# CHAPTER - 3

# TRIGONOMETRIC FUNCTIONS

## KEY POINTS

- 1 radian is an angle subtended at the centre of a circle by an arc whose length is equal to the radius of the circle.
- $\pi$ radian=180 degree

$$1 \text{ radian} = \left(\frac{180}{\pi}\right)^{\circ} = 57^{\circ}\ 16' \ 22'' \text{ (Appr.)}$$

- If an arc of length '$\ell$' makes an angle '$\theta$' radian at the centre of a circle of radius 'r', we have $\theta = \frac{\ell}{r}$.
- I degree is $\left(\frac{1}{360}\right)^{th}$ part of a circle. One degree is further divided into 60 parts called minutes and one minute is further divided into 60 parts called seconds.
- 360° = one complete revolution
- 1° = 60′ (minutes)
- 1′ = 60′′(second)

| Quadrant → | I | II | III | IV |
|---|---|---|---|---|
| t-functions which are postive | All | sinx<br>cosecx | tanx<br>cotx | cosx<br>secx |

| Function | Domain | Range |
|---|---|---|
| Sinx | R | [–1, 1] |
| Cosx | R | [–1, 1] |
| Tanx | $R - \left\{(2n+1)\frac{\pi}{2}\right\};\ n \in z$ | R |
| cosecx | $R - \{n\pi\};\ n \in z$ | R – (–1, 1) |
| Secx | $R - \left\{(2n+1)\frac{\pi}{2}\right\};\ n \in z$ | R – (–1, 1) |
| Cotx | $R - \{n\pi\};\ n \in z$ | R |

- **Trigonometric Identities:**

(i) $\sin(x + y) = \sin x \cos y + \cos x \sin y$

(ii) $\sin(x - y) = \sin x \cos y - \cos x \sin y$

(iii) $\cos(x + y) = \cos x \cos y - \sin x \sin y$

(iv) $\cos(x - y) = \cos x \cos y + \sin x \sin y$

(v) $\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x . \tan y}$

(vi) $\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x . \tan y}$

(vii) $\cot(x + y) = \frac{\cot x . \cot y - 1}{\cot y + \cot x}$

(viii) $\cot(x - y) = \frac{\cot x . \cot y + 1}{\cot y - \cot x}$

(ix) $\sin 2x = 2\sin x \cos x = \dfrac{2\tan x}{1+\tan^2 x}$

(x) $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$

(xi) $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

(xii) $\sin 3x = 3\sin x - 4\sin^3 x$

(xiii) $\cos 3x = 4\cos^3 x - 3\cos x$

(xiv) $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1-3\tan^2 x}$

(xv) $\cos x + \cos y = 2\cos\dfrac{x+y}{2}\cos\dfrac{x-y}{2}$

(xvi) $\cos x - \cos y = 2\sin\dfrac{x+y}{2}\sin\dfrac{y-x}{2}$

(xvii) $\sin x + \sin y = 2\sin\dfrac{x+y}{2}\cos\dfrac{x-y}{2}$

(xviii) $\sin x - \sin y = 2\cos\dfrac{x+y}{2}\sin\dfrac{x-y}{2}$

(xix) $2\sin x \cos y = \sin(x+y) + \sin(x-y)$

(xx) $2\cos x \sin y = \sin(x+y) - \sin(x-y)$

(xxi) $2\cos x \cos y = \cos(x+y) + \cos(x-y)$

(xxii) $2\sin x \sin y = \cos(x-y) - \cos(x+y)$

(xxiii) $\sin\dfrac{A}{2} = \pm\sqrt{\dfrac{1-\cos A}{2}}$

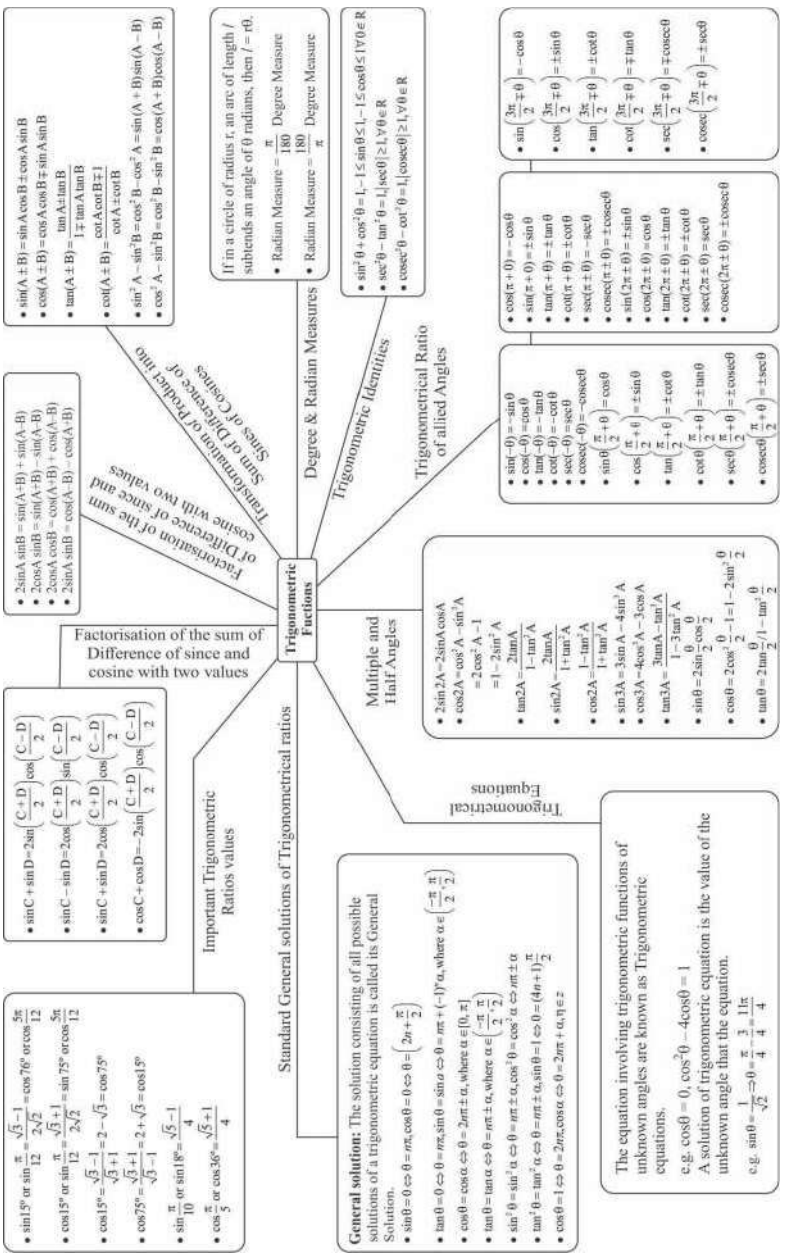
(xxiv) $\cos\dfrac{A}{2} = \pm\sqrt{\dfrac{1+\cos A}{2}}$

(xxv) $\tan\dfrac{A}{2} = \pm\sqrt{\dfrac{1-\cos A}{1+\cos A}}$

**sign '+' or '–' will be decided according to the quadrant in which angle lies.**

- Maximum and minimum values of the expression $A\cos\theta + B\sin\theta$ are $\sqrt{A^2+B^2}$ and $-\sqrt{A^2+B^2}$ respectively, where A and B are constants.

**MIND MAP**



## VERY SHORT ANSWER TYPE QUESTIONS

1. Write the radian measure of $5^\circ$ 37' 30".
2. Write the degree measure of $\frac{11}{16}$ radian.
3. Write the value of $\tan\left(\frac{19\pi}{3}\right)$.
4. What is the value of sin (–1125°).
5. Write the value of 2sin75° sin15°.
6. What is the maximum value of 3 – 7cos5x.
7. Express sin12$\theta$ + sin4$\theta$ as the product of sines and cosines.
8. Express 2cos4x sin2x as an algebraic sum of sines and cosines.
9. Write the maximum value of cos(cosx) and also write its minimum value.
10. Write is the value of $\tan\frac{\pi}{12}$.

## SHORT ANSWER TYPE QUESTIONS

11. Find the length of an arc of a circle of radius 5cm subtending a central angle measuring 15°.
12. If $\sin A = \frac{3}{5}$ and $\frac{\pi}{2} < A < \pi$ Find $\cos A$, $\sin 2A$.
13. What is the sign of cos x/2 – sin x/2 when

    (i) $0 < x < \pi/4$ (ii) $\frac{\pi}{2} < x < \pi$
14. Prove that cos 510° cos 330° + sin 390° cos 120° = –1.
15. Find the maximum and minimum value of $7 \cos x + 24 \sin x$.
16. Evaluate $\sin(\pi + x) \sin(\pi - x) \operatorname{cosec}^2 x$.
17. Find the angle in radians between the hands of a clock at 7 : 20 PM.

18. If $\cot\alpha = \frac{1}{2}$, $\sec\beta = \frac{-5}{3}$ where $\pi<\alpha< 3\pi/2$ and $\frac{\pi}{2} < \beta < \pi$. Find the value of $\tan(\alpha + \beta)$.

19. If $\cos x = \frac{-1}{3}$ and $\pi < x < \frac{3\pi}{2}$. Find the value of $\cos x/2$ , $\tan x/2$

20. If $\tan A = \frac{a}{a+1}$ and $\tan B = \frac{1}{2a+1}$ then find the value of $A + B$

21. A horse is tied to a post by a rope. If the horse moves along a circular path, always keeping the rope tight and describes 88 metres when it traces 72° at the centre, find the length of the rope.

22. Find the minimum and maximum value of $\sin^4 x + \cos^2 x$; $x \in R$

23. Find $x$ if $\tan(x - 15°) = \tan(x + 15°)$

24. If $\sec x = \sqrt{2}$ and $\frac{3\pi}{2} < x < 2\pi$ , find the value of $\frac{1 - \tan x - \operatorname{cosec} x}{1 - \cot x - \operatorname{cosec} x}$

25. If $f(x) = \frac{\cot x}{1 + \cot x}$ and $\alpha + \beta = \frac{5\pi}{4}$ then find $f(\alpha).f(\beta)$.

26. Prove that $\tan 70° = \tan 20° + 2\tan 50°$

27. Prove that $\tan 13x = \tan 4x + \tan 9x + \tan 4x \tan 9x \tan 13x$.

[Hint: $13x = 9x + 4x$]

## Prove the following Identities

**28.** $\frac{\tan 5\theta + \tan 3\theta}{\tan 5\theta - \tan 3\theta} = 4\cos 2\theta.\cos 4\theta$ .[Hint: Break into sin and cos]

**29.** $\frac{\cos x + \sin x}{\cos x - \sin x} - \frac{\cos x - \sin x}{\cos x + \sin x} = 2\tan 2x$ .

**30.** $\dfrac{\cos 4x \sin 3x - \cos 2x \sin x}{\sin 4x . \sin x + \cos 6x . \cos x} = \tan 2x$ .

[Hint: Transformation formula form product to sum or different]

**31.** $\dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = \tan\dfrac{\theta}{2}$ .[Hint: Use half angle formula]

**32.** $\tan\alpha.\ \tan(60° - \alpha).\ \tan(60° + \alpha) = \tan 3\alpha.$

**33.** $\sqrt{2+\sqrt{2+2\cos 4\theta}} = 2\cos\theta$ .[Hint: Use half angle formula]

**34.** $\dfrac{\cos x}{1-\sin x} = \tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)$.

**35.** $\cos 10° + \cos 110° + \cos 130° = 0.$

**36.** $\dfrac{\sin(x+y)-2\sin x+\sin(x-y)}{\cos(x+y)-2\cos x+\cos(x-y)} = \tan x$

**37.** $\sin x + \sin 2x + \sin 4x + \sin 5x = 4\cos\dfrac{x}{2}.\cos\dfrac{3x}{2}.\sin 3x$

[Hint: Use transformation formula from sum of product.]

**38.** $\dfrac{\sec 8\theta - 1}{\sec 4\theta - 1} = \dfrac{\tan 8\theta}{\tan 2\theta}$

**39.** Find the value of $\sqrt{3}$ cosec 20° – sec 20°

**40.** Draw the graph of cos$x$, sin$x$ and tan$x$ in [0, 2$\pi$].

41. Draw sin$x$, sin2$x$ and sin3$x$ on same graph and with same scale.

42. Evaluate: $\tan\left(\dfrac{13\pi}{12}\right)$

43. If $\tan A - \tan B = x$, $\cot B - \cot A = y$ prove that $\cot (A - B) = \frac{1}{x} + \frac{1}{y}$

44. If $\frac{\sin(x+y)}{\sin(x-y)} = \frac{a+b}{a-b}$ then prove that $\frac{\tan x}{\tan y} = \frac{a}{b}$.

45. Find the range of $5 \sin x - 12 \cos x + 7$.

46. Show that $\cos^2 + \cos^2\left(x + \frac{2\pi}{3}\right) + \cos^2\left(x - \frac{2\pi}{3}\right) = \frac{3}{2}$

[Hint: Use $\cos 2\theta = 2\cos^2\theta - 1$]

47. Show that $\sin\alpha + \sin\beta + \sin\gamma - \sin(\alpha + \beta + \gamma)$

$= 4\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\beta+\gamma}{2}\right)\sin\left(\frac{\alpha+\gamma}{2}\right)$

[Hint: Use transformation formula sum to product]

## Long Answer Type Questions

48. Find $\cos \pi/8$

49. Prove that $\sin 10° \sin 30° \sin 50° \sin 70° = \frac{1}{16}$.

50. $\cos\frac{\pi}{5}.\cos\frac{2\pi}{5}.\cos\frac{4\pi}{5}.\cos\frac{8\pi}{5} = \frac{1}{16}$

51. $\cos 20°.\cos 40°.\cos 80° = \frac{1}{8}$

52. Evaluate: $\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\left(\frac{5\pi}{8}\right) + \cos^4\left(\frac{7\pi}{8}\right)$

[Hint: Use $\cos 2\theta = 2\cos^2\theta - 1$]

53. If $\cos x = \cos\alpha.\cos\beta$ then prove that $\tan\left(\frac{x+\alpha}{2}\right).\tan\left(\frac{x-\alpha}{2}\right) = \tan^2\frac{\beta}{2}$

54. If $\tan(\pi\cos\theta) = \cot(\pi\sin\theta)$ then prove that $\cos\left(\theta - \frac{\pi}{4}\right) = \pm\frac{1}{2\sqrt{2}}$.

55. If $\sin(\theta+\alpha) = a$ and $\sin(\theta+\beta) = b$ then prove that

$$\cos 2(\alpha-\beta) - 4ab\cos(\alpha-\beta) = 1 - 2a^2 - 2b^2$$

56. If $\alpha$ and $\beta$ are the solution of the equation, $a\tan\theta + b\sec\theta = c$, then show that $\tan(\alpha+\beta) = \frac{2ac}{a^2-c^2}$.

57. Prove that

$$\cos^2 x + \cos^2 y - 2\cos x.\cos y.\cos(x+y) = \sin^2(x+y)$$

58. Prove that :

$$2\sin^2\beta + 4\cos(\alpha+\beta)\sin\alpha\sin\beta + \cos 2(\alpha+\beta) = \cos 2\alpha$$

59. Prove that : $\cos A\cos 2A\cos 4A\cos 8A = \frac{\sin 16A}{16.\sin A}$.

[Hint: Use transformation formula]

60. Evaluate: $\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)$

61. Prove that : $4\sin\alpha.\sin\left(\alpha+\frac{\pi}{3}\right).\sin\left(\alpha+\frac{2\pi}{3}\right) = \sin 3\alpha$.

[Hint: Use transformation formula of product to sum or diff.]

62. If sin A + sin B = $p$, cosA + cosB = q show that

(i) $\sin(A+B) = \frac{2pq}{p^2+q^2}$

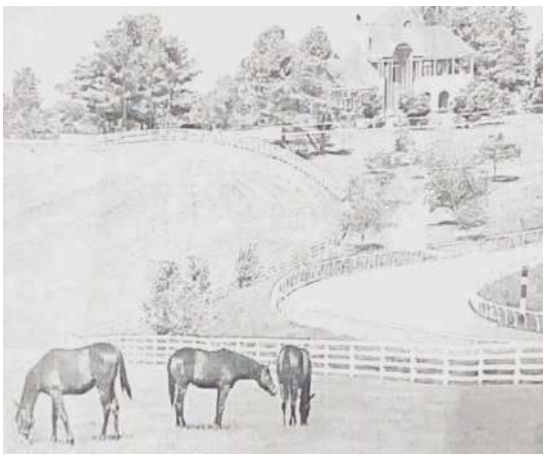
(ii) $\cos(A+B) = \frac{p^2-q^2}{p^2+q^2}$

(iii) $\tan(A+B) = \frac{p^2+q^2}{p^2-q^2}$

63. Show that $\sin^3 x + \sin^3\left(\frac{2\pi}{3} + x\right) + \sin^3\left(\frac{4\pi}{3} + x\right) = \frac{-3}{4}\sin 3x$

## CASE STUDY TYPE QUESTIONS

64. After retirement, Mr. D. N. Sharma purchased a farm house in shape of quadrilateral ABCD with $\angle A = 90°$, $\angle B = 72°$, $\angle C = 108°$ and $\angle D = 90°$. He also purchased a horse and cow. One day, he tied the horse with a rope at vertex B and oserved that it describes an arc of length 88 m when it moves along a circular path keeping the rope tight.



**Based on above information answer the following :-**

i. What is radian measure of $\angle B$ ?

ii. What is length of rope?

iii. What will be the length of arc described by horse if he doubles the rope length?

iv. What will be the length of arc described by cow if it is tied at vertex c with the rope of same length as horse?

65. While playing with this nephew Shashank, Mr. V.S. Malik observes a vertical pole in park. A wire is tied from top of pole to a point on ground level. Mr. Malik asks Shashank some

mathematics related questions. Mr. Shashank is Class-XI student and very intelligent in Maths. Using some tools he measure the distance of point at ground where wire is tied as 10 m. and angle between wire and ground level as 75°.

**Based on above information answer the following :-**

i. What is the value of tan75° ?

(a) $\frac{\sqrt{3}-1}{\sqrt{3}+1}$ (b) $\frac{\sqrt{3}+1}{\sqrt{3}-1}$

(c) $\frac{\sqrt{3}}{\sqrt{3}+1}$ (d) $\frac{\sqrt{3}}{\sqrt{3}-1}$

ii. What is the height of pole?

(a) $10\left(\sqrt{3}+1\right)$ (b) $10\left(\sqrt{3}-1\right)$

(c) $10\frac{\sqrt{3}+1}{\sqrt{3}-1}$ (d) $10\frac{\sqrt{3}-1}{\sqrt{3}+1}$

iii. What is the value of sin75° ?

(a) $\frac{\sqrt{3}+1}{\sqrt{3}-1}$ (b) $\frac{\sqrt{3}-1}{2\sqrt{2}}$ (c) $\frac{\sqrt{3}+1}{2\sqrt{2}}$ (d) $\frac{\sqrt{3}-1}{\sqrt{3}+1}$

iv. What is the length of wire?

(a) $10\sqrt{2}\left(\sqrt{3}+1\right)$ (b) $10\left(\sqrt{3}+1\right)$

(c) $10\sqrt{2}\left(\sqrt{3}-1\right)$ (d) $10\left(\sqrt{3}-1\right)$

iii. What is the value of sin105° ?

(a) $\frac{\sqrt{3}+1}{\sqrt{3}-1}$ (b) $\frac{\sqrt{3}-1}{2\sqrt{2}}$ (c) $\frac{\sqrt{3}+1}{2\sqrt{2}}$ (d) $\frac{\sqrt{3}-1}{\sqrt{3}+1}$

66. The greatest value of sinx cosx is -

(a) 1 (b) 2

(c) $\sqrt{2}$ (d) 1/2

67. The value of tan1° × tan2° × tan3° …….…. tan89° is -

(a) 0 (b) 1

(c) $\frac{1}{2}$ (d) Not defined.

68. The value of cos1° × cos2° × cos3° …….…. cos179° is -

(a) $\frac{1}{\sqrt{2}}$ (b) 0 (c)1 (d) –1.

69. The value of $\frac{1-\tan^2 15°}{1+\tan^2 15°}$ is -

(a) 1 (b) $\sqrt{3}$

(c) $\frac{\sqrt{3}}{2}$ (d) 2.

70. The value of sin50° – sin70° + sin10° is equal to -

(a) 1 (b) 0

(c) $\frac{1}{2}$ (d) 2.

71. If $\sin\theta + \cos\theta = 1$, then the value of $\sin 2\theta$ is equal to -

(a) 1 (b) $\frac{1}{2}$

(c) 0 (d) 2.

72. If $\alpha + \beta = \frac{\pi}{4}$, then value of $(1 + \tan\alpha) \cdot (1 + \tan\beta)$ is -

(a) 1 (b) 2

(c) –2 (d) Not defined.

73. If $\cos x = \frac{1}{2}\left(a + \frac{1}{a}\right)$, then $\cos 3x$ is -

(a) $\frac{1}{2}\left(a^3 + \frac{1}{a^3}\right)$ (b) $\frac{3}{2}\left(a^3 + \frac{1}{a^3}\right)$

(c) $\frac{1}{2}\left(a^3 - \frac{1}{a^3}\right)$ (d) $\frac{3}{2}\left(a^3 - \frac{1}{a^3}\right)$.

74. If $P = 2\sin^2 x - \cos^2 x$, then P lies in the interval -

(a) [1, 3] (b) [1, 2]

(c) [–1, 2] (d) None of these.

75. If $\frac{\pi}{4} < x < \frac{\pi}{2}$, then write the value of $\sqrt{1 - \sin 2x}$ is -

(a) $\cos x - \sin x$ (b) $\cos x + \sin x$

(c) $\sin x - \cos x$ (d) 2.

76. If $\sin\theta = -\frac{1}{\sqrt{2}}$ and $\tan\theta = 1$, then $\theta$ lies in which quadrant

(a) First (b) Second

(c) Third (d) Fourth

77. If $\sin(\alpha-\beta)=\frac{1}{2}$ and $\cos(\alpha+\beta)=\frac{1}{2}$, where, $\alpha$ and $\beta$ are positive acute angle, then

(a) $\alpha = 45°, \beta = 15°$ (b) $\alpha = 15°, \beta = 45°$

(c) $\alpha = 60°, \beta = 15°$ (d) None of these

78. If $\tan\theta=-\frac{1}{\sqrt{10}}$ and $\theta$ lies in the fourth quadrant, then $\cos\theta$ =

(a) $1/\sqrt{11}$ (b) $-1/\sqrt{11}$

(c) $\sqrt{\frac{10}{11}}$ (d) $-\sqrt{\frac{10}{11}}$

79. tan 15° =

(a) $\frac{1}{3}$ (b) $\sqrt{3}-2$

(c) $2-\sqrt{3}$ (d) None of these

80. If $\sin\alpha=\frac{-3}{5}$, where, $\pi < a < \frac{3\pi}{2}$, then $\cos\frac{1}{2}\alpha =$

(a) $\frac{-1}{\sqrt{10}}$ (b) $\frac{1}{\sqrt{10}}$

(c) $\frac{3}{\sqrt{10}}$ (d) $\frac{-3}{\sqrt{10}}$

81. Which of the following number(s) is/are rational

(a) sin 15° (b) cos 15°

(c) sin 15° cos 15° (d) sin 15° cos 75°

**Directions:** Each of these questions contains two statements, Assertion and Reason. Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.

(a) Assertion is correct, reason is correct: reason is a correct explanation for assertion.
(b) Assertion is correct, reason is correct; reason is not a correct explanation for assertion.
(c) Assertion is correct, reason is incorrect.
(d) Assertion is incorrect, reason is correct.

82. **Assertion:** $\frac{\cos(\pi + x).\cos(-x)}{\sin(\pi - x).\cos\left(\frac{\pi}{2} + x\right)} = \cot^2 x$

**Reason:** cos ($\pi$ + $\theta$) = – cos $\theta$ and cos (–$\theta$) = cos $\theta$. Also sin ($\pi$ – $\theta$) = sin $\theta$ and sin (– $\theta$) = – sin $\theta$.

83. **Assertion:** coses *x* is negative in third and fourth quadrants.

**Reason:** cot *x* decreases from 0 to – $\infty$ in first quadrant and increases from 0 to $\infty$ in third quadrant.

84. **Assertion:**The degree measure corresponding to (–2) radian is – 114° 19 min.

**Reason:** The degree measure of a given radian measure = $\frac{180}{\pi}$ × Radian measure.

85. **Assertion:** The ratio of the radii of two circles at the centres of which two equal arcs subtend angles of 30° and 70° is 21 : 10.

**Reason:** Number of radians in an angle subtended at the centre of a circle by an arc is equal to the ratio of the length of the arc to the radius of the circle.

# ANSWERS

1. $\frac{\pi}{32}$

2. 39° 22' 30"

3. $\sqrt{3}$

4. $-\frac{1}{\sqrt{2}}$

5. $\frac{1}{2}$

6. 10

7. 2 sin8θ cos4θ

8. $\sin 6x - \sin 2x$

9. 1 and –1

10. $\frac{\sqrt{3}-1}{\sqrt{3}+1}$

11. 70m

12. $\frac{-4}{5}$, $\frac{-24}{25}$ [Hint: For f(x) = a sin θ + b cos θ Max value = $\sqrt{a^2+b^2}$ – Min value = – $\sqrt{a^2+b^2}$ ]

13. (i) +ve (ii) –ve

15. Max value 25;
Min value –25
[Hint: For f(x) = a sin q + b cos q Max value = $\sqrt{a^2+b^2}$ Min value = $\sqrt{a^2+b^2}$ ]

16. –1

17. $\frac{5\pi}{9}$

18. $\frac{2}{11}$

19. $-1/\sqrt{3}$, $-2$

20. π/4

21. 70 m

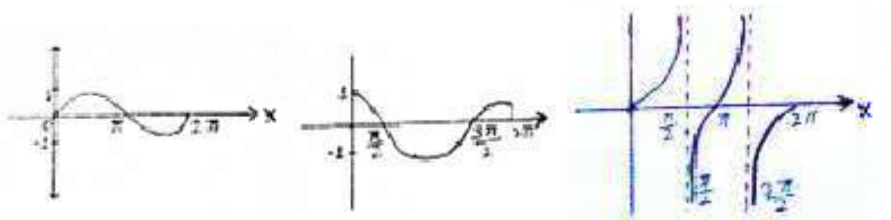
22. min = $\frac{3}{4}$, max = 1

23.30° [Hint: Break into sin and cos and use sin(A – B)]

24.1 [Hint: Break into sum and cos and rationalise]

25. $\frac{1}{2}$ 39. 4

40.



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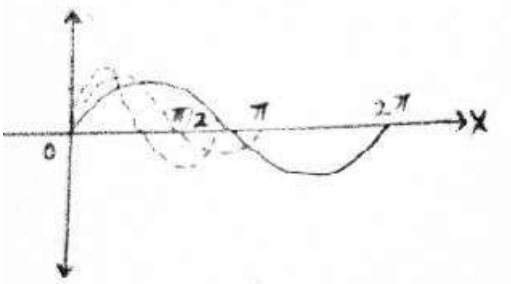

42.(i) $2-\sqrt{3}$ 52. $\frac{3}{2}$

45.[–6, 20] 48. $=\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$

60. $\frac{1}{8}$

64. i. $2\pi/5$ ii. 70 m iii. 176 m iv. 144 m

65. i. (b) ii. (c) iii. (c) iv. (a) v. (c)

66. (d) 67. (c) 68. (c)

69. (c) 70. (b) 72. (c)

73. (a) 74. (c) 75. (c)

76. (c) 77. (a) 78. (c)

79. (c) 80. (a) 81. (c)

82. (a) 83. (c) 84. (d)

85. (d)