INVERSE TRIGONOMETRIC FUNCTION

1. Let $\tan^{-1}(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, for $x \in \mathbb{R}$. Then the number of real solutions of the equation

$\sqrt{1+\cos(2x)} = \sqrt{2}\tan^{-1}(\tan x)$ in the set $\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ is equal to

[JEE(Advanced) 2023]

2. For any $y \in \mathbb{R}$, let $\cot^{-1}(y) \in (0, \pi)$ and $\tan^{-1}(y) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then the sum of all the solutions of the equation $\tan^{-1}\left(\frac{6y}{9-y^2}\right) + \cot^{-1}\left(\frac{9-y^2}{6y}\right) = \frac{2\pi}{3}$ for $0 < |y| < 3$, is equal to **[JEE(Advanced) 2023]**

(A) $2\sqrt{3}-3$ (B) $3-2\sqrt{3}$ (C) $4\sqrt{3}-6$ (D) $6-4\sqrt{3}$

3. Considering only the principal values of the inverse trigonometric functions, the value of

$$\frac{3}{2}\cos^{-1}\sqrt{\frac{2}{2+\pi^2}} + \frac{1}{4}\sin^{-1}\frac{2\sqrt{2}\pi}{2+\pi^2} + \tan^{-1}\frac{\sqrt{2}}{\pi}$$

is ________. **[JEE(Advanced) 2022]**

4. For any positive integer n, let $S_n : (0, \infty) \rightarrow \mathbb{R}$ be defined by $S_n(x) = \sum_{k=1}^{n}\cot^{-1}\left(\frac{1+k(k+1)x^2}{x}\right)$, where for any $x \in \mathbb{R}$, $\cot^{-1}x \in (0,\pi)$ and $\tan^{-1}(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then which of the following statements is (are) **TRUE** ? **[JEE(Advanced) 2021]**

(A) $S_{10}(x) = \frac{\pi}{2} - \tan^{-1}\left(\frac{1+11x^2}{10x}\right)$, for all $x > 0$

(B) $\lim_{n\to\infty}\cot(S_n(x)) = x$, for all $x > 0$

(C) The equation $S_3(x) = \frac{\pi}{4}$ has a root in $(0,\infty)$

(D) $\tan(S_n(x)) \le \frac{1}{2}$, for all $n \ge 1$ and $x > 0$

5. For non-negative integers n, let

$$f(n) = \frac{\sum_{k=0}^{n}\sin\left(\frac{k+1}{n+2}\pi\right)\sin\left(\frac{k+2}{n+2}\pi\right)}{\sum_{k=0}^{n}\sin^2\left(\frac{k+1}{n+2}\pi\right)}$$

Assuming $\cos^{-1} x$ takes values in $[0, \pi]$, which of the following options is/are correct ?

[JEE(Advanced) 2019]

(A) $\sin(7\cos^{-1} f(5)) = 0$

(B) $f(4) = \frac{\sqrt{3}}{2}$

(C) $\lim_{n\to\infty} f(n) = \frac{1}{2}$

(D) If $\alpha = \tan(\cos^{-1} f(6))$, then $\alpha^2 + 2\alpha - 1 = 0$

6. The value of $\sec^{-1}\left(\frac{1}{4}\sum_{k=0}^{10}\sec\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\sec\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right)\right)$ in the interval $\left[-\frac{\pi}{4}, \frac{3\pi}{4}\right]$ equals

7. The number of real solutions of the equation **[JEE(Advanced) 2019]**

$$\sin^{-1}\left(\sum_{i=1}^{\infty} x^{i+1} - x\sum_{i=1}^{\infty}\left(\frac{x}{2}\right)^i\right) = \frac{\pi}{2} - \cos^{-1}\left(\sum_{i=1}^{\infty}\left(-\frac{x}{2}\right)^i - \sum_{i=1}^{\infty}(-x)^i\right)$$

lying in the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$ is ______. **[JEE(Advanced) 2018]**

(Here, the inverse trigonometric functions $\sin^{-1}x$ and $\cos^{-1}x$ assume value in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $[0, \pi]$, respectively.)

8. Let $E_1 = \left\{x \in \mathbb{R} : x \neq 1 \text{ and } \frac{x}{x-1} > 0\right\}$

and $E_2 = \left\{x \in E_1 : \sin^{-1}\left(\log_e\left(\frac{x}{x-1}\right)\right) \text{ is a real number}\right\}$.

$\left(\text{Here, the inverse trigonometric function } \sin^{-1}x \text{ assumes values in } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].\right)$

Let $f : E_1 \to \mathbb{R}$ be the function defined by $f(x) = \log_e\left(\frac{x}{x-1}\right)$

and $g : E_2 \to \mathbb{R}$ be the function defined by $g(x) = \sin^{-1}\left(\log_e\left(\frac{x}{x-1}\right)\right)$ **[JEE(Advanced) 2018]**

LIST-I		LIST-II	
P.	The range of f is	**1.**	$\left(-\infty, \frac{1}{1-e}\right] \cup \left[\frac{e}{e-1}, \infty\right)$
Q.	The range of g contains	**2.**	$(0, 1)$
R.	The domain of f contains	**3.**	$\left[-\frac{1}{2}, \frac{1}{2}\right]$
S.	The domain of g is	**4.**	$(-\infty, 0) \cup (0, \infty)$
		5.	$\left(-\infty, \frac{e}{e-1}\right]$
		6.	$(-\infty, 0) \cup \left(\frac{1}{2}, \frac{e}{e-1}\right]$

The correct option is :

(A) P → 4; Q → 2; R → 1; S → 1
(B) P → 3; Q → 3; R → 6; S → 5
(C) P → 4; Q → 2; R → 1; S → 6
(D) P → 4; Q → 3; R → 6; S → 5

9. If $\alpha = 3\sin^{-1}\left(\frac{6}{11}\right)$ and $\beta = 3\cos^{-1}\left(\frac{4}{9}\right)$ where the inverse trigonometric functions take only the principal values, then the correct option(s) is(are) **[JEE(Advanced) 2015]**

(A) $\cos\beta > 0$ (B) $\sin\beta < 0$ (C) $\cos(\alpha+\beta) > 0$ (D) $\cos\alpha < 0$

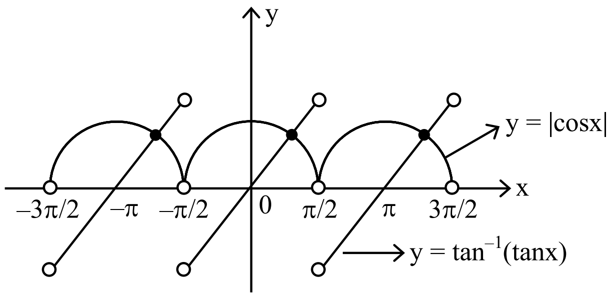
10. Let $f : [0,4\pi] \to [0,\pi]$ be defined by $f(x) = \cos^{-1}(\cos x)$. The number of points $x \in [0,4\pi]$ satisfying the equation $f(x) = \frac{10-x}{10}$ is **[JEE(Advanced) 2014]**

SOLUTIONS

1. Ans. (3)

Sol. $\sqrt{2}|\cos x| = \sqrt{2}.\tan^{-1}(\tan x)$

$|\cos x| = \tan^{-1}\tan x$



No. of solutions = 3

2. Ans. (C)

Sol. Case-I : $y \in (-3,0)$

$$\tan^{-1}\left(\frac{6y}{9-y^2}\right)+\pi+\tan^{-1}\left(\frac{6y}{9-y^2}\right)=\frac{2\pi}{3}$$

$$2\tan^{-1}\left(\frac{6y}{9-y^2}\right)=-\frac{\pi}{3}$$

$$y^2-6\sqrt{3}y-9=0 \Rightarrow y=3\sqrt{3}-6\ (\because\ y\in(-3,0))$$

Case-I : $y \in (0,3)$

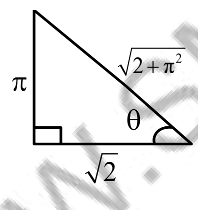
$$2\tan^{-1}\left(\frac{6y}{9-y^2}\right)=\frac{2\pi}{3} \Rightarrow \sqrt{3}y^2+6y-9\sqrt{3}=0$$

$y=\sqrt{3}$ or $y=-3\sqrt{3}$ (rejected)

sum $=\sqrt{3}+3\sqrt{3}-6=4\sqrt{3}-6$

3. Ans. (2.35 or 2.36)

Sol. $\cos^{-1}\sqrt{\dfrac{2}{2+\pi^2}}=\tan^{-1}\dfrac{\pi}{\sqrt{2}}$



$$\sin^{-1}\left(\frac{2\sqrt{2}\pi}{2+\pi^2}\right)=\sin^{-1}\left(\frac{2\times\frac{\pi}{\sqrt{2}}}{1+\left(\frac{\pi}{\sqrt{2}}\right)^2}\right)$$

$$=\pi-2\tan^{-1}\left(\frac{\pi}{\sqrt{2}}\right)$$

$$\left(\text{As, }\sin^{-1}\left(\frac{2x}{1+x^2}\right)=\pi-2\tan^{-1}x, x\ge 1\right)$$

and $\tan^{-1}\dfrac{\sqrt{2}}{\pi}=\cot^{-1}\left(\dfrac{\pi}{\sqrt{2}}\right)$

$\therefore$ Expression

$$=\frac{3}{2}\left(\tan^{-1}\frac{\pi}{\sqrt{2}}\right)+\frac{1}{4}\left(\pi-2\tan^{-1}\frac{\pi}{\sqrt{2}}\right)+\cot^{-1}\left(\frac{\pi}{\sqrt{2}}\right)$$

$$=\left(\frac{3}{2}-\frac{2}{4}\right)\tan^{-1}\frac{\pi}{\sqrt{2}}+\frac{\pi}{4}+\cot^{-1}\frac{\pi}{\sqrt{2}}$$

$$=\left(\tan^{-1}\frac{\pi}{\sqrt{2}}+\cot^{-1}\frac{\pi}{\sqrt{2}}\right)+\frac{\pi}{4}$$

$$=\frac{\pi}{2}+\frac{\pi}{4}=\frac{3\pi}{4}=2.35 \text{ or } 2.36$$

4. Ans. (A, B)

Sol. $S_n(x)=\displaystyle\sum_{k=1}^{n}\tan^{-1}\left(\frac{x}{1+kx(kx+x)}\right)$

$$=\sum_{k=1}^{n}\tan^{-1}\left(\frac{(kx+x)-(kx)}{1+(kx+x)(kx)}\right)$$

$S_n(x) = \tan^{-1}(nx + x) - \tan^{-1}x$

$$=\tan^{-1}\left(\frac{nx}{1+(n+1)x^2}\right)$$

(A) $S_{10}(x)=\tan^{-1}\dfrac{10x}{1+11x^2}$

$$=\frac{\pi}{2}-\tan^{-1}\left(\frac{1+11x^2}{10x}\right)\ (x>0)$$

(B) $\displaystyle\lim_{n\to\infty}\cot(S_n(x))=\lim_{n\to\infty}\frac{\frac{1}{n}+\left(1+\frac{1}{n}\right)x^2}{x}$

$= x\ (x > 0)$

(C) $S_3(x)=\tan^{-1}\dfrac{3x}{1+4x^2}=\dfrac{\pi}{4}$

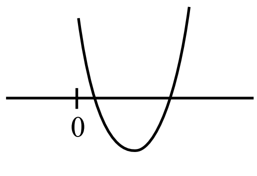
$\Rightarrow 4x^2 - 3x + 1 = 0 \Rightarrow x \notin \mathbb{R}$

(D) $\tan(S_n(x))=\dfrac{nx}{1+(n+1)x^2}$; $\forall\ n \ge 1$; $x > 0$

We need to check the validity of

$$\frac{nx}{1+(n+1)x^2}\le\frac{1}{2}\ \ \forall\ n\ge 1;\ x>0;\ n\in\mathbb{N}$$

$\Rightarrow 2nx \le (n + 1)x^2 + 1$



$\Rightarrow (n + 1)x^2 - 2nx + 1 \ge 0\ \forall\ n \ge 1$; $x > 0$; $n \in \mathbb{N}$

Discriminant of $y = (n + 1)x^2 - 2nx + 1$ is

$D = 4n^2 - 4(n + 1)$ and $n \in \mathbb{N}$

$D < 0$ for $n = 1$; true for $x > 0$

$D > 0$ for $n \geq 2 \Rightarrow \exists$ some $x > 0$

for which $y < 0$ as both roots of

$y = 0$ will be positive.

$y = (n + 1)x^2 - 2nx + 1, n \geq 2$

So, $y \geq 0\ \forall\ n \geq 1$; $\forall\ x > 0$; $n \in N$ is false.

5. Ans. (A, B, D)

Sol. $$f(n) = \frac{\sum_{k=0}^{n}\left(\cos\left(\frac{\pi}{n+2}\right) - \cos\left(\frac{2k+3}{n+2}\right)\pi\right)}{\sum_{k=0}^{n}\left(1 - \cos\left(\frac{2k+2}{n+2}\right)\pi\right)}$$

$$f(n) = \frac{(n+1)\cos\left(\frac{\pi}{n+2}\right) - \left(\sum_{k=0}^{n}\cos\left(\frac{2k+3}{n+2}\right)\pi\right)}{(n+1) - \left(\sum_{k=0}^{n}\cos\left(\frac{2k+2}{n+2}\right)\pi\right)}$$

$$f(n) = \frac{(n+1)\cos\frac{\pi}{n+2} - \left(\frac{\sin\left(\frac{(n+1)\pi}{n+2}\right)}{\sin\left(\frac{\pi}{n+2}\right)}.\cos\left(\frac{n+3}{n+2}\right)\pi\right)}{(n+1) - \left(\frac{\sin\left(\frac{(n+1)\pi}{n+2}\right)}{\sin\left(\frac{\pi}{n+2}\right)}.\cos\left(\frac{2(n+2)\pi}{2(n+2)}\right)\right)}$$

$$f(n) = \frac{(n+1)\cos\left(\frac{\pi}{n+2}\right) + \cos\left(\frac{\pi}{n+2}\right)}{(n+1)+1}$$

$$\Rightarrow g(x) = \cos\left(\frac{\pi}{n+2}\right)$$

(A) $\sin\left(7\cos^{-1}\cos\frac{\pi}{7}\right) = \sin\pi = 0$

(B) $f(4) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

(C) $\lim_{n\to\infty}\cos\left(\frac{\pi}{n+2}\right) = 1$

(D) $\alpha = \tan\left(\cos^{-1}\cos\frac{\pi}{8}\right) = \sqrt{2} - 1 \Rightarrow \alpha + 1 = \sqrt{2}$

$\alpha^2 + 2\alpha - 1 = 0$

6. Ans. (0.00)

Sol. $$\sec^{-1}\left(\frac{1}{4}\sum_{k=0}^{10}\frac{1}{\cos\left(\frac{7\pi}{12}+\frac{k\pi}{12}\right)\cos\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right)}\right)$$

$$= \sec^{-1}\left(\frac{1}{4}\sum_{k=0}^{10}\frac{\sin\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2} - \left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\right)}{\cos\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right).\cos\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right)}\right)$$

$$= \sec^{-1}\left(\frac{1}{4}\left(\sum_{k=0}^{10}\tan\left(\frac{7\pi}{12}+(k+1)\frac{\pi}{2}\right) - \tan\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\right)\right)$$

$$= \sec^{-1}\left(\frac{1}{4}\left(\tan\left(\frac{11\pi}{2}+\frac{7\pi}{12}\right) - \tan\left(\frac{7\pi}{12}\right)\right)\right)$$

$$= \sec^{-1}\left(\frac{1}{4}\left(-\cot\frac{7\pi}{12} - \tan\frac{7\pi}{12}\right)\right)$$

$$= \sec^{-1}\left(\frac{1}{4}\left(-\frac{1}{\sin\frac{7\pi}{12}\cos\frac{7\pi}{12}}\right)\right)$$

$$= \sec^{-1}\left(-\frac{1}{2}\times\frac{1}{\sin\frac{7\pi}{6}}\right) = \sec^{-1}(1) = 0.00$$

7. Ans. (2)

Sol. $$\sum_{i=1}^{\infty}x^{i+1} = \frac{x^2}{1-x}$$

$$\sum_{i=1}^{\infty}\left(\frac{x}{2}\right)^i = \frac{x}{2-x}$$

$$\sum_{i=1}^{\infty}\left(-\frac{x}{2}\right)^i = \frac{-x}{2+x}$$

$$\sum_{i=1}^{\infty}(-x)^i = \frac{-x}{1+x}$$

To have real solutions

$$\sum_{i=1}^{\infty}x^{i+1} - x\sum_{i=1}^{\infty}\left(\frac{x}{2}\right)^i = \sum_{i=1}^{\infty}\left(\frac{-x}{2}\right)^i - \sum_{i=1}^{\infty}(-x)^i$$

$$\frac{x^2}{1-x} - \frac{x^2}{2-x} = \frac{-x}{2+x} + \frac{x}{1+x}$$

$x(x^3 + 2x^2 + 5x - 2) = 0$

$\therefore$ $x = 0$ and let $f(x) = x^3 + 2x^2 + 5x - 2$

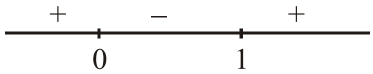
$f'(x) > 0 \Rightarrow f$ is $\uparrow$

$$f\left(\frac{1}{2}\right).f\left(-\frac{1}{2}\right) < 0$$

Hence two solutions exist

8. Ans. (A)

Sol. $E_1 : \dfrac{x}{x-1} > 0$



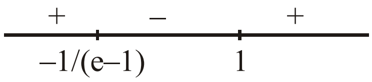
$\Rightarrow \quad E_1 : x \in (-\infty, 0) \cup (1, \infty)$

$E_2 : -1 \le \ell n\left(\dfrac{x}{x+1}\right) \le 1$

$\dfrac{1}{e} \le \dfrac{x}{x-1} \le e$

Now $\dfrac{x}{x-1} - \dfrac{1}{e} \ge 0$

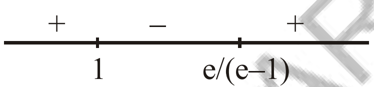
$\Rightarrow \quad \dfrac{(e-1)x+1}{e(x-1)} \ge 0$



$\Rightarrow \quad x \in \left(-\infty, \dfrac{1}{1-e}\right] \cup (1, \infty)$

also $\dfrac{x}{x-1} - e \le 0$

$\dfrac{(e-1)x-e}{x-1} \ge 0$



$\Rightarrow \quad x \in (-\infty, 1) \cup \left[\dfrac{e}{e-1}, \infty\right]$

So $E_2 : \left(-\infty, \dfrac{1}{1-e}\right] \cup \left[\dfrac{e}{e-1}, \infty\right]$

as Range $\dfrac{x}{x-1}$ of is $R^+ - \{1\}$

$\Rightarrow$ Range of f is $R - \{0\}$ or $(-\infty, 0) \cup (0, \infty)$

Range of g is $\left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right] \setminus \{0\}$ or

$\left[-\dfrac{\pi}{2}, 0\right) \cup \left(0, \dfrac{\pi}{2}\right]$

Now $P \to 4, Q \to 2, R \to 1, S \to 1$

9. Ans. (B, C, D)

Sol. $\alpha = 3\sin^{-1}\dfrac{6}{11}$ & $\beta = 3\cos^{-1}\dfrac{4}{9}$

$\because \dfrac{6}{11} > \dfrac{1}{2} \Rightarrow \sin^{-1}\dfrac{6}{11} > \sin^{-1}\dfrac{1}{2}$

$\Rightarrow 3\sin^{-1}\dfrac{6}{11} > 3\sin^{-1}\dfrac{1}{2} = \dfrac{\pi}{2}$

$\therefore \alpha > \dfrac{\pi}{2}$

$\therefore \cos\alpha < 0$

Now, $\beta = 3\cos^{-1}\dfrac{4}{9}$

$\because \dfrac{4}{9} < \dfrac{1}{2} \Rightarrow 3\cos^{-1}\dfrac{4}{9} > 3\cos^{-1}\dfrac{1}{2}$

$\therefore \beta > \pi$

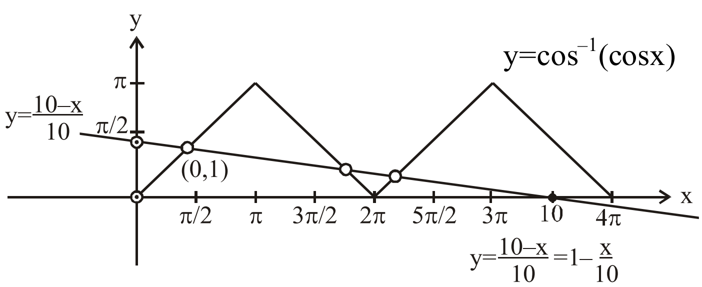
$\therefore \cos\beta < 0$ & $\sin\beta < 0$

Now, α is slightly greater than $\dfrac{\pi}{2}$ & β is slightly greater than π

$\therefore \cos(\alpha + \beta) > 0$

10. Ans. (3)

Sol.



from above figure it is clear that $y = \dfrac{10-x}{10}$ and $y = \cos^{-1}(\cos x)$ intersect at 3 distinct points, so number of solutions = 3