{2}-\cos^{-1}x$ C) $\sin^{-1}\{x\}$ D) $\cos^{-1}\{x\}$

Key. A,B,C

Sol. y = f(x) = sin {[x + 5] + {x – {x – {x}}}} = sin {x – {x – {x}}} = sin {x – {[x]} = sin {x – 0} = sin {x] = sin x

∵ $0<x<\frac{\pi}{4}$

∵ $x=\sin^{-1}y$ or $f^{-1}(x)=\sin^{-1}x$.

45. Range of $f(x)=\log_{\sqrt[3]{10}}\left(\sqrt{5}(2\sin x+\cos x)+5\right)$ is

A) [0, 1] B) [0, 3] C) $\left(-\infty,\frac{1}{3}\right]$ D) none of these

Key. D

Sol. we know that

$$-\sqrt{5} \le 2\sin x + \cos x \le \sqrt{5}, \ \forall x \in R$$

$\Rightarrow \quad -5 \le \sqrt{5}\left(2\sin x + \cos x\right) \le 5$

$\Rightarrow \quad 0 \le \sqrt{5}\left(2\sin x + \cos x\right) + 5 \le 10$

$\Rightarrow \quad -\infty < \log_{\sqrt[3]{10}}\left(\sqrt{5}\left(2\sin x + \cos x\right) + 5\right) \le 3$

Hence range is $\left(-\infty, 3\right]$

46. Which of the following functions $f : R \to R$ are bijective

(A) $f(x) = x\sin x$ (B) $f(x) = x - \sin^2 x$

(C) $f(x) = x + \sqrt{x^2}$ (D) $f(x) = -x + \cos^2 x$

Key. B,D

Sol. (a) f is not one-one as f cuts x axis twice in $(0, 2\pi]$

If fact f is continuous and achieves every real number infinite times.

(b) f is monotone as $f'(x) = 1 - \sin 2x \ge 0 \ \forall x$

$\lim\limits_{x \to -\infty} f \equiv \infty; \ \lim\limits_{x \to -\infty} f \equiv -\infty$ and f is continuous

Hence f is bijective.

(c) $f(x) = x + |x| = \begin{cases} 2x & if \quad x \ge 0 \\ 0 & if \quad x \le 0 \end{cases}$ is not one-one as f is a constant function for $x \le 0$

(d) f is monotone as $f'(x) \le 0 \ \ \forall x$ and as in (b) f is bijective.

Functions

Assertion Reasoning Type

1. Statement-1: Period of $f(x)=\sin 3x\cos[3x]-\cos 3x\sin[3x]$ where [ ] denotes the greatest integer function, is $\frac{2\pi}{3}$

Statement-2: Period of $\{x\}$ where {} denotes the fractional part of *x*, is 1

Key. D

Sol. $\{x\}=x-[x]$ which is periodic with period 1.
Statement 2 is true.
Consider Statement 1.
$f(x)=\sin(3x-[3x])=\sin(\{3x\})$
Using Statement 2, period of $f(x)$ is $\frac{1}{3}$.
Statement 1 is false.

2. Statement-1: $f(x)=\frac{x^2-5x-9}{3x^2+2x+7}, x\in R$ is not a one-one function.

Statement-2: $f(x)$ is not one-one, if for any $x_1, x_2 \in$ domain of $f(x)$ where $x_1 \neq x_2$, $f(x_1)=f(x_2)$.

Key. A

Sol. Statement-2 is true.
Consider Statement-1.
Let α and β denote the roots of the quadratic $x^2-5x-9=0$.
Then, $\alpha \neq \beta$, but $f(\alpha)=f(\beta)=0$
$\Rightarrow f(x)$ is not one one
$\Rightarrow$ Statement-1 is true.

A) Statement-1is True, Statement-2 is True; Statement-2 is a correctExplanation forStatement-1

B) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1

C) Statement-1 is True, Statement-2 is False

D) Statement-1 is False, Statement-2 is True

3. Statement 1: $A=\{x:x$ is a prime number, $x<30\}$ then number of distinct rational numbers whose numerator and denominator belong to 'A' is 93

Statement 2: $\frac{p}{q}\in Q \; \forall \; q\neq 0$ and $p,q\in I$

Key. D

Sol. $A=\{2,3,5,7,11,13,17,19,23,29\}$

Two different numbers for numbers and denominator from these can be obtained in

${}^{10}P_2=10.9=90$ ways and if $\frac{p}{q}=\frac{q}{q}=1$

So, number of ways (if numerator and denominator are same) $=90+1=91$

4. Statement 1: $A=\{x:x$ is a prime number, $x<30\}$ then number of distinct rational numbers whose numerator and denominator belong to $'A'$ is 93

Statement 2: $\frac{p}{q}\in Q\ \forall\ q\neq 0$ and $p,q\in I$

Key. D

Sol. $A=\{2,3,5,7,11,13,17,19,23,29\}$

Two different numbers for numbers and denominator from these can be obtained in

${}^{10}P_2=10.9=90$ ways and if $\frac{p}{q}=\frac{q}{q}=1$

So, number of ways (if numerator and denominator are same) $=90+1=91$

5. Statement-I: Consider the point A(0,1) and B(2,0) and P be a point on the line 4x+3y+9=0,then the coordinates of P such that $|PA-PB|$ is maximum is $\left(-\frac{24}{5},\frac{17}{5}\right)$

Statement-II: If A and B are two fixed points and P is any point in a plane then $|PA-PB|\geq AB$.

Key: C

Hint. $|PA-PB|\leq AB$

$|PA-PB|=AB$

P lies on extended line segment of AB solve $\overrightarrow{AB}$,4x+3y+9=0

6. STATEMENT–1: Reflection of function $y=\log_e x$ in the straight line x + y = 0 is y = $-e^{-x}$.

STATEMENT–2: The image of any point (α,β) in the line $x+y=0$ is $(-\beta,-\alpha)$.

Key: A

Hint: Since the point $(-\beta,-\alpha)$ lies on $y=\log_e x$

$\Rightarrow\quad -\alpha=\log_e(-\beta)\ \Rightarrow\ -\beta=e^{-\alpha}$

$\Rightarrow\quad \beta=-e^{-\alpha}\Rightarrow\qquad y=-e^{-x}$.

7. Statement-1: The linear fractional function $\frac{ax+b}{cx+d}(ad-bc\neq 0)$ cannot attain the value $\frac{a}{c}$.

Statement-2: The domain of the function

$g(y)=\frac{b-dy}{cy-a}$ is R ~ {a/c}.

Key: a

Hint: If $f(x)=y=\frac{ax+b}{cx+d}$ then $x=\frac{b-dy}{cy-a}$

Thus $f^{-1}(y)=g(y)=\frac{b-dy}{cy-a}$ and dom g = R ~ {a/c}

Hence f cannot attain the value a/c by definition.

8. Statement-I:- Let $f:R\to R$ be a real valued continuous function which assumes only rational values. If $f(\sqrt{3})=1729$. Then $f\left(\frac{2012}{2013}\right)=1729$.

Statement-II:- If a continuous function $f(x)$ assumes only rational values, then $f(x)$ is a constant function.

Key. A

Sol. If $f(x)$ is not a constant function then $f(x)$ takes both rational and Irrational values which is a contradiction to the hypothesis $\therefore f(x)$ must be constant.

9. If $f:\mathbf{R}\to\mathbf{R}$ is defined by $f(x)=\log\left(x+\sqrt{x^2+1}\right)$, then consider the two statements

STATEMENE I : $f(x)+f(-x)=0$

STATEMENE II: f is an odd function, and then which one of the following is correct?

1) Only I is correct 2) Only II is correct
3) I and II are correct and I is correct explanation of II
4) I and II are correct, I is not the correct explanation of II

Key. 3

Sol. $f(x)+f(-x)=0$

10. Let $f(x)=2+\cos x, x\in R$

STATEMENT-1: For each $t\in R$ $\exists$ a point c in $[t,t+\pi]$ such that $f^1(c)=0$.

STATEMENT-2: $f(t)=f(t+2\pi)\forall t\in R.$

Key: B

Hint: S_2 is true.

$f^1(x)=-\sin x$

$\therefore$ every interval of length π $[t,t+\pi]$ contains at least one point , 0, $\pm\pi,\pm 2\pi$

$\therefore$ $f^1(x)$ in $[t,t+\pi]$: $t\in R$

$\therefore S_1$ is true.

11. Statement-1: $f(x)=\frac{x^2-5x-9}{3x^2+2x+7}, x\in R$ is not a one-one function.

Statement-2: $f(x)$ is not one-one, if for any $x_1, x_2 \in$ domain of $f(x)$ where $x_1 \neq x_2$, $f(x_1)=f(x_2)$.

Key. A

Sol. Statement-2 is true.
Consider Statement-1.
Let α and β denote the roots of the quadratic $x^2-5x-9=0$.
Then, $\alpha \neq \beta$, but $f(\alpha)=f(\beta)=0$
$\Rightarrow f(x)$ is not one one
$\Rightarrow$ Statement-1 is true.

12. STATEMENT–1
The function f(x) = x [x], ([·] denotes greatest integer function) is a one–one function $\forall$ x $\geq$ 1.
because
STATEMENT–2
An increasing function is always one–one

Key. A

Sol. For x$\in$ [n, n + 1), let f(x) = f_n (x) , n$\in$N
$f_n(x)$ = nx which is increasing
Also f_{n+1} (x) = (n + 1)x >f_n (x) .
So, f(x) is an increasing function.

13. Statement – 1 : $f:R\to R$ defined by $f(x)=\frac{1}{2}x|x|+\cos x+1$ is invertible

Because
Statement – 2 : For $f:R\to R$, $f(x)$ is one – one - onto

Key. A

Sol. The given function $f:R\to R$ defined by
$f(x)=\frac{1}{2}x|x|+\cos x+1$ is one – one – onto function .
$\Rightarrow$ $f(x)$ is invertible

14. Let f(x) = $\sqrt{\cos^{-1}\sqrt{1-x^2}-\sin^{-1}x}$
STATEMENT–1
Range of f(x) is [0, $\sqrt{\pi}$]
because
STATEMENT–2
f(x) is an increasing function in its domain

Key. C

Sol. For x $\geq$ 0, f(x) = 0
for x < 0, f(x) = $\sqrt{-2\sin^{-1}x}$
$\Rightarrow$ range of f(x) is [0, $\sqrt{\pi}$].

15. STATEMENTS–1

The function f : R $\to$ R given $f(x)=\log_a(x+\sqrt{x^2+1})$ a > 0, a $\neq$ 1 is invertible.

because

STATEMENTS–2

f is many to one and into.

Key. C

Sol. f is injective since x $\neq$ y (x, y $\in$ R)

$\Rightarrow \log_a\left\{x+\sqrt{x^2+1}\right\}\neq\log_a\left\{y+\sqrt{y^2+1}\right\}$

$\Rightarrow$ f(x) $\neq$ f(y)

f is onto because $\log_a\left(x+\sqrt{x^2+1}\right)=y$

$\Rightarrow x=\dfrac{a^y-a^{-y}}{2}$.

16. Statement – I : $\underset{x\to 0}{Lt}\dfrac{e^{1/x}}{x}$ does not exist

Statement – II : $\lim\limits_{x\to 0^+}\dfrac{e^{1/x}}{x}$ does not exist

Key. A

Sol. Conceptual

17. **Assertion :** Function $f(x)=\sin(x+3\sin x)$ is periodic

Reason : $f(g(x))$ is periodic if g(x) is periodic.

Key. B

Sol. Period of $\sin(x+3\sin x)$ is 2π

18. **Assertion :** The number of solution of $(x^2+x+1)^2+(x^2+x+1)+1=x$ is zero

Reason : If f(x) doesn't cut y = x line. The equation $f^{-1}(x)=f(x)$ doesn't have any solution.

Key. A

Sol. Put $x^2+x+1=t$

$\Rightarrow t^2+t+1=x$

19. **Assertion :** The area bounded by the curve $|x|+|y|=a\,(a>0)$ is $2a^2$ and area bounded by $|px+qy|+|qx-py|=a$, where $p^2+q^2=1$ is also $2a^2$.

Reason : Since $lx+my=0$ is perpendicular to $mx-ly=0$ we can take one as x-axis and another as y-axis. Hence area bounded by $|lx+my|+|mx-ly|=a$ is $2a^2$ for all $l,m\in R, l\neq 0, m\neq 0$.

Key. C

Sol. $|x|+|y|=a$ fomes a square

20. **Assertion :** $f(x)=\begin{cases} g(x)+g(-x) & x>0 \\ a & x=0 \\ g(x)-g(-x) & x<0 \end{cases}$ is odd function iff a = 0

Reason : For an odd function $y = f(x)$ if x = 0 is in domain then f(0) must be equal to 0.

Key. D

Sol. $f(-x) = g(-x) + g(x), x > 0$

$\neq -g(x) - g(-x)$

21. Statement – 1: e^x can not be expressed as the sum of even and odd function.

Statement – 2: e^x is neither even nor odd function

Key. D

Sol. Statement – 1: Every function can be written as the sum of even and odd function

Statement – 2: $f(x) = e^x$

$f(-x) = e^{-x}$

Here neither f(x) = f(- x) nor f(- x) = - f(x)

So e^x is neither even nor odd function.

22. Statement – 1: If f(x) = sin x, then $f'(x) = \cos x$

Statement – 2: The derivative of an odd function is even and vice-versa

Key. A

Sol. (i) Let f(- x) = - f(x) $\forall x \in$ domain of f(x)

$\therefore$ $-f'(-x) = -f'(x)$ $\Rightarrow$ $f'(-x) = f'(x)$ $\Rightarrow$ $f'(x)$ is even

(ii) Let f(-x) = f(x) $\forall$ x $\in$ domain of f(x)

$\therefore$ $-f'(x) = f'(x)$ $\Rightarrow$ $f'(-x) = -f'(x)$ $\Rightarrow$ $f'(x)$ is odd

$\therefore$ statement – 2 is true, statement-1 is true

Statement – 2 is the correct explanation of statement 1

23. Statement – 1: The inverse of a strictly increasing exponential function is a logarithmic function that is strictly decreasing.

Statement – 2: lnx is inverse of e^x

Key. D

Sol. Statement – 1

Let g(x) is inverse of y = f(x) then

$f'(g(x)) = \dfrac{1}{g'(x)}$

$\therefore$ if f(x) is strictly increasing then g(x) is also strictly increasing.

Statement – 2

e^x is mirror image of in x.w.r.t. line

y = x

$\therefore$ lnx is inverse of e^x

24. Statement – 1: Fundamental period of sinx + tan x is 2π

Statement – 2: If the period of f(x) is T_1 and the period of g(x) is T_2, then the fundamental period of f(x) + g(x) is the L.C.M. of T_1 and T_2

Key. C

Sol. Statement – 1 Obvious

Statement – 2 f(x) = |sin x| + | cos x| fundamental period is $\frac{\pi}{2}$

25. Statement – 1: If a function y = f(x) is symmetric about y = x, then f(f(x)) = x

Statement – 2: If $f(x)=\begin{cases} x & : \ x \text{ is rational} \\ 1-x & : \ x \text{ is irrational}\end{cases}$, then f(f(x)) = x

Key. A

Sol. (i) y = f(x) is symmetric about y = x $\Rightarrow$ x = f(y)

$\therefore$ f(f(x)) = f(y) = x

$\therefore$ statement 1 is true

(ii) $f(x)=\begin{cases} x & , \ x \text{ is rational} \\ 1-x & , \ x \text{ is irrational}\end{cases}$ is

Symmetric about y = x

$\therefore$ f(f(x)) = x

26. Statement – 1: f(x) = sin x is periodic and g(x) = cos x is also periodic

Statement – 2: If the derivative of a function is periodic, then the function will also be periodic

Key. C

Sol. Statement – 1 Obvious

Statement – 2 Let f(x) = [x]

$f'(x)=\begin{cases} 0 \ ; & x \notin I \\ \text{Not Define} \ ; & x \in I\end{cases}$

Which is periodic but f(x) is not

27. Statement – 1: function f(x) = sin (x + 3 sin x) is periodic

Statement – 2: f(g(x)) is periodic if g (x) is periodic.

Key. B

Sol. Statement – 1 $f(x+2\pi)=f(x) \Rightarrow T=2\pi$ is period

Statement – 2 Obvious

28. Statement – 1: The function $y=\frac{ax+b}{cx+d}, (ad-bc \neq 0)$ cannot attain the value $\frac{a}{c}$

Statement – 2: The domain of $g(y)=\frac{b-dy}{cy-a}$ does not contain $\frac{a}{c}$

Key. A

Sol. Let $y=\frac{ax+b}{cx+d}$ on solving for x we get $x=\frac{b-dy}{cy-a}$

For existence of $x, y \neq a/c$

29. Statement – 1: Range of $\frac{1}{\{x\}}$ is $(1, \infty)$

(where {.} represents fractional part function)

Statement – 2: $0<\frac{1}{x}<1 \quad \Leftrightarrow \quad 1<x<\infty$

Key. A

Sol. Since $0 \leq \{x\} < 1 \ \therefore \ 1<\frac{1}{\{x\}}<\infty$

30. Statement – 1: Let $f: R-\{1,2,3\} \to R$ be a function defined by $f(x)=\frac{1}{x-1}+\frac{2}{x-2}+\frac{3}{x-3}$.
Then f is many-one function.
Statement – 2: If either $f'(x)>0$ or $f'(x)<0, \forall x \in$ domain of f, then y = f(x) is one-one function.

Key. C

Sol. From the graph it is clear that f(x) is not one-one
$\therefore$ statement – 1 is true.
Also $f'(x)<0, \forall x \in D$, but the function is not 1 – 1. So the statement – 2 is false

31. Consider the following statements
Statement – 1: $f: N \to R; f(x)=\sin x$ is a one-one function.
Statement – 2: The period of sinx is 2π and 2π is an irrational number

Key. A

Sol. f(x) = sin x, $x \in N$
Period of sin x is 2π
$f(x+2\pi)=\sin(x+2\pi)=\sin x$
but x + 2π will be irrational number.
So for one natural value of x, we will get only one value of y.
$\therefore$ f(x) = sin x: $N \to R$ is one-one function

Functions

Comprehension Type

Paragraph – 1

Consider the function $f: R \to (0,\infty)$ defined by $f(x) = 2^x + 2^{|x|}$

1. $f(x)$ is

A) One-One Onto B) One-One Into
C) Many-One Onto D) Many-One Into

Key. D

2. The number of solutions of the equation $f(x) = \ln\pi$ is

A) 0 B) 1 C) 2 D) 3

Key. A

3. The area of the region bounded by the curves $y = f(x), y = 0, x = -1$ and $x = 1$ is

A) $\ln\left(2^{7/2}\right)$ sq.units B) $\log_2\left(e^{7/2}\right)$ sq.units

C) $\dfrac{3}{\ln 2}$ sq.units D) $\dfrac{2}{\ln 2}$ sq.units

Key. B

Sol. 1-3

$f(x) = 2^x + 2^{|x|}$

For $x \ge 0, f(x) = 2.2^x$

And for $x < 0; f(x) = 2^x + 2^{-x}$

and in this case $f'(x) = 2^x \ln 2 - 2^{-x}\ln 2 = \ln 2\left(\dfrac{2^{2x}-1}{2^x}\right)$

$= \dfrac{\ln 2}{2^x}.\left(2^x+1\right)\left(2^x-1\right) < 0$

So; $f(x)$ is decreasing in $x \in (-\infty,0)$ and increasing in $x \in (0,\infty)$

So, $f(x)$ is many one

Also , Range of $f(x)$ is $[2,\infty)$

f is many –one Into

Since $\ln\pi < 2$

So, the equation $2^x + 2^{|x|} = \ln\pi$ has no solution

Now the required area = $\int_{-1}^{0}\left(2^x + 2^{-x}\right)dx + \int_{0}^{1} 2.2^x dx = \dfrac{7}{2\ln 2} = \dfrac{7}{2}.\log_2 e = \log_2\left(e^{7/2}\right)$

Paragraph – 2

Let $f : R - \{0,1\} \to R$ be a function satisfying the relation $f(x) + f\left(\dfrac{x-1}{x}\right) = x$ for all $x \in R - \{0,1\}$.

Based on this, answer the following questions.

4. $f(x)$ is equal to

A) $\frac{1}{2}\left[x+\frac{1}{1-x}-\frac{x-1}{x}\right]$ B) $\frac{1}{2}\left[x-\frac{1}{1-x}+\frac{x-1}{x}\right]$

C) $\frac{1}{2}\left[x-\frac{1}{1-x}-\frac{x-1}{x}\right]$ D) $\frac{1}{2}\left[x+\frac{1}{1-x}+\frac{x-1}{x}\right]$

Key. A

5. $f(-1)$ is equal to

A) $\frac{3}{4}$ B) $\frac{-3}{4}$ C) $\frac{5}{4}$ D) $\frac{-5}{4}$

Key. D

6. $f\left(\frac{1}{2}\right)$ is equal to

A) $\frac{5}{4}$ B) $\frac{-7}{4}$ C) $\frac{7}{4}$ D) $\frac{9}{4}$

Key. C

Sol. 4-6. Given that $f(x)+f\left(\frac{x-1}{x}\right)=x$ ---- (1)

for all $x \neq 0,1.$ Replacing x with $\frac{(x-1)}{x}$ both sides, we get that

$$f\left(\frac{x-1}{x}\right)+f\left(\frac{[(x-1)/x]-1}{(x-1)/x}\right)=\frac{x-1}{x}$$

That is $f\left(\frac{x-1}{x}\right)+f\left(\frac{1}{1-x}\right)=\frac{x-1}{x}$ ---- (2)

Again replacing *x* with $(x-1)/x$ in this, we get

$$f\left(\frac{1}{1-x}\right)+f(x)=\frac{1}{1-x} \quad \text{---- (3)}$$

Then by taking Eq. (1) + Eq. (3) – Eq. (2), we get that

$$2f(x)=x+\frac{1}{1-x}-\frac{x-1}{x}$$

or $$f(x)=\frac{1}{2}\left[x+\frac{1}{1-x}-\frac{x-1}{x}\right] \quad \text{---- (1.13)}$$

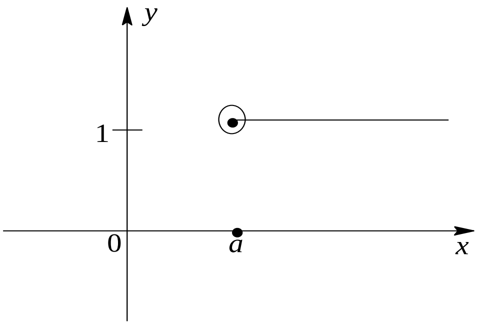
Substituting the values $x=-1$ and $1/2$ in Eq. (1.13) we get the solution for (ii) and (iii).

Paragraph – 3

The unit step function $u(x-a)$ is defined as $u(x-a)=\begin{cases}0 & , \ x<a \\ 1 & , \ x \geq a\end{cases}$.

The graph of $y=u(x-a)$ is as shown below:

Domain of $u(x-a)$ is *R* and its range is $\{0,1\}$.



Answer the following questions.

7. Let $f(x)=[x] \quad 0 \le x < 3$ where $[\ \ ]$ denotes the greatest integer function. The representation of $f(x)$ in terms of unit step function is

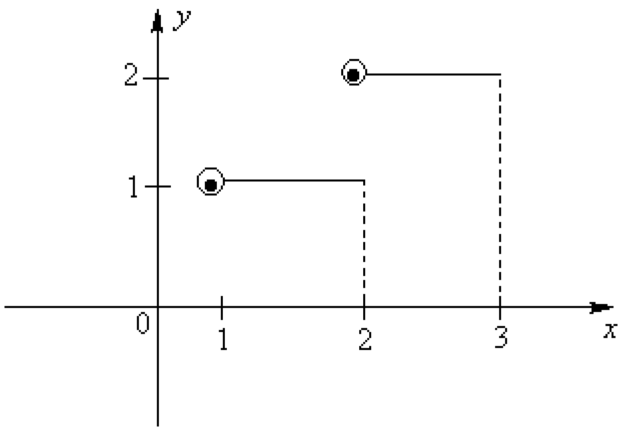
(A) $f(x)=u(x)+u(x-1)+u(x-2)+(x-3), 0 \le x \le 3$

(B) $f(x)=u(x-1)+u(x-2)+u(x-3), 0 \le x < 3$

(C) $f(x)=u(x-1)-u(x-2)+u(x-3), 0 \le x < 3$

(D) $f(x)=u(x-1)+2u(x-2)+3u(x-3), 0 \le x < 3$

Key. B

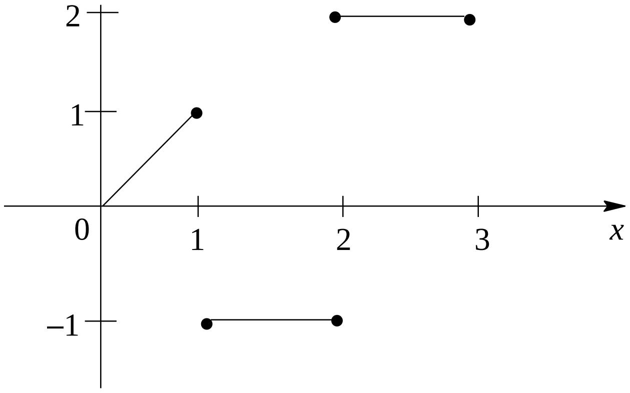


Sol.

Using the definition of $u(x)$, we can write $f(x)$ as

$f(x)=u(x-1)+u(x-2)+u(x-3), 0 \le x < 3$.

8. Graph of $y=f(x), 0 \le x < 3$ is shown below:



Representation of $f(x)$ in terms of the unit step function is given by

(A) $f(x) = x\{u(x) - u(x-1)\} - 2u(x-2)$

(B) $f(x) = x\{u(x) - u(x-1)\} - u(x-2) + 2u(x-3), 0 \le x < 3$

(C) $f(x) = x\{u(x) - u(x-1)\} - u(x-1) + 3u(x-2), 0 \le x < 3$

(D) $f(x) = xu(x) - u(x-1) + 2u(x-2), 0 \le x < 3$

Key. C

Sol. We have $f(x) = \begin{cases} x & 0 \le x < 1 \\ -1 & 1 \le x < 2 \\ 2 & 2 \le x < 3 \end{cases}$ using the definition of $u(x)$, $f(x)$ can be represented as

$$f(x) = x\{u(x) - u(x-1)\} - \{u(x-1) - u(x-2)\} + 2u(x-2)$$
$$= x\{u(x) - u(x-1)\} - u(x-1) + 3u(x-2) \quad 0 \le x < 3$$

9. Representation of the function $f(x) = \begin{cases} x^3, & 0 \le x < 1 \\ x-1, & 1 \le x < 3 \\ 0, & x \ge 3 \end{cases}$ in terms of the unit step function is

(A) $x^3[u(x) - u(x-1)] + (x-1)[u(x-1) - u(x-3)]$

(B) $x^3u(x) + (x-1)u(x-1)$

(C) $x^3[u(x) - u(x-1)] + (x-1)u(x-2) + u(x-3)$

(D) $x^3u(x) + (x-1)[u(x) - u(x-1)] + u(x-2)$

Key. A

Sol. Using the definition of $u(x)$, we may write $f(x)$ as

$$f(x) = x^3[u(x) - u(x-1)] + (x-1)[u(x-1) - u(x-3)].$$

Paragraph – 4

f(x) is a polynomial function $f:R\to R$ such that $f\left(2x\right)=f'\left(x\right).f''\left(x\right)$.

10. The value of f(3) is

(A) 4 (B) 12 (C) 15 (D) 18

Key. B

11. f(x) is

(A) one-one and onto (B) one-one and into

(C) many-one and onto (D) many-one and into

Key. A

12. Equation f(x) = x has

(A) Three real and distinct roots (B) one real root

(C) Four real and distinct roots (D) Two real and distinct roots

Key. A

Sol. 10-12. Suppose degree of f(x) is n, then the degree of f' is n - 1 and degree of f'' is n - 2.

So, n = (n - 1) + (n - 2) $\Rightarrow$ n = 3

Hence, f(x) = ax^3+bx^2+cx+d

From, $f\left(2x\right)=f'\left(x\right).f''\left(x\right)$, we have

$$8ax^3+4bx^2+2cx+d=\left(3ax^2+2bx+c\right)\left(6ax+2b\right)$$

Comparing coefficient of terms, we have

$a=4/9, b=0, c=0$ and d = 0.

$$\Rightarrow \quad f\left(x\right)=\frac{4x^3}{9}$$

(1) f(3) = 12

(2) one-one and onto

(3) $\frac{4x^3}{9}=x \Rightarrow x=0, \pm\frac{3}{2}$

Paragraph – 5

Suppose f(x) and g(x) are two continuous functions defined for $0 \le x \le 1$.

Given $f(x)=\int_0^1 e^{x+t}.f(t)dt$ and $g(x)=\int_0^1 e^{x+t}.g(t)dt+x$

13. The value of f(1) equals

A) 0 B) 1 C) $\frac{1}{e}$ D) e

14. The value of g(0) – f(0) equals

A) $\frac{2}{3+e^2}$ B) $\frac{3}{e^2-2}$ C) $\frac{-2}{e^2-3}$ D) 0

15. The value of $\frac{g(0)}{g(2)}$ equals

A) 0 B) $\frac{2}{3}$ C) $\frac{1}{3}$ D) $\frac{2}{e^2}$

Sol. 13. Ans. (a)

$f(x) = e^x \int_0^1 e^t f(t)dt$

Let $A = \int_0^1 e^t f(t)dt$

$\Rightarrow A = \int_0^1 e^t.Ae^t dt$

$\Rightarrow A = \int_0^1 e^{2t} dt \quad \Rightarrow A = 0$

as $\int_0^1 e^{2t}dt - 1 \neq 0$

hence $f(x) = 0 \Rightarrow f(\phi) = 0$

14. Ans. (c)

$g(x) = e^x \int_0^1 g(t)dt + x$

$g(x) = Be^x + x \ ---(1) \quad \Rightarrow g(t) = Be^t + t$

Where $B = \int_0^1 e^t g(t)dt,\ B = \int_0^1 e^t \left(Be^t + t\right)dt$

$B = B\ \int_0^1 e^{2t}dt + \int_0^1 e^t.tdt$

But $\int_0^1 e^{2t}dt = \frac{1}{2}\left(e^2 - 1\right),\ \int_0^1 t.e^t dt = 1$

$\therefore B = \frac{B}{2}\left(e^2 - 1\right) + 1 \Rightarrow B = \frac{2}{3.e^2}$

From (i)

$g(x) = \frac{2}{3 - e^2}.e^x + x$

$g(0) = \frac{2}{3 - e^2}$

15. Ans. (c)

$g(2) = \frac{2e^2}{3 - e^2} + 2 = \frac{6}{3 - e^2}$

$\Rightarrow \frac{g(0)}{g(2)} = \frac{1}{3}$

Paragraph – 6

Let f : N → N (N being the set of positive integers) be a function defined by f(x) = the biggest positive integer obtained by reshuffling the digits of x. For example f(296) = 962.

16. f is

(A) one-one and onto (B) one-one and into

(C) many one and onto (D) many-one and into

17. The biggest positive integer which divides f(n) – n, for all n $\in$ N, is

(A) 3 (B) 9

(C) 18 (D) 27

Sol. 16. (D)

As f(296) = f(926), f is many-one.

Further the numbers whose digits increase from left to right (for example) have no pre-image.

Hence f is into.

17. (B)

It is easy to see that the remainder, when a positive integer is divided by 9, is the same as the sum of the digits of the number (until the sum becomes a one digit number). Thus f(n) and n leave the same remainder, when divided by 9. Hence 9 divides f(n) – n. Further there is no reason to expect that the number is divisible by 27. The number f(n) – n is not divisible by 18 also, in case f(n) – n is odd.

Hence 9 is the biggest number.

By the definition of f, digits of f(x) are non-increasing from left to right.

Hence (B) is the correct answer.

Paragraph – 7

Let $f(x) = Min\{x - [x], -x - [-x]\}$, $-2 \le x \le 2$, $g(x) = |2 - |x - 2||$, $-2 \le x \le 2; h(x) = \frac{|\sin x|}{\sin x}$,

$-2 \le x \le 2$ x $\ne 0$ where[x] denotes the greatest integer function ≤ x

18. The number of solutions of the equation $x^2 + (f(x))^2 = 1$ is $\{-1 \le x \le 1\}$

(A)0 (B)2 (C)4 (D) 6

19. The sum of all the roots of the equation g(x)-h(x)=0 is {-2≤ x ≤ 2}

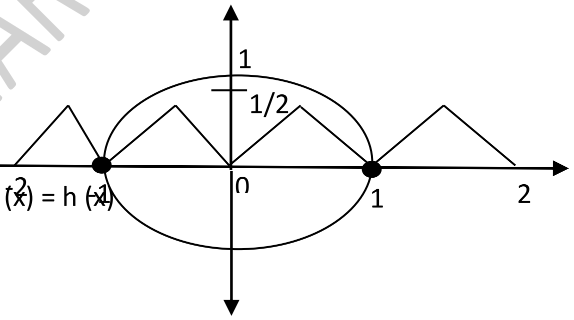
(A)positive (B)negative (C)zero (D) none of these

20. The value of $\int_{-2}^{2} f(x)dx$ = ________

(A) 0 B)2 (C)1 (D)8

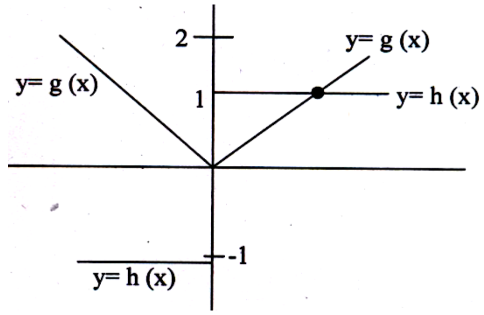
Sol. 18. (b)

Let $Y = f(x) \Rightarrow x^2 + y^2 = 1$



19. (a)

Let y = g (x) = h (x)



20. (c)

$$\int_{-2}^{2} f(x)\,dx = 4.\frac{1}{2}\times 1\times\frac{1}{2} = 1$$

Paragraph – 8

For a finite set A, $Let|A|$ denote the number of elements in the set A. Also let F denote the set of all functions $f:\{1,2,...,n\}\rightarrow\{1,2,....,k\}(n\geq 3, k\geq 2)$ satisfying

$f(i)\neq f(i+1)$ *for every* $i, 1\leq i\leq n-1$.

21. $|F| =$

a) $k^n(k-1)$ b) $k(k-1)^n$

c) $k^{n-1}(k-1)$ d) $k(k-1)^{n-1}$

22. If $\boldsymbol{c}(n,k)$ denote the number of functions in F satisfying

$f(n)\neq f(1), then,\ for\ n\geq 4, \boldsymbol{c}(n,k)$ =

a) $k(k-1)^{n-1}-\boldsymbol{c}(n-1,k)$ b) $k(k-1)^n-\boldsymbol{c}(n-1,k-1)$

c) $k^{n-1}(k-1)-\boldsymbol{c}(n-1,k)$ d) $k^n(k-1)-\boldsymbol{c}(n-1,k)$

23. For $n\geq k, \boldsymbol{c}(n,k)$, where $\boldsymbol{c}(n,k)$ has the same meaning as in Q.34, equals

a) $k^n+(-1)^n(k-1)$ b) $(k-1)^n+(-1)^{n-1}(k-1)$

c) $(k-1)^n+(-1)^n(k-1)$ d) $k^n+(-1)^{n-1}(k-1)$

Key: D-A-C

Hint 21. The image of the element 1 can be chosen in k ways and for each of the remaining (n - 1) elements, the image can be defined in (k-1) ways, since $f(i)\neq f(i+1)$

$\therefore$ Total number of mappings in F = k (k - 1)$^{n-1}$

22. Out of the total number of mappings in F, the number of mappings which satisfy f (n) = 1 is same as the number of maps which satisfy $f(n-1)\neq 1$ and this number is c(n – 1, k)

$\therefore c(n,k) = |F| - c(n-1,k)$

23. $c(n,k) = k(k-1)^{n-1} - c(n-1,k)$

$$= (k-1)^n + (k-1)^{n-1} - c(n-1,k)$$

$$\Rightarrow c(n,k) - (k-1)^n = (-1)\left(c(n-1,k) - (k-1)^{n-1}\right)$$

$$= (-1)^{n-3}\left(c(3,k) - (k-1)^3\right)$$

But c(3,k) = number of maps f in F for which $f(3) \neq f(1)$

=k(k-1) (k-2)

$$\therefore c(n,k) - (k-1)^n = (-1)^{n-1}(k-1)\left[k(k-2) - (k-1)^2\right]$$

$$= (-1)^n (k-1)$$

$$\therefore c(n,k) = (k-1)^n + (-1)^n (k-1)$$

Paragraph – 9

Let f : A $\to$ B be a function, the f is said to be one-one, if for any x, y$\in$A x$\neq$ y $\Rightarrow$ f(x) $\neq$ f(y) and f is said to be onto, if for any y$\in$B, there exist at least one x$\in$A, such that f(x) = y.

24. Let f : R $\to$ R define by f(x) = $\begin{cases} x^3, & \text{if} \quad x \in Q \\ -x^3, & \text{if} \quad x \in Q^c \end{cases}$, then f is

(A) one-one and onto (B) one-one and into

(C) many-one and onto (D) many-one and into

Key: A

Hint: Let x_1, x_2 $\in$R, such that $x_1 \neq x_2$

Now $x_1^3 = x_2^3$ or $x_1^3 = -x_2^3$ is possible only when $x_1 = x_2$ or $x_1 + x_2 = 0$, which is not possible, so f is one-one and f is onto also.

25. f : R $\to$ R

f(x) = $\begin{cases} x+7, & x \in Q \\ \sqrt{7} - x, & x \in Q^c \end{cases}$, then f is

(A) one-one & onto (B) one-one and into

(C) many-one & onto (D) many-one and into

Key: D

26. f :R $\to$ R

f(x) = $\begin{cases} x+\sqrt{5}, & x \in Q \\ \sqrt{5} - x, & x \in Q^C \end{cases}$, then f is

(A) one-one & onto (B) one-one and into

(C) many-one & onto (D) many-one and into

Key: A

Paragraph – 10

For a finite set A, $Let|A|$ denote the number of elements in the set A. Also let F denote the set of all functions $f:\{1,2,...,n\} \to \{1,2,....,k\}(n \geq 3, k \geq 2)$ satisfying

$f(i) \neq f(i+1)$ *for every* $i,\ 1 \leq i \leq n-1$.

27. $|F| =$

a) $k^n(k-1)$ b) $k(k-1)^n$ c) $k^{n-1}(k-1)$ d) $k(k-1)^{n-1}$

Key. D

28. If $\boldsymbol{c}(n,k)$ denote the number of functions in F satisfying

$f(n) \neq f(1)$, *then, for* $n \geq 4, \boldsymbol{c}(n,k)$ =

a) $k(k-1)^{n-1} - \boldsymbol{c}(n-1,k)$ b) $k(k-1)^n - \boldsymbol{c}(n-1,k-1)$

c) $k^{n-1}(k-1) - \boldsymbol{c}(n-1,k)$ d) $k^n(k-1) - \boldsymbol{c}(n-1,k)$

Key. A

29. For $n \geq k, \boldsymbol{c}(n,k)$, where $\boldsymbol{c}(n,k)$ has the same meaning as in Q.34, equals

a) $k^n + (-1)^n(k-1)$ b) $(k-1)^n + (-1)^{n-1}(k-1)$

c) $(k-1)^n + (-1)^n(k-1)$ d) $k^n + (-1)^{n-1}(k-1)$

Key. C

Sol. 27. The image of the element 1 can be chosen in k ways and for each of the remaining (n - 1) elements, the image can be defined in (k-1) ways, since $f(i) \neq f(i+1)$

$\therefore$ Total number of mappings in F = k (k - 1)$^{n-1}$

28. Out of the total number of mappings in F, the number of mappings which satisfy f (n) = 1 is same as the number of maps which satisfy $f(n-1) \neq 1$ and this number is c(n – 1, k)

$\therefore c(n,k) = |F| - c(n-1,k)$

29. $c(n,k) = k(k-1)^{n-1} - c(n-1,k)$

$= (k-1)^n + (k-1)^{n-1} - c(n-1,k)$

$\Rightarrow c(n,k) - (k-1)^n = (-1)\left(c(n-1,k) - (k-1)^{n-1}\right)$

$= (-1)^{n-3}\left(c(3,k) - (k-1)^3\right)$

But c(3,k) = number of maps f in F for which $f(3) \neq f(1)$

=k(k-1) (k-2)

$\therefore c(n,k) - (k-1)^n = (-1)^{n-1}(k-1)\left[k(k-2) - (k-1)^2\right]$

$= (-1)^n(k-1)$

$\therefore c(n,k) = (k-1)^n + (-1)^n(k-1)$

Paragraph – 11

Consider the function f : $R \to (0,\infty)$ defined by $f(x) = 2^x + 2^{|x|}$

30. $f(x)$ is

(A) One-One Onto (B) One-One Into

(C) Many-One Onto (D) Many-One Into

Key. D

31. The number of solutions of the equation $f(x) = \ln\pi$ is

(A) 0 (B) 1

(C) 2 (D) 3

Key. A

32. The area of the region bounded by the curves $y = f(x), y = 0, x = -1$ and $x = 1$ is

(A) $\ln(2^{7/2})$ sq.units (B) $\log_2(e^{7/2})$ sq.units

(C) $\frac{3}{\ln 2}$ sq.units (D) $\frac{2}{\ln 2}$ sq.units

Key. B

Sol. 30.31.32

$f(x) = 2^x + 2^{|x|}$

For $x \geq 0,\ f(x) = 2.2^x$

And for $x < 0; f(x) = 2^x + 2^{-x}$

and in this case $f'(x) = 2^x \ln 2 - 2^{-x}\ln 2 = \ln 2\left(\frac{2^{2x}-1}{2^x}\right)$

$= \frac{\ln 2}{2^x}.(2^x+1)(2^x-1) < 0$

So; $f(x)$ is decreasing in $x \in (-\infty, 0)$ and increasing in $x \in (0, \infty)$

So, $f(x)$ is many one

Also , Range of $f(x)$ is $[2, \infty)$

f is many –one Into

Since $\ln\pi < 2$

So, the equation $2^x + 2^{|x|} = \ln\pi$ has no solution

Now the required area = $\int_{-1}^{0}(2^x + 2^{-x})dx + \int_{0}^{1} 2.2^x dx = \frac{7}{2\ln 2} = \frac{7}{2}.\log_2 e = \log_2(e^{7/2})$

Paragraph – 12

Let $f(x)$ be a differentiable function for all $x \in R - \{0\}$ and satisfies the equation $(x^4 + x^2) f'(x) - 2x^3 f(x) + (x^2+1)^2 = 0$. (1, 2) is a point on the graph of $y = f(x)$.

33. Let $a \in R - \{0\}$. then

A) $f''(a) + af'(a) = 3$ B) $a.f''(a) + 2f'(a) = 2$

C) $a^2 f''(a) + f'(a) = 1$ D) $af'(a) + a^2 f''(a) = 1$

Key. B

34. Let $g(x) = \int_{0}^{x^2} \frac{(t-4)f'(t)}{1+t^2} dt\ \forall x \in R - \{0\}$ then

A) $g(x)$ is increasing in (-1, 1) and has maximum at x = 1

B) $g(x)$ is increasing in (-1, 1) and has neither maximum nor minimum at x = 1

C) $g(x)$ is decreasing in (1, 2) and has maximum at x = 1

D) $g(x)$ is decreasing in (1, 2) and has neither maximum nor minimum at x = 1

Key. B

35. Let $h(x)$ be the reflection of the graph of $f(x)$ with respect to the line $y = x$ then

A) $h(x)$ increases $\forall\ x < 0$

B) $h(x)$ decreases $\forall\ x < 1$

C) $h(x)$ increases $\forall\ x > 1$

D) $h(x)$ decreases $\forall\ x > 1$

Key. C

Sol. 33 – 35

$$f'(x) - \frac{2x}{x^2+1} f(x) = -\frac{(x^2+1)}{x^2} \Rightarrow I.F = e^{\int \frac{-2x}{x^2+1}dx} = \frac{1}{x^2+1}$$

$\therefore$ solution is $f(x) \times \frac{1}{x^2+1} = \int \frac{-(x^2+1)}{x^2} \times \frac{1}{x^2+1} dx = \frac{1}{x} + c$

As $f(1) = 2 \Rightarrow c = 0 \Rightarrow f(x) = x + \frac{1}{x}$

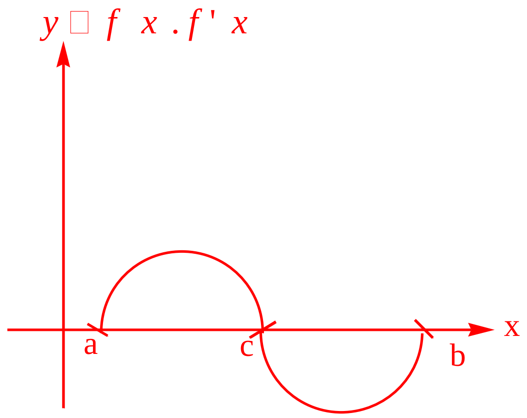
$$g'(x) = \frac{2(x+2)(x-2)(x+1)(x-1)(x^2+1)}{x^3(1+x^4)}$$

Paragraph – 13

f(x) is continuous and differentiable function. Given, f(x) assumes values of the form $\pm\sqrt{I}$ where 'I' denotes set of whole numbers whenever x = a or b; otherwise f(x) assumes real values. Also

$$f(c) = \frac{-3}{2} \ and \ |f(a)| \leq |f(b)|$$

The graph of $y = f(x).f'(x)$ is given below



Using the above data answer the following :

36. The number of rational values that $f(a) + f(b) + f(c)$ can assume is

(A) 4 (B) 2 (C) 3 (D) 5

Key. C

37. The number of values that $\left(f(a)\right)^2+\left(f(b)\right)^2+\left(f(c)\right)^2$ can assume is

(A) 4 (B) 3 (C) 2 (D) 7

Key. A

38. The possible number of triplets $\left(f(a),\ f(b),\ f(c)\right)$ is

(A) 4 (B) 5 (C) 6 (D) 7

Key. C

Sol. **36 to 38**

$f(x).f'(x)$³ 0 in [a, c] and £ 0 in [c, b]

$f(c) = -\sqrt{2}$ *and* $f(x)\ f'(x)$¹ 0 *in* (a,c)È(c,b)

$f(a),\ f(b)$£ 0

f(a)	f(b)	f(c)
0	0	$\frac{-3}{2}$
0	**-1**	$\frac{-3}{2}$
0	$-\sqrt{2}$	$\frac{-3}{2}$
-1	-1	$\frac{-3}{2}$
-1	$-\sqrt{2}$	$\frac{-3}{2}$
$-\sqrt{2}$	$-\sqrt{2}$	$\frac{-3}{2}$

Paragraph – 14

Let f be a real valued function such that

(i) $f(0)=\frac{1}{2}$

$f(x+y)=f(x)f(a-y)+f(y)f(a-x)$ for all $x,y\in R$, for some $a\in R$,then

39. $f(x^2+1)+f(x-1)=$

A) 0 B) $2x^2+x$

C) 1 D) x^2-x+2

Key. C

40. $f(1)+f(2)+f(3)+......+f(100)=$

A) 5050 B) 50 C) 100 D) 5000

Key. B

41. If $g(x)=\dfrac{x^2+2x}{2x^2+1}$ then $(fog)\left(\dfrac{1}{4}\right)+gof(4)=$

A) 0 B) 3/4 C) 4/3 D) 1

Key. C

Sol. 39, 40&41

$$f(x+y)=f(x)f(a-y)+f(y)f(a-x)$$

Put x = 0, y = 0 $\Rightarrow f(0)=2f(0)f(a)\Rightarrow f(a)=\dfrac{1}{2}$

Put y = 0, $f(x)=f(x)f(a)+f(0)f(a-x)$

$$\Rightarrow f(x)=\frac{f(x)+f(a-x)}{2}$$

$$f(x)=f(a-x)$$

put y = a – x

$$f(a)=f^2(x)+f^2(a-x) \text{ but } f(x)=f(a-x)$$

$$\Rightarrow \quad f(a)=2f^2(x)$$

$$\Rightarrow \quad f^2(x)=\frac{1}{4}$$

$$\Rightarrow \quad f(x)=\frac{1}{2} \qquad (\because f(0)=\frac{1}{2})$$

$\therefore f(x)=\dfrac{1}{2}$, a constant function

Paragraph – 15

A function $f(x)$ is said to be even iff $f(-x)=f(x)$ and odd iff $f(x)=-f(x)$. Also $f(x)$ is said to be periodic iff $f(x+T)=f(x), T>0$ and the least positive value T is called its period.

42. If $f(x)=\dfrac{x}{e^x-1}+\dfrac{x}{2}+1$ then $f(x)$ is

A. Even **B.** Odd
C. neither even nor odd **D.** Both even and odd

Key. A

43. $f(x)$ is an even polynomial function . Then sin $(f(x)-3x)$ is

A. an even function **B.** an odd function
C. neither even nor odd **D.** periodic with period $\dfrac{\pi}{2}$

Key. C

44. If $f(x)\, and\, g(x)$ be two functions with all real numbers as their domains, then $h(x)=[f(x)+f(-x)][g(x)-g(-x)]$ is

A. an even function when f is even and g is odd **B.** an even function when both f and g are odd
C. always an odd function **D.** always an even function

Key. C

Sol. 42. $f(x)=\frac{x}{e^x-1}+\frac{x}{2}+1$

$f(-x)=\frac{-x}{e^{-x}-1}-\frac{x}{2}+1=\frac{xe^x}{e^x-1}-\frac{x}{2}+1=\frac{x}{e^x-1}+\frac{x}{2}+1$

43. Let $g(x)=\sin[f(x)-3x]$

$g(-x)=\sin[f(-x)+3x]\ =\sin[f(x)+3x]$

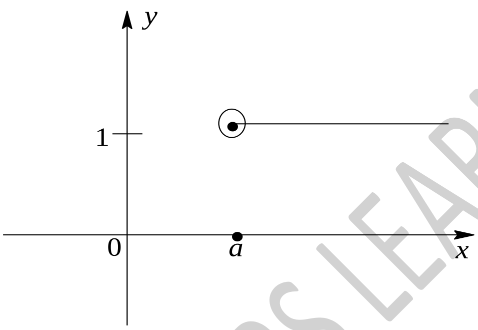
44. $h(-x)=[f(-x)+f(x)][g(-x)-g(x)]\quad =-h(x)$

Paragraph – 16

The unit step function $u(x-a)$ is defined as $u(x-a)=\begin{cases}0 & , & x<a \\ 1 & , & x\geq a\end{cases}$.

The graph of $y=u(x-a)$ is as shown below:

Domain of $u(x-a)$ is *R* and its range is $\{0,1\}$.



Answer the following questions.

45. Let $f(x)=[x]\quad 0\leq x<3$ where $[\ \ ]$ denotes the greatest integer function. The representation of $f(x)$ in terms of unit step function is

(A) $f(x)=u(x)+u(x-1)+u(x-2)+(x-3),0\leq x\leq 3$

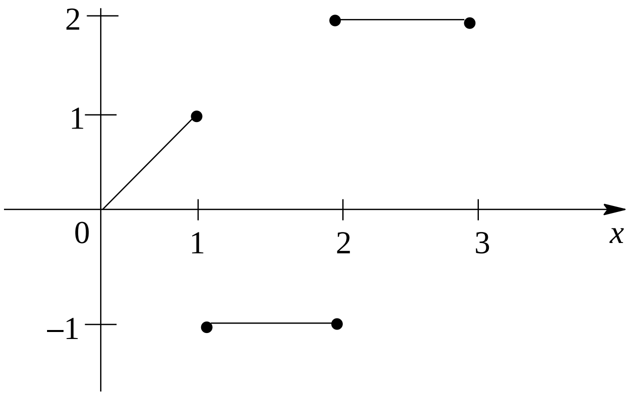
(B) $f(x)=u(x-1)+u(x-2)+u(x-3),0\leq x<3$

(C) $f(x)=u(x-1)-u(x-2)+u(x-3),0\leq x<3$

(D) $f(x)=u(x-1)+2u(x-2)+3u(x-3),0\leq x<3$

Key. B

46. Graph of $y = f(x), 0 \le x < 3$ is shown below:



Representation of $f(x)$ in terms of the unit step function is given by

(A) $f(x) = x\{u(x) - u(x-1)\} - 2u(x-2)$

(B) $f(x) = x\{u(x) - u(x-1)\} - u(x-2) + 2u(x-3), 0 \le x < 3$

(C) $f(x) = x\{u(x) - u(x-1)\} - u(x-1) + 3u(x-2), 0 \le x < 3$

(D) $f(x) = xu(x) - u(x-1) + 2u(x-2), 0 \le x < 3$

Key. C

47. Representation of the function $f(x) = \begin{cases} x^3, & 0 \le x < 1 \\ x-1, & 1 \le x < 3 \\ 0, & x \ge 3 \end{cases}$ in terms of the unit step function is

(A) $x^3[u(x) - u(x-1)] + (x-1)[u(x-1) - u(x-3)]$

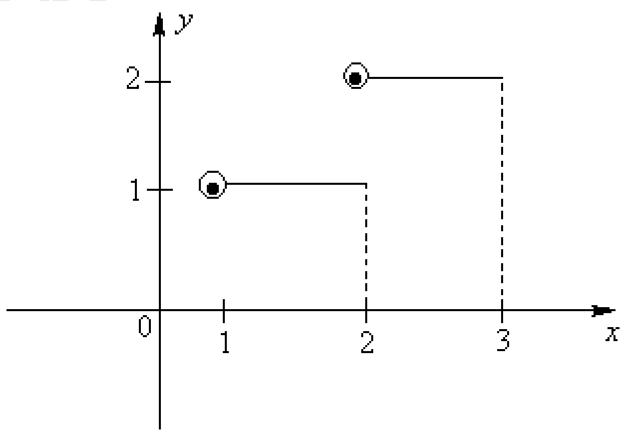
(B) $x^3u(x) + (x-1)u(x-1)$

(C) $x^3[u(x) - u(x-1)] + (x-1)u(x-2) + u(x-3)$

(D) $x^3u(x) + (x-1)[u(x) - u(x-1)] + u(x-2)$

Key. A

Sol. 45.



Using the definition of $u(x)$, we can write $f(x)$ as

$f(x) = u(x-1) + u(x-2) + u(x-3), 0 \le x < 3$.

46. We have $f(x) = \begin{cases} x & 0 \le x < 1 \\ -1 & 1 \le x < 2 \\ 2 & 2 \le x < 3 \end{cases}$ using the definition of $u(x)$, $f(x)$ can be represented as

$$f(x) = x\{u(x) - u(x-1)\} - \{u(x-1) - u(x-2)\} + 2u(x-2)$$
$$= x\{u(x) - u(x-1)\} - u(x-1) + 3u(x-2) \quad 0 \le x < 3$$

47. Using the definition of $u(x)$, we may write $f(x)$ as

$$f(x) = x^3[u(x) - u(x-1)] + (x-1)[u(x-1) - u(x-3)].$$

Paragraph – 17

Let f(x) be a real values and differentiable function on R such that f(x + y) = $\frac{f(x)+f(y)}{1-f(x).f(y)}$.

48. f(0) equals

(A) 1 (B) 0

(C) –1 (D) none of these

Key. B

49. f(x) is

(A) odd function (B) even function

(C) odd and even function simultaneously (D) neither even nor odd.

Key. A

50. $\frac{f'(x)}{f'(0)}$ equals

(A) $1 - f^2(x)$ (B) $2 + f^2(x)$

(C) $2 - f^2(x)$ (D) $1 + f^2(x)$

Key. D

Sol. 48–50 Put x = y = 0

$\Rightarrow f(0) = \frac{f(0)+f(0)}{1-[f(0)]^2} \Rightarrow f(0)[f^2(0)+1] = 0 \Rightarrow f(0) = 0$

(since $f^2(0) \neq -1$).

Now put y = –x, we get

$f(0) = \frac{f(x)+f(-x)}{1-f(x).f(-x)}$

$\Rightarrow f(x) + f(-x) = 0$

$\Rightarrow f(-x) = -f(x) \Rightarrow$ f(x) is an odd function.

Now $f(y-x) = \frac{f(y)+f(-x)}{1-f(-x).f(y)} \Rightarrow \frac{f(y-x)}{y-x} = \frac{f(y)-f(x)}{1+f(x).f(y)}.\frac{1}{(y-x)}$

Taking limit as $y \to x$, we get

$$\lim_{y\to x}\frac{f(y-x)-f(0)}{y-x} = \lim_{y\to x}\frac{f(y)-f(x)}{y-x}.\lim_{y\to x}\frac{1}{1+f(x).f(y)}$$

$$\Rightarrow \lim_{y\to x}\frac{f(y-x)}{y-x} = \lim_{y\to x}\frac{f(y)-f(x)}{y-x}\lim_{y\to x}\frac{1}{1+f^2(x)}$$

$\Rightarrow$ f′ (0) = f′(x) . $\frac{1}{1+f^2(x)}$ $\Rightarrow$ f′(x) = f′(0) [1 + f^2 (x)]

$\Rightarrow \frac{f'(x)}{f'(0)}$ = 1 +f^2(x).

Paragraph – 18

Paragraph : The function f(x) is defined for all x > 0 and satisfies the conditions :

(i) f(x) is strictly increasing on $(0,\infty)$

ii) $f(x) > -\frac{1}{x} \quad \forall \quad x > 0$

iii) $f(x) f\left(\frac{1}{x} + f(x)\right) = 1 \ \forall \ x > 0$.

51. The value of $f(1)$ is

a) $\frac{1+\sqrt{5}}{2}$ b) $\frac{1-\sqrt{5}}{2}$ c) $\frac{\sqrt{5}-1}{2}$ d) None

Key. B

52. A function $f(x)$ satisfying the above condition can be $(\lambda \ is \ a \ constant)$

a) $f(x) = \frac{\lambda}{x}$ b) $f(x) = \lambda x$ c) $f(x) = \lambda + x$ d) $f(x) = x^{\lambda}$

Key. A

53. $f'(x)$ is a

a) decreasing function for all x > 0 b) increasing for all x > 0

c) constant for all x > 0 d) monotonicity cannot be predicted

Key. A

Sol. 51.52. 53 . $Let \ f(1) = k, for \ x = 1,$ we have

$kf(k+1) = 1 \Rightarrow f(k+1) = \frac{1}{k}$

Now x = k + 1 gives

$f(k+1)\left(f\left(\frac{1}{k+1} + f(k+1)\right)\right) = 1$

$\Rightarrow f\left(\frac{1}{k} + \frac{1}{k+1}\right) = \frac{1}{f(k+1)} = k = f(1)$

Since f is increasing $\Rightarrow \frac{1}{k}+\frac{1}{k+1}=1 \Rightarrow k=\frac{1\pm\sqrt{5}}{2}$

By contradiction we can say that k can not be positive so, $k=\frac{1-\sqrt{5}}{2}$.

Generalized form $f(x)=\frac{k}{x}$.

Paragraph – 19

Paragraph : A pair of curves $y = g_1(x)$ and $y = g_2(x)$ are such that

(a) : The tangent drawn at points with equal abscissa intersect in y –axis

(b) the normal drawn at points with equal x- coordinates intersect on x –axis

(c) one curve $y = g_1(x)$ passes through (1, 1) and the other $y = g_2(x)$ passes through (2, 3) .

54. Which statement is true :

a) $y = g_1(x)$ is odd , $y = g_2(x)$ is even b) $y = g_1(x)$, $y = g_2(x)$ both are odd

c) $y = g_1(x)$ is even, $y = g_2(x)$ is odd d) None of these

Key. B

55. Which is true :

a) $g_1(1)+g_2(2)=3$ b) $g_1(1)-g_2(2)=-1$

c) $g_1(1)+g(2)=4$ d) $g_1(1)+g_2(2)=6$

Key. C

56. If both curves satisfy I[st] condition, then which is true

a) $g_1(x)-g_2(x)=x\left(g_1^1(x)-g_2^1(x)\right)$ b) $g_1(x)-g_2(x)=x\left(g_1^1(x)+g_2^1(x)\right)$

c) $g_1(x).g_1^1(x)=g_2(x)g_2^1(x)$ d) $g_1^2(x)+g_2^2(x)=const$

Key. A

Sol. 54. 55. 56. The equation of tangents with equal abscissa x, are

$Y-g_1(x)=g_1^1(x)(X-x)$ and

$Y-g_2(x)=g_2^1(x)(X-x)$ these intersect at y – axis,

$\Rightarrow -x\,g_1^1(x)+g_1(x)=-x\,g_2^1(x)+g_2(x)$

$g_1(x)-g_2(x)=x\left(g_1^1(x)-g_2^1(x)\right)$

Integrating

$\Rightarrow \log\left|g_1(x)-g_2(x)\right|=\log|x|+c-(i)$

Equation of normals with equal abscissa x, are $Y - g_1(x) = -\frac{1}{g_1^1(x)}(X - x)$ and

$Y - g_2(x) = -\frac{1}{g_2^1(x)}(X - x)$

Intersect on x – axis $\Rightarrow$.

$x + g_1(x)g_1^1(x) = x + g_2(x)g_2^1(x)$

$\Rightarrow\ g_1(x)g_1^1(x) = g_2(x)g_2^1(x)$

Integrating $g_1^2(x) - g_2^2(x) = c$

$\Rightarrow g_1(x) + g_2(x) = \frac{c}{g_1(x) - g_2(x)} = \pm\frac{c_2}{c_1^x} = \pm\frac{\lambda^2}{x}$ ------- (ii) (using (i))

From $(i)\, and\, (ii), weget$.

$2g_1x = \pm\left(\frac{\lambda_2}{x} + c_1x\right), 2g_2(x) = \pm\left(\frac{\lambda_2}{x} - c_1x\right)$

We have $g_1(1) = 1\ \ g_2(2) = 3$

$\Rightarrow\ \ g_1(x)\ = \frac{2}{x} - x\ \ and\ \ g_2(x) = \frac{2}{3} + x$

Paragraph – 20

Let us consider the two non empty sets X and Y such that X = {1, 2, 3, 4, 5, 6, 7} and Y = {1, 2, 3, 4, 5, 6}. And answer the following questions.

57. The number of into function from X to Y is

(A) $7^6 - 3\underline{|7}$ (B) $6^7 - 3\underline{|7}$

(C) $6^7 - \underline{|7}$ (D) $6^6 - 3\underline{|6}$

Key. B

58. The number of many one function from X to Y, is

(A) 6^7 (B) 7^6

(C) $7^6 - 6^7$ (D) $6^7 - \underline{|6}$

Key. A

59. The number of function from X to Y & such that $f(x_i) \neq x_i\ \forall x_i \in \{1, 2, 3, ... , 7\}$, is

(A) $6(5)^6$ (B) $5(6)^5$

(C) $5(7)^6$ (D) none of these

Key. A

Sol. 57. Number of into function = $6^7 - \frac{7!6!}{2!5!}$

$= 6^7 - 3\ .\ 7!$

58. Number of many-one function = 6^7.

59. 7 can be associated to $y \in Y$ in 6 ways

1, 2, 3, 4, 5, 6 ...... in 5 ways each.

$\therefore$ The total number of function required = $6(5)^6$.

Paragraph – 21

We know that any real number x can be expressed as following $x=[x]+\{x\}$ where [x] is an integer and $0\le\{x\}<1$. We define $[x]$ as the greatest integer less than or equal to x or integral part of x and $\{x\}$ as the fractional part of x.

Suppose for any real number x we write $x=(x)-<x>$ where (x) is integer and $0\le<x><1$. We define (x) as the least integer greater than or equal to x

60. The domain of the function $f(x)=\dfrac{1}{\sqrt{x-(x)}}$ is

(A) 1 (B) R – I (C) $(0,\infty)$ (D) ϕ

Key. D

61. Find the range of the function $f(x)=\dfrac{1}{\sqrt{(x)-[x]}}$

(A) ϕ (B) $\{1\}$ (C) $\left\{\dfrac{1}{\sqrt{n}},n\in N\right\}$ (D) $(1,\alpha)$

Key. B

62. If the function $f:R\to R$ be such that $f(x)=x-[x]$, (where [x] denotes the greatest integer less than or equal to x) then $f^{-1}(x)$ is

(A) $\dfrac{1}{x-[x]}$ (B) $[x]-x$ (C) not defined (D) $[x]+x$

Key. C

Sol. 60. $x-(x)\le0$

61. If $x\in I$ then $(x)=[x]$ so $f(x)$ is not defined

Let $x\notin I$ then $(x)-[x]=1$ so $f(x)=1$

Paragraph – 22

If $f(x)=\begin{cases}x-[x], & x\notin I\\ 1, & x\in I\end{cases}$ where I is an integer and [ ] represents the greatest integer function and

$g(x)=\underset{n\to\infty}{\mathrm{Lt}}\dfrac{\{f(x)\}^{2n}-1}{\{f(x)\}^{2n}+1}$ then

63. The range of $g(x)$ and $g(g(x))$ are respectively

a) $(0,1),\{1\}$ b) $(-1,0),\{0\}$

c) $\{-1,0\},\{0\}$ d) $\{-1,0\},\{1\}$

Key. C

64. The number of solutions of $[g[g(x)]]=x^4-2x^2-3|x^2-1|+3$ is

a) 4 b) 5 c) 6 d) 8

Key. B

65. The period of $g(x)$ and $g(g(x))$ are respectively

a) 1, 1 b) 1, 2 c) $(2,1)$ d) None of these

Key. D

Sol. 63. $g(x)=0,\ x\in I$

$=-1 \qquad x\notin I$

64. $g(g(x))=0$

65. $g(g(x))=0$ fundamental period is undefined.

Paragraph – 23

If f: [0, 2] $\to$ [0, 2] is a bijective function defined by $f(x)=ax^2+bx+c$, where

66. f(2) is equal to

A) 2 B) α where $a\in(0,2)$ C) 0 D) cannot be determined

Key. C

Sol. f(0) = $c\neq 0$

$\therefore$ f(2) = 0

67. Which of the following is one of the roots f(x) = 0 is

A) $\frac{1}{a}$ B) $\frac{1}{b}$ C) $\frac{1}{c}$ D) $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$

Key. A

Sol. Since f(0) = 2 $\Rightarrow$ c = 2

and f(2) = 0 $\Rightarrow$ 4a + 2b + c = 0 $\Rightarrow$ 4a + 2b + 2 = 0

$\Rightarrow$ 2a + b + 1 = 0 $\Rightarrow$ $2+\frac{b}{a}+\frac{1}{a}=0$

$\therefore$ $\frac{1}{a}$ is a root of $ax^2+bx+c=0$

68. Which of the following is not a value of a?

A) $-\frac{1}{4}$ B) $\frac{1}{2}$ C) $-\frac{1}{2}$ D) 1

Key. D

Sol. $-\frac{b}{2a}\le 0$ or $-\frac{b}{2a}\ge 2$

$\Rightarrow \frac{2a+1}{2a}\le 0$ or $\frac{2a+1}{2a}\ge 2$

$\Rightarrow -\frac{1}{2}\le a<0$ or $0<a\le\frac{1}{2}$

$\therefore$ a = 1 is not possible

Paragraph – 24

Let $f(x)=\begin{cases}2x+a & : & x\ge -1\\ bx^2+3 & : & x<-1\end{cases}$

and $g(x)=\begin{cases}x+4 & : & 0\le x\le 4\\ -3x-2 & : & -2<x<0\end{cases}$

functions

69. g(f(x)) is not defined if

A) $a \in (10, \infty), b \in (5, \infty)$ B) $a \in (4, 10), b \in (5, \infty)$

C) $a \in (10, \infty), b \in (0, 1)$ D) $a \in (4, 10), b \in (1, 5)$

Key. A

Sol. g(f(x)) is not define if

(i) – 2 + a > 8 and (ii) b + 3 > 8

a > 10 and b > 5

70. If domain of g(f(x)) is [- 1, 4], then

A) a = 1, b > 5 B) a = 2, b > 7 C) a = 2, b > 10 D) a = 0, b $\in$ R

Key. D

Sol. $x \in [-1, 2] \Rightarrow -1 \le x \le 2 \Rightarrow -2 \le 2x \le 4 \Rightarrow -2 + a \le 2x + a \le 4 + a$

$\Rightarrow$ - 2 + a $\le$ - 2 and 4 + a $\le$ 4 i.e. a = 0

B can take any value.

71. If a = 2 and b = 3 then range of g(f(x)) is

A) (- 2, 8] B) (0, 8] C) [4, 8] D) [- 1, 8]

Key. C

Sol. if a = 2, b = 3

$$f(x) = \begin{cases} 2x + 2 & ; \quad x \ge -1 \\ 3x^2 + 3 & ; \quad x < -1 \end{cases}$$

range of f(x) is $[0, \infty]$

Paragraph – 25

Let f: R $\to$ r is a function satisfying f(2 – x) = f(2 + x) and f(20 – x) = f(x), $\forall x \in R$. For this function fanswer the following.

72. If f(0) = 5, then minimum possible number of values of x satisfying f(x) = 5, for $x \in [0, 170]$, is

A) 21 B) 12 C) 11 D) 22

Key. A

Sol. Since period of f(x) is 2(10 – 2) = 16

$\therefore$ f(0) = f(16) = f(32) = … = f(160 = 5

$\therefore$ there are atleast 11values of x for which f(x) = 5

F(0) = f(4) = f(16)

Due to symmetry in one period length f(x) = 5 one solution other then 0, 16, 32, ________

$\therefore$ at least minimum possible number of values of x is 10 + 11 = 21

73. Graph of y = f(x) is

A) symmetrical about x = 18 B) symmetrical about x = 5

C) symmetrical about x = 8 D) symmetrical about x = 20

Key. A

Sol. Obvious by definition

74. If (2) $\neq$ f(6), then

A) fundamental period of f(x) is 1 B) fundamental period of f(x) may be 1

C) period of f(x) can't be 1 D) fundamental period of f(x) is 8

Key. C

Sol. If 1 is a period, then f(x) = f(x + 1), $\forall x \in R$

$\Rightarrow$ f(2) = f(3) = f(4) = f(5) = f(6)

Which contradicts the given hypotheses that f(2) $\neq$ f(6)

$\therefore$ 1 cannot be period of f(x)

Paragraph – 26

If $f:(0,\infty)\to(0,\infty)$ satisfy f(xf(y)) = $x^2y^a\,(a\in R)$, then

75. Value of a is

A) 4 B) 2 C) $\sqrt{2}$ D) 1

Key. A

Sol. $f:(0,\infty)\to(0,\infty)$

$f\left(x\left(f\left(y\right)\right)\right)=x^2y^2\left(a\in R\right)$

Put x = 1, we get f(f(y)) $=y^a$

Put $f\left(y\right)=\frac{1}{x}$, we get $f\left(1\right)=\frac{1}{\left(f\left(y\right)\right)^2}.\,y^a$ and put y = 1 we get $\left(f\left(1\right)\right)^3=1$

$\therefore$ f(1) = 1

for y = 1, we have f(x(f(1))) $=x^2$

$\therefore$ $f\left(x\right)=x^2$

thus a = 4

76. $\sum_{r=1}^{n}f\left(r\right){}^nC_r$ is

A) $n\,2^{n-1}$ B) $n\left(n-1\right)2^{n-2}$

C) $n\,2^{n-1}+n\left(n-1\right)2^{n-2}$ D) 0

Key. C

Sol. $\sum_{r=1}^{n}f\left(r\right){}^nC_r=\sum_{r=1}^{n}r^2\;{}^nC_r=\sum_{r=1}^{n}\left(r\left(r-1\right)\right){}^nC_r=n\left(n-1\right)2^{n-2}+n.2^{n-1}$

77. Number of solutions of $2\,f\left(x\right)=e^x$ is

A) 1 B) 2 C) 3 D) 4

Key. D

Sol. Since $f\left(x\right)=x^2$

$\therefore$ $2x^2=e^x$

Functions

Integer Answer Type

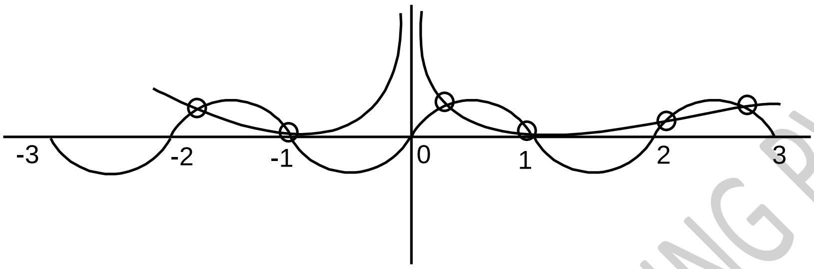
1. Number of distinct real solutions of $\sin \pi x = \left|\ln|x|\right| is$

Key. 6

Sol. No. of sol of $\sin \pi x = \left\|n\right|\left|x\right\|$

$y = \sin \pi x, y = \left\|n\right|\left|x\right\|$

No of sol = 6



No. of solutions 6

2. Let $f(x, y)$ be a periodic function satisfying the condition

$f(x, y) = f(2x+2y, 2y-2x) \forall x, y \in R$ and let $g(x)$ be a function defined as

$g(x) = f(2^x, 0)$. If T is period of $g(x)$ then find the value of T/4.

Key. 3

Sol. $f(x, y) = f(2x+2y, 2y-2x) = f(2(2x+2y)+2(2y-2x), 2(2y-2x)-2(2x+2y))$

$= f(8y, -8x)$

$= f(8(-8x), -8(8y)) = f(-64x, -64y)$

$= f(-64(-64x), -64(-64y))$

$= f(2^{12}x, 2^{12}y)$

$f(x, y) = f(2^{12}x, 2^{12}y)$

$f(x, 0) > f(2^{12}x, 0)$

$g(x) = f(2^x, 0) = f(2^{12}2^x, 0)$

$= f(2^{x+12}, 0) \quad = g(x+12)$

$\Rightarrow g(x) = g(x+12)$

$\Rightarrow$ period of $g(x)$ is $12 = T \quad \Rightarrow \frac{T}{4} = 3$

3. If $[\sin x]+\left[\frac{x}{2\pi}\right]+\left[\frac{2x}{5\pi}\right]=\frac{9x}{10\pi}$ when the number of solutions in the interval (30, 40) is (where [.] is GIF)

Key. 1

Sol. $[\sin x]=\frac{x}{2\pi}-\left[\frac{x}{2\pi}\right]+\frac{2x}{5\pi}-\left[\frac{2x}{5\pi}\right]$ No. of solutions = 1.

4. Let $f(x)$ be a real function not identically zero such that

$$f\left(x+y^{2n+1}\right)=f(x)+\{f(y)\}^{2n+1}$$

$n\in N$ and x, y are any real numbers and $f'(0)>0$, then the value of $f(5)$ is.

Key. 5

Sol. Given $f\left(x+y^{2n+1}\right)=f(x)+\{f(y)\}^{2n+1}$ - (1)

Diff w.r.t. 'x'

$$f'\left(x+y^{2n+1}\right)\left(1+(2n+1).y^{2n}.\frac{dy}{dx}\right)$$

$$= f'(x)+(2n+1).\{f(y)\}^{2n}.\frac{dy}{dx}$$

$\because$ x and y are independent real numbers

$$\frac{dy}{dx}=0$$

$$\therefore f'\left(x+y^{2n+1}\right)=f'(x) \quad - (1)$$

$\Rightarrow f(x)=x^{1/2n+1}$ and x = 0

(1) $f'(x)=f'(0)$ when $y=x^{1/2n+1}$ and $x=0$

integrating

$f(x)=f'(0)$ x + c putting x = 0, y = 1

$\therefore f(1)=0; c=0$ in the given equation

$\because f(x)=f'(0)x$

$\therefore f(x)=f'(0)x$

$f(1)=f'(0).1\Rightarrow f'(0)=1$

$\therefore f(x)=x$ Hence $f(5)=5$

5. Let $f(x)$ be a function such that $f(x-1)+f(x+1)=\sqrt{3}.f(x)\forall x\in R$. If $f(5)=10$. Then the value of $g'(x)$ where $g(x)=\sum_{r=0}^{99}f(5+12r)$ is.

Key. 0

Sol. $f(x-1)+f(x+1)=\sqrt{3}.f(x)$ - (1)

Putting $x=x+1$

$f(x)+f(x+2)=\sqrt{3}.f(x+1)$ - (2)

Again putting $x=x+1$

$f(x+1)+f(x+3)=\sqrt{3}.f(x+2)$ - (3)

+ (3) $\Rightarrow f(x-1)+f(x+3)=f(x+1)$ - (4)

Continning like this we get

$f(x)$ is a periodic function with period '12'

Then $g(x)=\sum_{r=0}^{99}f(5+12r)=100\times f(5)$

= 1000

$\Rightarrow g(x)=$ a constant function . Hence $g'(x)=0$

6. Let 'f' is a differentiable function such that $f'(x)=f(x)+\int_0^2 f(x)dx, f(0)=\frac{4-e^2}{3}$ then the value of $[f(2)]$ where [.] denotes the greatest integer $\le x$ is.

Key. 5

Sol. Given $f'(x)=f(x)+A$ where $A=\int_0^2 f(x)dx$

Solving – (1)

$f(x)=\lambda\left(e^x-1\right)+\frac{4-e^2}{3}$

Q $\int_0^2 f(x)dx=A\Rightarrow\lambda=1$ and $A=\frac{e^2-1}{3}$

$\therefore f(x)=e^x-1+\frac{4-e^2}{3}=e^x-\frac{1}{3}\left(e^2-1\right)$

$f(2)=\frac{2e^2+1}{3}$ $\therefore[f(2)]=5$

7. If 'f' is a polynomial function satisfying the condition

$f\left(\tan x\right)+f\left(\cot x\right)=f\left(\tan x\right).f\left(\cot x\right)$

$\forall x\in\left(\frac{-\pi}{2},\frac{\pi}{2}\right)-\{0\}$ and $f\left(2\right)=9$ then the value of $\frac{f'(2)}{6}$ is.

Key. 2

Sol. Putting $\tan x=t$, we get $f\left(t\right)$ is a polynomial function of the form, $f\left(t\right)=\pm t^{n}+1$

When t =2, $f\left(2\right)=9\Rightarrow n=3$ $\therefore f\left(t\right)=t^{3}+1$

$f'(2)=12$ $\therefore \frac{f'(2)}{6}=2$

8. f is a function such that $f(x)+2f(1-x)=x^2+1$. Value of $f(3)$ is

Key. 0

Sol. $f(x)+2f(1-x)=x^2+1$

$\therefore\ f(1-x)+2f(x)=(1-x)^2+1=x^2-2x+2$

Solving we get $3f(x)=x^2-4x+3$

$f(3)=0$

9. If n is a natural and $1\le n\le 100$, then the number of solutions of

$\left[\frac{n}{2}\right]+\left[\frac{n}{3}\right]+\left[\frac{n}{5}\right]=\frac{n}{2}+\frac{n}{3}+\frac{n}{5}$ (where [.] denotes the greatest integer function)

Key. 3

Sol. From the given equation, we have

$\left(\frac{n}{2}-\left[\frac{n}{2}\right]\right)+\left(\frac{n}{3}-\left[\frac{n}{3}\right]\right)+\left(\frac{n}{5}-\left[\frac{n}{5}\right]\right)=0$

$\Rightarrow\left\{\frac{n}{2}\right\}+\left\{\frac{n}{3}\right\}+\left\{\frac{n}{5}\right\}=0$; where {.} is a F.P.F.

But each of the fraction part functions is positive and their sum is zero. Hence each of the fraction part function is zero. Consequently, each of $\frac{n}{2},\frac{n}{3},\frac{n}{5}$ is an integer. The 1.c.m. of 2,3,5 is 30. Therefore we can take n = 30k where k is an integer.

Hence the number of solutions such that $1\le n\le 100$ is = 3 (viz. n = 30, 60 and 90)

10. The number of solutions of the equation $x^2+[x]-4x+3=0$ is ([.] $\rightarrow$ denotes the greatest integer function)

Key. 0

Sol. From the given equation, we have $x^2+(x-f)-4x+3=0$, where f = F. P. F. such that $0\le f<1$

$\Rightarrow\left(x^2-3x+3\right)-f=0\Rightarrow f=x^2-3x+3$.

But $0\le f<1$. Therefore $0\le x^2-3x+3<1$ ...(1)

Now, solving $x^2-3x+3=0$, we get $x=\frac{3\pm\sqrt{9-12}}{2}$ = imaginary

$\therefore \quad x^2-3x+3\geq 0$ for all $x\in R$.

Again from $x^2-3x+3<1$, we get $x^2-3x+2<0\Rightarrow(x-2)(x-1)<0$

$\Rightarrow \quad 1<x<2$.

Thus the inequality (1) is satisfied if 1 < x < 2 $\Rightarrow$ [x] = 1.

Putting {x} = 1 in the given equation, we get

$x^2+1-4x+3=0\Rightarrow x^2-4x+4=0\Rightarrow(x-2)^2=0\Rightarrow x=2$.

But x = 2 does not satisfy the inequality 1 < x < 2.

Hence no x is available to satisfy the equation. That is, the given equation has no solution.

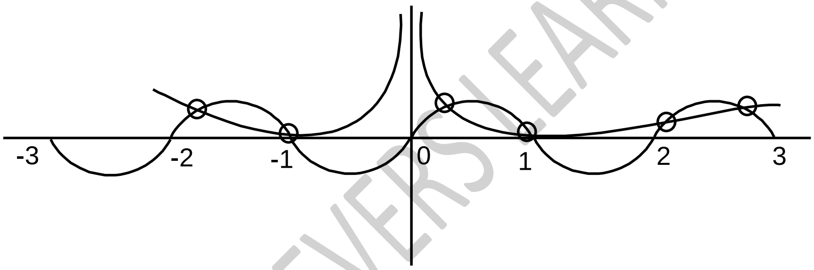
11. Number of distinct real solutions of $\sin\pi x=\left|\ln|x|\right| is$

Key. 6

Sol. No. of sol of $\sin\pi x=\left\|n\right||x\|$

$y=\sin\pi x, y=\left\|n\right||x\|$

No of sol = 6



No. of solutions 6

12. If f is a polynomial function satisfying $2+f(x).f(y)=\ f(x)+f(y)+f(xy)\forall x,y\in R$ and if $f(2)=5$ then the value of $f(f(1))$ is

Key. 5

Sol. $2+f(x).f(y)=f(x)+f(y)+f(xy)\forall x,y\in R.......(i)$ f $f(2)=5$

Put $y=\frac{1}{x}\Rightarrow 2+f(x).f\left(\frac{1}{x}\right)=f(x)+f\left(\frac{1}{x}\right)+f(1)....(ii)$

Put x = 1 $\Rightarrow 2+f(1).f(1)=f(1)+f(1)+f(1)$

$\Rightarrow [f(1)]^2-3.f(1)+2=0$

$\Rightarrow[f(1)-2][f(1)-1]=0$

As $f(2)=5\Rightarrow f(1)\neq 1\Rightarrow f(1)=2$

(ii) $\Rightarrow 2+f(x).f\left(\frac{1}{x}\right)=f(x)+f\left(\frac{1}{x}\right)+2$

$\Rightarrow f(x) = \pm x^n + 1$

$f(2) = 5 = 1 f(x) = x^2 + 1$

$f(f(1)) = f(2) = 5$

13. Let $f(x, y)$ be a periodic function satisfying the condition $f(x, y) = f(2x + 2y, 2y - 2x) \forall x, y \in R$ and let $g(x)$ be a function defined as $g(x) = f(2^x, 0)$. If T is period of $g(x)$ then the value of T/4.

Key. 3

Sol. $f(x, y) = f(2x + 2y, 2y - 2x) = f\,2(2x + 2y) + 2(2y + 2x), 2(2y - 2x) - 2(2x + 2y)$

$= f(8y, -8x)$

$= f(8(-8x), -8(8y)) = t(-64x, -64y)$

$= f(-64|-64x|, -64(-64y))$

$= f(2^{12}x, 2^{12}y)$

$f(x, y) = f(2^{12}x, 2^{12}y)$

$f(x, 0) > f(2^{12}x, 0)$

$g(x) = f(2^x, 0) = f(2^{12}2^x, 0)$

$= f(2^{x+12}, 0)$

$= g(x + 12)$

$\Rightarrow g(x) = g(x + 12)$

$\Rightarrow \text{period of g(x)} in 12 = T$

$\frac{T}{4} = 3$

14. Let $f\left(x + \frac{1}{y}\right) + f\left(x - \frac{1}{y}\right) = 2 f(x) f\left(\frac{1}{y}\right) \forall x, y \in R$, $y \neq 0$ and $f(0) = 0$ then the value of $f(1) + f(2) =$

Key. 0

Sol. Let $f\left(x + \frac{1}{y}\right) + f\left(x - \frac{1}{y}\right) .....(i) \; = 2 f(x) . f\left(\frac{1}{y}\right) \forall x, y \in R$

Given $f(0) = 0$

(i) put $x = 0, y = \frac{1}{x}$ we get

$f(x) + f(-x) = 2 f(0) f(x)$

$\Rightarrow f(x)+f(-x)=0 \Rightarrow f(x)=-f(-x)......(ii)$

(i) $\Rightarrow$ put $x=1, y=1$

$f(2)+f(0)=2[f(1)]^2$

$f(2)=2[f(1)]^2$ ......A)

(i) $\Rightarrow$ put $x=-1, y=-2$

$f(-2)=2f(-1)f(-1) \Rightarrow -f(2)=2(1)^2 ......B$

For A & B $f(2)=-f(2) \Rightarrow f(2)=0 \Rightarrow f(1)=0$

$\therefore f(1)=f(2)=0$

$\therefore f(1)+f(2)=0$

15. Let X = {1, 2, 3, ... 100} and Y be a subset of X such that the sum of no two elements in Y is divisible by 7. If the maximum possible number of element in Y is 40 + λ then λ is

Key: 5

Hint: Let Y_i be the subset of X such that $y_i = 7m + i$, $m \in I$

Y_0 = {7, 14, ... 98}, $n(Y_0) = 14$

Y_1 = {1, 8, 15 ...99}, $n(Y_1) = 15$

Y_2 = {2, 9, 16 ... , 100}, $n(Y_2) = 15$

Y_3 = {3, 10, 16 .... 94}, $nY_3) = 14$

Y_4 = {4, 11, 18 ... 95}, $n(Y_4) = 14$

Y_5 = {5, 12, ... 96}, $nY_5) = 14$

Y_6 = {6, 13, ... 97}, $n(Y_6) = 14$

The largest Y will consist of (!) an element of Y_0 (ii) Y_1 (iii) Y_2 (iv) Y_3 or Y_4

$\Rightarrow$ The maximum possible number of elements in Y = 1 + 15 + 15 + 14 = 45.

16. Let 'f' is a differentiable function such that

$f'(x)=f(x)+\int_0^2 f(x)dx, f(0)=\frac{4-e^2}{3}$ then the value of $[f(2)]$ where [.] denotes the greatest integer $\leq x$ is.

Key. 5

Sol.Given $f'(x)=f(x)+A$ where $A=\int_0^2 f(x)dx$

Solving – (1)

$f(x)=\lambda\left(e^x-1\right)+\frac{4-e^2}{3}$

$\because \int_0^2 f(x)dx = A \Rightarrow \lambda=1$ and $A=\frac{e^2-1}{3}$

$$\therefore f(x) = e^x - 1 + \frac{4-e^2}{3} = e^x - \frac{1}{3}\left(e^2 - 1\right)$$

$$f(2) = \frac{2e^2+1}{3} \qquad \therefore \left[f(2)\right] = 5$$

17. For non negative integers m, n define a function as follows

$$f(m,n) = \begin{cases} n+1 & \text{if} \quad m = 0 \\ f(m-1,1) & \text{if} \quad m \neq 0, n = 0 \\ f(m-1, f(m, n-1)) & \text{if} \quad m \neq 0, n \neq 0 \end{cases}$$

Then the value of $f(1,1)$ is

Key. 3

Sol. $f(1,1) = f(0, f(1,0)) = f(0, f(0,1)) = f(0,2) = 3$

18. If function f satisfies the relation f(x). f$'$ (–x)=f(–x). f$'$(x) for all x and f(0) = 3, now if f(3) = 3, then the value of f(–3) is

Key. 3

Sol. $f(x) \times f'(-x) = f(-x) \times f'(x)$

$\Rightarrow f'(x) \times f(-x) - f(x) \times f'(-x) = 0$

$\Rightarrow \frac{d}{dx}\left[f(x) f(-x)\right] = 0$

$\Rightarrow f(x) f(-x) = k$

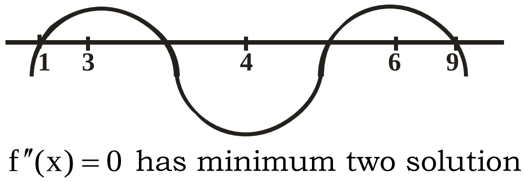
Give $\left(f(0)\right)^2 = k = 9 \qquad \Rightarrow k = 9$

Then $f(3) f(-3) = 9 \qquad \Rightarrow f(-3) = 3$

19. If f(x) is twice differentiable function such that f(1) = 0, f(3) = 2, f(4) = –5, f(6) = 2, f(9) = 0 then the minimum number of zero's of $g'(x) = x^2 f''(x) + 2xf'(x) + f''(x)$ in the interval (1,9) is

Key. 2

Sol. $f'(x) = 0$ has minimum three solution between (1,9)



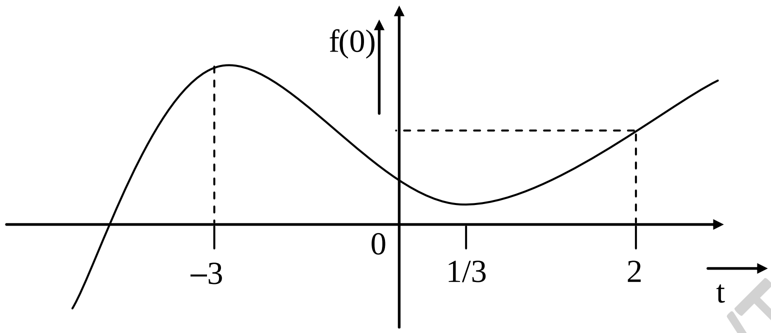
$f''(x) = 0$ has minimum two solution between (1,9)

Given equations $\frac{d}{dx}\left\{(x^2+1) f'(x)\right\} = 0$

20. The greatest value of $f(x)=\left(3-\sqrt{4-x^2}\right)^2+\left(1+\sqrt{4-x^2}\right)^3$ is A, then A/7 =

Key. 4

Sol. $f(x)=\left(3-\sqrt{4-x^2}\right)^2+\left(1+\sqrt{4-x^2}\right)^3$



Let t = $\sqrt{4-x^2}$, clearly $0\le t\le 2$

$\Rightarrow$ F(t) = $(3-t)^2+(1+t)^3$

For maxima and minima,

$\Rightarrow$ $F'(t)=0$

$\Rightarrow$ $-2(3-t)+3(1+t)^2=0$

$\Rightarrow$ $3t^2+8t-3=0$

$\Rightarrow$ $(3t-1)(t+3)=0$

$\Rightarrow$ t = $-3,\frac{1}{3}$

Also $F''(t)|_{t=-3}=-10$ $\Rightarrow$ maxima

And $F''(t)|_{t=1/3}=+10$ $\Rightarrow$ minima

As $t\ne -3$, hence maximum value of F(t) will occur at the end points for which

F(0) = 10, F(2) = 28

Hence, maximum value of F(t) = 28 for t = 2

$\Rightarrow$ maximum value of f(x) = 28 for x = 0.

21. If α is an integer satisfying $|\alpha|\le 5-|[x]|$, where x is a real number for which $2x\tan^{-1}x$ is greater than or equal to ln $(1+x^2)$, then the number of maximum non-negative possible values of α is
(where [.] denotes the greater integer function)

Key. 6

Sol. Let y = $2x\tan^{-1}x-\ln\left(1+x^2\right)$

$$y'=2\tan^{-1}x+\frac{2x}{1+x^2}-\frac{2x}{1+x^2}$$

$\Rightarrow$ $y'>0\,\forall x\in R^+, y'<0\,\forall x\in R^-$ $\Rightarrow y\ge 0,\ \forall x\in R$

$\therefore$ $5-|[x]|$ takes the values 0, 1, 2, 3, 4, 5. $\{\because |\alpha|\le 5-|[x]|\}$

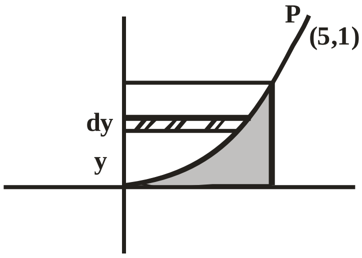
$\therefore$ $|\alpha|\le 5-|[x]|$ is satisfied by α = 0, ±1, ±2, ±3, ±4, ±5

22. Let y = g(x) be the inverse of a bijective mapping f : R → R defined as f (x) = $3x^3 + 2x$. The area bounded by the graph of g(x), the x-axis and the coordinate at x = 5 is 'A' then the value of (4A −7) is

Key. 6

Sol. Inverse of $y = 3x^3 + 2x$ is $x = 3y^3 + 2y$

Required area



$$A = \int_0^1 5 - (3y^3 + 2y)dy = \left[5y - \left(\frac{3y^4}{4} + y^2\right)\right]_0^1$$

$$= 5 - \left(\frac{3}{4} + 1\right) = \frac{20-7}{4} = \frac{13}{4}$$

$4A = 13$

$\therefore\ 4A - 7 = 6$

23. Let X = {1, 2, 3, ... 100} and Y be a subset of X such that the sum of no two elements in Y is divisible by 7. If the maximum possible number of element in Y is 40 + λ then λ is

Ans: 5

Hint Let Y_i be the subset of X such that $y_i = 7m + i$, m∈I

Y_0 = {7, 14, ... 98}, $n(Y_0) = 14$

Y_1 = {1, 8, 15 ...99}, $n(Y_1) = 15$

Y_2 = {2, 9, 16 ... , 100}, $n(Y_2) = 15$

Y_3 = {3, 10, 16 .... 94}, $nY_3) = 14$

Y_4 = {4, 11, 18 ... 95}, $n(Y_4) = 14$

Y_5 = {5, 12, ... 96}, $nY_5) = 14$

Y_6 = {6, 13, ... 97}, $n(Y_6) = 14$

The largest Y will consist of (!) an element of Y_0 (ii) Y_1 (iii) Y_2 (iv) Y_3 or Y_4

⇒ The maximum possible number of elements in Y = 1 + 15 + 15 + 14 = 45.

24. Number of real values of x, satisfying the equation $[x]^2 - 5[x] + 6 - \sin x = 0\,[.]$ denoting the greatest integer function is

Key: 1

Hint: $[x] = \dfrac{5 \pm \sqrt{25 + 4\sin x - 24}}{2.1}$

$$= \frac{5 \pm \sqrt{1 + 4\sin x}}{2}$$

$-1 \le \sin x \le 1$

$-4 \le 4\sin x \le 4$

$-3 \le 1 + 4\sin \le 5$

$0 \le 1 + 4\sin x \le 5$

$\Rightarrow [x]$ is an integer $\Leftrightarrow \sin x = 0$

$\Rightarrow [x] = 3$

$\Rightarrow x = \pi$

25. If f(x + 2) – 5(x + 1) + 6(x) = 0 for all , find f(x).

Sol. Let f(x) = e^{mx} be a trial solution.

Then $f(x+2) = e^{m(x+2)} = e^{mx}.e^{2m}, f(x+1) = e^{mx}.e^{m}$.

Therefore, f(x + 2) – 5f(x + 1) + 6f(x) = 0

$\Rightarrow \quad e^{mx}.e^{2m} - 5e^{mx}.e^{m} + 6e^{mx} = 0 \Rightarrow e^{mx}\left[e^{2m} - 5e^{m} + 6\right] = 0$

$\Rightarrow \quad e^{2m} - 5e^{m} + 6 = 0;\ e^{mx} \ne 0$

Let $e^m = u$. Then from the above equation,

$u^2 - 5u + 6 = 0 \Rightarrow$ (u – 2)(u – 3) = 0 u = 2 or u = 3.

$\therefore \quad e^m = 2$ or $e^m = 3$

$\Rightarrow \quad e^{mx} = 2^x$ or $e^{mx} = 3^x$

Hence $f(x) = A.2^x + B.3^x$.

26. If the function f(x) satisfies the equation f(x + 1) + f(x – 1) = $\sqrt{3}$ f(x) for all $x \in R$,

(i) show that f(x) is a periodic function and find its period.

(ii) If f(2) = 9, find the sum $\sum_{r=0}^{9} f(2+12r)$.

Sol. We have, f(x + 1 + f(x – 1) = $\sqrt{3}$ f(x) for all ...(1)

Replacing x by x – 1 and x + 1 successively in (1), we get f(x) + f(x – 2) = $\sqrt{3}$ f(x – 1) ...(2)

and f(x + 2) + f(x) = $\sqrt{3}$ f(x + 1) ...(3)

Adding (2) and (3), we get 2f(x) + f(x – 2) + f(x + 2) = $\sqrt{3}$ {f(x – 1) f(x + 1)}

= $\sqrt{3}$. $\sqrt{3}$ f(x); from (1)

= 3f(x)

$\Rightarrow$ f(x + 2) + f(x – 2) = f(x) ...(4)

Replacing x by x + 2 in (4), we get f(x + 4) + f(x) = f(x + 2) ...(5)

Adding (4) and (5), we get f(x + 4) + f(x – 2) = 0 ...(6)

Again replacing x by x + 6 in (6), we get f(x + 10) + f(x + 4) = 0 ...(7)

(6) – (7) $\Rightarrow$ f(x + 10) – f(x – 2) = 0

f(x + 10) = f(x – 2). ...(8)

Lastly, replacing x by x + 2 in (8), we get f(x + 12) = f(x).

Hence f(x) is a periodic function with period 12. This proves part (i).

(ii) $\sum_{r=0}^{9}$ f(2 + 12r) = f(2) + f(2 + 12) + f(2 + 24) + ... + f(2 + 12 × 9)

= f(2) + f(2) + f(2) + ... to 10 terms, since f(x) is periodic with period 12.

$= 10 \times f(2) = 10 \times 9 = 90.$

27. Let f be a function from the set of positive integers to the set of real numbers i.e. $f : N \to R$ such that (i) f(1) = 1(ii) f(1) + 2f(2) + 3f(3) + ... + nf(n) = n(n + 1) f(n); for $n \geq 2$.
Find the value of f(200).

Sol. Given f(1) + 2f(2) + 3f(3) + ... + nf(n) = n(n + 1) f(n) ...(1)
Replacing n by (n + 1), we get
f(1) + 2f(2) + 3f(3) + ... + nf(n) + (n + 1)f(n + 1) = (n + 1)(n + 2)f(n + 1) ...(2)
Subtracting (1) from (2), we get

$(n + 1) f(n + 1) = (n + 1)(n + 2) f(n + 1) – n(n + 1) f(n)$

$\Rightarrow \quad f(n + 1) = (n + 2)f(n + 1) – nf(n)$

$\Rightarrow \quad nf(n) = (n + 2)f(n + 1) – f(n + 1) = (n + 2 – 1) f(n + 1) = (n + 1)f(n + 1).$

Thus, we have 2f(2) = 3f(3) = ... = nf(n).
Hence from (1), f(1) + (n – 1) nf(n) = n(n + 1)f(n)

$\Rightarrow \quad f(n) \{n(n + 1) – n(n – 1)\} = f(1) = 1$

$\Rightarrow \quad \left(n^2 + n - n^2 + n\right) f(n) = 1 \Rightarrow 2nf(n) = 1$

$\Rightarrow \quad f(n) = \frac{1}{2n} \quad \therefore f(200) = \frac{1}{2 \times 200} = \frac{1}{400}$

28. If for all real values of u and v, 2f(u) cos v = f(u – v), prove that for all real values of x,
(i) f(x) + f(– x) = 2a cos x (ii) $f(\pi - x) + f(-x) = 0$ (iii) $f(\pi - x) + f(x) = 2b \sin x$
Deduce that f(x) = a cos x + b sin x where a and b are arbitrary constants.

Sol. Given f(u + v) + f(u – v) = 2f(u) cos v ...(1)
Putting u = 0 and v = x in (1), we get f(x) + f(– x) = 2f(0) cos x ...(2)
= 2a cos x where a = f(0).
This proves (i).

Again putting $u = \frac{\pi}{2} - x$ and $v = \frac{\pi}{2}$ in (1), we get

$f(\pi - x) + f(-x) = 0;\ Q \cos\frac{\pi}{2} = 0$.

This proves (ii).

Again, putting $u = \frac{\pi}{2} - x$ and $\upsilon = \frac{\pi}{2} - x$ in (1), we get

$$f(x - x) + f(x) = 2f\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2} - x\right) = 2b \sin x \quad ...(3)$$

This proves (iii),
Now, adding (2) and (3), we get 2f(x) + f(– x) + f(–x) = 2a cos x + 2b sin x

$\Rightarrow \quad 2f(x) = 0 = 2a \cos x + 2b \sin x$; from (3)

Hence f(x) = a cos x + b sin x.

29. Let $f(x+y+1) = \left(\sqrt{f(x)}+\sqrt{f(x)}\right)^2$ for all $x, y \in R$ and f(0) = 1. Find f(x).

Sol. Given $f(x+y+1) = \left(\sqrt{f(x)}+\sqrt{f(x)}\right)^2$

Putting y = 0, we get $f(x+1) = \left(\sqrt{f(x)}+1\right)^2$;

Putting x = 0, we have $f(1) = (1+1)^2 = 2^2$;

Putting x = 1, $f(2) = (2+1)^2 = 3^2$;

Putting x = 2, $f(3) = \left(\sqrt{f(2)}+1\right)^2 = (3+1)^2 = 4^2$ and so on.

Proceeding in this way, we get $f(x) = (x+1)^2$.

30. Find the natural number c for which $\sum_{r=1}^{n} f(c+r) = 16(2^n - 1)$ where the function satisfies the relation f(x + y) = f(x) .f(y) for all $x, y \in N$ (natural numbers) and f(1) = 2.

Sol. From the given functional equation f(x + y) = f(x) . f(y), we have $f(x) = e^{kx}$ ...(1)
Where k is a constant.

Putting x = 1, $f(1) = e^k \Rightarrow 2 = e^k$. Hence from (1), $f(x) = 2^x$.

This can also be obtained by putting y = 1 so that

$$F(x+1) = f(x).\ f(1) = 2f(x) \Rightarrow f(x) = 2f(x-1).$$

Putting successively x – 1, x – 2, x – 3, ... 2 for x in the above and multiplying them, we get $f(x) = 2^x$.

Now, $\sum_{r=1}^{n} f(c+r)$ = f(c + 1) + f(c + 2) + ... + f(c + n)

$$= 2^{c+1} + 2^{c+2} + \ldots + 2^{c+n} = 2^c.2 + 2^c.2^2 + \ldots + 2^c.2^n$$

$$= 2^c.\{1 + 2 + 2^2 + \ldots \text{ to n terms}\}$$

$$= 2^c.2.\frac{1(2^n-1)}{2-1} = 2^c.2(2^n-1)$$

$$\Rightarrow \quad 16(2^n-1) = 2^{c+1}(2^n-1) \Rightarrow 2^{c+1} = 16 = 2^4 \Rightarrow c+1 = 4 \therefore c = 3.$$

31. If the function f satisfies the relation f(x + y) + f(x – y) = 2f(x). f(y) for all $x, y \in R$ and f(0) ≠ 0, prove that f(x) is an even function.

Sol. Given f(x + y) + f(x – y) = 2f(x). f(y) ...(1)
Replacing x by y and y by x in (1), we have

$$F(y+x) + f(y-x) = 2f(y).\ f(x) \Rightarrow f(x+y) + f(y-x) = 2f(x).\ f(y) \quad ...(2)$$

From (1) and (2), we have f(x – y) = f(y – x).
Now, putting y = 2x we get f(- x) = f(x).
Hence f(x) is an even function.

32. If $f\left(\frac{x^2+y^2}{2}\right) = \frac{f\{f(x)\}^2 + \{f(y)\}^2}{2}$ for all $x, y \in R$, find f(x).

Sol. Differentiating both sides partially with respect to x, (keepiring y as constant), we get

$$f'\left(\frac{x^2+y^2}{2}\right)\times\frac{1}{2}(2x)=\frac{1}{2}\left[2f(x)f'(x)+0\right]\Rightarrow xf'\left(\frac{x^2+y^2}{2}\right)=f(x)f'(x) \quad ...(1)$$

Similarly differentiating both sides partially with respect to y (keeping x as constant) or interchanging x and y, we get

$$yf'\left(\frac{x^2+y^2}{2}\right)=f(y)f'(y)\text{b} \quad ...(2)$$

Now, dividing (2) by (1), we have $\frac{x}{y}=\frac{f(x)f'(x)}{f(y)f'(y)}$

$\Rightarrow \quad \frac{f(x)f'(x)}{x}=\frac{f(y)f'(y)}{y}=k\left(=f(1)f'(1)\right)$ say

$\Rightarrow \quad f(x)f'(x)=kx\Rightarrow\int f(x)f'(x)dx=k\int xdx$

$\Rightarrow \quad \frac{\{f(x)\}^2}{2}=k\frac{x^2}{2}+c\Rightarrow\{f(x)\}^2=kx^2+2c$.

When x = 0, y = 0, f(0) = 0 or 1.

$\Rightarrow \quad \{f(0)\}^2=0+2c\Rightarrow$ Either 0 = 2c or 1 = 2c.

$\therefore \quad c=0 \text{ or } c=\frac{1}{2}$.

Hence $f(x)=\pm kx^2 \text{ or } \pm\sqrt{kx^2+1}$.

33. Let $f:N\to N$ where f(x) = x + $(-1)^{x-1}$. Find the inverse of f.

Sol. We have, f(x) = x + $(-1)^{x-1}$.

Putting successively x = 1, 2, 3, … , we get

f(1) = 1 + 1 = 2; f(2) = 2 + (-1) = 1; f(3) = 3 + (- 1)^2 = 3 + 1 = 4; f(4) – 1 = 3

f(4) = 5 + 1 = 6; f(5) = 6 – 1 = 5; …..

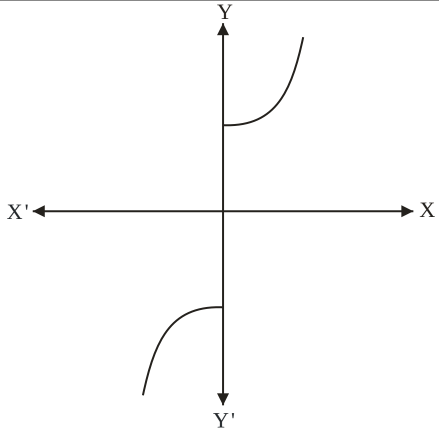
From above, we see that the graph of f(x) consists of the points (1, 2), (2, 1), (3, 4), (4, 3), (5, 6), (6, 5),… .

Thus if (a, b) is a point on the graph, then (b, a) is also a point on the graph.

Hence the inverse of is f itself i.e. $f^{-1}(x)=x+(-1)^{x-1};\ x\in N$.

34. We have, $y=(1+x^2)\text{sgn}(x)=\begin{cases}1+x^2 & ;x>0\\ 0 & ;x=0\\ -(1+x^2) & ;x<0.\end{cases}$

The graph of the function is shown as below:



From the graph, we see that f is one-one and hence invertible.

Now, for the interval x > 0, we have $y = 1 + x^2 \Rightarrow x^2 = y - 1 \Rightarrow x = \sqrt{y-1}; y > 1$.

Hence its inverse is $y = \sqrt{x-1}; x > 1$.

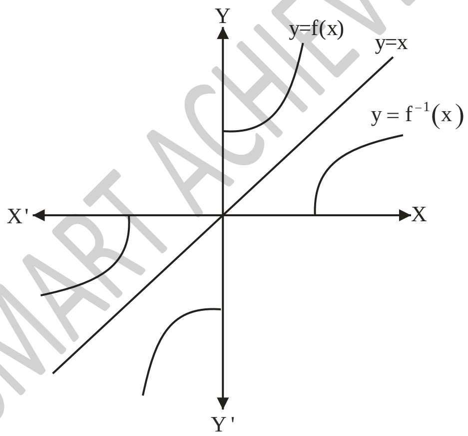
For the interval x < 0 we have $y = -\left(1 + x^2\right); y < -1 \Rightarrow x^2 + 1 = -y \Rightarrow x^2 = -y - 1$

$\Rightarrow \qquad x = \sqrt{-(y+1)};\ y < -1.$

Hence its inverse is $y = -\sqrt{-(x+1)}; x < -1$.

Thus the inverse of the given function is $y = \begin{cases} \sqrt{x-1} & ; x > 1 \\ 0 & ; x = 0 \\ -\sqrt{(x+1)} & ; x < -1. \end{cases}$

The graph of the inverse of f(x) is the mirror image of th given function in the line y = x as shown below:



35. Let $f(x) = ax^3 + bx^2 + cx + d$ sin x. Find the condition that f(x) is always one-one function.

Sol. Given: $f(x) = ax^3 + bx^2 + cx + d\sin x$

For f(x) to be one-one, f should be strictly monotonic i.e. either f is strictly increasing or strictly decreasing.

Differentiating (1), we get $f'(x) = 3ax^2 + 2bx + c + d\cos x$

$\geq 3ax^2 + 2bx + c - |d|$; for |cos x| $\leq$ 1.

If f(x) > 0 for all $x \in R$, we must have a > 0 and $4b^2 - 4.3a(c-|d|) < 0$

(from the quadratic expression)

$\Rightarrow \quad b^2 < 3a(c-|d|)$.

$\therefore f(x)$ is strictly increasing if a > 0 and $b^2 < 3a(c - |d|)$.

Also. $f'(x) = 3ax^2 + 2bx + c + d\cos x$

$\leq 3ax^2 + 2bx + c + |d|$

If $f'(x) < 0$ for all $x \in R$, a < 0 and $4b^2 - 12a(c+|d|) < 0 \Rightarrow b^2 < 3a(c+|d|)$.

Hence the required condition is (i) a > 0, $b^2 < 3a(c-|d|)$ (ii) a < 0, $b^2 < 3a(c-|d|)$.

36. Let $f:[0,1] \to [0,1]$ defined by $f(x) = \dfrac{1-x}{1+x}$, for $0 \le x \le 1$ and let $g:[0,1] \to [0,1]$ defined by $g(x) = 4x(1-x), 0 \le x \le 1$. If range of $fog(x)$ is $[\alpha.\beta]$, then $\alpha + \beta =$

Key. 1

Sol. $fog(x) = f(g(x)) = f(4x(1-x))$

$\Rightarrow \dfrac{1-4x(1-x)}{1+4x(1-x)}$ when $0 \le 4x(1-x) \le 1$ and $0 \le x \le 1$

But $4x - 4x^2 \ge 0 \Rightarrow 0 \le x \le 1$

$4x - 4x^2 \le 1 \Rightarrow (2x-1)^2 \ge 0 \Rightarrow x \in R$

Hence $fog(x) = \dfrac{1-4x+4x^2}{1+4x-4x^2}, 0 \le x \le 1$

Let $y = \dfrac{4x^2 - 4x + 1}{-(4x^2 - 4x) + 1}, \ 0 \le x \le 1$

put $0 \le x \le 1 \ t \in [-1, 0]$

$y = \dfrac{1+t}{1-t}, \dfrac{dy}{dt} = \dfrac{1-t+1+t}{(1-t)^2} > 0$

Range of $fog(x) = [0,1]$

$\Rightarrow \alpha + \beta = 1$

37. If the function $f(x) = \dfrac{x-1}{c - x^2 + 1}$ does not take any value in the interval $\left[-1, -\dfrac{1}{3}\right]$, then the largest integral value that c can attain is equal to

Key. 0

Sol. Let $y = f(x) = \dfrac{x-1}{c - x^2 + 1}$

Take $y = -t$, where $t \in \left[\dfrac{1}{3}, 1\right]$

$\therefore\ -t = \dfrac{x-1}{c-x^2+1}$

$\Rightarrow\ x^2 - c - 1 = \dfrac{x-1}{t} \Rightarrow x^2 - \dfrac{1}{t}x + \dfrac{1}{t} - c - 1 = 0$

As $-t \in \left[-1, -\dfrac{1}{3}\right]$, hence the above must not possess real solution

$\therefore\ \left(\dfrac{1}{t}\right)^2 - 4\left(\dfrac{1}{t} - c - 1\right) < 0 \Rightarrow \dfrac{1}{t^2} - \dfrac{4}{t} + 4 \le -4c$

$\Rightarrow c < -\dfrac{1}{4}\left(\dfrac{1}{t} - 2\right)^2$

Now, $\dfrac{1}{3} \le t \le l \Rightarrow 1 \le \dfrac{1}{t} - 2 \le 1 \Rightarrow -\dfrac{1}{4} \le \dfrac{1}{4}\left(\dfrac{1}{t} - 2\right)^2 \le 0$

Hence, $c \in \left(-\infty, -\dfrac{1}{4}\right)$

38. If $f(x) = \cos[\pi^2]x + \cos[-\pi^2]x$ where [x] stands for greatest integer function, then $f(2\pi) - f\left(\dfrac{\pi}{2}\right) =$

Key. 3

Sol. $f(x) = \cos 9x + \cos 10x$

$f(2\pi) - f\left(\dfrac{\pi}{2}\right) = 1 + 1 - 0 + 1 = 3$

39. If $f(x+y, x-y) = xy$, then the arithmetic mean of $f(x,y) and\ f(y,-x)$ is _______

Key. 0

Sol. $x + y = a, x - y = b \Rightarrow x = \dfrac{a+b}{2}, y = \dfrac{a-b}{2}$

$f(a,b) = \left(\dfrac{a+b}{2}\right)\left(\dfrac{a-b}{2}\right) = \dfrac{a^2 - b^2}{2}$

$A.M = \dfrac{f(x.y) + f(y,-x)}{2} = \dfrac{x^2 - y^2 + y^2 - x^2}{4} = 0$

40. If $f(x) = 1 + \alpha x, \alpha \ne 0$ is the inverse of itself then $[|\alpha|]$=
([.] denotes greatest integer function)

Key. 1

Sol. $1 + \alpha x = \dfrac{x-1}{\alpha} \Rightarrow \alpha = -1$

41. $f: R \to R$ is given by $f(x) = \dfrac{a^x}{a^x + \sqrt{a}} \;\forall\; x \in R,$ then

$$f\left(\frac{1}{11}\right) + f\left(\frac{2}{11}\right) + ... + f\left(\frac{9}{11}\right) + f\left(\frac{10}{11}\right) =$$

Key. 5

Sol. $f(1-x) = 1 - f(x) \Rightarrow f(x) + f(1-x) = 1$

$$\Rightarrow f\left(\frac{1}{11}\right) + f\left(\frac{10}{11}\right) = 1$$

42. If the function $f(x) = \begin{cases} x+1 & \text{if} \quad x \le 1 \\ 2x+1 & \text{if} \quad 1 < x \le 2 \end{cases}$

and $g(x) = \begin{cases} x^2 & -1 \le x \le 2 \\ x+2 & 2 \le x \le 3 \end{cases}$ then the number

of roots of the equation $f(g(x)) = 2$ is

Key. 2

Sol. $f(g(x)) = \begin{cases} x^2+1 & -1 \le x \le 1 \\ 2x^2+1 & 1 < x \le \sqrt{2} \end{cases}$

$f(g(x)) = 2 \Rightarrow x^2 + 1 = 2$ & $2x^2 + 1 = 2$

$x = \pm 1$ $\quad x = \pm \dfrac{1}{\sqrt{2}}$

No. of sol = 2

43. The number of points (p,q) such that $p, q \in \{1,2,3,4\}$ and the equation $px^2 + qx + 1 = 0$ has real roots is

Key. 7

Sol. $px^2 + qx + 1 = 0$ has real roots $\Rightarrow q^2 - 4p \ge 0 \Rightarrow 4p \le q^2$

$\Rightarrow (1,2)(1,3)(1,4)(2,3)(2,4)(3,4)(4,4)$

$\therefore$ number of points = 7

44. Solve the following equation for x (where [x] & {x} denotes integral and fractional part of x)
|x – 1| = 2 [x] – 3{x}

Ans. $x = 3/2, \dfrac{11}{4}$

Sol. $\left\{\dfrac{3}{2}, \dfrac{11}{4}\right\}$

$|x-1| = 2[x] - 3\{x\}$

(i) if $x \ge 1$, then x – 1 = 2 [x] - 3 {x}

[x] + {x} – 1 = 2[x] – 3{x}

4{x} = 1 + [x]

∴ $0 \le 1 + [x] < 4$

∴ $-1 \le [x] < 3$

∴ possible value of [x] are 1 and 2

if [x] = 1, then $\{x\} = \frac{1}{2}$ ∴ $x = 3/2$

if [x] = 2, then {x} = 3/4 ∴ $x = \frac{11}{4}$

(ii) x < 1, then the equation becomes
1 – x = 2 [x] – 3 {x}
1 – [x] – {x} = 2[x] – 3{x}
2 {x} = 3[x] – 1

∴ $0 \le 3[x] - 1 < 2$

i.e. $1 \le 3[x] < 3$

i.e. $\frac{1}{3} \le [x] < 1$ which is not possible

∴ $x = 3/2, \frac{11}{4}$ are the only solutions

45. Find the period of the following functions

(i) $f(x) = \tan\frac{\pi}{2}[x]$, where [.] denotes greatest integer function

(ii) $f(x) = \frac{\sin x + \sin 3x}{\cos x + \cos 3x}$

Ans. π

Sol. (i) $f(x) = \tan\frac{\pi}{2}[x]$ $f(x+1) = \tan\frac{\pi}{2}[x+1] = -\cot\frac{\pi}{2}[x]$

$f(x+2) = \tan\frac{\pi}{2}[x+2] = \tan\frac{\pi}{2}(2+[x])$

$= \tan\left(\pi + \frac{\pi}{2}[x]\right) = \tan\frac{\pi}{2}[x] = f(x)$

∴ 2 is period of f(x)
1 is not a period of f(x)

∴ 2 is the fundamental period

(ii) $f(x) = \frac{\sin x + \sin 3x}{\cos x + \cos 3x} = \frac{2\sin 2x \cos x}{2\cos 2x.\cos x} = \tan 2x$ where $x \ne (2n+1)\frac{\pi}{4}, (2n+1)\frac{\pi}{2}$

domain is

$f\left(x + \frac{\pi}{2}\right) = \tan 2\left(x + \frac{\pi}{2}\right) = \tan(2x+\pi) = \tan 2x$ it seems that $\frac{\pi}{2}$ is a period but it is not because f(0) is defined where as $f\left(\frac{\pi}{2}\right)$ is not defined.

∴ Period is π

46. Find domain and range of the following:

(i) $f(x) = \cos^{-1}\sqrt{\log_{[x]}\frac{[x]}{x}}$, where [.] denotes the greatest integer function

(ii) $f(x) = \log_{[x-1]}\sin x$, where [ ] denotes greatest integer function.

Ans. $D: [3,\pi) \cup \bigcup_{n=1}^{\infty}(2n\pi, 2n\pi+\pi)$; $R: (-\infty, 0]$

Sol. (i) for domain (i) [x] > 0 and [x] $\neq$ 1 so $[x] \geq 2$, so $x \in [2, \infty)$

for range if $x \in [2, \infty)$, then $\frac{|x|}{x} = 1$ so $f(x) = \cos^{-1} 0 = \frac{\pi}{2}$

Range of $f(x) = \frac{\pi}{2}$ Ans. $D : [2, \infty)$; $R : \{\pi/2\}$ c

(ii) $f(x) = \log_{[x-1]} \sin x$

$\sin x > 0$ $\Rightarrow$ $x \in (2n\pi, (2n+1)\pi)$

here [x – 1] > 0 & [x – 1] $\neq$ 1 $\Rightarrow$ $x \in [3, \infty]$

Domain $x \in [3, \pi] \cup \left[\bigcup_{n=1}^{\infty} (2n\pi, (2n+1)\pi\right)$.

For range $\sin x \in (0, 1]$ and $[x-1] \in [2, \infty)$ so range $\in (-\infty, 0]$

{Ans. : $D : [3, \pi) \cup \bigcup_{n=1}^{\infty} (2n\pi, 2n\pi + \pi)$; $R : (-\infty, 0]$ }

47. If f(x) + f(y) + f(xy) = 2 + f(x). f(y), for all real values of x & y and f(x) is a polynomial function with f(4) = 17, then find the value of f(5)

Sol. Let x = y = 1

F(x) + f(y) + f(xy) = 2 + f(x). f(y)

$3f(1) = 2 + (f(1))^2$ $\Rightarrow$ f(1) = 1, 2. But given that

f(1) $\neq$ 1 so f(1) = 2

Now put $y = \frac{1}{x}$

$$f(x) + f\left(\frac{1}{x}\right) + f(1) = 2 + f(x).\, f\left(\frac{1}{x}\right)$$

so f(x) = $\pm$ xn + 1

Now f(4) = 17 $\Rightarrow$ $\pm(4)^n + 1 = 17 \Rightarrow n = 2$

$f(x) = +(x)^2 + 1$.

48. Determine a function f satisfying the functional relation $f(x) + f\left(\frac{1}{1-x}\right) = \frac{2(1-2x)}{x(1-x)}$.

Ans. $f(x) = \frac{x+1}{x-1}$

Sol. $f(x) + f\left(\frac{1}{1-x}\right) = \frac{2}{x} - \frac{2}{1-x}$

Put $x = \frac{1}{1-t}$ $1 - x = 1 - \frac{1}{1-t} = -\frac{1}{1-t}$

$\therefore$ $f\left(\frac{1}{1-t}\right) + f\left(\frac{t-1}{t}\right) = 2(1-t) - \frac{2(1-t)}{-t}$

$\therefore$ $f\left(\frac{1}{1-x}\right) + f\left(\frac{x-1}{x}\right) = 2(1-x) + 2\left(\frac{1-x}{x}\right)$

Put $\frac{x-1}{x} = t$

x – 1 = xt $x = \frac{1}{1-t}$

$$1-x = 1-\frac{1}{1-t} = \frac{-t}{1-t}$$

$$\frac{1}{1-x} = \frac{t-1}{t}$$

$\therefore \quad f\left(\frac{t-1}{x}\right) + f(t) = \frac{2t}{t-1} + \left(\frac{2t}{t-1}\right)\left(\frac{1-t}{1}\right)$

$\therefore \quad f\left(\frac{x-1}{x}\right) + f(x) = \frac{2x}{x-1} - 2x$

$\therefore \quad 2f(x) = \frac{2}{x} - \frac{2}{1-x} - 2(1-x) - \frac{2(1-x)}{x} + \frac{2x}{x-1} - 2x$

$$= \frac{2x}{x-1} + \frac{2}{x-1} 2\left(\frac{x+1}{x-1}\right)$$

$\therefore \quad f(x) = \frac{x+1}{x-1}$

49. Find the integral solutions to the equation [x] [y] = x + y. Show that all the non-integral solutions lie on exactly two lines. Determine these lines. Here[.] denotes greatest integer function.

Ans. Integral solution (0, 0); (2, 2). X + y = 6, x + y = 0

Sol. [x] [y] = x + y

let $x = I_1 + f_1$

$y = I_2 + f_2$

then $I_1 I_2 = I_1 + I_2 + f_1 + f_2$

(i) if x, y $\in$ I

then

xy = x + y

or $y = \frac{x}{x-1}$

$\Rightarrow$ (x, y) is (0, 0), (2, 2)

(ii) if x, y $\notin$ I

$x = I_1 + f_1$

$y = I_2 + f_2$

then $I_1 + I_2 + f_1 + f_2 = I_1 I_2$

$\Rightarrow$ $f_1 + f_2 \in I$

$0 < f_1 + f_2 < 2 \quad \Rightarrow \quad f_1 + f_2 = 1$

$I_1 + I_2 + 1 = I_1 I_2$

$I_1 = \frac{I_2 + 1}{I_2 - 1} = 1 + \frac{2}{I_2 - 1}$

$I_2 - 1 = \pm 1, \pm 2, \quad I_2 = 2, 0, 3, -1$

$I_1 = 3, -1, 2, 0$

$I_1 I_2 = 6, 0$

$x + y = I_1 I_2$

$\Rightarrow$ x + y = 0 or x + y = 6

Ans. Integral solution (0, 0); (2, 2). X + y = 6, x + y = 0

50. If domain of $f\left(x\right)=\dfrac{\sin^{-1}\left(\sin x\right)}{\sqrt{-\log_{\left(\frac{x+4}{2}\right)}\log_2\left(\dfrac{2x-1}{3+x}\right)}}$ is $\left(a,b\right)\cup\left(c,\infty\right)$, then find the value of a + b + 3c.

Ans. a = - 4, b = - 3, c = 4 and so a + b + 3c = 5

Sol. Domain of $\sin^{-1}\left(\sin x\right)$ is which of R

$$-\log_{\frac{x+4}{2}}\log_2\left(\frac{2x-1}{3+x}\right)>0$$

i.e. $\log_{\frac{x+4}{2}}\left(\log_2\left(\dfrac{2x-1}{3+x}\right)\right)<0$

case-I $0<\dfrac{x+4}{2}<1$ i.e. - 4 < x < - 2

then $\log_2\dfrac{2x-1}{3+x}>1$ i.e. $\dfrac{2x-1}{3+x}>2$ i.e. $\dfrac{2x-1-6-2x}{3+x}>0$

i.e. x + 3 < 0 i.e. x < - 3

∴ - 4 < x < - 3 …(i)

case-II if $\dfrac{x+4}{2}>1$ i.e. x > - 2

then $0<\log_2\dfrac{2x-1}{3+x}<1$ i.e. $1<\dfrac{2x-1}{3+x}<2$

i.e. $\dfrac{2x-1-3-x}{x+3}>0$ and $\dfrac{2x-1-6-2x}{x+3}<0$

i.e. $\dfrac{x-4}{x+3}>0$ and $\dfrac{-7}{x+3}<0$

i.e. {x < - 3 or x > 4} and x > - 3

i.e. x > 4 …(iii)

from (i) and (ii) $x\in\left(-4,-3\right)\cup\left(4,\infty\right)$

∴ a = - 4, b = - 3, c = 4 and so a + b + 3c = 5 Ans.

Functions

Matrix-Match Type

1. Column – I | Column – II

A) Domain of $f(x) = \log\left[ax^3+(b+a)x^2+(b+c)x+c\right]$ If $b^2-4ac<0, a>0$ is

B) Domain of $f(x) = \ln\left(\tan^{-1}((x^3-6x^2+11x-6)x(e^x-1))\right)$ is

C) range of $f(x)=\dfrac{x^2-3x+2}{x^2+x-6}$

D) range of $f(x)=\sin^2\dfrac{x}{4}+\cos\dfrac{x}{4}$ is

P) $\left[-1,\dfrac{5}{4}\right]$

Q) $R-\left\{\dfrac{1}{5},1\right\}$

R) $(-1,\infty)$

S) $(1,2)\cup(3,\infty)$

Key. A – p,r; B – q,r; C – r,s; D – q,r

Sol. A) $f(x)=\log(x(ax^2+bx+c)+(ax^2+bx+c)$

$=\log(x+1)(ax^2+bx+c)$

$\therefore$ domain of $f(x)=(-1,\infty)$

as $ax^2+bx+c>0\ \forall x$ since $a>0, \Delta<0$

B) For $ln(\tan^{-1}(x-1)(x-2)(x-3)x(e^x-1))$ to be defined

$x(e^x-1)(x-1)(x-2)(x-3)>0$

$x\in(1,2)\cup(3,\infty)$

C) $y=f(x)=\dfrac{x-1}{x+3}$ for $x\neq 2,-3$

and $x=\dfrac{3y+1}{1-y}$ $y\neq 1$ and $x\to 2\Rightarrow y\to\dfrac{1}{5}$

$\therefore$ range of $f(x)=R-\left\{1.\dfrac{1}{5}\right\}$

D) $f(x)=1-\cos^2\dfrac{x}{4}+\cos\dfrac{x}{4}=\dfrac{5}{4}-\left(\cos\dfrac{x}{4}-\dfrac{1}{2}\right)^2 \qquad \Rightarrow -1\leq f(x)\leq\dfrac{5}{4}$

2. Let $f, g : R \to R$ be the functions defined by $f(x) = x^2 + 1$ and $g(x) = 2[x] - 1$ where $[x]$ is the largest integer $\leq x$. Then match the items given in Column I with those in Column II.

Column-1		Column-2	
(A)	$(gof)\left(\frac{1}{2}\right)$	(P)	3
(B)	$(fog)\left(\frac{3}{2}\right)$	(Q)	0
(C)	$(fogof)\left(\frac{3}{4}\right)$	(R)	2
(D)	$(gofog)\left(\frac{2}{3}\right)$	(S)	1

Key. A-s; B-r; C-r; D-p

Sol. (A) $(gof)\left(\frac{1}{2}\right) = g\left[f\left(\frac{1}{2}\right)\right] = g\left(\frac{5}{4}\right) = 2(1) - 1 = 1$

(B) $(fog)\left(\frac{3}{2}\right) = f\left[g\left(\frac{3}{2}\right)\right] = f(1) = 1^2 + 1 = 2$

(C) $(fogof)\left(\frac{3}{4}\right) = (fog)\left(f\left(\frac{3}{4}\right)\right) = (fog)\left(\frac{25}{16}\right) = f\left[g\left(\frac{25}{16}\right)\right]$

$= f(1) = 1^2 + 1 = 2$

(D) $(gofog)\left(\frac{2}{3}\right) = (gof)\left[g\left(\frac{2}{3}\right)\right] = (gof)(-1) = g[f(-1)]$

$= g(2) = 2 \times 2 - 1 = 3.$

3.

COLUMN-I		COLUMN-II	
(A)	If f is a function such that $f(0) = 2, f(1) = 3$ and $f(x+2) = 2f(x) - f(x+1)$, then $f(5)$ is equal to	(p)	4
(B)	If $f(x) = \begin{cases} x^2, & for\ x \geq 0 \\ x, & for\ x < 0 \end{cases}$, then $f(\sqrt{13}) =$	(q)	3
(C)	If $f(x) + 2f(1-x) = x^2 + 2$ for all $x \in R$, then $f(5)$ is	(r)	12

(D)	If $f(x)=\dfrac{4^x}{4^x+2}$ for all $x\in R$, then $\sum_{k=1}^{6} f\left(\dfrac{k}{7}\right)=$	(s)	11
		(t)	13

Key. A-s; B-s; C-q; D-q

Sol. (A) Given $f(0)=2,\ f(1)=3$

and $f(x+2)=2f(x)-f(x+1)$

Put $x=0$

$f(2)=2f(0)-f(1)=2\times2-3=1$

Put $x=1$

$f(3)=2f(1)-f(2)=2\times3-1=5$

Put $x=2$

$f(4)=2f(2)-f(3)=2\times1-5=-3$

Put $x=3$

$f(5)=2f(3)-f(4)=2\times5-(-3)=13$

(B) $\sqrt{13}\ge0\Rightarrow f\left(\sqrt{13}\right)=\left(\sqrt{13}\right)^2=13$

(C) Given $f(x)+2f(1-x)=x^2+2,\forall x\in R \longrightarrow(1)$

Replace x by $1-x$

$f(1-x)+2f(x)=(1-x)^2+2 \longrightarrow(2)$

Multiply Eq (2) by 2

$2f(1-x)+4f(x)=2(1-x)^2+4 \longrightarrow(3)$

$(3)-(1)\Rightarrow 3f(x)=2(1-x)^2-x^2+2$

Put $x=5$

$3f(5)=2\times16-25+2=9$

$\Rightarrow f(5)=3$

(D) $f(x)=\dfrac{4^x}{4^x+2}$

$$\Rightarrow f(1-x)=\frac{4^{1-x}}{4^{1-x}+2}=\frac{\frac{4}{4^x}}{\frac{4}{4^x}+2}$$

$$=\frac{4}{4+2.4^x}=\frac{2}{4^x+2}$$

$\therefore f(x)+f(1-x)=1$

$$\Rightarrow f\left(\frac{1}{7}\right)+f\left(\frac{6}{7}\right)=1$$

$$f\left(\frac{2}{7}\right)+f\left(\frac{5}{7}\right)=1$$

$$f\left(\frac{3}{7}\right)+f\left(\frac{4}{7}\right)=1$$

(+) ________________

$$\sum_{k=1}^{6} f\left(\frac{k}{7}\right) = 1+1+1 = 3.$$

4. Match the following

	Column–1		Column–2
(A)	If $f(x) = \sqrt{4^x + 8^{\frac{2}{3}(x-1)} - 72 - 4^{x-\frac{3}{2}}}$ is defined $\forall$ x ≥ a, then f(a) =	(P)	0
(B)	If $2f(x^2) + 3f\left(\frac{1}{x^2}\right) = x^2 - 1, \ (x \neq 0)$ then f(1) =	(Q)	4
(C)	$f(x) = [x] + \sum_{r=1}^{398} \frac{\{x+r\}}{398}$, where [·] denotes the greatest integer function and {.} denotes the fractional part of x, then f(3) =	(R)	3
(D)	The number of points of discontinuity of $f(x) = [\tan x[\cot x]]; x \in \left[\frac{\pi}{12}, \frac{\pi}{2}\right)$ where [.] denotes the greatest integer function.	(S)	2

Key. (A) → P, (B) → P, (C) → R, (D) → R

Sol. (A) $4^x + 8^{\frac{2}{3}(x-1)} - 72 - 4^{x-\frac{3}{2}} \geq 0$

$\Rightarrow \ 2^{2x} + 2^{2x-2} - 72 - 2^{2x-3} \geq 0$

$\Rightarrow \ 2^{2x}\left[1 + \frac{1}{4} - \frac{1}{8}\right] \geq 72$

$2^{2x} \geq 64$ and $x \geq 3$

$\Rightarrow$ f(3) = 0

(C) $\sum_{r=1}^{398} \frac{\{x+r\}}{398} = \frac{398}{398}\{x\} = \{x\}$

$\therefore$ f(x) = [x] + {x} = x

$\Rightarrow$ f(3) = 3

(D) $\because$ f(x) is discontinuous when $\cot x \in$ Integer

As $\frac{\pi}{12} \leq x < \frac{\pi}{2}$

$\therefore \ 0 < \cot x \leq 2 + \sqrt{3}$

$\therefore$ Number of points of discontinuous = 3.

5. Match the following.

$f: R-\{0\} \to R$. Let $f(x) = ax + \frac{b}{x}$ $(ab \neq 0)$,

	Column I		Column II
(A)	$a>0, b>0$	(P)	$f(x)$ is a bijective
(B)	$a>0, b<0$	(Q)	$f(x)$ is onto function
(C)	$a<0, b>0$	(R)	$f(x)$ is one-one function
(D)	$a<0, b<0$	(S)	$f(x)$ is neither onto nor one-one function

Key. (A) $\to$ (S); (B) $\to$ (Q); (C) $\to$ (Q); (D) $\to$ (S)

Sol. Conceptual

6. Match the following.

Column – I (Function) Here [ ] g.i.f, { } denote fractional part		Column – II (fundamental Period)	
A	$f(x) = e^{\cos^4 \pi x} + x - [x] + \cos^2 \pi x$	P	$\frac{1}{3}$
B	$f(x) = \cos(2\pi\{2x\}) + \sin(2\pi\{2x\})$	Q	$\frac{1}{4}$
C	$f(x) = \sin(3\pi\{x\}) + \tan(\pi[x])$	R	$\frac{1}{2}$
D	$f(x) = 3x - [3x+a] - b$ where $a, b \in R^+$	S	1

Key. A) S; B) R; C) S; D) P

Sol. A) $f(x) = e\cos^4 \pi x + x - [x] + \cos^2 \pi x$

$\cos^4 \pi x, \cos^2 \pi x, x - [x]$ are periodic with period 1

$\therefore$ period of $f(x)$

B) $f(x) = \cos 2\pi\{2x\} + \sin 2\pi\{2x\}$

We know period of $\{2x\}$ periodic with period 1/2

$\Rightarrow$ period of $f(x) is \frac{1}{2}$

C) $f(x) = \sin 3\pi\{x\} + \tan \pi[x]$

Clearly $\tan \pi[x] = 0 \forall x \in R$ and period of $\{x\}$

$\Rightarrow \sin 3\pi\{x\}$ in period with partial -1

D) for $f(x) = x - [x+a] - b = (x+a) - [x+a] - (a+b)$

$= \{x+a\} - (a+b)$

Which in periodic with period 1

$\therefore$ for $f(x) = 3x - [3x+a] - b$

$= (3x+a) - [3x+a] - (a+b)$ = $\{3x+a\} - (a+b)$

Which periodic with period 1/3

7. Let A = {1, 3, 5, 7} and B = {2, 4, 6, 8} and f:A$\rightarrow$B, then the number of functions 'f' possible for

	Column-I		Column-II
(A)	$i+f(i)<10, \forall i \in \{1,3,5,7\}$	(p)	16
(B)	$f(i)-i>2, \forall i \in \{1,3,5,7\}$	(q)	24
(C)	$f(i)\geq 6, \forall i \in \{1,3,5,7\}$	(r)	0
(D)	$f(i)\neq i+1, \forall i \in \{1,3,5,7\}$	(s)	81

Key: $A\rightarrow q; B\rightarrow r; C\rightarrow p; D\rightarrow s$

Hint. A(q), B(r), C(p), D(s)
(A) For 1, 3, 5, 7$\in$A we have 4, 3, 2, 1, choices respectively.
(B) Image of f(7) should be greater than 9.
(C) For 1, 3, 5, 7$\in$A we have 2, 2, 2, 2 choices respectively.
(D) for any i$\in$A, we have 3 choices.

8. Match the following.

$f: R-\{0\}\rightarrow R$. Let $f(x)=ax+\frac{b}{x}\ (ab\neq 0)$,

	Column I		Column II
(A)	$a>0, b>0$	(P)	$f(x)$ is a bijective
(B)	$a>0, b<0$	(Q)	$f(x)$ is onto function
(C)	$a<0, b>0$	(R)	$f(x)$ is one-one function
(D)	$a<0, b<0$	(S)	$f(x)$ is neither onto nor one-one function

KEY : A – S, B – Q, C – Q, D – S

Hint Conceptual

9. Match the following

	Column–1		Column–2
(A)	If $f(x)=\sqrt{4^x+8^{\frac{2}{3}(x-1)}-72-4^{x-\frac{3}{2}}}$ is defined $\forall$ x $\geq$ a, then f(a) =	(P)	0
(B)	If $2f(x^2)+3f\left(\frac{1}{x^2}\right)=x^2-1,\ (x\neq 0)$ then f(1) =	(Q)	4
(C)	$f(x)=[x]+\sum_{r=1}^{398}\frac{\{x+r\}}{398}$, where [·] denotes the greatest integer function and {.} denotes the fractional part of x, then f(3) =	(R)	3
(D)	The number of points of discontinuity of $f(x)=[\tan x[\cot x]]; x\in\left[\frac{\pi}{12},\frac{\pi}{2}\right)$ where [.]	(S)	2

	denotes the greatest integer function.		

Key: A

Hint: A) $\to$ P, (B) $\to$ P, (C) $\to$ R, (D) $\to$ R

(A) $4^x + 8^{\frac{2}{3}(x-1)} - 72 - 4^{x-\frac{3}{2}} \ge 0$

$\Rightarrow$ $2^{2x} + 2^{2x-2} - 72 - 2^{2x-3} \ge 0$

$\Rightarrow$ $2^{2x}\left[1+\frac{1}{4}-\frac{1}{8}\right] \ge 72$

$2^{2x} \ge 64$ and $x \ge 3$

$\Rightarrow$ f(3) = 0

(C) $\sum_{r=1}^{398} \frac{\{x+r\}}{398} = \frac{398}{398}\{x\} = \{x\}$

$\therefore$ f(x) = [x] + {x} = x

$\Rightarrow$ f(3) = 3

(D) Q f(x) is discontinuous when $\cot x \in \text{Integer}$

As $\frac{\pi}{12} \le x < \frac{\pi}{2}$

$\therefore$ $0 < \cot x \le 2 + \sqrt{3}$

$\therefore$ Number of points of discontinuous = 3.

10.

	Column I		Column II
A	The interval containing the complete set of solution of the equation $\left\|\frac{1-x^2}{x}\right\| + \|x\| = \left\|\frac{1}{x}\right\|$ are	P	$\left[0, \frac{1}{2}\right]$
B	The interval containing the complete set of values of 'a', for which $(a+1)x + ay - 1 = 0$ is a normal to the curve xy =1, are	Q	$[-1,1]$
C	Complete set of values of 'a' for which equation $a\sin^2 x + \|\cos x\| - 2a = 0$ has atleast one solution belongs to the interval	R	$[0,1]$
D	The interval containing the range of the function $f(x) = \frac{1}{1+2\cos^2 x + 3\cos^4 x + 4\cos^6 x + ...\infty}$	S	$[-1,0]$
		T	$\left[-1, \frac{1}{2}\right]$

Key: A – Q; B – Q, S, T;
C – P, Q, R,T; D – Q, R

Hint: $(a)\left|\frac{1-x^2}{x}\right| + |x| = \left|\frac{1-x^2}{x} + x\right| = \left|\frac{1}{x}\right| \Rightarrow \frac{1-x^2}{x}.x \ge 0 \Rightarrow x^2 - 1 \le 0, x \ne 0 \Rightarrow x \in [-1,1] - \{0\}$

$(b)\, slope > 0 \Rightarrow \frac{-(a+1)}{a} > 0 \Rightarrow a \in (-1,0)$

$(c)\, a - a\cos^2 x + |\cos x| - 2a = 0 \Rightarrow a = \dfrac{|\cos x|}{1+\cos^2 x} = \dfrac{1}{|\cos x| + \dfrac{1}{|\cos x|}}$

$|\cos x| \in [0,1] \Rightarrow |\cos x| + \dfrac{1}{|\cos x|} \in [2,\infty) \Rightarrow a \in \left(0, \dfrac{1}{2}\right]$

$(d)\, if \quad x \neq n\pi$

$S = 1 + 2\cos^2 x + 3\cos^4 x + \ldots\ldots\ldots\ldots(1)$

$\cos^2 xS = \cos^2 x + \cos^4 x + \ldots\ldots\ldots(2), \qquad eq.(1) - eq.(2)$

$\sin^2 xS = 1 + \cos^2 x + \cos^4 x + \ldots\ldots\ldots$

$\sin^2 xS = \dfrac{1}{1-\cos^2 x} \Rightarrow S = \dfrac{1}{\sin^4 x} \Rightarrow f(x) = \sin^4 x$

$for\ x = n\pi\ f(x) is\ not\ difined,\ Range of\ \ f(x) = (0,1]$

11. Column I (Function) — Column II (Type)

(A) f : R → R, $f(x) = x^3 + 3x - 7$ — (p) one-one

(B) f : R → R, $f(x) = x^3 - 7x$ — (q) Onto

(C) f : R → [2, 6] f(x) = $\sqrt{3}$ sinx - cosx + 4 — (r) non periodic

(D) f : R → R, f(x) = ln $\left(x + \sqrt{1+x^2}\right)$ — (s) odd

Key: (A-p, q, r), (B-q, r,s), (C-q), (D-p, q, r, s)

Hint: $\sin\theta = \dfrac{2\tan\theta/2}{1+\tan^2\theta/2} \Rightarrow \tan\left(\dfrac{A-B}{2}\right) = 3, \dfrac{1}{3}$

by Napeir's rule

$\tan\left(\dfrac{A-B}{2}\right) = \dfrac{a-b}{a+b}\cot\dfrac{C}{2}$

$\Rightarrow \cot\dfrac{C}{2} = 1, 9$

If $\cot\dfrac{C}{2} = 1 \Rightarrow$ area of $\Delta = b^2$

$r = (s - c)\tan\dfrac{c}{2} \Rightarrow r + c - s = 0$

12. Match the following: -

Column I		Column II	
(A)	Solution of $\lvert x^2-1+\sin x\rvert = \lvert x^2-1\rvert + \lvert \sin x\rvert$ in $[-2\pi, 2\pi]$	(p)	$(-\infty, 1)$
(B)	Domain of f(x) $= \dfrac{1}{\sqrt{\log_{1/2}\left(x^2-7x+13\right)}}$	(q)	$\left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$
(C)	Domain of single valued function $y = f(x)$ given by $10^x + 10^y = 10$ is	(r)	$(3, 4)$
(D)	Let $x \in \left(0, \dfrac{\pi}{2}\right)$, then solution of f(x) $= \dfrac{1}{\sqrt{-\log_{\sin x} \tan x}}$ is	(s)	$x \in [-2\pi, -\pi] \cup [-1, 0] \cup [1, \pi] \cup [2\pi]$

Key : A→s; B→r; C→p; D→DELETED

Sol : A. $\lvert x^2-1+\sin x\rvert = \lvert x^2-1\rvert + \lvert \sin x\rvert$

exist only if $\left(x^2-1\right)$ and (sin x) are of the same sign

$\left(\because \lvert a+b\rvert = \lvert a\rvert + \lvert b\rvert\right)$ only if (ab $\geq$ 0)

$\therefore$ $(x^2-1)\sin x \geq 0$

− + − + − + − +

-2π -π -1 0 1 π 2π

$\Rightarrow (x-1)(x+1)\sin x \geq 0.$

thus $x \in [-2\pi, -\pi] \cup [-1, 0]$

$x \in [-2\pi, -\pi] \cup [-1, 0] \cup [1, \pi] \cup \{2\pi\}$

B. $f(x) = \dfrac{1}{\sqrt{\log_{\frac{1}{2}}\left(x^2-7x+13\right)}}$

exists if $\log_{\frac{1}{2}}\left(x^2-7x+13\right) > 0$

$x^2-7x+13 < 1$ and $x^2-7x+13 > 0$

$\Rightarrow x^2-7x+13 > 0 \Rightarrow \left(x-\dfrac{7}{2}\right)^2 + \dfrac{3}{4} > 0$

which is true for all x $\in$ R

again $x^2-7x+13 < 1 \Rightarrow x^2-7x+12 < 0$

$\Rightarrow$ $(x-3)(x-4) < 0$

$\Rightarrow 3 < x < 4$

thus D_f is (3,4)

C. since $10^x + 10^y = 10$

$\Rightarrow 10^y = 10 - 10^x$

$\Rightarrow y = \log_{10}\left(10 - 10^x\right)$

Now y is defined if $10 - 10^x > 0$

$\Rightarrow 10^1 > 10^x$

$1 > x$ i.e. $x < 1$

$\therefore\ D_f = \left(-\infty, 1\right)$

D. Here $x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow 0 < \sin x < 1$

again $\log_a x < b \Rightarrow x > a^b$, if $0 < a < 1$ and $x < a^b$ if $a > 1$

Thus $f\left(x\right) = \frac{1}{\sqrt{-\log_{\sin x}\tan x}}$ exists

if $-\log_{\sin x}\tan x > 0$

$\Rightarrow \log_{\sin x}\tan x < 0$

$\Rightarrow \tan x > \left(\sin x\right)^0 = 1$

$\Rightarrow \tan x > 1$

$\Rightarrow x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)\left[Q\ x \in \left(0, \frac{\pi}{2}\right)\right]$

$\therefore$ reqd. sol. is $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

13. For any $0 < a \in R$, let $f_a\left(x\right) = \frac{a^x}{a^x + \sqrt{a}}$ for all $x \in R$. Then match the items in Column I with those in Column II.

Column-1		Column-2	
(A)	$\sum_{k=1}^{1997} f_9\left(\frac{k}{1998}\right) =$	(P)	998.5
(B)	$\sum_{k=1}^{1997} f_4\left(\frac{k}{1998}\right) =$	(Q)	1004.5
(C)	$\sum_{k=1}^{2009} f_{16}\left(\frac{k}{2010}\right) =$	(R)	993
(D)	$\sum_{k=1}^{2008} f_{25}\left(\frac{k}{2009}\right) =$	(S)	1004

Key. A-p; B-p; C-q; D-s

Sol. $f_a\left(x\right) = \frac{a^x}{a^x + \sqrt{a}} \Rightarrow f_a\left(\frac{1}{2}\right) = \frac{\sqrt{a}}{\sqrt{a} + \sqrt{a}} = \frac{1}{2}$

$$\Rightarrow f_a(1-x)=\frac{a^{1-x}}{a^{1-x}+\sqrt{a}}=\frac{\frac{a}{a^x}}{\frac{a}{a^x}+\sqrt{a}}=\frac{a}{a+a^x\sqrt{a}}=\frac{\sqrt{a}}{a^x+\sqrt{a}}$$

Now $f_a(x)+f_a(1-x)=\frac{a^x+\sqrt{a}}{a^x+\sqrt{a}}=1$

(A) $\sum_{k=1}^{1997} f_9\left(\frac{k}{1998}\right)=f_9\left(\frac{1}{1998}\right)+f_9\left(\frac{2}{1998}\right)+.......+f_9\left(\frac{1996}{1998}\right)+f_9\left(\frac{1997}{1998}\right)$

$$=t_1+t_2+t_3+.......+t_{1997}$$

Here $t_1+t_{1997}=1, t_2+t_{1996}=1$ and so on.

$$t_{999}=f_9\left(\frac{999}{1998}\right)=f_9\left(\frac{1}{2}\right)=\frac{1}{2}$$

$\therefore$ Ans = $1\times 998+\frac{1}{2}=998.5$

(B) Similar to (A)

(C) Similar to (A) & (B) Ans $=1\times 1004+\frac{1}{2}=1004.5$

(D) Similar to (A), (B) & (C) Ans $=1\times 1004=1004$

14. Match the following:

Column – I	Column - II
A) Let $f:R\to R$ satisfies $f(x+y)+f(x-y)=2f(x)f(y)\forall\ x,y\in R$ and $f(0)\neq 0$, then $f(x)$ is	P) Even
B) Let $f:R\to R$ is defined by $f(x)=\frac{e^{\lvert x\rvert}-e^{-x}}{e^x+e^{-x}}$, then $f(x)$ is	Q) Odd
C) Let $f:R\to R$ be a polynomial function satisfying $f(x)f\left(\frac{1}{x}\right)=f(x)+f\left(\frac{1}{x}\right)$ and $f(3)=28$ then $f(x)$ is	R) Into
D) let $f:R\to R$ is defined by $f(x)=2x+\sin x$, then $f(x)$ is	S) Many - one
	T) Bijective

Key. A-PRS, B-R,S; C-T; D-QT

Sol. Put $x=y=0,$ then $2f(0)=2\{f)0)\}^2\Rightarrow f(0)=1.$ Now put $x=0$, then

$f(y)+f(-y)=2f(0)f(y)$

$\therefore f(y)=f(-y)\Rightarrow f(x)$ is even $\Rightarrow$ $f(x)$ is many one

Again $f(0)=1 \Rightarrow f(x)$ cannot be onto

If $x \le 0$ then $f(x) = 0 \Rightarrow f(x)$ is many one and neither even nor odd. Clearly $f(x)$ is not onto, for example $f(x) \ne -1$ for any x

$f(x)$ is either $-x^n + 1$ or $x^n + 1$. But $f(3) = 28$

$\therefore f(x) = x^3 + 1$ which is neither even nor odd but one-one and onto both.

Obviously $f(x)$ is odd and $f'(x) = 2 + \cos x > 0,$ so $f(x)$ is one-one.
Also,
$f(x) \to \infty$ if $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$ so $f(x)$ is onto

15. Match the following: -

	COLUMN-I		COLUMN-II
(A)	If f is a function such that $f(0)=2, f(1)=3$ and $f(x+2)=2f(x)-f(x+1)$, then $f(5)$ is equal to	(p)	4
(B)	If $f(x)=\begin{cases} x^2, & for\ x \ge 0 \\ x, & for\ x<0 \end{cases}$, then $f\left(\sqrt{13}\right)=$	(q)	3
(C)	If $f(x)+2f(1-x)=x^2+2$ for all $x \in R$, then $f(5)$ is	(r)	12
(D)	If $f(x)=\dfrac{4^x}{4^x+2}$ for all $x \in R$, then $\sum_{k=1}^{6} f\left(\dfrac{k}{7}\right)=$	(s)	11
		(t)	13

Key. A-t; B-t; C-q; D-q

Sol. (A) Given $f(0)=2, f(1)=3$

and $f(x+2)=2f(x)-f(x+1)$

Put $x=0$

$f(2)=2f(0)-f(1)=2\times 2-3=1$

Put $x=1$

$f(3)=2f(1)-f(2)=2\times 3-1=5$

Put $x=2$

$f(4)=2f(2)-f(3)=2\times 1-5=-3$

Put $x=3$

$f(5)=2f(3)-f(4)=2\times 5-(-3)=13$

(B) $\sqrt{13} \ge 0 \Rightarrow f\left(\sqrt{13}\right)=\left(\sqrt{13}\right)^2=13$

(C) Given $f(x)+2f(1-x)=x^2+2, \forall x \in R \quad \longrightarrow (1)$

Replace x by $1-x$

$f(1-x)+2f(x)=(1-x)^2+2 \quad \longrightarrow (2)$

Multiply Eq (2) by 2

$2f(1-x)+4f(x)=2(1-x)^2+4 \quad \longrightarrow (3)$

$(3)-(1) \Rightarrow 3f(x) = 2(1-x)^2 - x^2 + 2$

Put $x = 5$

$3f(5) = 2\times16 - 25 + 2 = 9$

$\Rightarrow f(5) = 3$

(D) $f(x) = \dfrac{4^x}{4^x+2}$

$\Rightarrow f(1-x) = \dfrac{4^{1-x}}{4^{1-x}+2} = \dfrac{\frac{4}{4^x}}{\frac{4}{4^x}+2}$

$= \dfrac{4}{4+2.4^x} = \dfrac{2}{4^x+2}$

$\therefore f(x) + f(1-x) = 1$

$\Rightarrow f\left(\dfrac{1}{7}\right) + f\left(\dfrac{6}{7}\right) = 1$

$f\left(\dfrac{2}{7}\right) + f\left(\dfrac{5}{7}\right) = 1$

$f\left(\dfrac{3}{7}\right) + f\left(\dfrac{4}{7}\right) = 1$

(+) ___________________

$\sum_{k=1}^{6} f\left(\dfrac{k}{7}\right) = 1+1+1 = 3$.

16. Let A = {1, 3, 5, 7} and B = {2, 4, 6, 8} and f:A $\rightarrow$ B, then number of function of f possible for

Column – 1		Column – 2	
(A)	$i + f(i) < 10,\ \forall i = \{1,3,5,7\}$	(p)	128
(B)	$f(i) - i > 2,\ \forall i = \{1,3,5,7\}$	(q)	24
(C)	$f(i) \geq 6,\ \forall i = \{1,3,5,7\}$	(r)	0
(D)	$f(i) \neq i+1,\ \forall i = \{1,3,5,7\}$	(s)	81

Key. A - (q); B - (r); C - (p); D - (s)

Sol. (A) For 1, 3, 5, 7 $\in$ A we have 4, 3, 2, 1, choices respectively.
(B) Image of f(7) should be greater than 9.
(C)For 1, 3, 5, 7 $\in$ A we have 2, 4, 4, 4 choices respectively.
(D) For any i $\in$ A , we have 3 choices.

17. Let f(x) = g(x) + h(x)

Where g(x) = $\dfrac{1}{\pi}$ $(\sin^{-1}x + \tan^{-1}x)$

and h(x) = $\dfrac{x+1}{x^2+2x+5}$

	Column I		Column II
(A)	If domain of f(x) is [a, b], then a + b equals	(p)	4
(B)	maximum value of g(x) is	(q)	1/4
(C)	Maximum value of h(x) is	(r)	0
(D)	If range of f(x) is [*l*, m], then *l* + m equals	(s)	3/4

Key. (A- r); (B – s); (C – q); (D – q)

Sol. f(x) can be re–written as

$$f(x) = \frac{1}{\pi}\left(\sin^{-1}x + \tan^{-1}x\right) + \frac{1}{(x+1)+\frac{4}{x+1}}$$

= g(x) + h(x)

Domain of f(x) is [–1, 1] ⇒ a + b = 0

Maximum value of g(x) is g(1) = 3/4

maximum value of h(x) occurs when (x + 1) + $\frac{4}{(x+1)}$ is minimum at x = 1.

⇒ maximum value of h(x) = $\frac{1}{4}$

∴ Range of f(x) is [–3/4, 1]

⇒ *l* + m = 1 – 3/4 = 1/4

18. The correct matching of the functions in Column-I with their respective inverses in Column-II is

Column – I	Column – II
a) $f(x) = \frac{2x}{x^2+1}, \lvert x\rvert \le 1$	p) $f^{-1}(x) = -\sqrt{1+\sqrt{x-1}}$
b) $f(x) = \frac{2x}{x^2+1}, x \ge 1$	q) $f^{-1}(x) = -\sqrt{1-\sqrt{x-1}}$
c) $f(x) = x^4 - 2x^2 + 2, x \le -1$	r) $f^{-1}(x) = \frac{1+\sqrt{1-x^2}}{x}$
d) $f(x) = x^4 - 2x^2 + 2, -1 \le x \le 0$	s) $f^{-1}(x) = \frac{1-\sqrt{1-x^2}}{x}$

Key. a) s; b) r; c) p; d) q

Sol. Conceptual

19. Match the following

	COLUMN – I		COLUMN - II
A	Odd function	P	$x-[x]\ \forall x \in R$
B	Even function	Q	$\log\left(x+\sqrt{1+x^2}\right)\forall x \in R$

C	Neither even nor odd	R	$x\log\left(\frac{1+x}{1-x}\right)\ \ \forall x\in(-1,1)$
D	Periodic function	S	$\frac{2^{x/2}}{1+2^{x/2}}\ \ \forall x\in R$

Key. A$\to$ q; B $\to$ r; C $\to$ p,s; D $\to$ p

Sol. $x-[x]$ is neither even nor odd function. It is a periodic function with period '1'

$\log\left(x-\sqrt{1+x^2}\right)$ is odd function

$xlong\left(\frac{1+x}{1-x}\right),\frac{2^{\frac{x}{2}}}{1+2^{\frac{x}{2}}}$ are even functions.

20. Let the range of the function f(x) be [a,b] column I gives the function *f(x)* and column II gives the values of b – a

	COLUMN – I $f(x)$		COLUMN – II $b-a$
A	$\log_2\left(\frac{\sin x-\cos x+3\sqrt{2}}{\sqrt{2}}\right)$	P	$\frac{8}{3}$
B	$\frac{1}{3-2\sin 2x}$	Q	2
C	$\sin\log_e\left(\frac{\sqrt{4-x^2}}{1-x}\right)$	R	$\frac{4}{5}$
D	$\frac{x^2-x+1}{x^2+x+1}$	S	$\frac{3}{4}$

Key. A$\to$ q; B $\to$ r; C $\to$ q; D $\to$ p

Sol. b) $\frac{1}{3-2\sin 2x}\Rightarrow$ Range is $\left[\frac{1}{5},1\right]\Rightarrow b-a=\frac{4}{5}$

c) $\sqrt{4-x^2}>0\Rightarrow -2<x<2$ & $1-x>0\Rightarrow x<1$ $\therefore$ Domain $(-2,1)$

Range is $[-1,1]\Rightarrow b+a=2$

21. Match the following: -

	Column – I		**Column – II**
(A)	The number of possible values of k if fundamental period of $\sin^{-1}(\sin kx)$ is $\frac{\pi}{2}$, is	(p)	1
(B)	Numbers of elements in the domain of $f(x)=\tan^{-1}x\sin^{-1}x+\sec^{-1}x$ is	(q)	2
(C)	Period of the function $f(x)=\sin\left(\frac{\pi x}{2}\right).\mathrm{cosec}\left(\frac{\pi x}{2}\right)$ is	(r)	3
(D)	If the range of the function $f(x)=\cos^{-1}[5x]$ is {a, b, c} and a + b + c $=\frac{\lambda\pi}{2}$, then λ is equal to (where [.] denotes	(s)	4

	greatest integer)		
		(t)	0

Key. A $\to$ q; B $\to$ q; C $\to$ q; D $\to$ r

Sol. (A) Fundamental period of $\sin^{-1}(\sin kx)$ is $\frac{2\pi}{|k|} = \frac{\pi}{2}$ i.e. |k| = 4 i.e. $k = \pm 4$

(B) domain of $\tan^{-1} x$ is R, domain of $\sin^{-1} x$ is [- 1, 1], domain of $\sec^{-1} x$ is $(-\infty, -1] \cup [1, \infty)$

$\therefore$ domain of f(x) is {- 1, 1}

(C) π is a period of sin x. cosec x

$\therefore$ $\pi \times \frac{2}{\pi}$ is a period of $\sin\frac{\pi x}{2}.\operatorname{cosec}\frac{\pi x}{2}$

i.e. 2 is a period of $\sin\frac{\pi x}{2}\operatorname{cosec}\frac{\pi x}{2}$

$$f(x+1) = \sin\frac{\pi}{2}(x+1)\operatorname{cosec}\frac{\pi}{2}(x+1) = \cos\frac{\pi}{2}x.\ \sec\frac{\pi}{2}x \neq f(x)$$

(D) $f(x) = \cos^{-1}[5x]$

[5x] can take the values – 1, 0, 1

$\therefore$ range $= \left\{\pi, \frac{\pi}{2}, 0\right\}$

$\therefore$ $a + b + c = \pi + \frac{\pi}{2} + 0 = \frac{3\pi}{2}$

22. Match the following: -

	Column – I		Column – II
(A)	Function $f : \left[0, \frac{\pi}{3}\right] \to [0,1]$ defined by $f(x) = \sqrt{\sin x}$ is	(p)	One to one function
(B)	Function $f : (1,\infty) \to (1,\infty)$ defined by $f(x) = \frac{x+3}{x-1}$ is	(q)	May – one function
(C)	Function $f : \left[-\frac{\pi}{2}, \frac{4\pi}{3}\right] \to [-1,1]$ defined by f(x) = sin x is	(r)	Into function
(D)	Function $f : (2,\infty) \to [8, \infty)$ defined by $f(x) = \frac{x^2}{x-2}$ is	(s)	Onto function

Key. A $\to$ q; B $\to$ q; C $\to$ q; D $\to$ r

Sol. (A) $f'(x) = \frac{1}{2\sqrt{\sin x}}\cos x$

$f'(x)$ is positive if $x \in \left[0, \frac{\pi}{3}\right]$

f is one to one function

Since $0 \le x \le \frac{\pi}{3}$

$0 \le \sin x \le \frac{\sqrt{3}}{2}$

$0 \le \sqrt{\sin x} x \le \sqrt{\frac{\sqrt{3}}{2}} < 1$

f is into function

(B) $f(x)=\frac{x+3}{x-1}$

$f'(x)=\frac{(x-1)1-(x+3).1}{(x-1)^2}$

$f'(x)=\frac{-4}{(x-1)2}$

$f'(x)<0$ Hence f(x) is one to one

Since x > 1

$\therefore$ Range of $y=\frac{x+3}{x-1}$ is $(1,\infty)$

f is onto function

(C) $-\frac{\pi}{3}\le x\le\frac{4\pi}{3}$

F(x) = sin x

from graph f(x) is many-one and onto

(D) $f(x)=\frac{x^2}{x-2}$

$f'(x)=\frac{(x-2).\,2x-x^2}{(x-2)^2}$

$f'(x)=\frac{x^2-4x}{(x-2)^2}$

$\therefore$ $f'(x)<0$ if $2 < x < 4$

f(x) is many-one

f(4) = 8 (is the least value of f(x))

$\therefore$ range $=[8,\infty)$

$\therefore$ f(x) is onto.

23. Match the following: -

	Column – I		Column – II
(A)	If smallest positive integral value of x for which $x^2-x-\sin^{-1}(\sin 2)<0$ λ, then $3+\lambda$ is equal to	(p)	4
(B)	Number of solution of 2[x] = x + 2 {x} is (where [.] {.} are greatest integer and least integer functions respectively)	(q)	1
(C)	If $x^2+y^2=1$ and maximum value of x + y is $\frac{\sqrt{2}\lambda}{3}$, then λ is equal to	(r)	2
(D)	$f\left(x+\frac{1}{2}\right)+f\left(x-\frac{1}{2}\right)=f(x)$ for all $x\in R$, then period of f(x) is	(s)	0
		(t)	3

Key. A $\to$ q; B $\to$ q; C $\to$ q; D $\to$ r

Sol. (A) $x^2-x+\pi-2<0$

$x=\frac{1\pm\sqrt{4\pi-7}}{2}$

$\therefore \quad \frac{1-\sqrt{4\pi-7}}{2} < x < \frac{1+\sqrt{4\pi-7}}{2}$

$\therefore \quad \lambda = 1$

(B) 2[x] = x + 2 {x}

(i) x is an integer, then the equation because 2x = x + 0

i.e. x = 0 is a solution

(ii) If $x \notin I$, the equation becomes

2[x] = [x] + {x} + 2{x} i.e. $\{x\} = \frac{1}{3}[x]$

$\therefore \quad 0 < \frac{[x]}{3} < 1 \quad \Rightarrow \quad 0 < [x] < 3$

$\therefore$ possible values of [x] are 1,2

if [x] = 1, then $\{x\} = \frac{1}{3}$ $\quad \therefore \quad x = 1 + \frac{1}{3} = \frac{4}{3}$

if [x] = 2, then $\{x\} = \frac{2}{3}$ $\quad \therefore \quad x = \frac{8}{3}$

$\therefore$ there are 3 solutions

(C) Let $x = \cos\theta,\ y = \sin\theta$

$\therefore \quad x + y = \cos\theta + \sin\theta$

$\therefore$ maximum value of x + y is $\sqrt{2}$

(D) $f\left(x+\frac{1}{2}\right) + f\left(x-\frac{1}{2}\right) = f(x)$

$\Rightarrow \quad f(x+1) + f(x) = f\left(x+\frac{1}{2}\right) \quad \Rightarrow \quad f(x+1) + f\left(x-\frac{1}{2}\right) = 0$

$\Rightarrow \quad f\left(x+\frac{3}{2}\right) = -f(x) \quad \Rightarrow \quad f(x+3) = -f\left(x+\frac{3}{2}\right) = f(x)$

$\therefore$ f(x) is periodic with period 3

24. Match the following: -

	Column – I		Column – II
(A)	If function f(x) is defined in [- 2, 2], then domain of f(|x| + 1) is	(p)	$(-\infty, -4)$
(B)	Range of the function $f(x) = \frac{\sin^{-1}x + \cos^{-1}x + \tan^{-1}x}{\pi}$ is	(q)	[- 1, 1]
(C)	Range of the function f(x) = 3|sin x| - 4 | cos x| is	(r)	[- 4, 3]
(D)	Range of $f(x) = (\sin^{-1}x)\sin x$ is	(s)	$\left[0, \frac{\pi}{2}\sin 1\right]$
		(t)	$\left[\frac{1}{4}, \frac{3}{4}\right]$

Key. A $\rightarrow$ q; B $\rightarrow$ q; C $\rightarrow$ q; D $\rightarrow$ r

Sol. (A) $|x| + 1 \le 2 \quad \Rightarrow \quad -1 \le x \le 1$

(B) Domain of f(x) is [- 1, 1]

$f(x) = \frac{\pi}{2} + \tan^{-1}x \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

$\therefore \quad \frac{f(x)}{\pi} \in \left[\frac{1}{4}, \frac{3}{4}\right]$

(C) Range f(x) = 3 |sin x| - 4 | cos x| for $x \in \left[0, \frac{\pi}{2}\right]$

F(x) = 3 sin x – 4 cos x ∈ [- 4, 3]

Q f(x) is increasing

∴ Range f() is [- 4, 3] because f(x) is even & periodic with period $\frac{\pi}{2}$.

(D) Obvious

25. Match the following: -

	Column – I		Column – II
(A)	Domain of $f(x) = \sin^{-1}\left(\frac{2-x}{2x}\right)$ is	(p)	$[-2, \infty)$
(B)	Range of $f(x) = \frac{2x^2 - 2}{3x^2 + 1}$ is	(q)	$(-\infty, -1] \cup [1, \infty)$
(C)	Set of all values of ρ for which the function f(x) = px + sin x is bijective is	(r)	$(-\infty, -2] \cup [2/3, \infty)$
(D)	If $f: (-\infty, 1] \to A$ is defined by $f(x) = x^2 - 3x$, then set A for which f(x) becomes invertiable, is	(s)	$[-2, 2/3)$
		(t)	$(-\infty, 0)$

Key. A → q; B → q; C → q; D → r

Sol. (A) $f(x) = \sin^{-1}\left(\frac{2-x}{2x}\right)$

$-1 \le \frac{2-x}{2x} \le 1 \Rightarrow -\frac{1}{2} \le \frac{1}{x} \le \frac{3}{2} \Rightarrow x \in (-\infty, -2] \cup [2/3, \infty)$

(B) $y = \frac{2x^2 - 2}{3x^2 + 1} \Rightarrow (3x^2 + 1)y = 2x^2 - 2 \Rightarrow x^2(3y - 2) = -y - 2$

$\Rightarrow x^2 = \frac{-y-2}{3y-2}$

$\therefore x^2 \ge 0 \Rightarrow \frac{-y-2}{3y-2} \ge 0 \Rightarrow \frac{y+2}{3y-2} \le 0 \Rightarrow y \in [-2, 2/3)$

(C) f(x) = px + sin x

$f'(x) = p + \sin x$

either $f'(x) \le 0$ or $f'(x) \ge 0$

$p + \sin x \le 0$ or $p + \sin x \ge 0$

$p \le -1$ or $p \ge 1$

$p \in (-\infty, -1] \cup [1, \infty)$

(D) From graph range of function $[-2, \infty)$

Also f(x) is one-one

∴ $A = [-2, \infty)$

26. Match the following: -

	Column – I		Column – II
(A)	$f(x) = xe^{x(1-x)},\ x \in [0,1]$	(p)	[0, 2]
(B)	$f(x) + \lvert 3-x \rvert + \lvert 2+x \rvert,\ x \in [0,4]$	(q)	[5, 7]
(C)	$f(x) = x^4 + 2x^2 + 5,\ x \in [-1,1]$	(r)	[0, 1]
(D)	$f(x) = x^4 .\ e^{-x^2},\ x \in [-1, 0]$	(s)	[5, 8]

Key. A $\to$ q; B $\to$ q; C $\to$ q; D $\to$ r

Sol. (A) $f'(x) = (1-x)(2x+1)e^{x(1-x)} \geq 0$ in [0, 1]

(B) $f(x) = \begin{cases} 1-2x & x < -2 \\ 5 & -2 \leq x < 3 \\ 2x-1 & x \geq 3 \end{cases}$

Min. of f(x) = 5

Max. of f(x) = 2(4) – 1 = 7

(C) $f(x) = (x^2+1)^2 + 4$ Minimum at x = 0

(D) $f'(x) = 2x^2(2-x^2)e^{-x^2}$ $\Rightarrow$ Decreasing in [- 1, 0]

27. Match the following

	Column – I		Column – II
a)	If $f^1(x^2) = \frac{1}{x}\ \forall x > 0$ and $f(1) = 1$ then $f(4)$ =	p)	0
b)	If $f^1(\sin^2 x) = \cos^2 x\ \forall x \in R$ and $f(1) = 1$ then $2f(0)$ =	q)	1
c)	If $f^1(\sin x) = \cos^2 x\ \forall x \in R$ and $f(1) = 1$ then $3f(0)$ =	r)	2
d)	If $f^1(\log x) = \begin{cases} 1 & \text{for} \quad 0 < x \leq 1 \\ x & \text{for} \quad x > 1 \end{cases}$ and $f(0) = 0$ then $f(\ln 2)$ =	s)	3

Key. a) s; b) q; c) q; d) q

Sol. a) Put $x^2 = t$. Then $x = \sqrt{t}$

b) Put $\sin^2 x = t, 0 \leq t \leq 1$. Then $\cos^2 x = 1 - t$

c) Put $\sin x = t, 0 \leq t \leq 1$. Then $\cos^2 x = 1 - t^2$

d) Put $\log x = t$