COMPOUND ANGLE

1. Let α and β be real numbers such that $-\frac{\pi}{4} < \beta < 0 < \alpha < \frac{\pi}{4}$. If $\sin(\alpha+\beta) = \frac{1}{3}$ and $\cos(\alpha-\beta) = \frac{2}{3}$, then the greatest integer less than or equal to

$$\left(\frac{\sin\alpha}{\cos\beta} + \frac{\cos\beta}{\sin\alpha} + \frac{\cos\alpha}{\sin\beta} + \frac{\sin\beta}{\cos\alpha} \right)^2$$

is _____. **[JEE(Advanced) 2022]**

2. Let α and β be nonzero real numbers such that $2(\cos\beta - \cos\alpha) + \cos\alpha \cos\beta = 1$. Then which of the following is/are true ? **[JEE(Advanced) 2017]**

(A) $\tan\left(\frac{\alpha}{2}\right) - \sqrt{3}\tan\left(\frac{\beta}{2}\right) = 0$

(B) $\sqrt{3}\tan\left(\frac{\alpha}{2}\right) - \tan\left(\frac{\beta}{2}\right) = 0$

(C) $\tan\left(\frac{\alpha}{2}\right) + \sqrt{3}\tan\left(\frac{\beta}{2}\right) = 0$

(D) $\sqrt{3}\tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\beta}{2}\right) = 0$

3. The value of $\displaystyle\sum_{k=1}^{13} \frac{1}{\sin\left(\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right)\sin\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)}$ is equal to **[JEE(Advanced) 2016]**

(A) $3 - \sqrt{3}$ (B) $2(3 - \sqrt{3})$ (C) $2(\sqrt{3} - 1)$ (D) $2(2 + \sqrt{3})$

SOLUTIONS

1. **Ans. (1)**

Sol. $\alpha \in \left(0, \dfrac{\pi}{4}\right), \beta \in \left(-\dfrac{\pi}{4}, 0\right) \Rightarrow \alpha + \beta \in \left(-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right)$

$\sin(\alpha + \beta) = \dfrac{1}{3}, \cos(\alpha - \beta) = \dfrac{2}{3}$

$\left(\dfrac{\sin\alpha}{\cos\beta} + \dfrac{\cos\alpha}{\sin\beta} + \dfrac{\cos\beta}{\sin\alpha} + \dfrac{\sin\beta}{\cos\alpha}\right)^2$

$\left(\dfrac{\cos(\alpha - \beta)}{\cos\beta\sin\beta} + \dfrac{\cos(\beta - \alpha)}{\sin\alpha\cos\alpha}\right)^2$

$= 4\cos^2(\alpha - \beta)\left(\dfrac{1}{\sin 2\beta} + \dfrac{1}{\sin 2\alpha}\right)^2$

$= 4\cos^2(\alpha - \beta)\left(\dfrac{2\sin(\alpha + \beta)\cos(\alpha - \beta)}{\sin 2\alpha \sin 2\beta}\right)^2 \ldots(1)$

$= \dfrac{16\cos^4(\alpha - \beta)\sin^2(\alpha + \beta) \times 4}{\left(\cos 2(\alpha - \beta) - \cos 2(\alpha + \beta)\right)^2}$

$= \dfrac{64\cos^4(\alpha - \beta)\sin^2(\alpha + \beta)}{\left(2\cos^2(\alpha - \beta) - 1 - 1 + 2\sin^2(\alpha + \beta)\right)^2}$

$= 64 \times \dfrac{16}{81} \times \dfrac{1}{9} \dfrac{1}{\left(2 \times \dfrac{4}{9} - 1 - 1 + \dfrac{2}{9}\right)^2}$

$= \dfrac{64 \times 16}{81 \times 9} \cdot \dfrac{81}{64} = \dfrac{16}{9}$

$\left[\dfrac{16}{9}\right] = 1$

2. **Ans. (Bonus)**

3. **Ans. (C)**

Sol. We have,

$= 2.\displaystyle\sum_{k=1}^{13} \dfrac{\sin\left(\left(\dfrac{k\pi}{6} + \dfrac{\pi}{4}\right) - \left((k-1)\dfrac{\pi}{6} + \dfrac{\pi}{4}\right)\right)}{\sin\left(\dfrac{\pi}{4} + (k-1)\dfrac{\pi}{6}\right).\sin\left(\dfrac{\pi}{4} + \dfrac{k\pi}{6}\right)}$

$= 2\displaystyle\sum_{k=1}^{13} \left(\cot\left((k-1)\dfrac{\pi}{6} + \dfrac{\pi}{4}\right) - \cot\left(\dfrac{k\pi}{6} + \dfrac{\pi}{4}\right)\right)$

$= 2\left[\cot\dfrac{\pi}{4} - \cot\left(\dfrac{13\pi}{6} + \dfrac{\pi}{4}\right)\right]$

$= 2\left(1 - \cot\left(\dfrac{5\pi}{12}\right)\right)$

$= 2\left(1 - (2 - \sqrt{3})\right) = 2(\sqrt{3} - 1)$