SOLUTION OF TRIANGLE

Paragraph for Question No. 1 and 2

Consider on obtuse angled triangle ABC in which the difference between the largest and the smallest angle is $\frac{\pi}{2}$ and whose sides are in arithmetic progression. Suppose that the vertices of this triangle lie on a circle of radius 1.

1. Let a be the area of the triangle ABC. Then the value of $(64a)^2$ is **[JEE(Advanced) 2023]**

2. Then the inradius of the triangle ABC is **[JEE(Advanced) 2023]**

3. Let PQRS be a quadrilateral in a plane, where QR = 1, $\angle PQR = \angle QRS = 70^\circ$, $\angle PQS = 15^\circ$ and $\angle PRS = 40^\circ$. If $\angle RPS = \theta^\circ$, PQ = α and PS = β, then the interval(s) that contain(s) the value of $4\alpha\beta \sin\theta^\circ$ is/are **[JEE(Advanced) 2022]**

(A) $(0, \sqrt{2})$ (B) (1, 2)

(C) $(\sqrt{2}, 3)$ (D) $(2\sqrt{2}, 3\sqrt{2})$

4. In a triangle ABC, let $AB = \sqrt{23}$, $BC = 3$ and CA= 4. Then the value of $\frac{\cot A + \cot C}{\cot B}$ is_____. **[JEE(Advanced) 2021]**

5. Consider a triangle PQR having sides of lengths p, q and r opposite to the angles P, Q and R, respectively. Then which of the following statements is (are) **TRUE**? **[JEE(Advanced) 2021]**

(A) $\cos P \geq 1 - \frac{p^2}{2qr}$ (B) $\cos R \geq \left(\frac{q-r}{p+q}\right)\cos P + \left(\frac{p-r}{p+q}\right)\cos Q$

(C) $\frac{q+r}{p} < 2\frac{\sqrt{\sin Q \sin R}}{\sin P}$ (D) If p < q and p < r, then $\cos Q > \frac{p}{r}$ and $\cos R > \frac{p}{q}$

6. Let x, y and z be positive real numbers. Suppose x, y and z are lengths of the sides of a triangle opposite to its angles X, Y and Z, respectively. If

$$\tan\frac{X}{2} + \tan\frac{Z}{2} = \frac{2y}{x+y+z},$$

then which of the following statements is/are TRUE? **[JEE(Advanced) 2020]**

(A) 2Y = X + Z (B) Y = X + Z

(C) $\tan\frac{X}{2} = \frac{x}{y+z}$ (D) $x^2 + z^2 - y^2 = xz$

7. In a non-right-angled triangle ΔPQR, let p, q, r denote the lengths of the sides opposite to the angles at P, Q, R respectively. The median from R meets the side PQ at S, the perpendicular from P meets the side QR at E, and RS and PE intersect at O. If $p = \sqrt{3}$, q = 1, and the radius of the circumcircle of the ΔPQR equals 1, then which of the following options is/are correct ? **[JEE(Advanced) 2019]**

(A) Area of $\Delta SOE = \frac{\sqrt{3}}{12}$ (B) Radius of incircle of $\Delta PQR = \frac{\sqrt{3}}{2}(2-\sqrt{3})$

(C) Length of $RS = \frac{\sqrt{7}}{2}$ (D) Length of $OE = \frac{1}{6}$

8. In a triangle PQR, let $\angle PQR = 30°$ and the sides PQ and QR have lengths $10\sqrt{3}$ and 10, respectively. Then, which of the following statement(s) is (are) TRUE? **[JEE(Advanced) 2018]**

(A) $\angle QPR = 45°$

(B) The area of the triangle PQR is $25\sqrt{3}$ and $\angle QRP = 120°$

(C) The radius of the incircle of the triangle PQR is $10\sqrt{3}-15$

(D) The area of the circumcircle of the triangle PQR is 100π.

9. In a triangle XYZ, let x,y,z be the lengths of sides opposite to the angles X,Y,Z, respectively and $2s = x + y + z$. If $\frac{s-x}{4}=\frac{s-y}{3}=\frac{s-z}{2}$ and area of incircle of the triangle XYZ is $\frac{8\pi}{3}$, then-

[JEE(Advanced) 2017]

(A) area of the triangle XYZ is $6\sqrt{6}$

(B) the radius of circumcircle of the triangle XYZ is $\frac{35}{6}\sqrt{6}$

(C) $\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2}=\frac{4}{35}$

(D) $\sin^2\left(\frac{X+Y}{2}\right)=\frac{3}{5}$

10. In a triangle the sum of two sides is x and the product of the same two sides is y. If $x^2 - c^2 = y$, where c is a third side of the triangle, then the ratio of the in-radius to the circum-radius of the triangle is -

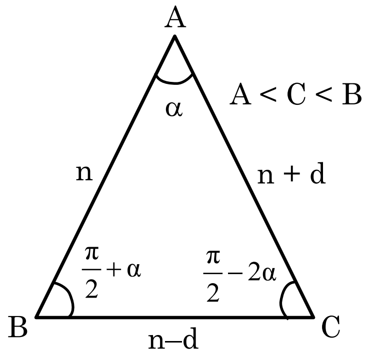
[JEE(Advanced) 2014]

(A) $\frac{3y}{2x(x+c)}$ (B) $\frac{3y}{2c(x+c)}$ (C) $\frac{3y}{4x(x+c)}$ (D) $\frac{3y}{4c(x+c)}$

SOLUTIONS

1. Ans. (1008.00)

Sol.



$n - d = 2\sin\alpha$ ....(1)

$n + d = 2\sin\left(\frac{\pi}{2}+\alpha\right)$

$\Rightarrow n + d = 2\cos\alpha$ ....(2)

$n = 2\sin\left(\frac{\pi}{2}-2\alpha\right)$

$\Rightarrow n = 2\cos 2\alpha$ ....(3)

$\Rightarrow 2\cos 2\alpha = \sin\alpha + \cos\alpha$

$\Rightarrow 2(\cos\alpha - \sin\alpha) = 1$

$\Rightarrow \sin 2\alpha = \frac{3}{4}$

Then,

$a = \frac{1}{2}.n.(n+d).\sin\alpha = \frac{1}{2}.2\cos 2\alpha.2\cos\alpha.\sin\alpha$

$= \sin 2\alpha.\cos 2\alpha$

$= \frac{3}{4}\times\frac{\sqrt{7}}{4} = \frac{3\sqrt{7}}{16}$

$(64a)^2 = \left(64\times\frac{3\sqrt{7}}{16}\right)^2 = 16\times 9\times 7 = 1008$

2. Ans. (0.25)

Sol. From above equation in Ques. 1

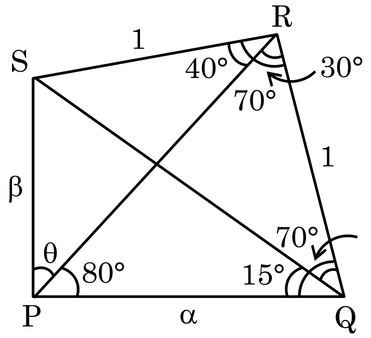
$r = \frac{\Delta}{s} = \frac{1}{2}\frac{n(n+d)\sin\alpha}{\left(\frac{3n}{2}\right)}$

$= \frac{(n+d).\sin\alpha}{3} = \frac{2\cos\alpha.\sin\alpha}{3}$ (from (2))

$r = \frac{\sin 2\alpha}{3} = \frac{1}{4}$

3. Ans. (A, B)

Sol.



$\angle PRQ = 70° - 40° = 30°$

$\angle RQS = 70° - 15° = 55°$

$\angle QSR = 180° - 55° - 70° = 55$

$\therefore$ QR = RS = 1

$\angle QPR = 180° - 70° - 30° = 80°$

Apply sine-rule in ΔPRQ :

$\frac{\alpha}{\sin 30°} = \frac{1}{\sin 80°} \Rightarrow \alpha = \frac{1}{2\sin 80°}$ ....(1)

Apply sine-rule in ΔPRS

$\frac{\beta}{\sin 40°} = \frac{1}{\sin\theta} \Rightarrow \beta\sin\theta = \sin 40°$ ....(2)

$4\alpha\beta\sin\theta = \frac{4\sin 40°}{2\sin 80°} = \frac{4\sin 40°}{2(2\sin 40°\cos 40°)}$

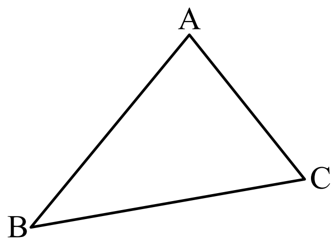
$= \sec 40°$

Now $\sec 30° < \sec 40° < \sec 45°$

$\Rightarrow \frac{2}{\sqrt{3}} < \sec 40° < \sqrt{2}$

4. Ans. (2)

Sol.



Given $c = \sqrt{23}$; a = 3; b = 4

$\cot A = \frac{\cos A}{\sin A} = \frac{b^2+c^2-a^2}{2bc\sin A}$

$= \frac{b^2+c^2-a^2}{2.2\Delta}$ $\left\{\Delta = \frac{1}{2}bc\sin A\right\}$

$\cot A = \frac{b^2+c^2-a^2}{4\Delta}$

Similarly, $\cot B = \dfrac{a^2 + c^2 - b^2}{4\Delta}$

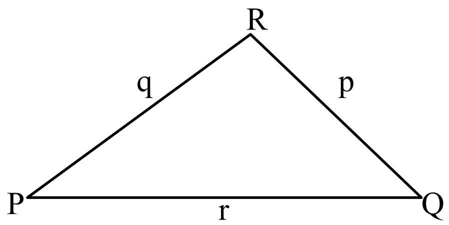
& $\cot C = \dfrac{a^2 + b^2 - c^2}{4\Delta}$

$\therefore \dfrac{\cot A + \cot C}{\cot B} = \dfrac{b^2 + c^2 - a^2 + a^2 + b^2 - c^2}{a^2 + c^2 - b^2}$

$= \dfrac{2b^2}{a^2 + c^2 - b^2} = \dfrac{32}{16} = 2$

5. Ans. (A, B)

Sol.



(A)

$\text{Cos}P = \dfrac{q^2 + r^2 - p^2}{2qr} = \dfrac{q^2 + r^2}{2qr} - \dfrac{p^2}{2qr} \geq 1 - \dfrac{p^2}{2qr}$

(as $p^2 + q^2 \geq 2qr$ (AM $\geq$ GM)),

So (A) is correct

(B) $(p + q)\cos R \geq (q - r)\cos P + (p - r)\cos Q$

$\Rightarrow$ $(p\cos R + r\cos P) + (q\cos R + r\cos Q)$

$\geq q\cos P + p\cos Q$

$\Rightarrow$ $q + p \geq r$

So (B) is correct

(C) $\dfrac{q + r}{p} = \dfrac{\sin Q + \sin R}{\sin P} \geq \dfrac{2\sqrt{\sin Q \times \sin R}}{\sin P}$

So (C) is incorrect

(D) $\cos Q > \dfrac{p}{r}$ $\Rightarrow \sin R \cos Q > \sin P$

$\Rightarrow \sin P + \sin(R - Q) > 2\sin P$

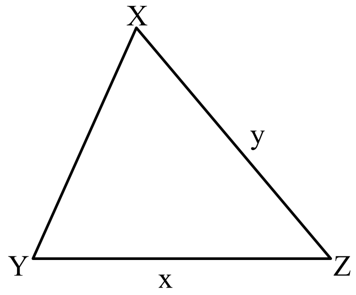
$\Rightarrow \sin(R - Q) > \sin P$

need not necessarily hold true if $R < Q$

Hence (A), (B)

6. Ans. (B, C)

Sol.



$\tan\dfrac{X}{2} + \tan\dfrac{Z}{2} = \dfrac{2y}{x + y + z}$

$\dfrac{\Delta}{S(S - x)} + \dfrac{\Delta}{S(S - z)} = \dfrac{2y}{2S}$

$\dfrac{\Delta}{S}\left(\dfrac{2S - (x + z)}{(S - x)(S - z)}\right) = \dfrac{y}{S}$

$\Rightarrow \dfrac{\Delta y}{S(S - x)(S - z)} = \dfrac{y}{S}$

$\Rightarrow \Delta^2 = (S - x)^2 (S - z)^2$

$\Rightarrow S(S - y) = (S - x)(S - z)$

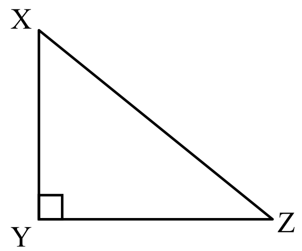
$\Rightarrow (x + y + z)(x + z - y) = (y + z - x)(x + y - z)$

$\Rightarrow (x + z)^2 - y^2 = y^2 - (z - x)^2$

$\Rightarrow (x + z)^2 + (x - z)^2 = 2y^2$

$\Rightarrow x^2 + z^2 = y^2 \Rightarrow \angle Y = \dfrac{\pi}{2}$

$\Rightarrow \angle Y = \angle X + \angle Z$



$\tan\dfrac{X}{2} = \dfrac{\Delta}{S(S - x)}$

$\tan\dfrac{X}{2} = \dfrac{\frac{1}{2}xz}{\frac{(y + z)^2 - x^2}{4}}$

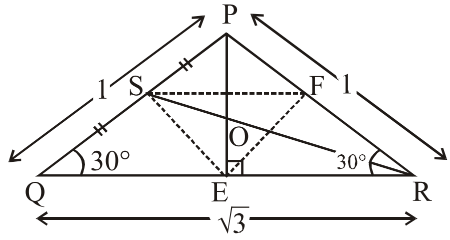
$\tan\dfrac{X}{2} = \dfrac{2xz}{y^2 + z^2 + 2yz - x^2}$

$\tan\dfrac{X}{2} = \dfrac{2xz}{2z^2 + 2yz}$ (using $y^2 = x^2 + z^2$)

$\tan\dfrac{X}{2} = \dfrac{x}{y + z}$

7. Ans. (B, C, D)

Sol.



$$\frac{\sin P}{\sqrt{3}} = \frac{\sin Q}{1} = \frac{1}{2R} = \frac{1}{2}$$

$$\Rightarrow P = \frac{\pi}{3} \text{ or } \frac{2\pi}{3} \text{ and } Q = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$$

Since $p > q \Rightarrow P > Q$

So, if $P = \frac{\pi}{3}$ and $Q = \frac{\pi}{6} \Rightarrow R = \frac{\pi}{2}$

(not possible)

Hence, $P = \frac{2\pi}{3}$ and $Q = R = \frac{\pi}{6}$

$$r = \frac{\Delta}{s} = \frac{\frac{1}{2}(1)(\sqrt{3})\left(\frac{1}{2}\right)}{\left(\frac{\sqrt{3}+2}{2}\right)} = \frac{\sqrt{3}}{2}(2-\sqrt{3})$$

Now, area of ΔSEF = $\frac{1}{4}$ area of ΔPQR

$\Rightarrow$ area of ΔSOE = $\frac{1}{3}$ area of ΔSEF = $\frac{1}{12}$

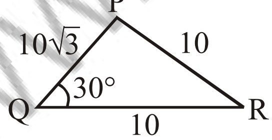
area of ΔPQR = $\frac{1}{12}.\frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{48}$

$$RS = \frac{1}{2}\sqrt{2(3)+2(1)-1} = \frac{\sqrt{7}}{2}$$

$$OE = \frac{1}{3}PE = \frac{1}{3}.\frac{1}{2}\sqrt{2(1)^2+2(1)^2-3} = \frac{1}{6}$$

8. Ans. (B, C, D)

Sol.



$$\cos 30° = \frac{(10\sqrt{3})^2 + (10)^2 - (PR)^2}{2 \times 10\sqrt{3} \times 10}$$

$\Rightarrow$ PR = 10

$\because$ QR = PR $\Rightarrow$ $\angle PQR = \angle QPR$

$\angle QPR = 30°$

(B) area of Δ PQR

$$= \frac{1}{2} \times 10\sqrt{3} \times 10 \times \sin 30° = \frac{1}{2} \times 10 \times 10\sqrt{3} \times \frac{1}{2}$$

$= 25\sqrt{3}$

$\angle QRP = 180° - (30° + 30°) = 120°$

(C) $$r = \frac{\Delta}{s} = \frac{25\sqrt{3}}{\left(\frac{10+10+10\sqrt{3}}{2}\right)} = \frac{25\sqrt{3}}{10+5\sqrt{3}}$$

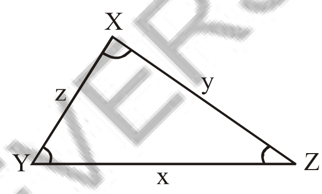
$$= 5\sqrt{3}.(2-\sqrt{3}) = 10\sqrt{3} - 15$$

(D) $$R = \frac{a}{2\sin A} = \frac{10}{2\sin 30°} = 10$$

$\therefore$ Area = $\pi R^2 = 100\pi$

9. Ans. (A, C, D)

Sol.



Let $\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2} = k$

$s - x = 4k \Rightarrow y + z - x = 8k$

$s - y = 3k \Rightarrow x + z - y = 6k$

$s - z = 2k \Rightarrow x + y - z = 4k$

$\Rightarrow x = 5k, y = 6k, z = 7k$

$\Rightarrow 3s - (x + y + z) = 9k$

$s = 9k$

Let r be inradius

$$\Rightarrow \pi r^2 = \frac{8\pi}{3}$$

$$\pi\left(\frac{\Delta}{s}\right)^2 = \frac{8\pi}{3}$$

$$\Delta = \sqrt{\frac{8}{3}}.s$$

$$\sqrt{s(s-x)(s-y)(s-z)} = \sqrt{\frac{8}{3}}.s$$

$$\sqrt{9k.4k.3k.2k} = \sqrt{\frac{8}{3}}9k$$

$$\sqrt{24.9}k^2 = \sqrt{\frac{8}{3}}.9k$$

k = 1

$\Rightarrow \quad x = 5, y = 6, z = 7$

$$\Delta = \sqrt{\frac{8}{3}}.9k = \sqrt{\frac{8}{3}}.9 = 6\sqrt{6}$$

$$R = \text{circumradius} = \frac{xyz}{4\Delta} = \frac{5.6.7}{4.6\sqrt{6}} = \frac{35}{4\sqrt{6}}$$

$$\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2} = \frac{r}{4R} = \frac{\frac{\Delta}{s}}{\frac{xyz}{\Delta}} = \frac{\Delta^2}{s.xyz}$$

$$= \frac{36.6}{9.5.6.7} = \frac{4}{35}$$

$$\cos Z = \frac{x^2+y^2-z^2}{2xy} = \frac{25+36-49}{2.5.6} = \frac{1}{5}$$

$$\sin^2\left(\frac{X+Y}{2}\right) = \cos^2\frac{Z}{2} = \frac{1+\cos Z}{2} = \frac{1+\frac{1}{5}}{2} = \frac{3}{5}$$

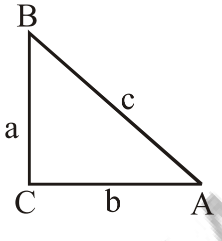
10. Ans. (B)

Sol. $a + b = x$

$ab = y$

$(a+b)^2 - c^2 = ab$

$\Rightarrow a^2 + b^2 - c^2 = -ab$



$$\cos c = -\frac{1}{2} \Rightarrow c = \frac{2\pi}{3}$$

$$\text{Now, } \frac{r}{R} = \frac{\frac{\Delta}{S}}{\frac{abc}{4\Delta}} = \frac{4\Delta^2}{\frac{(a+b+c)}{2}abc}$$

$$= \frac{8\left(\frac{1}{2}ab\frac{\sqrt{3}}{2}\right)^2}{(x+c)(yc)} = \frac{\frac{1}{2}y^2.3}{(x+c)cy} = \frac{3y}{2c(x+c)}$$