

COMPLEX NUMBER

1. Let $A = \left\{ \frac{1967 + 1686i \sin \theta}{7 - 3i \cos \theta} : \theta \in \mathbb{R} \right\}$. If A contains exactly one positive integer n, then the value of n is
 [JEE(Advanced) 2023]
2. Let z be complex number satisfying $|z|^3 + 2z^2 + 4\bar{z} - 8 = 0$, where \bar{z} denotes the complex conjugate of z. Let the imaginary part of z be nonzero.
 [JEE(Advanced) 2023]

Match each entry in **List-I** to the correct entries in **List-II**.

List-I	List-II
(P) $ z ^2$ is equal to	(1) 12
(Q) $ z - \bar{z} ^2$ is equal to	(2) 4
(R) $ z ^2 + z + \bar{z} ^2$ is equal to	(3) 8
(S) $ z + 1 ^2$ is equal to	(4) 10
	(5) 7

The correct option is :

- (A) (P) → (1) (Q) → (3) (R) → (5) (S) → (4)
 (B) (P) → (2) (Q) → (1) (R) → (3) (S) → (5)
 (C) (P) → (2) (Q) → (4) (R) → (5) (S) → (1)
 (D) (P) → (2) (Q) → (3) (R) → (5) (S) → (4)
3. Let $A_1, A_2, A_3, \dots, A_8$ be the vertices of a regular octagon that lie on a circle of radius 2. Let P be a point on the circle and let PA_i denote the distance between the points P and A_i for $i = 1, 2, \dots, 8$. If P varies over the circle, then the maximum value of the product $PA_1 \cdot PA_2 \cdot \dots \cdot PA_8$, is
 [JEE(Advanced) 2023]

4. Let z be a complex number with non-zero imaginary part. If

$$\frac{2 + 3z + 4z^2}{2 - 3z + 4z^2}$$

is a real number, then the value of $|z|^2$ is _____.
 [JEE(Advanced) 2022]

5. Let \bar{z} denote the complex conjugate of a complex number z and let $i = \sqrt{-1}$. In the set of complex numbers, the number of distinct roots of the equation

$$\bar{z} - z^2 = i(\bar{z} + z^2)$$

is _____.
 [JEE(Advanced) 2022]

6. Let \bar{z} denote the complex conjugate of a complex number z. If z is a non-zero complex number for which both real and imaginary parts of

$$(\bar{z})^2 + \frac{1}{z^2}$$

are integers, then which of the following is/are possible value(s) of $|z|$?
 [JEE(Advanced) 2022]

- (A) $\left(\frac{43 + 3\sqrt{205}}{2} \right)^{\frac{1}{4}}$ (B) $\left(\frac{7 + \sqrt{33}}{4} \right)^{\frac{1}{4}}$
 (C) $\left(\frac{9 + \sqrt{65}}{4} \right)^{\frac{1}{4}}$ (D) $\left(\frac{7 + \sqrt{13}}{6} \right)^{\frac{1}{4}}$

7. Let $\theta_1, \theta_2, \dots, \theta_{10}$ be positive valued angles (in radian) such that $\theta_1 + \theta_2 + \dots + \theta_{10} = 2\pi$. Define the complex numbers $z_1 = e^{i\theta_1}$, $z_k = z_{k-1}e^{i\theta_k}$ for $k = 2, 3, \dots, 10$, where $i = \sqrt{-1}$. Consider the statements P and Q given below : [JEE(Advanced) 2021]

$$P : |z_2 - z_1| + |z_3 - z_2| + \dots + |z_{10} - z_9| + |z_1 - z_{10}| \leq 2\pi$$

$$Q : |z_2^2 - z_1^2| + |z_3^2 - z_2^2| + \dots + |z_{10}^2 - z_9^2| + |z_1^2 - z_{10}^2| \leq 4\pi$$

Then,

- (A) P is **TRUE** and Q is **FALSE** (B) Q is **TRUE** and P is **FALSE**
 (C) both P and Q are **TRUE** (D) both P and Q are **FALSE**
8. For any complex number $w = c + id$, let $\arg(w) \in (-\pi, \pi]$, where $i = \sqrt{-1}$. Let α and β be real numbers such that for all complex numbers $z = x + iy$ satisfying $\arg\left(\frac{z+\alpha}{z+\beta}\right) = \frac{\pi}{4}$, the ordered pair (x, y) lies on the circle

$$x^2 + y^2 + 5x - 3y + 4 = 0.$$

Then which of the following statements is (are) **TRUE** ? [JEE(Advanced) 2021]

- (A) $\alpha = -1$ (B) $\alpha\beta = 4$ (C) $\alpha\beta = -4$ (D) $\beta = 4$
9. Let S be the set of all complex numbers z satisfying $|z^2 + z + 1| = 1$. Then which of the following statements is/are **TRUE** ? [JEE(Advanced) 2020]
- (A) $\left|z + \frac{1}{2}\right| \leq \frac{1}{2}$ for all $z \in S$ (B) $|z| \leq 2$ for all $z \in S$
 (C) $\left|z + \frac{1}{2}\right| \geq \frac{1}{2}$ for all $z \in S$ (D) The set S has exactly four elements
10. For a complex number z , let $\text{Re}(z)$ denote the real part of z . Let S be the set of all complex numbers z satisfying $z^4 - |z|^4 = 4iz^2$, where $i = \sqrt{-1}$. Then the minimum possible value of $|z_1 - z_2|^2$, where $z_1, z_2 \in S$ with $\text{Re}(z_1) > 0$ and $\text{Re}(z_2) < 0$, is _____. [JEE(Advanced) 2020]
11. Let S be the set of all complex numbers z satisfying $|z - 2 + i| \geq \sqrt{5}$. If the complex number z_0 is such that $\frac{1}{|z_0 - 1|}$ is the maximum of the set $\left\{\frac{1}{|z - 1|} : z \in S\right\}$, then the principal argument of $\frac{4 - z_0 - \bar{z}_0}{z_0 - \bar{z}_0 + 2i}$ is [JEE(Advanced) 2019]

$$\frac{1}{|z_0 - 1|} \text{ is the maximum of the set } \left\{ \frac{1}{|z - 1|} : z \in S \right\}, \text{ then the principal argument of } \frac{4 - z_0 - \bar{z}_0}{z_0 - \bar{z}_0 + 2i} \text{ is}$$

- (A) $\frac{\pi}{4}$ (B) $-\frac{\pi}{2}$ (C) $\frac{3\pi}{4}$ (D) $\frac{\pi}{2}$
12. Let $\omega \neq 1$ be a cube root of unity. Then the minimum of the set $\{a + b\omega + c\omega^2\}^2 : a, b, c \text{ distinct non-zero integers}$ equals _____. [JEE(Advanced) 2019]

13. For a non-zero complex number z , let $\arg(z)$ denotes the principal argument with $-\pi < \arg(z) \leq \pi$. Then, which of the following statement(s) is (are) **FALSE**? **[JEE(Advanced) 2018]**

(A) $\arg(-1 - i) = \frac{\pi}{4}$, where $i = \sqrt{-1}$

(B) The function $f : \mathbb{R} \rightarrow (-\pi, \pi]$, defined by $f(t) = \arg(-1 + it)$ for all $t \in \mathbb{R}$, is continuous at all points of \mathbb{R} , where $i = \sqrt{-1}$

(C) For any two non-zero complex numbers z_1 and z_2 , $\arg\left(\frac{z_1}{z_2}\right) - \arg(z_1) + \arg(z_2)$ is an integer multiple of 2π

(D) For any three given distinct complex numbers z_1, z_2 and z_3 , the locus of the point z satisfying the condition $\arg\left(\frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}\right) = \pi$, lies on a straight line

14. Let s, t, r be the non-zero complex numbers and L be the set of solutions $z = x + iy$ ($x, y \in \mathbb{R}, i = \sqrt{-1}$) of the equation $sz + t\bar{z} + r = 0$, where $\bar{z} = x - iy$. Then, which of the following statement(s) is (are) **TRUE** ? **[JEE(Advanced) 2018]**

(A) If L has exactly one element, then $|s| \neq |t|$

(B) If $|s| = |t|$, then L has infinitely many elements

(C) The number of elements in $L \cap \{z : |z - 1 + i| = 5\}$ is at most 2

(D) If L has more than one element, then L has infinitely many elements

15. Let a, b, x and y be real numbers such that $a - b = 1$ and $y \neq 0$. If the complex number $z = x + iy$ satisfies

$\operatorname{Im}\left(\frac{az + b}{z + 1}\right) = y$, then which of the following is(are) possible value(s) of x ? **[JEE(Advanced) 2017]**

(A) $-1 - \sqrt{1 - y^2}$ (B) $1 + \sqrt{1 + y^2}$ (C) $1 - \sqrt{1 + y^2}$ (D) $-1 + \sqrt{1 - y^2}$

16. Let $z = \frac{-1 + \sqrt{3}i}{2}$, where $i = \sqrt{-1}$, and $r, s \in \{1, 2, 3\}$. Let $P = \begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix}$ and I be the identity matrix

of order 2. Then the total number of ordered pairs (r, s) for which $P^2 = -I$ is **[JEE(Advanced) 2016]**

17. Let $a, b \in \mathbb{R}$ and $a^2 + b^2 \neq 0$. Suppose $S = \left\{z \in \mathbb{C} : z = \frac{1}{a + ibt}, t \in \mathbb{R}, t \neq 0\right\}$, where.

If $z = x + iy$ and $z \in S$, then (x, y) lies on **[JEE(Advanced) 2016]**

(A) the circle with radius $\frac{1}{2a}$ and centre $\left(\frac{1}{2a}, 0\right)$ for $a > 0, b \neq 0$

(B) the circle with radius $-\frac{1}{2a}$ and centre $\left(-\frac{1}{2a}, 0\right)$ for $a < 0, b \neq 0$

(C) the x -axis for $a \neq 0, b = 0$

(D) the y -axis for $a = 0, b \neq 0$

18.

Column-I		Column-II	
(A)	In \mathbb{R}^2 , if the magnitude of the projection vector of the vector $\alpha\hat{i} + \beta\hat{j}$ on $\sqrt{3}\hat{i} + \hat{j}$ is $\sqrt{3}$ and if $\alpha = 2 + \sqrt{3}\beta$, then possible value(s) of $ \alpha $ is (are)	(P)	1
(B)	Let a and b be real numbers such that the function $f(x) = \begin{cases} -3ax^2 - 2, & x < 1 \\ bx + a^2, & x \geq 1 \end{cases}$ is differentiable for all $x \in \mathbb{R}$. Then possible value(s) of a is (are)	(Q)	2
(C)	Let $\omega \neq 1$ be a complex cube root of unity. If $(3 - 3\omega + 2\omega^2)^{4n+3} + (2 + 3\omega - 3\omega^2)^{4n+3} + (-3 + 2\omega + 3\omega^2)^{4n+3} = 0$ then possible value(s) of n is (are)	(R)	3
(D)	Let the harmonic mean of two positive real number a and b be 4, If q is a positive real number such that a, 5, q, b is an arithmetic progression, then the value(s) of $ q - a $ is (are)	(S)	4
		(T)	5

19. For any integer k, let $\alpha_k = \cos\left(\frac{k\pi}{7}\right) + i \sin\left(\frac{k\pi}{7}\right)$, where $i = \sqrt{-1}$. The value of the expression

$$\frac{\sum_{k=1}^{12} |\alpha_{k+1} - \alpha_k|}{\sum_{k=1}^3 |\alpha_{4k-1} - \alpha_{4k-2}|}$$
 is

[JEE(Advanced) 2015]

20. The quadratic equation $p(x) = 0$ with real coefficients has purely imaginary roots. Then the equation $p(p(x)) = 0$ has

[JEE(Advanced) 2014]

- (A) only purely imaginary roots (B) all real roots
 (C) two real and two purely imaginary roots (D) neither real nor purely imaginary roots.

21. Let $z_k = \cos\left(\frac{2k\pi}{10}\right) + i \sin\left(\frac{2k\pi}{10}\right)$; $k = 1, 2, \dots, 9$.

[JEE(Advanced) 2014]

List-I

List-II

- P. For each z_k there exists a z_j such that $z_k \cdot z_j = 1$
 Q. There exists a $k \in \{1, 2, \dots, 9\}$ such that $z_1 \cdot z = z_k$ has no solution z in the set of complex numbers.

1. True
 2. False

R. $\frac{|1 - z_1| |1 - z_2| \dots |1 - z_9|}{10}$ equals

3. 1

S. $1 - \sum_{k=1}^9 \cos\left(\frac{2k\pi}{10}\right)$ equals

4. 2

Codes :

	P	Q	R	S
(A)	1	2	4	3
(B)	2	1	3	4
(C)	1	2	3	4
(D)	2	1	4	3

SOLUTIONS

1. **Ans. (281)**

Sol.
$$A = \frac{1967 + 1686i \sin \theta}{7 - 3i \cos \theta}$$

$$= \frac{281(7 + 6i \sin \theta)}{7 - 3i \cos \theta} \times \frac{7 + 3i \cos \theta}{7 + 3i \cos \theta}$$

$$= \frac{281(49 - 18 \sin \theta \cos \theta + i(21 \cos \theta + 42 \sin \theta))}{49 + 9 \cos^2 \theta}$$

for positive integer

$\text{Im}(A) = 0$

$21 \cos \theta + 42 \sin \theta = 0$

$\tan \theta = \frac{-1}{2}; \sin 2\theta = \frac{-4}{5}, \cos^2 \theta = \frac{4}{5}$

$\text{Re}(A) = \frac{281(49 - 9 \sin 2\theta)}{49 + 9 \cos^2 \theta}$

$$= \frac{281 \left(49 - 9 \times \frac{-4}{5} \right)}{49 + 9 \times \frac{4}{5}} = 281 \text{ (+ve integer)}$$

2. **Ans. (B)**

Sol. $\because |z|^3 + 2z^2 + 4\bar{z} - 8 = 0 \quad \dots (1)$

Take conjugate both sides

$\Rightarrow |z|^3 + 2\bar{z}^2 + 4z - 8 = 0 \quad \dots (2)$

By (1) - (2)

$\Rightarrow 2(z^2 - \bar{z}^2) + 4(\bar{z} - z) = 0$

$\Rightarrow \boxed{z + \bar{z} = 2} \quad \dots (3)$

$\Rightarrow |z + \bar{z}| = 2 \quad \dots (4)$

Let $z = x + iy$

$\therefore \boxed{x = 1} \quad \therefore z = 1 + iy$

Put in (1)

$\Rightarrow (1 + y^2)^{3/2} + 2(1 - y^2 + 2iy) + 4(1 - iy) - 8 = 0$

$\Rightarrow (1 + y^2)^{3/2} = 2(1 + y^2)$

$\Rightarrow \sqrt{1 + y^2} = 2 = |z|$

Also $\boxed{y = \pm\sqrt{3}}$

$\therefore z = 1 \pm i\sqrt{3}$

$\Rightarrow z - \bar{z} = \pm 2i\sqrt{3}$

$\Rightarrow |z - \bar{z}| = 2\sqrt{3}$

$\Rightarrow |z - \bar{z}|^2 = 12$

Now $z + 1 = 2 + i\sqrt{3}$

$|z + 1|^2 = 4 + 3 = 7$

$\therefore P \rightarrow 2; Q \rightarrow 1; R \rightarrow 3; S \rightarrow 5$

\therefore Option (B) is correct.

3. **Ans. (512)**

Sol. $z^8 - 2^8 = (z - 2)(z - \alpha)(z - \alpha^2) \dots (z - \alpha^7)$

Put $z = 2e^{i\theta}$

$2^8(e^{i8\theta} - 1) = (2e^{i\theta} - 2)(2e^{i\theta} - \alpha) \dots (2e^{i\theta} - \alpha^7)$

Take mod

$2^8 |e^{i8\theta} - 1| = PA_1 PA_2 \dots PA_8$

$2^8 |2\sin 4\theta| = PA_1 PA_2 \dots PA_8$

$(PA_1 \cdot PA_2 \dots PA_8)_{\max} = 512$

4. **Ans. (0.50)**

Sol. Given that

$z \neq \bar{z}$

Let $\alpha = \frac{2 + 3z + 4z^2}{2 - 3z + 4z^2} = \frac{(2 - 3z + 4z^2) + 6z}{2 - 3z + 4z^2}$

$\therefore \alpha = 1 + \frac{6z}{2 - 3z + 4z^2}$

If α is a real number, then

$\alpha = \bar{\alpha}$

$\Rightarrow \frac{z}{2 - 3z + 4z^2} = \frac{\bar{z}}{2 - 3\bar{z} + 4\bar{z}^2}$

$\therefore 2(z - \bar{z}) = 4z\bar{z}(z - \bar{z})$

$\Rightarrow (z - \bar{z})(2 - 4z\bar{z}) = 0$

As $z \neq \bar{z}$ (Given)

$\Rightarrow z\bar{z} = \frac{2}{4} = \frac{1}{2}$

$\Rightarrow |z|^2 = 0.50$

5. **Ans. (4.00)**

Sol. Given ,

$\bar{z} - z^2 = i(\bar{z} + z^2)$

$\Rightarrow (1 - i)\bar{z} = (1 + i)z^2$

$\Rightarrow \frac{(1 - i)}{(1 + i)}\bar{z} = z^2$

$\Rightarrow \left(-\frac{2i}{2}\right)\bar{z} = z^2$

$$\therefore z^2 = -i\bar{z}$$

Let $z = x + iy$,

$$\therefore (x^2 - y^2) + i(2xy) = -i(x - iy)$$

$$\text{so, } x^2 - y^2 + y = 0 \quad \dots(1)$$

$$\text{and } (2y + 1)x = 0 \quad \dots(2)$$

$$\Rightarrow x = 0 \text{ or } y = -\frac{1}{2}$$

Case I : When $x = 0$

$$\therefore (1) \Rightarrow y(1 - y) = 0 \Rightarrow y = 0, 1$$

$$\therefore (0, 0), (0, 1)$$

Case II : When $y = -\frac{1}{2}$

$$\therefore (1) \Rightarrow x^2 - \frac{1}{4} - \frac{1}{2} = 0 \Rightarrow x^2 = \frac{3}{4}$$

$$\Rightarrow x = \pm \frac{\sqrt{3}}{2}$$

$$\therefore \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right), \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$$

\Rightarrow Number of distinct 'z' is equal to 4.

6. **Ans. (A)**

Sol. Let $(\bar{z})^2 + \frac{1}{z^2} = m + in$, $m, n \in \mathbb{Z}$

$$(\bar{z})^2 + \frac{\bar{z}^2}{|z|^4} = m + in$$

$$\Rightarrow (x^2 - y^2) \left(1 + \frac{1}{|z|^4}\right) = m \quad \dots(1)$$

$$\& -2xy \left(1 + \frac{1}{|z|^4}\right) = n \quad \dots(2)$$

Equation (1)² + (2)²

$$\left(1 + \frac{1}{|z|^4}\right)^2 [(x^2 + y^2)^2] = m^2 + n^2$$

$$\left(1 + \frac{1}{|z|^4}\right)^2 (|z|^4) = m^2 + n^2$$

$$\Rightarrow |z|^4 + \frac{1}{|z|^4} + 2 = m^2 + n^2$$

Now for option (A)

$$|z|^4 = \frac{43 + 3\sqrt{205}}{2}$$

$$\Rightarrow m^2 + n^2 = 45$$

$$\Rightarrow m = \pm 6, n = \pm 3$$

Option (B)

$$|z|^4 + \frac{1}{|z|^4} + 2 = \frac{7 + \sqrt{33}}{4} + \frac{7 - \sqrt{33}}{4} + 2$$

$$= \frac{7}{2} + 2 = \frac{11}{2}$$

Option (C)

$$|z|^4 + \frac{1}{|z|^4} + 2 = \frac{9 + \sqrt{65}}{4} + \frac{9 - \sqrt{65}}{4} + 2$$

$$= \frac{9}{2} + 2 = \frac{13}{2}$$

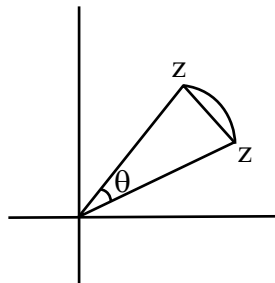
Option (D)

$$|z|^4 + \frac{1}{|z|^4} + 2 = \frac{7 + \sqrt{13}}{6} + \frac{7 - \sqrt{13}}{6} + 2$$

$$= \frac{7}{3} + 2 = \frac{13}{2}$$

7. **Ans. (C)**

Sol.



$$|z_1| = |z_2| = \dots = |z_{10}| = 1$$

$$\text{angle} = \frac{\text{arc}}{\text{rad}}$$

$$\theta_2 = \text{arc}(z_1 z_2) > |z_2 - z_1|$$

$$P : |z_2 - z_1| + \dots + |z_1 - z_{10}| \leq \theta_1 + \theta_2 + \dots + \theta_{10}$$

$$\Rightarrow |z_2 - z_1| + \dots + |z_1 - z_{10}| \leq 2\pi \text{ P is true}$$

$$z_1^2 = e^{i2\theta_1}, z_k^2 = z_{k-1}^2 \cdot e^{i2\theta_k}$$

Let $2\theta_k = \alpha_k$

$$z_1^2 = e^{i\alpha_1}, z_k^2 = z_{k-1}^2 \cdot e^{i\alpha_k}$$

$$\alpha_1 + \alpha_2 + \dots + \alpha_k = 4\pi$$

one similar sense

$$|z_1^2 - z_2^2| + \dots + |z_1^2 - z_{10}^2| \leq 4\pi$$

Q is also true

8. Ans. (B, D)

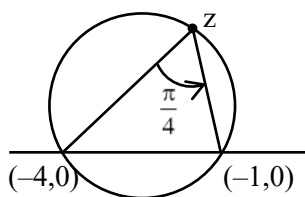
Sol. $\arg\left(\frac{z+\alpha}{z+\beta}\right) = \frac{\pi}{4}$ implies z is

on arc and $(-\alpha, 0)$ & $(-\beta, 0)$ subtend $\frac{\pi}{4}$ on z .

And z lies on $x^2 + y^2 + 5x - 3y + 4 = 0$

So put $y = 0$;

$x^2 + 5x + 4 = 0 \Rightarrow x = -1 ; x = -4$



Now, $\arg\left(\frac{z+\alpha}{z+\beta}\right) = \frac{\pi}{4}$

$\Rightarrow z + \alpha = (z + \beta) \cdot r \cdot e^{i\frac{\pi}{4}}$

So, $z + \beta = z + 4 \Rightarrow \beta = 4$ & $z + \alpha = z + 1$

$\Rightarrow \alpha = 1$

9. Ans. (B, C)

Sol. $|z^2 + z + 1| = 1$

$\Rightarrow \left| \left(z + \frac{1}{2}\right)^2 + \frac{3}{4} \right| = 1$

$\Rightarrow \left| \left(z + \frac{1}{2}\right)^2 + \frac{3}{4} \right| \leq \left| z + \frac{1}{2} \right|^2 + \frac{3}{4}$

$\Rightarrow 1 \leq \left| z + \frac{1}{2} \right|^2 + \frac{3}{4} \Rightarrow \left| \left(z + \frac{1}{2}\right)^2 \right| \geq \frac{1}{4}$

$\Rightarrow \left| z + \frac{1}{2} \right| \geq \frac{1}{2}$

also $|(z^2 + z) + 1| = 1 \geq ||z^2 + z| - 1|$

$\Rightarrow |z^2 + z| - 1 \leq 1$

$\Rightarrow |z^2 + z| \leq 2$

$\Rightarrow ||z^2| - |z|| \leq |z^2 + z| \leq 2$

$\Rightarrow |r^2 - r| \leq 2$

$\Rightarrow r = |z| \leq 2 ; \forall z \in S$

Also we can always find root of the equation

$z^2 + z + 1 = e^{i\theta} ; \forall \theta \in \mathbb{R}$

Hence set 'S' is infinite

10. Ans. (8)

Sol. Let $z = x + iy$

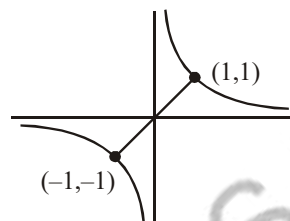
$z^4 - |z|^4 = 4iz^2$

$\Rightarrow z^4 - (z\bar{z})^2 = 4iz^2$

$\Rightarrow z = 0$ or $z^2 - (\bar{z})^2 = 4i$

$\Rightarrow 4ixy = 4i$

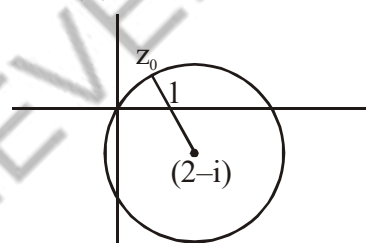
$\Rightarrow xy = 1$



$|z_1 - z_2|_{\min}^2 = 8$

11. Ans. (B)

Sol.



$\arg\left(\frac{4 - (z_0 + \bar{z}_0)}{(z_0 - \bar{z}_0) + 2i}\right)$

$= \arg\left(\frac{4 - 2\operatorname{Re}z_0}{2i\operatorname{Im}z_0 + 2i}\right) = \arg\left(\frac{2 - \operatorname{Re}z_0}{(1 + \operatorname{Im}z_0)i}\right)$

$= \arg\left(-\left(\frac{2 - \operatorname{Re}z_0}{1 + \operatorname{Im}z_0}\right)i\right)$

$= \arg(-ki) ; k > 0$ (as $\operatorname{Re}z_0 < 2$ & $\operatorname{Im}z_0 > 0$)

$= -\frac{\pi}{2}$

12. Ans. (3.00)

Sol. $|a + b\omega + c\omega^2|^2 = (a + b\omega + c\omega^2)(\overline{a + b\omega + c\omega^2})$

$= (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

$= a^2 + b^2 + c^2 - ab - bc - ca$

$= \frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2]$

$\geq \frac{1+1+4}{2} = 3$ (when $a = 1, b = 2, c = 3$)

13. Ans. (A, B, D)

Sol. (A) $\arg(-1 - i) = -\frac{3\pi}{4}$,

(B) $f(t) = \arg(-1 + it)$
 $= \begin{cases} \pi - \tan^{-1}(t), & t \geq 0 \\ -\pi + \tan^{-1}(-t), & t < 0 \end{cases}$

Discontinuous at $t = 0$.

(C) $\arg\left(\frac{z_1}{z_2}\right) - \arg(z_1) + \arg(z_2)$
 $= \arg z_1 - \arg(z_2) + 2n\pi - \arg(z_1) + \arg(z_2)$
 $= 2n\pi$.

(D) $\arg\left(\frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}\right) = \pi$
 $\Rightarrow \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$ is real.
 $\Rightarrow z, z_1, z_2, z_3$ are concyclic.

14. Ans. (A, C, D)

Sol. Given

$sz + t\bar{z} + r = 0$ (1)

on taking conjugate $\bar{s}\bar{z} + \bar{t}z + \bar{r} = 0$ (2)

from (1) and (2) eliminating \bar{z}

$z(|s|^2 - |t|^2) = \bar{r}t - r\bar{s}$

- (A) If $|s| \neq |t|$ then z has unique value
- (B) If $|s| = |t|$ then $\bar{r}t - r\bar{s}$ may or may not be zero so L may be empty set
- (C) locus of z is null set or singleton set or a line in all cases it will intersect given circle at most two points.
- (D) In this case locus of z is a line so L has infinite elements

15. Ans. (A, D)

Sol. $\text{Im}\left(\frac{az + b}{z + 1}\right) = y$ and $z = x + iy$

$\therefore \text{Im}\left(\frac{a(x + iy) + b}{x + iy + 1}\right) = y$

$\Rightarrow \text{Im}\left(\frac{(ax + b + iay)(x + 1 - iy)}{(x + 1)^2 + y^2}\right) = y$

$\Rightarrow -y(ax + b) + ay(x + 1)$

$= y((x + 1)^2 + y^2)$
 $\Rightarrow (a - b)y = y((x + 1)^2 + y^2)$
 $\therefore y \neq 0$ and $a - b = 1$

$\Rightarrow (x + 1)^2 + y^2 = 1$

$\Rightarrow x = -1 \pm \sqrt{1 - y^2}$

16. Ans. (1)

Sol. $z = \omega$

$P = \begin{bmatrix} (-\omega)^r & \omega^{2s} \\ \omega^{2s} & \omega^r \end{bmatrix}, P^2 = -I$

$\Rightarrow P^2 = \begin{bmatrix} \omega^{2r} + \omega^{4s} & \omega^{r+2s}((-1)^r + 1) \\ \omega^{r+2s}((-1)^r + 1) & \omega^{4s} + \omega^{2r} \end{bmatrix} = -I$

$\Rightarrow (-1)^r + 1 = 0 \Rightarrow r$ is odd $\Rightarrow r = 1, 3$

also $\omega^{2r} + \omega^{4s} = -1 \therefore r \neq 3$

by $r = 1 \Rightarrow \omega^2 + \omega^{4s} = -1 \Rightarrow s = 1$

$(r, s) = (1, 1)$

only 1 pair

17. Ans. (A, C, D)

Sol. $x + iy = \frac{1}{a + ibt}$

$x + iy = \frac{a - ibt}{a^2 + b^2t^2}$

Let $a \neq 0$ & $b \neq 0$

$x = \frac{a}{a^2 + b^2t^2}$ (1)

$y = \frac{-bt}{a^2 + b^2t^2}$ (2)

$\frac{y}{x} = \frac{-bt}{a} \Rightarrow t = -\frac{ay}{bx}$

put in (1)

$x \left\{ a^2 + b^2 \cdot \frac{a^2y^2}{b^2x^2} \right\} = a$

$a^2(x^2 + y^2) = ax$

$x^2 + y^2 - \frac{1}{a}x = 0$

$\left(x - \frac{1}{2a}\right)^2 + y^2 = \frac{1}{4a^2}$

⇒ option (A) is correct

for $a \neq 0, b = 0$

$$x + iy = \frac{1}{a}$$

$$x = \frac{1}{a}, y = 0 \Rightarrow z \text{ lies on x-axis}$$

⇒ option (C) is correct

for $a = 0, b \neq 0$

$$x + iy = \frac{1}{ibt}$$

$$y = -\frac{1}{bt}i, x = 0$$

⇒ z lies on y-axis.

⇒ option (D) is correct

18. **Ans.** (A) → (P,Q); (B) → (P,Q);
(C) → (P,Q,S,T); (D) → (Q,T)

Sol. (A) $\left| \frac{\alpha\sqrt{3} + \beta}{2} \right| = \sqrt{3} \Rightarrow \alpha\sqrt{3} + \frac{\alpha - 2}{\sqrt{3}} = \pm 2\sqrt{3}$

$$\Rightarrow \alpha \left(\frac{4}{\sqrt{3}} \right) = \frac{2}{\sqrt{3}} \pm 2\sqrt{3}$$

$$\alpha = 2, -1 \Rightarrow |\alpha| = 1, 2$$

(B) By continuity $-3a - 2 = b + a^2$

By differentiability $-6a = b$

$$a^2 - 3a + 2 = 0 \Rightarrow a = 1, 2$$

(C) $\left((-3 + 2\omega + 3\omega^2)\omega \right)^{4n+3} +$

$$\left((-3 + 2\omega + 3\omega^2)\omega^2 \right)^{4n+3} +$$

$$\left((-3 + 2\omega + 3\omega^2) \right)^{4n+3} = 0$$

$$\Rightarrow (-3 + 2\omega + 3\omega^2)^{4n+3} [\omega^{4n+3} + \omega^{8n+6} + 1] = 0$$

$$\Rightarrow \omega^n + \omega^{2n} + 1 = 0$$

⇒ n is not a multiple of 3.

(D) $\frac{2ab}{a+b} = 4, 2(5-a) = b-5$

$$b = 15 - 2a$$

$$2a(15 - 2a) = 4(15 - a) \Rightarrow 15a - 2a^2$$

$$= 30 - 2a$$

$$2a^2 - 17a + 30 = 0$$

$$\Rightarrow 2a^2 - 12a - 5a + 30 = 0$$

$$2a(a - 6) - 5(a - 6) = 0$$

$$a = \frac{5}{2}, 6$$

$$\Rightarrow |q - a| = |10 - 2a| = 5 \text{ or } 2$$

19. **Ans.** (4)

Sol. α_k are vertices of 14 sided regular polygon

$|\alpha_{k+1} - \alpha_k|$ length of a side of the regular polygon

$|\alpha_{4k-1} - \alpha_{4k-2}|$ length of a side of the regular polygon

$$\Rightarrow \frac{12(S)}{3(S)} = 4$$

20. **Ans.** (D)

Sol. Let $p(x) = ax^2 + b + c$

$$p(x) = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

so $b = 0$ as roots are purely imaginary

so equation will be $ax^2 + c = 0$

Now $p(p(x)) = 0$

$$\Rightarrow ap^2(x) + c = 0 \Rightarrow p(x) = \pm \sqrt{-\frac{c}{a}}$$

$$ax^2 + c = \pm \sqrt{-\frac{c}{a}} \Rightarrow x \notin \mathbb{R}$$

if $x = i\beta$ then

$$-a\beta^2 + c = \pm \sqrt{-\frac{c}{a}} \text{ not possible}$$

(real) (imaginary)

So, neither real nor purely imaginary roots.

21. **Ans.** (C)

Sol. (P) $e^{\frac{i2k\pi}{10}} \cdot e^{\frac{i2j\pi}{10}} = 1$

$$e^{i\frac{\pi}{10} \times 2(k+j)} = 1$$

$$\frac{\pi}{10} (2(k+j)) = 2n\pi$$

$$(k+j) = 10$$

Possible

(Q) $e^{\frac{i2\pi}{10}} \cdot z = e^{\frac{i2\pi k}{10}}$

$$z = \frac{e^{\frac{i2\pi k}{10}}}{e^{\frac{i2\pi}{10}}} \text{ is possible}$$

(R) $z^{10} - 1 = (z - 1)(z - z_1)(z - z_2)\dots(z - z_9)$

put $z = 1$

$$\lim_{z \rightarrow 1} \frac{z^{10} - 1}{(z - 1)} = (1 - z_1)(1 - z_2)\dots(1 - z_9)$$

$$\lim_{z \rightarrow 1} \frac{10z^9}{1} = (1 - z_1)(1 - z_2)\dots(1 - z_9)$$

$$= |(1 - z_1)(1 - z_2)\dots(1 - z_9)| = 10$$

(S) $1 + \cos \frac{2\pi}{10} + \cos \frac{4\pi}{10} + \dots + \cos \frac{18\pi}{10} = 0$

since they are sum of ten, tenth roots of unity

$$\sum_{k=1}^9 \cos\left(\frac{2k\pi}{10}\right) = -1$$

$$1 + 1 = 2$$