VECTOR

1. Let P be the plane $\sqrt{3}x+2y+3z=16$ and let

$$S=\left\{\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k}:\alpha^2+\beta^2+\gamma^2=1 \text{ and the distance of } (\alpha,\beta,\gamma) \text{ from the plane P is } \frac{7}{2}\right\}.$$

Let $\vec{u},\vec{v}$ and $\vec{w}$ be three distinct vectors in S such that $|\vec{u}-\vec{v}|=|\vec{v}-\vec{w}|=|\vec{w}-\vec{u}|$.

Let V be the volume of the parallelepiped determined by vectors $\vec{u},\vec{v}$ and $\vec{w}$. Then the value of $\frac{80}{\sqrt{3}}V$ is

[JEE(Advanced) 2023]

2. Let the position vectors of the points P,Q,R and S be $\vec{a}=\hat{i}+2\hat{j}-5\hat{k}$, $\vec{b}=3\hat{i}+6\hat{j}+3\hat{k}$, $\vec{c}=\frac{17}{5}\hat{i}+\frac{16}{5}\hat{j}+7\hat{k}$ and $\vec{d}=2\hat{i}+\hat{j}+\hat{k}$, respectively. Then which of the following statements is true? **[JEE(Advanced) 2023]**

(A) The points P,Q,R and S are **NOT** coplanar

(B) $\frac{\vec{b}+2\vec{d}}{3}$ is the position vector of a point which divides PR internally in the ratio 5 : 4

(C) $\frac{\vec{b}+2\vec{d}}{3}$ is the position vector of a point which divides PR externally in the ratio 5 : 4

(D) The square of the magnitude of the vector $\vec{b}\times\vec{d}$ is 95

3. Let $\hat{i},\hat{j}$ and $\hat{k}$ be the unit vectors along the three positive coordinate axes. Let

$$\vec{a}=3\hat{i}+\hat{j}-\hat{k},$$

$$\vec{b}=\hat{i}+b_2\hat{j}+b_3\hat{k}, \qquad b_2, b_3 \in \mathbb{R},$$

$$\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}, \qquad c_1, c_2, c_3 \in \mathbb{R}$$

be three vectors such that $b_2b_3>0$, $\vec{a}\cdot\vec{b}=0$ and

$$\begin{pmatrix}0 & -c_3 & c_2\\ c_3 & 0 & -c_1\\ -c_2 & c_1 & 0\end{pmatrix}\begin{pmatrix}1\\ b_2\\ b_3\end{pmatrix}=\begin{pmatrix}3-c_1\\ 1-c_2\\ -1-c_3\end{pmatrix}.$$

Then, which of the following is/are TRUE ? **[JEE(Advanced) 2022]**

(A) $\vec{a}\cdot\vec{c}=0$ (B) $\vec{b}\cdot\vec{c}=0$ (C) $|\vec{b}|>\sqrt{10}$ (D) $|\vec{c}|\le\sqrt{11}$

4. Let $\vec{u},\vec{v}$ and $\vec{w}$ be vectors in three-dimensional space, where $\vec{u}$ and $\vec{v}$ are unit vectors which are not perpendicular to each other and $\vec{u}.\vec{w}=1$, $\vec{v}.\vec{w}=1$, $\vec{w}.\vec{w}=4$

If the volume of the parallelopiped, whose adjacent sides are represented by the vectors $\vec{u},\vec{v}$ and $\vec{w}$, is $\sqrt{2}$, then the value of $|3\vec{u}+5\vec{v}|$ is____. **[JEE(Advanced) 2021]**

5. Let O be the origin and $\overrightarrow{OA}=2\hat{i}+2\hat{j}+\hat{k}$, $\overrightarrow{OB}=\hat{i}-2\hat{j}+2\hat{k}$ and $\overrightarrow{OC}=\frac{1}{2}\left(\overrightarrow{OB}-\lambda\overrightarrow{OA}\right)$ for some $\lambda>0$. If $\left|\overrightarrow{OB}\times\overrightarrow{OC}\right|=\frac{9}{2}$, then which of the following statements is (are) **TRUE**? **[JEE(Advanced) 2021]**

(A) Projection of $\overrightarrow{OC}$ on $\overrightarrow{OA}$ is $-\frac{3}{2}$

(B) Area of the triangle OAB is $\frac{9}{2}$

(C) Area of the triangle ABC is $\frac{9}{2}$

(D) The acute angle between the diagonals of the parallelogram with adjacent sides $\overrightarrow{OA}$ and $\overrightarrow{OC}$ is $\frac{\pi}{3}$

6. In a triangle PQR, let $\vec{a}=\overrightarrow{QR},\vec{b}=\overrightarrow{RP}$ and $\vec{c}=\overrightarrow{PQ}$. If $|\vec{a}|=3$, $|\vec{b}|=4$ and $\frac{\vec{a}.(\vec{c}-\vec{b})}{\vec{c}.(\vec{a}-\vec{b})}=\frac{|\vec{a}|}{|\vec{a}|+|\vec{b}|}$, then the value of $\left|\vec{a}\times\vec{b}\right|^2$ is ______

[JEE(Advanced) 2020]

7. Let a and b be positive real numbers. Suppose $\overrightarrow{PQ}=a\hat{i}+b\hat{j}$ and $\overrightarrow{PS}=a\hat{i}-b\hat{j}$ are adjacent sides of a parallelogram PQRS. Let $\vec{u}$ and $\vec{v}$ be the projection vectors of $\vec{w}=\hat{i}+\hat{j}$ along $\overrightarrow{PQ}$ and $\overrightarrow{PS}$, respectively. If $|\vec{u}|+|\vec{v}|=|\vec{w}|$ and if the area of the parallelogram PQRS is 8, then which of the following statements is/are TRUE ? **[JEE(Advanced) 2020]**

(A) $a+b=4$

(B) $a-b=2$

(C) The length of the diagonal PR of the parallelogram PQRS is 4

(D) $\vec{w}$ is an angle bisector of the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PS}$

8. Three lines

$L_1:\ \vec{r}=\lambda\hat{i},\ \lambda\in\mathbb{R}$,

$L_2:\ \vec{r}=\vec{k}+\mu\hat{j},\ \mu\in\mathbb{R}$ and

$L_3:\ \vec{r}=\hat{i}+\hat{j}+\nu\hat{k},\ \nu\in\mathbb{R}$

are given. For which point(s) Q on L_2 can we find a point P on L_1 and a point R on L_3 so that P, Q and R are collinear? **[JEE(Advanced) 2020]**

(A) $\hat{k}+\hat{j}$

(B) $\hat{k}$

(C) $\hat{k}+\frac{1}{2}\hat{j}$

(D) $\hat{k}-\frac{1}{2}\hat{j}$

9. Let $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$ be two vectors. Consider a vector $\vec{c} = \alpha\vec{a} + \beta\vec{b}$, $\alpha, \beta \in \mathbb{R}$. If the projection of $\vec{c}$ on the vector $(\vec{a} + \vec{b})$ is $3\sqrt{2}$, then the minimum value of $\left(\vec{c} - \left(\vec{a} \times \vec{b}\right)\right).\vec{c}$ equals

[JEE(Advanced) 2019]

10. Let $\vec{a}$ and $\vec{b}$ be two unit vectors such that $\vec{a}.\vec{b} = 0$. For some x, y $\in \mathbb{R}$, let $\vec{c} = x\vec{a} + y\vec{b} + (\vec{a} \times \vec{b})$. If $|\vec{c}| = 2$ and the vector $\vec{c}$ is inclined at the same angle α to both $\vec{a}$ and $\vec{b}$, then the value of $8\cos^2\alpha$ is ______. **[JEE(Advanced) 2018]**

11. Let O be the origin and let PQR be an arbitrary triangle. The point S is such that

$\overrightarrow{OP}.\overrightarrow{OQ} + \overrightarrow{OR}.\overrightarrow{OS} = \overrightarrow{OR}.\overrightarrow{OP} + \overrightarrow{OQ}.\overrightarrow{OS} = \overrightarrow{OQ}.\overrightarrow{OR} + \overrightarrow{OP}.\overrightarrow{OS}$

Then the triangle PQR has S as its **[JEE(Advanced) 2017]**

(A) incentre (B) orthocenter

(C) circumcentre (D) centroid

Paragraph for Question No. 12 and 13

Let O be the origin, and $\overline{OX}, \overline{OY}, \overline{OZ}$ be three unit vectors in the directions of the sides $\overline{QR}, \overline{RP}, \overline{PQ}$, respectively, of a triangle PQR.

12. $\left|\overline{OX} \times \overline{OY}\right| =$ **[JEE(Advanced) 2017]**

(A) sin(Q + R) (B) sin(P + R) (C) sin 2R (D) sin(P + Q)

13. If the triangle PQR varies, then the minimum value of cos(P + Q) + cos(Q + R) + cos(R + P) is

[JEE(Advanced) 2017]

(A) $\frac{3}{2}$ (B) $-\frac{3}{2}$ (C) $\frac{5}{3}$ (D) $-\frac{5}{3}$

14. Let $\hat{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$ be a unit vector in $\mathbb{R}^3$ and $\hat{w} = \frac{1}{\sqrt{6}}(\hat{i} + \hat{j} + 2\hat{k})$. Given that there exists a vector in $\mathbb{R}^3$ such that $|\hat{u} \times \vec{v}| = 1$ and $\hat{w}.(\hat{u} \times \vec{v}) = 1$. Which of the following statement(s) is(are) correct?

[JEE(Advanced) 2016]

(A) There is exactly one choice for such $\vec{v}$

(B) There are infinitely many choice for such $\vec{v}$

(C) If $\hat{u}$ lies in the xy-plane then $|u_1| = |u_2|$

(D) If $\hat{u}$ lies in the xz-plane then $2|u_1| = |u_3|$

15. Let ΔPQR be a triangle. Let $\vec{a} = \overrightarrow{QR}, \vec{b} = \overrightarrow{RP}$ and $\vec{c} = \overrightarrow{PQ}$. If $|\vec{a}| = 12, |\vec{b}| = 4\sqrt{3}$ and $\vec{b}.\vec{c} = 24$, then which of the following is (are) true? **[JEE(Advanced) 2015]**

(A) $\frac{|\vec{c}|^2}{2} - |\vec{a}| = 12$ (B) $\frac{|\vec{c}|^2}{2} + |\vec{a}| = 30$

(C) $|\vec{a} \times \vec{b} + \vec{c} \times \vec{a}| = 48\sqrt{3}$ (D) $\vec{a}.\vec{b} = -72$

16. **Column-I** **Column-II**

(A) In a triangle ΔXYZ, let a,b and c be the lengths of the sides opposite to the angles X,Y and Z, respectively. If $2(a^2 - b^2) = c^2$ and $\lambda = \dfrac{\sin(X-Y)}{\sin Z}$, then possible values of n for which $\cos(n\pi\lambda) = 0$ is (are) — (P) 1; (Q) 2

(B) In a triangle ΔXYZ, let a,b and c be the lengths of the sides opposite to the angles X,Y and Z, respectively. If $1 + \cos 2X - 2\cos 2Y = 2\sin X \sin Y$, then possible value(s) of $\dfrac{a}{b}$ is (are)

(C) In $\mathbb{R}^2$, Let $\sqrt{3}i + \hat{j}$, $\hat{i} + \sqrt{3}\hat{j}$ and $\beta\hat{i} + (1-\beta)\hat{j}$ be the position vectors of X, Y and Z with respect to the origin O, respectively. If the distance of Z from the bisector of the acute angle of $\overrightarrow{OX}$ and $\overrightarrow{OY}$ is $\dfrac{3}{\sqrt{2}}$, then possible value(s) of $|\beta|$ is (are) — (R) 3

(D) Suppose that $F(\alpha)$ denotes the area of the region bounded by $x = 0$, $x = 2$, $y^2 = 4x$ and $y = |\alpha x - 1| + |\alpha x - 2| + \alpha x$, where $\alpha \in \{0,1\}$. Then the value(s) of $F(\alpha) + \dfrac{8}{3}\sqrt{2}$, when $\alpha = 0$ and $\alpha = 1$, is(are) — (S) 5; (T) 6

[JEE(Advanced) 2015]

17. Suppose that $\vec{p}, \vec{q}$ and $\vec{r}$ are three non-coplanar vectors in $\mathbb{R}^3$. Let the components of a vector $\vec{s}$ along $\vec{p}, \vec{q}$ and $\vec{r}$ be 4, 3 and 5, respectively. If the components of this vector $\vec{r}$ along $(-\vec{p}+\vec{q}+\vec{r})$, $(\vec{p}-\vec{q}+\vec{r})$ and $(-\vec{p}-\vec{q}+\vec{r})$ are x,y and z, respectively, then the value of $2x + y + z$ is

[JEE(Advanced) 2015]

18. Let $\vec{x}, \vec{y}$ and $\vec{z}$ be three vectors each of magnitude $\sqrt{2}$ and the angle between each pair of them is $\dfrac{\pi}{3}$. If $\vec{a}$ is a nonzero vector perpendicular to $\vec{x}$ and $\vec{y} \times \vec{z}$ and $\vec{b}$ is nonzero vector perpendicular to $\vec{y}$ and $\vec{z} \times \vec{x}$, then **[JEE(Advanced) 2015]**

(A) $\vec{b} = (\vec{b}.\vec{z})(\vec{z} - \vec{x})$

(B) $\vec{a} = (\vec{a}.\vec{y})(\vec{y} - \vec{z})$

(C) $\vec{a}.\vec{b} = -(\vec{a}.\vec{y})(\vec{b}.\vec{z})$

(D) $\vec{a} = (\vec{a}.\vec{y})(\vec{z} - \vec{y})$

19. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three non-coplanar unit vectors such that the angle between every pair of them is $\dfrac{\pi}{3}$. If $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} = p\vec{a} + q\vec{b} + r\vec{c}$, where p,q and r are scalars, then the value of $\dfrac{p^2 + 2q^2 + r^2}{q^2}$ is

[JEE(Advanced) 2014]

20.

List-I		List-II	
P.	Let $y(x) = \cos(3\cos^{-1} x)$, $x \in [-1, 1]$, $x \neq \pm\frac{\sqrt{3}}{2}$. Then $\frac{1}{y(x)}\left\{(x^2-1)\frac{d^2y(x)}{dx^2} + x\frac{dy(x)}{dx}\right\}$ equals	**1.**	1
Q.	Let $A_1, A_2, \ldots, A_n$ $(n > 2)$ be the vertices of a regular polygon of n sides with its centre at the origin. Let $\overrightarrow{a_k}$ be the position vector of the point A_k, k = 1, 2, ... n. If $\left\|\sum_{k=1}^{n-1}\left(\overrightarrow{a_k} \times \overrightarrow{a_{k+1}}\right)\right\| = \left\|\sum_{k=1}^{n-1}\left(\overrightarrow{a_k} . \overrightarrow{a_{k+1}}\right)\right\|$, then the minimum value of n is	**2.**	2
R.	If the normal from the point P(h, 1) on the ellipse $\frac{x^2}{6} + \frac{y^2}{3} = 1$ is perpendicular to the line x + y = 8, then the value of h is	**3.**	8
S.	Number of positive solutions satisfying the equation $\tan^{-1}\left(\frac{1}{2x+1}\right) + \tan^{-1}\left(\frac{1}{4x+1}\right) = \tan^{-1}\left(\frac{2}{x^2}\right)$ is	**4.**	9

[JEE(Advanced) 2014]

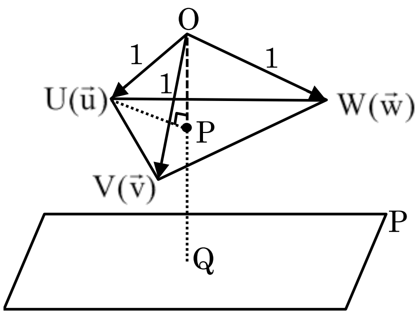
Codes :

	P	Q	R	S
(A)	4	3	2	1
(B)	2	4	3	1
(C)	4	3	1	2
(D)	2	4	1	3

SOLUTIONS

1. Ans. (45)

Sol.



Given $|\vec{u}-\vec{v}| = |\vec{v}-\vec{w}| = |\vec{w}-\vec{u}|$

$\Rightarrow \Delta UVW$ is an equilateral Δ

Now distances of U, V, W from $P = \frac{7}{2}$

$\Rightarrow PQ = \frac{7}{2}$

Also, Distance of plane P from origin

$\Rightarrow OQ = 4$

$\therefore OP = OQ - PQ \Rightarrow OP = \frac{1}{2}$

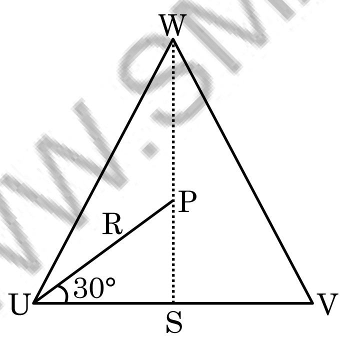
Hence, $PU = \sqrt{OU^2 - OP^2} \Rightarrow PU = \frac{\sqrt{3}}{2} = R$

Also, for ΔUVW, P is circumcenter

$\therefore$ for ΔUVW : $US = R\cos 30°$

$\Rightarrow UV = 2R\cos 30°$

$\Rightarrow UV = \frac{3}{2}$



$\therefore \quad Ar(\Delta UVW) = \frac{\sqrt{3}}{4}\left(\frac{3}{2}\right)^2 = \frac{9\sqrt{3}}{16}$

$\therefore$ Volume of tetrahedron with coterminous edges $\vec{u}, \vec{v}, \vec{w}$

$= \frac{1}{3}(Ar.\Delta UVW) \times OP = \frac{1}{3} \times \frac{9\sqrt{3}}{16} \times \frac{1}{2} = \frac{3\sqrt{3}}{32}$

$\therefore$ parallelopiped with coterminous edges

$\vec{u}, \vec{v}, \vec{w} = 6 \times \frac{3\sqrt{3}}{32} = \frac{9\sqrt{3}}{16} = V$

$\therefore \frac{80}{\sqrt{3}} V = 45$

2. Ans. (B)

Sol. $P\left(\hat{i} + 2\hat{j} - 5\hat{k}\right) = P(\vec{a})$

$Q\left(3\hat{i} + 6\hat{j} + 3\hat{k}\right) = Q(\vec{b})$

$R\left(\frac{17}{5}\hat{i} + \frac{16}{5}\hat{j} + 7\hat{k}\right) = R(\vec{c})$

$S\left(2\hat{i} + \hat{j} + \hat{k}\right) = S(\vec{d})$

$\frac{\vec{b} + 2\vec{d}}{3} = \frac{7\hat{i} + 8\hat{j} + 5\hat{k}}{3}$

$\frac{5\vec{c} + 4\vec{a}}{9} = \frac{21\hat{i} + 24\hat{j} + 15\hat{k}}{9}$

$\Rightarrow \frac{\vec{b} + 2\vec{d}}{3} = \frac{5\vec{c} + 4\vec{a}}{9}$

so [B] is correct.

option (D)

$\left|\vec{b} \times \vec{d}\right|^2 = \left|\vec{b}\right|\left|\vec{d}\right|^2 - \left(\vec{b}.\vec{d}\right)^2$

$= (9 + 36 + 9)\ (4 + 1 + 1) - (6 + 6 + 3)^2$

$= 54 \times 6 - (15)^2$

$= 324 - 225$

$= 99$

3. Ans. (B, C, D)

Sol. $\vec{a} = 3\hat{i} + \hat{j} - \hat{k}$

$\vec{b} = \hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

$$\begin{pmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 3-c_1 \\ 1-c_2 \\ -1-c_3 \end{pmatrix}$$

multiply & compare

$b_2c_3 - b_3c_2 = c_1 - 3 \quad \ldots(1)$

$c_3 - b_3c_1 = 1 - c_2 \quad \ldots(2)$

$c_2 - b_2c_1 = 1 + c_3 \quad \ldots(3)$

$(1)\hat{i} - (2)\hat{j} + (3)\hat{k}$

$\hat{i}(b_2c_3 - b_3c_2) - \hat{j}(c_3 - b_3c_1) + \hat{k}(c_2 - b_2c_1)$

$= c_1\hat{i} + c_2\hat{j} + c_2\hat{k} - 3\hat{i} - \hat{j} + \hat{k}$

$\vec{b} \times \vec{c} = \vec{c} - \vec{a}$

Take dot product with $\vec{b}$

$0 = \vec{c}.\vec{b} - \vec{a}.\vec{b}$

$\vec{b}.\vec{c} = 0$

$\vec{b} \perp \vec{c}$

$\vec{b} \hat{} \vec{c} = 90°$

Take dot product with $\vec{c}$

$0 = |\vec{c}|^2 - \vec{a}.\vec{c}$

$\vec{a}.\vec{c} = |\vec{c}|^2$

$\vec{a}.\vec{c} \neq 0$

$\vec{b} \times \vec{c} = \vec{c} - \vec{a}$

Squaring

$|\vec{b}|^2 |\vec{c}|^2 = |\vec{c}|^2 + |\vec{a}|^2 - 2\vec{c}.\vec{a}$

$|\vec{b}|^2 |\vec{c}|^2 = |\vec{c}|^2 + 11 - 2|\vec{c}|^2$

$|\vec{b}|^2 |\vec{c}|^2 = 11 - |\vec{c}|^2$

$|\vec{c}|^2 \left(|\vec{b}|^2 + 1\right) = 11$

$|\vec{c}|^2 = \dfrac{11}{|\vec{b}|^2 + 1}$

$|\vec{c}| \leq \sqrt{11}$

given $\vec{a}.\vec{b} = 0$

$b_2 - b_3 = -3$ also

$b_2^2 + b_3^2 - 2b_2b_3 = 9 \quad b_2b_3 > 0$

$b_2^2 + b_3^2 = 9 + 2b_2b_3$

$b_2^2 + b_3^2 = 9 + 2b_2b_3 > 9$

$b_2^2 + b_3^2 > 9$

$|\vec{b}| = \sqrt{1 + b_2^2 + b_3^2}$

$|\vec{b}| > \sqrt{10}$

4. Ans. (7)

Sol. Given, $|\vec{u}| = 1$; $|\vec{v}| = 1$; $\vec{u}.\vec{v} \neq 0$; $\vec{u}.\vec{w} = 1$; $\vec{v}.\vec{w} = 1$;

$\vec{w}.\vec{w} = |\vec{w}|^2 = 4 \Rightarrow |\vec{w}| = 2$;

$[\vec{u} \ \ \vec{v} \ \ \vec{w}] = \sqrt{2}$

and $[\vec{u} \ \ \vec{v} \ \ \vec{w}]^2 = \begin{vmatrix} \vec{u}.\vec{u} & \vec{u}.\vec{v} & \vec{u}.\vec{w} \\ \vec{v}.\vec{u} & \vec{v}.\vec{v} & \vec{v}.\vec{w} \\ \vec{w}.\vec{u} & \vec{w}.\vec{v} & \vec{w}.\vec{w} \end{vmatrix} = 2$

$\Rightarrow \begin{vmatrix} 1 & \vec{u}.\vec{v} & 1 \\ \vec{u}.\vec{v} & 1 & 1 \\ 1 & 1 & 4 \end{vmatrix} = 2$

$\Rightarrow \vec{u}.\vec{v} = \dfrac{1}{2}$

So, $|3\vec{u} + 5\vec{v}| = \sqrt{9|\vec{u}|^2 + 25|\vec{v}|^2 + 2.3.5\vec{u}.\vec{v}}$

$= \sqrt{9 + 25 + 30\left(\dfrac{1}{2}\right)} = \sqrt{49} = 7$

5. Ans. (A, B, C)

Sol. $\overrightarrow{OB} \times \overrightarrow{OC} = \dfrac{1}{2}\overrightarrow{OB} \times \left(\overrightarrow{OB} - \lambda\overrightarrow{OA}\right)$

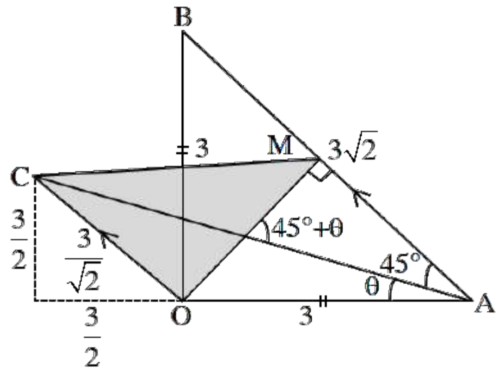
$= \dfrac{\lambda}{2}\left(\overrightarrow{OA} \times \overrightarrow{OB}\right)$

$|\overrightarrow{OB}| \times |\overrightarrow{OC}| = \dfrac{|\lambda|}{2}|\overrightarrow{OA}| \times |\overrightarrow{OB}|$

(Note $\overrightarrow{OA}$ & $\overrightarrow{OB}$ are perpendicular)

$\Rightarrow \dfrac{9\lambda}{2} = \dfrac{9}{2} \Rightarrow \lambda = 1$ (given $\lambda > 0$)

So $\overrightarrow{OC} = \dfrac{\overrightarrow{OB} - \overrightarrow{OA}}{2} = \dfrac{\overrightarrow{AB}}{2}$



M is mid point of AB

Note projection of $\overrightarrow{OC}$ on $\overrightarrow{OA} = -\frac{3}{2}$

$\tan\theta = \frac{1}{3}$

Area of $\Delta ABC = \frac{9}{2}$

Acute angle between diagonals is

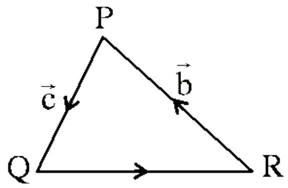
$$\tan^{-1}\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right) = \tan^{-1} 2$$

6. Ans. (108.00)

Sol. We have $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

$\Rightarrow \vec{c} = -\vec{a}-\vec{b}$

Now, $\frac{\vec{a}.(-\vec{a}-2\vec{b})}{(-\vec{a}-\vec{b}).(\vec{a}-\vec{b})} = \frac{3}{7}$



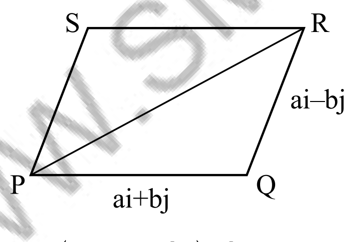
$\Rightarrow \frac{9+2\vec{a}.\vec{b}}{9-16} = \frac{3}{7}$

$\Rightarrow \vec{a}.\vec{b} = -6$

$\Rightarrow |\vec{a}\times\vec{b}|^2 = a^2b^2 - (\vec{a}.\vec{b})^2 = 9 \times 16 - 36 = 108$

7. Ans. (A, C)

Sol.



$\vec{u} = \left((i+j).\widehat{PQ}\right)\widehat{PQ}$

$\vec{u} = \left|(i+j).\widehat{PQ}\right|$

$|\vec{u}| = \left|(i+j).\frac{(ai+bj)}{\sqrt{a^2+b^2}}\right| = \frac{a+b}{a^2+b^2}$

$\vec{v} = (i+j).\widehat{PS}$

$|\vec{v}| = \left|\frac{(i+j).(ai-bj)}{\sqrt{a^2+b^2}}\right| = \frac{a-b}{\sqrt{a^2+b^2}}$

$|\vec{u}|+|\vec{v}| = |\vec{w}|$

$\frac{|(a+b)|+|(a-b)|}{\sqrt{a^2+b^2}} = \sqrt{2}$

For $a \geq b$

$2a = \sqrt{2}.\sqrt{a^2+b^2}$

$4a^2 = 2a^2 + 2b^2$

$a^2 = b^2 \ \therefore\ a = b$ ...(1)

$(a > 0, b > 0)$

similarly for $a \leq b$ we will get $a = b$

Now area of parallelogram

$= |(ai+bj)\times(ai-bj)|$

$= 2ab$

$\therefore\ 2ab = 8$

$ab = 4$ ...(2)

from (1) and (2)

$a = 2, b = 2$ $\quad\therefore\ a + b = 4$ option (A)

length of diagonal is

$|2a\hat{i}| = |4\hat{i}| = 4$

so option (C)

8. Ans. (C, D)

Sol. Let P(λ, 0, 0), Q(0, μ, 1), R(1, 1, ν) be points. L_1, L_2 and L_3 respectively

Since P, Q, R are collinear, $\overrightarrow{PQ}$ is collinear with $\overrightarrow{QR}$

Hence $\frac{-\lambda}{1} = \frac{\mu}{1-\mu} = \frac{1}{\nu-1}$

For every $\mu \in R - \{0, 1\}$ there exist unique $\lambda, \nu \in R$

Hence Q cannot have coordinates (0, 1, 1) and (0, 0, 1).

9. Ans. (18.00)

Sol. $\vec{c} = (2\alpha + \beta)\hat{i} + \hat{j}(\alpha + 2\beta) + \hat{k}(\beta - \alpha)$

$\dfrac{\vec{c}.(\vec{a}+\vec{b})}{|\vec{a}+\vec{b}|} = 3\sqrt{2}$

$\Rightarrow \quad \alpha + \beta = 2 \qquad ....(1)$

$(\vec{c} - (\vec{a}\times\vec{b})).(\alpha\vec{a} + \beta\vec{b})$

$= |\vec{c}|^2 = \alpha^2 |\vec{a}|^2 + \beta^2 |b|^2 + 2\alpha\beta(\vec{a}.\vec{b})$

$= 6(\alpha^2 + \beta^2 + \alpha\beta)$

$= 6(\alpha^2 + (2-\alpha)^2 + \alpha(2-\alpha))$

$= 6((\alpha-1)^2 + 3)$

$\Rightarrow$ Min. value = 18

10. Ans. (3)

Sol. $\vec{c} = x\vec{a} + y\vec{b} + \vec{a}\times\vec{b}$

$\vec{c}.\vec{a} = x$ and $x = 2\cos\alpha$

$\vec{c}.\vec{b} = y$ and $y = 2\cos\alpha$

Also, $|\vec{a}\times\vec{b}| = 1$

$\therefore \quad \vec{c} = 2\cos\alpha(\vec{a}+\vec{b}) + \vec{a}\times\vec{b}$

$\vec{c}^2 = 4\cos^2\alpha(\vec{a}+\vec{b})^2 + (\vec{a}\times\vec{b})^2 +$
$2\cos\alpha\ (\vec{a}+\vec{b}).(\vec{a}\times\vec{b})$

$4 = 8\cos^2\alpha + 1$

$8\cos^2\alpha = 3$

11. Ans. (B)

Sol. Let position vector of $P(\vec{p})$, $Q(\vec{q})$, $R(\vec{r})$ & $S(\vec{s})$ with respect to $O(\vec{o})$

Now, $\overrightarrow{OP}.\overrightarrow{OQ} + \overrightarrow{OR}.\overrightarrow{OS} = \overrightarrow{OR}.\overrightarrow{OP} + \overrightarrow{OQ}.\overrightarrow{OS}$

$\Rightarrow \quad \vec{p}.\vec{q} + \vec{r}.\vec{s} = \vec{r}.\vec{p} + \vec{q}.\vec{s}$

$\Rightarrow \quad (\vec{p}-\vec{s}).(\vec{q}-\vec{r}) = 0 \qquad ....(1)$

Also, $\overrightarrow{OR}.\overrightarrow{OP} + \overrightarrow{OQ}.\overrightarrow{OS} = \overrightarrow{OQ}.\overrightarrow{OR} + \overrightarrow{OP}.\overrightarrow{OS}$

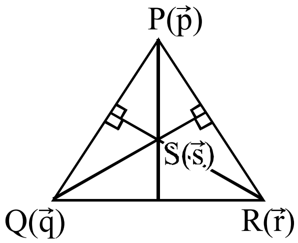
$\Rightarrow \quad \vec{r}.\vec{p} + \vec{q}.\vec{s} = \vec{q}.\vec{r} + \vec{p}.\vec{s}$

$\Rightarrow \quad (\vec{r}-\vec{s}).(\vec{p}-\vec{q}) = 0 \qquad ....(2)$

Also, $\overrightarrow{OP}.\overrightarrow{OQ} + \overrightarrow{OR}.\overrightarrow{OS} = \overrightarrow{OQ}.\overrightarrow{OR} + \overrightarrow{OP}.\overrightarrow{OS}$

$\Rightarrow \quad \vec{p}.\vec{q} + \vec{r}.\vec{s} = \vec{q}.\vec{r} + \vec{p}.\vec{s}$

$\Rightarrow \quad (\vec{q}-\vec{s}).(\vec{p}-\vec{r}) = 0 \qquad ....(3)$



$\Rightarrow$ Triangle PQR has S as its orthocentre

$\therefore$ option (B) is correct.

12. Ans. (D)

Sol. $\overrightarrow{OX} = \dfrac{\overrightarrow{QR}}{QR}$

$\overrightarrow{OY} = \dfrac{\overrightarrow{RP}}{RP}$

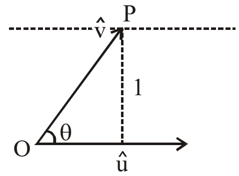
$\left|\overrightarrow{OX}\times\overrightarrow{OY}\right| = \sin R = \sin(P+Q)$

13. Ans. (B)

Sol. $-(\cos P + \cos Q + \cos R) \geq -\dfrac{3}{2}$ as we know $\cos P + \cos Q + \cos R$ will take its maximum value when $P = Q = R = \dfrac{\pi}{3}$

14. Ans. (B, C)

Sol. $|\hat{w}||\hat{u}\times\hat{v}|\cos\phi = 1 \Rightarrow \phi = 0$



$\Rightarrow \quad \hat{u}\times\vec{v} = \hat{w}$ also $|\vec{v}|\sin\theta = 1$

$\Rightarrow$ there may be infinite vectors $\vec{v} = \overrightarrow{OP}$

such that P is always 1 unit dist. from $\hat{u}$

For option (C) : $\hat{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & 0 \\ v_1 & v_2 & v_3 \end{vmatrix}$

$\hat{w} = (u_2v_3)\hat{i} - (u_1v_3)\hat{j} + (u_1v_2 - u_2v_1)\hat{k}$

$u_2v_3 = \dfrac{1}{\sqrt{6}}, -u_1v_3 = \dfrac{1}{\sqrt{6}}$

$\Rightarrow \quad |u_1| = |u_2|$

for option (D) : $\hat{u}\times\vec{v}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ u_1 & 0 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}$

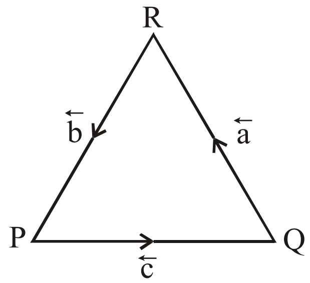
$$\hat{w}=(-v_2u_3)\hat{i}-(u_1v_3-u_3v_1)\hat{j}+(u_1v_2)\hat{k}$$

$$-v_2u_3=\frac{1}{\sqrt{6}},\ u_1v_2=\frac{2}{\sqrt{6}}$$

$\Rightarrow 2|u_3|=|u_1|$ So, (D) is wrong

15. Ans. (A, C, D)

Sol.



$$|\vec{a}|=12,\ |\vec{b}|=4\sqrt{3}$$

$\vec{a}+\vec{b}+\vec{c}=0$ ....(1)

$a^2=b^2+c^2+2\vec{b}.\vec{c}$

$144=48+c^2+48$

$c^2=48 \quad\Rightarrow\quad c=4\sqrt{3}$

Also $c^2=a^2+b^2+2\vec{a}.\vec{b}$

$48=144+48+2\vec{a}.\vec{b}$

$\Rightarrow \vec{a}.\vec{b}=-72$

Also by (1)

$\vec{a}\times\vec{b}=\vec{c}\times\vec{a}$

$\Rightarrow |\vec{a}\times\vec{b}+\vec{c}\times\vec{a}|=2|\vec{a}\times\vec{b}|$

$=2\sqrt{a^2b^2-(\vec{a}.\vec{b})^2}$

$=2\sqrt{12^2.48-(72)^2}$

$=2.12\sqrt{48-36}=48\sqrt{3}$

$\therefore$ A, C, D are correct & B incorrect.

16. Ans. (A) $\rightarrow$ (P,R,S); (B) $\rightarrow$ (P); (C) $\rightarrow$ (P,Q); (D) $\rightarrow$ (S,T)

Sol. (A) $2\left(\sin^2 x-\sin^2 y\right)=\sin^2 z$

$$2\sin(x-y)\sin(x+y)=\sin^2 z$$

$\because x+y+z=\pi$

$$\frac{\sin(x-y)}{\sin z}=\frac{1}{2}\Rightarrow\lambda=\frac{1}{2}$$

$$\Rightarrow \cos\left(\frac{n\pi}{2}\right)=0 \Rightarrow n=1,3,5$$

(B) $1+1-2\sin^2x-2(1-2\sin^2y)$

$=2\sin x\sin y$

$\Rightarrow -2a^2+4b^2=2ab$

$\Rightarrow a^2+ab-2b^2=0$

$$\left(\frac{a}{b}\right)^2+\left(\frac{a}{b}\right)-2=0\Rightarrow\frac{a}{b}=-2,1$$

$\Rightarrow \frac{a}{b}=1$ as -2 rejected

(C) Angle bisector of $\overrightarrow{OX}$ & $\overrightarrow{OY}$ is along the line y = x and its distance from $(\beta,1-\beta)$ is

$$\left|\frac{\beta-(1-\beta)}{\sqrt{2}}\right|=\frac{3}{\sqrt{2}}\Rightarrow 2\beta-1=\pm3$$

$\Rightarrow \beta=2,-1$

$\Rightarrow |\beta|=1,2$

(D) 7

$$6-\int_0^2 2\sqrt{x}\,dx \qquad 5-\int_0^2 2\sqrt{x}\,dx$$

$$6-\frac{8}{3}\sqrt{2}\quad ...(1)\qquad 5-\frac{8}{3}\sqrt{2}\quad ...(2)$$

By (1) & (2) $F(\alpha)+\frac{8}{3}\sqrt{2}$

can be 5 or 6.

17. Ans. (Bonus)

Sol. Although the language of the question is not appropriate (incomplete information) and it must be declare as bonus but as per the theme of problem it must be as follows

$\vec{s}=4\vec{p}+3\vec{q}+5\vec{r}$

$$\vec{s}=x(-\vec{p}+\vec{q}+\vec{r})+y(\vec{p}-\vec{q}+\vec{r})+z(-\vec{p}-\vec{q}+\vec{r})$$

$$\vec{s}=\vec{p}(-x+y-z)+\vec{q}(x-y-z)+\vec{r}(x+y+z)$$

$-x+y-z=4$

$x-y-z=3$

$x+y+z=5$

$\Rightarrow x=4,\ y=\frac{9}{2}, z=-\frac{7}{2}$

$\Rightarrow 2x+y+z=8-\frac{7}{2}+\frac{9}{2}=9$

18. Ans. (A, B, C)

Sol. Given that $|\vec{x}|=|\vec{y}|=|\vec{z}|=\sqrt{2}$

and angle between each pair is $\frac{\pi}{3}$

$\therefore \quad \vec{x}.\vec{y}=\vec{y}.\vec{z}=\vec{z}.\vec{x}=\sqrt{2}.\sqrt{2}.\frac{1}{2}=1$

Now $\vec{a}$ is $\perp$ to $\vec{x}$ & $(\vec{y}\times\vec{z})$

Let $\quad \vec{a}=\lambda(\vec{x}\times(\vec{y}\times\vec{z}))$

$=\lambda((\vec{x}.\vec{z})\vec{y}-(\vec{x}.\vec{y})\vec{z})=\lambda(\vec{y}-\vec{z})$

$\vec{a}.\vec{y}=\lambda(\vec{y}.\vec{y}-\vec{y}.\vec{z})=\lambda(2-1)=\lambda$

$\Rightarrow \vec{a}=(\vec{a}.\vec{y})(\vec{y}-\vec{z})$

Now let $\vec{b}=\mu\left(\vec{y}\times(\vec{z}\times\vec{x})\right)=\mu(\vec{z}-\vec{x})$

$\vec{b}.\vec{z}=\mu(2-1)=\mu$

$\Rightarrow \quad \vec{b}=(\vec{b}.\vec{z})(\vec{z}-\vec{x})$

Now $\vec{a}.\vec{b}=(\vec{a}.\vec{y})(\vec{y}-\vec{z}).(\vec{b}.\vec{z})(\vec{z}-\vec{x})$

$=(\vec{a}.\vec{y})(\vec{b}.\vec{z})(\vec{y}.\vec{z}-\vec{y}.\vec{x}-\vec{z}.\vec{z}+\vec{z}.\vec{x})$

$=(\vec{a}.\vec{y})(\vec{b}.\vec{z})(1-1-2+1)$

$=-(\vec{a}.\vec{y})(\vec{b}.\vec{z})$

19. Ans. (4)

Sol. We know $\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]^2=\begin{vmatrix}\vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c}\\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c}\\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}.\vec{c}\end{vmatrix}$

$$=\begin{vmatrix}1 & \frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & 1 & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2} & 1\end{vmatrix}=\frac{5}{4}-\frac{3}{4}=\frac{1}{2}$$

$\therefore \left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]=\frac{1}{\sqrt{2}} \qquad ...(1)$

as given $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}=p\vec{a}+q\vec{b}+r\vec{c}$

take dot product with $\vec{a}$

$\Rightarrow \vec{a}.\left(\vec{a}\times\vec{b}\right)+\vec{a}.\left(\vec{b}\times\vec{c}\right)=p\vec{a}^2+q\vec{b}.\vec{a}+r\vec{c}.\vec{a}$

$\Rightarrow 0+\frac{1}{\sqrt{2}}=p+\frac{q}{2}+\frac{r}{2} \qquad ...(2)$

Now, take dot product with $\vec{b}$ & $\vec{c}$

$0=\frac{p}{2}+q+\frac{r}{2} \qquad ...(3)$

& $\frac{1}{\sqrt{2}}=\frac{p}{2}+\frac{q}{2}+r \qquad ...(4)$

equation (2) – equation (4)

$\Rightarrow \frac{p}{2}-\frac{r}{2}=0 \Rightarrow p=r \Rightarrow p+q=0$ by equation (3)

$\therefore \frac{p^2+2q^2+r^2}{q^2}=\frac{p^2+2p^2+p^2}{p^2}=4$

20. Ans. (A)

Sol. (P) $\quad y=\cos(3\cos^{-1}x)=(4x^3-3x)$

$\frac{dy}{dx}=12x^2-3, \frac{d^2y}{dx^2}=24x$

then $\frac{1}{y}\left[\left(x^2-1\right)\frac{d^2y}{dx^2}+\frac{xdy}{dx}\right]$

$\frac{1}{4x^3-3x}\left[\left(x^2-1\right).24x+x\left(12x^2-3\right)\right]$

$=9$

(Q) $\quad$ let $\vec{a}_1=\hat{i}$,

then $\vec{a}_2=\cos\frac{2\pi}{n}\hat{i}+\sin\frac{2\pi}{n}\hat{j}$

$\vec{a}_3=\cos\frac{4\pi}{n}\hat{i}+\sin\frac{4\pi}{4}\hat{j}...$

now

$\left|\vec{a}_1\times\vec{a}_2+\vec{a}_2\times\vec{a}_3+.....+\vec{a}_{n-1}\times\vec{a}_n\right|$

$=\left|\vec{a}_1.\vec{a}_2+\vec{a}_2.\vec{a}_3+......+\vec{a}_{n-1}.\vec{a}_n\right|$

$=\left|(n-1)\sin\frac{2\pi}{n}\hat{k}\right|=\left|(n-1)\cos\frac{2\pi}{n}\right|$

$\Rightarrow \tan\frac{2\pi}{n} = 1$

$\Rightarrow$ for minimum n $\frac{2\pi}{n} = \frac{\pi}{4} \Rightarrow n = 8$

(R) $\frac{x^2}{6} + \frac{y^2}{3} = 1 \Rightarrow \frac{dy}{dx} = -\frac{x}{2y} \Rightarrow \frac{2y}{x} = 1$

$\Rightarrow \frac{4y^2}{6} + \frac{y^2}{3} = 1 \Rightarrow y = \pm 1$ & $x = \pm 2$

as normal passes through (–2,–1) and (h,1) slope of normal

$= \frac{2}{h+2} = 1 \Rightarrow h = 0$

OR

if normal passes through (2,1) then

h = 2

(S) $\tan^{-1}\left(\frac{1}{2x+1}\right) + \tan^{-1}\left(\frac{1}{4x+1}\right) = \tan^{-1}\left(\frac{2}{x^2}\right)$

$\Rightarrow \tan^{-1}\frac{\frac{1}{2x+1}+\frac{1}{4x+1}}{1-\frac{1}{2x+1}\cdot\frac{1}{4x+1}} = \tan^{-1}\frac{2}{x^2}$

$\Rightarrow x = 0, -\frac{2}{3}, 3$

but only +ve integral x = 3