DIFFERENTIAL EQUATION

1. Let $f:[(1,\infty)\to\mathbb{R}$ be a differentiable function such that $f(1)=\frac{1}{3}$ and $3\int_1^x f(t)\,dt = xf(x)-\frac{x^3}{3}, x\in[1,\infty)$.

Let e denote the base of the natural logarithm. Then the value of f(e) is **[JEE(Advanced) 2023]**

(A) $\frac{e^2+4}{3}$ (B) $\frac{\log_e 4+e}{3}$

(C) $\frac{4e^2}{3}$ (D) $\frac{e^2-4}{3}$

2. For $x \in \mathbb{R}$, let y(x) be a solution of the differential equation **[JEE(Advanced) 2023]**

$(x^2-5)\frac{dy}{dx} - 2xy = -2x(x^2-5)^2$ such that y(2) = 7.

Then the maximum value of the function y(x) is

3. If y(x) is the solution of the differential equation

$$xdy - (y^2 - 4y)dx = 0 \text{ for } x > 0,\ y(1) = 2,$$

and the slope of the curve y = y(x) is never zero, then the value of $10y(\sqrt{2})$ is ______.

[JEE(Advanced) 2022]

4. For $x \in \mathbb{R}$, let the function y(x) be the solution of the differential equation

$$\frac{dy}{dx}+12y=\cos\left(\frac{\pi}{12}x\right),\ y(0)=0.$$

Then, which of the following statements is/are TRUE? **[JEE(Advanced) 2022]**

(A) y(x) is an increasing function

(B) y(x) is a decreasing function

(C) There exists a real number β such that the line y = β intersects the curve y = y(x) at infinitely many points

(D) y(x) is a periodic function

5. For any real numbers α and β, let $y_{\alpha,\beta}(x)$, $x \in \mathbb{R}$, be the solution of the differential equation

$$\frac{dy}{dx} + \alpha y = xe^{\beta x},\ y(1) = 1$$

Let $S = \{y_{\alpha,\beta}(x) : \alpha, \beta \in \mathbb{R}\}$. Then which of the following functions belong(s) to the set S?

[JEE(Advanced) 2021]

(A) $f(x) = \frac{x^2}{2}e^{-x} + \left(e-\frac{1}{2}\right)e^{-x}$ (B) $f(x) = -\frac{x^2}{2}e^{-x} + \left(e+\frac{1}{2}\right)e^{-x}$

(C) $f(x) = \frac{e^x}{2}\left(x-\frac{1}{2}\right)+\left(e-\frac{e^2}{4}\right)e^{-x}$ (D) $f(x) = \frac{e^x}{2}\left(\frac{1}{2}-x\right)+\left(e+\frac{e^2}{4}\right)e^{-x}$

6. Let Γ denote a curve y = y(x) which is in the first quadrant and let the point (1, 0) lie on it. Let the tangent to Γ at a point P intersect the y-axis at Y_P. If PY_P has length 1 for each point P on Γ, then which of the following options is/are correct ? **[JEE(Advanced) 2019]**

(A) $y=\log_e\left(\frac{1+\sqrt{1-x^2}}{x}\right)-\sqrt{1-x^2}$

(B) $xy'-\sqrt{1-x^2}=0$

(C) $y=-\log_e\left(\frac{1+\sqrt{1-x^2}}{x}\right)+\sqrt{1-x^2}$

(D) $xy'+\sqrt{1-x^2}=0$

7. Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ be two non-constant differentiable functions.

If $f'(x) = (e^{(f(x)-g(x))})g'(x)$ for all $x \in \mathbb{R}$, and $f(1) = g(2) = 1$, then which of the following statement(s) is (are) TRUE? **[JEE(Advanced) 2018]**

(A) $f(2) < 1 - \log_e 2$

(B) $f(2) > 1 - \log_e 2$

(C) $g(1) > 1 - \log_e 2$

(D) $g(1) < 1 - \log_e 2$

8. Let $f:(0,\pi)\to\mathbb{R}$ be a twice differentiable function such that

$$\lim_{t\to x}\frac{f(x)\sin t-f(t)\sin x}{t-x}=\sin^2 x \text{ for all } x\in(0,\pi).$$

If $f\left(\frac{\pi}{6}\right)=-\frac{\pi}{12}$, then which of the following statement(s) is (are) TRUE ? **[JEE(Advanced) 2018]**

(A) $f\left(\frac{\pi}{4}\right)=\frac{\pi}{4\sqrt{2}}$

(B) $f(x)<\frac{x^4}{6}-x^2$ for all $x\in(0,\pi)$

(C) There exists $\alpha\in(0,\pi)$ such that $f'(\alpha)=0$

(D) $f''\left(\frac{\pi}{2}\right)+f\left(\frac{\pi}{2}\right)=0$

9. Let $f:\mathbb{R}\to\mathbb{R}$ be a differentiable function with $f(0) = 0$. If $y = f(x)$ satisfies the differential equation

$$\frac{dy}{dx}=(2+5y)(5y-2),$$

then the value of $\lim_{x\to-\infty} f(x)$ is _____ . **[JEE(Advanced) 2018]**

10. If y = y(x) satisfies the differential equation

$$8\sqrt{x}\left(\sqrt{9+\sqrt{x}}\right)dy=\left(\sqrt{4+\sqrt{9+\sqrt{x}}}\right)^{-1}dx,\quad x>0$$

and $y(0)=\sqrt{7}$, then y(256) = **[JEE(Advanced) 2017]**

(A) 80 (B) 3 (C) 16 (D) 9

11. If $f:\mathbb{R}\to\mathbb{R}$ is a differentiable function such that $f'(x) > 2f(x)$ for all $x\in\mathbb{R}$, and $f(0) = 1$, then **[JEE(Advanced) 2017]**

(A) $f(x) > e^{2x}$ in $(0,\infty)$

(B) $f(x)$ is decreasing in $(0,\infty)$

(C) $f(x)$ is increasing in $(0,\infty)$

(D) $f'(x) < e^{2x}$ in $(0,\infty)$

12. A solution curve of the differential equation $(x^2 + xy + 4x + 2y + 4)\dfrac{dy}{dx} - y^2 = 0, x > 0$, passes through the point (1,3). The the solution curve- **[JEE(Advanced) 2016]**

(A) intersects y = x + 2 exactly at one point

(B) intersects y = x + 2 exactly at two points

(C) intersects $y = (x + 2)^2$

(D) does NOT intersect $y = (x + 3)^2$

13. Let $f : (0, \infty) \to \mathbb{R}$ be a differentiable function such that $f'(x) = 2 - \dfrac{f(x)}{x}$ for all $x \in (0,\infty)$ and $f(1) \neq 1$. Then **[JEE(Advanced) 2016]**

(A) $\lim\limits_{x\to 0^+} f'\left(\dfrac{1}{x}\right) = 1$

(B) $\lim\limits_{x\to 0^+} x f\left(\dfrac{1}{x}\right) = 2$

(C) $\lim\limits_{x\to 0^+} x^2 f'(x) = 0$

(D) $|f(x)| \leq 2$ for all $x \in (0,2)$

14. Let y(x) be a solution of the differential equation $(1 + e^x)y' + ye^x = 1$. If y(0) = 2, then which of the following statements is(are) true ? **[JEE(Advanced) 2015]**

(A) y(–4) = 0

(B) y(–2) = 0

(C) y(x) has a critical point in the interval (–1,0)

(D) y(x) has no critical point in the interval (–1,0)

15. Consider the family of all circles whose centers lie on the straight line y = x. If this family of circles is represented by the differential equation Py'' + Qy' + 1 = 0, where P,Q are functions of x,y and y' (here $y' = \dfrac{dy}{dx}, y'' = \dfrac{d^2y}{dx^2}$), then which of the following statements is (are) true? **[JEE(Advanced) 2015]**

(A) P = y + x

(B) P = y – x

(C) $P + Q = 1 - x + y + y' + (y')^2$

(D) $P - Q = x + y - y' - (y')^2$

16. The function y = f(x) is the solution of the differential equation $\dfrac{dy}{dx} + \dfrac{xy}{x^2 - 1} = \dfrac{x^4 + 2x}{\sqrt{1 - x^2}}$ in (–1, 1) satisfying $f(0) = 0$. Then $\displaystyle\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} f(x)dx$ is **[JEE(Advanced) 2014]**

(A) $\dfrac{\pi}{3} - \dfrac{\sqrt{3}}{2}$

(B) $\dfrac{\pi}{3} - \dfrac{\sqrt{3}}{4}$

(C) $\dfrac{\pi}{6} - \dfrac{\sqrt{3}}{4}$

(D) $\dfrac{\pi}{6} - \dfrac{\sqrt{3}}{2}$

SOLUTIONS

1. Ans. (C)

Sol. Diff. wr.t 'x'

$3f(x) = f(x) + xf'(x) - x^2$

$\frac{dy}{dx} - \left(\frac{2}{x}\right)y = x$

$IF = e^{-2\ell nx} = \frac{1}{x^2}$

$y\left(\frac{1}{x^2}\right) = \int x.\frac{1}{x^2}dx$

$y = x^2\ell nx + cx^2$

$\therefore\ y(1) = \frac{1}{3} \Rightarrow c = \frac{1}{3}$

$y(e) = \frac{4e^2}{3}$

2. Ans. (16)

Sol. $\frac{dy}{dx} - \frac{2x}{x^2-5}y = -2x(x^2-5)$

$IF = e^{-\int\frac{2x}{x^2-5}dx} = \frac{1}{(x^2-5)}$

$y.\frac{1}{x^2-5} = \int -2x.dx + c$

$\Rightarrow \frac{y}{x^2-5} = -x^2 + c$

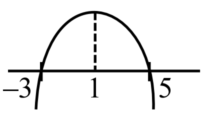
$x = 2, y = 7$

$\frac{7}{-1} = -4 + c \Rightarrow c = -3$

$y = -(x^2-5)(x^2+3)$

put $x^2 = t \geq 0$

$y = -(t-5)(t+3)$



$y_{max} = 16$ when $x^2 = 1$

$y_{max} = 16$

3. Ans. (8)

Sol. $xdy - (y^2 - 4y)dx = 0, x > 0$

$\int\frac{dy}{y^2-4y} = \int\frac{dx}{x}$

$\int\left(\frac{1}{y-4} - \frac{1}{y}\right)dy = 4\int\frac{dx}{x}$

$\log_e|y-4| - \log_e|y| = 4\log_e x + \log_e c$

$\frac{|y-4|}{|y|} = cx^4 \xrightarrow{(1,2)} c = 1$

$|y-4| = |y|x^4$

C-1 and C-2

$y - 4 = yx^4$ $\quad$ $y - 4 = -yx^4$

$y = \frac{4}{1-x^4}$ $\quad$ $y = \frac{4}{1+x^4}$

y(1) = ND (rejected) $\quad$ y(1) = 2

$y(\sqrt{2}) = \frac{4}{5} \Rightarrow 10y(\sqrt{2}) = 8$

4. Ans. (C)

Sol. $\frac{dy}{dx} + 12y = \cos\left(\frac{\pi}{12}x\right)$

Linear D.E.

$I.F. = e^{\int 12.dx} = e^{12x}$

Solution of DE

$y.e^{12x} = \int e^{12x}.\cos\left(\frac{\pi}{12}x\right)dx$

$y.e^{12x} = \frac{e^{12x}}{(12)^2 + \left(\frac{\pi}{12}\right)^2}\left(12\cos\frac{\pi}{12}x + \frac{\pi}{12}\sin\frac{\pi}{12}x\right) + C$

$\Rightarrow y = \frac{(12)}{(12)^4 + \pi^2}\left((12)^2\cos\left(\frac{\pi x}{12}\right) + \pi\sin\left(\frac{\pi x}{12}\right)\right) + \frac{C}{e^{12x}}$

$y = \frac{(12)}{(12)^4 + \pi^2}\left((12)^2\cos\left(\frac{\pi x}{12}\right) + \pi\sin\left(\frac{\pi x}{12}\right)\right) + \frac{C}{e^{12x}}$

Given y(0) = 0

$\Rightarrow 0 = \frac{12}{12^4 + \pi^2}(12^2 + 0) + C$

$\Rightarrow C = \frac{-12^3}{12^4 + \pi^2}$

$\therefore\ y = \frac{12}{12^4 + \pi^2}\left[(12)^2\cos\left(\frac{\pi x}{12}\right) + \pi\sin\left(\frac{\pi x}{12}\right) - 12^2.e^{-12x}\right]$

Now,

$$\frac{dy}{dx}=\frac{12}{12^4+\pi^2}\left[\underbrace{-12\pi\sin\left(\frac{\pi x}{12}\right)+\frac{\pi^2}{12}\cos\left(\frac{\pi x}{12}\right)}_{\text{min. value}}+12^3e^{-12x}\right]$$

$$\left(-\sqrt{144\pi^2+\frac{\pi^4}{144}}=-12\pi\sqrt{1+\frac{\pi^2}{12^4}}\right)$$

$\Rightarrow \frac{dy}{dx}>0\ \forall\ x\le 0$ & may be negative/positive for $x>0$

So, f(x) is neither increasing nor decreasing

For some $\beta\in R$, $y=\beta$ intersects $y=f(x)$ at infinitely many points

So, option (C) is correct

5. Ans. (A, C)

Sol. Integrating factor = $e^{\alpha x}$

So $ye^{\alpha x}=\int xe^{(\alpha+\beta)x}dx$

Case-I

If $\alpha+\beta=0$ $ye^{\alpha x}=\frac{x^2}{2}+c$

It passes through (1, 1) $\Rightarrow C=e^{\alpha}-\frac{1}{2}$

So $ye^{\alpha x}=\frac{x^2-1}{2}+e^{\alpha}$

for $\alpha=1$

$$y=\frac{x^2}{2}e^{-x}+\left(e-\frac{1}{2}\right)e^{-x}\to(A)$$

Case-II

If $\alpha+\beta\ne 0$

$$ye^{\alpha x}=\int\frac{x.e^{(\alpha+\beta)x}}{\alpha+\beta}-\frac{1}{\alpha+\beta}e^{(\alpha+\beta)x}\,dx$$

$$\Rightarrow ye^{\alpha x}=\frac{xe^{(\alpha+\beta)x}}{\alpha+\beta}-\frac{e^{(\alpha+\beta)x}}{(\alpha+\beta)^2}+c$$

$$\Rightarrow \text{So } c=e^{\alpha}-\frac{e^{\alpha+\beta}}{\alpha+\beta}+\frac{e^{\alpha+\beta}}{(\alpha+\beta)^2}$$

$$y=\frac{e^{\beta x}}{(\alpha+\beta)^2}\left((\alpha+\beta)x-1\right)+e^{-\alpha x}\left(e^{x}-\frac{e^{\alpha+\beta}}{\alpha+\beta}+\frac{e^{\alpha+\beta}}{(\alpha+\beta)^2}\right)$$

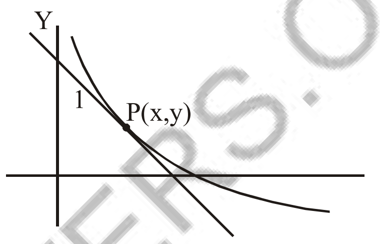
If $\alpha=\beta=1$

$$y=\frac{e^x}{4}(2x-1)+e^{-x}\left(e-\frac{e^2}{2}+\frac{e^2}{4}\right)$$

$$y=\frac{e^x}{2}\left(x-\frac{1}{2}\right)+e^{-x}\left(e-\frac{e^2}{4}\right)\to(c)$$

6. Ans. (A, D)

Sol.



$Y-y=y'(X-x)$

So, $Y_P=(0,\ y-xy')$

So, $x^2+(xy')^2=1\Rightarrow\frac{dy}{dx}=-\sqrt{\frac{1-x^2}{x^2}}$

[$\frac{dy}{dx}$ can not be positive i.e. $f(x)$ can not be increasing in first quadrant, for $x\in(0,1)$]

Hence, $\int dy=-\int\frac{\sqrt{1-x^2}}{x}dx$

$\Rightarrow y=-\int\frac{\cos^2\theta\,d\theta}{\sin\theta}$; put $x=\sin\theta$

$\Rightarrow y=-\int\text{cosec}\theta\,d\theta+\int\sin\theta\,d\theta$

$\Rightarrow y=ln(\text{cosec}\theta+\cot\theta)-\cos\theta+C$

$$\Rightarrow y=\ell n\left(\frac{1+\sqrt{1-x^2}}{x}\right)-\sqrt{1-x^2}+C$$

$$\Rightarrow y=\ell n\left(\frac{1+\sqrt{1-x^2}}{x}\right)-\sqrt{1-x^2}$$

(as y(1) = 0)

7. Ans. (B, C)

Sol. $f'(x) = e^{(f(x) - g(x))} g'(x) \ \forall \ x \in \mathbb{R}$

$\Rightarrow \quad e^{-f(x)}. f'(x) - e^{-g(x)}g'(x) = 0$

$\Rightarrow \quad \int\left(e^{-f(x)}f'(x) - e^{-g(x)}.g'(x)\right)dx = C$

$\Rightarrow \quad -e^{-f(x)} + e^{-g(x)} = C$

$\Rightarrow \quad -e^{-f(1)} + e^{-g(1)} = -e^{-f(2)} + e^{-g(2)}$

$\Rightarrow \quad -\frac{1}{e} + e^{-g(1)} = -e^{-f(2)} + \frac{1}{e}$

$\Rightarrow \quad e^{-f(2)} + e^{-g(1)} = \frac{2}{e}$

$\therefore \quad e^{-f(2)} < \frac{2}{e}$ and $e^{-g(1)} < \frac{2}{e}$

$\Rightarrow \quad -f(2) < \ln 2 - 1$ and $-g(1) < \ln 2 - 1$

$\Rightarrow \quad f(2) > 1 - \ln 2$ and $g(1) > 1 - \ln 2$

8. Ans. (B, C, D)

Sol. $\lim_{t \to x} \frac{f(x)\sin t - f(t)\sin x}{t - x} = \sin^2 x$

by using L'Hopital

$\lim_{t \to x} \frac{f(x)\cos t - f'(t)\sin x}{1} = \sin^2 x$

$\Rightarrow$ f(x)cosx –f'(x)sinx = $\sin^2 x$

$\Rightarrow -\left(\frac{f'(x)\sin x - f(x)\cos x}{\sin^2 x}\right) = 1$

$\Rightarrow -d\left(\frac{f(x)}{\sin x}\right) = 1$

$\Rightarrow \frac{f(x)}{\sin x} = -x + c$

Put $x = \frac{\pi}{6}$ & $f\left(\frac{\pi}{6}\right) = -\frac{\pi}{12}$

$\therefore$ c = 0 $\Rightarrow$ f(x) = –xsinx

(A) $f\left(\frac{\pi}{4}\right) = \frac{-\pi}{4}\frac{1}{\sqrt{2}}$

(B) f(x) = –x sinx

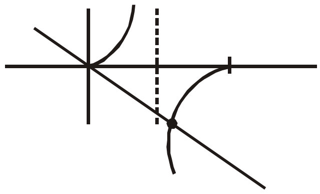
as $\sin x > x - \frac{x^3}{6}$, $-x \sin x < -x^2 + \frac{x^4}{6}$

$\therefore$ $f(x) < -x^2 + \frac{x^4}{6} \ \forall \ x \in (0, \pi)$

(C) f'(x) = –sinx – x cosx

f'(x) = 0 $\Rightarrow$ tanx = –x

$\Rightarrow$ there exist $\alpha \in (0, \pi)$ for which $f'(\alpha) = 0$



(D) f''(x) = –2cosx + xsinx

$f''\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$, $f\left(\frac{\pi}{2}\right) = -\frac{\pi}{2}$

$f''\left(\frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) = 0$

9. Ans. (0.4)

Sol. $\frac{dy}{dx} = 25y^2 - 4$

So, $\frac{dy}{25y^2 - 4} = dx$

Integrating, $\frac{1}{25} \times \frac{1}{2 \times \frac{2}{5}} \ln\left|\frac{y - \frac{2}{5}}{y + \frac{2}{5}}\right| = x + c$

$\Rightarrow \ \ln\left|\frac{5y - 2}{5y + 2}\right| = 20(x + c)$

Now, c = 0 as f(0) = 0

Hence $\left|\frac{5y - 2}{5y + 2}\right| = e^{(20x)}$

$\operatorname{let}_{x \to -\infty} \left|\frac{5f(x) - 2}{5f(x) + 2}\right| = \operatorname{let}_{x \to -\infty} e^{(20x)}$

Now, RHS = 0 $\Rightarrow$ $\operatorname{let}_{x \to -\infty} (5f(x) - 2) = 0$

$\Rightarrow \ \operatorname{let}_{x \to -\infty} f(x) = \frac{2}{5}$

10. Ans. (B)

Sol. $y = \frac{1}{8}\int \frac{dx}{\sqrt{4 + \sqrt{9 + x}}.\sqrt{x}.\sqrt{9 + \sqrt{x}}}$

put $\sqrt{9 + \sqrt{x}} = t \Rightarrow \frac{dx}{\sqrt{x}.\sqrt{9 + \sqrt{x}}} = 4dt$

$\therefore y = \frac{4}{8}\int \frac{dt}{\sqrt{4+t}}$

$\Rightarrow y = \sqrt{4+t} + C$

$\Rightarrow y(x) = \sqrt{4+\sqrt{9+\sqrt{x}}} + C$

at $x = 0$: $y(0) = \sqrt{7}$ $\Rightarrow$ $C = 0$

$\therefore y(x) = \sqrt{4+\sqrt{9+\sqrt{x}}}$

$\Rightarrow$ $y(256) = 3$

11. Ans. (A, C)

Sol. Given that,

$f'(x) > 2f(x)\ \forall\ x \in R$

$\Rightarrow$ $f'(x) - 2f(x) > 0\ \ \forall\ x \in R$

$\therefore$ $e^{-2x}(f'(x) - 2f(x)) > 0\ \forall\ x \in R$

$\Rightarrow$ $\frac{d}{dx}\left(e^{-2x}f(x)\right) > 0\ \forall\ x \in R$

Let $g(x) = e^{-2x}f(x)$

Now, $g'(x) > 0\ \forall\ x \in R$

$\Rightarrow$ $g(x)$ is strictly increasing $\forall\ x \in R$

Also, $g(0) = 1$

$\therefore$ $\forall\ x > 0$

$\Rightarrow$ $g(x) > g(0) = 1$

$\therefore$ $e^{-2x}.f(x) > 1\ \forall\ x \in (0, \infty) \Rightarrow f(x) > e^{2x}\ \forall\ x \in (0,\infty)$

$\therefore$ option (A) is correct

As, $f'(x) > 2\ f(x) > 2e^{2x} > 2\ \forall\ x \in (0, \infty)$

$\Rightarrow$ $f(x)$ is strictly increasing on $x \in (0, \infty)$

$\Rightarrow$ option (C) is correct

As, we have proved above that

$f'(x) > 2.e^{2x}\ \forall\ x \in (0, \infty)$

$\Rightarrow$ option (D) is incorrect

$\therefore$ options (A) and (C) are correct

12. Ans. (A, D)

Sol. $(x^2 + xy + 4x + 2y + 4)\frac{dy}{dx} - y^2 = 0$

$((x+2)^2 + y(x+2))\frac{dy}{dx} = y^2$

Let $x + 2 = X$, $y = Y$

$(X)(X+Y)\frac{dY}{dX} = Y^2$

$-X^2dY = XYdY - Y^2dX$

$-X^2dY = Y(XdY - YdX)$

$-\frac{dY}{Y} = \frac{XdY - YdX}{X^2}$

$-\ell n\,|Y| = \left(\frac{Y}{X}\right) + C$

$-\ \ell n\,|y| = \frac{y}{x+2} + C$

$\because$ it is passing through (1, 3)

$-\ell n3 = 1 + C$

$C = -1 - \ell n3$

$\therefore$ curve $\frac{y}{x+2} + \ell n\,|y| - 1 - \ell n3 = 0, x > 0$ ....(i)

put $y = x + 2$ in equation (i)

then $\frac{x+2}{x+2} + \ell n\,|x+2| - 1 - \ell n3 = 0$

$x = 1, -5$(reject)

$\therefore$ curve intersect $y = x + 2$ at point (1, 3)

for option (C), put $y = (x + 2)^2$, we will get

$x + 2 + 2\ell n(x + 2) = 1 + \ell n3$

Clearly left hand side is an increasing function

Hence, it is always greater than $2 + 2\ell n2$

therefore no solution

for option (C) put $y = (x + 3)^2$ in equation (i)

$\frac{(x+3)^2}{x+2} + \ell n(x+3)^2 - 1 - \ell n3 = 0$

$\frac{(x+3)^2}{x+2} + \ell n\frac{(x+3)^2}{3} - 1 = 0$

$\because$ $x > 0 \Rightarrow x + 3 > x + 2$ and $x + 3 > 3$

So $\frac{(x+3)^2}{x+2} + \ell n\frac{(x+3)^2}{3} > 1$

$\therefore$ $\frac{(x+3)^2}{x+2} + \ell n\frac{(x+3)^2}{3} - 1 = 0$

has no solution

$\Rightarrow$ curve $y = (x + 3)^2$ does not intersect

13. Ans. (A)

Sol. Let $y = f(x)$

$$\frac{dy}{dx} + \frac{y}{x} = 2 \text{ (linear differential equation)}$$

$$\therefore\ y.e^{\int\frac{dx}{x}} = 2\int e^{\int\frac{dx}{x}} = 2\int e^{\int\frac{dx}{x}}dx + c$$

$$\Rightarrow yx = 2\int x dx + c$$

$$\therefore\ yx = x^2 + c$$

$$\Rightarrow f(x) = x + \frac{c}{x} \text{ ; As } f(1) \neq 1 \Rightarrow c \neq 0$$

$$\Rightarrow\ f'(x) = 1 - \frac{c}{x^2}, c \neq 0$$

(A) $\lim\limits_{x\to 0^+} f'\left(\frac{1}{x}\right) = \lim\limits_{x\to 0^+}\left(1 - cx^2\right) = 1$

(B) $\lim\limits_{x\to 0^+} x f\left(\frac{1}{x}\right) = \lim\limits_{x\to 0^+} x\left(\frac{1}{x} + cx\right) = \lim\limits_{x\to 0^+}\left(1 + cx^2\right) = 1$

(C) $\lim\limits_{x\to 0^+} x^2 f'(x) = \lim\limits_{x\to 0^+} x^2\left(1 - \frac{c}{x^2}\right) = \lim\limits_{x\to 0^+}\left(x^2 - c\right) = -c$

(D) $f(x) = x + \frac{c}{x}, c \neq 0$

for $c > 0$

$$\therefore\ \lim_{x\to 0^+} f(x) = \infty$$

$\Rightarrow$ function is not bounded in (0, 2)

14. Ans. (A, C)

Sol. $y' + e^x y' + ye^x = 1$

$\Rightarrow\quad dy + d(e^x y) = dx$

$\Rightarrow\quad y + e^x y = x + c$

$\because\quad y(0) = 2 \quad\Rightarrow\quad c = 4$

$$\Rightarrow\quad y = \frac{x+4}{1+e^x}$$

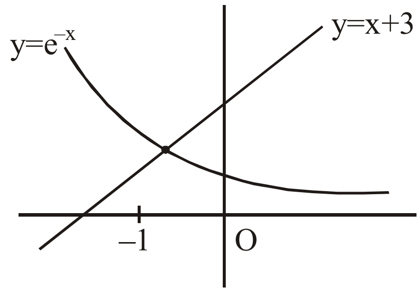
$\therefore\quad y(-4) = 0$

for critical point given

$$\frac{dy}{dx} = \frac{1 - ye^x}{1+e^x} = \frac{1 - \left(\frac{x+4}{1+e^x}\right)e^x}{1+e^x} = \frac{1-(x+3)e^x}{(1+e^x)^2}$$

$\because\quad \frac{dy}{dx} = 0 \quad\Rightarrow\quad e^x(x+3) - 1 = 0$

$\Rightarrow\quad x + 3 = e^{-x}$



y(x) has a critical point in the interval (–1, 0)

15. Ans. (B, C)

Sol. Let Circle

$$x^2 + y^2 - 2ax - 2ay + c = 0$$

On differentiation

$$2x + 2yy' - 2a - 2ay' = 0$$

$$\Rightarrow\quad x + yy' - a(1 + y') = 0$$

$$\Rightarrow\quad a = \frac{x + yy'}{1 + y'}$$

again differentiation

$$\frac{(1 + (y')^2 + yy'')(1 + y') - (x + yy')(y'')}{(1 + y')^2} = 0$$

$$\Rightarrow\quad 1 + y'((y')^2 + y' + 1) + y''(y - x) = 0$$

$\therefore\quad P = y - x$

$\quad\quad Q = 1 + y' + (y')^2$

16. Ans. (B)

Sol. $\left(\frac{dy}{dx} - \frac{xy}{(1-x^2)}\right)\sqrt{1-x^2} = x^4 + 2x$

$$\Rightarrow\quad \sqrt{1-x^2}\,dy + d\left(\sqrt{1-x^2}\right)y = (x^4 + 2x)dx$$

$$\Rightarrow\quad y\sqrt{1-x^2} = \frac{x^5}{5} + x^2 + c$$

by (0, 0) c = 0

$$y = \frac{x^5}{5\sqrt{1-x^2}} + \frac{x^2}{\sqrt{1-x^2}}$$

$$= \int_{-\sqrt{3}/2}^{\sqrt{3}/2}\left(\frac{x^5}{5\sqrt{1-x^2}} + \frac{x^2}{\sqrt{1-x^2}}\right)dx$$

$$= 2\int_0^{\sqrt{3}/2}\frac{x^2}{\sqrt{1-x^2}}dx \qquad \text{(put } x = \sin\theta)$$

$$= 2\int_0^{\pi/3}\sin^2\theta\, d\theta = \int_0^{\pi/3}(1 - \cos 2\theta)d\theta$$

$$= \left(\theta - \frac{\sin 2\theta}{2}\right)_0^{\pi/3} = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$$