CHAPTER - 4

COMPLEX NUMBERS AND QUADRATIC EQUATIONS

KEY POINTS

- The imaginary number $\sqrt{-1} = i$, is called iota
- For any integer k, $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$
- $i^2 = -1$; $i^4 = i^\circ = 1$
- $\sqrt{a} \times \sqrt{b} \neq \sqrt{ab}$ if both a and b are negative real numbers
- $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$, if atleast one number is positive.
- A number of the form z = a + ib, where a, b $\in$ R is called a complex number.

  a is called the real part of z, denoted by Re(z) and b is called the imaginary part of z, denoted by Im(z)
- a + ib = c + id $\Leftrightarrow$ a = c, and b = d
- $z_1 = a + ib$, $z_2 = c + id$.

  In general, we cannot compare and say that $z_1 > z_2$ or $z_1 < z_2$ but if b, d = 0 and a > c then $z_1 > z_2$

  i.e. we can compare two complex numbers only if they are purely real.
- 0 + *i* 0 is additive identity of a complex number.
- –z = –a –ib is called the Additive Inverse or negative of z = a + ib

- 1 + i 0 is multiplicative identity of complex number.
- $z^{-1} = \frac{1}{z} = \frac{a - ib}{a^2 + b^2} = \frac{\bar{z}}{|z|^2}$ is called the multiplicative Inverse of z = a + ib (a ≠ 0, b ≠ 0)
- $\bar{z} = a - ib$ is called conjugate of *z* = *a* + *ib*
- The coordinate plane that represents the complex numbers is called the complex plane or the Argand plane
- $|z_1 + z_2| \le |z_1| + |z_2|$; $|z_1 - z_2| \ge |z_1| - |z_2|$
- $|z_1 z_2| = |z_1|.\ |z_2|$; $\left|\frac{z_1}{z_2}\right| = \frac{|Z_1|}{|Z_2|}$
- $|z^n| = |z|^n$; $|z| = |\bar{z}| = |-z| = |-\bar{z}|$
- $(\overline{z_1 \pm z_2}) = \bar{z}_1 \pm \bar{z}_2$; $\left(\overline{\frac{Z_1}{Z_2}}\right) = \frac{\bar{Z}_1}{\bar{Z}_2}$
- $(\overline{z^n}) = (\bar{z})^n$
- $z\,.\,\bar{z} = |z|^2$
- For the quadratic equation $ax^2 + bx + c = 0$, a, b, c ∈ R, a ≠ 0,if $b^2 - 4ac < 0$ then it will have complex roots given by,

$$x = \frac{-b \pm i\sqrt{4ac - b^2}}{2a}$$

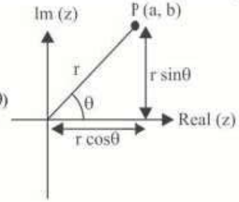
W. R. Hamilton
(1805-1865)

MIND MAP

Trigonometric Fuctions

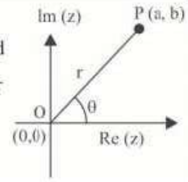
Polar Representation

Let $a = r\cos\theta$
$b = r\sin\theta$
where,
$and\ \theta = \arg(z)$
$\therefore z = a + ib = r(\cos\theta + i\sin\theta)$
The arument 'θ' of complex number $z_1 = a+ib$ is called principal argument of z if $-\pi < \theta \le \pi$.



Modulus & Argument of Complex Number

If $z = a + ib$ is a complex number
(i) Distance of z from origin is called as modulus of complex number z.
It is denoted by $r = |z| = \sqrt{a^2 + b^2}$
(ii) Angle 0 made by OP with +ve direction of X-axis is called argument of z.

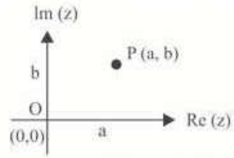


Powers of 'i'

$i = \sqrt{-1},\ i^2 = -1$
In general, $i^{4l+r} = \begin{cases} 1; r = 0 \\ i; r = 1 \\ -1; r = 2 \\ -i; r = 3 \end{cases}$

Argand Plane

A complex number $z = a + ib$ can be represented by a unique point $P(a,b)$ in the argand plane



$z = a + ib$ is represented by a poin $P(a,b)$

Square root of Complex Number

Let $x + iy = \sqrt{a + ib}$, squaring both side, we get $(x + iy)^2 = a + ib$ i.e. $x^2 - y^2 = a, 2xy = b$ solving these equations, we get square root of z.

Multiplicative Inverse of complex number

For a non-zero complex number $z = a + ib (a \ne 0, b \ne 0)$, there exists a complex number $\frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2}i$ denoted by $\frac{1}{z}$ or z^{-1}, called multiplicative inverse of Z

Such that: $(a + ib)\left(\frac{a}{a^2 + b^2} + i\frac{-b}{a^2 + b^2} = 1 + 0i = 1\right)$

Solution of Quadratic Equation

General form of quadratic equation in x is $ax^2 + bx + c = 0$, whered $a, b, c \in R$ and $a \ne 0$
The solutions of given quadratic equation are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
$\because b^2 - 4ac < 0$
Note: • A polynomial equation has atleast one root.
• A polynomial equation of degree n has n roots.

Definition of complex numbers

A number of the form $a + ib$, where $a, b \in R$ and $i = \sqrt{-1}$ is called a comples number and denoted by 'i'
$z = a + ib \rightarrow$ Imaginary part
↓
Real part
Conjugate of a complex number : For a given complex number $r = a + ib$, its conjugate is defined as $\bar{z} = a - ib$

Algebra of complex Numbers

Let: $z_1 = a + ib$ and $z_1 = c + id$ be two complex numbers, where $a, b, c, d \in R$ and $i = \sqrt{-1}$
1. Addition: $z_1 + z_2 = (a + ib) + (c + id) = (a + c) + i(b + d)$
2. Subtraction: $z_1 - z_2 = (a + ib) - (c + id) = (a - c) + (b - d)i$
3. Multiplication: $z_1.z_2 = (a + ib)(c + id)$
$= a(c + id) + ib(c + id)$
$= (ac - bd) + (ad + bc)i \quad (\because i^2 = -1)$
4. Division: $\frac{z_1}{z_1} = \frac{a + ib}{c + id} \cdot \frac{c - id}{c - id}$
$= \left(\frac{ac + bd}{c^2 + d^2}\right) + \left(\frac{bc - ad}{c^2 + d^2}\right)i$
Note : If $a + ib = c + id$
$\Rightarrow d = c \ \&\ d = d$

VERY SHORT ANSWER TYPE QUESTIONS

1. Write the value of $i + i^{10} + i^{20} + i^{30}$

2. Write the additive Inverse of $6i - i\sqrt{-49}$

3. Write the multiplicative Inverse of $1 + 4\sqrt{3}\ i$

4. Write the conjugate of $\dfrac{2-i}{(1-2i)^2}$

5. Write in the form of $a + ib$: $\dfrac{1}{-2+\sqrt{-3}}$

6. Multiply $2 - 3\,i$ by its conjugate.

7. What is the least integral value of k which makes the roots of the equation $x^2 + 5x + k = 0$ imaginary?

8. Find the real value of 'a' for which $3i^3 - 2ai^2 + (1-a)i$ is real.

9. Find the value of $\left(-\sqrt{-1}\right)^{4n-3}$, when n $\in$ N.

10. If a complex number lies in the third quadrant, then find the quadrant of it's conjugate.

11. Find the value of $\sqrt{-25} \times \sqrt{-9}$

12. Evaluate :

    (i) $\sqrt{-16} + 3\sqrt{-25} + \sqrt{-36} - \sqrt{-625}$

    (ii) $i\sqrt{-16} + i\sqrt{-25} + \sqrt{49} - i\sqrt{-49} + 14$

    (iii) $\left(i^{77} + i^{70} + i^{87} + i^{414}\right)^3$

    (iv) $\dfrac{(3+\sqrt{5i})(3-\sqrt{5}i)}{(\sqrt{3}+\sqrt{2}i)-(\sqrt{3}-\sqrt{2}i)}$

13. Find x and y if (x + iy) (2 – 3i) = 4 + i.

14. If n is any positive integer, write value of $\frac{i^{4n+1} - i^{4n-1}}{2}$

15. If $z_1 = \sqrt{2}(\cos 30° + i \sin 30°)$, $z_2 = \sqrt{3}(\cos 60 + i \sin 30°)$

Find Re $(z_1 z_2)$

16. If $|z+4| \leq 3$ then find the greatest and least values of $|z+1|$.

17. Find the real value of a for which $3i^3 - 2ai^2 + (1-a)i + 5$ is real.

SHORT ANSWER TYPE QUESTIONS

18. If x + iy = $\sqrt{\frac{1+i}{1-i}}$ prove that $x^2 + y^2 = 1$

19. Find real value of θ such that, $\frac{1+i\cos\theta}{1-2i\cos\theta}$ is a real number.

20. If $\left|\frac{z-5i}{z+5i}\right| = 1$ show that z is a real number.

21. Find real value of x and y if $\frac{(1+i)x-2i}{3+i} + \frac{(2-3i)y+i}{3-i} = i$.

22. If $(1+i)(1+2i)(1+3i).......(1+ni) = x+iy$.

Show, $2.5.10..........(1+n^2) = x^2 + y^2$

23. If z = 2 – 3i show that $z^2 - 4z + 13 = 0$, hence find the value of $4z^3 - 3z^2 + 169$.

24. If $\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = a + ib$, find a and b.

25. For complex numbers $z_1 = 6 + 3i$, $z_2 = 3 - i$ find $\frac{z_1}{z_2}$.

26. If $\left(\frac{2+2i}{2-2i}\right)^n = 1$, find the least positive integral value of n

27. If $(x+iy)^{\frac{1}{3}} = a+ib$ prove $\left(\frac{x}{a}+\frac{y}{b}\right) = 4\left(a^2-b^2\right)$.

28. Solve

(i) $x^2 - \left(3\sqrt{2}-2i\right)x - 6\sqrt{2}i = 0$ (ii) $ix^2 - 4x - 4i = 0$

29. Solve $|z + 1| = z + 2(1 + i)$

30. If $|z^2-1| = |z|^2+1$, then show that z lies on imaginary axis.

[Hint: Take $z = x + iy$]

31. Show that $\left|\frac{z-2}{z-3}\right| = 2$ represent a circle find its centre and radius.

32. Find all non-zero complex number z satisfying $\bar{z} = iz^2$.

33. If $iz^3 + z^2 - z + i = 0$ then show that $| z | = 1$.

34. If z_1, z_2 are complex numbers such that, $\frac{2z_1}{3z_2}$ is purely imaginary number then find $\left|\frac{z_1-z_2}{z_1+z_2}\right|$.

35. If z_1 and z_2 are complex numbers such that,

$\left|1-\bar{z}_1 z_2\right|^2 - \left|z_1 - z_{2^2}\right|^2 = k\left(1-\left|z_1\right|^2\right)\left(1-\left|z_2\right|^2\right)$. Find value of k.

LONG ANSWER TYPE QUESTIONS

36. Find number of solutions of $z^2 + |z|^2 = 0$.

37. If z_1, z_2 are complex numbers such that $\left|\frac{z_1 - 3z_2}{3 - z_1.\bar{z}_2}\right| = 1$ *and* $|z_2| \neq 1$ then find $|z_1|$.

38. Evaluate $x^4 - 4x^3 + 4x^2 + 8x + 44$, When $x = 3 + 2i$

39. If $z = x + iy$ and $w = \frac{1 - iz}{z - i}$ show that if $|w| = 1$ then z is purely real.

40. If $\left(\frac{1+i}{1+2^2 i}\right) \times \left(\frac{1+3^2 i}{1+4^2 i}\right) \times \ldots\ldots.. \times \left(\frac{1+(2n-1)^2 i}{1+(2n)^2 i}\right) = \frac{a+ib}{c+id}$ then show that $\frac{2}{17} \times \frac{82}{257} \times \ldots\ldots. \times \frac{1+(2n-1)^4}{1+(2n)^4} = \frac{a^2+b^2}{c^2+d^2}$.

41. Find the values of x and y for which complex numbers $-3 + ix^2y$ and $x^2 + y + 4i$ are conjugate to each other.

42. Show that the complex number z_1, z_2 and z_3 satisfying $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2}$ are the vertices of a equilateral triangle.

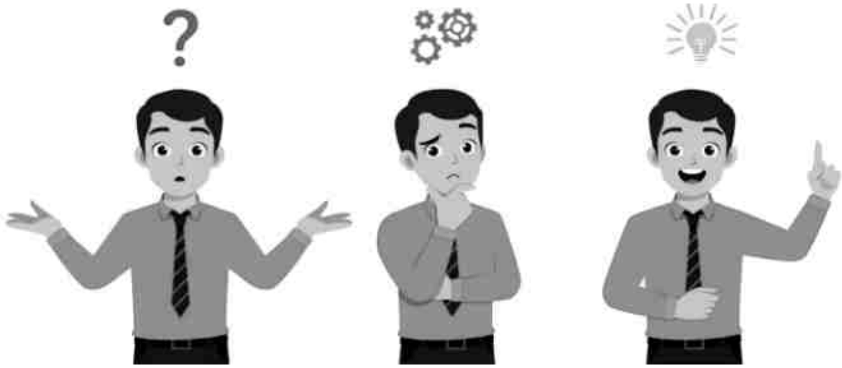
43. If $f(z) = \frac{7 - z}{1 - z^2}$ where $z = 1 + 2i$ then show that $|f(z)| = \frac{|z|}{2}$.

44. If z_1, z_2, z_3 are complex numbers such that

$|z_1| = |z_2| = |z_3| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1$ then find the value of $|z_1 + z_2 + z_3|$

CASE STUDY TYPE QUESTIONS

45. While solving a typical equation a person finds that one of the root of the equation is a complex number $z = \frac{1+2i}{1-3i}$, help him to find



i. The standard form of z

(a) $-\frac{1}{2}+\frac{i}{2}$ (b) $\frac{1}{2}-\frac{i}{2}$ (c) $-\frac{1}{2}-\frac{i}{2}$ (d) $\frac{1}{2}+\frac{i}{2}$

ii. If $z = 2x+(4-y)i,$ then

(a) $x=\frac{1}{4},\ y=\frac{7}{2}$ (b) $x=-\frac{1}{4},\ y=\frac{7}{2}$

(c) $x=\frac{1}{4},\ y=-\frac{7}{2}$ (d) $x=-\frac{1}{4},\ y=-\frac{7}{2}$

iii. Conjugate of Z is

(a) $\frac{1-2i}{1-3i}$ (b) $\frac{1+2i}{1+3i}$ (c) $\frac{1+2i}{1-3i}$ (d) $\frac{1-2i}{1+3i}$

iv. The modulus of z is

(a) 1/3 (b) 1/2 (c) $1/\sqrt{2}$ (d) $1/\sqrt{3}$

v. z lies in

(a) I quadrant (b) II quadrant

(c) III quadrant (d) IV quadrant

Multiple Choice Questions

46. $(\sqrt{-2})(\sqrt{3})$ is equal to

(a) $\sqrt{6}$ (b) $-\sqrt{6}$

(c) $i\sqrt{6}$ (d) None of these

47. If $\frac{(a^2+1)^2}{2a-i} = x+iy$, x^2+y^2 is equal to

(a) $\frac{(a^2+1)^4}{4a^2+1}$ (b) $\frac{(a+1)^2}{4a^2+1}$

(c) $\frac{(a^2-1)^2}{(4a^2-1)^2}$ (d) None of these

48. If z = $\frac{1}{1-\cos\theta - i\sin\theta}$, then Re(z) =

(a) 0 (b) ½

(c) cot θ/2 (d) ½ cot θ /2

49. If f(z) = $\frac{7-z}{1-z^2}$, where z = 1 + 2i, then |f(z)| is

(a) $\frac{|z|}{2}$ (b) |z|

(c) 2|z| (d) None of these

50. The value of $(1+i)^4+(1-i)^4$ is

(a) 8 (b) 4

(c) – 8 (d) – 4

51. The equation $|z+1-i|=|z-1+i|$ represent a

(a) Straight line (b) Circle

(c) Parabola (d) Hyperbola

52. The value of $\frac{i^{4n+1}-i^{4n-1}}{2}$ is

(a) 2i (b) –2i

(c) i (d) –i

53. If three complex number z_1, z_2 and z_3 are in A.P, then points representing them lies on

(a) Circle (b) Parabola

(c) Hyperbola (d) Straight line

54. The sum of series $i+i^2+i^3+...$ up to 1000 terms is

(a) 0 (b) *i*

(c) –*i* (d) None of these

55. If $z_1=\sqrt{3}+i\sqrt{3}, z_2=\sqrt{3}+i$, then the point $\frac{z_1}{z_2}$ lies in

(a) I quadrant (b) II quadrant

(c) III quadrant (d) IV quadrant

56. If $i=\sqrt{-1}$ then $1 + i^2 + i^3 - i^6 + i^8$ is equal to

(a) $2 - i$ (b) 1

(c) 3 (d) – 1

57. The complex number $\frac{1+2i}{1-i}$ lies in which of the complex plane

(a) First (b) Second

(c) Third (d) Fourth

58. If $\frac{c+i}{c-i} = a + ib$ where a, b, c are real, than $a^2 + b^2 =$

(a) 1 (b) – 1

(c) c^2 (d) $-c^2$

59. If the conjugate of $(x + iy)(1 - 2i)$ be $1 + i$, then

(a) $x = \frac{1}{5}$ (b) $y = \frac{3}{5}$

(c) $x + iy = \frac{1-i}{1-2i}$ (d) $x - iy = \frac{1-i}{1+2i}$

60. If $|z^2 - 1| = |z|^2 + 1$, then z lies on

(a) An ellipse (b) The imaginary axis

(c) A circle (d) The rea axis

61. If $z = 1 + i$, then the multiplicative inverse of z^2 is (where $i = \sqrt{-1}$)

(a) $2i$ (b) $1 - i$

(c) $-i/2$ (d) $i/2$

Directions: Each of these questions contains two statements, Assertion and Reason. Each of these questions also has four alternative choices, only one which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.

(a) Assertion is correct, reason is correct, reason is a correct explanation for assertion.

(b) Assertion is correct, reason is correct, reason is not a correct explanation for assertion.

(c) Assertin is correct, reason is incorrect.

(d) Assertion is incorrect, reason is correct.

62. **Assertion:** If$|z_1 + z_2|^2 = |z_1|^2$ |+ $|z_2|^2$, then $\frac{z_1}{z_2}$ is purely imaginary.

Reason: If z is purely imaginary, then $z + \overline{z} = 0$

63. **Assertion:** Consider z_1 and z_2 are two complex numbers such that $|z_1| = |z_2| + |z_1 - z_2|$, then $\text{Im}\left(\frac{z_1}{z_2}\right) = 0.$

Reason: arg(z) = 0 $\Rightarrow$ z is purely real.

64. **Assertion:** If P and Q are the points in the plant XOY representing the complex numbers z_1 and z_2 respectively then distance |PQ| = $|Z_2 - Z_1|$

Reason: Locus of the point P(z) satisfying |z – (2 + 3i)| = 4 is a straight line.

65. **Assertion:** The equation $ix^2 - 3ix + 2i = 0$ has non-real roots.

Reason: If *a*, *b*, *c* are real and $b^2 - 4ac \geq 0$, then the roots of the equation $ax^2 + bx + c = 0$ are real and if $b^2 - 4ac < 0$, then roots of $ax^2 + bx + c = 0$ are non-real.

ANSWERS

1. $-1 + i$
2. $-7 - 6i$
3. $\frac{1}{49} - \frac{4\sqrt{3}i}{49}$
4. $\frac{-2}{25} + \frac{11i}{25}$
5. $-\frac{2}{7} - \frac{i\sqrt{3}}{7}$
6. 13
7. 7
8. –2
9. $-i$
10. First
11. –15
12. (i) 0 (ii) 19

(iii) –8 (iv) $\frac{-7}{\sqrt{2}}i$

13. $x = \frac{5}{13}, \quad y = \frac{14}{13}$

14. i

15. 0 (zero)

16. 6 and zero

17. a = –2

19. $\theta = (2n+1)\frac{\pi}{2}$

21. x = 3, y = –1

23. zero

24. a = 0, b = –2

25. $\frac{z_1}{z_2} = \frac{3(1+i)}{2}$

26. n = 4

28. (i) $3\sqrt{2}$ and $-2i$ (ii) –2i, –2i

29. ½, –2i

31. radius= $\frac{2}{3}$

32. $z = 0,\ i,\ \frac{\sqrt{3}}{2} - \frac{1}{2}i,\ -\frac{\sqrt{3}}{2} - \frac{1}{2}i$

34. 1

35. K=1

36. Infinitely many solutions of the form $z = 0 \pm iy;\ y \in R$

37. $|z_1| = \sqrt{x^2 + y^2}$ [Hint: $|z|^2 = z \times \bar{z}$]

38. 5

40. When x = 1, y = –4 or x = –1, y = –4

41. 1 (one)

44. 1 [Hint: $|z|^2 = z \times \bar{z}$, $|\bar{z}| = |z|$]

45. i. (a)	ii. (b)	iii. (d)	iv. (c)	v. (b)

46. (b)	47. (a)	48. (b)	49. (a)
50. (c)	51. (a)	52. (c)	53. (d)
54. (a)	55. (d)	56. (a)	57. (b)
58. (a)	59. (c)	60. (b)	61. (c)
62. (b)	63. (a)	64. (c)	65. (d)